Variational Formulation & the Galerkin Method

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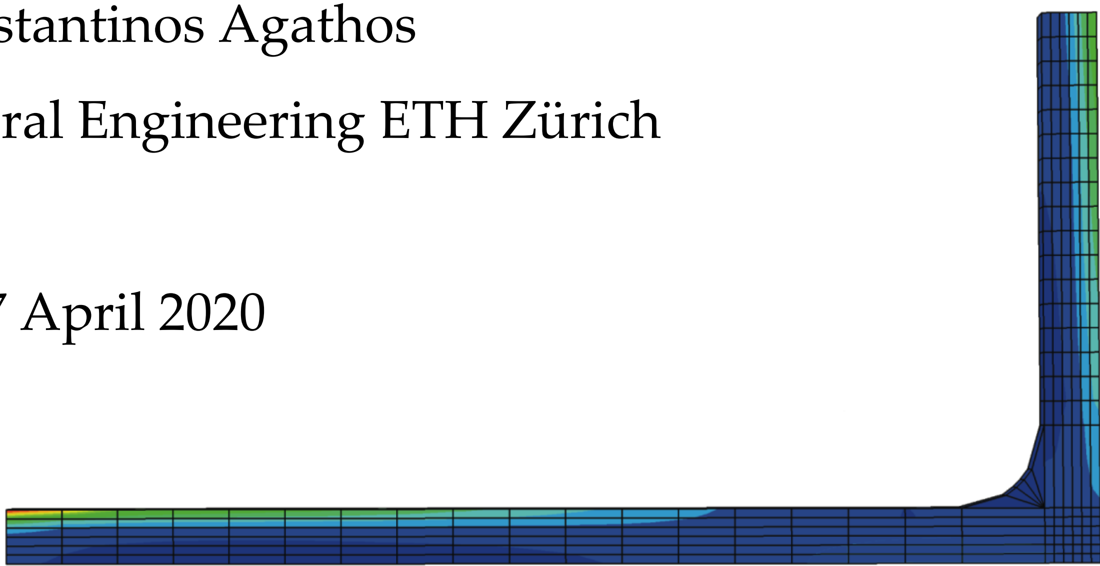
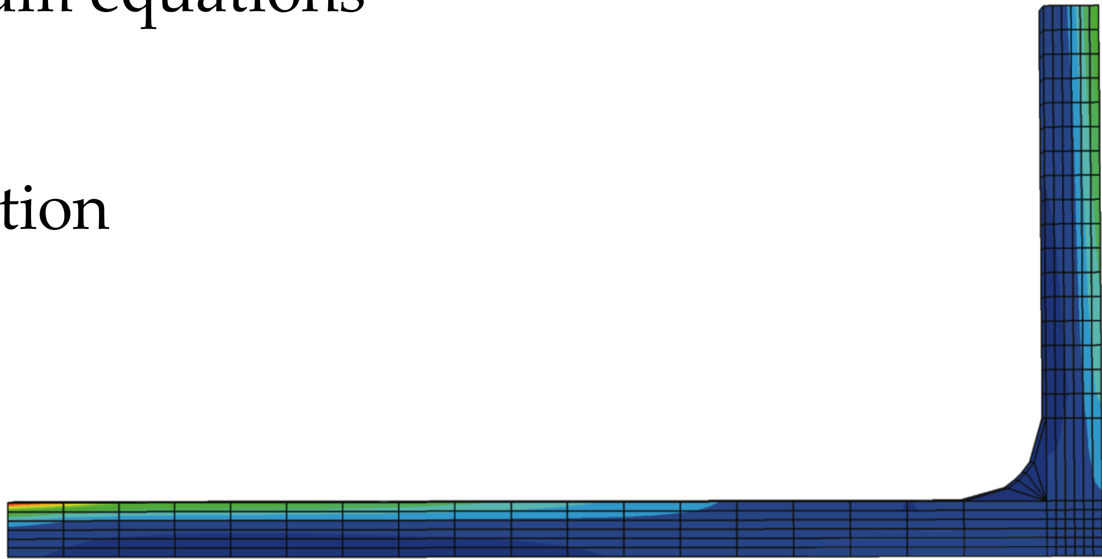


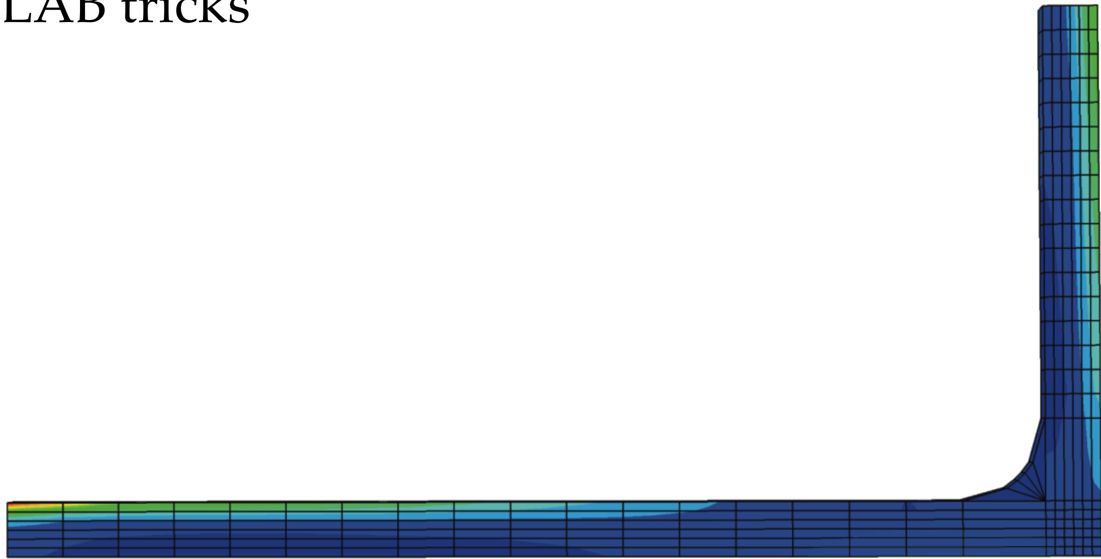
Today's Lecture Contents:

- Introduction – reminder of previous lecture
- FE discretization
- Discretized equilibrium equations
- MATLAB demonstration



Learning goals:

- Understanding how the weak form can be discretized using finite elements
- Understanding some of the advantages of FE discretization
- Learning some cool MATLAB tricks



FEM for PDEs

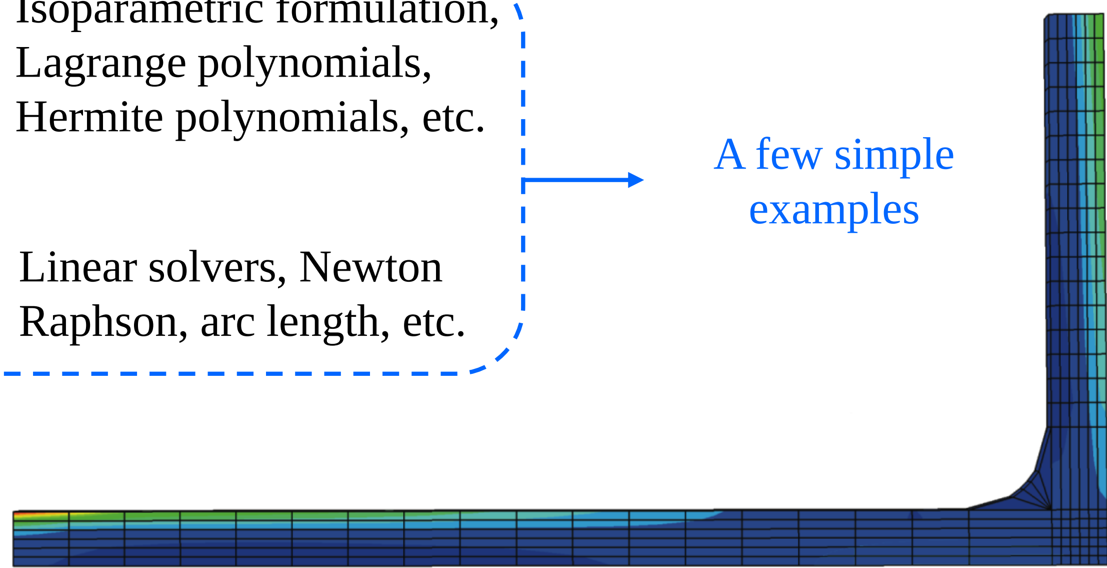
A series of steps is typically followed by all FE methods:

- **Strong form** → Elasticity, heat conduction, Navier stokes, etc.
- **Weak form** → Potential minimization, principle of virtual work, Galerkin method, etc.

→ Last lecture's main focus!

- **Discretization** → Isoparametric formulation, Lagrange polynomials, Hermite polynomials, etc.
- **Solution** → Linear solvers, Newton Raphson, arc length, etc.

→ A few simple examples



Strong form

General form of 2D second order partial differential equations (PDEs)

$$A(x,y)\frac{\partial^2 u}{\partial x^2} + 2B(x,y)\frac{\partial^2 u}{\partial x \partial y} + C(x,y)\frac{\partial^2 u}{\partial y^2} = \phi\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right)$$

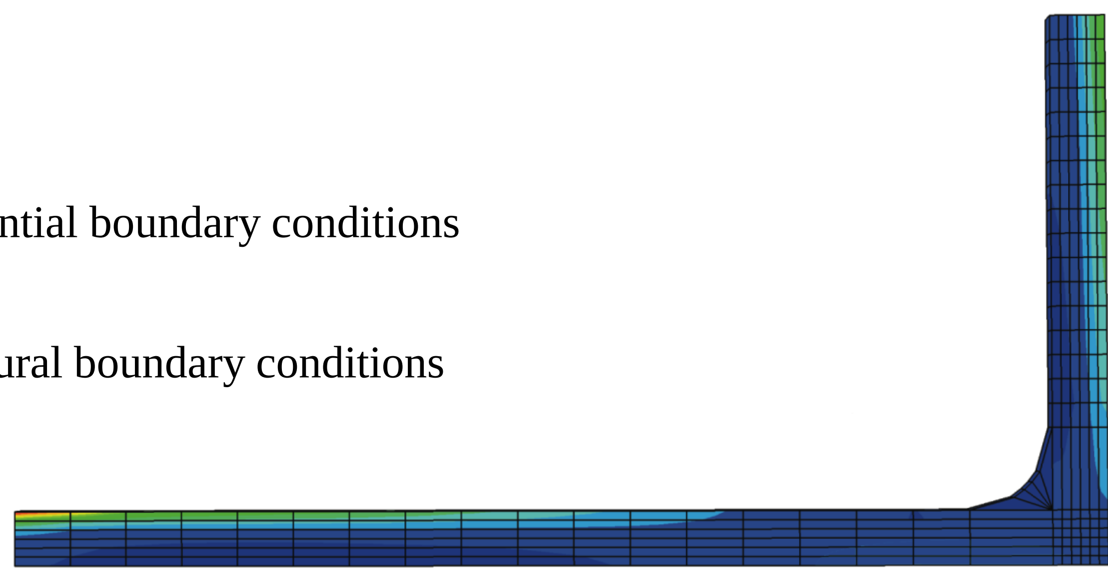
Categorization:

$$B^2 - AC = \begin{cases} < 0 \rightarrow \text{elliptic} \\ = 0 \rightarrow \text{parabolic} \\ > 0 \rightarrow \text{hyperbolic} \end{cases}$$

Boundary conditions (BCs):

$u(x_0, y_0) = u_0$ Dirichlet or essential boundary conditions

$\dfrac{\partial^{m+n} u}{\partial x^m \partial y^n} = \bar{u}(x,u)$ Neumann of natural boundary conditions



FEM for PDEs

The wide range of applicability is attributed to the fact that FEM is essentially a tool for solving partial differential equations (PDEs), for instance:

Bernoulli-Euler beam equilibrium equations:

$$EI_z \frac{d^4 w}{dx^4} = f_y(x)$$

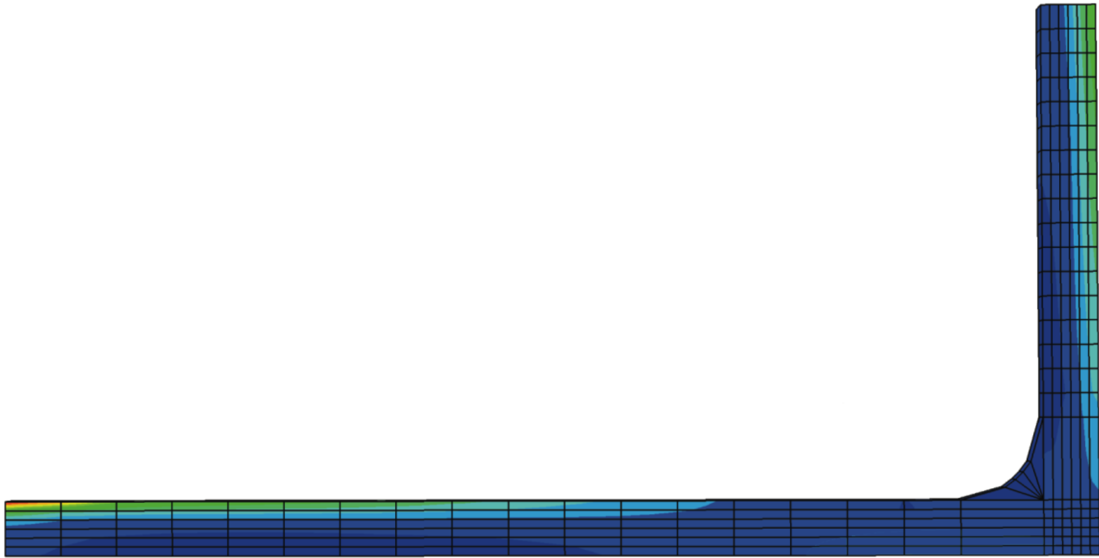
2D elasticity (Navier equations):

$$\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + F_x = 0$$

$$\frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + F_y = 0$$

Laplace equation:

$$\frac{\partial^2 \phi(x,y)}{\partial x^2} + \frac{\partial^2 \phi(x,y)}{\partial y^2} = 0$$



Weak form

Principle of stationary potential energy:

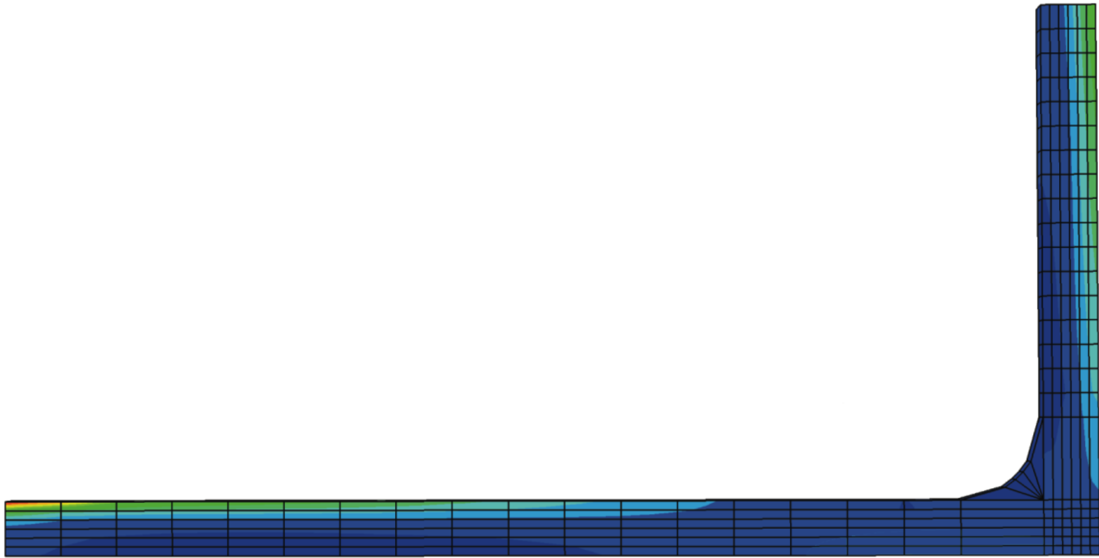
$$\int_0^L EA\frac{du}{dx}\delta\frac{du}{dx}\,dx = \int_0^L \delta u\,F\,dx + \delta u(L)R$$

Principle of Virtual Work:

$$\int_0^L EA\frac{du}{dx}\frac{d\delta u}{dx}\,dx = \int_0^L \delta u\,F\,dx + \delta u(L)R$$

Galerkin method:

$$\int_0^L \frac{dw}{dx}EA\frac{du}{dx}\,dx = \int_0^L w\,F dx + w\left(L\right)R$$



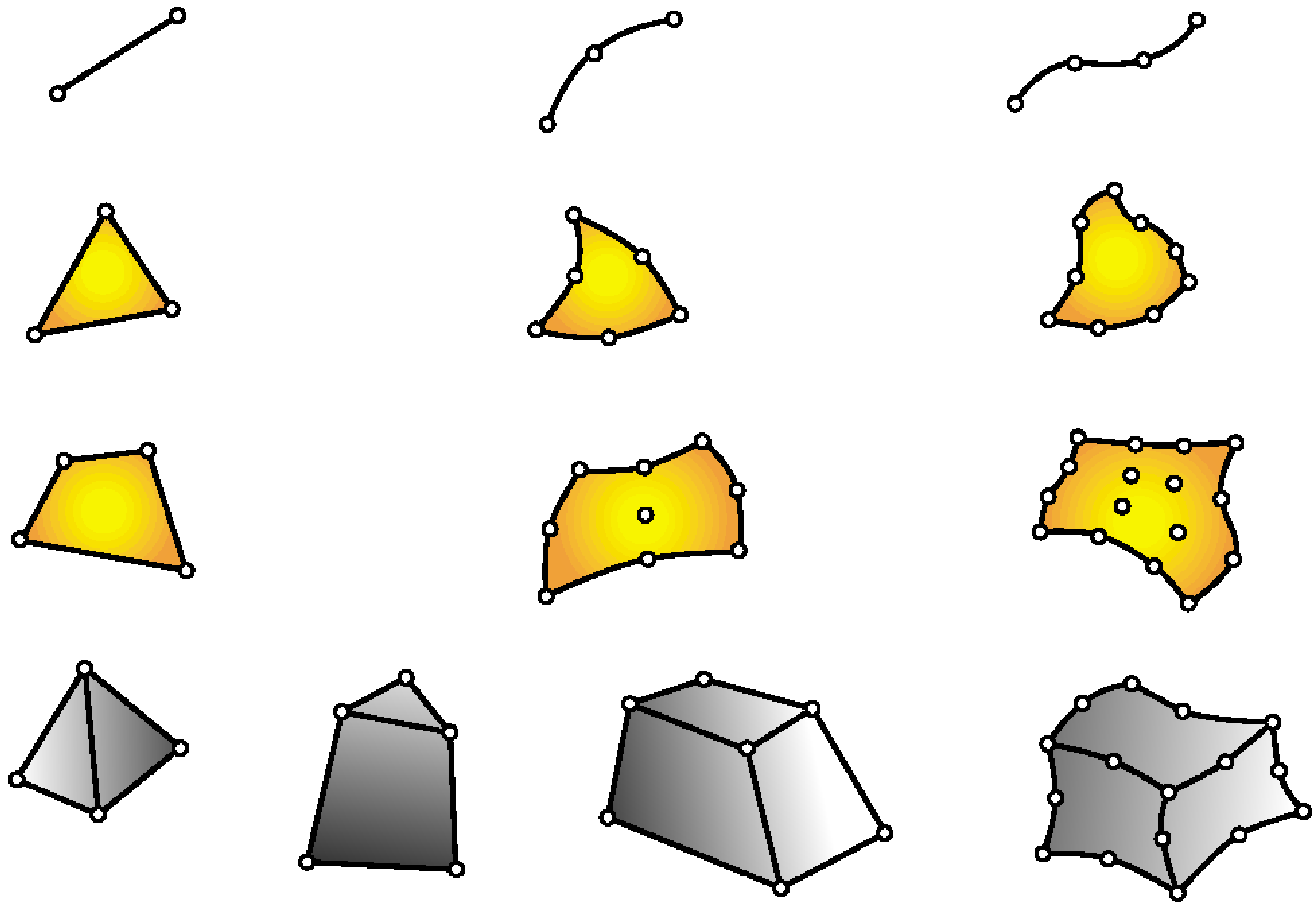
FEM discretization

1D

2D

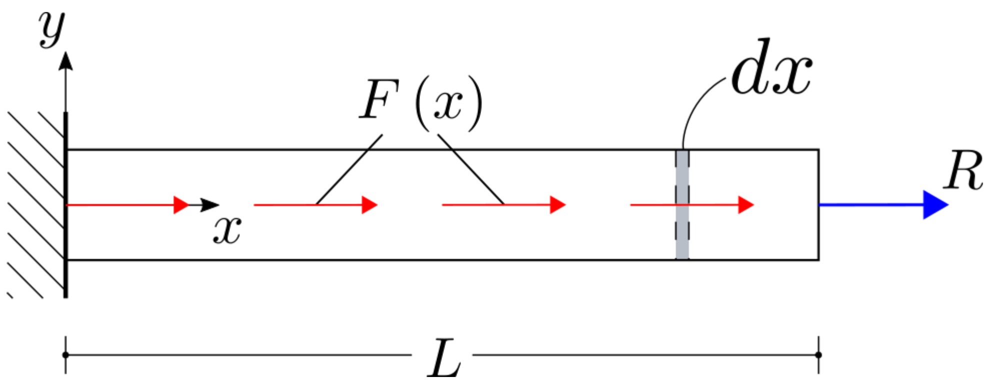
2D

3D



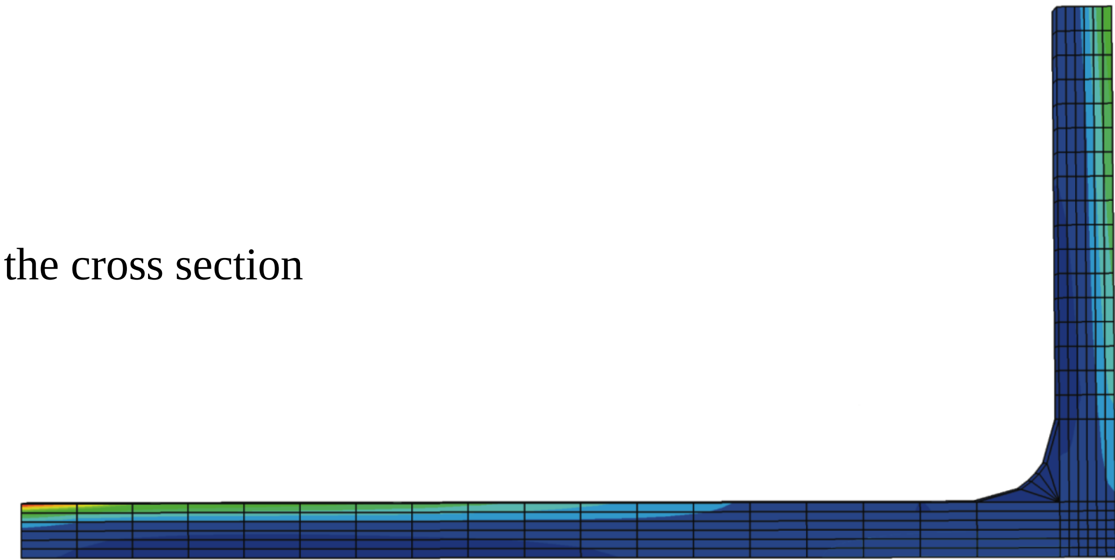
Reminder – 1D bar

Illustrative example – 1D bar with a distributed and an end load

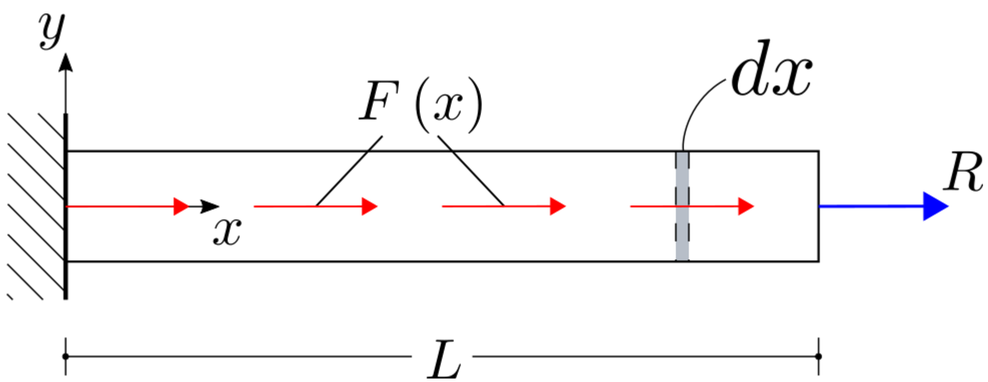


Assumptions:

- Constant cross section
- Linear elastic material
- Loads applied at the centroid of the cross section
- Arbitrary distributed load



Strong form



Equilibrium Equation

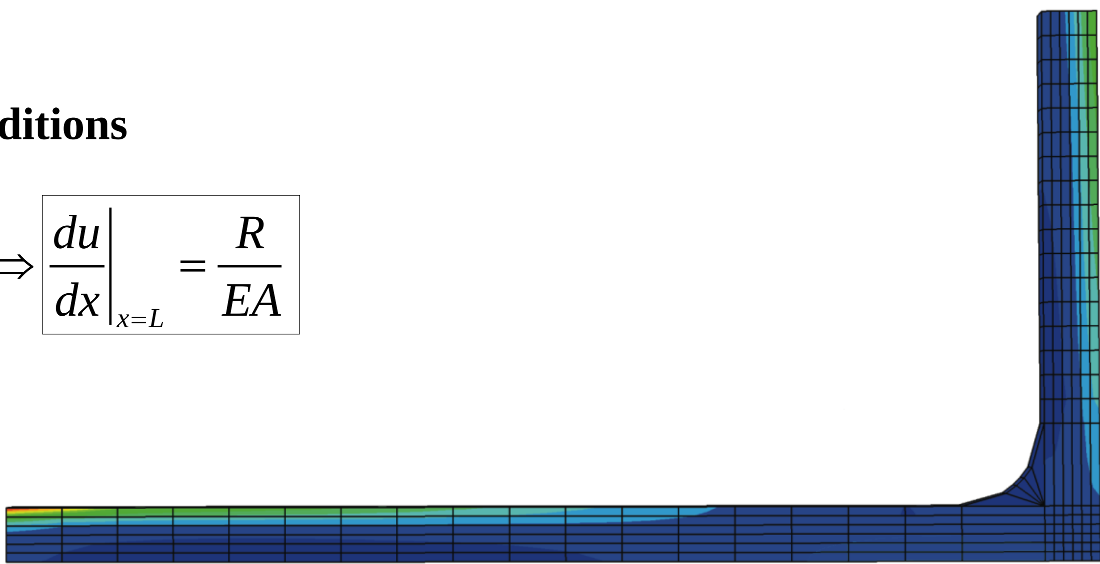
$$\boxed{EA\frac{d^2u}{dx^2} = -F(x)}$$

Dirichlet (essential) boundary conditions

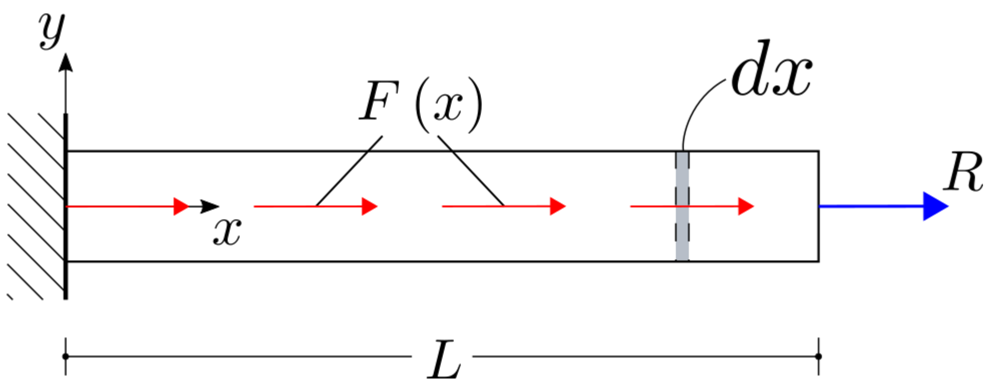
$$\boxed{u(x=0)=0}$$

Neumann (natural) boundary conditions

$$A\sigma(x=L)=R \Rightarrow AE\left.\frac{du}{dx}\right|_{x=L} = R \Rightarrow \boxed{\left.\frac{du}{dx}\right|_{x=L} = \frac{R}{EA}}$$



Strong form solution



Assuming no distributed load:

$$F(x) = 0$$

$$\boxed{\begin{aligned} &EA\frac{d^2u}{dx^2} = 0 \\ &u(x=0) = 0 \\ &\left.\frac{du}{dx}\right|_{x=L} = \frac{R}{EA} \end{aligned}}$$

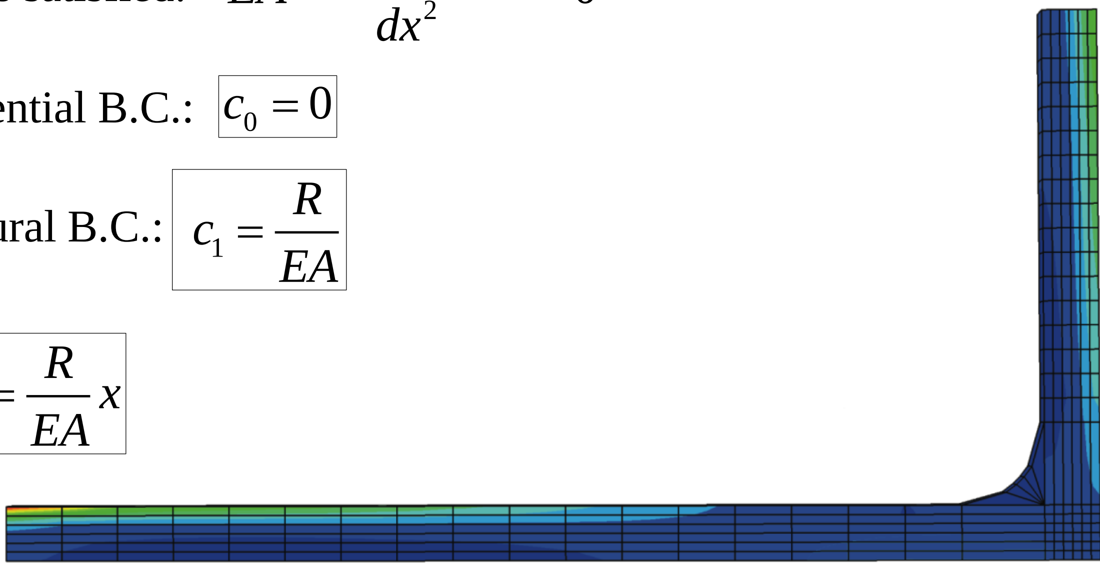
The solution should be of the form: $u(x) = c_0 + c_1 x$

Equilibrium is satisfied: $EA\dfrac{d^2(c_0 + c_1 x)}{dx^2} = 0$

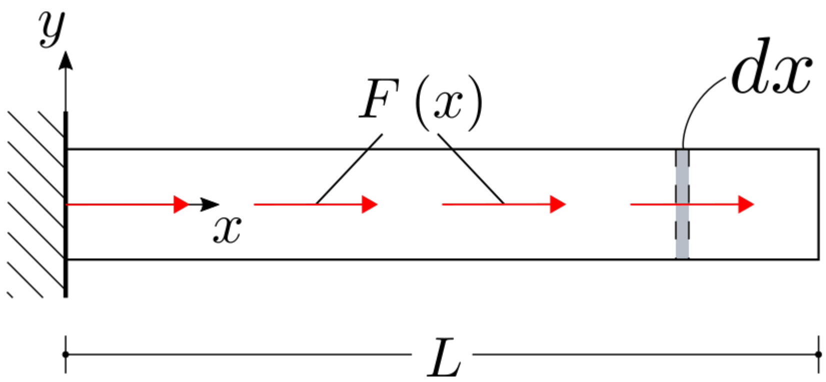
From the essential B.C.: $\boxed{c_0 = 0}$

From the natural B.C.: $\boxed{c_1 = \dfrac{R}{EA}}$

Final form of the solution: $\boxed{u(x) = \dfrac{R}{EA}x}$



Strong form solution



Similarly, assuming a linear distributed load and no end load:

$$F(x) = ax$$

$$R = 0$$

$$\boxed{\begin{gathered} EA\frac{d^2u}{dx^2} = -ax \\ u(x=0) = 0 \\ \left.\frac{du}{dx}\right|_{x=L} = 0 \end{gathered}}$$

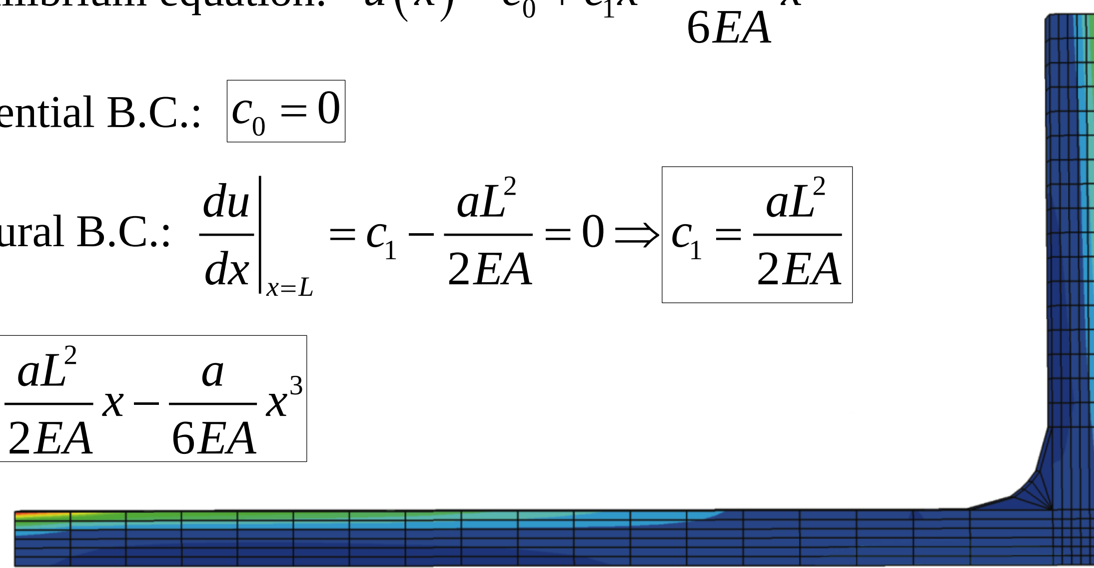
The solution should be of the form: $u(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3$

From the equilibrium equation: $u(x) = c_0 + c_1 x - \frac{a}{6EA}x^3$

From the essential B.C.: $\boxed{c_0 = 0}$

From the natural B.C.: $\left.\frac{du}{dx}\right|_{x=L} = c_1 - \frac{aL^2}{2EA} = 0 \Rightarrow \boxed{c_1 = \frac{aL^2}{2EA}}$

Final form of the solution: $\boxed{u(x) = \frac{aL^2}{2EA}x - \frac{a}{6EA}x^3}$



Weak form

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Principle of Virtual Work:

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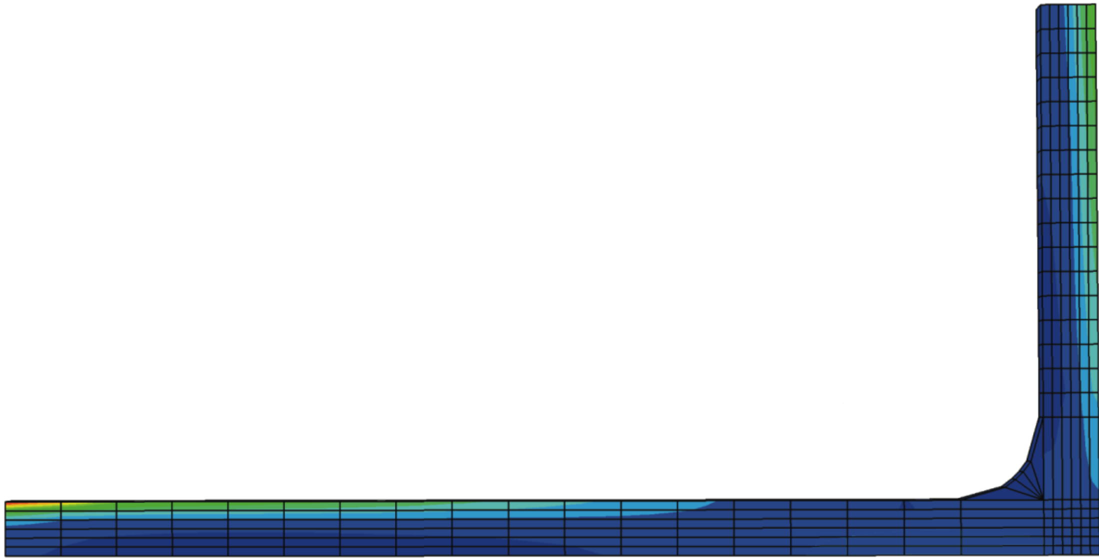


Weak vs Strong form

Weak form:

$$\int_0^L \frac{dw}{dx} EA \frac{du}{dx} dx = \int_0^L w F dx + w(L) R$$
$$u(0) = 0$$

- Natural BCs are part of the weak form
- Highest derivative is of order 1
- Degree of continuity required: C_0
- Equilibrium is satisfied in an integral, "**weak**" sense

Strong form:

$$EA \frac{d^2u}{dx^2} = -F$$
$$\left.\frac{du}{dx}\right|_{x=L} = \frac{R}{EA}$$
$$u(x=0) = 0$$

- Natural BCs are explicitly imposed
- Highest derivative is of order 2
- Degree of continuity required: C_1
- Equilibrium is satisfied everywhere in a "**strong**" sense

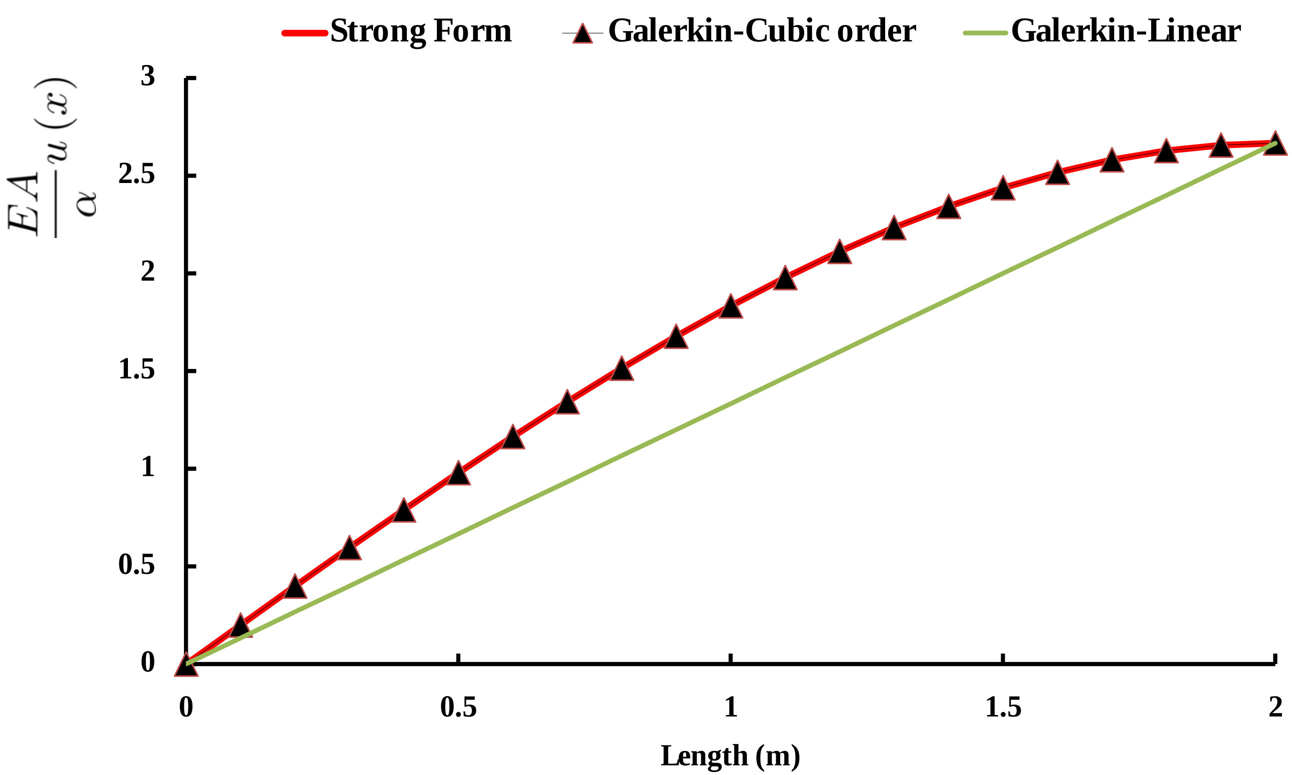
Weak form solution - Approximate

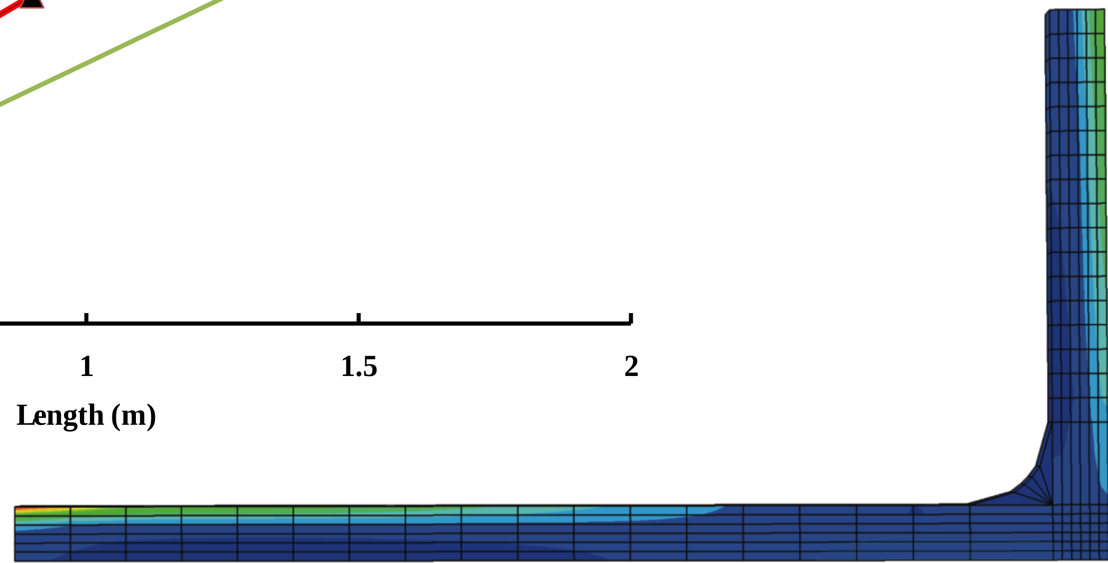
Exact solution:

$$u(x) = \frac{aL^2}{2EA}x - \frac{a}{6EA}x^3$$

Approximate solution:

$$\bar{u}(x) = \frac{aL^2}{3EA}x$$

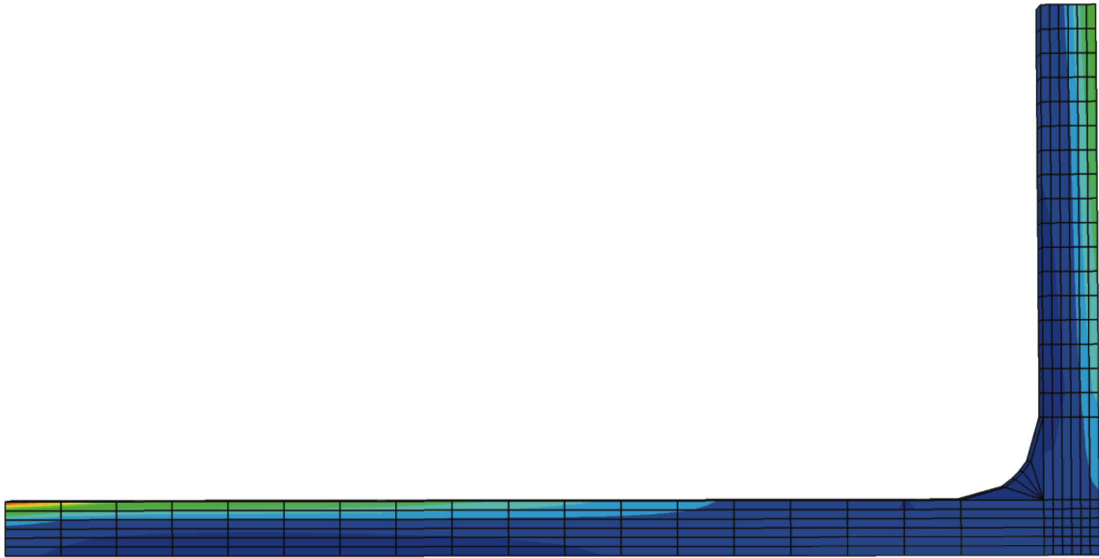




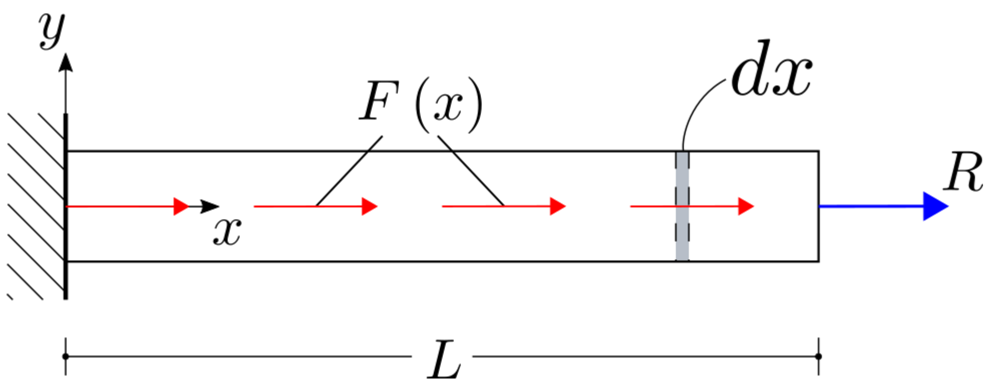
FE discretization

A piecewise polynomial solution is assumed

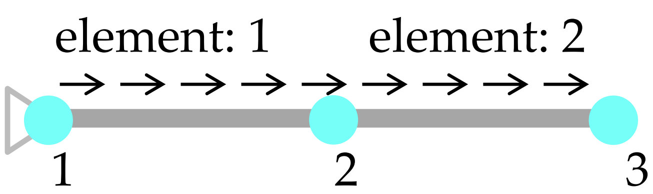
- The problem domain is divided in subdomains (elements) of simple geometry
- A polynomial solution is assumed for each subdomain
- Continuity between subdomains should be imposed



FE discretization



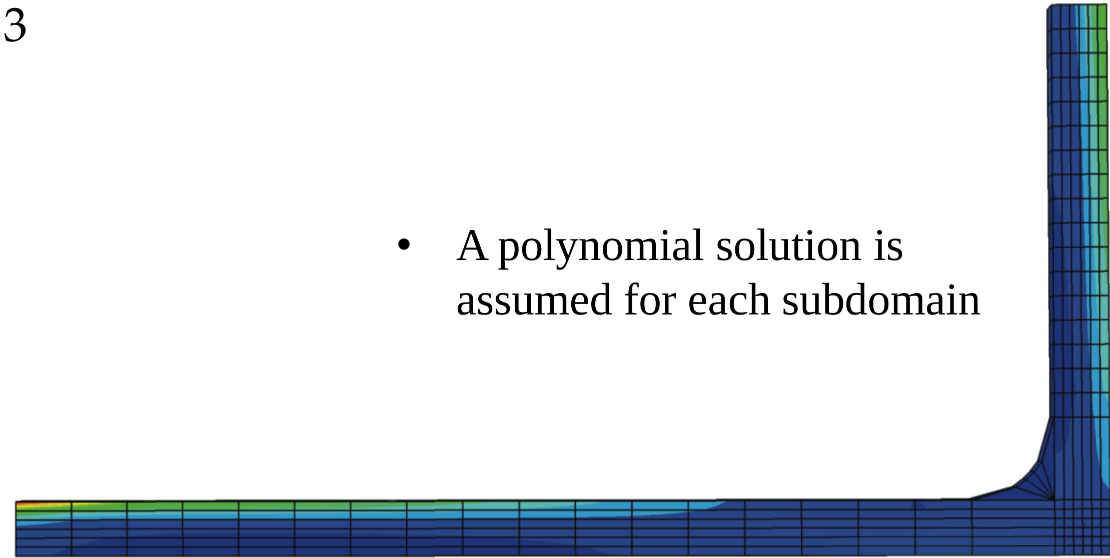
1D bar case:



- The domain is divided in subdomains

$$u(x) = a_0 + a_1 x$$

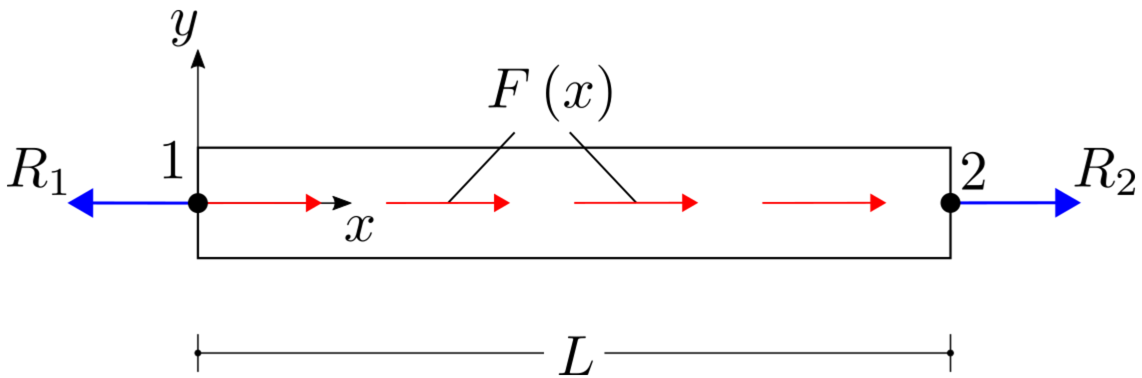
- A polynomial solution is assumed for each subdomain



FE discretization

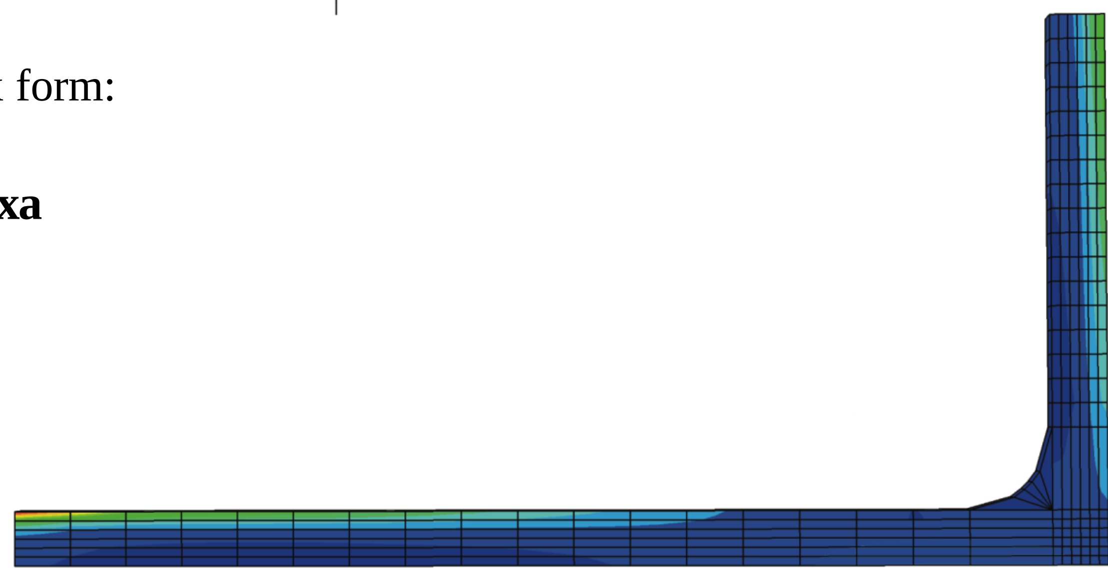
For both the interpretation of results and imposition of continuity it is preferable to write displacements in terms of their nodal values. To do so:

- We consider a single element



- We write displacements in matrix form:

$$u(x) = a_0 + a_1 x = \underbrace{\begin{bmatrix} 1 & x \end{bmatrix}}_{\mathbf{x}} \underbrace{\begin{bmatrix} a_0 \\ a_1 \end{bmatrix}}_{\mathbf{a}} = \mathbf{xa}$$



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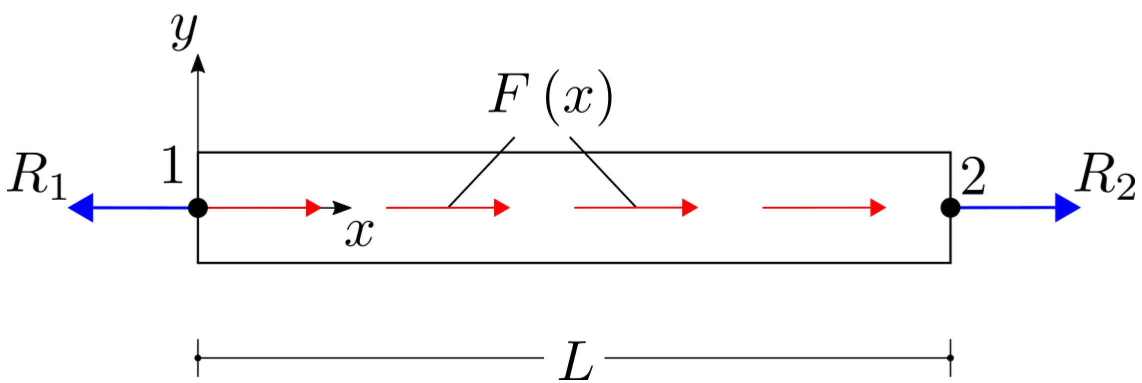
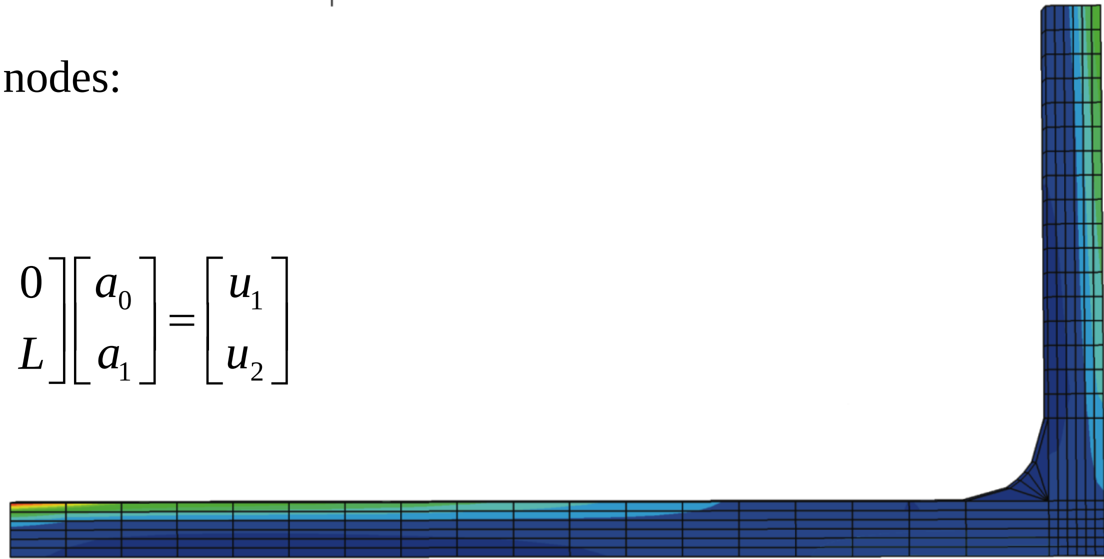


- We evaluate displacements at the nodes:

$$\left.\begin{aligned} u(x=0) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = u_1 \\ u(x=L) &= \begin{bmatrix} 1 & L \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = u_2 \end{aligned}\right\} \begin{bmatrix} 1 & 0 \\ 1 & L \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$



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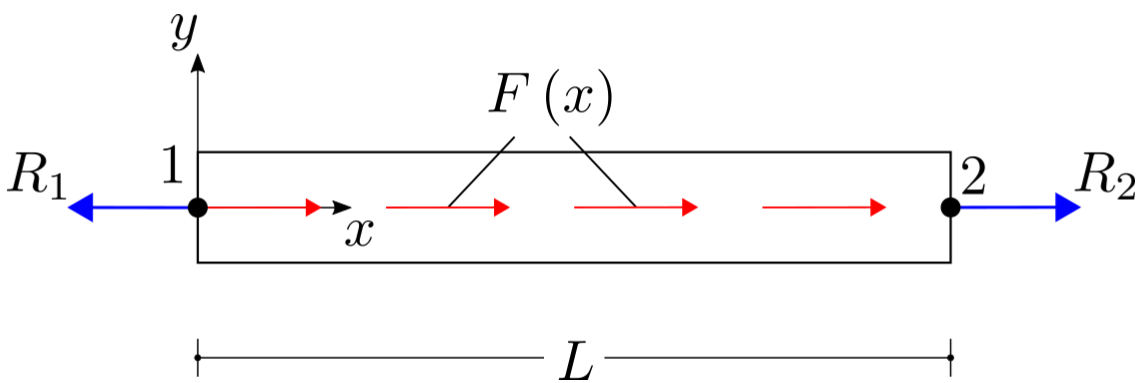
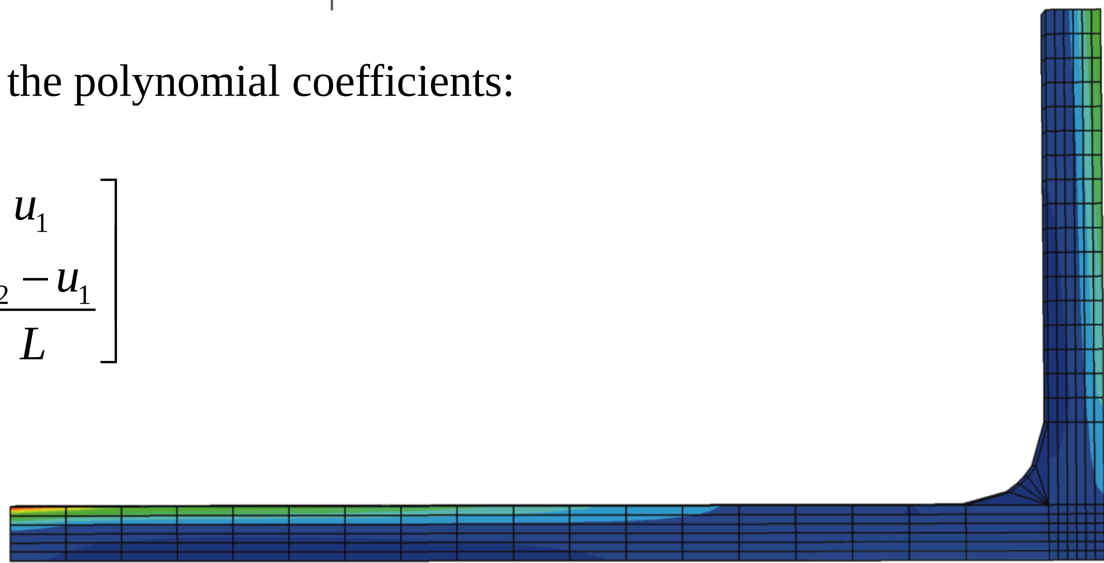


- We solve the resulting system for the polynomial coefficients:

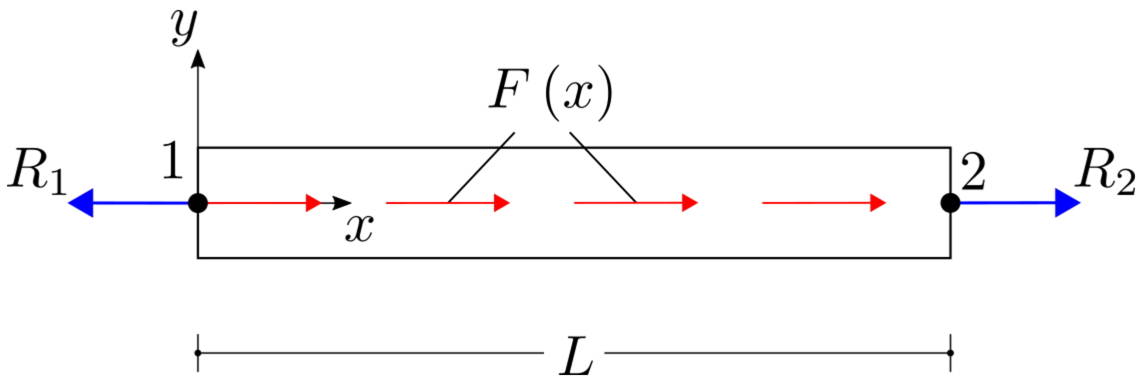
$$\begin{bmatrix} 1 & 0 \\ 1 & L \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \Rightarrow \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} u_1 \\ \dfrac{u_2 - u_1}{L} \end{bmatrix}$$



FE discretization

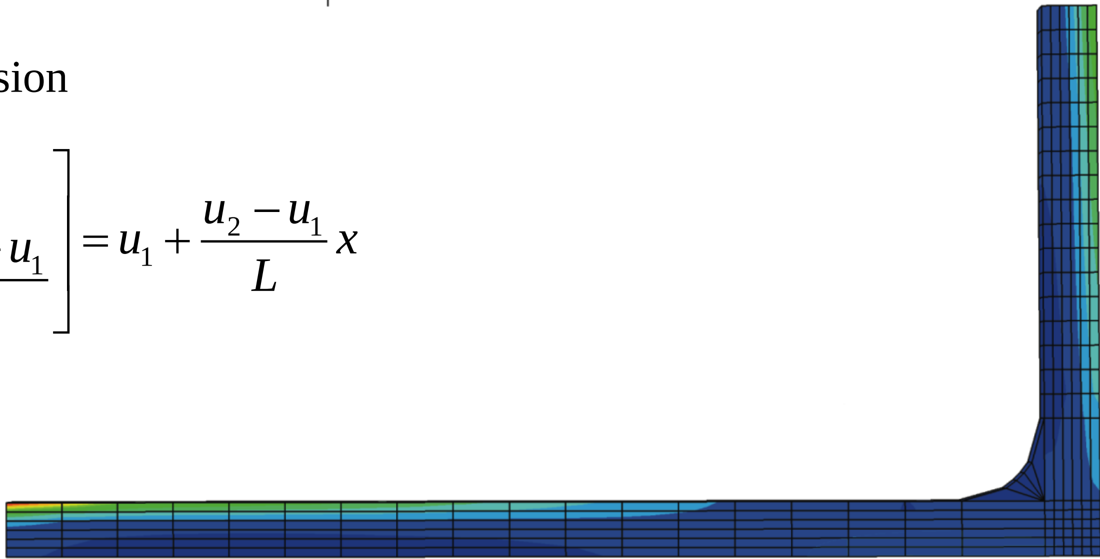
For both the interpretation of results and imposition of continuity it is preferable to write displacements in terms of their nodal values. To do so:

- We consider a single element



- We substitute in the initial expression

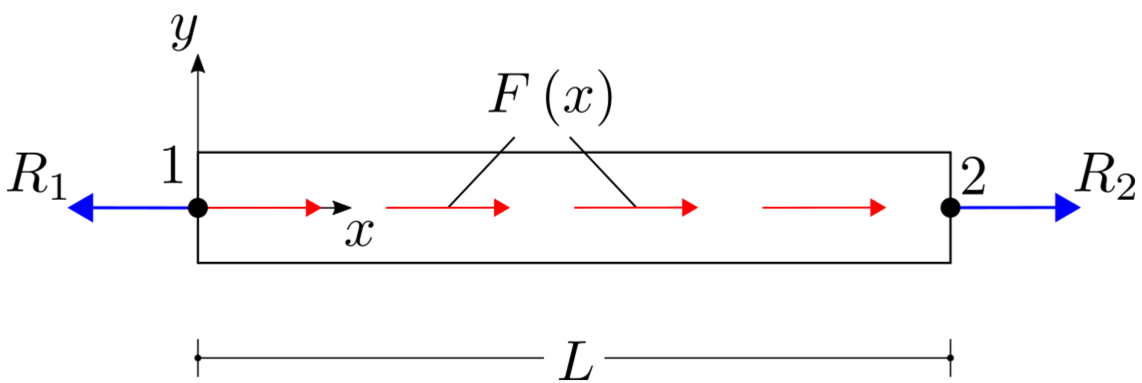
$$u(x) = \begin{bmatrix} 1 & x \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \begin{bmatrix} 1 & x \end{bmatrix} \begin{bmatrix} u_1 \\ \dfrac{u_2 - u_1}{L} \end{bmatrix} = u_1 + \frac{u_2 - u_1}{L} x$$



FE discretization

For both the interpretation of results and imposition of continuity it is preferable to write displacements in terms of their nodal values. To do so:

- We consider a single element



- We write the resulting expression in terms of the nodal displacements

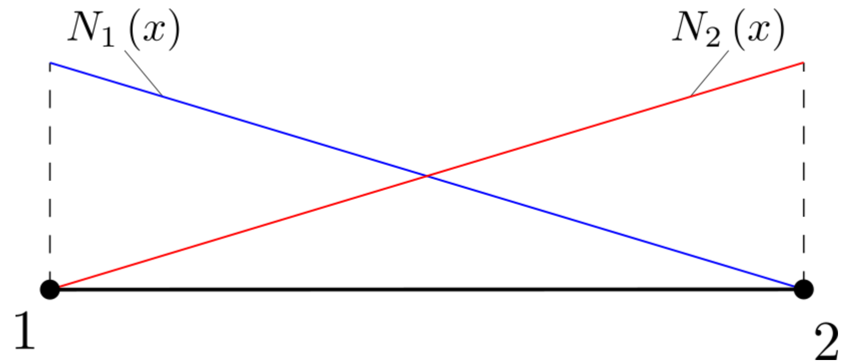
$$u(x) = u_1 + \frac{u_2 - u_1}{L} x = \begin{bmatrix} 1 - \frac{x}{L} & \frac{x}{L} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

FE discretization

The expressions resulting from the above procedure are the shape functions of a bar element:

$$u(x)=\begin{bmatrix}\underbrace{1-\frac{x}{L}}_{N_1} & \underbrace{\frac{x}{L}}_{N_2}\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix}=\begin{bmatrix}N_1 & N_2\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix}=\sum_{i=1}^{2}N_i u_i$$

- They assume a value of 1 at the node they refer to
- They assume a value of 0 at all other nodes

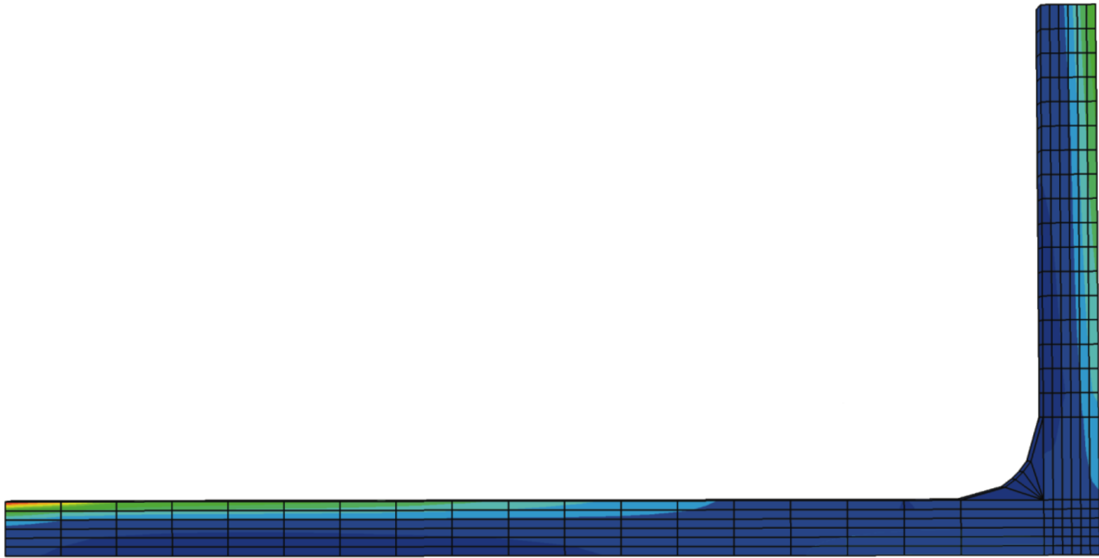


FE formulation

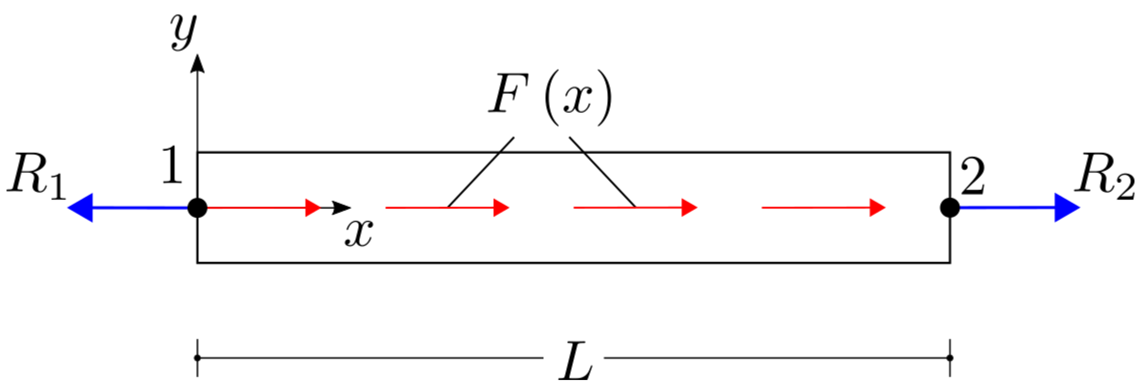
The weak form can be solved using the described FE discretization in the same way as done for a polynomial displacement assumption:

- Assume a general form for the solution
- Plug into weak form
- Obtain unknown coefficients

The unknown coefficients in this case will be the nodal displacements!



FE formulation



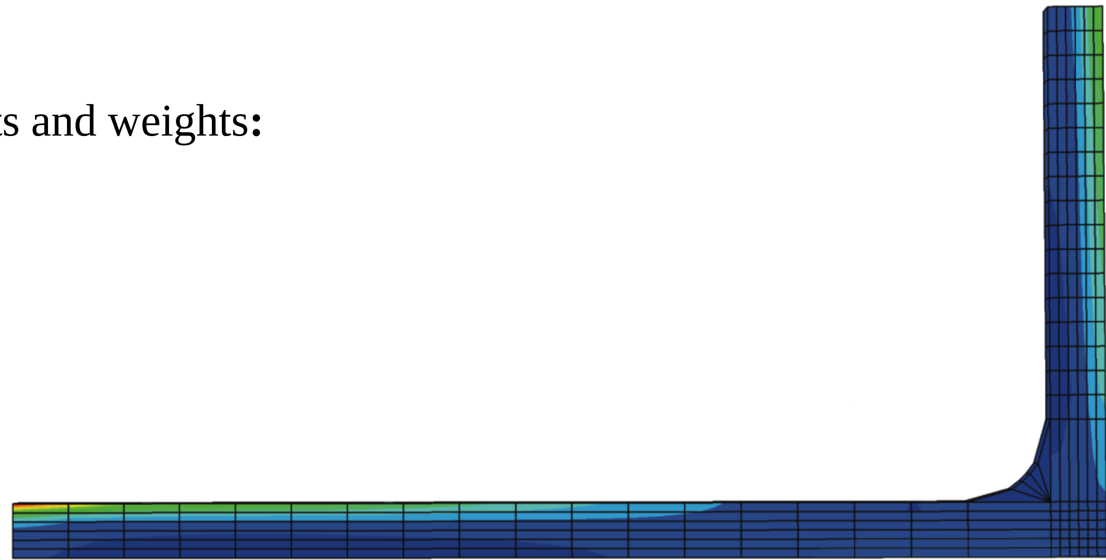
Galerkin weak form:

$$\int_0^L \frac{dw}{dx} EA \frac{du}{dx}\, dx = \int_0^L w F dx$$

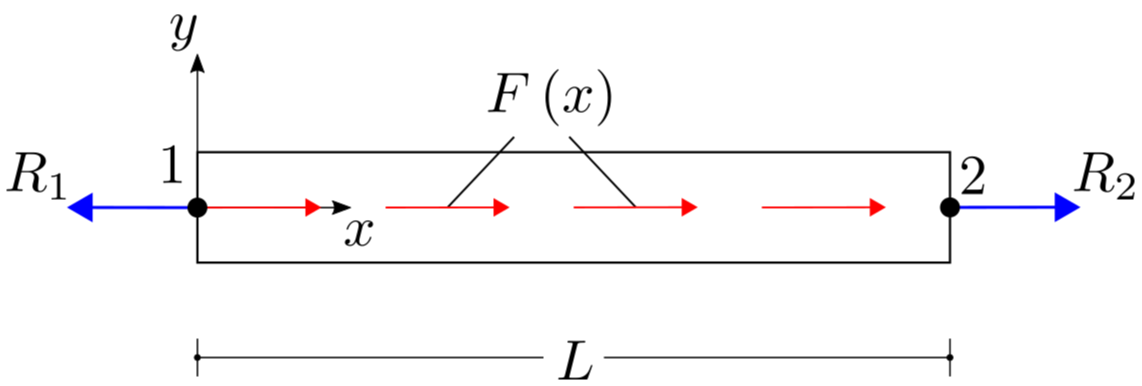
FE discretization for displacements and weights:

$$u(x) = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$w(x) = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$$



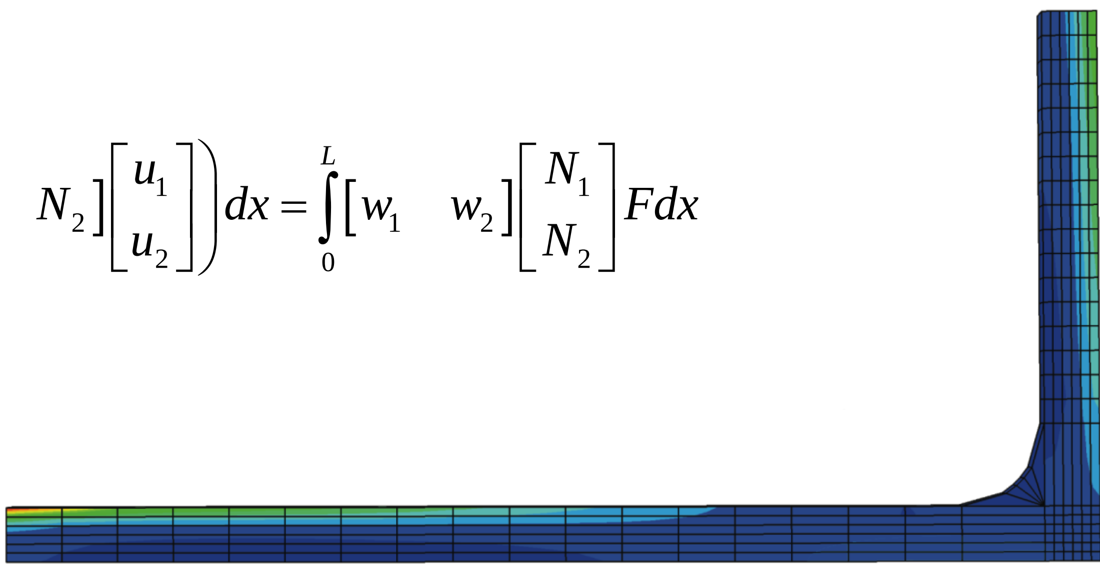
FE formulation



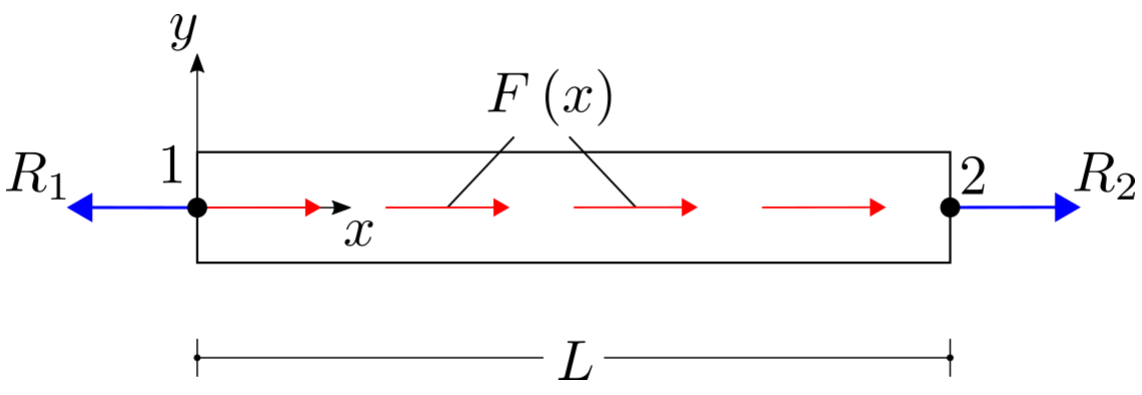
Galerkin weak form & FE shape functions:

$$\int_0^L \frac{dw}{dx} EA \frac{du}{dx} dx = \int_0^L w F dx \Leftrightarrow$$

$$\int_0^L \frac{d}{dx}\left(\begin{bmatrix} w_1 & w_2 \end{bmatrix}\begin{bmatrix} N_1 \\ N_2 \end{bmatrix}\right) EA \frac{d}{dx}\left(\begin{bmatrix} N_1 & N_2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) dx = \int_0^L \begin{bmatrix} w_1 & w_2 \end{bmatrix}\begin{bmatrix} N_1 \\ N_2 \end{bmatrix} F dx$$



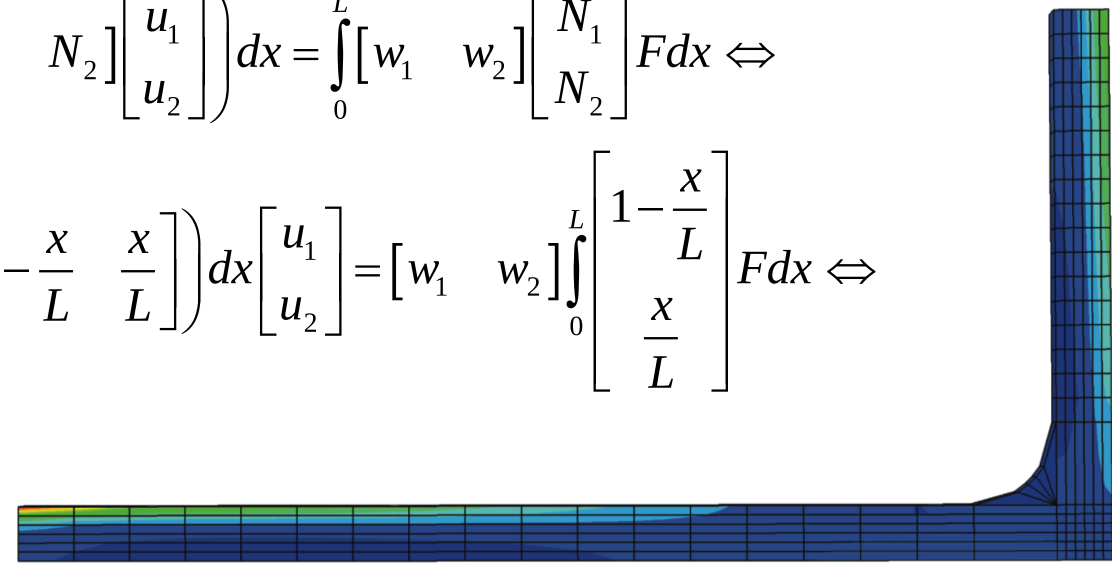
FE formulation



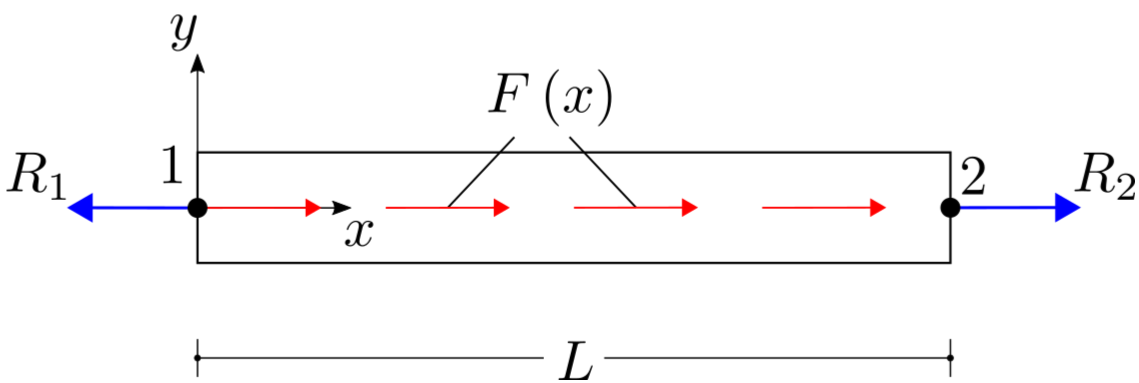
Galerkin weak form & FE shape functions:

$$\int_0^L \frac{d}{dx}\left(\begin{bmatrix} w_1 & w_2 \end{bmatrix}\begin{bmatrix} N_1 \\ N_2 \end{bmatrix}\right) EA \frac{d}{dx}\left(\begin{bmatrix} N_1 & N_2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right) dx = \int_0^L \begin{bmatrix} w_1 & w_2 \end{bmatrix}\begin{bmatrix} N_1 \\ N_2 \end{bmatrix} F dx \Leftrightarrow$$

$$\begin{bmatrix} w_1 & w_2 \end{bmatrix}\int_0^L \frac{d}{dx}\left(\begin{bmatrix} 1-\frac{x}{L} \\ \frac{x}{L} \end{bmatrix}\right) EA \frac{d}{dx}\left(\begin{bmatrix} 1-\frac{x}{L} & \frac{x}{L} \end{bmatrix}\right) dx \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} w_1 & w_2 \end{bmatrix}\int_0^L \begin{bmatrix} 1-\frac{x}{L} \\ \frac{x}{L} \end{bmatrix} F dx \Leftrightarrow$$



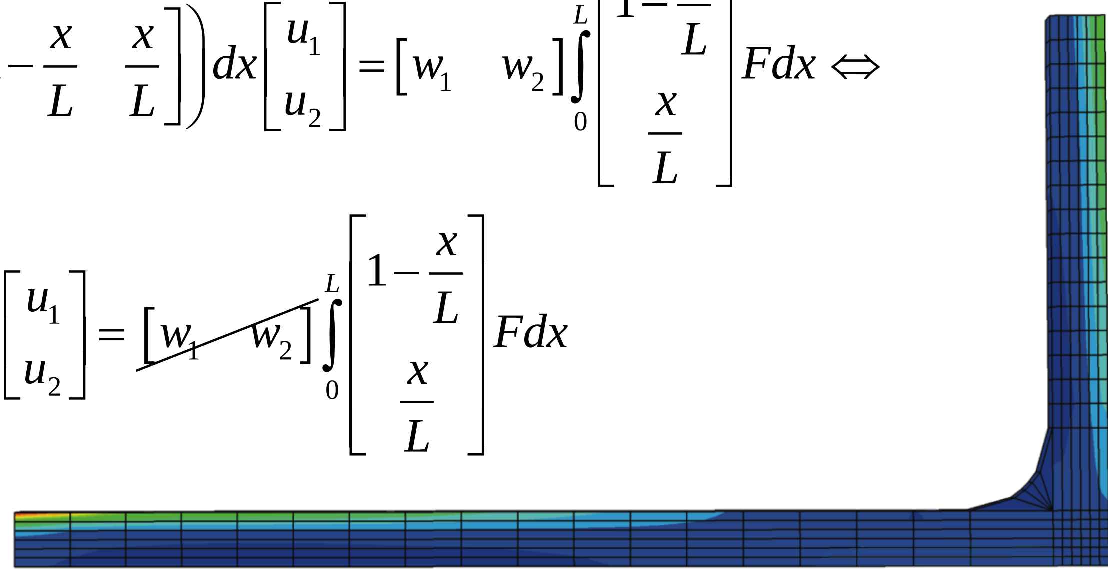
FE formulation



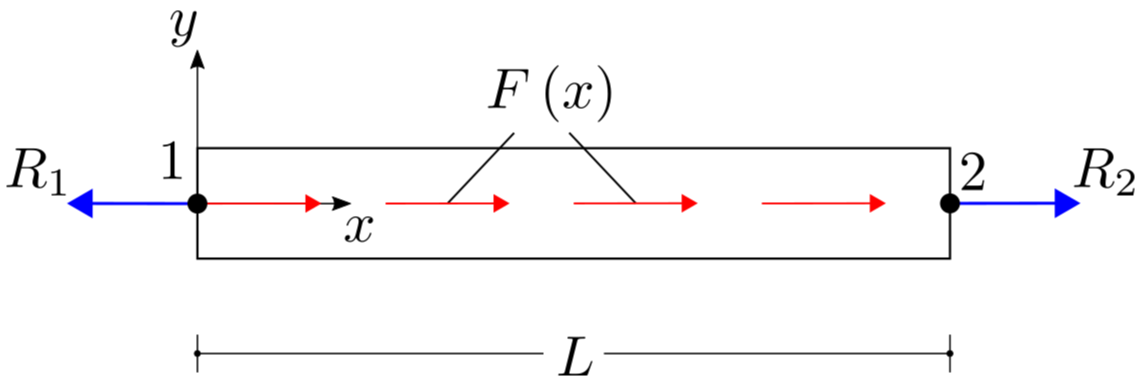
Galerkin weak form & FE shape functions:

$$\begin{bmatrix} w_1 & w_2 \end{bmatrix} \int_0^L \frac{d}{dx}\left(\begin{bmatrix} 1-\frac{x}{L} \\ \frac{x}{L} \end{bmatrix} \right) EA \frac{d}{dx}\left(\begin{bmatrix} 1-\frac{x}{L} & \frac{x}{L} \end{bmatrix} \right) dx \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} w_1 & w_2 \end{bmatrix} \int_0^L \begin{bmatrix} 1-\frac{x}{L} \\ \frac{x}{L} \end{bmatrix} F dx \Leftrightarrow$$

$$\cancel{\begin{bmatrix} w_1 & w_2 \end{bmatrix}} \int_0^L \begin{bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{bmatrix} EA \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} dx \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \cancel{\begin{bmatrix} w_1 & w_2 \end{bmatrix}} \int_0^L \begin{bmatrix} 1-\frac{x}{L} \\ \frac{x}{L} \end{bmatrix} F dx$$



FE formulation

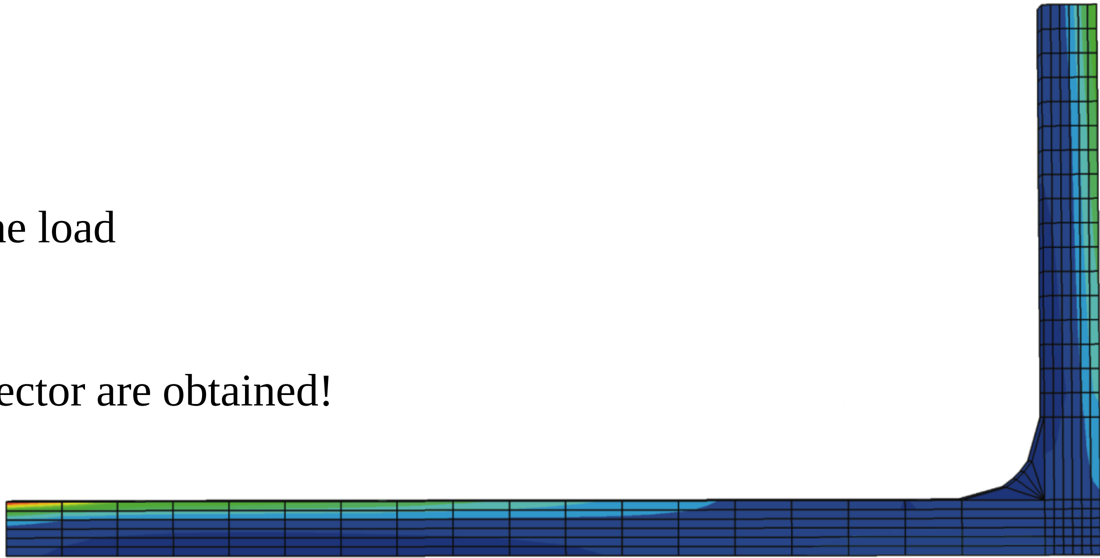


After carrying out the integrations (assuming a constant distributed load):

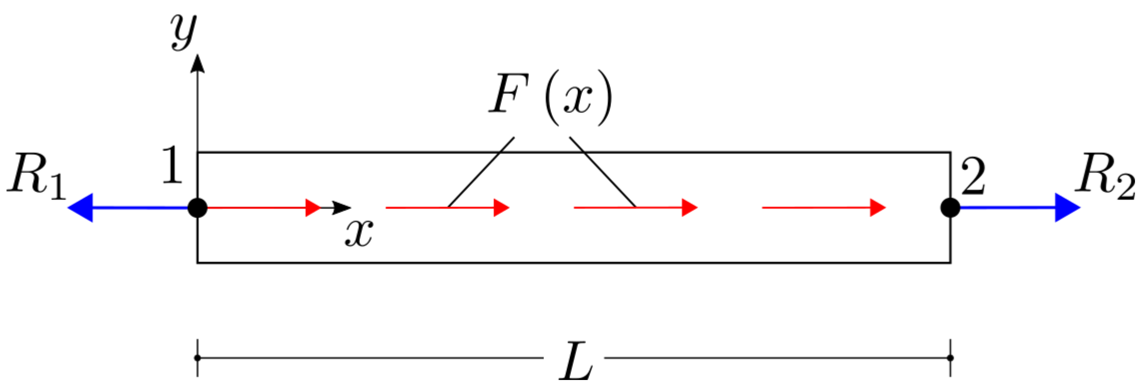
$$\underbrace{\frac{EA}{L}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}}_{\mathbf{K}}\underbrace{\begin{bmatrix}u_1\\ u_2\end{bmatrix}}_{\mathbf{u}}=\underbrace{\frac{FL}{2}\begin{bmatrix}1\\ 1\end{bmatrix}}_{\mathbf{f}}$$

where F is the constant value of the load

The bar stiffness matrix and load vector are obtained!



FE formulation

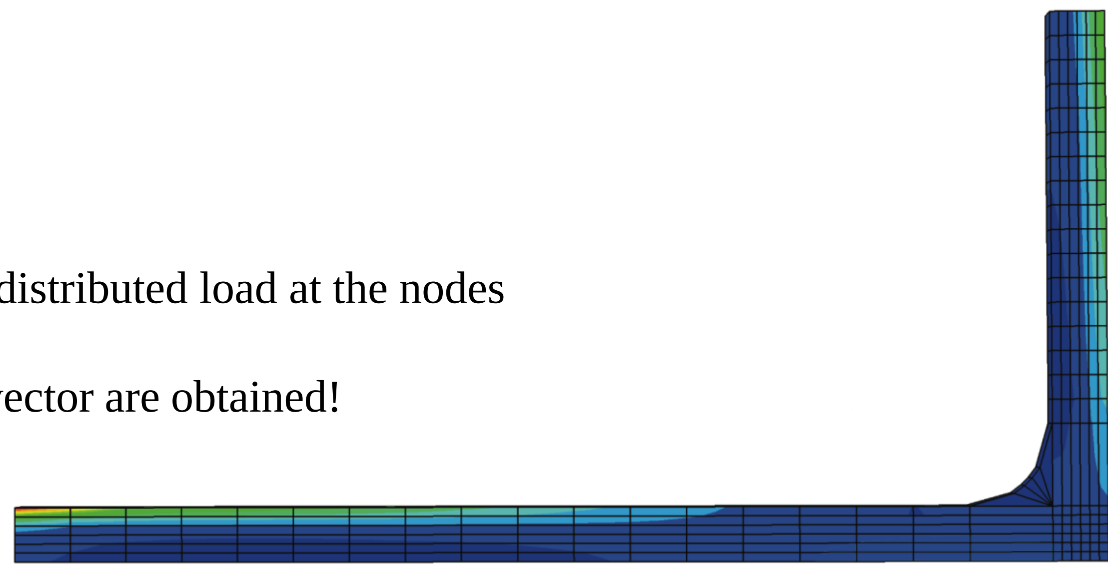


After carrying out the integrations (assuming a linear distributed load):

$$\underbrace{\frac{EA}{L}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}_{\mathbf{K}}\underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}_{\mathbf{u}} = \underbrace{\begin{bmatrix} \dfrac{F_1 L}{3} + \dfrac{F_2 L}{6} \\ \dfrac{F_2 L}{3} + \dfrac{F_1 L}{6} \end{bmatrix}}_{\mathbf{f}}$$

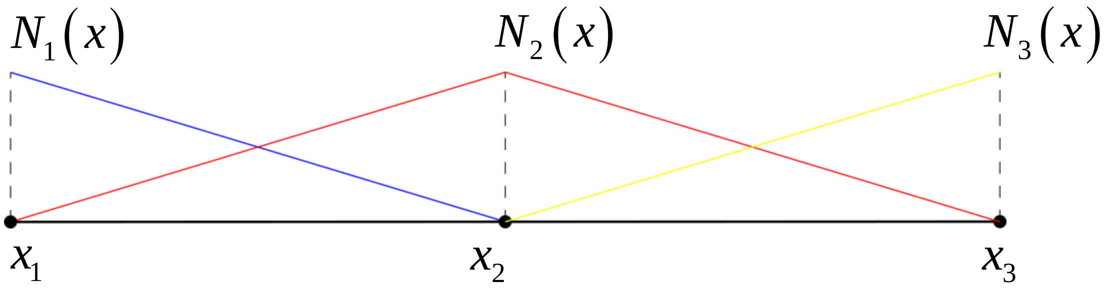
where F_1, F_2 are the values of the distributed load at the nodes

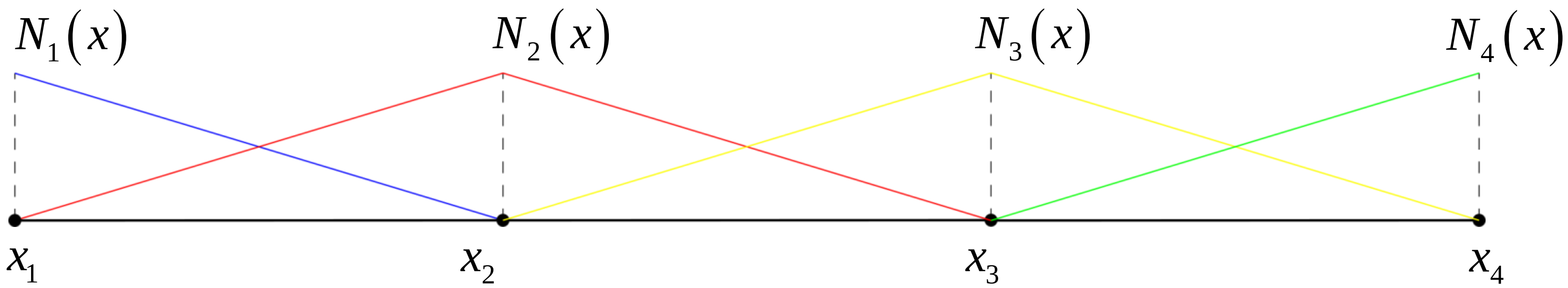
The bar stiffness matrix and load vector are obtained!



FE formulation

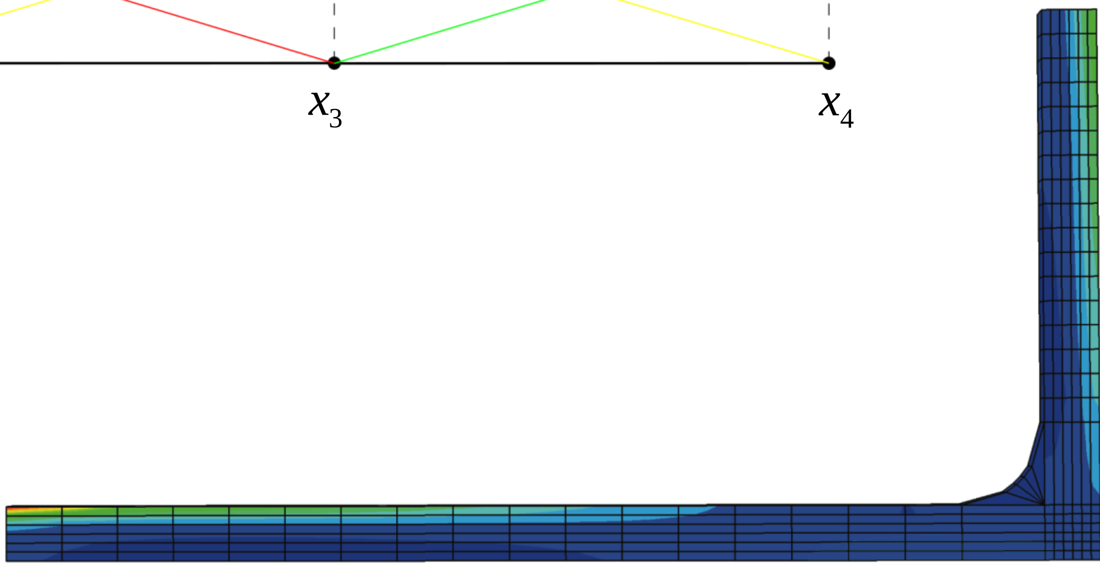
Consider an assembly of multiple elements:





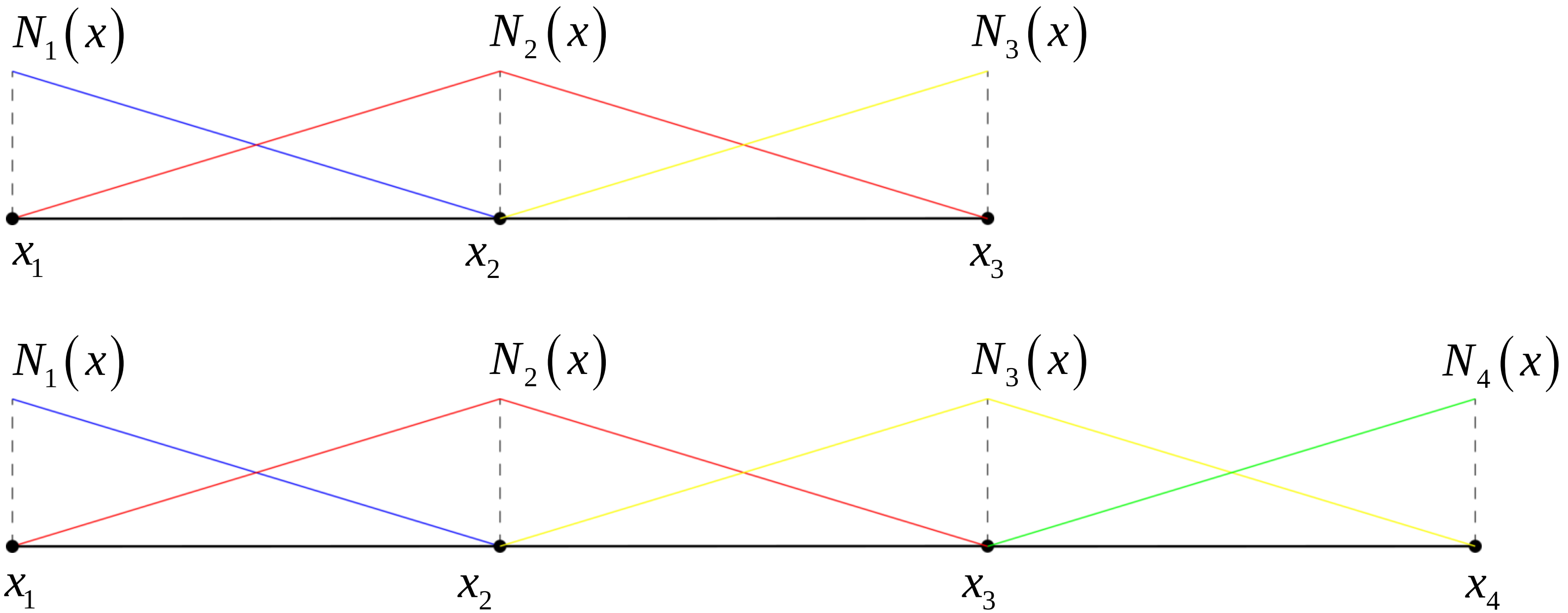
$$u(x) = \sum_{i=1}^{n} N_i u_i$$

$$w(x) = \sum_{i=1}^{n} N_i w_i$$



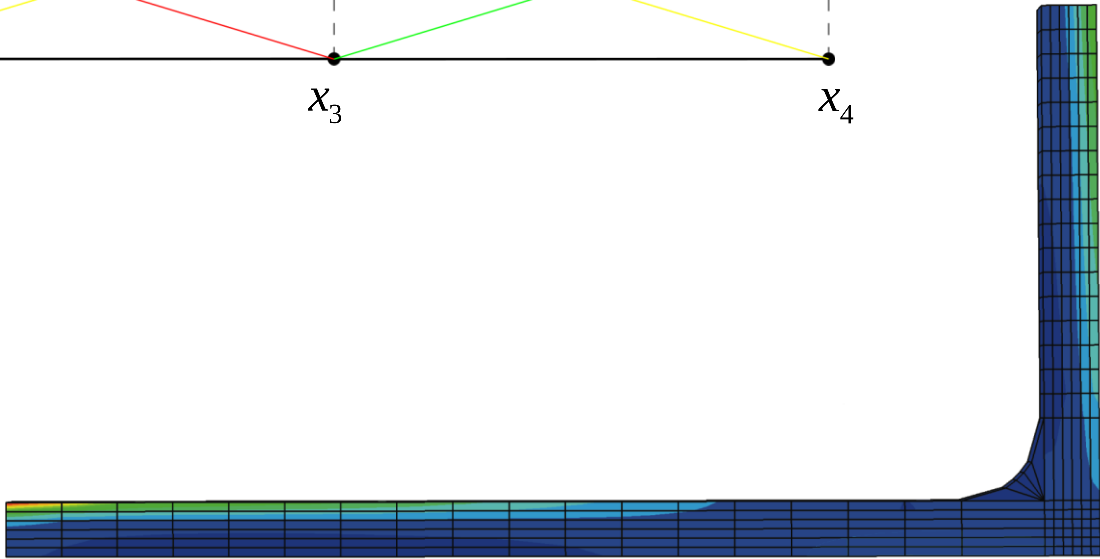
FE formulation

Consider an assembly of multiple elements**:**



We observe that:

$$N_i = 0 \text{ for } x < x_{i-1} \text{ or } x > x_{i+1}$$

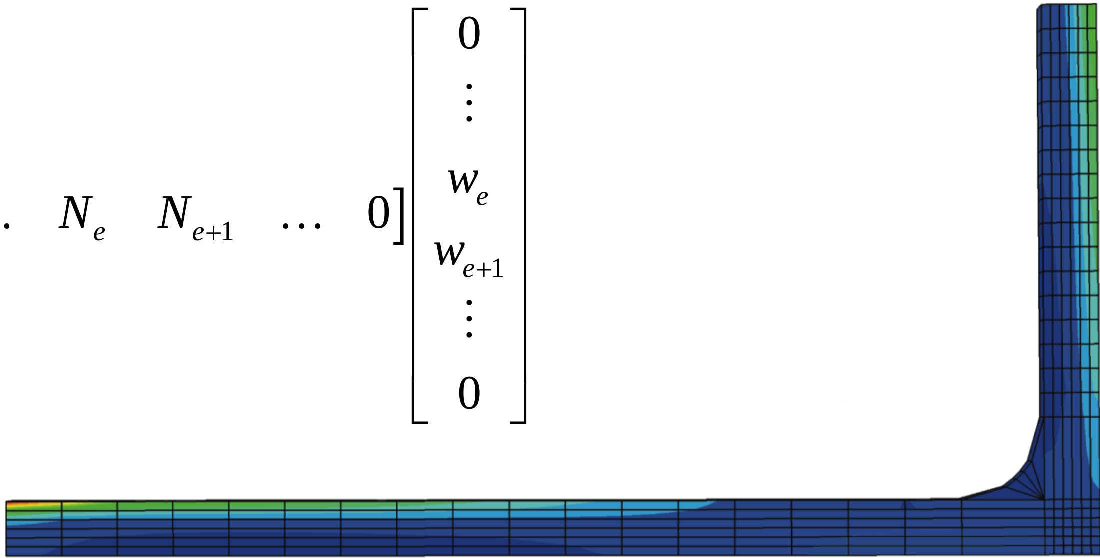


FE formulation

Using this observation we can write displacements and weights elementwise:

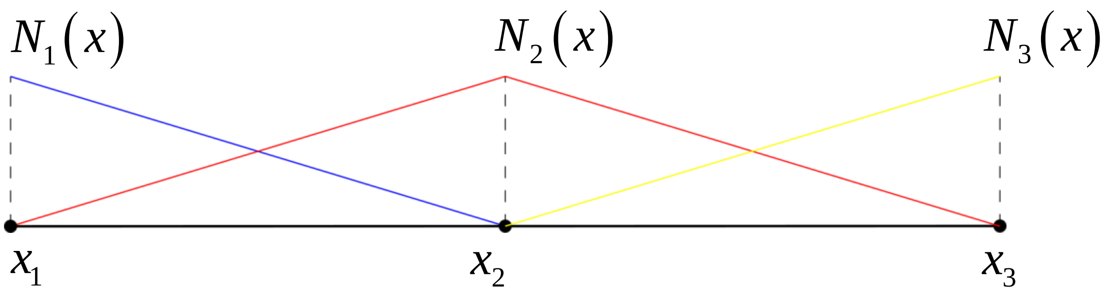
$$u^e(x) = \begin{bmatrix} N_e & N_{e+1} \end{bmatrix} \begin{bmatrix} u_e \\ u_{e+1} \end{bmatrix} = \begin{bmatrix} 0 & \dots & N_e & N_{e+1} & \dots & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ u_e \\ u_{e+1} \\ \vdots \\ 0 \end{bmatrix}$$

$$w^e(x) = \begin{bmatrix} N_e & N_{e+1} \end{bmatrix} \begin{bmatrix} w_e \\ w_{e+1} \end{bmatrix} = \begin{bmatrix} 0 & \dots & N_e & N_{e+1} & \dots & 0 \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ w_e \\ w_{e+1} \\ \vdots \\ 0 \end{bmatrix}$$



FE formulation

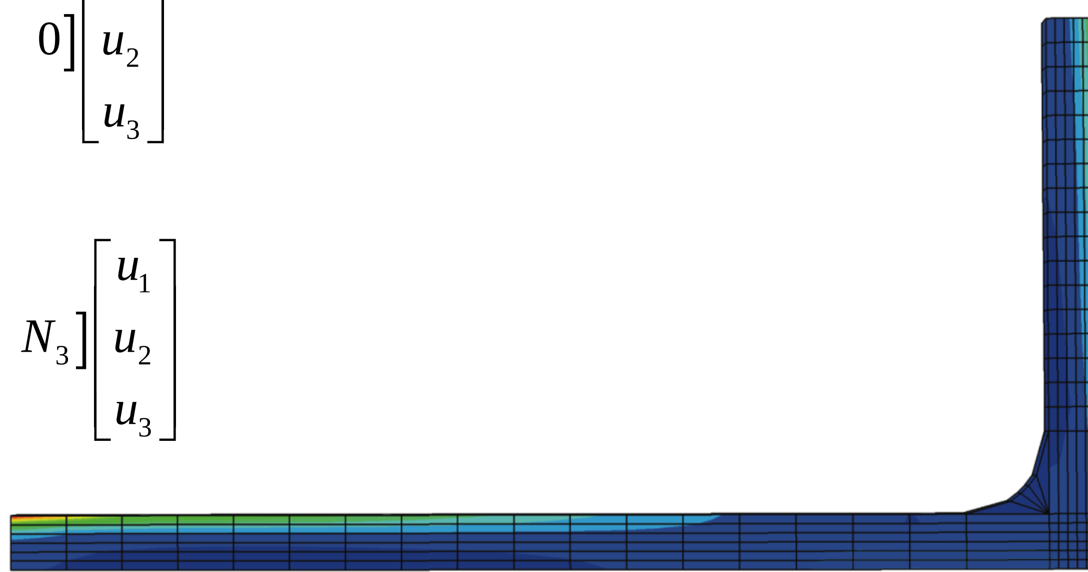
For the 2 element case:



Displacements assume the form:

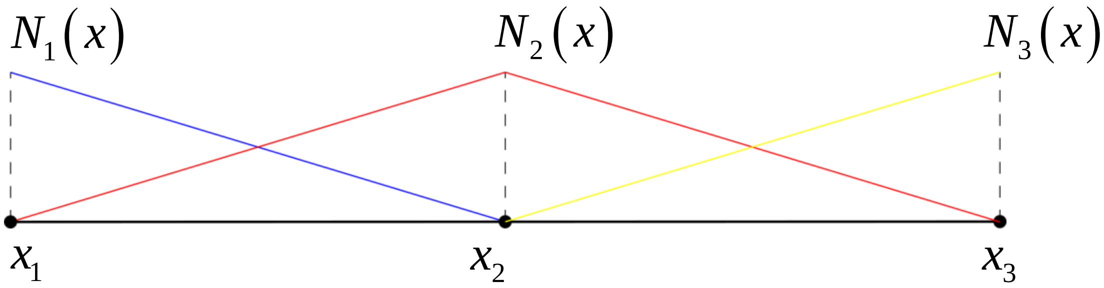
$$u^1(x) = \begin{bmatrix} N_1 & N_2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} N_1 & N_2 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$u^2(x) = \begin{bmatrix} N_2 & N_3 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 & N_2 & N_3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$



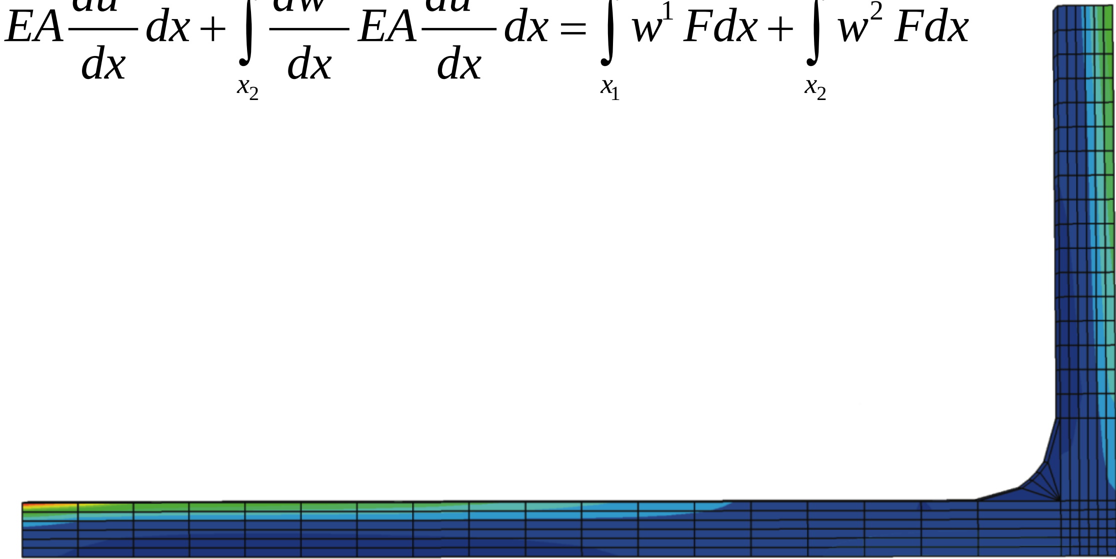
FE formulation

For the 2 element case:



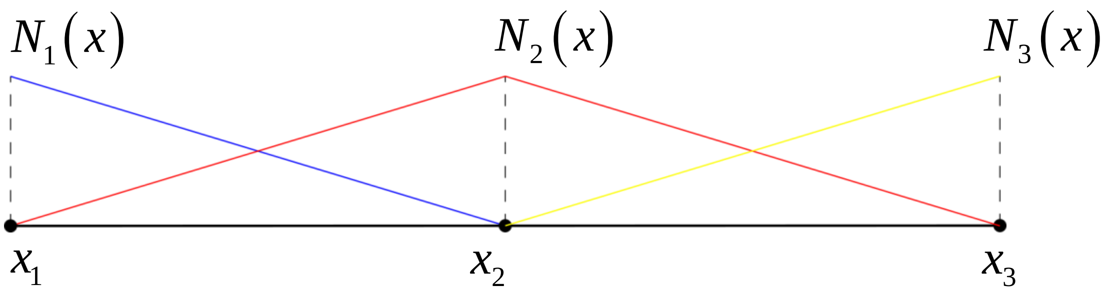
Substituting in the weak form for the two element case:

$$\int_{x_1}^{x_3} \frac{dw}{dx} EA \frac{du}{dx} dx = \int_{x_1}^{x_3} w F dx \Leftrightarrow \int_{x_1}^{x_2} \frac{dw^1}{dx} EA \frac{du^1}{dx} dx + \int_{x_2}^{x_3} \frac{dw^2}{dx} EA \frac{du^2}{dx} dx = \int_{x_1}^{x_2} w^1 F dx + \int_{x_2}^{x_3} w^2 F dx$$



FE formulation

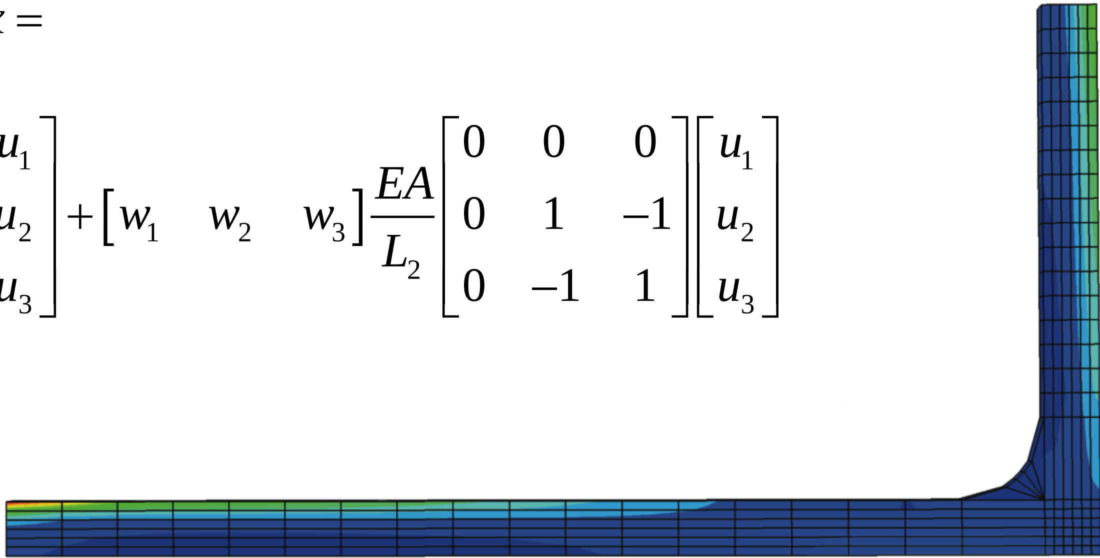
For the 2 element case:



Carrying out the derivations/integrations:

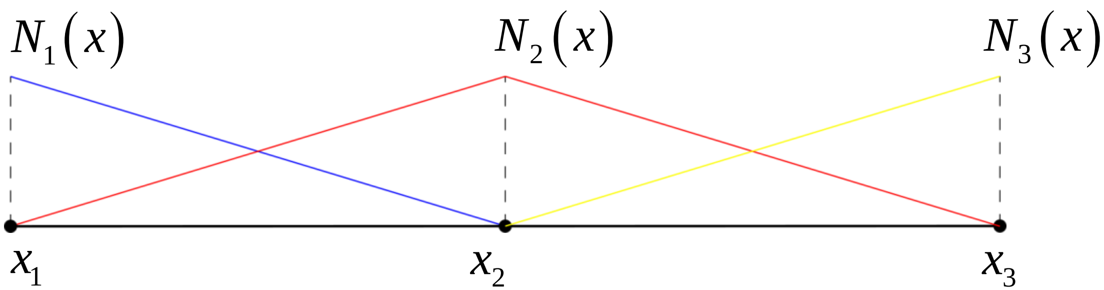
$$\int_{x_1}^{x_2} \frac{dw^1}{dx} EA \frac{du^1}{dx} dx + \int_{x_2}^{x_3} \frac{dw^2}{dx} EA \frac{du^2}{dx} dx =$$

$$= \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \frac{EA}{L_1} \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \frac{EA}{L_2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$



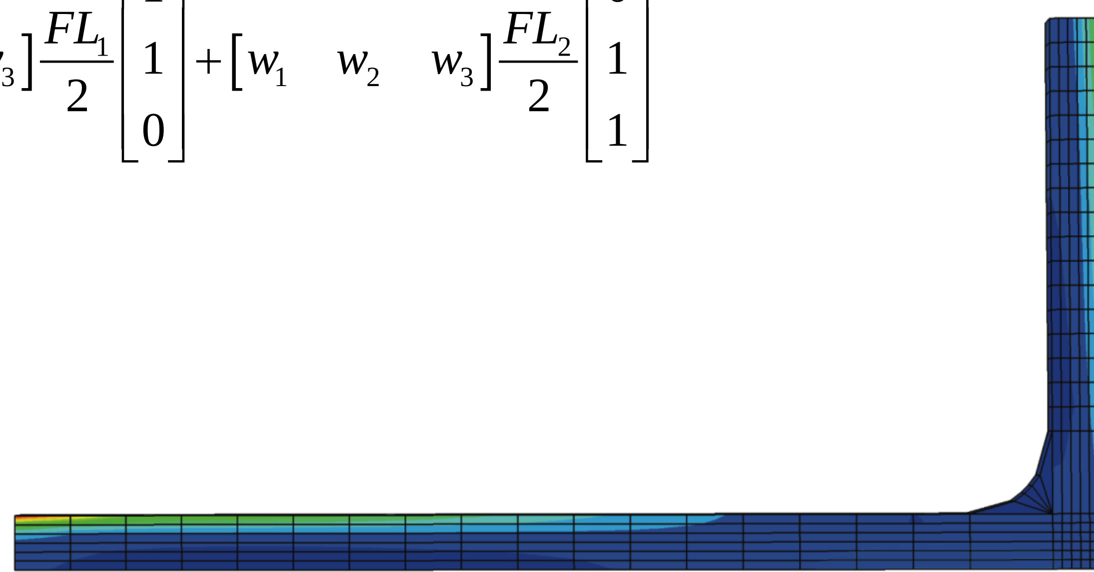
FE formulation

For the 2 element case:



Carrying out the derivations/integrations:

$$\int_{x_1}^{x_2} w^1 \, F dx + \int_{x_2}^{x_3} w^2 \, F dx = \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \frac{FL_1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix} \frac{FL_2}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$



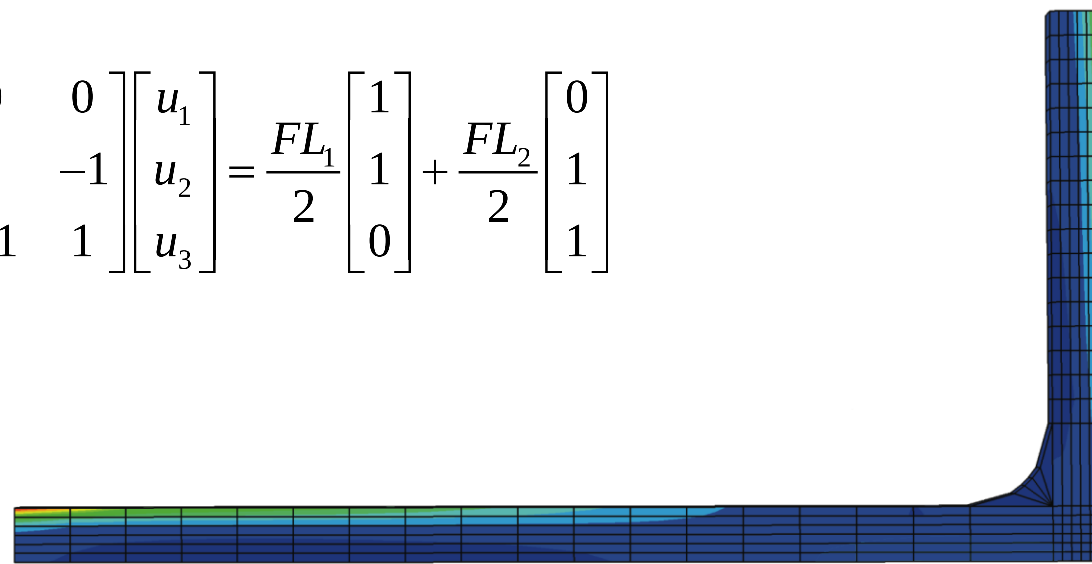
FE formulation

Equating the two sides**:**

$$\cancel{\begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix}} \frac{EA}{L_1}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \cancel{\begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix}} \frac{EA}{L_2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} =$$

$$\cancel{\begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix}} \frac{FL_1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \cancel{\begin{bmatrix} w_1 & w_2 & w_3 \end{bmatrix}} \frac{FL_2}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\frac{EA}{L_1}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \frac{EA}{L_2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \frac{FL_1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{FL_2}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$



FE formulation

Writing the equation in more general terms:

$$\frac{EA}{L_1}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \frac{EA}{L_2}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \frac{FL_1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{FL_2}{2}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} k_{11}^1 & k_{12}^1 & 0 \\ k_{21}^1 & k_{22}^1 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & k_{11}^2 & k_{12}^2 \\ 0 & k_{21}^2 & k_{22}^2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f_1^1 \\ f_2^1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ f_1^2 \\ f_2^2 \end{bmatrix}$$

$$\begin{bmatrix} k_{11}^1 & k_{12}^1 & 0 \\ k_{21}^1 & k_{22}^1 + k_{11}^2 & k_{12}^2 \\ 0 & k_{21}^2 & k_{22}^2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} f_1^1 \\ f_2^1 + f_1^2 \\ f_2^2 \end{bmatrix}$$

The assembly process is obtained!