Notes - Math 135b

May 7, 2014

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Limit Theorems

Given a sequence of random variables, $Y_1, Y_2, \ldots$, we want to show that "when n is large", Y_n is approximately $f(n)$ for some simple (deterministic) function, $f(n)$.

Definition 1. *A sequence, $Y_1, Y_2, \ldots$, of random variables converges to a number, a, in probability if $P(|Y_n - a| \le \epsilon)$ converges to 1 for any fixed ϵ. This is equivalent to $P(|Y_n - a| > \epsilon)$ converges to 0.*

Example:

1. Let $X_1, X_2, \ldots$ be a sequence of independent and identically distributed (iid) numbers.
   $EX_i = \mu$ and $Var(X_i) = \sigma^2$.
   $S_n = X_1 + \ldots + X_n$.
   When n is large, $\frac{S_n}{n\mu}$ converges to 1 in probability.

2. Toss a fair coin n times (independently).
   Let R_n be the "longest runs of heads" or longest substring of consecutive results of heads. (Results look like HHTTHHHTHTHTHH. $R_n = 3$ in this case.)

$$\begin{aligned} P(R_n \ge 1) &= P(\text{at least } k \text{ consecutive result of heads somewhere in the result string}) \\ &= P(\bigcup_{i=1}^{n-k+1})\ \{i \text{ is the first head in an interval of at least } k \text{ heads}\} \\ &\le n\frac{1}{2^k} \end{aligned}$$

We discovered the upper bound. Now, for the lower bound.
Divide the string of size n instead disjoint strings of size ki.

$$\begin{aligned} P(R_n \ge k) &\le P(\text{at least one of the blocks consists of all heads}) \\ &= 1 - (1 - \frac{1}{2^k})^{\lfloor \frac{n}{k} \rfloor} \end{aligned}$$

Theorem 2. $\frac{R_n}{\log_2 n}$ *converges to 1 in probability*

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From last time, Let there be n tosses of a fair coin and R_n be the longest interval of heads.

$$P(R_n \geq k) \leq n\frac{1}{2^k}$$

$$P(R_n \geq k) \geq 1 - \left(1 - \frac{1}{2^k}\right)^{\lfloor \frac{n}{k} \rfloor}$$

Theorem 3.

$$\frac{R_n}{\log_2 n} \to 1$$

in probability

Proof. Need to show that for any $\epsilon > 0$

$$\begin{aligned} P(R_n \geq (1+\epsilon)\log_2 n) &\to 0 \qquad (1) \\ P(R_n \geq (1-\epsilon)\log_2 n) &\to 1 \qquad (2) \end{aligned}$$

$$\begin{aligned} P\left(\left|\frac{R_n}{\log_2 n} - 1\right| \geq \epsilon\right) &= P\left(\frac{R_n}{\log_2 n} \geq 1+\epsilon \text{ or } \frac{R_n}{\log_2 n} \leq 1-\epsilon\right) \\ &= P\left(\frac{R_n}{\log_2 n} \geq 1+\epsilon\right) + P\left(\frac{R_n}{\log_2 n} \leq 1-\epsilon\right) \end{aligned}$$

Plug in $k = (1+\epsilon)\log_2 n$ to equation (1), you get

$$\begin{aligned} \ldots &\leq n\frac{1}{2^{(1+\epsilon)\log_2 n}} \\ &= n\frac{1}{n^{1+\epsilon}} \\ &= \frac{1}{n^\epsilon} \end{aligned}$$

Plug in $k = (1-\epsilon)\log_2 n$ to equation (2), you get

$$\begin{aligned} \ldots &\geq 1 - e^{-\frac{1}{2^k}\lfloor \frac{n}{k} \rfloor} \\ &\geq 1 - e^{\frac{-1}{2^k}(\frac{n}{k}-1)} \\ &\geq 1 - exp(-n^{-(1-\epsilon)}(\frac{n}{(1-\epsilon)\log_2 n) - 1} \\ &= 1 - exp\left(-n^\epsilon\frac{1}{(1-\epsilon)\log_2 n} + \frac{1}{n^{1-\epsilon}}\right) \\ &\to 1 \end{aligned}$$

□

Theorem 4 (Markov's inequality)**.** *If $X \geq 0$ is any random variables, $P(X > a) \leq \frac{1}{a}EX$*

Proof. $I\{X \geq a\}$ where this is 1 when the event happens and 0 otherwise. $I\{X \geq a\} \leq \frac{1}{a}X$ by Bernoulli □

Theorem 5 (Chebyshev's inequality). *$EX = \mu$ and $Var(X) = \mu^2$. Otherwise, X is aribitrary. Then,*

$$\begin{aligned} P(|X - \mu| \geq k) &\leq \frac{\sigma^2}{k^2} \\ P((X - \mu)^2 \geq k^2) &\leq \frac{\sigma^2}{k^2} \end{aligned}$$

Proof. We need to prove

$$\begin{aligned} P\left(\left|\frac{S_n}{n} - \mu\right| \geq \epsilon\right) &\to 0 \\ E\left(\frac{S_n}{n}\right) &= \frac{1}{n}n\mu = \mu \\ Var\left(\frac{S_n}{n}\right) & \end{aligned}$$

$$S_n = X_1 + \ldots + X_n \text{ where } X_i \text{ is aribtrary}$$

$$ES_n = EX_1 + \ldots + EX_n$$

$$\begin{aligned} Var(S_n) &= Var(X_1) + \ldots + Var(X_n) + \sum_{i \neq j} Cov(X_i, X_j) \\ Cov(X_1, X_j) &= E(X_i X_j) - EX_i EX_j \\ Var(aX) = a^2(n & \end{aligned}$$

□

Theorem 6 (Weak law of large numbers). *Let $X_1, X_2, \ldots$ be a sequence that is iid with $EX_1 = \mu$ and $Var(X_1) = \sigma^2 < \infty$.*
$S_n = X_1 + \ldots + X_n$
Then, $\frac{S_n}{n} \to \mu$ in probability.

Proof. Rest on the two inequalities:

$$\begin{aligned} P\left(\left|\frac{S_n}{n} - \mu\right| \geq \epsilon\right) &\leq \frac{1}{\epsilon^2} \cdot \frac{\sigma^2}{n} \\ &\to 0 \end{aligned}$$

□

example: Two investment choices at the beginning of each year:

- a risk-free "bond" which returns 6% per year
- a risky "stock" which increases your investment by 50% with probability .8 and wipes away your money with probability .2

If your money is s, then after a year, bond will give you $1.06s$ and stock $\begin{cases} 1.5s & \text{with probability } .8 \\ 0 & \text{with probabiliy } .2 \end{cases}$

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"hedging" - invest a fixed proportion x into the stock and $1-x$ into the bond. (We know that when $x=1$, you will eventually lose everything.)
Average growth is

$$\lambda = \lim_{n\to\infty} \frac{1}{n} \log \frac{x_n}{x_0}$$

where $x_n \approx x_0 e^{\lambda n}$.
x_n is your capital at the end of year n.
$I_i = I\{$ stocks goes up in year $i\}$.
They are independent indicators with $EI_i = .8$.

$$\begin{aligned}
x_n &= x_{n-1}(1-x)\cdot 1.06 + x_{n-1}\cdot x \cdot 1.5 \cdot I_n \\
&= x_{n-1}(1.06(1-x) + 1.5x \cdot I_n) \\
&= x_0(1.06(1-x)+1.5x)^{s_n}((1-x)1.06)^{n-s_n} \text{ where } s_n = I_1 + \dots I_n
\end{aligned}$$

The last step unrolled the recurrence.

$$\begin{aligned}
\frac{1}{n}\log\frac{x_n}{x_0} &= \frac{s_n}{n}\log(1.06+.44x) + (1-\frac{S_n}{n})\log(1.06(1-x)) \\
&\to_{n\to\infty} .8\log(1.06+.44x) + .2\log(1.06(1-x))
\end{aligned}$$

To maximize this,

$$\frac{.8\cdot .44}{1.06+.44x} = \frac{.2}{1-x}$$

where $x \in (0,1)$ Therefore, $\lambda = \lim_{n\to\infty}\frac{1}{n}\log\frac{x_n}{x_0}$ where $x_n \approx x_0 e^{n\lambda}$.

$e^\lambda = 1+u$
Solve $x = \frac{7}{22}$ and $\lambda \approx 8.1\%$ and $u = 8.4\%$

Example: Distribute n balls independently at random into n boxes. N_n is the number of empty boxes.
$N_n = I_1 + \ldots + I_n$ where $I_i = I\{i\text{th box is empty}\}$
You would use weak law of large number, but they're not independent.

$$EI_i = \left(\frac{n-1}{n}\right)^n = \left(1-\frac{1}{n}\right)^n \to e^{-1}$$

$$E(N_n^2) = EN_n + \sum_{i\neq j} E(I_iI_j)$$

$$E(I_iI_j) = P(\text{both box } i \text{ and } j \text{ are empty}) = \left(\frac{n-2}{n}\right)^n$$

$$Var(N_n) = E(N_n^2) - (EN_n)^2 = n\left(1-\frac{1}{n}\right)^n + n(n-1)\left(1-\frac{2}{n}\right)^n - n^2\left(1-\frac{1}{n}\right)^{2n}$$

By chebyshev's inequality,

$$\begin{aligned}
P\left(\left|\frac{1}{n}N_n - \frac{1}{n}EN_n\right| \geq \epsilon\right) &\leq \frac{1}{\epsilon^2}\frac{1}{n^2}VarN_n \\
&= \frac{1}{\epsilon^2}\left(\frac{1}{n}\left(1-\frac{1}{n}\right)^n + \frac{n-1}{n}\left(1-\frac{2}{n}\right)^n - \left(1-\frac{1}{n}\right)^{2n}\right) \\
&\to_{n\to\infty} \frac{1}{\epsilon^2}(0+e^{-2}-e^{-2}) \\
&= 0
\end{aligned}$$

This implies that $P\left(\left|\frac{1}{n}N_n - e^{-1}\right| > \epsilon\right) \to 0$

Moment Generating functions and Central Limit Theorems

If X is a random variable, then its moment generating function is

$$\phi(t) = \phi_X(t) = E(e^{tX}) = \begin{cases} \sum_x e^{tx}P(X=x) & \text{discrete} \\ \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx & \text{continuous} \end{cases}$$

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Moment generating function

$$\phi(t) = E[e^{tX}] = \begin{cases} \sum_x e^{tX}P(X=x) & \text{discrete} \\ \int_{-\infty}^{\infty} e^{tX} f(x)\,dx & \text{continuous} \end{cases}$$

Notes:

1. It doesn't always exists, i.e.

$$f_X(x) = \begin{cases} e^{-x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

   Therefore,

$$\phi(t) = \int_0^{\infty} e^{tx}e^{-x}\,dx = \frac{1}{1-t}$$

   only when $t < 1$.

2.

$$\begin{aligned}
E(e^{tX}) &= E[1 + tX + \frac{1}{2}t^2x^2 + \frac{1}{6}t^3x^3 + \ldots] \\
&= 1 + tE(X) + \frac{1}{2}t^2E(X^2) + \ldots
\end{aligned}$$

   Each of the $E(X^i)$ are called *moments*.

Example: Poisson(λ)

$$\begin{aligned}\phi(t) &= \sum_{n=0}^{\infty} e^{tn} \cdot \frac{\lambda^n}{n!} e^{-\lambda} \\ &= e^{-\lambda} \sum_{n=0}^{\infty} \frac{(e^t\lambda)^n}{n!} \\ &= e^{-\lambda+\lambda e^t} \\ &= e^{\lambda(e^t-1)}\end{aligned}$$

Example: Normal

$$\begin{aligned}\phi_X(t) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{-x^2/2}\, dx \\ &= \frac{1}{\sqrt{2\pi}} e^{\frac{1}{2}t^2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-t)^2}\, dx \\ &= e^{\frac{1}{2}t^2}\end{aligned}$$

$$-\frac{1}{2}(2tx + x^2) = \frac{1}{2}((x-t)^2 - t^2)$$

Lemma 7. *If $X_1, X_2, \ldots, X_n$ are independent and $S_n = X_1 + \ldots + X_n$, then*

$$\phi_{S_n}(t) = E[e^{tS_n}] = E[e^{tX_1} \cdot e^{tX_2} \ldots e^{tX_n}] = \phi_{X_1}(t) \ldots \phi X_n(t)$$

If X_i is identically distributed as X, then

$$\phi_{S_n}(t) = [\phi_X(t)]^n$$

Theorem 8. *1. If $\phi_X(t) = \phi_Y(t)$ for all t, then $P(X \le x) = P(Y \le x)$ for all x*

2. If $\phi_{X_n}(t) \to \phi_X(t)$ for all t, then $P(X_n \le x) \to P(X \le x)$ for all x.

Example: Show that the sum of independent Poisson random variables is Poisson.

$$\begin{array}{lll} X_1 & Poisson(\lambda_1) & e^{\lambda_1}(e^t-1) \\ X_2 & Poisson(\lambda_2) & e^{\lambda_2}(e^t-1) \\ & \vdots & \\ X_n & Poisson(\lambda_n) & e^{\lambda_n}(e^t-1) \end{array}$$

$$\phi_{X_1+\ldots+X_n}(t) = e^{(\lambda_1+\ldots+\lambda_n)(e^t-1)}$$

where $X_1 + \ldots X_n$ is $Poisson(\lambda_1 + \ldots + \lambda_n)$

Example: Moment generating functions of $Binomial(n,p)$
$S_n = \sum_{k=1}^{n} I_k$ where I_k are independent and $I_k = I\{\text{success on } k\text{th trial}\}$

$$\phi_{S_n}(t) = (e^t p + 1 - p)^n \text{ (from the fact} E[e^{tX}] = \phi_X(t))$$

Central limit theorem

Assume that X is a random variable with $EX = \mu$ and $Var(X) = \sigma^2$.
Let $S_n = X_1 + \ldots + X_n$, where $X_1, \ldots, X_n$ are iid as X.
Then,

$$P\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} \leq \sqrt{n}\right) \rightarrow_{n\rightarrow\infty} P(Z \leq x)$$

where $Z \in N(0,1)$

Proof. $Y_i = \frac{X_i - \mu}{\sigma}$
$E(Y_i) = 0$ and $Var(Y_i) = 1$
Let $\frac{S_n - n\mu}{\sigma} = Y_1 + \ldots + Y_n$.
We want to show that $\phi_{(Y_1+\ldots+Y_n)/\sqrt{n}}(t) \rightarrow \phi_Z(t)$

$$\begin{aligned}
\phi_{(Y_1+\ldots+Y_n)/\sqrt{n}}(t) &= E[e^{t\frac{Y_1+\ldots+Y_n}{\sqrt{n}}}] \\
&= E[e^{\frac{t}{\sqrt{n}}}(Y_1 + \ldots + Y_n)] \\
&= E[e^{\frac{t}{\sqrt{n}}Y_1}] \ldots E[e^{\frac{t}{\sqrt{n}}Y_n}] \\
&= E[e^{\frac{t}{\sqrt{n}}Y_1}]^n \\
&= \left(1 + \frac{t}{\sqrt{n}}EY + \frac{1}{2}\frac{t^2}{n}E(Y^2) + \frac{1}{6}\frac{t^3}{n^{3/2}}E(Y^3) + \ldots\right)^n \\
&= 1 + 0 + \frac{1}{2}\frac{t^2}{n}(1) + \frac{1}{6}\frac{t^3}{n^{3/2}}E(Y^3) + \ldots)^n \\
&\approx (1 + \frac{t^2}{2}\frac{1}{n})^n \\
&\rightarrow e^{\frac{t^2}{2}}
\end{aligned}$$

□

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Conditioning

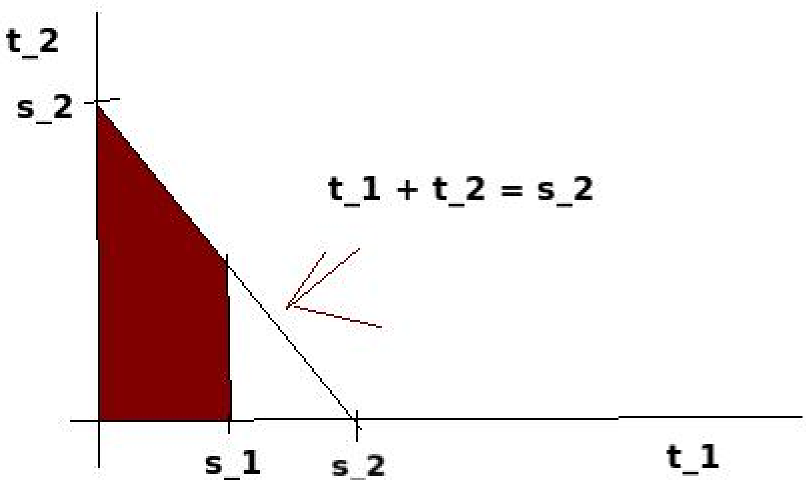
Conditional probability: $P(E|F) = \frac{P(E \bigcap F)}{P(F)}$
Conditional pmf: $p_{X|Y}(x|y) = \frac{p(x,y)}{p_Y(y)} = P(X = x|Y = y)$
Conditional density: $f_{X|Y}(x|y) = \frac{f(x,y)}{f_Y(y)}$

Example: X, Y are independent Poisson: $EX = \lambda_1$, $EY = \lambda_2$. Conditional probability mass function of X given $X + Y = n$.

$$
\begin{aligned}
P(X=k|X+Y=N) &= \frac{P(X=k, X+Y=n)}{P(X+Y=n)} \\
&= \frac{P(X=k)P(Y=n-k)}{\frac{(\lambda_1+\lambda_2)^n}{n!}e^{-(\lambda_1+\lambda_2)}} \\
&= \frac{\frac{\lambda_1^k}{k!}e^{-\lambda_1} \cdot \frac{\lambda_2^{n-k}}{(n-k)!}e^{-\lambda_2}}{\frac{(\lambda_1+\lambda_2)^n}{n!}e^{-(\lambda_1+\lambda_2)}} \\
&= \binom{n}{k}\left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k\left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n-k}
\end{aligned}
$$

We have $Binomial(n, \frac{\lambda_1}{\lambda_1+\lambda_2})$.

Example: T_1, T_2 are two independent exponential(λ).
Let $S_1 = T_1$, $S_2 = T_1 + T_2$.
Compute $f_{S_1|S_2}(s_1|s_2)$ or $\frac{f(s_1,s_2)}{f_{S_2}(s_2)}$.



First,

$$
\begin{aligned}
P(S_1 \leq s_1, S_2 \leq s_2) &= P(T_1 \leq S_1, T_1 + T_2 \leq S_2) \\
&= \int_0^{s_1} dt_1 \int_0^{s_2-t_1} dt_2 f_{T_1,T_2}(t_1,t_2) \\
f(s_1,s_2) &= \frac{\sigma}{\sigma s_2}\int_0^{s_2-s_1} dt_2 f_{T_1,T_2}(s_1,t_2) \\
&= f_{T_1,T_2}(s_1, s_2-s_1) \\
&= f_{T_1}(s_1)f_{T_2}(s_2-s_1) \\
&= \lambda e^{-\lambda s_1}\lambda e^{-\lambda(s_2-s_1)} \\
&= \lambda^2 e^{-\lambda s_2}
\end{aligned}
$$

Therefore,

$$
f(s_1,s_2) = \begin{cases} \lambda^2 e^{-\lambda s_2} & \text{if } 0 \leq s_1 \leq s_2 \\ 0 & \text{otherwise} \end{cases}
$$

$f_{s_2}(s) = \lambda^2 s_2 e^{-\lambda s_2}$ for $s_2 \geq 0$.
Therefore,

$$
f_{S_1|S_2}(s_1|s_2) = \frac{\lambda^2 e^{-\lambda s_2}}{\lambda^2 s_2 e^{-\lambda s_2}} = \frac{1}{s_2}
$$

give that $0 \leq s_1 \leq s_2$ and 0 otherwise.
Therefore, given that if we know when the second bulb fails, the first bulb can fails follow the uniform distribution between 0 and s_2.

Computing expectataions and probability by conditioning

Example: Roll a die. Assume the number rolled is N. Then, continue rolling until you either match or exceed N. What is the expected number of additional rolls?
This is geometric if we know N, so let's condition on the value of N.
Let X be the number of additional rolls.

$$\begin{aligned} E(X|N=n) &= \frac{6}{7-n} \\ E(X) &= \sum_{n=1}^{6} E(X|N=n)P(N=n) \\ &= \frac{1}{6}\sum_{n=1}^{6}\frac{6}{7-n} \\ &= \sum_{n=1}^{6}\frac{1}{7-n} \\ &= 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{6} \end{aligned}$$

Example: the number N customers entering a store on a given day is $Poisson(\lambda)$. Each of them buys something independent with probability p.
Compute the probability that k people buy something.

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Let N be the number of people entering the store. It follows Poisson(λ)
Each buys something with probability p, indepedently.
Let X be the number of people who buy something.
Why should X be Poisson? Approximate:
Let n be the population of the town
Let ϵ be the probability that a person enteres the store.
$\lambda = n\epsilon$.
Let $N \approx Binomial(n, \epsilon)$.
Let $X \approx Binomial(n, p\epsilon)$.

$$
\begin{aligned}
P(X=k) &= \sum_{n=k}^{\infty} P(X=k|N=n)P(N=n) \\
&= \sum_{n=k}^{\infty} \binom{n}{k} p^k (1-p)^{n-k} \frac{\lambda^n e^{-\lambda}}{n!} \\
&= e^{-\lambda} \sum_{n=k}^{\infty} \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} \frac{\lambda^n}{n!} \\
&= \frac{e^{-\lambda} p^k \lambda^k}{k!} \sum_{n=k}^{\infty} \frac{(1-p)^{n-k} \lambda^{n-k}}{(n-k)!} \\
&= \frac{e^{-\lambda} (p\lambda)^k}{k!} \sum_{l=0}^{\infty} \frac{((1-p)\lambda)^l}{l!} \\
&= \frac{e^{-\lambda} (p\lambda)^k}{k!} e^{(1-p)\lambda} \\
&= \frac{e^{-p\lambda} (p\lambda)^k}{k!}
\end{aligned}
$$

Therefore, it is in fact, Poisson($p\lambda$).
Example: Toss a coin repeatedly with probability of getting heads p.
What is the expected number of times need to get k sucessive heads?
(Note: If you remove "successive", the answer is $\frac{k}{p}$)
Let N_k be the number of tosses required.
Let's condition on N_{k-1}.

$$
\begin{aligned}
E[N_k|N_{k-1}=m] &= p(m+1) + (1-p)(m+1+E(N_k)) \\
&= pm + p + (1-p)m + 1 - p + m_k(1-p) \\
&= m + 1 + m_k(1-p) \\
m_k = E(N_k) &= \sum_{m=k-1}^{\infty} E[N_k|N_{k-1}=m]P(N_{k-1}=m) \\
&= \sum_{m=k-1}^{\infty} [m+1+m_k(1-p)]P(N_{k-1}) \\
&= m_{k-1} + 1 + m_k(1-p) \\
&= \frac{1}{p} + \frac{m_{k-1}}{p} \\
m_1 &= \frac{1}{p} \\
m_2 &= \frac{1}{p} + \frac{1}{p^2} \\
m_3 &= \frac{1}{p} + \frac{1}{p^2} + \frac{1}{p^3} \\
&\vdots
\end{aligned}
$$

1. $p(m+1)$ is the probability that you have heads on the $m+1$ try times its value.

2. Otherwise, it's tails and however long it is to get it again

Example: Gambler's ruin
We have a random walk on the integers.
For a given spot, you have a probability of p to walk one step forward.
Otherwise, you walk one step backwards.
What is your probability reaching N before reaching 0?
Call this probability, P_i "One step conditioning",

$$\begin{aligned}
P_0 &= 0 \\
P_N &= 1 \\
P_i &= pP_{i+1} + (1-p)P_{i-1} \\
P_{i+1} - P_i &= \frac{1-p}{p}(P_i - P_{i-1}) \\
P_2 - P_1 &= \frac{1-p}{p}(P_1 - P_0) \\
&= \frac{1-p}{p}P_1 \\
P_3 - P_2 &= \left(\frac{1-p}{p}\right)^2 (P_1) \\
&\vdots \\
P_N - P_{N-1} &= \left(\frac{1-p}{p}\right)^{N-1} P_1 \\
P_i &= P_1\left(1 + \left(\frac{1-p}{p}\right) + \left(\frac{1-p}{p}\right)^2 + \ldots + \left(\frac{1-p}{p}\right)^N\right)
\end{aligned}$$

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Gambler's ruin

Every round, you either win a dollar with probability, p. otherwise, you lose a dollar.
What is my probability that I will reach N before I hit 0?
Let $P_i = P($ reaching N before reaching 0).
Let $X_1 = \begin{cases} 1 & \text{win a dollar on 1st round} \\ -1 & \text{lose a dollar on 1st round} \end{cases}$
Let

$$\begin{aligned}
P_i &= P(\text{reach } N \text{ before reaching } 0 | X_1 = 1)P(X_1 = 1) + P(\text{reach } N \text{ before reaching } 0 | X_1 = -1)P(X_1 = -1) \\
&= P_{i+1}p + P_{i-1}(1-p)
\end{aligned}$$

This is a recurrence relation.

$$
\begin{aligned}
P_2 - P_1 &= \frac{1-p}{p} P_1 \\
P_3 - P_2 &= \frac{1-p}{p}(P_2 - P_1) = \left(\frac{1-p}{p}\right)^2 P_1 \\
&\vdots \\
P_i - P_{i-1} &= \left(\frac{1-p}{p}\right)^{i-1} P_1 \text{ where } i = 1, \ldots, N
\end{aligned}
$$

We know that $P_0 = 0$ and $P_N = 1$.
We also know that

$$
\begin{aligned}
P_i - P_1 &= \left(\left(\frac{1-p}{p}\right) + \left(\frac{1-p}{p}\right)^2 + \ldots + \left(\frac{1-p}{p}\right)^{i-1}\right) P_1 \\
P_i &= \left(1 + \left(\frac{1-p}{p}\right) + \left(\frac{1-p}{p}\right)^2 + \ldots + \left(\frac{1-p}{p}\right)^{i-1}\right) P_1 \\
P_i &= \frac{1 - \left(\frac{1-p}{p}\right)^i}{1 - \frac{1-p}{p}} P_1 \text{ where } p \neq \frac{1}{2} \\
P_i &= i p_1 \text{ where } p = \frac{1}{2}
\end{aligned}
$$

Let's use $P_N = 1$

$$
P_i = \begin{cases} \frac{1-\left(\frac{1-p}{p}\right)^i}{1-\left(\frac{1-p}{p}\right)^N} & p \neq \frac{1}{2} \\ \frac{i}{N} & p = \frac{1}{2} \end{cases}
$$

Let $p = \frac{1}{2}$, $N = 10$, $i = 5$, $P_i = \frac{1}{2}$.
Let $p = .6$, $N = 10$, $i = 5$, $P_i = .87$.
Let $p = \frac{18}{38}$, $N = 1000$, $i = 900$, $P_i = 3 \cdot 10^{-5}$
Best Strategy: "Bold Play"
You have amount, x and want to get to N. The strategy is as follows:

1. Bet x if $x \leq \frac{N}{2}$
2. Bet $N - x$ if $x \geq \frac{N}{2}$

You can assume that $N = 1$.
Let $P(X) = P(\text{get to } N \text{ before } 0)$

$$
P(X) = \begin{cases} p \cdot P(2x) & x \in [0, \frac{1}{2}] \\ p \cdot 1 + (1-p) \cdot P(2x-1) & x \in [\frac{1}{2}, 1] \end{cases}
$$

We can compute by solving a linear system on $\frac{k}{2^n}$, where $k = 0, \ldots, 2^n$
Let $n = 1$. $p = P\left(\frac{1}{2}\right)$
Let $n = 2$. $p^2 = P\left(\frac{1}{4}\right)$, $P\left(\frac{3}{4}\right) = p + (1-p)p$

Let $n = 3$. $p^3 = P\left(\frac{1}{4}\right)$, $P\left(\frac{3}{8}\right) = pP\left(\frac{3}{4}\right) = p^2 + p^2(1-p)$, $P\left(\frac{5}{8}\right) = p + p^2(1-p)$, $P\left(\frac{7}{8}\right) = p + p(1-p) + p(1-p)^2$.

$P(.9) = .88$, for $p = \frac{12}{38}$
$P(X) = x$ when $p = \frac{1}{2}$.

Remark: Look at the function, $P(x)$. Graph it. It's nowhere differentiable, but continuous in its domain. It's highly irregular. It's strictly increasing. You can even solve the recurrence. Let $P(x) = P(Y \leq x)$.

$$Y = \sum_{j=1}^{\infty} D_j \frac{1}{2^j} \text{ where } D_j \text{ is the } j\text{th digit}$$

$$P(D_j = 1) = p$$

$$P(D_j = 0) = 1 - p$$

Example: Assume that you keep flipping a fair coin until you get heads. Each time you flip tails, roll a die, and collect as many dollars as number shown on the die.
Let Y be the amount you win. Calculate EY, $Var(Y)$.
We have a sum of iid with random number of terms.
Let $X_1, X_2, \ldots$ be iid with $EX_1 = \mu$ and $Var(X_1) = \sigma^2$
Let N be independent of all X's.
Take

$$S = \sum_{k=1}^{N} X_i$$

Compute ES, $Var(S)$.

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Sums with random number of terms.

$$X_1, X_2, \ldots$$

Say the terms are iid.
Let N be the another (nonnegative integer) random variable, independent of X_i.
We want to find out $E(S)$ and $Var(S)$ if $S = \sum_{i=0}^{N} X_i$

$$E[S|N = n] = nE(X_1)$$

Then,

$$\begin{aligned} E[S] &= \sum_{n=0}^{\infty} nE(X_1)P(N = n) \\ &= E(X_1)E(N) \end{aligned}$$

Now, the variance

$$\begin{aligned}
E(S^2) &= \sum_{n=0}^{\infty} E[S^2|N=n]P(N=n) \\
&= \sum_{n=0}^{\infty} E((\sum_{i=1}^{n} X_i)^2)P(N=n) \\
&= \sum_{n=0}^{\infty}((Var(\sum_{i=1}^{n} X_i)) + (E(\sum_{i=1}^{n} X_i))^2)P(N=n) \\
&= \sum_{n=0}^{\infty}[nVar(X_1) + n^2(EX_1)^2]P(N=n) \\
&= Var(X_1)E(N) + E(X_1)^2E(N^2)
\end{aligned}$$

Therefore,

$$\begin{aligned}
Var(S) &= E(S^2) - (ES)^2 \\
&= Var(X_1)E(N) + (EX_1)^2E(N^2) - (EX_1)^2(EN)^2 \\
&= Var(X_1)E(N) + (EX_1)^2Var(N)
\end{aligned}$$

Example: Toss a fair coin until 1st heads. Each time you toss tails, roll a die, collect as many dollar as the number on the die. Let Y be your total winnings. Compute EY and $Var(Y)$.

$$EX_1 = \frac{7}{2}$$

$$E(X_1^2) = \frac{1}{6}(1^2 + \ldots + 6^2)$$

$$Var(X_1) = \ldots = \frac{35}{12}$$

Let N be the number of failures before successs, which is a geometric random variable - 1.
$EN = 2 - 1 = 1$.
$Var(N) = \frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2} = 2$ Plug in and get $Var(Y) = \frac{59}{12}$.

Markov Chains

Consider the following experiments with a book:

1. Pick letters at random and do it random in sucession.
2. Start by a random letter. Next, pick a letter at random such that it succeeds the previous letter, and repeat.
3. Start the same way. Next, pick a letter at random from letters which are at or after, in alphabetical order, the position of the previous letter.

Definition 9. *Let $X_0, X_1, \ldots$ be a sequence of random variables. This is called a Markov Chain if the distribution of X_{n+1} (where $n = 0, 1, \ldots$ only depends on the value of X_n.*

Note: We will often assume that the possible values of X_n be non-negative integers. This means that X_n has countably many values.
The information about the process is given by the transition probabilities,

$$P_{ij} = P(X_{n+1} = j | X_n = i)$$

"Given at stage n, we are in state i, we move to state j in the next state."
Note: P_{ij} does not depend on n.
Note: $P_{ij} \geq 0$, $\sum_j P_{ij} = 1$.

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Example: Have a book. Choose a random letter. Then, at each step, either

1. Pick another random letter (Markov)
2. choose a random occurence of the person's letter, then pick next letter (next letter to last is the 1st letter). (is also Markov)
3. Choose a random occurence of the last two letters, in order, then pick next letter (not Markov, but becomes Markov if you keep track of two letters)
4. Choose a random occurences of all previously chosen letters, in order, then pick next letter (not Markov)
5. In step n, choose a random letter with probability $\frac{1}{n}$ and do (2) with probability, $1 - \frac{1}{n}$

Let $X_0, X_1, \ldots$ be a sequence of discrete random variables with random variables with values, most often, $0, 1, 2, \ldots$. The information about the Markov chain is given by the transition probabilities,

$$P_{ij} = P(X_{n+1} = j | X_n = i)$$

There is also a transition matrix represented by $\begin{bmatrix} P_{11} & P_{12} & P_{13} & \ldots \\ P_{21} & P_{22} & P_{23} & \ldots \\ & \vdots & & \end{bmatrix}$.

This is also known as a stochastic matrix. Row sums are 1.

Example: The random moves to the right (by 1) with probability, p, and to the left with probability $1-p$, except when it at 0 or 4. These two states are absorbing once there, the walker does not move.

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1-p & 0 & p & 0 & 0 \\ 0 & 1-p & 0 & p & 0 \\ 0 & 0 & 1-p & 0 & p \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Example: Same as the previous example except that 0 or 4 are reflecting. From 0, always move to 1. From 4, always move to 3.

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1-p & 0 & p & 0 & 0 \\ 0 & 1-p & 0 & p & 0 \\ 0 & 0 & 1-p & 0 & p \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Example: A general random walk on a graph, walker moves to a randomly chosen neighbor. I have a graph with 4 nodes, a, b, c, d. Here's an adjacency matrix of their movements.

$$\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

It has the following transition matrix

$$P = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}$$

Example: I have a graph with two nodes. Node 0 transitions to node 0 with probability α
Node 0 transitions to node 1 with probability $1 - \alpha$
Node 1 transitions to node 1 with probability β
Node 1 transitions to node 0 with probability $1 - \beta$
Here is the transition matrix

$$P = \begin{bmatrix} \alpha & 1-\alpha \\ 1-\beta & \beta \end{bmatrix}$$

Example: "Changeover" Keep track of two toss sequences in an infinite sequence of coin tosses with probability p of heads.

$$\begin{bmatrix} States & HH & HT & TH & TT \\ HH & p & 1-p & 0 & 0 \\ HT & 0 & 0 & p & 1-p \\ TH & p & 1-p & 0 & 0 \\ TT & 0 & 0 & p & 1-p \end{bmatrix}$$

The horizontal axis is the current flip and the previous flip. The vertical axis is the previous two flips.

Example: Random walk on $\mathbb{Z}$.

Example: Birth-death chain. Let's say my population is X. Assume it absorbs at 0. We have a probability p_x be the probability of birth and q_x be the probability of death and r_x be the probability of death and birth. The transition matrix is pretty much a diagonal.

4/17(discussion)

Prove

$$\lim_{n\to\infty} e^{-n} \sum_{n\to\infty} \frac{n^k}{k!} = \frac{1}{2}$$

We're going to use Central limit theorem. to use cental limit theorem, you need to realize

$$X_n = \sum_{i=1}^{n} Y_i$$

The former is Poisson(n), while Y_i is Poisson(1).
Fact: A and B are Poisson with parameters a and b respectively, then $A+B$ is Poisson with Poisson($a+b$).

Proof.

$$\begin{aligned}
P(A+B \le n) &= \sum_{l=0}^{n} P(A=l, B=n-l) \\
&= \sum_{l=0}^{n} P(A=l)P(B=n-l) \text{ (Remember, iid)} \\
&= \sum_{l=0}^{n} e^{-a}\frac{a^l}{l!} \cdot e^{-b}\frac{b^{n-l}}{(n-l)!} \\
&= e^{-a-b}\frac{(a+b)^n}{n!}
\end{aligned}$$

□

Therefore, $EX_n = n$ and $Var(X_n) = n$ since $EY_i = 1$ and $Var(Y_i) = 1$.
By the central limit theorem

$$P\left(\frac{X_n - n}{\sqrt{n}} \le 0\right) \to \Phi(0) = \frac{1}{2}$$

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Let X be a random variable describing total time to exit Idea: Define random variable,

$$I = \begin{cases} 1 & \text{left} \\ 0 & \text{right} \end{cases}$$

$$\begin{aligned}
E(x) &= E(E(X|I)) \\
&= E(X|I=0)P(I=0) + E(X|I=1)P(I=1) \\
&= \sum_x xP(X=x|I=0)P(I=0) + \sum_x xP(X=x|I=1)P(I=1) \\
&= \sum_x x\frac{P(X=x, I=0)}{P(I=0)}P(I=0) + \sum_x x\frac{P(X=x, I=1)}{P(I=1)}P(I=1) \\
&= \sum_x xP(X=x, I=0) + \sum_x xP(X=x, I=1)
\end{aligned}$$

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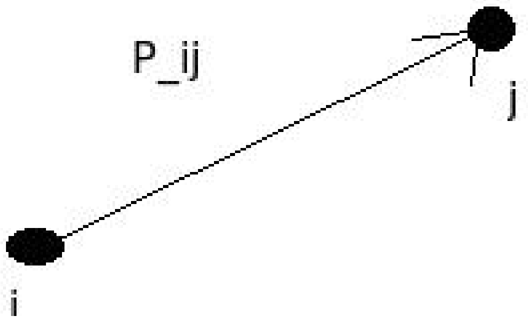
Markov Chain

Determined by

- state space: a countable set, often represented by positive numbers.
- transition probabilities
  $$P_{ij} = P(X_{n+1} = j | X_n = i)$$
  which satisfy $P_{ij} \ge 0$, $\sum_j P_{ij}, \forall i$

- initial state

Graphically, one often represents a Markov Chain like this:



$$P_{ij}^n = P(X_n = j|X_0 = i)(= P(X_{n+m} = j|X_m = i))$$
$$P_{ij}^1 = Pij$$

This is called the *n-step probabilities.*

$$\begin{aligned}
P_{ij}^{n+m} &= P(X_{n+m} = j|X_0 = i) \\
&= \sum_k P(X_{n+m} = j, X_n = k|X_0 = i) \text{ } (X_0 = i \text{ isn't really a condition. It's just the starting state}) \\
&= P(X_{n+m} = j|X_n = k)P(X_n = k|X_0 = i) \\
&= P(X_m = j|X_0 = k)P(X_n = k|X_0 = i) \text{ (doesn't matter how you got there.)} \\
&= \sum_k P_{kj}^m P_{ik}^n
\end{aligned}$$

Let $P^{(n)}$ be the matrix, P_{ij}^n

$$\begin{aligned}
P^{(m+n)} &= P^{(n)}P^{(m)} \\
P^{(n)} &= P^{(1)}P^{(1)} \dots P^{(1)} \\
&= P^n
\end{aligned}$$

Conclusion: Computing n-step transition probability is the same as computing powers of P.

Example: Two state Markov Chain defined by

$$P = \begin{bmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{bmatrix}$$

Then,

$$P^2 = \begin{bmatrix} \alpha^2 + (1-\alpha)\beta & \alpha(1-\alpha) + (1-\alpha)(1-\beta) \\ \alpha\beta + (1-\beta)\beta & \beta(1-\alpha) + (1-\beta)^2 \end{bmatrix}$$

Assume now, that initial distribution is given by

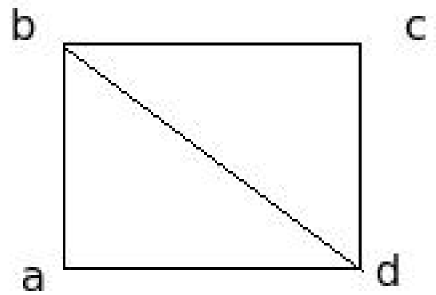
$$\alpha_i = P(X_0 = i), \forall i, (\sum_i \alpha_i = 1, \alpha_i \geq 0)$$

$$
\begin{aligned}
P(X_n = j) &= \sum_i P(X_n = j | X_0 = i) P(X_0 = i) \\
&= \sum_i \alpha_i P_{ij}^n
\end{aligned}
$$

The row of probabilities is given by

$$P(X_n = i) = [\alpha_0, \alpha_1, \ldots] P^n$$

Example: Random walk on the graph below:



Choose starting point at random. What is the probability that a random walk is at a after 2 steps. At each vertex, you are equally likely to choose any of the edges it is connected to.

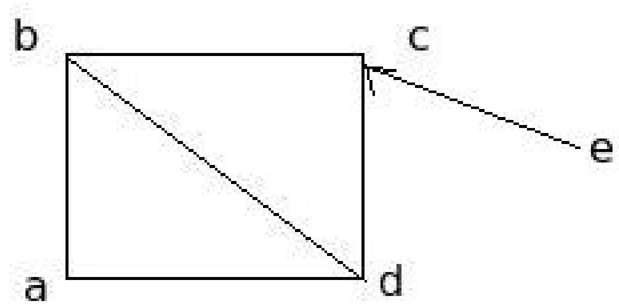
$$P = \begin{bmatrix} 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} P^2 = \begin{bmatrix} P(X_2 = 1) & P(X_2 = 2) & P(X_2 = 3) & P(X_2 = 4) \end{bmatrix}$$

Classification of states

- We say that a state j is accessible from state i if $P_{ij}^n > 0$ for some $n > 0$. (there is a possibility of reaching j from i in any number of steps).
- If i is accessible from j, and j is accessible from i, then we say that i and j are coummunicate, $i \leftrightarrow j$
- "$\leftrightarrow$" is an equivalence relation.
  - $i \leftrightarrow i$
  - $i \leftrightarrow j \Rightarrow j \leftrightarrow i$
  - $i \leftrightarrow j$ and $j \leftrightarrow k \Rightarrow i \leftrightarrow k$

Example:



Any state, a, b, c, d is accessible from e, but e is not accessible from a, b, c, d.
e is accessible from e by definition.

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If j is not accessible from i, then $P_{ij}^n = 0\ \forall n$.

$$\begin{aligned} P(\text{ ever visit } j | X_0 = i) &= P(\bigcup_{n=0}^{\infty} \{X_n = j\} | X_0 = i) \\ &\leq \sum_{n=0}^{\infty} P(X_n = j | X_0 = i) = 0 \end{aligned}$$

Also, note that accessibility, etc..., the site of elements in P does not matter, all that matters is which are positive and which 0.
If the chain has M states, then j is accessible from i iff for some $n \leq M (P + P^2 + \ldots + P^m)_{ij} > 0$
The communication relation "$\leftrightarrow$" is an equivalence relation

1. $i \leftrightarrow i$
2. $i \leftrightarrow j$ implies $j \leftrightarrow i$
3. $i \leftrightarrow j$ and $j \leftrightarrow i$ implies $i \leftrightarrow k$

To prove (3), it is enough to prove

$$(i \to j) \text{ and } (j \to k) \text{ implies } i \to k$$

This holds because $\exists n$ so that $P_{ij}^n > 0$
$\exists n$ so that $P_{ij}^m > 0$
Then, $P_{ik}^{m+n} > 0$
This relation divides states into classes within a class, all states communicate to each other. This chain is irreducible if there is only one class. (If we have m states, that means all entries of $I + P + \ldots + P^M$ are positive)
Example:

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{3} & \frac{2}{3} \end{bmatrix}$$

As we can see, $0 \leftrightarrow 1$, $1 \leftrightarrow 2$, so irreducible.
Example:

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

For any state i, denote

$$f_i = P(\text{ ever reenter } i | X_0 = i)$$

We call a state recurrent if $f_i = 1$.
We call a state transient if $f_i < 1$.

From previous example, $f_2 = \frac{1}{4}$, so 2 is transient.
$f_3 = 1$, so 3 is recurrent. $f_0 = 1$ because the only possibility that it isn't recurrent is if we go to state 1 and stay there forever. As $n \to \infty$, $\left(\frac{1}{2}\right)^n = 0$, so the probability of it staying at 1 is 0. Therefore, it's 1.
$f_1 = 1$ by similar logic as above.
Therefore, f_0 and f_1 are recurrent states.

The Markov Chain visits a recurrent state infinitely many times, or not at all. (starting from an arbitrary state).
On the other hand,

$$P(\text{reenter } i \text{ exactly } n \text{ times}|X_0 = i) = f_i^{n-1}(1 - f_i)$$

Therefore, the number of times spent at i is a geometric random variable with success(which is never returning in this case) probability of $1 - f_i$, so our expectation is $\frac{1}{1-f_i}$.

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$f_i = P(\text{ ever reenter } i|X_0 = i)$
The state i is recurrent if $f_i = 1$.
The state i is transient if $f_i < 1$.
$P(\text{ visit } i\ n \text{ times } |X_0 = i) = f_i^{n-1}(1 - f_i)$

$$E(\text{number of visits to } i|X_0 = i) = \frac{1}{1 - f_i}$$

$$E(\text{number of visits to } i|X_0 = i) = E(\sum_{n=0}^{\infty} I_n|X_0 = i)$$

where $I_n = I\{X_n = i\}$, $n = 0, 1, 2, \ldots$

$$\begin{aligned} I_n &= \sum_{n=0}^{\infty} P(X_n = i|X_0 = i) \\ &= \sum_{n=0}^{\infty} P_{ii}^n \end{aligned}$$

Theorem 10. *1 is recurrent* $\Leftrightarrow \sum_{n=1}^{\infty} P_{ii}^n = \infty$

If a Markov chain only has finitely many states, then there must be at least one recurrent state. Why?
If all states are transient, then all the states will be visited only finitely many times. That is impossible.

Proposition 11. *If i is recurrent, and $i \leftrightarrow j$, then also, j is recurrent. If the chain is irreducible, then either all states are recurrent or all transient.*

Proof. $P_{ij}^k > 0, P_{ji}^m > 0$ for some k, m.
For any $n \geq 0$:

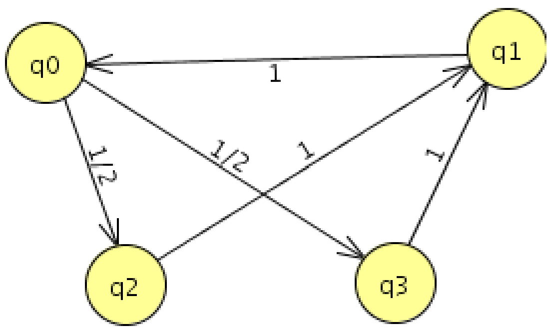
$$P_{ij}^{m+n+k} \geq P_{ji}^m P_{ii}^n P_{ij}^k$$

If a state is recurrent, then $f_i = 1$.
Then, $E(\text{number of visits to } i|X_0 = i) = \frac{1}{1-f_i} = \infty$.

If $\sum_{n=0}^{\infty} P_{ii} = \infty$, then $\sum_{n=0}^{\infty} P_{jj}^{m+n+k} = \infty$, so $\sum_{l=0}^{\infty} P_{jj}^{l} = \infty$.
If i is recurrent, then so is j. $\square$

Example:

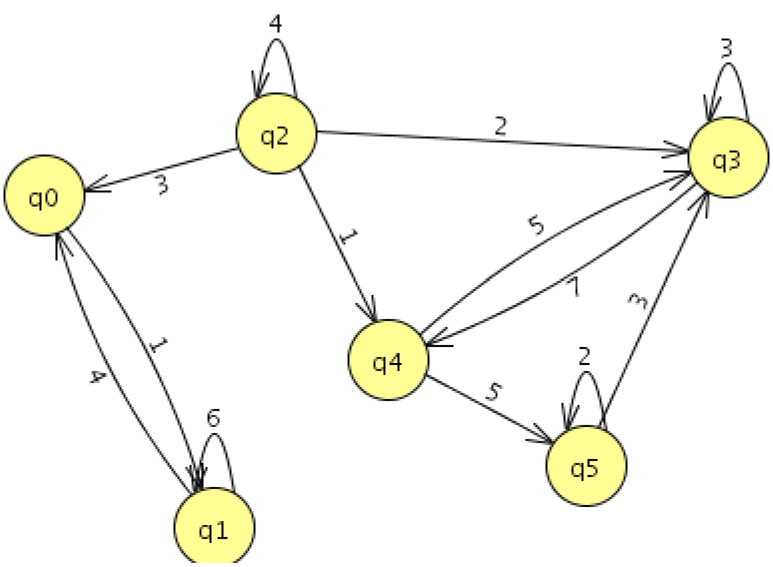
$$P = \begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$



As we can see every element can reach each other, so the chain is irreducible. We can also see that state 0 whether it chooses to go to state 2 and 3 will end up at state 1 and back to state 0. Therefore, it is recurrent and the entire thing is also recurrent.

Example:

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 4 & 6 & 0 & 0 & 0 & 0 \\ 3 & 0 & 4 & 2 & 1 & 0 \\ 0 & 0 & 0 & 3 & 7 & 0 \\ 0 & 0 & 0 & 5 & 0 & 5 \\ 0 & 0 & 0 & 3 & 0 & 2 \end{bmatrix}$$



As we can see, nothing can reach 2 besides 2 itself. Therefore, 2 is in a group of its own. 0 and 1 can reach other and nothing else, so it is in a group of its own. 3, 4, and 5 can reach each other.
As we can see, 2 can only reach itself sometimes. Therefore, $j_2 < 1$ making it transient.
When we are at state 1 and choose to go to 0, it will always come back to 1. Also, state 1 can choose to stay. Therefore, $j_1 = 1$, which then means that $j_0 = 1$ making both of them recurrent.

We can also see that $\{3, 4, 5\}$ is recurrent.

Example: Simple random walk on $\mathbb{Z}$. When $p = \frac{1}{2}$, this is a simple symmetric random walk.
Simple random walk on $\mathbb{Z}^2$.
Simple random walk on $\mathbb{Z}^3$.
Higher dimensions makes for lower probability to return to origin.
In the 1 dimension case,

$$P_{00}^{2n-1} = 0 \text{ (because if you walk out } n \text{ steps, you need to walk back } n \text{ steps)}$$

$P_{00}^{2n} = \binom{2n}{n} p^n (1-p)^n$($n$ steps to the left and n steps to the right. Pick n steps out of $2n$ steps to be left.)
Stirling's formula:

$$n! \approx n^n e^{-n} \sqrt{2\pi n}$$

So, according to Stirling's formula,

$$\begin{aligned} \binom{2n}{n} &= \frac{(2n)!}{(n!)^2} \\ &= \frac{(2n)^{2n} e^{-2n} \sqrt{2\pi 2n}}{n^{2n} e^{-2n} 2\pi n} \\ &= \frac{2^{2n}}{n\pi} \end{aligned}$$

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Continued from last time

$$\begin{aligned} P_{00}^{2n-1} &= 0 \\ P_{00}^{2n} &= \binom{2n}{n} p^n (1-p)^n \\ &\approx \frac{2^{2n}}{\sqrt{n\pi}} p^n (1-p)^n \\ &= \frac{1}{\sqrt{n\pi}} (4p(1-p))^n \end{aligned}$$

When $p = \frac{1}{2}$, $P_{00}^{2n} \approx \frac{1}{\sqrt{n\pi}}$, $\sum_{n=0}^{\infty} P_{00}^{2n} = \infty$, so random walk is continuous.
When $p \neq \frac{1}{2}$, $P_{00}^{2n} \approx \frac{1}{\sqrt{n\pi}}$, $\sum_{n=0}^{\infty} P_{00}^{2n} < \infty$, so random walk is transient.
Question: What is the $f_0 = P($ ever reenter $0 \,|X_0 = 0)$?
Gambler's ruin!
Recall that

$$P(\text{reach } N \text{ before } 0|S_0^{(1)} = 1) = \begin{cases} \frac{1}{N} & p = \frac{1}{2} \\ \frac{1-\frac{1-p}{p}}{1-\left(\frac{1-p}{p}\right)^N} & \text{otherwise} \end{cases}$$

Also, that $P(\text{reach } 0 \text{ before } N \,|S_0^{(1)} = 1) = 1 - P(\text{reach } N \text{ before } 0 \,|S_0^{(1)} = 1)$.
Let's send $N \to \infty$,

$$P(\text{ever reach } 0|S_n^{(1)} = 1) = \begin{cases} 1 & p = \frac{1}{2} \\ 1 & p < \frac{1}{2} \\ \frac{1-p}{p} & p > \frac{1}{2} \end{cases}$$

Assume that $p > \frac{1}{2}$. Then,

$$\begin{aligned}
f_0 &= P(\text{1st jump to 1 and it returns eventually}) + P(\text{1st jump to -1 and it returns eventually}) \\
&= p\frac{1-p}{p} + (1-p)1 \\
&= 2(1-p)
\end{aligned}$$

2 dimensional case

Drunk at Chicago problem. Drunk goes wandering around Chicago. What is the probability that the drunk comes back home?
Let $S_n^{(2)}$ be the sequence of random walk.
Let k be the number of times walk moves in the x-direction.

$$P(\text{back at home(0,0) in } 2n \text{ steps}) = \sum_{k=0}^{n} \binom{2n}{2k} \frac{1}{2^{2n}} P(S_{2k}^{(1)} = 0) P(S_{2(n-k)}^{(1)} = 0)$$

With overwhelming probability, $k \approx \frac{n}{2}$.
Then,

$$\begin{aligned}
P(S_{2k}^{(1)} = 0) &\approx \frac{1}{\sqrt{n}} \\
P(S_{2(n-k)}^{(1)} = 0) &\approx \frac{1}{\sqrt{n}} \\
P(\text{back at home(0,0) in } 2n \text{ steps}) &\approx \frac{1}{n} \sum_{k=0}^{n} \binom{2n}{2k} \frac{1}{2^{2n}} \\
&\approx \frac{1}{n} \\
P(\text{even number of steps in horizontal direction}) &\to \frac{1}{2} \\
\sum_{n=0}^{\infty} P(\text{back to (0,0) n(2n) step}) &= \infty
\end{aligned}$$

So, the chain is still recurrent. In fact,

$$P(\text{back to 0 in } 2n \text{ steps}) = P(S_{2n}^{(1)} = 0)^2$$

Random walk in 2 dimension with diagonals

Different random walk, but now that different. Rotate the lattice.
As the two coordinates evolve independently.

5/1

Random walk in 3 dimensions

Squirrels walk around in a 3 dimensional maze. In this walk, you will walk $2k$ times in the z direction. Otherwise, it's running around in the x and y direction.

It has $\frac{1}{3}$ probability of moving in the z-direction. In those $2k$ steps, it has a probability, $P(S_{2k}^1 = 0)$ to come back to 0 in the z-direction.

$$
\begin{aligned}
P(S_{2n}^3 = 0) &= \sum_{k=n}^{n} \binom{2n}{2k} \left(\frac{1}{3}\right)^{2k} \left(\frac{2}{3}\right)^{2(n-k)} P(S_{2k}^1 = 0) P(S_{2(n-k)}^{(2)} = 0) \\
&\approx \sum_{k=0}^{n} \binom{2n}{2k} \left(\frac{1}{3}\right)^{2k} \left(\frac{2}{3}\right)^{2(n-k)} \frac{1}{\sqrt{n}} \frac{1}{n} \\
&= \frac{1}{n^{3/2}} P(\text{ random walk makes even number of steps in z direction}) \\
&\approx \frac{1}{n^{3/2}} \frac{1}{2} \text{ (solve the binomial as } n \to \infty) \\
\sum_n P(S_{2n}^3 = 0) &< \infty \text{ (so, the 3d random walk is transient)}
\end{aligned}
$$

Another way:

$$
\begin{aligned}
f_0 &= \text{probability of return to 0} \\
\frac{1}{1-f_0} &= \sum_{n=0}^{\infty} (P_{2n}^{(3)} = 0) \\
&= 1 + \frac{1}{6} + \ldots \\
&= \frac{1}{(2\pi)^3} \int_{(-\pi,\pi)^3} \frac{dx\, dy\, dz}{1 - \frac{1}{3}(\cos(x) + \cos(y) + \cos(z))} \text{ (take an analysis course for why this is)}
\end{aligned}
$$

Aside, Let $X = Binomial(n, p)$ random variables.

$$p_n = P(X \text{ is even}) = \frac{1}{2}$$

Why?

$$
\begin{aligned}
p_0 &= 1 \\
p_{n+1} &= p_n(1-p) + (1-p_n)p \\
&= p + p_n(1-2p)
\end{aligned}
$$

Since we know it's $\frac{1}{2}$,

$$p_n = \frac{1}{2} + c(1-2p)^n$$

$$1 = \frac{1}{2} + c, c = \frac{1}{2}$$

So,

$$p_n = \frac{1}{2} + \frac{1}{2}(1-2p)^n$$

Plug this is all into the stuff and we get

$$f_0 \approx .3405$$

Remark: A Markov chain can never leave a recurrent class, which is therefore often called closed.

Proof. Let C be a recurrent class, $i \in C$ and $j \notin C$. We need to show that $p_{ij} = 0$.

Assume not, $p_{ij} > 0$. As j does not communicate with i, the chain never reaches i from j (i is not accessible from j).

If the chain starts at i, it returns to there infinitely many times, so it eventually jumps to j and never returns. CONTRADICTION! □

Branching processes

Unlike random walks problem, it has no spatial constraints.

Consider a population of individuals which evolves according to the following rule: Each individual generation, n produces a random number of offsprings in the next generation independently from other individuals.

$$P_i = P(\text{ number of offspring} = i)$$

$i = 0, 1, 2, \ldots$

Problem introduced by F. Galton (late 1800s).

We will start with a single individual and they will produce offsprings. The graph of this structure is called a "Family tree".

Let X_n be the number of individuals in generation n and p_i be a probability mass function of the number of offsprings individual i will produce. Notes:

1. If X_n reaches 0, it's an absorbing state.
2. $P(X_{n+1} = 0 | X_n = k) > 0$ provided $p_0^k > 0$
3. Therefore, all states other than 0 are transient.

$$P(X_{n+1} = i | X_n = k) \quad = \quad P(S_1 + S_2 + \ldots + S_k = i)$$

5/4

Branching Process

Given $X_n = l$, $X_{n+1} = S_1 + \ldots + S_l$ where S_i are iid with probability mass function given by P_k.
Let π_0 be the probability that the process goes extinct and $P(X_n = 0)$ the population is extinct by time

n. $\pi_0 = P(X_n = 0 \text{ for some } n) = \lim_{n\to\infty} P(X_n = 0)$ Compute $M_n = E(X_n)$ and $V_n = Var(X_n)$ Using the equation above,

$$M_{n+1} = M_n \mu$$

and

$$V_{n+1} = M_n \sigma^2 + V_n \mu^2$$

where $\mu = \sum_{k=0}^{\infty} kP_k = EX_1$, $\sigma^2 = Var(X_1)$, $M_0 = 1$, and $V_0 = 0$. If we solve the recurrence,

$$M_n = \mu^n$$

Then, In conclusion, $P(X_n \geq 1) \leq EX_n = \mu^n$, so if $\mu < 1$, then $P(X_n \geq 1) \to 0$ and so $P(X_n = 0) \to 1$.
Interpretation: If each family does not have on average, one child, you will go extinct over time.
As for the Variance,

$$\begin{aligned}
V_{n+1} &= M_n \sigma^2 + V_n \mu^2 \\
&= \mu^n \sigma^2 + V_n \mu^2 \\
V_n &= A\mu^n \\
A &= \frac{\sigma^2}{\mu(1-\mu)} \text{ given that } \mu \neq 1 \\
V_n &= \frac{\sigma^2}{\mu(1-\mu)} \mu^n + B\mu^{2n}
\end{aligned}$$

From $V_0 = 0$, $B = -A$, $V_n = \frac{\sigma^2}{\mu(1-\mu)}\mu^n - \frac{\sigma^2}{\mu(1-\mu)}\mu^{2n}$, $u \neq 1$.
If $\mu = 1$, then $V_n = n\sigma^2$ and $M_n = 1$.

Moment generation function

$$\phi_{X_n}(s) = \sum_{k=0}^{\infty} P(X_n = k)s^k = E[s^{X_n}]$$

where $0 \leq s \leq 1$.
Example: The previous branching process example (probability of species going extinct).

$$\phi(s) = \phi_{X_1}(s) = \sum_{k=0}^{\infty} P_k s^k$$

$$\begin{aligned}
\phi_{X_2}(s) &= E[s^{X_2}] \\
&= \sum_{k=0}^{\infty} E[s^{X_2} | X_1 = k] P(X_1 = k) \\
&= \sum_{k=0}^{\infty} E[s^{S_1 + \ldots + S_k}] P(X_1 = k) \\
&= \sum_{k=0}^{\infty} (E(s^{S_1}) E(s^{S_2}) \ldots E(s^{S_k}) P(X_1 = k) \\
&= \sum_{k=0}^{\infty} \phi(s)^k P_k \\
&= \phi(\phi(s))
\end{aligned}$$

As we can see,

$$\phi_{X_n}(s) = \phi(\phi \dots (\phi(s)) \dots) = \phi(\phi_{X_{n-1}}(s))$$

Example: Iteration of maps, "cobwebbing" Given $y = f(x)$ and $y = x$, keep iterating through. Take If $x_{n+1} = f(x_n)$, then if x_n converges, it must converge to some point of f.
Famous chaotic iterative maps, $f(x) = 4x(1-x)$

5/6

$$\phi(s) = \phi_{X_1}(s) = \sum_{k=0}^{\infty} P_k s^k$$

$$\pi_0 = P(\text{ branching process ever dies out}) = \lim_{n \to \infty} d_n$$

Note: d_n is an incrasing sequence

Properties of ϕ

1. $\phi(0) = P_0 > 0$
2. $\phi(1) = 1$ because $\phi(1) = \sum_{k=0}^{\infty} P_k = 1$
3. $$\phi'(s) = \sum_{k=0}^{\infty} k P_k s^{k-1} \geq 0$$

   Therefore, ϕ is increasing
4. $$\phi\text{"}(s) = \sum_{k=1}^{\infty} k(k-1) P_k s^{k-2} \geq 0$$

   Therefore, ϕ is convex

two cases of geometry of ϕ

1. ϕ is always above the diagonal, $\pi_0 = \lim_{n \to \infty} d_n = 1$. We can tell if this is the case for ϕ if the $\phi'(1) < 1$.
2. ϕ is not always above the diagonal. In this case, $\pi_0 = \lim_{n \to \infty} d_n$, d_n converges to the smallest solution to $s = \phi(s)$. We can tell if this is the case for ϕ if the i $\phi'(1) > 1$.

To calculate $\phi'(1)$, we just do $\sum_{k=1}^{\infty} k P_k = \mu$.
In summary,

$$\mu = \sum_{k=1}^{\infty} k P_k$$

If $\mu \leq 1$, $\pi_0 = 1$.
If $\mu > 1$, $\phi(s) = sum_{k=0}^{\infty} P_k s^k$ and find the smallest solution to $s = \phi(s)$.

Example:

$$P_k = p^k (1-p)$$

Note: Not geometric. It's more like geometric - 1 because geometric cannot have 0.

$$\mu = \frac{1}{1-p} - 1 = \frac{p}{1-p}$$

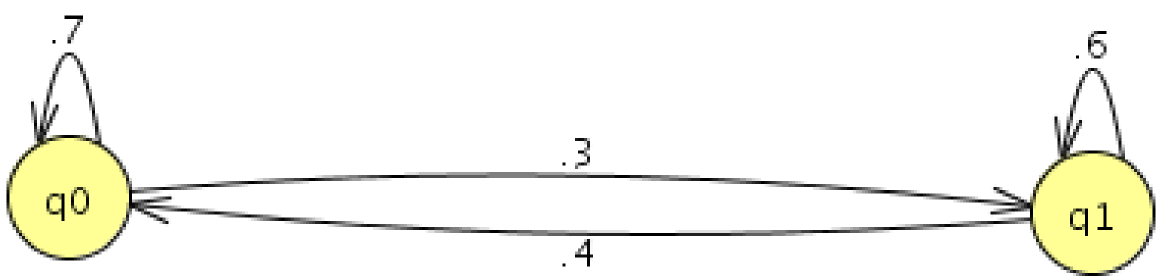
If $p \leq \frac{1}{2}$, $\pi_0 = 1$.
Suppose that $p > \frac{1}{2}$.Then,

$$\begin{aligned} \phi(s) &= \sum_{k=0}^{\infty} s^k p^k (1-p) \\ &= \frac{1-p}{1-ps} \\ &= s \end{aligned}$$

$(ps - 1 + p)(s - 1) = 0$, so $\pi_0 = \frac{1-p}{p}$.

Example: $\begin{array}{c|cccc} k & 0 & 1 & 2 & 3 \\ P_k & \frac{1}{8} & \frac{3}{8} & \frac{3}{8} & \frac{1}{8} \end{array}$ $\mu = \frac{3}{2} > 1$.
$\pi_0 : \phi(s) = \frac{1}{8} + \frac{3}{8}s + \frac{3}{8}s^2 + \frac{1}{8}s^3 = s$
Solving this gets us $s = 1, -\sqrt{5} - 2, \sqrt{5} - 2$.
Take the smallest non-negative solution, which is $\sqrt{5} - 2$.

Limiting probabilities



$$\begin{aligned} P &= \begin{bmatrix} .7 & .3 \\ .4 & .6 \end{bmatrix} \\ P^4 &= \begin{bmatrix} .5749 & .4251 \\ .5668 & .4332 \end{bmatrix} \\ P^8 &= \begin{bmatrix} .572 & .428 \\ .570 & .430 \end{bmatrix} \end{aligned}$$

The matrix elements appear to converge and the rows become almost identical. Why?

Definition 12. *State i has period d if $P_{ii}^n > 0$, then $d | n$ and d is the largest of such positive integer.*

Example: Simple symmetrical random walk on $\mathbb{Z}$.
The period of any state is 2 because you can return to original position if you can walk forth k steps and walk back k steps where $k \geq 1$.

Example: Random walk on a square. Again, 2 because you can walk around the square or walk forth and back.

Example: Random walk on a triangle. 1 because you can return in 2 steps (walk out and back) or 3 steps (walk around the triangle).

It can be shown that a period is the same for all states in the same class. If a state has period 1, it is called aperiodic. If the chain is irreducible, we call it aperiodic if all state have period 1.

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Periods

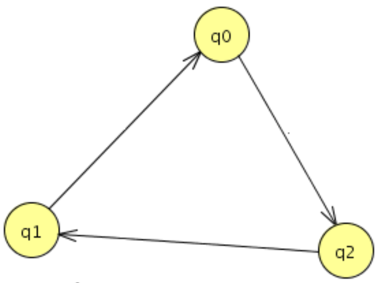

Figure 3

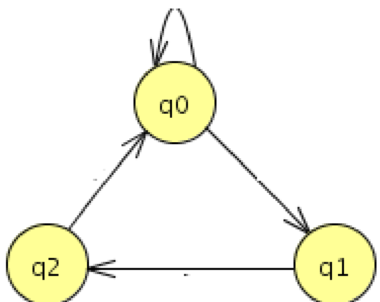

Figure 4

Example: Random walk on a undirected square. You have a period of 2.
Example: Random walk on a deterministic Chain (figure 3) You have a period of 3.
Example: Random walk on a undirected triangle. You have a period of 1.
Example: Random walk on a deterministic Chain with a chain(figure 4). Period is 1.

Let $R_i = \min\{n \geq 1 | X_n = i\}$ or the first time after time 0, that the chain is in i.
Also, let $f_i^n = P(R_i = n | X_0 = i)$ or the probability that the return time to i equals n.
Let $f_i = \sum_{n=1}^{\infty} f_i^n$, which is the $P(\text{ever reenter } i | X_0 = i)$, so if the chain is recurrent, $\sum_{n=1}^{\infty} f_i^n = 1$.

$$\begin{aligned} m_i &= E[R_i | X_0 = i] \\ &= \sum_{n=1}^{\infty} n f_i^n \end{aligned}$$

If $m_i < \infty$, then the series above converges and i is positive recurrent.
In this case, the chain is at i every m_i times on average.

Theorem 13. *Assume that the chain is irreducible and positive recurrent. Let $N_n(i)$ be the number of visits to i in the time interval between 0 and n. Then,*

$$\lim_{n \to \infty} \frac{N_n(i)}{n} = \frac{1}{m_i}$$

the former expression is the proportion of time spent at i.

Note: Very trivial theorem because you need $N_n(i)$ to get m_i.
Fact: Positive recurrent in class property. An irreducible chain is positive recurrent if each of its states is. A finite reducible chain is always positive recurrent

Theorem 14. *An irreducible positive recurrent Markov Chain has a unique invariant (aka Stationary distribution. This is a vector of possibilities, π_i where $i = \{0, 1, 2, \ldots\}$.*
$\sum \pi_i = 1$ so that

$$\sum_{\pi_i} P_{ij} = \pi_j \Leftrightarrow [\pi_0, \pi_1, \ldots] \cdot = [\pi_0, \pi_1, \ldots]$$

So, $[\pi_0, \pi_1, \ldots]$ is the left eigenvector of P (for eigenvalue 1). This means that $P(X_0 = i) = \pi_i$ implies $P(X_n = i) = \pi_i$. Moreover,

$$\pi_i = \frac{1}{m_i}$$

In fact, an irreducible chain is positive recurrent $\Leftrightarrow$ a stationary distribution exists.

Theorem 15. *If a Markov Chain is irreducible/aperiodic and positive recurrent, then*

$$\begin{aligned} \lim_{n\to\infty} P_{ij}^n &= \lim_{n\to\infty} P(X_n = j | X_0 = i) \\ &= \pi_j \text{ (no matter what } i \text{ is)} \end{aligned}$$

In summary, for finite chains,

1. $\frac{1}{n} N_n(i) \to \pi_i \; \forall i$
2. If the chain is aperiodic,
$$P_{ij}^n \to \pi_j$$
In other words, the matrix P^n eventually looks like
$$\begin{bmatrix} \pi_0 & \pi_1 & \pi_2 & \ldots \\ \pi_0 & \pi_1 & \pi_2 & \ldots \\ \vdots & & & \end{bmatrix}$$
3. If the chain is aperiodic with period d, then $\forall i, j \; \exists r$
$$\lim_{m\to\infty} P_{ij}^{md+r} = d\pi_j$$
For all n such that $n \neq r(\mod d)$, $P_{ij}^n = 0$

Example:

$$P = \begin{bmatrix} \alpha & 1-\alpha \\ \beta & 1-\beta \end{bmatrix}$$

where $0 < \alpha, \beta < 1$

1. $$\lim_{n\to\infty} P_{01}^n = \lim_{n\to\infty} P(X_n = 1 | X_0 = 0) = \pi_1$$

2.

$$\begin{aligned}
\pi_0\alpha + \pi_1\beta &= \pi_0 \\
\pi_0(1-\alpha) + \pi_1(1-\beta) &= \pi_1 \\
\pi_0 + \pi_1 &= 1 \\
&\text{with a little algebra} \\
\pi_0 &= \frac{\beta}{1+\beta-\alpha} \\
\pi_1 &= \frac{1-\alpha}{1-\beta+\alpha}
\end{aligned}$$

3. What is the time spent at 1 if $X_0 = 1$? π_1

4. Given $X_0 = 1$, what is the expected return time to 1? $\frac{1}{\pi_1}$

5. Proportion of time the chain is at 1, while the previous time it was at 0? $\pi_0 P_{01}$. Look at transitions.

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For irreducible, finite state Markov Chains

- $\frac{N_n(i)}{n} \to \pi_i$ (irreducible)
- $P_{ij}^n \to \pi_j$ (irreducible, aperiodic)

where $[\pi_1 \ldots \pi_n]P = [\pi_1 \ldots \pi_n]$.

Example: The state of Gary
(figure 5)
Let Cheerful be state 0, so-so be state 1, and glum be state 2. Then, his transition matrix is

$$P = \begin{pmatrix} .5 & .4 & .1 \\ .3 & .4 & .3 \\ .2 & .3 & .5 \end{pmatrix}$$

From an initial observation, we can see it's irreducible and aperiodic.
In the long run, what proportion of time is Gary in each state?

$$\begin{aligned}
\pi_c + \pi_s + \pi_g &= 1 \\
(\pi_c, \pi_s, \pi_g)P &= (\pi_c, \pi_s, \pi_g)
\end{aligned}$$

Solve for $Kern(P-I)^T$. This implies that $\pi_c = \frac{21}{62}$, $\pi_s = \frac{23}{62}$, $\pi_g = \frac{18}{62}$.

Example: The Hardy Weinberg Principle in Genetics.
(Figure 6)
Assume that the population has proportion, p of being AA, q of being aa, and r of being Aa.
Assume mating proceeds by choosing a random individual from population, producing 1 offspring.
Let X_n be genetic state of the n^{th} generation.

Write down P and determine limiting probabilities.
Let AA be state 0, aa be state 1, and Aa be state 2. Then,

$$P = \begin{pmatrix} p+\frac{r}{2} & 0 & q+\frac{r}{2} \\ 0 & q+\frac{r}{2} & p+\frac{r}{2} \\ \frac{p}{2}+\frac{r}{4} & \frac{q}{2}+\frac{r}{2} & \frac{p}{2}+\frac{q}{2}+\frac{r}{2} \end{pmatrix}$$

Let $p, q, r = \frac{1}{3}$. Then,

$$P = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{pmatrix}$$

Solving for n, $\pi_{AA} = \frac{1}{4}$, $\pi_{aa} = \frac{1}{4}$, $\pi_{Aa} = \frac{1}{2}$.

Example: Random walk on a cube. Let each vertex of the cube be a state.
Calculate $\pi_1, \ldots, \pi_8$.
This is irreduicble with Conjecture:

$$\pi_1 = \pi_2 = \ldots = \pi_8 = \frac{1}{8}$$

Our transition matrix is

$$P = \begin{pmatrix} 0 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} & 0 & 0 \\ & \vdots & & & & & & \end{pmatrix}$$

The pattern... if i and j are connected by an edge, then $P_{ij} = P_{ji}$, then $P_{ij} = P_{ji} = \frac{1}{3}$. Then, it's doubly stochastic. not connected by an edge, $P_{ij} = P_{ji} = 0$

Definition 16. *Let M be a square matrix. It is doubly stochastic if the sum of the rows is 1 and the sum of the column is 1.*

Fact: For every doubly stochastic matrix, the invariant measure is $\pi_1 = \ldots = \pi_8 = \frac{1}{8}$.

Proof. We have a set of DSMs = convex hull of $\{B_1, \ldots, B_{8!}\}$, the permutation matrix. Let $M = c_1B_1 + c_2B_2 + \ldots + c_{8!}B_{8!}$ where $\sum_i c_i = 1$ where $c_i \geq 0$. $\square$

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Markov Chain examples continued

Example: A production process can be in 4 states labeled 1, 2, 3, 4.
On 1 or 2, it's up. Otherwise, it's down.
Suppose that the transition matrix is

$$P = \begin{pmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & 0 & \frac{1}{2} \end{pmatrix}$$

1. Compute average length of time system remains up.
2. Compute proportion of time system up, then down the next time.

1. If you solve
$$[\pi_1, \pi_2, \pi_3, \pi_4]P = [\pi_1, \pi_2, \pi_3, \pi_4]$$
with $\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1$, you'll get $\pi_1 = \frac{3}{16}$, $\pi_2 = \frac{1}{4}$, $\pi_3 = \frac{14}{48}$, $\pi_4 = \frac{13}{48}$.
Let μ be the average stretch of time system is up and d be the average is down.
Proportion of time up $= \frac{\mu}{\mu+d} = \pi_1 + \pi_2$.
2. $\pi_1(P_{13} + P_{14}) + \pi_2(P_{23} + P_{24}) = \frac{9}{32}$ is the breakdown rate. Since there is a single breakdown in stretch of time up then down.
3. Now, solve for $\mu = \frac{N}{9}$ and $d = 2$.

Example: Random walk on a graph.
(figure 7)

$$P = \begin{pmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{3} & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix}$$

By symmetry, we expect $\pi_1 = \pi_4$ and $\pi_2 = \pi_3$. What do we mean by symmetry? Let

$$Q_n = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} Q_n^{-1}$$

By Symmetry, we have $Q_n^{-1}PQ_n = P$.
Then, $\pi P = \pi \Rightarrow \pi Q_n^{-1}PQ_n = \pi \Rightarrow \pi Q_n^{-1}P = \pi Q_n^{-1}$.
By uniqueness of π, $\pi = \pi Q_n - 1 = \pi Q_n$. then, $(\pi_1, \pi_2, \pi_3, \pi_4) = (\pi_4, \pi_2, \pi_3, \pi_1)$, which implies that $\pi_1 = \pi_4$. Likewise, for $\pi_2 = \pi_3$.

$$\begin{aligned} \pi_1 &= \frac{1}{3}\pi_2 + \frac{1}{3}\pi_3 \\ \pi_2 &= \frac{1}{2}\pi_1 + \frac{1}{3}\pi_3 + \frac{1}{2}\pi_4 \\ \pi_3 &= \frac{1}{2}\pi_1 + \frac{1}{3}\pi_2 + \frac{1}{2}\pi_4 \\ \pi_1 &= \frac{1}{3}\pi_2 + \frac{1}{3}\pi_3 \end{aligned}$$

$$\pi_1 = \frac{2}{3}\pi_2$$

$$1 = 2\pi_1 + 2\pi_2$$

$$\pi_2 = \frac{3}{10}$$

$$\pi_1 = \frac{1}{5}$$

Question:

$$E(\text{returned to } 1|X_0 = 1) = \frac{1}{\pi_1} = 5$$

Renewal theory

Suppose $f_k \geq 0$ with $k \in \{1, \dots, N\}$ such that $\sum_{k=1}^{N} f_k = 1$.
Define a Markov Chain (a renewal chain) (Figure 8)

$$\begin{aligned}
P_{0k} &= \frac{f_k}{1 - \sum_{l=1}^{k} f_l} \forall k > 0 \\
P_{00} &= f_1 \\
P_{01} &= 1 - f_1 \\
P_{k(k+1)} &= \frac{P_{(k-1)k} - f_{k+1}}{P_{(k-1)k}}
\end{aligned}$$

Suppose that f_k is chosen to make chain aperiodic.
For aperiocity, we require that $gcd(k, f_k > 0) = 1$.
Calculate if $P($ first return to 0 is from $k) = f_k$, then $m_0 = \sum_{k=1}^{N} k f_k = \mu$.
Calculate P_{00}^{n} return to 0 after exactly n steps.
We get by conditioning to 0,

$$P_{00}^{n} = \sum_{k=1}^{n} f_k P_{00}^{n-k} \text{ for } n \leq N$$

Otherwise,

$$P_{00}^{n} = \sum_{k=1}^{N} f_k P_{00}^{n-k}$$

Chain irreducible, aperiodic implies that

$$\lim_{n \to \infty} P_{00}^{n} = \frac{1}{m_0} = \frac{1}{\mu}$$

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Theorem 17 (Erdos, Feller, Pollard)**.** *Given* $f_1, \dots f_n \geq 0$ *such that* $\sum_{i=1}^{N} f_i = 1$. *Define* $\mu = \sum_{i=1}^{n} i f_i$ *Assuming* $gcd(k|f_k > 0) = 1$, *then* $\lim_{n\to\infty} u_n = \frac{1}{\mu}$.

Example: Roll a fair die forever. Let S_m be sum of first m rolls.
Let $P_n = P(S_m$ even equals$n)$.
Estimate P_{10000}.

$$
\begin{aligned}
P_n &= 0 \\
P_0 &= 1 \\
&\vdots \\
P_n &= \frac{1}{6}(P_{n-1} + \ldots + P_{n-6}) \\
f_k &= \frac{1}{6} \text{ for } k \in \{1, \ldots, 6\} \\
\mu &= \sum_{i=1}^{6} \frac{i}{6} \\
&= \frac{7}{2}
\end{aligned}
$$

Then,

$$\lim_{n\to\infty} P_n = \frac{1}{\mu} = \frac{2}{7}$$

Parrondo's Paradox

Consider games, A, B, C with parameters, p, p_1, p_2, γ, which are each probabilities themselves and $M \geq 2$.
Game A: assymetric simple random walk

- + 1 to capital with probability p
- - 1 to capital wiht probability $1 - p$

Losing game if $p < \frac{1}{2}$.
Game B: Winning probabilities depend on whether your capital is divisible by M.
If $M | capital$, then + 1 with probability p_1 and - 1 with probability $1 - p_1$.
If $M \not| capitals$, then +1 with probability p_2 and -1 with probability $1 - p_2$.
Game C: At each step with probability γ play game A and probability $1 - \gamma$ to play game B.
Question: If A and B are losing, can C be winning? Yes!
First, we analyze game B.

Analysis of Game B

By Gambler's ruin, $M < x < 2M$,

$$P_x(\text{walk hits } 2M \text{ before M}) = \frac{1 - \left(\frac{1-P_2}{P_2}\right)^x}{1 - \left(\frac{1-P_2}{P_2}\right)^M}$$

Starting on a multiple of M,

$$P_M = P_1 \frac{1 - \left(\frac{1-P_2}{P_2}\right)}{1 - \left(\frac{1-P_2}{P_2}\right)^M}$$

Call the equation above, (1).
The probability that you decrease your capital by M before increasing by M or returning to initial point is

$$(1-p_1)\frac{\left(\frac{1-P_2}{P_2}\right)^{M-1}-\left(\frac{1-P_2}{P_2}\right)^{M}}{1-\left(\frac{1-P_2}{P_2}\right)^{M}}$$

Call the equation above, (2).
So, game B is losing if (2)/(1) >1 or equivalently,

$$\frac{(1-p_1)(1-p_2)^{M-1}}{P_1P_2^{M-1}}>1$$

Analysis of Game C

Observe that it is the same as game B with probability

$$\begin{aligned} q_1 &= \gamma p+(1-\gamma)p_1 \\ q_2 &= \gamma p+(1-\gamma)p_1 \end{aligned}$$

So, C is winning if

$$\frac{(1-q_1)(1-q_2)^{M-1}}{q_1q_1^{M-1}}<1$$

Choosing $p=\frac{5}{11}$, $p_1=\frac{1}{11^2}$, $p_2=\frac{10}{11}$, $\gamma=\frac{1}{2}$, and $M=3$ gives $\frac{6}{5}>1$ for game B. Basically, you lose.
For game C, $\frac{217}{300}<1$, so win.

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pattern in coin tosses

Coin with heads probability, p. Toss is repeatedly to get a sequence of 1s and 0s. Fix a pattern, say 1011101.
Waiting Game: What is the expected number of tosses before you get the pattern?
Horse race: Two patterns say 1001 and 0100. What is the probability that 1001 appears first?
Markov Chain: Fix an l. State spaces all patterns of length l. Transition rule: the Markov Chain keeps track of the last l symbols in coin tosses.
Start with some distribution of patterns of length l.
Invariant distribution is very simple. Would be annoying to write the transition matrix (2^l states!).
Let $n\le l$, the tosses $n+1$, $n+2$, ..., $n+l$ are independent of the starting pattern.
Therefore, for any pattern, A, of length l,

$$\pi_A=p^k(1-p)^{l-k}$$

where k is the number of 1s in A.
Assume that B and A are arbitrary patterns. Let $N_{B\to A}$ be the number of times that the coin needs to be tossed from B to get A. You have to toss at least once.

Example: Let $B=111001$ and $A=110$.
$111001\to110011\to100110$, so $N_{B\to A}=2$

$111001 \to 110010 \to 100100 \to 001001 \to 010011 \to 100111 \to 001110$, so $N_{B\to A} = 6$

Example: We can reformulate the waiting game to

$$E(\emptyset \to 1011101)$$

For any pattern A, we know that

$$E(A \to A) = \frac{1}{\pi_A}$$

Example: $E(1011101 \to 1011101) = E(101 \to 1011101) = \frac{1}{\pi_{1011101}}$
The 101 above that is both at the beginning and the ending is 101 is called an overlap.

$$E(\emptyset \to 1011101) = E(\emptyset \to 101) + E(101 \to 1011101)$$

$$\begin{aligned} E(\emptyset \to 101) &= E(\emptyset \to 1) + E(1 \to 101) \\ &= \frac{1}{p} + \frac{1}{\pi_{101}} \\ E(\emptyset \to 1011101) &= \frac{1}{\pi_1} + \frac{1}{\pi_1 01} + \frac{1}{\pi_{1011101}} \\ &= \frac{1}{p} + \frac{1}{p^2(1-p)} + \frac{1}{5} \end{aligned}$$

Horse Race: Let A and B patterns.

$$P_A = P(A \text{ win}) \text{ and } P_B = P(B \text{ wins})$$

Let N be the time that the first of the two patterns appears.

$$E(\emptyset \to A) = E(N) + p_B E(N_{B\to A})$$

$(N_{\emptyset\to A} = N + I\{B \text{ wins}\}N_{B\to A})$
$E(\emptyset \to B) = E(N) + p_A E(A \to B)$
Then, $1 = p_a + p_b$.
For $p = \frac{1}{2}$ and $A = 1001$ and $B = 0100$.

$$E(\emptyset \to A) = 16 + 2 = 18$$
$$E(\emptyset \to B) = 16 + 2 = 18$$
$$E(B \to A) = E(100 \to 1001) = E(\emptyset \to 1001) - E(\emptyset \to 100) = 18 - 8 = 10$$
$$E(A \to B) = 18 - 4 = 14$$

Solve the system:

$$p_A = \frac{5}{12}, p_B = \frac{7}{12}, E(N) = \frac{73}{6}$$

Example: $A = 1010$, $B = 0100$

$$\begin{aligned} E(\emptyset \to A) &= 20 \\ E(\emptyset \to B) &= 18 \\ p_A &= \frac{9}{14} \end{aligned}$$

A wins at the horse race, but loses at the waiting game. Why? A probably wins more, but when loses, loses badly.

Example: $A = 011$, $B = 100$, $C = 001$

$$P(A \text{ beats} B) = \frac{1}{2} P(B \text{ beats} C) = \frac{3}{4} P(C \text{ beats} A) = \frac{2}{3}$$

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Example: Random walk on the graph: The walker chooses a neighbor at random, then moves there. What is the proportion of time spent at 0.

Reversible Markov Chain

Assume that a chain starts in the invariant distribution, $X_0, X_1, \ldots, X_n$ has the same statistics on the time-reversal as $X_n, X_{n-1}, \ldots, X_0$, then we call the chain reversible. Formally, we can define such chains as follows:

$$P(X_m = j | X_{m-1} = i) = P(X_{m-1} = j | X_m = i)$$

Moving forwards = Moving backwards.

Let π_i be the invariant distribution and let π_i be the initial distribution, then equation says that $P_{ij} = \frac{P(X_{m-1})=j, X_m=i}{P(X_m=i)} = P(X_{m-1} = j) P(X_m = i | X_{m-1} = j) = \frac{\pi_j P_{ji}}{\pi_i}$.

A Markov Chain is reversible with respect to invariant distribution π_i when $\pi_i P_{ij} = \pi_j P_{ji}$.

Proposition 18. *If a probability mass function, π_i satisfies $\pi_i P_{ij} = \pi_j P_{ji}$, then it is automatically invariant.*

Proof. Need to check $\pi_j = \sum_i \pi_i P_{ij}$.
We know that $\sum_i \pi_i P_{ij} = \sum_i \pi_j Pji = \pi_j \sum_i^n P_{ji} = \pi_j D$
More general problem: Assign nonnegative weights to any edge in a complete graph, $w_{ij} = w_{ji}$ is the weight of the edge between i and j.
When in i, the walker goes to j with probability proportional to w_{ij}, so $P_{ij} = \frac{w_{ij}}{\sum_j w_{ij}}$.

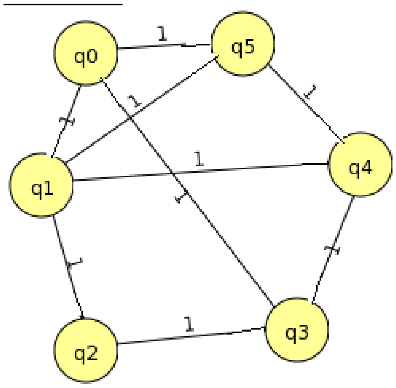
$$\pi_i = \frac{\sum_j w_{ij}}{\sum_{i,j} w_{ij}}$$

is a reversible measure. why? $\pi_i P_{ij} = \pi_j P_{ji}$. This is equivalent to $\sum_j w_{ij} \frac{w_{ij}}{\sum_j w_{ij}} = \sum_i w_{ij} \frac{w_{ij}}{\sum_i w_{ij}}$.

$\square$

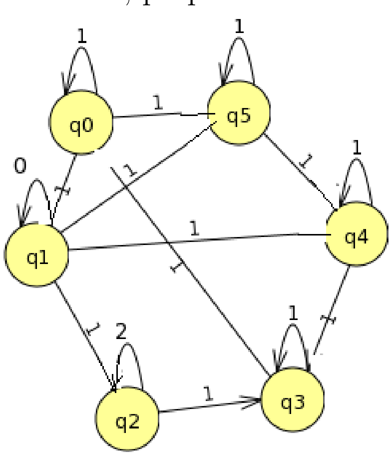
Remark: This chain is irreducible exactly when the graph with edge$\{i,\ j\}$ whenever $w_{ij} > 0$ is connected. Aperiodic $\Leftrightarrow$ not bipartite.

Example: Back to the walker problem.



$$\begin{aligned}
\pi_0 &= \frac{3}{18} \\
\pi_1 &= \frac{3}{18} \\
\pi_2 &= \frac{3}{18} \\
\pi_3 &= \frac{3}{18} \\
\pi_4 &= \frac{3}{18} \\
\pi_5 &= \frac{3}{18}
\end{aligned}$$

Therefore, proportion of time spent at 0 is $\frac{1}{6}$.



Example: Ehrenfest chain
M balls, two urns. Each time, pick a ball at random, move it from one urn to another.
Let X_n be the number of balls in urn 1.

$$
\begin{aligned}
P_{0,1} &= 1 \\
P_{M,M-1} &= 1 \\
P_{i,i-1} &= \frac{i}{M} \\
P_{i,i+1} &= \frac{M-i}{M}
\end{aligned}
$$

With some intuitive guessing, you can see $\pi_i = \binom{M}{i}\frac{1}{2^M}$.
Let's prove this. First, check for reversibility. Are the following true?

$$
\begin{aligned}
\pi_0 P_{01} &= \pi_1 P_{10} \\
\pi_i P_{i,i+1} &= \pi_{i+1} P_{i+1,i} \\
\pi_i P_{i,i-1} &= \pi_{i-1} P_{i-1,i} \\
\pi_M P_{M,M-1} &= \pi_{M-1} P_{M-1,i}
\end{aligned}
$$

For the first equation

$$
\begin{aligned}
\pi_0 P_{01} &= \pi_1 P_{10} \\
\frac{1}{2^M} \cdot 1 &= M\frac{1}{2^M} \cdot \frac{1}{M} \\
1 &= 1
\end{aligned}
$$

You can check the other equations, it matches.
Also, note that this chain is irreducible, but not aperiodic. It has period 2.

5/27

Poisson Process

A counting process, $N(t) \geq 0$ is a random process with

1. $N(t)$ is nonnegative integer values
2. $N(t)$ increases in t

$N(t) - N(s) =$ Number of events in $[s, t]$.
$N(t)$ is right-continuous?)

Definition 19. *A Poisson process with rate (aka intensity) $\lambda > 0$ is a counting process $N(t)$ such that*

1. *$N(0) = 0$*
2. *It has independent increments: if $(s_1, t_1] \bigcap (s_2, t_2] = \emptyset$, then $N(t_1) - N(s_1)$ and $N(t_2) - N(s_2)$ are independent*
3. *Number of events in any interval of length, t, is Poisson(λt), i.e. $P(N(t+s) - N(s) = k) = e^{-\lambda t}\frac{(\lambda t)^k}{k!}$, $i = 0, 1, 2, \ldots$. $E(N(t+s) - N(s)) = \lambda t$*

Review of Exponential random variables

X is $exp(\lambda)$ means that

$$\begin{aligned} f_X(x) &= \begin{cases} \lambda e^{-\lambda x} & x \geq 0 \\ 0 & x < 0 \end{cases} \\ P(X \geq x) &= e^{-\lambda x}, x > 0 \\ E(X) &= \frac{1}{\lambda} \\ Var(X) &= \frac{1}{\lambda^2} \\ P(X > s + t | X > t) &= P(X > s) \end{aligned}$$

The last equation is known as the "memoryless" property.

Review of Poisson random variables

X is $Poisson(\lambda)$ means that

$$\begin{aligned} P(X = k) &= e^{-\lambda} \frac{\lambda^k}{k!}, k = 0, 1, \ldots \\ EX &= \lambda \\ Var(X) &= \lambda \end{aligned}$$

If X_i is $Poisson(\lambda_i)$ and indpendent, then $X_1 + \ldots + X_n$ is $Poisson(\lambda_1 + \ldots + \lambda_n)$. (Superposition property)
If X is $Poisson(\lambda)$ and I_i are independent indicators with $P(I_i = 1) = p$, then $\sum_{i=1}^{x} I_i$ is $Poisson(\lambda p)$. (thinning property)
Figure 12
$P(N(h) = 1) = e^{-\lambda h} \lambda h \approx \lambda h$ as $h \to 0$.
$P(N(h) \geq 2) = O(h^2) < \lambda h$
This is why λ is called rate.

In fact, we could construct a Poisson process in the following way: Any integer multiple of $\frac{1}{n}$.
Toss a coins with heads probability $\frac{\lambda}{n}$. Mark the location of heads.
Let $N_n(t)$ be the number of heads on $[0, t]$.
When $n \to \infty$, $N_n(t)$ converges to a Poisson process with rate λ.
Number of heads is $Binomial(nt \pm 1, \frac{\lambda}{n}) \approx Poisson(\lambda t)$.

Let $T_1, T_2, \ldots$ be the interarrival times where T_n be the time elapses between $(n-1)'s$ and $n'th$ event. (like buses coming).

Proposition 20. $T_1, T_2, \ldots$ *are independent and distributed* $exp(\lambda)$.

Proof. $P(T_1 > t) = P(N(t) = 0) e^{-\lambda t}$
$P(T_2 > t | T_1 = s) = P(\text{no events in } (s, s+t] | T_1 = s) = P(N(t) = 0) = e^{-\lambda t}$. It does not matter what happen between 0 and s.

□

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Poisson process

"Tossing low probability coins very fast."
Mark an x at each multiple of $\frac{1}{n}$ with probability $\frac{\lambda}{n}$ as $n \to \infty$ P(number of events in interval of length $k \ldots$
T_n are independently and identically distributed $exp(\lambda)$ and $ET_n = \frac{1}{x}$.
Let $S_n = T_1 + \ldots + T_n$ be the waiting time for n^{th} event.
Compute the density of S_n.
(We know that $ES_n = \frac{n}{\lambda}$)

$$\begin{aligned}
P(S_n > t) &= P(N(t) < n) = \sum_{j=0}^{n-1} e^{-\lambda t} \frac{(\lambda t)^j}{j!} \\
-f_{S_n}(t) &= \sum_{j=0}^{n-1} \frac{1}{j} (-\lambda e^{-\lambda t} (\lambda t)^j + e^{-\lambda t} j (\lambda t)^{j-1} \lambda) \\
&= \lambda e^{-\lambda t} \sum_{j=0}^{k-1} (-\frac{(\lambda t)^j}{j!} + \frac{(\lambda t)^{j-1}}{(j-1)!} \\
&= -\lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} f_{S_n}(t) = \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!}
\end{aligned}$$

Above result is called Gamma distrubtion.

example: Poisson process with rate λ.

$$\begin{aligned}
E(\text{time of 10th event}) &= \frac{10}{\lambda} \\
P(\text{10th event occurs 2 or more time units after 9th event}) &= e^{-2\lambda} \\
P(\text{10th event occurs later than time 20}) &= \int_{20}^{\infty} \lambda e^{-\lambda t} \frac{(\lambda t)^9}{9!} \, dt \\
&= P(S_{10} > 20) \\
&= P(N(20) < 10) \\
&= \sum_{j=0}^{9} e^{-20\lambda} \frac{(20\lambda)^j}{9!}
\end{aligned}$$

$$
\begin{aligned}
P(2 \text{ events in } [1,4) \text{ and } 3 \text{ events in } [3,5]) &= \sum_{k=0}^{2} P(\ldots | k \text{ events in } [3,4]) P(k \text{ events in } [3,4]) \\
&= \sum_{k=0}^{2} P(2-k \text{ events in } (1,3] \text{ and } 3-k \text{ events in } [4,5]) \\
&\quad P(k \text{ events in } [3,4]) \\
&= \sum_{k=0}^{2} e^{-2\lambda} \frac{(2\lambda)^{2-k}}{(2-k)!} e^{-\lambda} \frac{\lambda^{3-k}}{(3-k)!} e^{-\lambda} \frac{\lambda^k}{k!} \\
&= e^{-4\lambda}(\frac{1}{3}\lambda^5 + \lambda^4 + \lambda^3)
\end{aligned}
$$

0.1 Superposition

Two independent Poisson processes with λ_1 and λ_2 can be combined into a single Poisson process with rate $\lambda_1 + \lambda_2$.
$N_1(t) + N_2(t) = N(t)$.

Thinning

Flip a coin with probability p for heads for every event in a Poisson process. Call Type I events for those with heads and Type II events for those with tails.
Let $N_1(t)$ be the number of Type I events in $[0, t]$.
Let $N_2(t)$ be the number of Type II events in $[0, t]$.
These are both Poisson processes with parameter λp and $\lambda(1-p)$, respectively.
Further, these two processes are independent.
Idea: Discretize.
construct Type I process by flipping $\frac{p\lambda}{n}$, coins at multiples of $\frac{1}{n}$. Then, construct type II process by flipping $\frac{\lambda(1-p)}{n}$ coins on the remaining points. Send $n \to \infty$.

Example: Customer arrive at a store at rate 10/hr. Each are either male or female with probability $\frac{1}{2}$. Assume that you know that exactly 10 women entered within some hour (say 10 - 11am). (a) Compute the probability that exactly 10 men also entered.

They're completely independent Poisson process. Parameter is $\frac{1}{2} \cdot 10 = 5$, so our answer is

$$e^{-5}\frac{5^{10}}{10!}$$

Compute the probability that at least 20 customers have entered.

$$= \sum_{k=10}^{\infty} P(k \text{ men entered}) = \sum_{10}^{\infty} e^{-5}\frac{5^k}{k!}$$

Example: Assume that cars arrive at rate, 10 / hr. Assume each car will pick up a hitchhiker with probability $\frac{1}{10}$. You are 2nd in line. What is the probability tht you'll have to wait for more than 2 hours?

Cars that pick up are a Poisson process with rate $10 \cdot \frac{1}{10} = 1$.
$\lambda = 2$ because average is 1 and time is 2.

$$P(T_1 + T_2 > 2) = P(N(2) \leq 1) = e^{-\lambda t}(1 + \lambda t) = e^{-2}(1+2) = 3e^{-2}$$

6/1

Assume that you have two independent Poisson process $N_1(t)$ with rate λ_1 and $N_2(t)$ with rate λ_2. What is the probability that n events occur in the first process before m events occur in the 2nd process.

Example: $\lambda_1 = 5$, $\lambda_2 = 1$, $n = 5$, $m = 2$.
Let T_i^1 be the interarrival times for the 1st process.
Let T_i^2 be the interarrival times for the 2nd process.
Let $S_n^1 = T_1^1 + \dots T_n^1$
Let $S_m^2 = T_1^2 + \dots T_m^2$
Start with a continued Poisson process with $\lambda_1 + \lambda_2$, then independently decide for each event whether it belongs to the 1st process with probability $\frac{\lambda_1}{\lambda_1+\lambda_2}$ or the 2nd process with probability $\frac{\lambda_2}{\lambda_1+\lambda_2}$. The obtained processes are independent and have the correct rates.

$$\begin{aligned}
P(S_n^1 < S_m^1) &= P(\text{among first } m+n-1 \text{ events (in the combined process), } n \text{ or more events belong to the 1st} \\
&= \sum_{k=n}^{n+m-1} \binom{n+m-1}{k} \left(\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^k \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{n+m-1-k} \\
&= \sum_{k=5}^{6} \binom{6}{k} \left(\frac{5}{6}\right)^k \left(\frac{1}{6}\right)^{6-k} \\
&= \frac{11 \cdot 5^5}{6^6}
\end{aligned}$$

If you change $m = 1$, then our answer is $\left(\frac{5}{6}\right)^5$

Example: You have 3 friends, A, B, C. Each will call you on exponential amount of time with $exp(30min), exp(1hr), exp(2.5hr)$ respectively. You will go out with the first one that calls. What is the probability that you go out with A?

Intepret each call as the first event in a Poisson process (rates $2, 1, \frac{5}{2}$ (the units are an hour)). Note that rates are inverses of your mean.

Let the rates be $\lambda_1, \lambda_2, \lambda_3$ respectively.
Start with $(\lambda_1 + \lambda_2 + \lambda_3)$ Poisson process, distribute the events with probability $\frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3}$, $\frac{\lambda_2}{\lambda_1+\lambda_2+\lambda_3}$, and $\frac{\lambda_3}{\lambda_1+\lambda_2+\lambda_3}$ respectively.
Probability of A calling first is $\frac{\lambda_1}{\lambda_1+\lambda_2+\lambda_3} = \frac{2}{2+1+\frac{5}{2}}$.

Theorem 21. *Given that $N(t) = k$, the conditional distribution of the arrival times, $S_1, \dots, S_k$, is distributed as order statistics of k independent uniform variables: the set $\{S_1, \dots, S_k\} = \{U_1, \dots, U_i\}$,* *where U_i are iid uniform on $[0, t)$.*

Theorem 22. *Given that $S_k = t$, $S_1, \ldots, S_{k-1}$ are distributed as order statistics of $k-1$ uniform $(0, t)$ random variables.*

Proof. Why? N independently coins with heads probability p. Condition on k heads.

$$P(k \text{ heads occurs on specifics subset of } k \text{ coins} | k \text{ heads}) \quad = \quad \text{same for all subsets of size } k$$

$\square$

Example: Assume that passangers arrive as a Poisson process with rate λ.

1. The only bus departs after time T. Compute the expected combined walking time, W for all passangers.

   If $S_1, S_2, \ldots,$ are the arrival times in $[0, 1]$, then the combined waiting times, $T - S_1 + T - S_3 + \ldots$ Let $N(t)$ be the number of arrivales in $[0, t]$.

$$\begin{aligned} E(W) &= \sum_{k=0}^{\infty} E[W|N(T) = k]P(N(T) = k) \\ &= \sum_{k=0}^{\infty} \ldots \end{aligned}$$

2. now, two buses depart, are at T, one at $S < T$, EW
   $\lambda \frac{s^2}{2} + \lambda \frac{(T-S)^2}{2}$

3. Now assume T(the only bus time) is *exponential*(μ), indpendent of the passengers.

$$\begin{aligned} EW &= E(W|T = t)E(T = t) \\ &= \int_0^{\infty} \lambda \frac{t^2}{2} f_T(t) dt \\ &= \frac{\lambda}{2} E(T^2) \\ &= \frac{\lambda}{2}(Var(T) + (ET)^2) \\ &= \frac{\lambda}{2} \frac{2}{\mu^2} \\ &= \frac{\lambda}{\mu^2} \end{aligned}$$

4. Two busses now arrive as first two events in a *poisson*(μ) process $EW = 2\frac{\lambda}{\mu^2}$

6/3

Exercise 7.9 You have two machines. M_1 has lifetime $exp(\lambda_1)$ and M_2 has lifetime $exp(\lambda_2)$. Let their lifetimes be T_1 and T_2.
Machine 1 starts at time 0 and Machine 2 starts at time t.

$$\begin{aligned}
P(\text{Machine 1 is the 1st to fail}) &= P(T_1 < T_2 + t) \\
&= P(T_1 < t) + P(T_1 \geq t, T_1 < T_2 + t) \\
&= P(T_1 < t) + P(T_1 < T_2 + t |]T_1 \geq t) P(T_1 \geq t) \\
&= 1 - e^{-\lambda_1 t} + P(T_1 - t < T_2 | T_1 \geq t) e^{-\lambda_1 t} \\
&= 1 - e^{-\lambda_1 t} + P(T_1 < T_2) e^{-\lambda_1 t} \text{ (Memoryless probability.)} \\
&= 1 - e^{-\lambda_1 t} + \frac{\lambda_1}{\lambda_1 + \lambda_2} e^{-\lambda_1 t} \text{ (Poisson process)}
\end{aligned}$$

What if we replace t with T, exponential (and independent of T_1, T_2) with parameter μ?

$$\begin{aligned}
P(T_1 < T_2 + T) &= P(T_1 < T) + P(T_1 \geq T, T_1 < T_2 + T) \\
&= \frac{\lambda_1}{\lambda_1 + \mu} + \frac{\lambda_1}{\lambda_1 + \lambda_2} \frac{\mu}{\lambda_1 + \lambda_2}
\end{aligned}$$

Example: Hitchhiker's problem. You have A and B. Cars that would pick up a hitchhiker approximately at a Poisson process at rate λ_0. A is first in line.
After $exp(\lambda_A)$ time, A quits.
After $exp(\lambda_B)$ time, B quits.

$$\begin{aligned}
P(\text{A gets picked}) &= \frac{\lambda_c}{\lambda_A + \lambda_c} \\
P(\text{B gets picked}) &= P(\text{at least 2 cars arrives before B quits} \bigcup \text{A quits and at least one car arrives before B qu} \\
&= P(\text{at least 2 cars arrives before B quits}) + P(\text{A quits and at least one car arrives before B} \\
&= \left(\frac{\lambda_c}{\lambda_B + \lambda_c}\right)^2 + \left(\frac{\lambda_A}{\lambda_A + \lambda_B + \lambda_c}\right)\left(\frac{\lambda_c}{\lambda_A + \lambda_B + \lambda_c}\right) - \left(\frac{\lambda_A}{\lambda_A + \lambda_B + \lambda_c}\right)\left(\frac{\lambda_c}{\lambda_A + \lambda_B + \lambda_c}\right)^2 \\
&= \frac{\lambda_A + \lambda_c}{\lambda_A + \lambda_B + \lambda_C} \cdot \frac{\lambda_C}{\lambda_B + \lambda_C}
\end{aligned}$$

Better way:

$$P(\text{B gets picked}) = P(\text{first event is either } A \text{ or } C \text{ and next event among } B \text{ and } C \text{ only has to be C})$$