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PREFACE TO THE SECOND EDITION

This edition is divided into two parts. The first part is essentially the revised and expanded version of the first edition and consists of 15 chapters. The second part is entirely new and is made of 7 chapters on more recent and more complex material.

Overall, the additions amount to nearly doubling the content of the first edition. The first 15 chapters are revised for typos and other errors and are supplemented by several new sections. The major novelty, however, is in the 7 chapters contained in the second part of the book. These chapters use a similar approach adopted in the first part and deal with mathematical tools for fixed-income sector and interest rate products. The last chapter is a brief introduction to stopping times and American-style instruments.

The other major addition to this edition are the Exercises added at the ends of the chapters. Solutions will appear in a separate solutions manual.

Several people provided comments and helped during the process of revising the first part and with writing the seven new chapters. I thank Don Chance, Xiangrong Jin, Christina Yunzal, and the four anonymous referees who provided very useful comments. The comments that I received from numerous readers during the past three years are also greatly appreciated.