Lepp triangulations and applications to science, engineering, medicine, visualization and terrain modeling

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Outline

- Triangulations: introduction, applications, Delaunay
- Triangulation problems
- Longest edge algorithms for adaptive / multigrid finite element methods
- Lepp algorithms
  - improved, simple, more general
  - quality triangulation
- Recent results on Lepp centroid algorithms
  - 2D optimal size quality triangulations
  - Simple and efficient 3D improvement algorithms
- Conclusions on further research

Discretization

Key concept for

- numerical methods
- computational science / engineering
- visualization
- medical applications
- computer graphics / computer games
- computer animation

Given a geometric (continium) object, use a finite set of points to construct a geometric model

Triangulations

- The most used techniques for modeling geometric objects
- Flexible, easily adapted to complex object
- Suitable for non-uniform representations as required in finite element methods and other complex applications

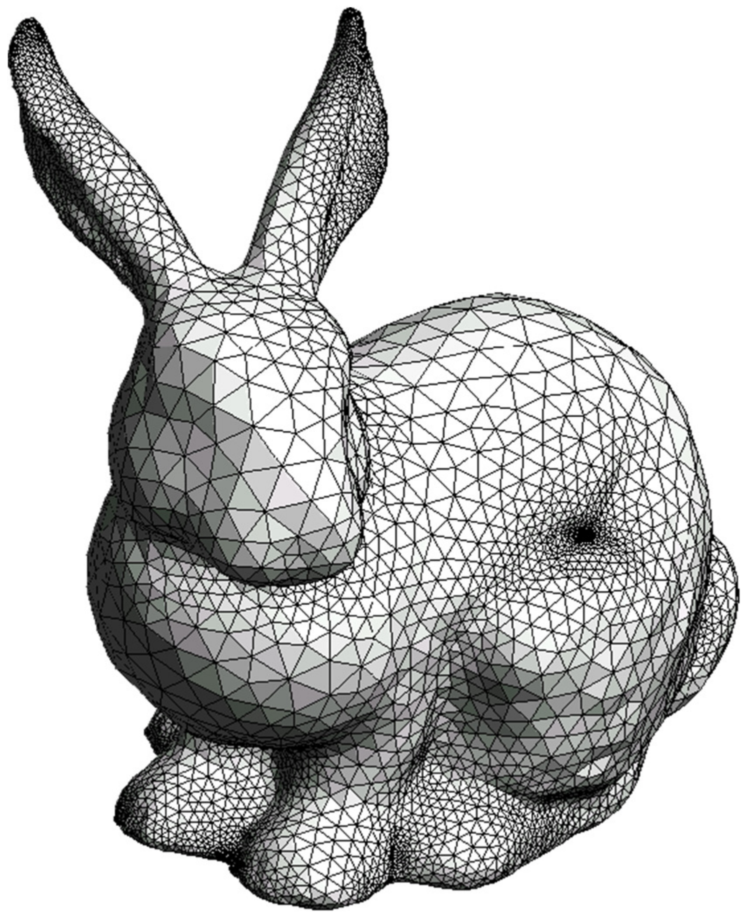
Algorithms

- Initially developed for computational engineering and CAD applications
- Later for computer graphics, computer games, 3D animation

Utah tea pot!

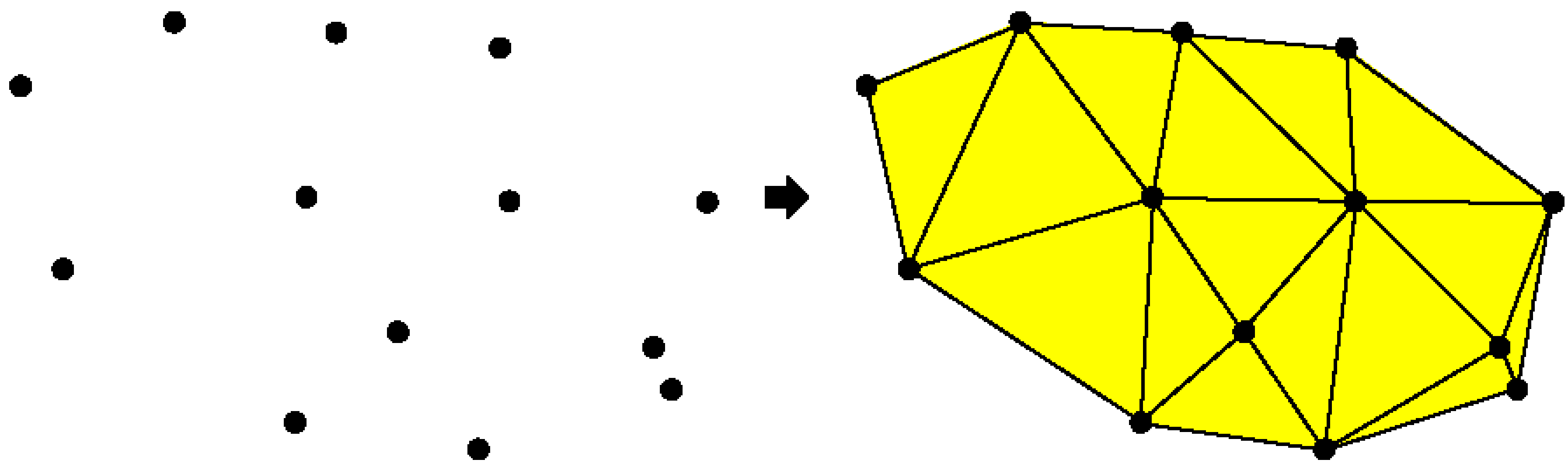
http://mech.fsv.cvut.cz/~dr/papers/Habil03/node51.html

Stanford rabbit



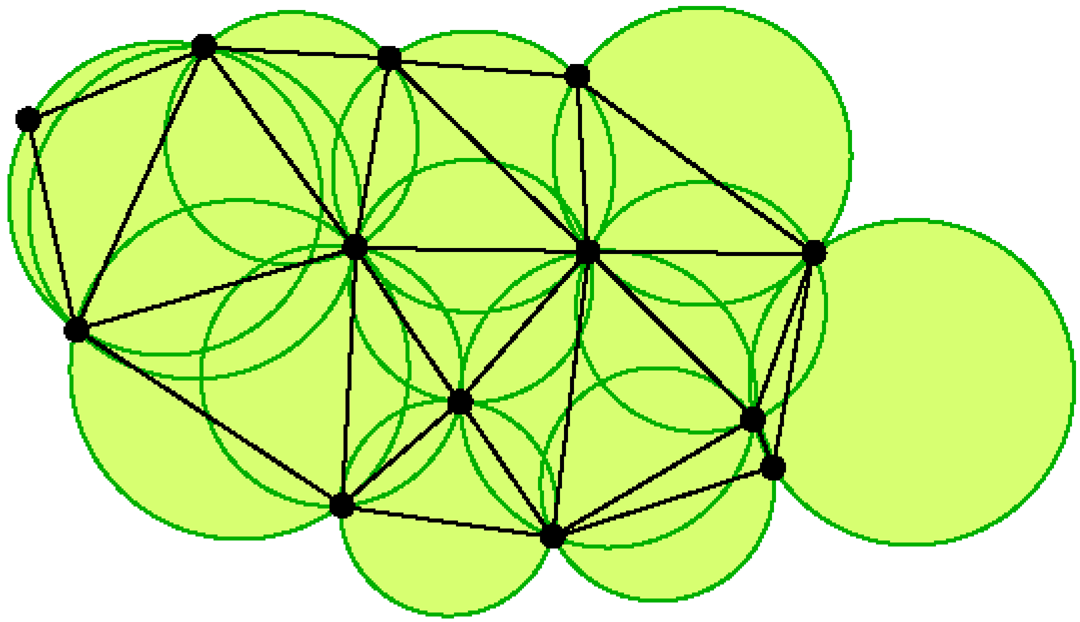
http://mech.fsv.cvut.cz/~dr/papers/Habil03/node51.html

Delaunay triangulation



Property: the most equilateral triangulation

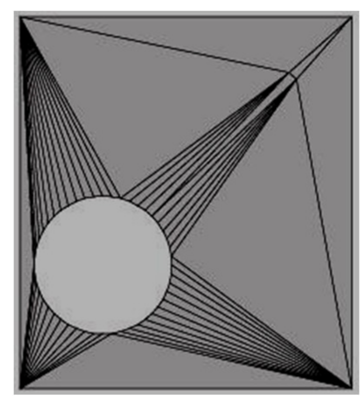
Delaunay: circumcircle property



Triangulation quality

- Delaunay triangulation is the most equilateral triangulation given the point distribution.
- To improve triangulation is necessary to add points

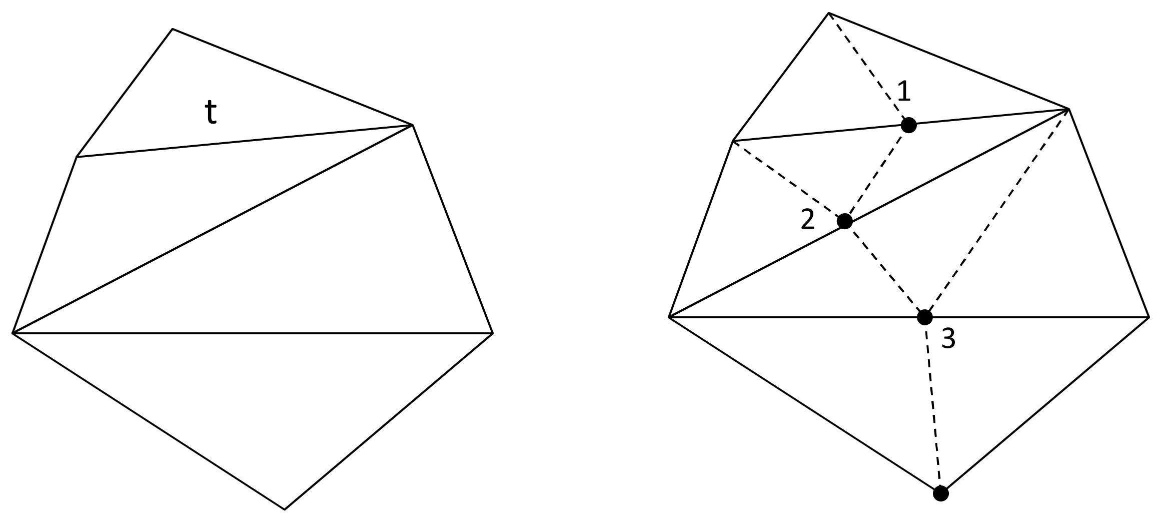
Constrained Delaunay triangulation



Longest edge bisection algorithms for adaptive / multigrid finite element methods

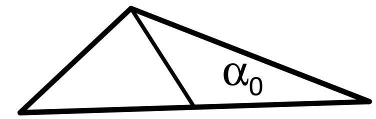
Rivara 84, 87, 92

- Local refinement for good input triangulations
- Only longest edge bisections are performed



Iterative longest edge bisection of traingles

Rosenberg-Stenger 1975, Stynes 1979, 1980



- angles $\geq \alpha_0 / 2$
- produces finite number of nonsimilar triangles
- Special set of quasiequilateral triangles
    - at most four nonsimilar triangles
    - set is closed
- Non quasiequilateral triangles converge towards quasiequilateral triangles (triangles tend to be improved)

Result: longest edge algorithms maintain triangulation quality

Example: adaptive triangulation obtained with the refinement algorithm

Rivara 1984

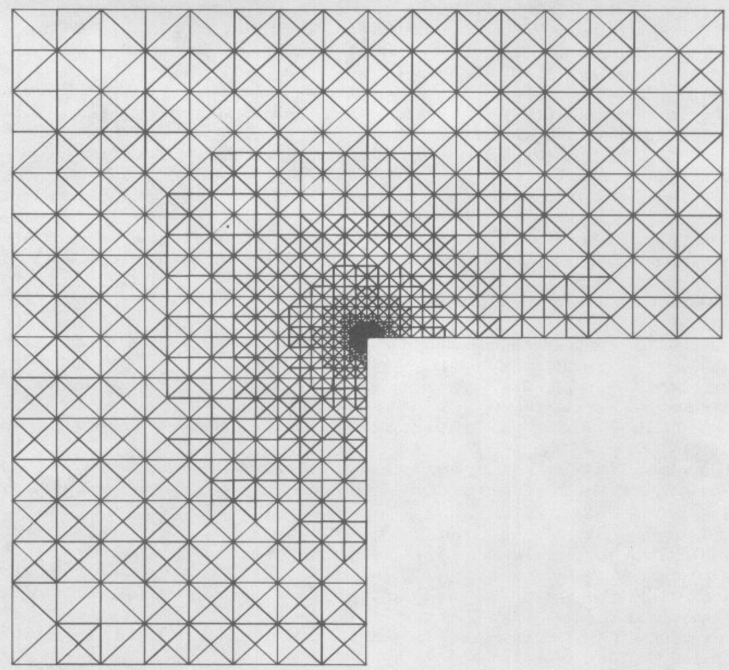
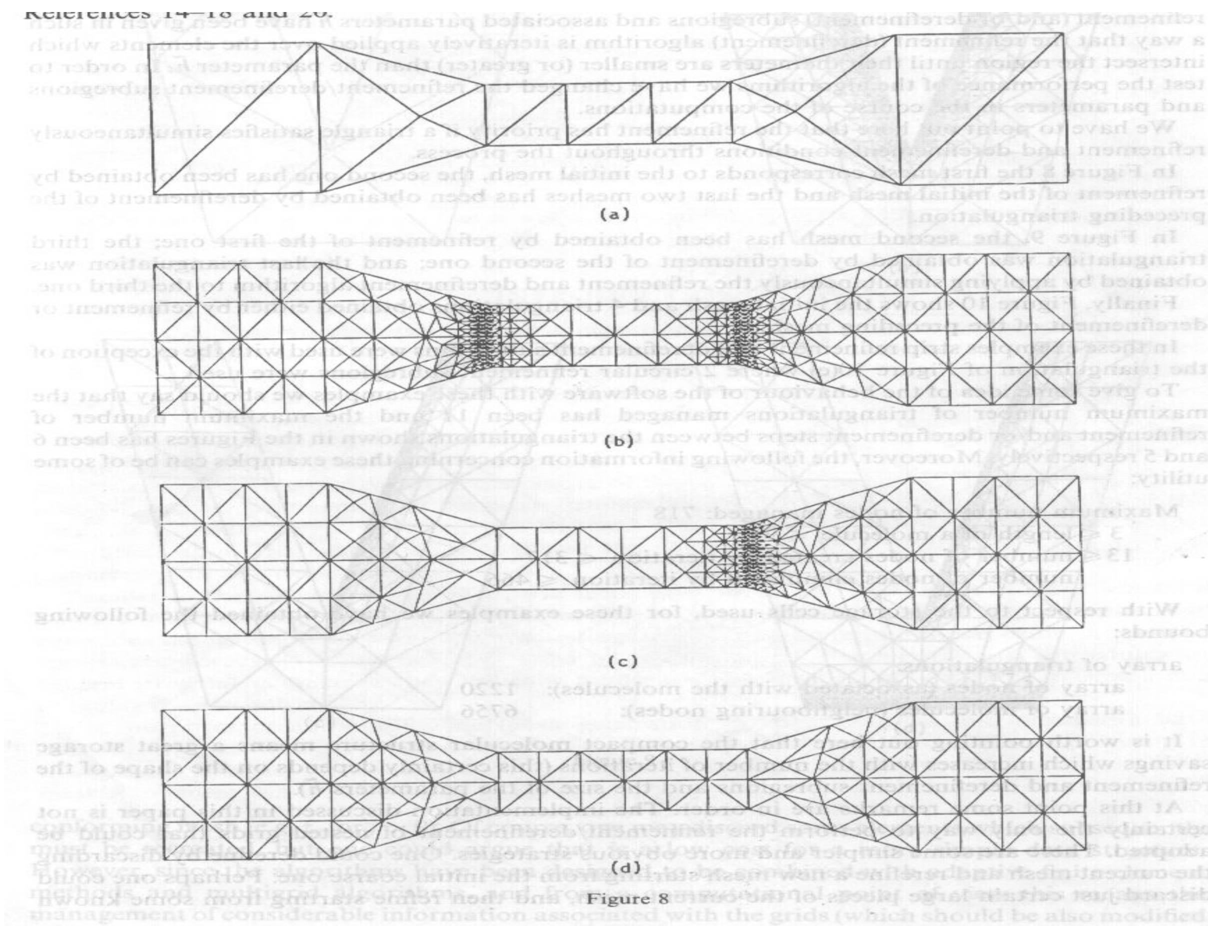


Fig. 13. Last adaptive triangulation obtained for Problem 2.

Refinement / Derefinement algorithm

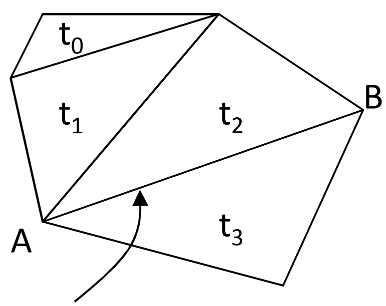
Rivara, Spencer 1989



Lepp algorithms: improved and more general longest-edge based algorithms

Rivara, Simpson, Hitschfeld, others, since 1997

- Longest-Edge Propagation Path (t) Lepp(t)



Terminal-edge

$Lepp(t_0) = \{t_0, t_1, t_2, t_3\}$

t_2, t_3 terminal triangles

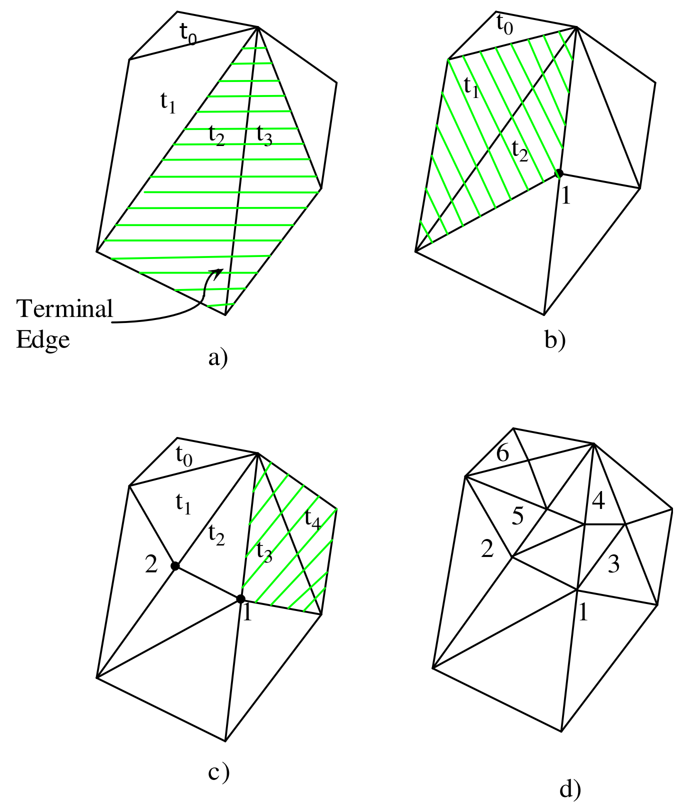
local mesh operations over terminal triangles

Lepp improvement / refinement algorithms

- Given target triangle t to be refined / improved
- Find Lepp (t), associated terminal triangles t_1*, t_2* and terminal edge E
- Select point P inside terminal triangles
- Insert point p in mesh.

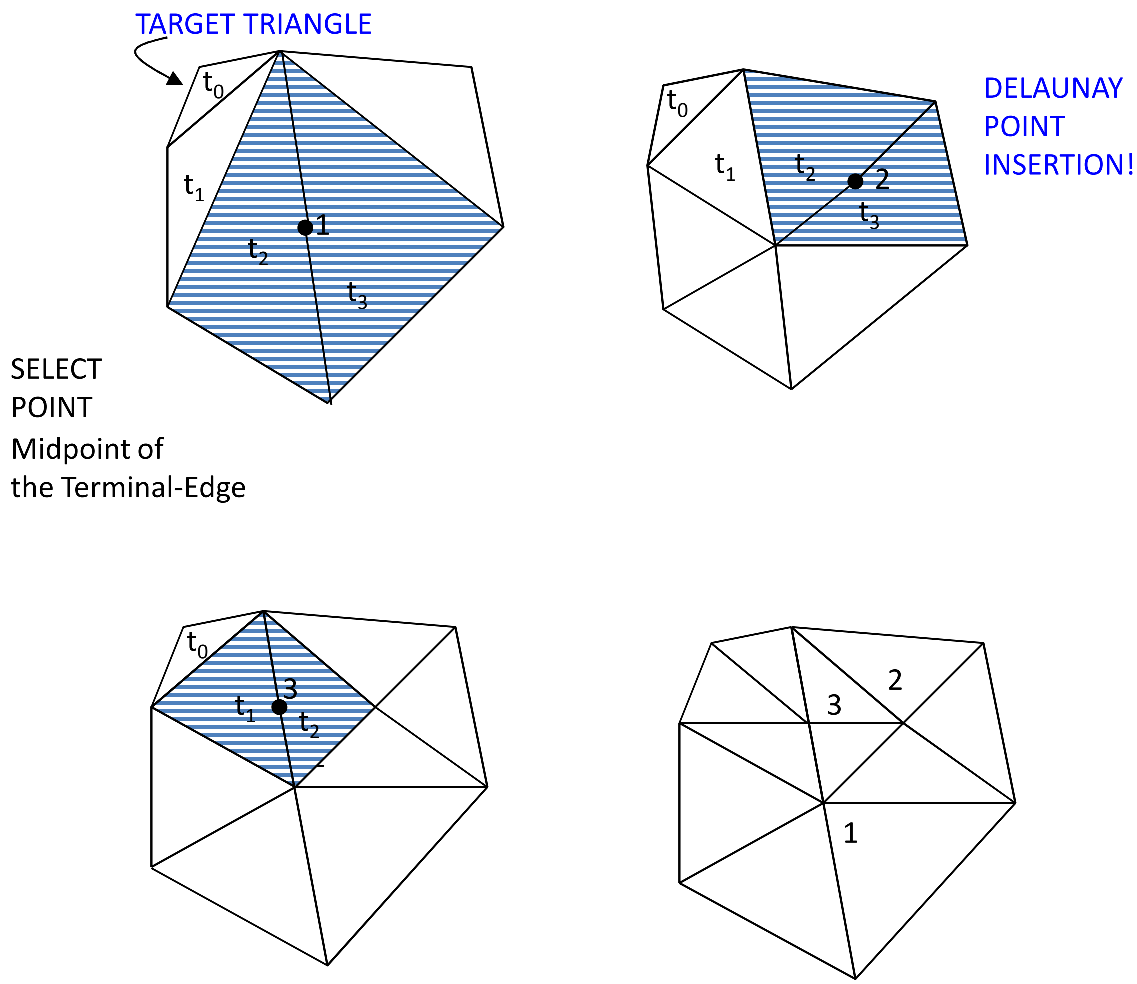
Point selection: midpoint of E / centroid of t_1*, t_2*
Point insertion: longest edge bisection / Delaunay

Lepp-bisection Refinement Algorithm



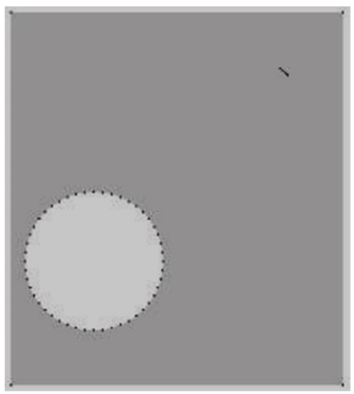
- Algorithm maintains mesh quality!
- Average Lepp size is 2!

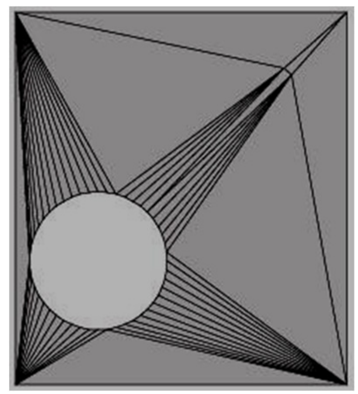
Lepp- Delaunay algorithm

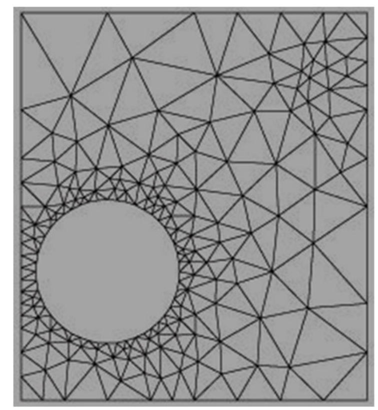


Lepp- Delaunay
Automatic quality triangulation algorithm
2 and 3 dimensions

Rivara, Palma 1997



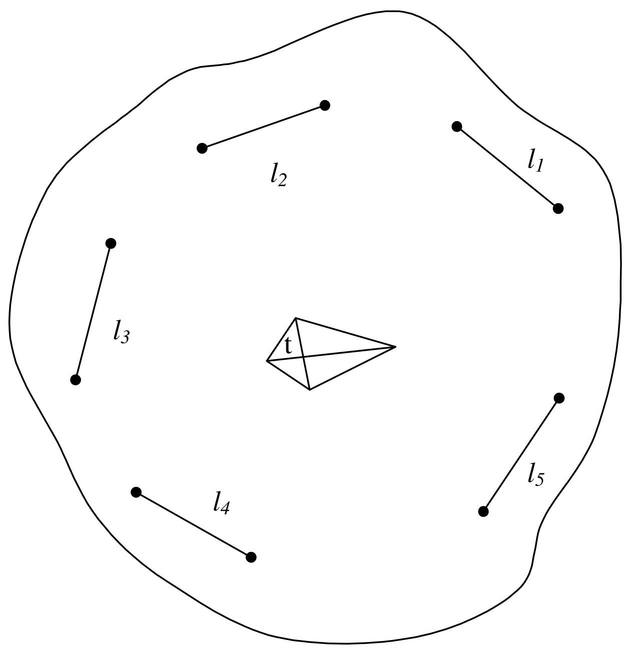




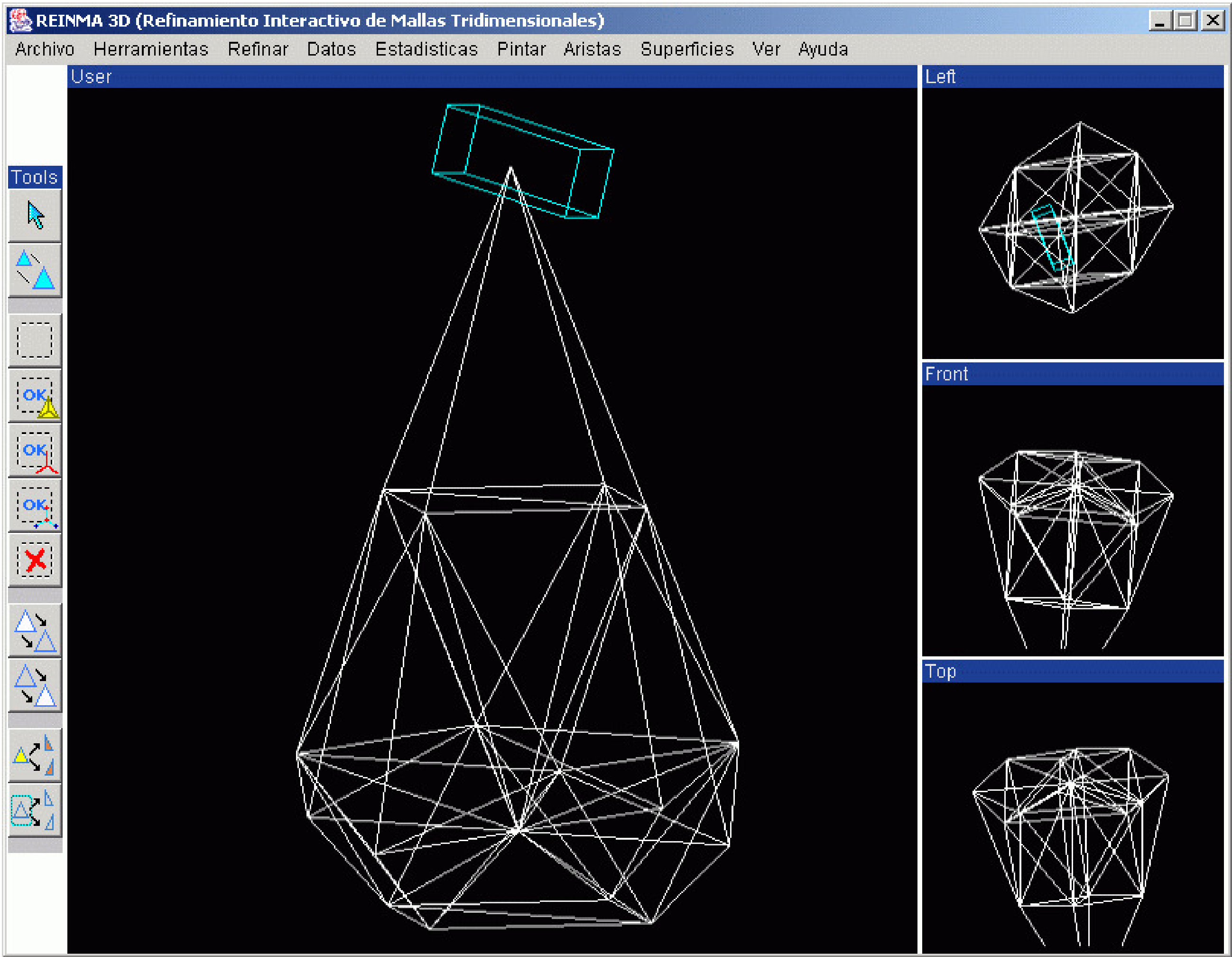
Min angle $\geq 30º$

Lepp(t) in 3-dimensions

- A submesh associated to an individual tetrahedron
- Allows the identification of a set of terminal-edges



- Each Terminal-edge is shared by a set of tetrahedra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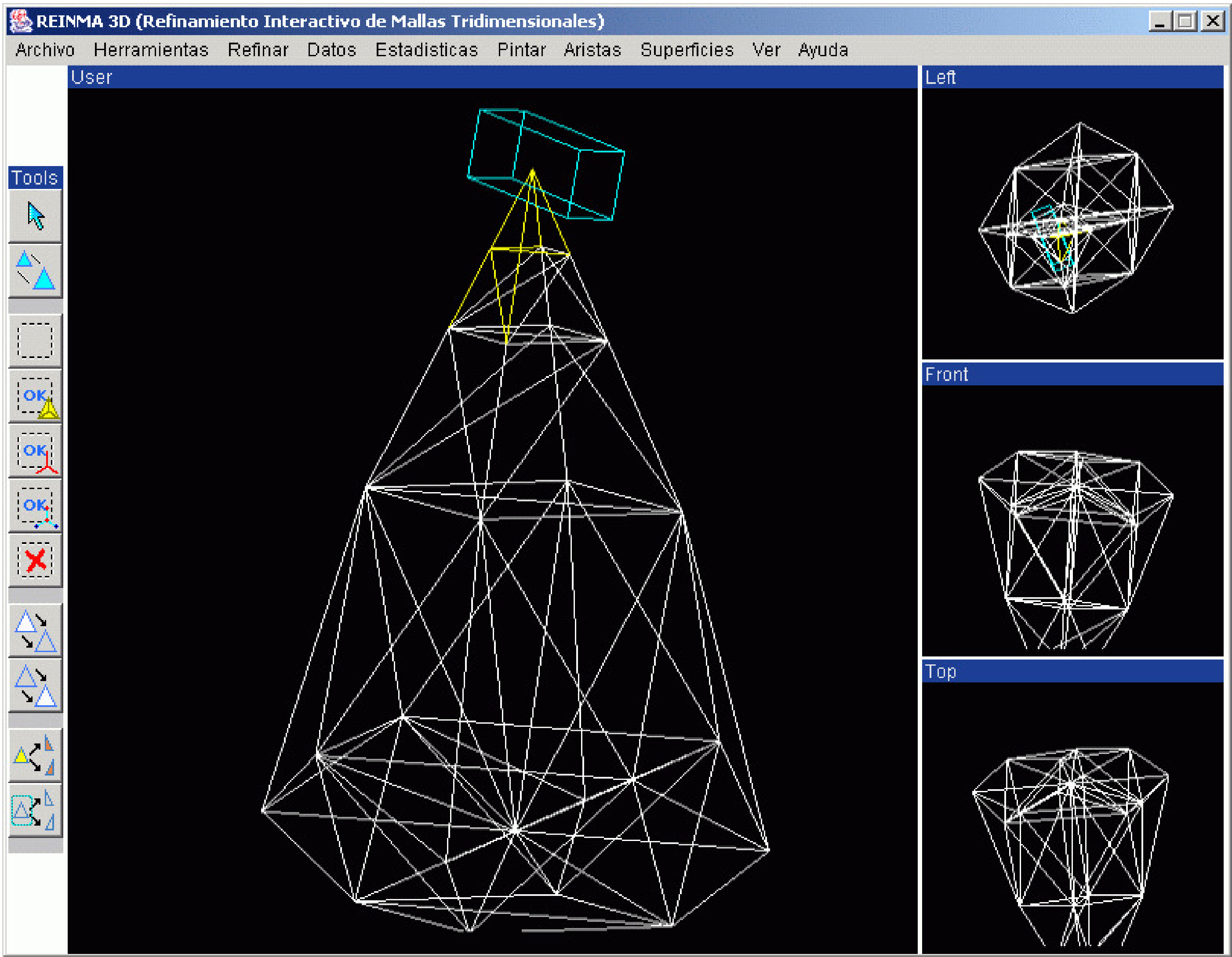
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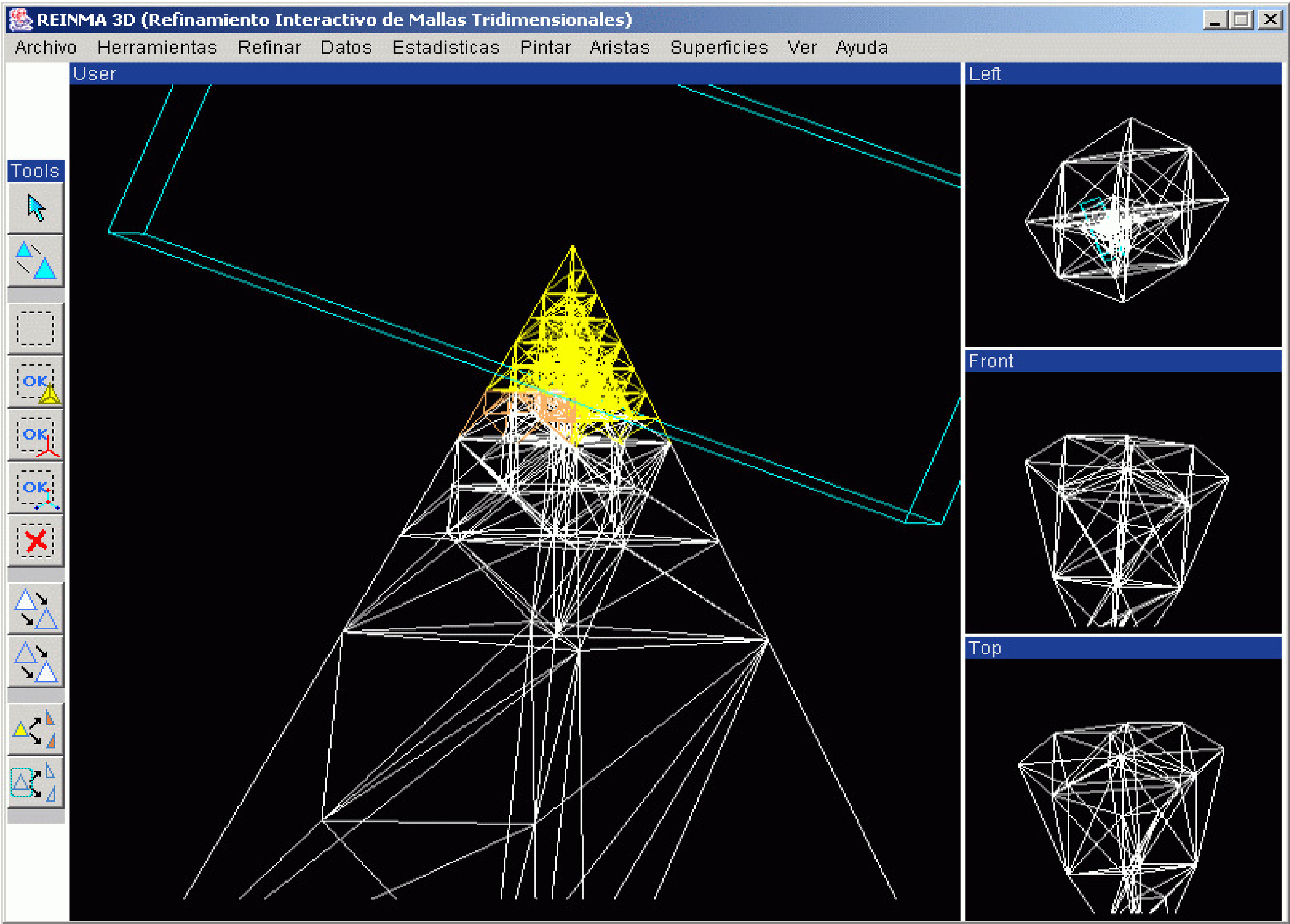
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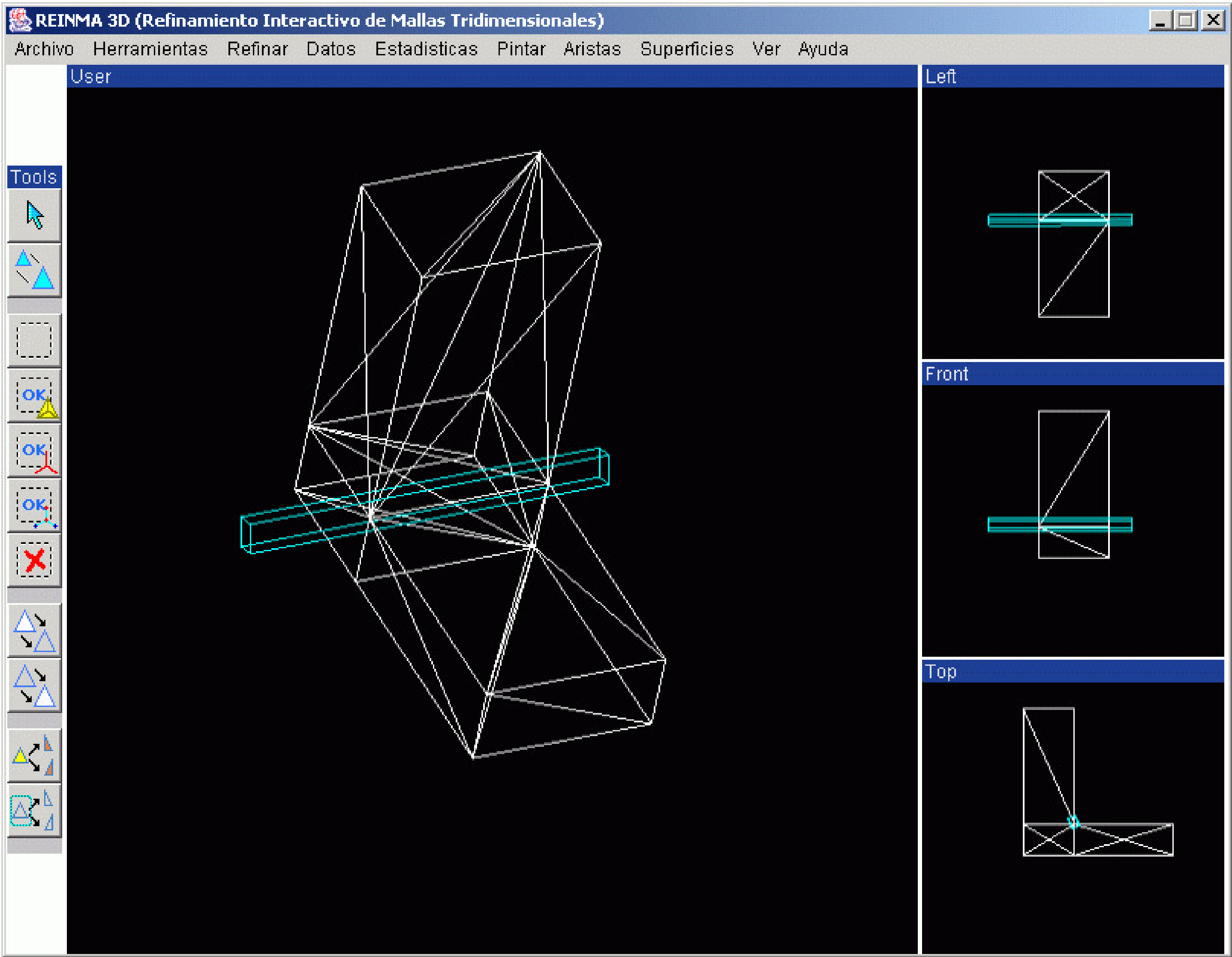
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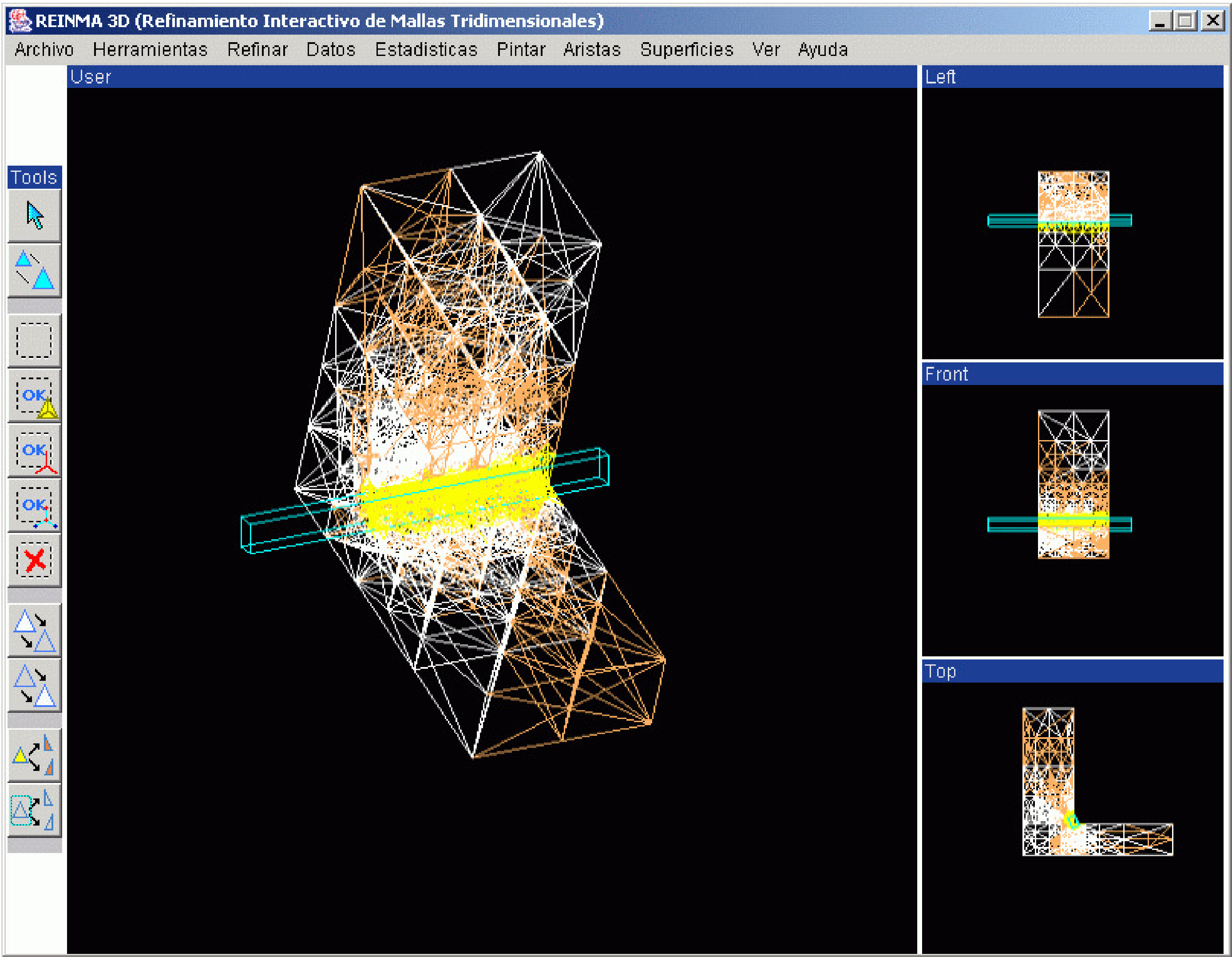
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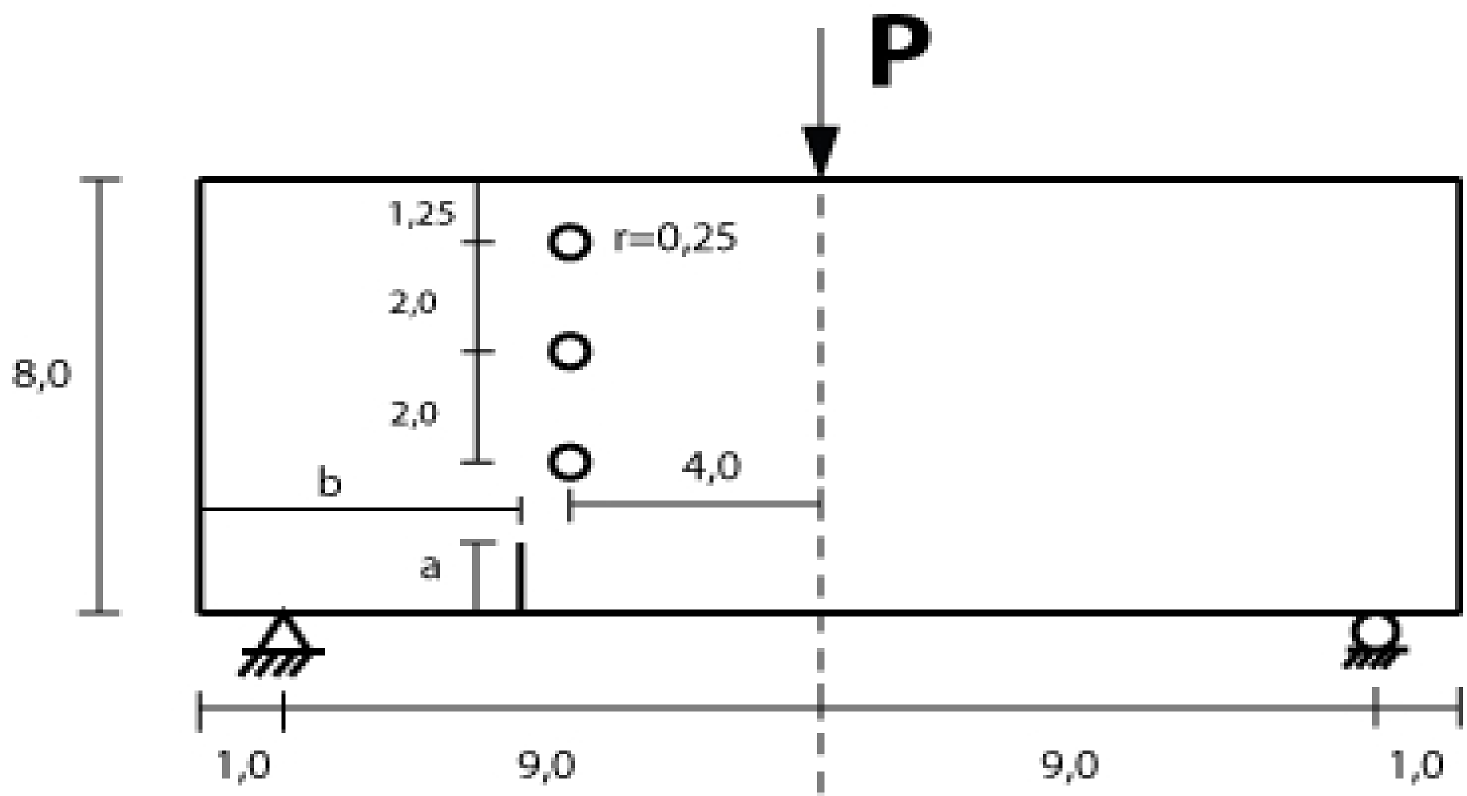
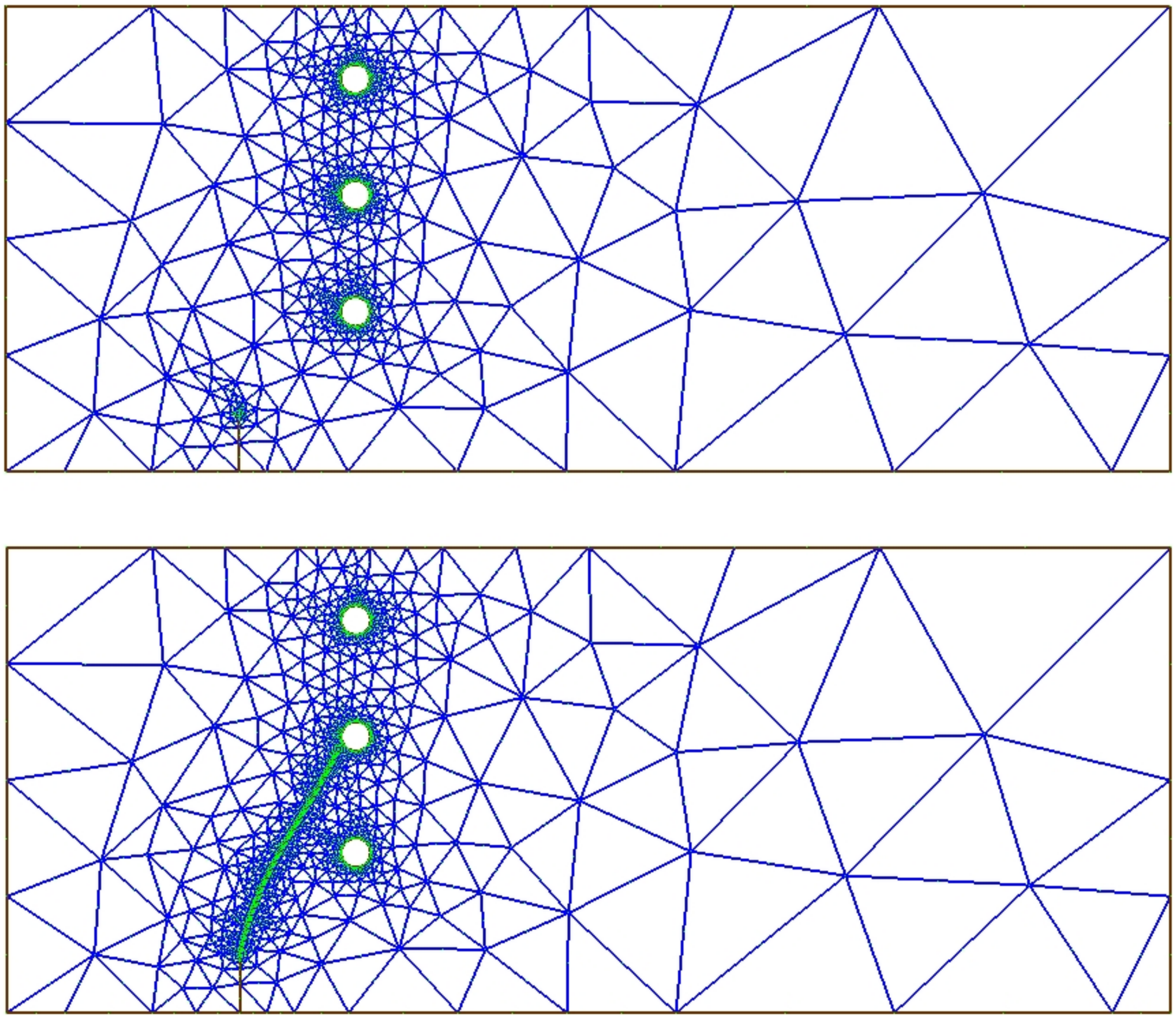
Center-cracked beam with three holes



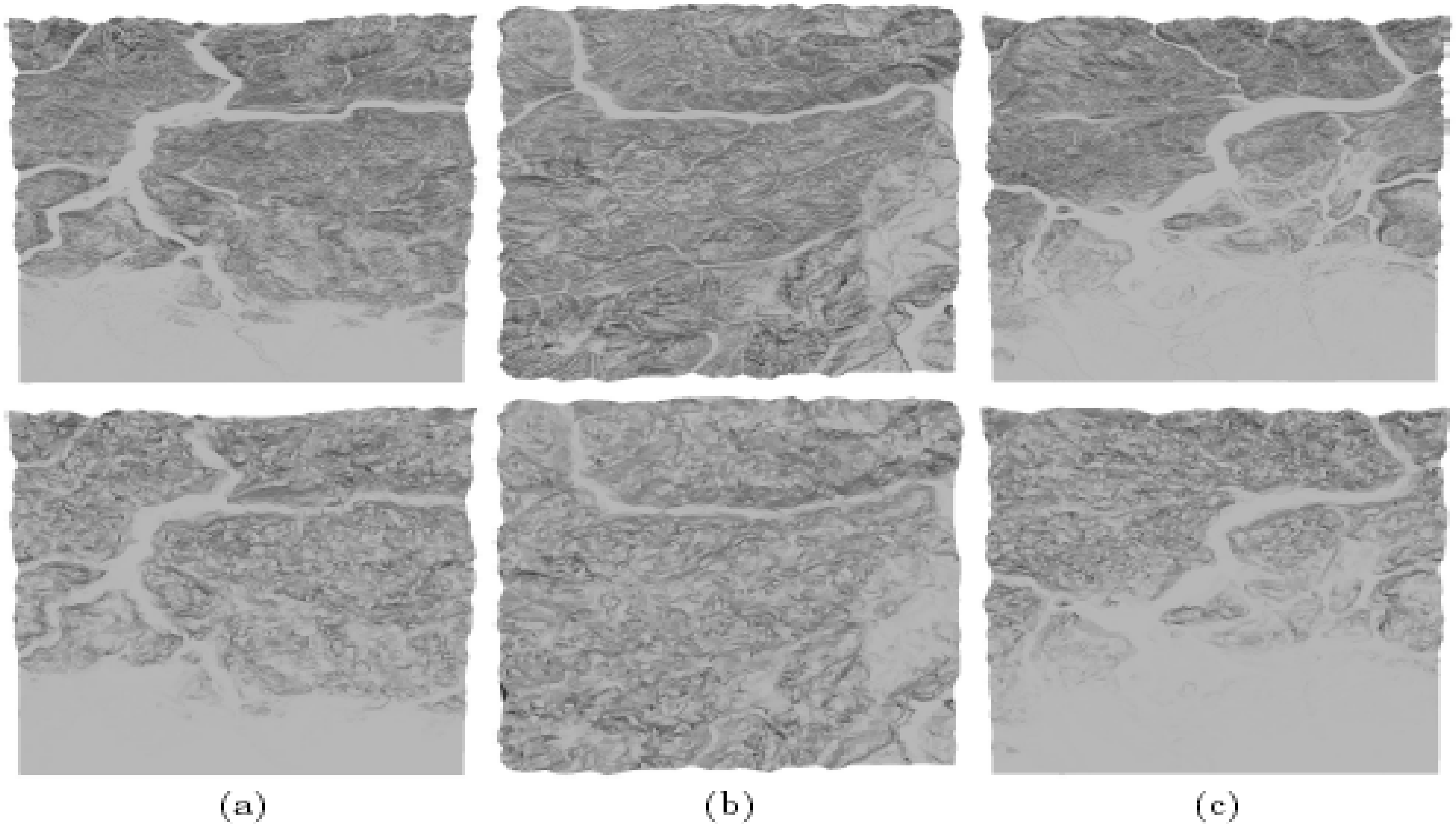
CASO	a	b
I	1,5	5,0
II	1,0	4,0

Mesh evolution

Azócar, Elgueta, Rivara 2010

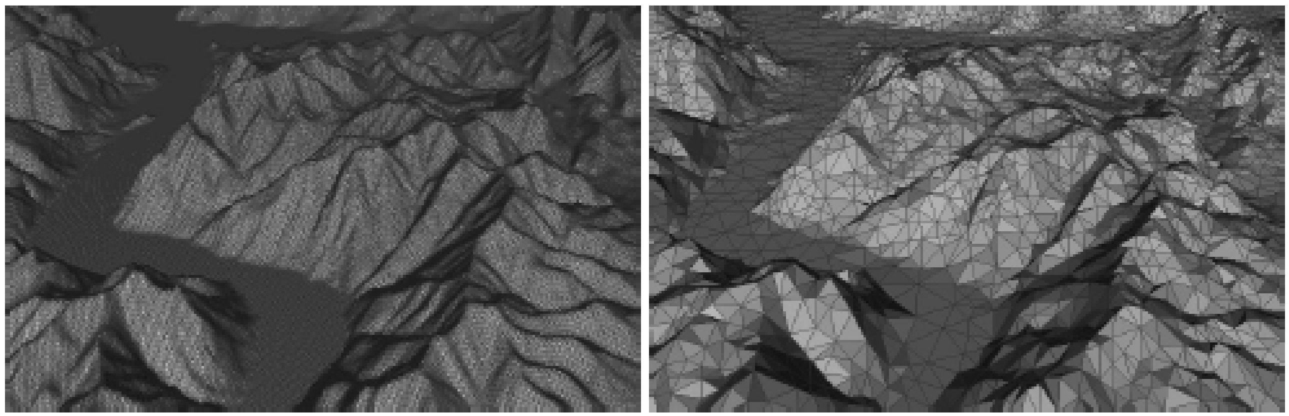


Lepp surface simplification algorithm



(a) Cono lake; (b) Dolomit; (c) Maggiere lake
Fila de abajo: 1% de los datos de arriba

Coll, Guerrieri, Sellares 2011



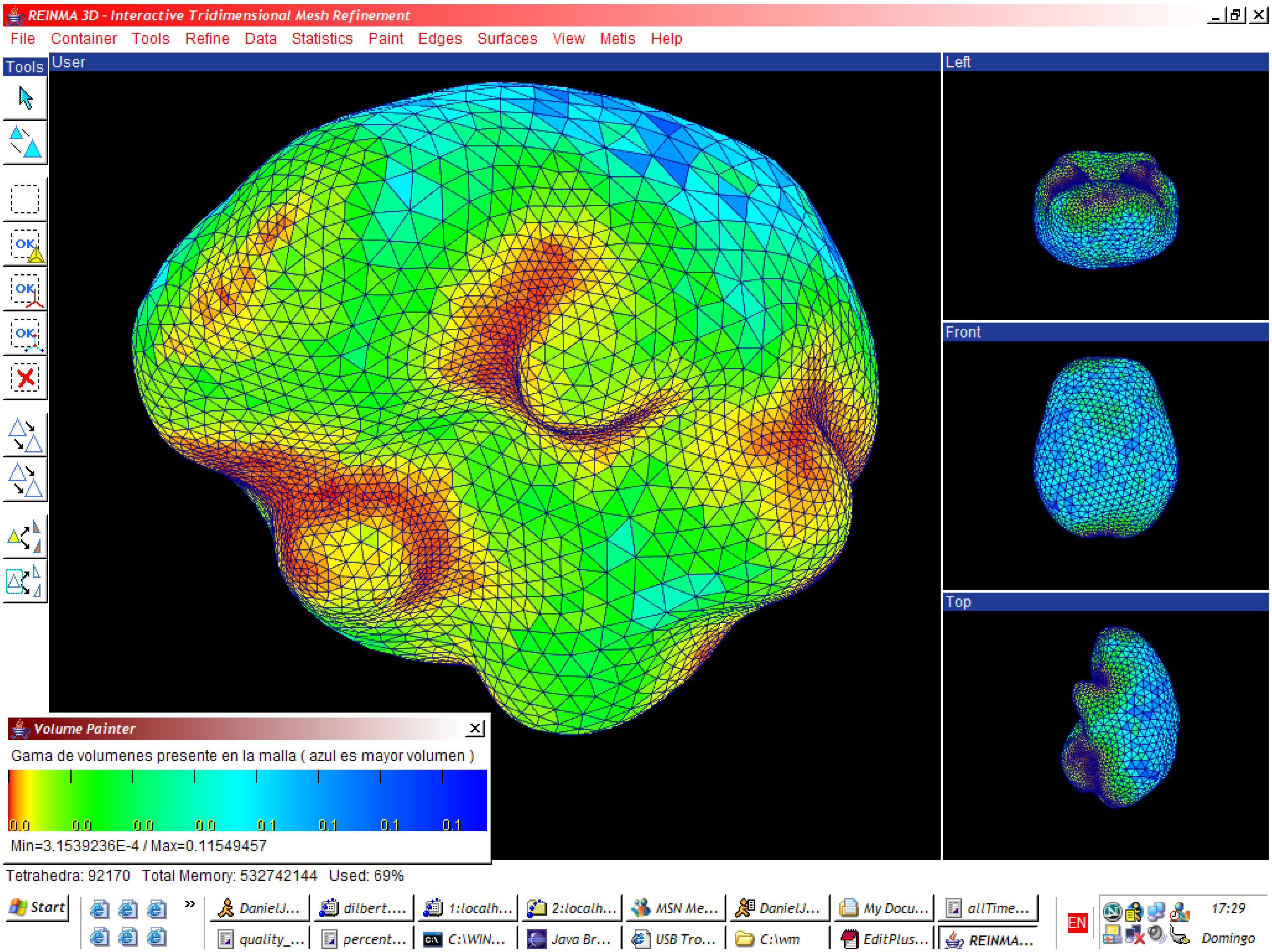
(a) (b)

Fig. 7. Detail of the Como lake. In (a) the original model and in (b) the refined model (5%).

Simplified mesh: 5% input data

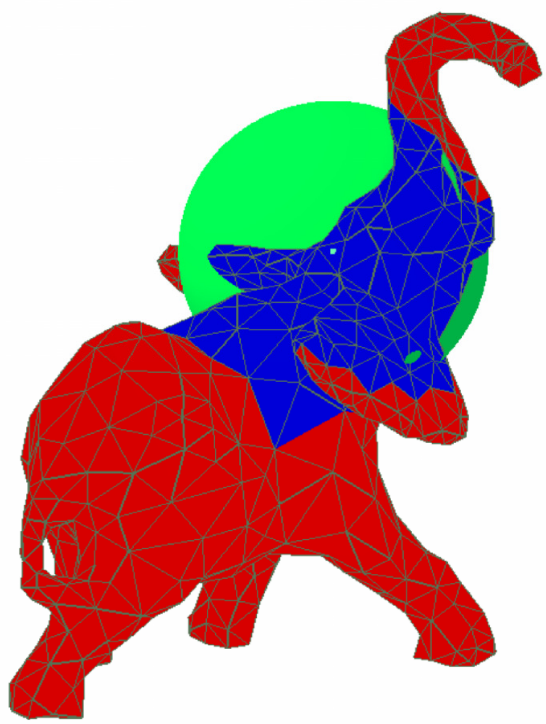
Simple parallel Lepp bisection algorithm

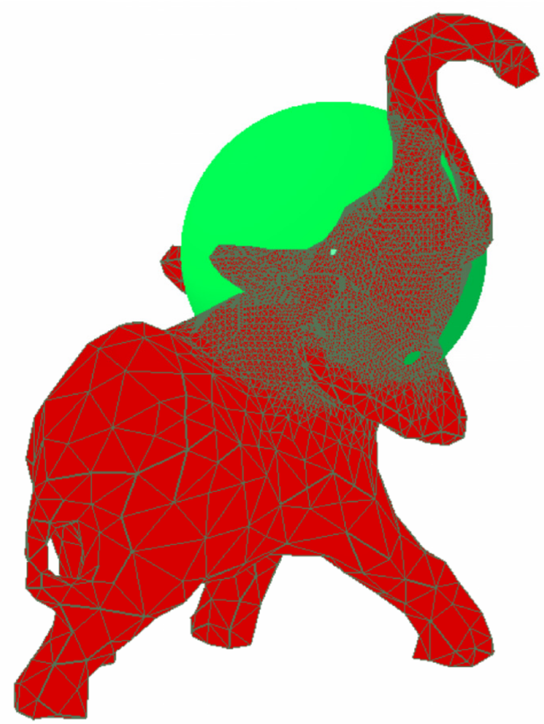
Rivara, Pizarro, Chrisochoides



Multithread parallel algorithm in 3-dimensions

Rodriguez, Rivara 2013





Recent results on 2D and 3D algorithms

2D Lepp Delaunay centroid algorithm theoretical results

Rivara, Rodríguez 2019

Input: bad quality Delaunay triangulation

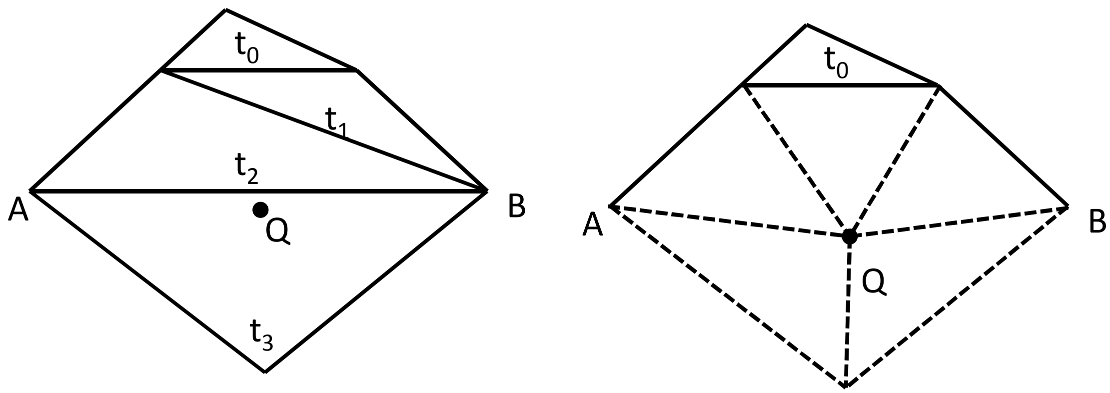
angle tolerance $\theta_{tol} = 30°$

- algorithm terminates
- optimal size triangulation
- quality triangulation τ_f with angles $\geq 30°$
- order independent algorithm (mesh size does not depend of triangle processing order)
- Simple algorithm

Algorithm at a glance

For each bad triangle t_0 ($\alpha < 30°$)

- algorithm finds local largest edge (terminal edge), terminal triangles $\widetilde{t_1}$, $\tilde{t}_2$ **using Lepp path**



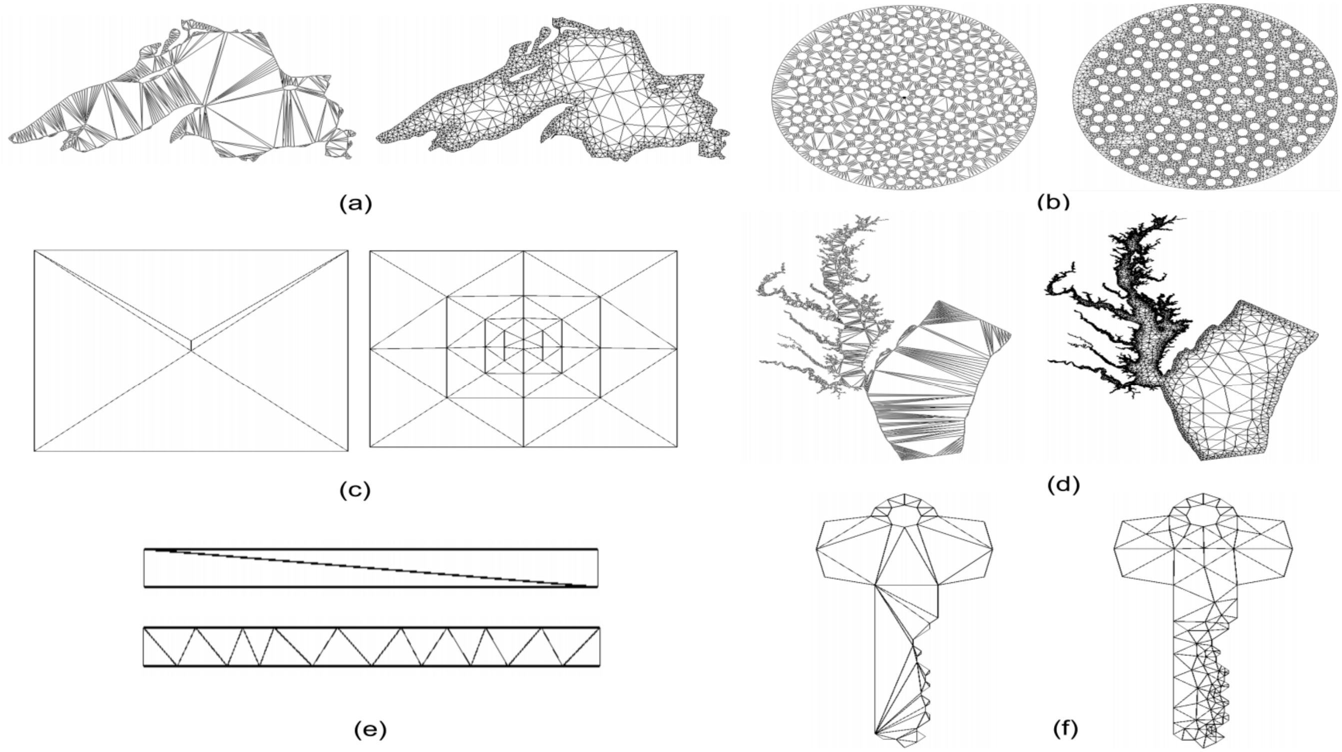
- Centroid Q of terminal quad is Delaunay inserted (non constrained case)

REPEADED until t_0 is destroyed.

Comparison with Ruppert's algorithm

- Reported by Shewchuck 2002 without off-center strategy (key geometry)

	Del centroid algorithm	Ruppert's algorithm [16]	
Triangle processing	without order	without order	Ordering triangles
Final Mesh size	229	450	249


(a)
(b)
(c)
(d)
(e)
(f)

Mesh size

	Superior lake	Neuss geometry	Square	Chesapeake bay	Long rectangle	Key geometry
	size(τ_0)	size(τ_0)	size(τ_0)	size(τ_0)	size(τ_0)	size(τ_0)
	528	3,070	9	14,262	2	54
θ_{tol}	size(τ_f)	size(τ_f)	size(τ_f)	size(τ_f)	size(τ_f)	size(τ_f)
30	1,835	8,338	54	36,803	19	170
33	2,273	9,939	65	45,883	22	229
34	2,512	11,054	70	52,027	25	262
35	3,017	12,742	81	63,138	27	349

Comparison with Triangle Software

θ_{tol}	Superior lake	Neuss geometry	Square	Chesapeake bay	Long rectangle	Key geometry
30	0.44	13.18	24.07	4.82	-15.79	22.94
33	-5,28	12,01	16.92	2.41	0.00	10.92
34	-5.29	-2.70	20.00	3.61	-68.00	-8.78
35	$-\infty$	$-\infty$	24.69	$-\infty$	-207.41	5.44

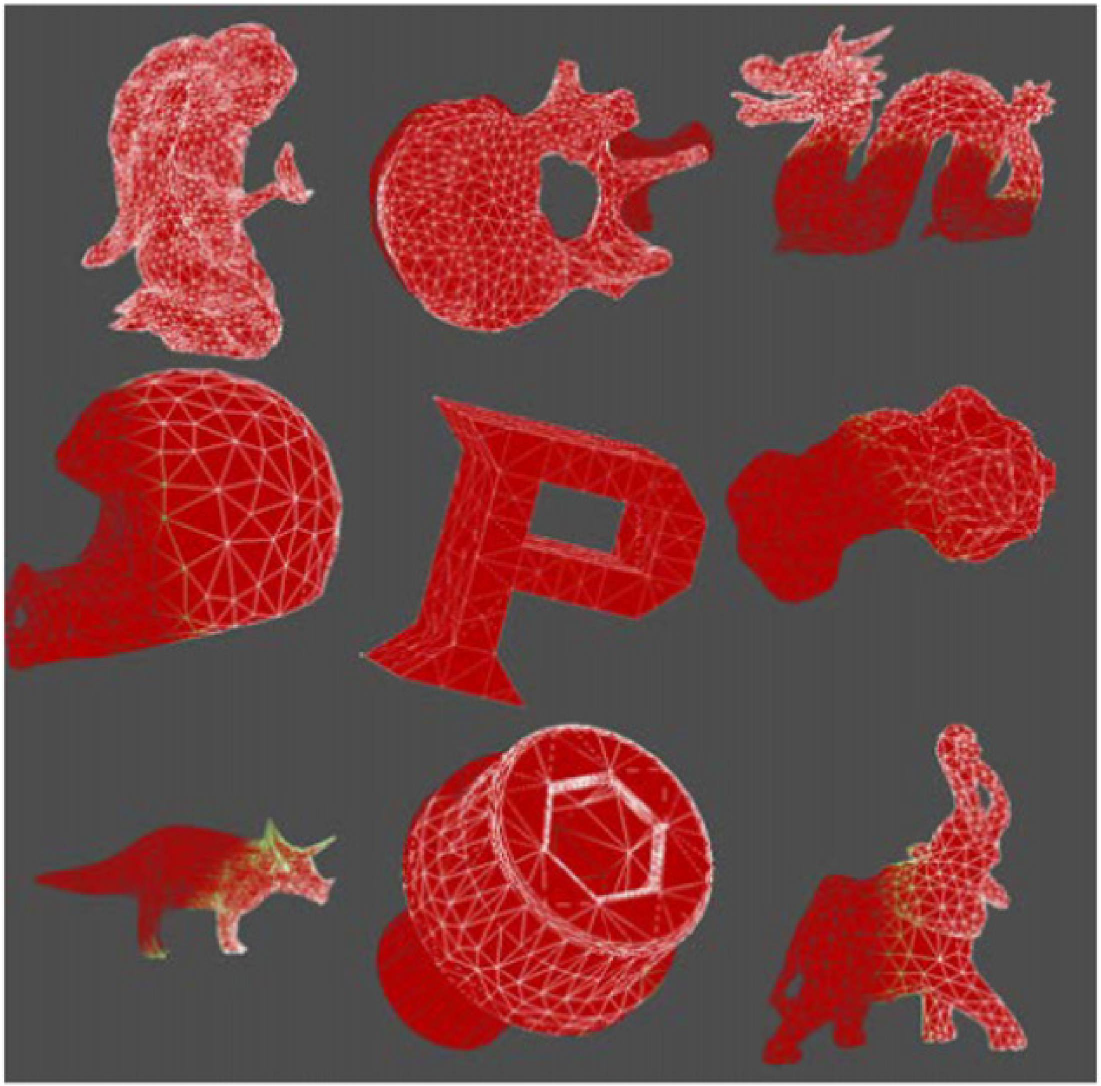
Triangle processes skinny and oversized triangles in order and uses boundary preprocess step

3-dimensions
Mesh improvement algorithm

Balboa, Rivara 2019

- Based on finding terminal edges (local largest edges) in the mesh by using Lepp search.
- Performing operations inside terminal star (set of terminal tetrahedra)
    - Simple insertion of centroid Q
    - Swapping of the terminal edge
    - Longest edge bisection

Test problems



Test problems

Comments on the test problems

- elephant, retinal, angel initial meshes have small number of interior points.
- P is good mesh

Mesh Sizes

Initial and final meshes

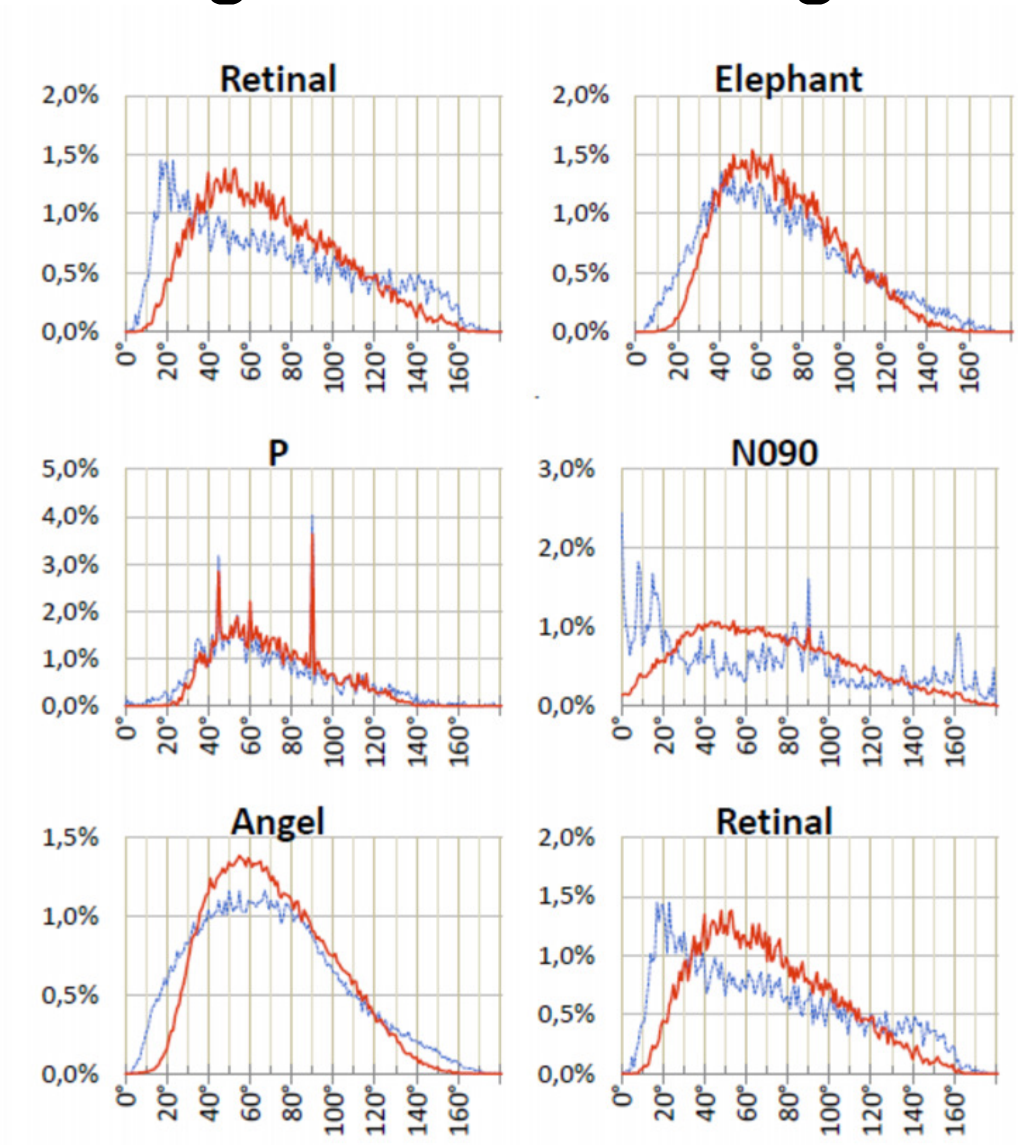
Mesh sizes (# tetrahedra)

Mesh		Retinal	Elephant	P	N090	Angel
Size	Initial	1374	1905	926	2623	13509
	Final	2663	2834	1080	10014	24822
Mesh		**Helmet**	**Dragon**	**Spine**	**Triceratops**	**Rand2000**
Size	Initial	1268	7209	3089	46202	13016
	Final	2105	12104	7338	85379	31030

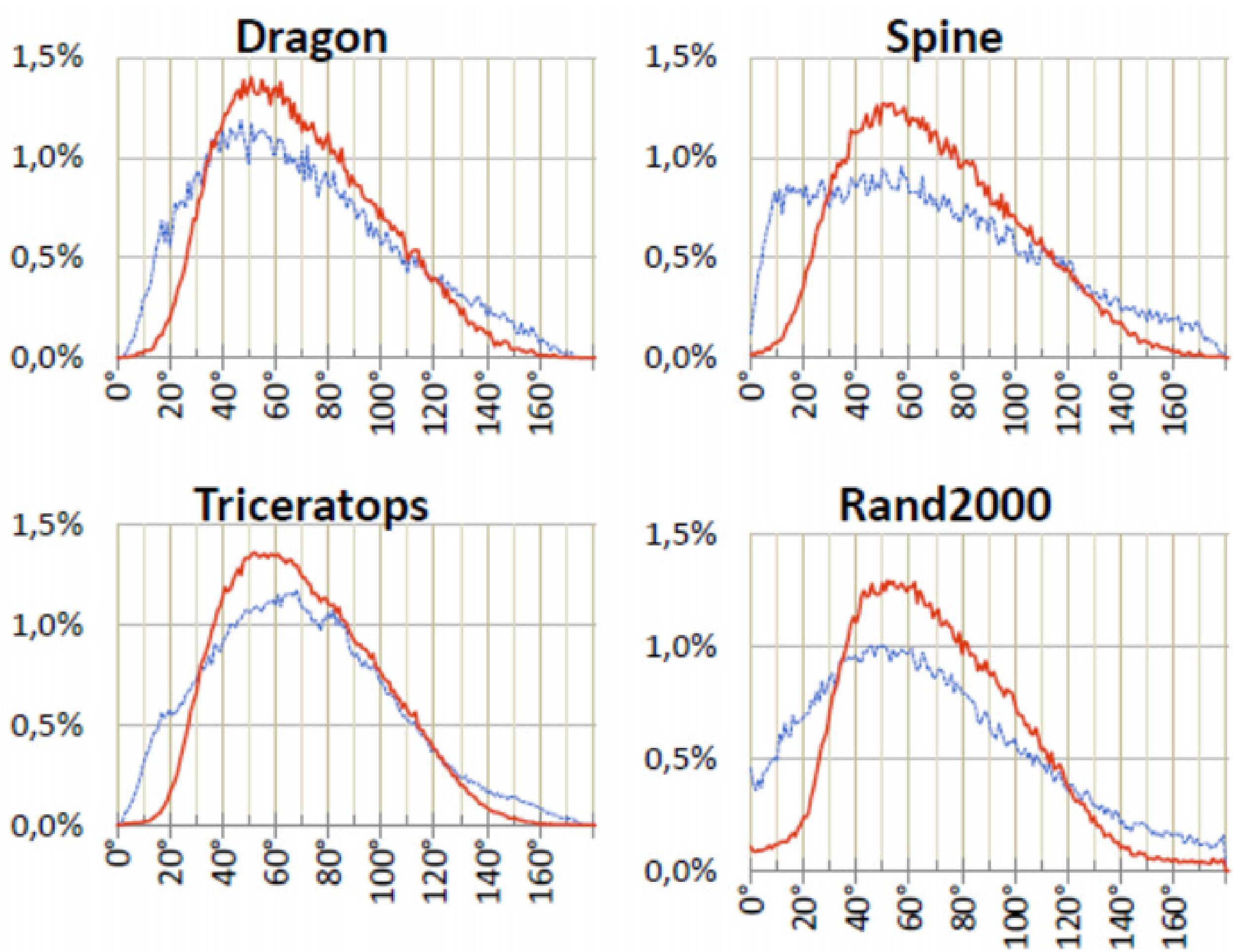
Distribution of extreme dihedral angles

%Dihedral Angles								
Mesh		<5°	<10°	<20°	<30°	>150°	>160°	>170°
Retinal	Initial	0.06	1.8	10.46	22.31	3.71	0.92	0.13
	Final	0	0.06	1.87	8.12	0.99	0.12	0
Elephant	Initial	0.008	0.54	3.9	10.51	1.6	0.54	0.07
	Final	0	0	0.39	3.63	0.16	0.017	0
P	Initial	0.25	0.41	1.94	5.45	0.76	0.41	0.14
	Final	0	0	0.062	1.33	0.016	0	0
N090	Initial	6.46	12.73	24.22	31.48	10.08	6.58	2.16
	Final	0.73	2.24	6.93	13.95	2.99	1.33	0.29
Angel	Initial	0.04	0.64	5.01	12.1	1.44	0.43	0.047
	Final	0.005	0.034	0.56	4.41	0.21	0.039	0.007
Helmet	Initial	0.005	0.47	3.4	9.96	0.85	0.25	0.026
	Final	0	0.056	0.46	4.18	0.16	0.055	0
Dragon	Initial	0.053	0.74	5.64	13.19	1.95	0.61	0.067
	Final	0.007	0.075	0.91	5.07	0.43	0.091	0.007
Spine	Initial	1.56	5	13.19	21.36	4.49	2.48	0.77
	Final	0.1	0.36	2.15	7.98	0.84	0.29	0.075
Triceratops	Initial	0.15	0.99	5.52	11.7	1.88	0.72	0.15
	Final	0.015	0.07	0.59	4.27	0.28	0.067	0.015
Rand2000	Initial	1.98	4.34	10.55	18.13	4.41	2.7	1.23
	Final	0.36	0.99	2.49	6.65	1.37	0.84	0.4

Histogram dihedral angles



Histogram dihedral angles



3D Lepp centroid (non Delaunay) algorithm

- Simple and effective improvement algorithm
- For meshes having bad / good distribution of interior points in 3-dimensions
- Low computational cost as compared with previous methods
- Appropriate for parallelization

Conclusions

Algorithms appropiate for complex applications in different fields

- adaptive techniques
- moving meshes / crack problems
- refinement / derefinement, great interest recently
- parallelization for big problems
- further theoretical studies 2D and 3D