- (a) What is the best choice of  $k_t$  for period t = 4?
- (b) What is the best choice of  $k_t$  for period t = 3?
- (c) From these, can you iterate and find the best choice of  $k_t$  for t = 1?
- (d) Determine the value function  $V_t$  that gives the optimal payoff for t = 1, 2, 3, 4
  - (e) Plot the value function V, and interpret it.



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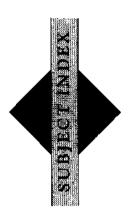
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