INTRODUCTION

This book is intended as background reading for modern asset pricing theory as outlined by Jarrow (1996), Hull (1999), Duffie (1996), Ingersoll (1987), Musiela and Rutkowski (1997), and other excellent sources.

Pricing models for financial derivatives require, by their very nature, utilization of continuous-time stochastic processes. A good understanding of the tools of stochastic calculus and of some deep theorems in the theory of stochastic processes is necessary for practical asset valuation.

There are several excellent technical sources dealing with this mathematical theory. Karatzas and Shreve (1991), Karatzas and Shreve (1999), and Revuz and Yor (1994) are the first that come to mind. Others are discussed in the references. Yet even to a mathematically well-trained reader, these sources are not easy to follow. Sometimes, the material discussed has no direct applications in finance. At other times, the practical relevance of the assumptions is difficult to understand.

The purpose of this text is to provide an introduction to the mathematics utilized in the pricing models of derivative instruments. The text approaches the mathematics behind continuous-time finance informally. Examples are given and relevance to financial markets is provided.

Such an approach may be found imprecise by a technical reader. We simply hope that the informal treatment provides enough intuition about some of these difficult concepts to compensate for this shortcoming. Unfortunately, by providing a descriptive treatment of these concepts, it is difficult to emphasize technicalities. This would defeat the purpose of the book. Further, there are excellent sources at a technical level. What seems to be missing is a text that explains the assumptions and concepts behind these mathematical tools and then relates them to dynamic asset pricing theory.

1 Audience

The text is directed toward a reader with some background in finance. A strong background in calculus or stochastic processes is not needed, although previous courses in these fields will certainly be helpful. One chapter will review some basic concepts in calculus, but it is best if the reader has already fulfilled some minimum calculus requirements. It is hoped that strong practitioners in financial markets, as well as beginning graduate students, will find the text useful.

2 New Developments

During the past two decades, some major developments have occurred in the theoretical understanding of how derivative asset prices are determined and how these prices move over time. There were also some recent institutional changes that indirectly made the methods discussed in the following pages popular.

The past two decades saw the freeing of exchange and capital controls. This made the exchange rates significantly more variable. In the meantime, world trade grew significantly. This made the elimination of currency risk a much higher priority.

During this time, interest rate controls were eliminated. This coincided with increases in the government budget deficits, which in turn led to large new issues of government debt in all industrialized nations. For this reason (among others), the need to eliminate the interest-rate risk became more urgent. Interest-rate derivatives became very popular.

It is mainly the need to hedge interest-rate and currency risks that is at the origin of the recent prolific increase in markets for derivative products. This need was partially met by financial markets. New products were developed and offered, but the conceptual understanding of the structure, functioning, and pricing of these derivative products also played an important role. Because theoretical valuation models were directly applicable to these new products, financial intermediaries were able to "correctly" price and successfully market them. Without such a clear understanding of the conceptual framework, it is not evident to what extent a similar development might have occurred.

As a result of these needs, new exchanges and marketplaces came into existence. Introduction of new products became easier and less costly.

Trading became cheaper. The deregulation of the financial services that gathered steam during the 1980s was also an important factor here.

Three major steps in the theoretical revolution led to the use of advanced mathematical methods that we discuss in this book:

- The *arbitrage theorem*[1] gives the formal conditions under which "arbitrage" profits can or cannot exist. It is shown that if asset prices satisfy a simple condition, then arbitrage cannot exist. This was a major development that eventually permitted the calculation of the arbitrage-free price of any "new" derivative product. Arbitrage pricing must be contrasted with *equilibrium pricing*, which takes into consideration conditions other than arbitrage that are imposed by general equilibrium.
- The *Black–Scholes model* (Black and Scholes, 1973) used the method of arbitrage-free pricing. But the paper was also influential because of the technical steps introduced in obtaining a closed-form formula for options prices. For an approach that used abstract notions such as Ito calculus, the formula was accurate enough to win the attention of market participants.
- The methodology of using *equivalent martingale measures* was developed later. This method dramatically simplified and generalized the original approach of Black and Scholes. With these tools, a general method could be used to price any derivative product. Hence, arbitrage-free prices under more realistic conditions could be obtained.

Finally, derivative products have a property that makes them especially suitable for a mathematical approach. Despite their apparent complexity, derivative products are in fact extremely simple instruments. Often their value depends only on the underlying asset, some interest rates, and a few parameters to be calculated. It is significantly easier to model such an instrument mathematically[2] than, say, to model stocks. The latter are titles on private companies, and in general, hundreds of factors influence the performance of a company and, hence, of the stock itself.

3 Objectives

We have the following plan for learning the mathematics of derivative products.

[1]This is sometimes called "the Fundamental Theorem of Finance."

[2]This is especially true if one is armed with the arbitrage theorem.

3.1 *The Arbitrage Theorem*

The meaning and the relevance of the *arbitrage theorem* will be introduced first. This is a major result of the theory of finance. Without a good understanding of the conditions under which arbitrage, and hence infinite profits, is ruled out, it would be difficult to motivate the mathematics that we intend to discuss.

3.2 *Risk-Neutral Probabilities*

The arbitrage theorem, by itself, is sufficient to introduce some of the main mathematical concepts that we discuss later. In particular, the arbitrage theorem provides a *mathematical framework* and, more important, justifies the existence and utilization of risk-neutral probabilities. The latter are "synthetic" probabilities utilized in valuing assets. They make it possible to bypass issues related to risk premiums.

3.3 *Wiener and Poisson Processes*

All of these require an introductory discussion of Wiener processes from a practical point of view, which means learning the "economic assumptions" behind notions such as Wiener processes, stochastic calculus, and differential equations.

3.4 *New Calculus*

In doing this, some familiarity with the *new* calculus needs to be developed. Hence, we go over some of the basic results and discuss some simple examples.

3.5 *Martingales*

At this point, the notion of martingales and their uses in asset valuation should be introduced. *Martingale measures* and the way they are utilized in valuing asset prices are discussed with examples.

3.6 *Partial Differential Equations*

Derivative asset valuation utilizes the notion of arbitrage to obtain *partial differential equations* (PDEs) that must be satisfied by the prices of these products. We present the mathematics of partial differential equations and their numerical estimation.

3.7 *The Girsanov Theorem*

The Girsanov theorem permits changing means of random processes by varying the underlying probability distribution. The theorem is in the background of some of the most important pricing methods.

3.8 *The Feynman–Kac Formula*

The Feynman–Kac formula and its simpler versions give a correspondence between classes of partial differential equations and certain conditional expectations. These expectations are in the form of discounted future asset prices, where the discount rate is *random*. This correspondence is useful in pricing interest-rate derivatives.

3.9 *Examples*

The text gives as many examples as possible. Some of these examples have relevance to financial markets; others simply illustrate the mathematical concept under study.