Notes

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Continuous Functions

Definition: A set on which a function, f, is defined is called the domain of f and is denoted $dom(f)$.
$x \in dom(f) \subset \mathbb{R}$ real-valued functions
$x \to f(x) \in \mathbb{R}$ natural domain

Ex) $f(s) = \sqrt{4 - x^2}$ for $x \in (-2, 2)$. (-2,2) is the largest set for the domain of this function

Ex) $g(x) = sin(\frac{1}{x})$. Its domain is $dom(g) = \mathbb{R} - \{0\}$

Consider a function f and $x_0 \in dom(f)$. By informal definition of Continuous Function, If f is continuous and $x \in dom(f)$ is very close to x_0, then $f(x)$ is very close to $f(x_0)$.
Definition: Let f be a real-valued function and $x_0 \in dom(f)$. We say that f is *continuous* at x_0 if for every sequence, $x_{n n \geq 1}$ of points in $dom(f)$ converging to x_0, we have $\lim_{n \to \infty} f(x_n) = f(x_0)$.
What does it mean to be very close? It means that I have these really small numbers, ϵ and δ , such that $|f(x) - f(x_0)| < \epsilon$ and $|x - x_0| < \delta$.

Theorem: Let f be a real-valued function whose domain is a subset of the $\mathbb{R}$. The function f is continuous at $x_0 \in dom(f)$ iff if the following holds:

$$(\forall \epsilon > 0)(\exists \delta(\epsilon) > 0) \text{ such that if } |x - x_0| < \epsilon \text{ and } x \in dom(f), \text{ we have } |f(x) - f(x_0)| < \epsilon. \tag{1}$$

Proof: Suppose that the theorem holds, prove that f is continuous at $x_0 \in dom(f)$.

We need to show that for every sequence $\{x_0\}_{n \geq 1} \subset dom(f)$ such that $x_n \to x_0$ as $n \to \infty$, we have $\lim_{f(x_n) = f(x_0)}$.

Definition of Limit: $(\forall \epsilon > 0)(\exists N(\epsilon)$ such that $\forall n \geq N$ we have $|f(x_0) - f(x) < \epsilon$

From (1), there exists $\delta(\epsilon) > 0)$ such that if $|x_n - x_0| < \delta$, then $|f(x_n) - f(x_0)| < \epsilon$. $(\exists N)$ such that for $n \geq N$ ...

We know that f is continuous at $x_0 \in dom(f)$. This has proven that $\lim_{n \to \infty} x_n = x_0 \implies \lim_{n \to \infty} f(x_n) = f(x_0)$.Let's prove the converse.

We'll do a proof by contradiction. Let's assume that equation (1) is false, which is basically $(\exists \epsilon > 0)(\forall \delta > 0)$ there is $x \in dom(f)$ satisfying $|x - x_0| < \delta$ and $|f(x) - f(x_0)| \geq \epsilon$.

For $\delta_n = \frac{1}{n}$ where $n = \{1, 2, 3, ...\}$. Choose $|x_n - x_0| < \delta_n = \frac{1}{n}$ and $f(x_n) - f(x_0)| \geq \epsilon$. We know that $\lim_{n \to \infty} x_n = x_0$. CONTRADICTION!!!!

Problem

$f : [0, 1] \to \mathbb{R}$

$$f(x) = \begin{cases} 0 & \text{if x is irrational} \\ \frac{1}{n} & \text{if } x = \frac{m}{n} \text{ where m and n are integers} \end{cases}$$

Prove that f is not continuous at rational points.

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Example

$$f(x) = \begin{cases} sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Claim: f is not continuous at $x = 0$. The opposite means that $x_n \to 0 \implies f(x_n) \to f(0)$. For $x_n = \frac{1}{\pi n}$

For $x'_n = \frac{1}{2\pi n + \frac{\pi}{2}}$

$x'_n \to 0$ as $n \to \infty$.

$f(x'_n) = \sin(\frac{1}{x_n}) = \sin(2\pi n + \frac{\pi}{2}) = \sin(\frac{\pi}{2}) = 1$

$$f(x) = \begin{cases} x\sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Claim: f is continuous at x = 0.

Well... $|f(x)| = |x||sin\frac{1}{x}| \leq |x|$. We also know that $|f(x)| \leq |x| \Leftrightarrow -x \leq f(x) \leq x$, so it's sandwiched between those two functions. By the sandwich theorem, since $\lim_{n\to 0} -x = 0$ and $\lim_{n\to 0} x = 0$, then we get that $\lim_{n\to 0} |x||sin\frac{1}{x}| = 0$. Since at 0, we have that $f(x)$ is 0 at 0, it's continuous.

$f(x) = 2008x^3 + 7$ with $x_0 \in \mathbb{R}$

$x_n \to x_0$ as $n \to \infty \implies f(x_n) \to f(x_0)$?

$\lim_{n\to infty} f(x_n) = 2008(\lim_{n\to\infty} x_n)^3 + 7 = f(x)$

Proof of $(\forall\epsilon > 0)(\exists\delta(\epsilon) > 0)(|x - x_0| < \delta$ and $x \in dom(f) \implies |f(x) - f(x_0) < \epsilon$:

$$\begin{aligned} |2008x^3 + 7 - 2008x_0^3 - 7| &< \epsilon \\ 2008|x^3 - x_0^3| &< \epsilon \\ 2008|x - x_0||x^2 + xx_0 + x_0^2| &< \epsilon \end{aligned}$$

For now, let's ignore the $x^2 + xx_0 + x_0^2$ term.

We know(?)$x - x_0 < 1 \implies |x| \leq |x_0| + 1$.

$|x^2 + xx_0 + x_0^2| \leq |x|^2 + (|x||x_0|) + |x_0|^2 \leq (|x_0| + 1)^2 + (|x_0| + 1)|x_0| + |x_0|^2$.

$|x - x_0| < \frac{\epsilon}{2008(|x_0|+1)^2+(|x_0|+1)|x_0|+x_0^2}$

$|f(x) - f(x_0)| = 2008|x - x_0||x^2 + xx_0 + x_0^2| \leq \epsilon$

<u>Theorem</u>: Let f(x) be a real-valued function continouous at x_0. Let c be an arbitrary constant. Then, $g(x) = |f(x)|$ and $h(x) = cf(x)$ are also continuous at x_0.

<u>Proof</u>: $c = 0$ is trivial since $h(x) = 0$. Therefore we can assume that $c \neq 0$

Let $x_0 \in dom(f)$. For every $\epsilon > 0$, there exists a δ such that $|x = x_0| < \delta \implies |h(x) - h(x_0) < \epsilon$.

$$\begin{aligned} |h(x) - h(x_0)| &= |cf(x) - cf(x_0)| \\ &= |c||f(x) - f(x_0)| \\ &< \epsilon \text{ (We want this!)} \end{aligned}$$

Even easier:

$$\begin{aligned} \lim_{n\to\infty} h(x_n) &= \lim_{n\to\infty} cf(x_n) \\ &= c\lim_{n\to\infty} f(x) \\ &= c(f(x_0)) \end{aligned}$$

Part 2:

$$\begin{aligned} |g(x) - g(x_0)| &= ||f(x)| - |f(x_0|| \\ &\leq |f(x) - f(x_0)| \end{aligned}$$

I claim that $-|z-w| \leq |z| - |w| \leq |z-w|$. This is triangle inequality.

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Let f, g be continuous functions. Then, $f(x)/pmg(x)$, $f(x)g(x)$, and $\frac{f(x)}{g(x)}$ are continuous.
Theorem: Let f and g be real-valued functions that are continuous at $x_0 \in \mathbb{R}$. Then,

1. $f(x) + g(x)$ is continuous at x_0
2. $f(x)g(x)$ is continuous at x_0.
3. $\frac{f(x)}{g(x)}$ continuous at x_0 if $g(x_0) \neq 0$.

$x_n \in dom(f+g) = dom(f) \cap dom(g)$
$dom(\frac{f}{g}) = dom(f) \cap \{x \in dom(g) | g(x) \neq 0\}$
Proof: sum/difference identity: We need $f(x_n) + g(x_n) \to_{n\to\infty} f(x_0) + g(x_0)$
We know $f(x_n) \to f(x_0)$ and $g(x_n) \to g(x_0)$ as $x \to \infty$
$\lim_{n\to\infty}(f(x_n) + g(x_n)) = \lim_{n\to\infty} f(x_n) + \lim_{n\to\infty} g(x_n) = f(x_0) + g(x_0)$
product identity: We need $f(x_n)g(x_n) \to_{n\to\infty} f(x_0)g(x_0)$

$$\begin{aligned}
\lim_{n\to\infty} f(x_n)g(x_n) &= \lim_{n\to\infty} f(x) \lim_{n\to\infty} g(x) \\
&= f(x_0)g(x_0)
\end{aligned}$$

quotient identity: Same idea as the previous two.

Why is this important? Well... now we know that all polynomial is continuous using induction.
Examples of Continuous from these identities Ex) $p(x) = a_n x^n + \ldots + a_1 x + a_0$
Ex) $f(x) = \frac{p(x)}{q(x)}$ where p and q are polynomials. It is continuous if $q(x_0) \neq 0$.

Prove that $f(x) = \sin(x)$ is continuous.
We need to show that $(\forall x_n \to_{n\to\infty} 0)(\sin(x_n) \to 0)$
We know that $|\sin(x)| \leq |x|$. We know that as $n \to 0$, $|x_n| \to 0$. Since $|\sin(x)| \leq |x|$ and $|x_n| \to 0$, $|\sin(x)| \to 0$ as $n \to 0$. Therefore, $\sin(x)$ is continuous at $x = 0$.

Prove that $f(x) = \cos(x)$ is continuous at 0.
We need to show that $(\forall x_n \to_{n\to\infty} 0)(1 - \cos(x_n) \to 0)$
We know that $|1 - \cos(x)| \leq |x|$. We know that as $n \to 0$, $|x_n| \to 0$. Since $|1 - \cos(x)| \leq |x|$ and $|x_n| \to 0$, $|1 - \cos(x)| \to 0$ as $n \to 0$. Then, $\lim_{n\to 0} 1 - \cos(x) = 0$, so $\lim_{n\to 0}\cos(x) = 1$.
Therefore, $\cos(x)$ is continuous at $x = 0$.

Let $x_n \to x_0$ as $n \to \infty$.
We need $\sin(x_n) \to \sin(x_0)$.

$$\begin{aligned}
\sin(x_n) &= \sin(x_0 + (x_n - x_0)) \\
&= \sin(x_0)\cos(x_n - x_0) + \cos(x_0)\sin(x_n - x_0)
\end{aligned}$$

As $n \to \infty$, we get $\sin(x_0)(1) + \cos(x_0) * 0$, which is $\sin(x_0)$

Corollary

$\tan(x) = \frac{\sin(x)}{\cos(x)}$ is continuous when $\cos(x) \neq 0$.

Theorem

If $f(x)$ is continuous at x_0 and $g(x)$ is continuous at $f(x_0)$, then $g \circ f(x) = g(f(x))$ is continuous at x_0.
Proof: $x_n \to x_0$ as $n \to \infty$.
$x_n \in dom(g \circ f) = \{x | x \in dom(f) \text{ and } f(x) \in dom(g)\}$
Since f is continuous at x_0, we have $f(x_n) \to f(x_0)$ as $n \to \infty$.
$g(x_n) = g(f(x_n)) \to_{n\to\infty} g(f(x_0))$. Since $g(x)$ is continuous at $f(x)$.

This is continuous. Why? $max(f(x), g(x)) = \frac{f(x)+g(x)}{2} + \frac{|g(x)-f(x)|}{2}$

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Continuous functions:
Seqeunce version: f is not continuous at x_0 if $\forall x_n$, as $x_n \to x_0$, $f(x_n) \to f(x_0)$
ϵ δ version: $(\forall \epsilon > 0)(\exists \delta > 0)(\forall x)$ satisfying $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \epsilon$.

Ex) Let $a_n = 1 - \frac{1}{n}$. Now, $a_n \to 1$ as $n \to \infty$. But $f(a_n) = 0 \forall n \geq 1$ as $f(a_n) \to 0 \neq 1$.

Ex) Let $f : [0, 1] \to [0, 1]$.
$f(x) = \begin{cases} 0 & x \text{ is irrational} \\ \frac{1}{q} & \text{otherwise} \end{cases}$
$x_0 \in [0, 1]$ $\mathbb{Q}$.

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Theorem: Let f be a continuous real-valued function on a closed interval $[a, b]$. Then f is bounded on $[a, b]$.
Proof: Let us assume that f is not bounded. $\{x_n\}$, $f(x_n) \geq n$, $n = 1, 2, 3, \ldots$, $a \leq x_n \leq b$, i.e. $\{x_n\}$ is a bounded sequence. By Bolzano-Weierstrass theorem, we have $x_{n_k}, \to_{k\to\infty} x_0$ (i.e. $\{x_0\}$ has a converging subsequence) $\implies f(x_{n_k}) \to_{k\to\infty} f(x_0)$ since f is continouous. Then, $f(x_{n_k}) > n_k \to \infty$. This is a contradiction.

$\infty > M = \sup f(x)$, smallest possible upper bound where $x \in [a, b]$.
We want to prove that $\exists x \in [a, b]$ such that $f(x_0) = M = \sup_{x\in[a,b]} f(x)$. In other words, $\sup_{x\in[a,b]} f(x) = \max_{x\in[a,b]} f(x)$.
Proof by contradiction: Suppose this is not true, i.e. $\forall x \in [a, b]$ $f(x) < M \implies g(x)\frac{1}{M-f(x)}$, which is continuous on $[a, b]$
$f(x)$ is bounded on $[a, b]$ by the first part of the proof. Let $D \geq g(x) > 0$. Then, $D \geq \frac{1}{M-f(x)} \implies f(x) \leq M - \frac{1}{D}$.
$M = \sup_{x\in[a,b]} f(x)$. $M - \frac{1}{n} \leq f(x_n) \leq M \implies \exists$ converging subsequence.
$x_{n_k} \to_{k\to\infty} x_0$.
$M\frac{1}{n_k} \leq f(x_k) \leq M$
$f(x_{n_k} \to_{k\to\infty} f(x_0)$
$M - 0 \leq f(x_0) \leq M$.
Therefore, $f(x_0) = M$.

Theorem: (Intermediate Value Theorem) Let f be continuous on $[a, b]$. Then, for any value, d, between

$f(a)$ and $f(b)$. There exists $x_0 \in [a,b]$ such that $f(x_0) = d$.
If $d = f(a)$ or $d = f(b)$, then we're done.

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Figure 2
proof: Without loss of generality, we can assume that $f(a) < f(b)$ Let $S = \{x \in [a,b] | f(x) < y\}$. Also, let $x_0 = \sup S$ (It's well-defined because $\mathbb{R}$ is complete). So, there exists a sequence $\{S_n\} \subseteq S, S_n \to x_0$ as $n \to \infty$ where $x_0 \in [a,b]$. However, f is continuous, $f(S_n) \to f(x_0)$. Hence, $f(x_0) \leq y$. We know that $x_0 \neq b$. Otherwise, $f(x_0) = f(b) \leq y < f(b)$. A contradiction. Let $t_n = x_0 + \frac{1}{n}$ Then, as $t_n \to x_0$, $f(t_n) \geq y$. Hence $f(x_0) \geq y$. Therefore, $f(x_0) = y$.

Example Let $f : [0,1] \to [0,1]$ be continuous. Then, f has a fix point ($f(x) = x$)
Figure 3.
Let $g(x) = f(x) - x$ where g is continuous and $g(x) \geq 0$. $g(1) \leq 0$. Suppose $g(0) > 0$ and $g(1) < 0$.
By the Intermediate Value Theorem, there exists $0 < x < 1$, $g(x) = 0$, but for this x, $f(x) = x$.

Corollary Let I be an interval, $f : I \to \mathbb{R}$. Suppose f is continuous. Then, $f(I) = \{f(x), x \in I\}$ is an interval or a single point.
Proof: Suppose $f(I) = J$ is not a single point. Then, $\inf(J) < \sup(J)$. Take $y \in \mathbb{R}$ such that $\inf J < y < \sup J$. Then, there exists $y_0, y_1 \in J$ such that $y_0 < y < y_1$.
Let $x_0, x_1 \in I$ be such that $f(x_0) = y_0, f(x_1) = y_1$. By the intermediate value theorem, there exists x between x_0 and x_1 such that $f(x) = y$ where $y \in J$.

Theorem: Let I be an interval, $f : I \to \mathbb{R}$. Suppose f is continuous and strictly increasing on I.
Then,

1. f^{-1} is well-defined function on $J = f(I)$ and strictly increasing.
2. f^{-1} is continuous. This statement comes from the following theorem.

Theorem: Let J be an interval in $\mathbb{R}$ and $g : J \to \mathbb{R}$. Suppose g is strictly increasing such that $g(J)$ is an interval on J. Then, g is continuous on J.

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Proof: Take $x_0 \in J$, which is not an endpoint. Then, $g(x_0)$ is not an endpoint either. Take $\epsilon_0 > 0$.. Then, $g(x_0) - \epsilon g(x_0) + \epsilon) \subseteq g(J)$. For any ϵ such that $0 < \epsilon < \epsilon_0$, $g(x_0) + \epsilon \in g(J)$. So, there are some powers $x_1, x_2 \in J$ such that $g(x_1) = g(x_0) - \epsilon$ and $g(x_2) = g(x_0) + \epsilon$.
Let $\delta = \min\{x_0 - x_1, x_2 - x_0\}$.
Then, $x_0 - \delta < x < x_0 + \delta \Rightarrow g(x_0) - \epsilon < g(x) < g(x_0) + \epsilon$.
In other words, for any $\epsilon > 0$, $\epsilon < \epsilon_0$. There exists $\delta > 0$ such that $|x - x_0| < \delta \Rightarrow |g(x) - g(x_0)| < \epsilon$.

Then, by Theorem 17.2, when x_0 is a an endpoint, we can prove that in a similar way.
$\forall x_0 \forall \epsilon > 0 \exists \delta > 0 \forall x \in S$ with $|x_0 - x| < \delta$, $|f(x_0) - f(x)| < \epsilon$.

Theorem 18.6: Suppose I is an interval and that $f : I \to \mathbb{R}$, which is continuous and one to one. Then, f is strictly increasing or decreasing.
Proof: Take two points $a, b \in I$. Without loss of generality let $f(a) < f(b)$. Take any two points, x_1, x_2 such that $a < x_1 < x_2 < b$. Suppose for contradiction that $f(x_1) > f(b) > f(a)$. By the Intermediate Value Theorem, then there exists c such that $c \in [a,b]$ and that $f(c) = f(b)$. This is a contradiction because f is supposed to be one to one. Hence, we know that $f(x_1) < f(b)$. Similarly, $f(a) < f(x_1)$. In a similar way,

$f(x_1) < f(x_2) < f(b)$. Hence, f is strictly increasing on $[a, b]$. If we take $[a', b'] \supset [a, b]$, f is strictly increasing. $\square$.

Theorem: Suppose $S \subseteq \mathbb{R}$ such that f is uniformly continuous on S. If $\forall \epsilon > 0\ \exists \delta > 0\ \forall x, y \in S$ such that $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.
original defintion: f is continuous on S if $\forall x_0 \in S\ \forall \epsilon > 0\ \exists \delta > 0\ \forall x$ such that $|x - x_0| < \delta$. Then, $|f(x) - f(x_0)| < \epsilon$

Ex) $f(x) = \frac{1}{x^2}$ on $(0, \infty)$. f is uniformly continuous on $(1, \infty)$, but not on $(0, \infty)$. For uniformly continuous, δ depends on choice of ϵ and x_0.
Proof: $f(x) - f(y) = \frac{(y-x)(y+x)}{x^2 y^2}$.
Then, $\frac{y+x}{x^2 y^2} = \frac{1}{x^2 y} + \frac{1}{x y^2} \leq 1 + 1 = 2$. Then, $|y - x| < \frac{\epsilon}{2} = \delta \Rightarrow |f(x) - f(y)| < \epsilon$

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What is the negation of the definition of uniform continuity?
$(\exists \epsilon > 0)(\forall \delta > 0)(\exists x, y \in S with |x - y| < \delta$ and $|f(x) - f(y)| \geq \epsilon$
Theorem 19.2: $f : [a, b] \to \mathbb{R}$ and f is continuous $\Rightarrow f$ is uniformly continuous on $[a, b]$
Proof: Suppose for contradiction, f is not uniformly continuous. $(\exists \epsilon > 0)(\forall \frac{1}{N})(\exists x_N, y_N \in [a, b])(|f(x_N) - f(y_N)| \geq \epsilon)$.
$\{x_N\}_{N=1}^{\infty} \subseteq [a, b]$.
By the Bolzano-Weierstrass Theorem, there exists $\{x_{n_k}\}_{k=1}^{\infty} \subseteq \{x_N\}_{N=1}^{\infty}$ and $x_0 \in \mathbb{R}$.
$x_{N_k} \to x_0$ as $k \to \infty$. However, $[a, b]$ is closed, so $x_0 \in [a, b]$. $y_{N_k} \to x_0$ as $k \to \infty$. $\lim_{k\to\infty} f(x_{n_k}) = f(x_0) = \lim_{k\to\infty} f(y_{N_k})$ because f is continuous.
This is a contradiction because $\lim_{k\to\infty}(f(x_{N_k}) - f(y_{N_k})) = 0$ and $\lim_{k\to\infty}(f(x_{N_k}) - f(y_{N_k})) \geq \epsilon > 0$.
Theorem 19.4: $f : S \to \mathbb{R}$ is uniformly continuous on S. then, for a Cauchy sequence $\{x_n\} \subseteq S$, $\{f(x_n)\}$ is also a Cauchy Sequence.
Proof: Since f is uniformly continuous, $(\forall \epsilon > 0)(\exists \epsilon' > 0)(\forall x, y \in S)(|x - y| < \epsilon')(|f(x) - f(y)| < \epsilon$.
Since $\{x_n\}$ is a Cauchy sequence for the above ϵ', there exists N, with $m, n > N$, $|x_m - x_n| < \epsilon'$.
This means that $(\forall \epsilon > 0)(\exists N)(\forall m, n > N)(|f(x_m) - f(x_n)| < \epsilon)$.
$f(x) = \frac{1}{x^2}$ is not uniformly continuous on $(0, \infty)$.
$\{x_n = \frac{1}{n}\} \subseteq (0, \infty)$, a Cauchy sequence.
Suppose for a contradiction that f is uniformy continuous $(0, \infty)$.
$\{f(x_n) = n^2\}$ is a Cauchy Sequence. A Contradiction!

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Theorem: A real-valued function f on (a, b) is uniformly continuous on (a, b) if and only if it can be extended to a continuous function on $[a, b]$.
Example: $g(x) = \frac{1-\cos(x)}{x^2}$, $x \in (0, 2\pi)$ is uniformly continuous on $(0, 2\pi)$.
$g(0) = \frac{1}{2} = \lim_{x\to 0} g(x)$
Theorem: Let f be a continuous function on an interval I. Let I^o be I without its endpoints. If f is differentiable on I^o and $f'(x)$ is bounded on I^o, then f is uniformly continuous on I.
Example: $f(x) = \frac{1}{x}$. Then, $f'(x) = \frac{-1}{x^2}$
$|f'(x)| = \frac{1}{x^2} \leq 1$.
Mean Value Theorem: Let a function f be differentiable on $[a, b]$. Then, there exists a point, c, $f'(c) = \frac{f(b)-f(a)}{b-a}$

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a) $\lim_{x\to\frac{2}{\pi}} \sin(\frac{1}{x}) = \sin(\frac{1}{\frac{2}{\pi}}) = \sin(\frac{\pi}{2}) = 1$
Remark: $f : S \to \mathbb{R}$ is continuous at $a \in S$ iff $\lim_{x\to a^S} f(x) = f(a)$.

b) $\lim_{x\to+\infty} x\sin(\frac{1}{x}) = 1$. Why? Let $y = \frac{1}{x}$. Then, it will $x\sin(\frac{1}{x}) = \frac{\sin(y)}{x}$ and we know as $x \to \infty$, $y \to 0$, and function to 1. c) $\lim_{x\to 1} \frac{x^{\frac{1}{2008}}-1}{x-1} = \lim_{x\to 1} \frac{x^{\frac{1}{2008}}-1}{(x^{\frac{2007}{2008}}+x^{\frac{2006}{2008}}+...+x^{\frac{1}{2008}}+1)(x^{\frac{1}{2008}}-1)}$
Let $x = y^2 008$... e) $\lim_{x\to 0^+} x^{x^2} = \lim_{x\to 0^+} (e^{\ln x})^{x^2} = \lim_{x\to 0^+} e^{x^2\log(x)}$.
$\lim_{x\to 0^+} x^2\log(x) = \lim_{x\to 0^+} \log x \frac{\log(X)}{\frac{1}{x^2}} = \lim_{x\to\infty} \frac{\frac{1}{x}}{\frac{-2}{x^3}} = \lim_{x\to 0^+} ...$

Theorem Let f_1 and f_2 be functions for which the limits $\lim_{x\to a^S} f_1(x) = L_1$ and $\lim_{x\to a^S} f_2(x)$ exists and are finite. Then, $\lim_{x\to a^S}(f_1(x)+f_2(x)) = L_1 + L_2$ and $\lim_{x\to a^S} f_1 * f_2(x) = L_1L_2$.
Theorem Let f be a function for which the limit, $L = \lim_{x\to a^s} f(x)$ exists and is finite. If g is a function defined on $\{f(x)|x \in S\} \cup L$ and is continuous at L, $\lim_{x\to a^S} g\dot{f}(x) = g(L)$.

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Theorem: Let f be a function for which the limit, $\lim_{x\to a} = L$ exists and is finite. Let the function g be defined on $\{x \in S\} \cup \{L\}$. IF $g(y)$ is continuous at L, then $\lim_{x\to a^S} g(f(x)) = g(L)$.

Ex) $\lim_{x\to 0} f(x) = 0$
$\lim_{y\to 0} g(x) = 200 \neq g(0)$.
Then, $\lim_{x\to 0} g(f(x))$ does not exist.

Let $\{x_n\}_{n\geq 1}^{\infty}$ be an arbitrary sequence on S such that $x_n \to a$ as $n \to \infty \Rightarrow f(x_n) \to L$ as $n \to \infty$ and $g(f(x)) \to g(L)$ as $n \to \infty$ because g is continuous at L.

Theorem: Let f be a function defined on some subset S of the real line $\mathbb{R}$. Let a be a real number that is equal to the limit of some subsequence on S. Then, $\lim_{x\to 0^+} f(x) = L \Leftrightarrow \forall\epsilon > 0 \exists\delta(\epsilon, a) > 0$ such that if $|x-a| < \delta$ and $x \in S$, then $|f(x) - L| < \epsilon$

Suppose that $\forall\epsilon > 0\exists\delta(\epsilon, a) > 0$ such that if $|x - a| < \delta$ and $x \in S$, then $|f(x) - L| < \epsilon$ is true. We have to prove that $\forall\{x_0\}_{n\geq 1} \subset S$ such that $\lim_{n\to\infty} x_n = a$, we have $\lim_{n\to\infty} f(x) = L$.
We have to show that $\forall\epsilon > 0\exists N(\epsilon)$ such that $\forall n \geq N$ $|f(x) - L| < \epsilon$.
Take $\delta(\epsilon, a)$ from our hypothesis. Then, $|x_n - a| < \delta \Rightarrow |f(x_n) - L| < \epsilon$. Since $\lim_{n\to\infty} = a$, we have that $\exists N(\delta(\epsilon, a)) = N(\epsilon)$ such that if $n \geq N$, then $|x_n - a| < \delta$.

We have $\lim_{x\to a^S} f(x) = L$. We need to prove the previous hypothesis. By contradiction, assume that the previous hypothesis is not true. Then, $\exists\epsilon > 0$ $\forall\delta > 0$ $\exists x \in S$ and $|x - a| < \delta$, but $|f(x) - L| > \epsilon$.
Example:$\delta_n = \frac{1}{n}$. Then, $\exists x_n$ such that $|x_n - a| < \delta = \frac{1}{n}$, but $f(x_n) - L| > \epsilon$. $\lim_{n\to x_n} x_n = a$. However, $\lim_{n\to\infty} f(x_n) \neq L$.
theorem $\lim_{x\to a} f(x)$ exists iff both $\lim_{x\to a^+}$ and $\lim_{x\to a^+} f(x)$ exists and equal to each other. Then, $\lim_{x\to a} f(x) = \lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x)$.

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Theorem: The power series converges for $|x| < R$ and diverges for $|x| > R$.
The Root Test. $\sum_{n=0}^{\infty} a_n x^n$.
$\limsup |a_n x^n|^{\frac{1}{n}} = \limsup |a_n|^{\frac{1}{x}}|x| = |x|\limsup |a_n|^{\frac{1}{n}} = |x|\rho = \frac{|x|}{R}$
If $\frac{|x|}{R} < 1$, then the series converges.
If $\frac{|x|}{R} > 1$, then the series diverges.
Remark: $\lim n \to \infty|\frac{a_{n+1}}{a_n}| = \rho \Rightarrow |a_n|^{\frac{1}{n}} = \rho$
Examples:
a) $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$. This converges on $|x| < 1$ and diverge on $|x| \geq 1$.
b) $\sum_{n=1}^{\infty} \frac{1}{n} x^n$. $\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} \frac{n}{n+1} = 1$

The series converge for $|x| < 1$ and diverges for $|x| > 1$.
For $x = 1$, we get the harmonic series, which diverges.
c) $\sum_{n=0}^{\infty} (-1)^n n^2 008 x^{n+1} = \sum_{n=1}^{\infty} (-1)^{n-1}(n-1)^{2008} x^n$.
$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} (\frac{n}{n-1})^{2008} = 1$.
Converges to $|x| < 1$ and diverges on $|x| \geq 1$.
d) $\sum_{n=0}^{\infty} \frac{1}{n!} x^n$. $\lim (\frac{1}{n!})^{\frac{1}{n}} = \lim_{n\to\infty} (\frac{1}{n!})^{\frac{1}{n}} = 0$.
$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = \lim_{n\to\infty} \frac{n!}{(n+1)!} = \lim_{n\to\infty} \frac{1}{n} = 0$.
e) $\sum_{n=0}^{\infty} (nx)^n = \lim_{n=0}^{\infty} n^n x^n$
Let $a_n = n^n$.
$\limsup |a_n|^{\frac{1}{n}} = \limsup (n^n)^{\frac{1}{n}} = \limsup (n^{n * \frac{1}{n}}) = \limsup n = \infty$.
That means that $\rho = \infty$. Then, $R = \frac{1}{\rho} = 0$.

Let $f(x) = \sum_{n=0} a_n x^n = \lim_{N\to 0} \sum_{n=0} N a_n x^n$

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<u>(Theorem)???</u>: Let $f_n(x)$ converges to $f(x)$ for all $x \in (a,b)$. Suppose $f_n(x)$ is continuous for all n. Then, $f(x)$ is continuous.
The above statement is wrong!

Consider $f_n(x) = \begin{cases} \text{(some line from 0 to 1)} & x \in [-\frac{1}{n}, \frac{1}{n}] \\ 0 & x < \frac{-1}{n} \\ 1 & x > \frac{1}{n} \end{cases}$

Then, $f(x)$ is a step function with a vertical line.

Pointwise Convergence

$f_n(x) \to f(x)$ as $n \to \infty$ $\forall x \in S$. $\forall \epsilon > 0$ $\exists N(\epsilon, x)$ such that if $n > N$, then $|f_n(x) - f(x)| < \epsilon$.

Uniform Convergence

<u>Definition</u>: Let f_n be a sequence of real-valued functions defined on S. We say that $f_n(x)$ uniformly converges to $f(x)$ on S if $\forall \epsilon > 0$ $\exists N(\epsilon)$ such that if $x \in S$ and $n \geq N$, then $|f_n(x) - f(x)| < \epsilon$.
<u>Theorem</u>: The uniform limit of continuous functions is continuous. In other words, let $\{f_n(x)\}_{n\geq 1}$ be a sequence of functions continuous on S. Assume that $f_n(x)$ converge uniformly to $f(x)$ on S. Let $S = dom(f)$. Then, $f(x)$ is continuous on S.
<u>Proof</u>: Let $x_0 \in S$. We have to prove that $f(x)$ is continuous at x_0. Then, we need to prove$\forall \epsilon > 0$ $\exists \delta(\epsilon)$ such that if $|x - x_0| < \epsilon$ and $x \in S \Rightarrow |f(x) - f(x_0)| < \epsilon$.
Let $\epsilon = \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3}$.
Then, $\forall \frac{\epsilon}{3} > 0$ $\exists N(\frac{\epsilon}{3})$ such that if $x \in S$ and $n \geq N$, then $|f_n(x) - f(x)| < \frac{\epsilon}{3}$.
Take N to be the N above. $f(x) - f(x_0) = (f(x) - f_n(x)) + (f_n(x) - f_n(x_0)) + (f_n(x_0) - f(x_0))$.
We know that $|f(x) - f(x_0)| \leq |f(x) - f_n(x)| + |f_n(x) - f_n(x_0)| + |f_n(x_0) - f(x_0)|$.
Choose $n = N(\frac{\epsilon}{3}) + 17$. Then, $|f(x) - f_n(x)| < \frac{\epsilon}{3}$ and $|f_n(x_0) - f(x)| < \frac{\epsilon}{3}$. We know that f)

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Ex) $g_n(x) = \frac{x^3}{1+nx^2}$. $g_n(x) \to g(x) \equiv 0$, so pointwise as $n \to \infty$
If $x = 0$, then $g_n(0) = 0 \Rightarrow 0 = g(0)$.
If $x \not 0$, $g_n(x) \to 0$, $\sup_{x\to\mathbb{R}} |g_n(x)| = \infty$.
$g_n(x) = \frac{x}{\frac{1}{x^2} + n} \to \frac{\infty}{n}$
Ex) $h_n(x) = (1 + \frac{x}{n})^n$

$h_n(x) \to_{n\to\infty} e^x$
$\lim_{n\to\infty}[\sup_{x\in\mathbb{R}} |h_n(x) - h(x)|] = \sup_{x\in\mathbb{R}} |(x + \frac{x}{n})^n - e^x| = \infty$
$e^x - (1 + \frac{x}{n})^n = e^x - (1 + \frac{x}{n})^n = e^x(1 - \frac{(1+\frac{x}{n})^n}{e^x})$.

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Let $\{f_n\}_{n\geq 1}$ be a sequence of continuous functions on $[a, b]$.
$f_n \to f$ pointwise on $[a, b]$, then $f_n(x) = \begin{cases} 0 & \text{if } \frac{1}{n} \leq x \leq 1 \\ 4n^2(x) & \text{if } 0 \leq x\frac{1}{2n} \\ 4n - 4nx & \end{cases}$
1) $\forall x \in [0, 1]$ $f_n(x) \to f(x) \equiv 0$ as $n \to \infty$.
If $x = 0$, then $f_n(0) = 0 \to f(0) = 0$ as $n \to \infty$.
$\frac{1}{n} < x \Leftrightarrow n > \frac{1}{x}$.
If $n > \frac{1}{x}$, then $f_n(x) = 0$.

More on Uniform Convergence

Theorem: Let $\{f_n\}_{n\geq 1}$ be a sequence of continuous functions on $[a, b]$. Suppose that $f_n \to f$ uniformly on $[a, b]$. Then, $\int_a^b f_n(x)dx \to \int_a^b f(x)dx$ as $n \to \infty$.
$\forall \epsilon > 0$ $\exists N(\epsilon)$ such that $|f_n(x) - f(x)| < \frac{\epsilon}{b-a}$ $\forall x \in [a, b]$ and $\forall n \geq N$.
$f_n(x) - \frac{\epsilon}{b-a} < f(x) < f_n(x) + \frac{\epsilon}{b-a} \Rightarrow \int_a^b (f_n(x) - \frac{\epsilon}{b-a})dx < \int_a^b f(x)dx < \int_a^b (f_n(x) + \frac{\epsilon}{b-a}dx)$.
Then, we get $\int_a^b (f_n(x))dx - \epsilon < \int_a^b f(x)dx < \int_a^b (f_n(x))dx + \epsilon$.
Then, $\forall\epsilon$ $\exists N$ such that $|\int_a^b f_n(x)dx - \int_a^b f(x)dx| < \epsilon$ $\forall n \geq N$.
Then, $\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx$.

Let's revisit Cauchy Sequences. The problem with current knowledge is that you often have to know the limit in order to prove that it converges, etc... Cauchy allows us to show convergence without knowing the limits.
Definition: A sequence $\{f_n(x)\}_{n\geq 1}$ on $S \subseteq \mathbb{R}$ is uniformly Cauchy if ($\forall\epsilon > 0$ $\exists N$ such that $|f_n(x) - f_m(x)| < \epsilon$ $\forall x \in S$ $\forall n, m \geq N$.
Theorem: $\{f_n(x)\}_{n\geq 1}$ is uniformly Cauchy on s iff $f_n(x) \to$ some function $f(x)$ uniformly on S.
$|f_n(x) - f(x)| < \frac{\epsilon}{2}$ $\forall n \geq N$.
$|f_m(x) - f(x)| < \frac{\epsilon}{2}$ $\forall m \geq N$
$|f_n(x) - f_m(x)| = |f_n(x) - f(x) - f_m(x) + f(x)| < \epsilon$.

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Other direction of Cauchy sequence

Assume that $\{f_n(x)\}$ is uniformly Cauchy. Let's fix x_0.
Then, $\{f_n(x_0)\}_{n\geq 1}$ is a Cauchy sequence.
This implies that there exists $\lim f_n(x_0) = f(x_0)$
We know that $f_n(x) \to f(x)$ pointwise.
$(\forall\frac{\epsilon}{2})(\exists N)(|f_n(x) - f_m(x)| < \frac{\epsilon}{2})$ $(\forall n, m \geq N)(\forall x \in S)$.
Let's fix x_0. Then, $|f_n(x_0) - f_m(x_0)| < \frac{\epsilon}{2}$
Then, $f_m(x_0) - \frac{\epsilon}{2} < f_n(x_0) < f_m(x_0) + \frac{\epsilon}{2}$
$\lim f_m(x_0) - \frac{\epsilon}{2} \leq f_n(x) \leq \lim f_m(x_0) + \frac{\epsilon}{2}$
Then, $f(x_0) - \frac{\epsilon}{2} \leq f_n(x_0) \leq f(x_0) + \frac{\epsilon}{2}$
$|f_n(x) - f(x)| \leq \epsilon < \epsilon$ $\forall n \geq N$.

Ex) We have the equation $\sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n g_n(x)$ with $g(x) = g(x+2)$.
$g(x) = \begin{cases} x & \text{if } 0 \leq x < 1 \\ 2-x & \text{if } 1 \leq x \leq 2 \end{cases}$
$S(x) = \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n g_n(x)$.
$S(x)$ is continuous. $S_n(x) = \sum_{k=1}^{n} \left(\frac{3}{4}\right)^k g_k(x)$.
$\left|\left(\frac{3}{4}\right)^n g_n(x)\right| = \left(\frac{3}{4}\right)^n$
$\{S_n(x)\}_{n\geq 1}$ is uniformly Cauchy.

$\epsilon > |S_n(x) - S_m(x)| = \left|\sum_{k=0}^{n} \left(\frac{3}{4}\right)^k g_k(x)\right| \; (\forall n \geq m \geq N)$

Uniform limit of continuous functions are continuous.
Sums of continuous functions are continuous.

Let $f_n(x) \to f(x)$ uniformly, then lots of good things happen
Suppose $g(x) = \sum_{n=1} \infty g_n(x)$
Then, $S_n(x) = \sum_{k=1} n g_k(x)$.
Then, $S_n(x) \to S(x)$ uniformly.

Ex) Let $f_n(x) \to f(x)$ pointwise.
$|f_n(x) - f(x)| < \frac{1}{\sqrt{n}} < a_n \to 0$.
$(\forall \epsilon > 0)(\exists N)(\forall n)(n \geq N \implies \frac{1}{n} < \epsilon)$
This implies that $(\forall n \geq N)(\forall x \in S)(|f_n(x) - f(x)| < \epsilon)$
Then, $f_n(x) \to f(x)$ uniformly on S as $n \to \infty$.
If $\sup_{x\in S} |f_n(x) - f(x)| = a_n \to 0, \ldots$

Weierstrass M-Test

Let $\{M_n\}_{n\geq 1}$ be a sequence of non-negative numbers such that $\sum_{n=1}^{\infty} M_n < \infty$. If $|g_n(x)| \leq M_n$ $(\forall x \in S)$, then $\sum_{n=1}^{\infty} g_n(x)$ converges uniformly on S.

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Theorem: Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series with a radius of convergence, $R > 0$. Let $0 < r < R$. Then, $\sum_{n=0} \infty a_n x^n$ converges uniformly on $[-r, r]$.
$\sum_{a_n} x^n$ and $\sum |a_n| x^n$ have the same radius of convergence, $\rho = \limsup |a_n|^{\frac{1}{n}}$, so $R = \frac{1}{\rho}$.
$\sum_{n=0}^{\infty} |a_n| r^n < \infty$.
Proof: Let $x \in [-r, r]$.
By the Weierstrass M-test, $|a_n x^n| \leq \frac{a_n r^n}{M_n}$.
If $\sum_{n=1}^{\infty} M_n < \infty$, then the series converges uniformly....

Corollary: $f(x) = \sum_{n=1}^{\infty} a_n x^n$ is continuous on $(-R, R)$.
Example: $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$
$\sum_{n=0}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2}$. Derivative of above.
$\sum_{n=0}^{\infty} (n)(n-1) x^{n-2} = \frac{1}{(1-x)^3}$

Proposition: If the power series, $\sum_{n=0}^{\infty} a_n x^n$ has radius of convergence, R. Then, $\sum_{n=1}^{\infty} n a_n x^{n-1}$ and $\sum_{n=0}^{\infty} a_n \frac{x^{n+1}}{n+1}$ has the same radius of convergence, R.
$x \sum_{n=1}^{\infty} n a_n x^n$
Then, $\rho = \limsup (n|a_n|)^{\frac{1}{n}} = \limsup n^{\frac{1}{n}} |a_n|^{\frac{1}{n}} = \limsup |a_n|^{\frac{1}{n}}$
Now, $\lim n^{\frac{1}{n}} = \lim e \ldots$

Theorem: Suppose that $\sum_{n=0}^{\infty} a_n x^n$ has a radius of convergence, $R > 0$. Then, $\int_0^x f(t)dt = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ for $|x| < R$
Example: $\int_0^n \sum_{n=0}^{\infty} a_n t^n dt = \sum_{n=0}^{\infty} \int_0^n a_n t^n dt = \sum a_n \frac{a^{n+1}}{n+1}$

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Theorem: Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ has radius of convergence $R > 0$.

Example: Let $p = .47 = \frac{1}{x}$
$x_i = \begin{cases} 1 & \text{if we win with probability } p = .47 \\ -1 & \text{if we lose with probability } p = .47 \end{cases}$
$S_n = \sum_{i=1}^{n} x_i$
Then, $S_n = n(2p-1) + \sqrt{n}$. In this case, $S_n = -.006n + \sqrt{n}$. In other words, as $n \to \infty$, you lose money.
A Probability Equation: $P(x) = \frac{1}{\sqrt{n}} \int_0^x e^{\frac{-t^2}{2}} dt$
Then, $P'(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{2\pi} n! 2^n} x^2 n$
$P(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{e^n n!} \frac{x^{2n+1}}{2n+1}$

$f(x) = \sum_{n=0} \infty a_n x^n$ with radius of convergence $R > 0$.
Abel's Theorem: Let $f(x) = \sum_{n=0}^{\infty}$ be a power series with finite positive radius of convergence, R. If the series converges at R (i.e. $\sum_{n=0}^{\infty} a_n R^n < \infty$), then f(x) is continuous at $x = R$.
Also, if the series converges at $-R$, then $f(x)$ is continuous at $x = -R$
$\int_a^b f(x)g'(x) = f(b)g(b) - f(a)g(b) - \int_a^b f'(x)g(x)dx$
Let $R = 1$ and $f_n(x) = \sum_{k=0}^{n} a_n x^n$. then, $s_n = f_n(1) = \sum_{k=0}^{n} a_1 1^k = \sum k = 0^n a_k$
So, $\sum_{k=0}^{\infty} a_k 1^k \to f(1) = S$ because the series converges at $x = 1$.
$S_n = \sum k = 0^n a_k$ We know that

$$\begin{aligned} a_1 &= (a_0 + a_1) - a_0 = S_1 - S_0 \\ a_2 &= (a_0 + a_1 + a_2) - a_0 - a_1 = S_2 - S_2 \\ &\vdots \\ a_n &= S_n - S_{n-1} \end{aligned}$$

$$
\begin{aligned}
f_n(x) &= a_n + \sum_{k=1}^{n} a_k x^k \\
&= S_0 + \sum_{k=1}^{n} (S_k - S_{k-1}) x^k \\
&= S_0 + \sum_{k=1}^{\infty} S_k x^k - x \sum_{k=1}^{\infty} S_{k-1} x^{k-1} \\
&= S_0 + \sum_{k=1}^{\infty} S_k x^k - x \sum_{k=0}^{\infty} S_k x^k \\
&= S_0 + \sum_{k=1}^{\infty} S_k x^k S_n x^k - x(\sum_{k=0}^{\infty} S_k x^k
\end{aligned}
$$

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Basic Properties of Derivatives

<u>Defintion</u>: Let $f(x)$ be defined in some open interval around point a. Then, we say that $f(x)$ is differentiable at $x = a$ if $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists. The limit is called the derivative of f at $x = a$.
<u>Example</u>: Let $f(x) = x^9$.
$f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{x^9-a^9}{x-a} = \lim_{x\to a} x^8 + x^7 a + \ldots x a^7 + a^8 = 9a^8$
<u>Theorem</u>: Let $f(x)$ and $g(x)$ be differentiable at $x = a$.
Then,

1. $(cf(x))' = cf'(x)$ for any constant c.
2. $(f(x) + g(x))' = f'(x) + g'(x)$
3. $f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$
4. $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x)-f(x)g'(x)}{g'(x)}$ if $g(x) \neq 0$.

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<u>Theorem</u>: Let $f(x)$ be differentiable at $x = a$. Then, $f(x)$ is continuous at $x = a$.
<u>Proof</u>: We know that $f'(a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$ exists. We also know that $f(x) = \left(\frac{f(x)-f(a)}{x-a}\right)(x-a) + f(a)$. Then, $\lim x \to a \left(\frac{f(x)-f(a)}{x-a}\right)(x-a) = \lim x \to a \left(\frac{f(x)-f(a)}{x-a}\right) \lim_{x\to a}(x-a) = f'(a) * 0 = 0$. Therefore, $\lim_{x\to a} f(x) = \lim_{x\to a} \lim x \to a \left(\frac{f(x)-f(a)}{x-a}\right) \lim_{x\to a}(x-a) + f(a) = f(a)$.

<u>Theorem(Chain Rule)</u>: If f is differentiable at a and g is differentiable at $f(a)$, then $g(f(x))$ is also differentiable at a and $(g \circ f)'(a) = g'(f(a))f'(a)$.
<u>Proof</u>: $(g \circ f)'(a) = \lim_{x\to a} \frac{(g\circ f)(x)-(g\circ f)(a)}{x-a} = \lim_{x\to a} \frac{g(f(x))-g(f(a))}{x-a} = \lim_{x\to a} \frac{g(f(x))-g(f(a))}{f(x)-f(a)} \frac{f(x)-f(a)}{x-a}$. Let $y = f(x)$. Then, the previous expression is now $\lim_{x\to a} \frac{g(y)-g(f(a))}{y-f(a)} f'(a)$ ....

<u>Example</u>: $f(x) = exp(\sin^2(x))$. Then, $f'(x) = exp(\sin^2(x))(\sin^2(x))' = exp(\sin^2(x)) * 2\sin(x)\cos(x)$.

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There exists $c \in (a, b)$ such that $\gamma = \frac{f(b)-f(a)}{b-a} = f'(c)$. Threfore, $f(b) - f(a) = \gamma(b-a)$ and $f(b) - f(a) = f'(c)(b-a)$.

$\int_a^b f'(x)dx = f(b) - f(a) = \gamma(b-a)$.
Three cases:
1) $f'(x) > \gamma \forall x \in (a,b)$
2) $f'(x) < \gamma \forall x \in (a,b)$
3) $\exists x_1, x_2 \in (a,b)$ such that $f'(x_1) > \gamma$ and $f'(x_2) < \gamma$

Case 1: If $f'(x) > \gamma \forall x \in (a,b)$, then

$$\begin{aligned}\int_a^b f'(x)dx > \int_a^b \gamma dx = \gamma(b-a) &= \frac{f(b)-f(a)}{b-a}(b-a) \\ &= f(b) - f(a)\end{aligned}$$

Something is wrong with this.
If g is continuous on $[a,b]$, then $G(x) = \int_a^b g(t)dt$ is differentiable and $G'(x) = g(x)$.
$g(x) = f'(x)$ and $\int_a^b f'(x)dx = f(b) - f(a)$. We're also assuming that the derivative is continuous.
Example: $f(x) = \begin{cases} x^n \sin(x^{-1000}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$
$f'(x) = 2x\sin(x^{-2008}) + x^2\cos(x^{-2008})(-2008x^{-2009})$
This function is not continuous on 0.

Lemma: Suppose $f(x)$ is differentiable on (a,b) and $f(x)$ attains its maximum(or minimum) at $c \in (a,b)$. Then, $f'(c) = 0$.
Proof: Let us assume that $f'(c) \neq 0$.
Without loss of generality, we can assume that $f'(c) > 0$. (Otherwise, you can take the negative of the function.)
Then, $\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = f'(c) > 0$.
$(\forall \epsilon > 0)(\exists \delta))(|x-c| < \delta \Rightarrow |\frac{f(x)-f(c)}{x-c} - f'(c)| < \epsilon$
$f'(c) - \epsilon < \frac{f(x)-f(c)}{x-c} < f'(c) + \epsilon$ for x such that $0 < |x-c| < \delta$.
Now, choose $\epsilon = \frac{f'(c)}{2}$
$\frac{f'(c)}{2} < \frac{f(x)-f(c)}{x-c}$.
Then, if $x > c$, we have $f(x) > f(c)$.
Then, $f(c)$ is not a maximum! CONTRADICTION!!!

Theorem: (special case of the mean value theorem. Rolle's Theorem) Suppose that $f(x)$ is continuous on $[a,b]$, is differentiable on (a,b), and $f(a) = f(b)$. Then, there exists $c \in (a,b)$ such that $f'(c) = 0$.
Since f is continuous on $[a,b]$, it attains both its maximum and its minimum on $[a,b]$.
Mean Value Theorem (again): Suppose that $f(x)$ is continuous on $[a,b]$ and is differentiable on (a,b). Then, there exists $c \in (a,b)$ such that $f'(c) = \frac{f(b)-f(a)}{b-a} = \gamma$
Let $g(x) = f(x) - \gamma x$.
Then, $g(a) = g(b)$.
Then, there exists $c \in (a,b)$ such that $g'(c) = 0 = f'(c) - \gamma c$

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Definition: Let $f(x)$ be defined on I. We say that f is increasing on I if $x_1, x_2 \in I$ and $x_1 < x_2$ implies that $f(x_1) \leq f(x_2)$. We say that f is strictly increasing if $x_1, x_2 \in I$ and $x_1 < x_2$ implies that $f(x_1) < f(x_2)$.
Theorem: Let f be differentiable on an interval (a,b). Then,

1. f is strictly increasing on (a,b) if $f'(x) > 0$ for all $x \in (a,b)$
2. f is strictly decreasing on (a,b) if $f'(x) < 0$ $\forall x \in (a,b)$.

3. f is increasing on (a,b) if $f'(x) \geq 0 \; \forall x \in (a,b)$
4. f is decreasing on (a,b) if $f'(x) \leq 0 \; \forall x \in (a,b)$

Proof:

1. If $x_1 < x_2$, then $f(x_2) - f(x_1) = f'(c)(x_2 - x_1) > 0$

Intermediate Value Theorem for Derivatives: Let f be differentiable on (a,b). Let $a < x_1 < x_2 < b$. Finally, let us assume that c is between $f'(x_1)$ and $f'(x_2)$. Then, $\exists x_0 \in (x_1, x_2)$ such that $f'(x_0) = c$
Proof: Let $g(x) = f(x) - cx$ is continuous $[x_1, x_2] \Rightarrow g$ attains its minimum on $[x_1, x_2]$.
Consider x_1, $g'(x_1) = f'(x_1) - c < 0$.
Consider x_2, $g'(x_2) = f'(x_2) - c > 0$.
Suppose you have f, which is 1 to 1 on I.
f is differentiable. Then, f is strictly monotone. $f(a,b) \to (f(a), f(b))$.
Can we differentiate an inverse function?
We know that $f^{-1} : J \to I$ and $f^{-1}(f(x)) = x \; \forall x \in I$
By the chain rule, $(f^{-1})'(f(x)) * f'(x) = 1$. Then, $(f^{-1})'(f(x)) = \frac{1}{f'(x)}$

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Theorem: f is one to one and differentiable on $I \subseteq (a,b)$ and f' is strictly monotone. $f : I \to y$ Also, I and y are both open intervals.
$0 \neq f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}$
We know that $\lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{f'(x_0)}$
$\lim_{x \to x_0} \frac{x - x_0}{y - y_0} = \lim_{x \to x_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0}$
Now, we know that $\lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = f^{-1}(y_0)$
If $y \to y_0$, then $x \to x_0$ since $x = (f^{-1})'(y)$
If $x \to x_0$, then $y \to y_0$.

Theorem: Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for $|x| < R$, where $R > 0$.
We know that $a_0 = f(0)$, but what about a_1?
$a_1 = f'(0)$. Why? $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = a_1 + 2a_2 x + 3a_3 x^2 + \ldots$
Let $x = 0$. Then, $f'(0) = a_1$.
What about a_2?
$a_2 = \frac{f^{(2)}(0)}{2!}$. In fact, $a_n = \frac{f^{(n)}(0)}{n!}$
If $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$
Ex) $f(x) = \begin{cases} e^{-\frac{1}{x}} & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}$
When is $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(x)$?
$R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k$.
Defintion: Let $f(x)$ be defined on some interval around 0. If f has derivates of all orders at 0, we call $\sum_{n=0} \infty \frac{f^{(n)}(0)}{n!} x^n$
Taylor's Theorem: Let f be defined on (a,b) where $a < 0 < b$. Suppose f has nth derivative on (a,b). Then, $\forall x \in (a,b)$, $\exists y \in (0,x)$ such that $f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x)}{k} x^k + \frac{f^{(n)}(y)}{n!} x^n$
In other words, $R_n(x) = \frac{f^{(n)}(y)}{n!} x^n$

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Proof of Taylor's Theorem: Fix $x \in (a,b)$ such that $x \neq 0$. Choose $M = M(x) \in \mathbb{R}$ such that $f(x) = \sum_{k=0}^{n-1} \frac{f^{(k)}(x)x^k}{k!} + Mx^n$, so that $M = \frac{f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k}{x^n}$
(We want to show that $M = \frac{f^{(n)}(?)}{n!}$)
Consider $g(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} t^k + Mt^n - f(t)$ on $[0,x]$ (or $[x,0]$ if $x < 0$)
Clearly, $g(x) = 0$. Also, $g(0) = f(0) - f(0) = 0$
So, $g(0) = g(x)$. Then, $\exists x_1 \in (0,x)$ such that $g^{(1)}(x_1) = 0$.
$\frac{dg}{dt}(t) = \sum_{k=1}^{n-1} \frac{f^{(k)}(0)}{(k-1)!} t^{k-1} + nMt^{n-1} - f'(t)$
We know that $g'(x_1) = 0$ Also, $g'(0) = 0$.
Then, $g'(0) = g'(x_1)$.
Therefore, $\exists x_2 \in (0, x_1)$ such that $g"(x_2) = 0$
$g"(t) = \sum_{k=2}^{n-1} \frac{f^{(k)}(x)}{(k-2)!} t^{k-2} + n(n-1)Mt^{n-2} - f"(t)$
$g"(x_2) = 0$. Also, $g"(0) = 0$.
Indeed, $g"(0) = f"(0) - f"(0) = 0$
$g"(0) = g"(x_2)$. Then, $\exists x_3$ such that $g'''(x_3) = 0$
Continue on... and you get you get $\exists x_n$ such that $g^{(n)}(x_n) = 0$
$g(t) = \sum_{(k=0)}^{n-1} \frac{f^{(k)}(0)}{k!} t^k + M^n t^n - f(t)$
$g^{(n)}(t) = Mn! - f^{(n)}(t)$. $g^{(n)}(z) = 0 \Rightarrow M = \frac{f^{(n)}(z)}{n!}$

Theorem: $R_n(x) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(0)}{k!} x^k = \int_0^x \frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t)dt$
Note:

$$\begin{aligned} f(x) &= f(0) + \int_0^x f'(t)dt \\ &= f(0) + \int_0^x -(x-t)' f'(t)dt \end{aligned}$$

Using integration by parts, let $u(t) = (x-t)$ and $v(t) = f'(t)$
$\int_0^x -u'v dt = \int_0^x uv' dt - [uv]_0^x$
Then,

$$\begin{aligned} &= f(0) + \int_0^x (x-t) f"(t)dt - [(x-t)f'(t)]_0^x \\ &= f(0) + \int_0^x (x-t)f"(t)dt - f'(0)x \\ &= f(0) + f'(0)x + \int_0^x (x-t)f"(t)dt \end{aligned}$$

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Metric Spaces

Definition: Let S be a set and $d : SXS \to \mathbb{R}$. The function d is called a metric on S if it satisfies the following:

1. $d(x,x) = 0 \forall x \in S$
2. $d(x,y) > 0 \forall x,y \in S$ such that $x \neq y$
3. $d(x,y) = d(y,x) \forall x,y \in S$
4. $d(x,z) \leq d(x,y) + d(y,z) \forall x,y,z \in S$

Definition: (S,d) is called a metric space.
Example:

1. Let S be an arbitrary set $d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$

2. Let $S = \mathbb{R}^n$. Then, $x = (x_1, \dots, x_n)$, $y = (y_1, \dots, y_n)$.

Typically, $d(x,y) = \sqrt{\sum_{k=1}^n (x_k - y_k)^2}$
Special cases: $n = 1$ and $S = \mathbb{R}$
$d(x,y) = \sqrt{(x-y)^2} = |x-y|$
If $n = 2$, $d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$
Since $d(x,z) \leq d(x,y) + d(y,z)$.
$\sqrt{\sum_{i=1}^n (x_i - z_i)^2} \leq \sqrt{\sum_{i=1}^n (x_i - y_i)^2} + \sqrt{\sum_{i=1}^n (y_i - z_i)^2}$
Open Sets: Suppose we have a metric space, (S,d) and $E \subseteq S$. Then, $x \in E$ is called an interior point to E if $\exists \epsilon$ such that $\forall y$ satisfying $d(x,y) < \epsilon$, we know that $y \in E$.
Let E^o be the collection of all interior points of E.
Definition: $E \subseteq S$ is called an open set if $E^o = E$. In other words, E is an open set if any point of E is its interior point (i.e. $(\forall x \in E)(\exists \epsilon > 0)(d(x,y) < \epsilon \implies y \in E)$)
Theorem:

1. S is an open itself, $\emptyset$ is open

2. If $\{S_\alpha : \alpha \in \mathcal{A}\}$ is a collection of open set, then $\bigcup_{\alpha \in \mathcal{A}}$ is also open.

3. If $\{E_n, 1 \leq n \leq N\}$ is a finite collection of open sets, then $\bigcap_{n=1}^N E_n$ is open.

Example: $E_n = (-\frac{1}{n}, \frac{1}{n})$. $S = \mathbb{R}$
$\bigcap_{n=1}^\infty E_n = \{0\}$ is not open!
Example: $\bigcap_{n=1}^N E_n = \bigcap_{n=1}^N (-\frac{1}{n}, \frac{1}{n}) = (-\frac{1}{N}, \frac{1}{N})$

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Definition: A set $E \subseteq S$ is closed if its complement S E is open.
Definition: Let E be an arbitrary subset of S. The closure $\bar{E}$ of E is the intersection of all closed sets containing E.
$E^o \subseteq E \subseteq \bar{E}$ where E^o is the interior of E.
Proposition:

1. $\emptyset$ is a closed set. S is a closed set

2. Intersection of any number of closed sets is closed

3. Union of a finite number of closed sets is closed

Note: $\bigcup_{\alpha \in \mathcal{A}} \bigcup_\alpha$ is open if each U_α is open.
What about closed sets?
$E_\alpha = S \ \bigcup_\alpha$.
$\bigcap_{\alpha \in \mathcal{A}} \cdots$
Proposition: Let E be a subset of S. Then,

1. E is closed iff $\bar{E} = E$

2. The set E is closed iff it contains the limit of any converging sequence in E.

3. An element belongs to $\bar{E}$ iff it is the limit of a converging sequence on E

4. A Point in the boundary of E (recall Boundary(E) $= \dots$

The real material

Let (S, d) and $(S*, d*)$ be metric spaces. $f : S \to S^*$ is continuous at $s_o in S$ if $\forall \epsilon > 0$ $\exists \delta(\epsilon, s_o)$ such that if $|s - s_o| < \delta$, then $|f(s) - f(s_o)| < \epsilon$.
Theorem: Let (S, d) and $(S*, d*)$ be metric spaces. A function $f : S \to S^*$ is continuous iff for any open subset U of S^*, its preimage, $f^{-1}(U)$ is also open.
Then, $f^{-1}(U) = \{x \in S | f(x) \in U\}$
Proof: $\Rightarrow$
Suppose $f : S \to S^*$ is continuous. We need to prove that if $U \subseteq S^*$ is open, then $f'(U)$ is open on S

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Lemma: Let $V = B_\epsilon(y)$ is an open set. Let $x \in B_\epsilon(y)$. We have to show that $\exists \epsilon > 0$ such that $B_\epsilon(x) \subseteq V = B_\epsilon(y)$. Choose $\tilde{\epsilon} < \epsilon - d(y, x)$ We have to show that $d(x, y) \ldots$
Consider the preimage of $f'(B_\epsilon(f(s)))$. Since $f'(B_\epsilon(f(s)))$ is open, $\exists \delta > 0$ such that $B\epsilon(s) \subset f'(B_\epsilon(f(s)))$.
Definition: Let (S, d) be a metric space and $E \subseteq S$.
Let $\mathcal{U} = \{U_\alpha\}_{\alpha \in \mathcal{A}}$ be a collection of open sets such that $E \subseteq \bigcup_{\alpha \in \mathcal{A}} U_\alpha$. then, we say that $\mathcal{U} = \{U_\alpha | \alpha \in \mathcal{A}\}$ is called an cover of E.
Definition: E is compact if any open cover of E contians finite subcover.
$f\{U_1, \ldots, U_m\} \in \mathcal{U}$ such that $E \subseteq \bigcup_{i=1}^m U_i$
Theorem: (Heine-Borel) Let $(\mathbb{R}, |x - y|)$ be the metric space. A set $E \subseteq \mathbb{R}$ is compact iff it is bounded and closed.
Let $f : [a, b] \to \mathbb{R}$. This is closed because $R\ [a, b] = (-\infty, a) \bigcup (b, \infty)$ is open.
Let $f : [a, b] \to \mathbb{R}$ be continuous. Then,

1. f is bounded (i.e. $\exists M$ such that $|f(x)| \leq M$ $\forall x \in [a, b]$)
2. f attains its maximum and minimum on $[a, b]$
3. f is uniformly continuous $[a, b]$.

Theorem: Consider metric spaces (S, d) and (S^*, d^*) and a continuous function $f : S \to s^*$. Let E be a compact subset of S. Then,

1. $f(E) = \{y \in S^*$ such that $\exists x \in E$ such that $y = f(x)$ is compact in S^*
2. f is uniformly continuous on E