HW 2.1
ANLY590-02
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Part I. Analytic Assignment

1) Compute the gradient vector for a plane in 3D space.

$$z = f(x,y) = ax + by + c$$

$$\boxed{\begin{aligned} f_x &= a \\ f_y &= b \\ f_z &= -1 \end{aligned} \Rightarrow \nabla f(x,y) = \begin{bmatrix} a \\ b \\ -1 \end{bmatrix}}$$

2) Compute the gradient vector for a hyperplane

$$z = f(x) = f(x_1, x_2, \cdots x_N) = \sum_{i=1}^{N} a_i(x_i - b_i) + S = a_1x_1 + a_2x_2 + \cdots + a_N x_N + d$$

$$\boxed{\begin{aligned} f_x &= a_1 + a_2 + a_3 + \cdots + a_N \\ f_z &= -1 \end{aligned} \Rightarrow \nabla f(x) = \begin{bmatrix} a_1 + a_2 + a_3 + \cdots + a_N \\ -1 \end{bmatrix}}$$

3) Compute the partial derivative of the paraboloid function.

$$z = f(x,y) = A(x-x_0)^2 + B(y-y_0)^2 + C$$

$$\boxed{\begin{aligned} f_x(x,y) &= \frac{\partial f(x,y)}{\partial x} = 2Ax - 2Ax_0 \\ f_y(x,y) &= \frac{\partial f(x,y)}{\partial y} = 2By - 2By_0 \end{aligned}}$$

4) Given the following matrices and vectors. Compute the following quantities and specify the shape of the output, If an operation is not defined then just say "not defined"

① $x^T = (3 \ 1 \ 4)$ $[1 \times 3]$

② $y^T = \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}$ $[3 \times 1]$

③ $B^T = \begin{bmatrix} 3 & 5 & 1 \\ 5 & 2 & 4 \end{bmatrix}$ $[2 \times 3]$

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④

$$x \cdot x = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$$

$$= 9+1+16 = 26$$

[1x1]

⑤

$$x \cdot y^T = \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} \begin{pmatrix} 2 \\ 5 \\ 1 \end{pmatrix}$$

$$= 6+5+4 = 15$$

[1x1]

⑥

$$x \times y = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \begin{bmatrix} 2 & 5 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 15 & 3 \\ 2 & 5 & 1 \\ 8 & 20 & 4 \end{bmatrix}$$

[3x3]

⑦

$$y \times x = \begin{bmatrix} 2 & 5 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$$

$$= 6+5+4 = 15$$

[1x1]

⑧

$$A \times X = \begin{bmatrix} 4 & 5 & 2 \\ 3 & 1 & 5 \\ 6 & 4 & 3 \end{bmatrix} \times \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4\times3+5\times1+2\times4 \\ 3\times3+1\times1+5\times4 \\ 6\times3+4\times1+3\times4 \end{bmatrix} = \begin{bmatrix} 25 \\ 30 \\ 34 \end{bmatrix}$$

[3x1]

⑨

$$A \times B = \begin{bmatrix} 4 & 5 & 2 \\ 3 & 1 & 5 \\ 6 & 4 & 3 \end{bmatrix} \times \begin{bmatrix} 3 & 5 \\ 5 & 2 \\ 1 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4\times3+5\times5+2\times1 & 4\times5+5\times2 \\ 3\times3+1\times5+5\times1 & 3\times5+1\times2+ \\ 6\times3+4\times5+3\times1 & 6\times5+4\times2 \end{bmatrix}$$

$$= \begin{bmatrix} 39 & 38 \\ 19 & 37 \\ 41 & 50 \end{bmatrix}$$

[3x2]

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B. reshape (1,6) = $\begin{pmatrix} 3 & 5 & 5 & 2 & 1 & 4 \end{pmatrix}$

[1X6]

5) Linear Least Squares (LLS): Single-variable

$$L(p) = L(m,b) = \sum_{i=1}^{N} (\hat{y}_i - M(\hat{x}_i, m, b))^2$$

$$= (y_1 - (mx_1+b))^2 + (y_2 - (mx_2+b))^2 + \dots + (y_N - (mx_N+b))^2$$

$$= y_1^2 - 2y_1(mx_1+b) + (mx_1+b)^2 + y_2^2 - 2y_2(mx_2+b) + (mx_2+b)^2 + \dots$$

$$+ y_N^2 - 2y_N(mx_N+b) + (mx_N+b)^2$$

$$= y_1^2 - 2x_1y_1m - 2y_1b + x_1^2m^2 + 2x_1mb + b^2 + \dots + y_N^2 - 2x_Ny_Nm$$

$$- 2y_Nb + x_N^2m^2 + 2x_Nmb + b^2$$

$$= (y_1^2 + y_2^2 + \dots + y_N^2) - 2m(x_1y_1 + x_2y_2 + \dots + x_Ny_N) - 2b(y_1 + y_2 + \dots + y_N)$$

$$+ (x_1^2 + x_2^2 + \dots + x_N^2)m^2 + 2(x_1 + x_2 + \dots + x_N)mb + Nb^2$$

$$= N\cdot\overline{y^2} - 2mN\cdot\overline{xy} - 2b\cdot N\bar{y} + m^2\cdot N\cdot\overline{x^2} + 2mb\cdot N\cdot\bar{x} + Nb^2$$

$$\frac{dL}{dm} = -2N\overline{xy} + 2N\overline{x^2}m + 2bN\bar{x} = 0$$

$$\Rightarrow \overline{x^2}m + b\bar{x} - \overline{xy} = 0$$

$$\frac{dL}{db} = -2N\bar{y} + 2mN\bar{x} + 2Nb = 0$$

$$\Rightarrow m\bar{x} - \bar{y} + b = 0$$

$$\begin{cases} \overline{x^2}m + b\bar{x} - \overline{xy} = 0 \\ m\bar{x} - \bar{y} + b = 0 \end{cases} \Rightarrow \begin{array}{l} \overline{x^2}m + (\bar{y} - m\bar{x})\bar{x} - \overline{xy} = 0 \\ \overline{x^2}m + \bar{x}\bar{y} - m(\bar{x})^2 - \overline{xy} = 0 \end{array}$$

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$$\begin{cases} m = \dfrac{\bar{x}\bar{y} - \overline{xy}}{(\bar{x})^2 - \overline{x^2}} \\ \\ b = \bar{y} - m\bar{x} = \bar{y} - \dfrac{\bar{x}\bar{y} - \overline{xy}}{(\bar{x})^2 - \overline{x^2}}\bar{x} \end{cases}$$

Since $var(X) = \frac{1}{N}\sum_{i=1}^{n}(x_i - \bar{x})^2 = \frac{1}{N}\left(x_1^2 - 2x_1\bar{x} + (\bar{x})^2 + \cdots + x_n^2 - 2\right.$

$$= \frac{1}{N}\left(N\cdot\overline{x^2} - 2N\cdot\bar{x}\cdot\bar{x} + N(\bar{x})^2\right)$$

$$= \overline{x^2} - (\bar{x})^2$$

$cov(X,Y) = \frac{1}{N}\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y}) = \frac{1}{N}\left(x_i y_i - x_i\bar{y} - y_i\bar{x} + \bar{x}\right.$

$$= \frac{1}{N}\left(N\cdot\overline{xy} - N\bar{x}\bar{y} - N\bar{y}\bar{x} + N\bar{x}\bar{y}\right)$$

$$= \frac{1}{N}\left(N\cdot\overline{xy} - N\bar{x}\bar{y}\right)$$

$$= \overline{xy} - \bar{x}\bar{y}$$

$$m = \frac{-cov(X,Y)}{-var(X)} = \frac{cov(X,Y)}{var(X)}$$

$$b = \bar{y} - \frac{cov(X,Y)}{var(X)}\bar{x}$$