The Euclidean space $\mathbb{R}^n$

Galatia Cleanthous

Assistant Professor, National University of Ireland, Maynooth

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The Euclidean plane $\mathbb{R}^2$

The **Euclidean plane**, is defined as the set of all *points* with *two* real coordinates;

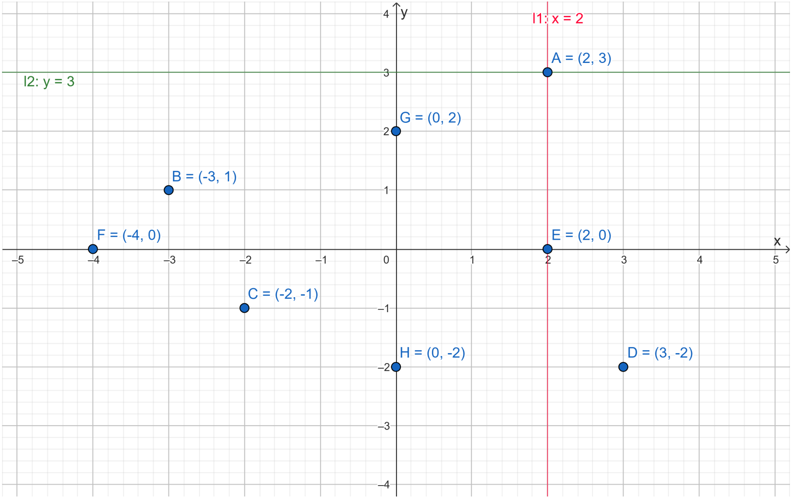
$$\mathbb{R}^2 := \{(x, y) : x, y \in \mathbb{R}\}. \tag{1}$$

Every point can be represented as the corresponding pair of Cartesian coordinates (René Descartes):

- The x-axis and y-axes are orthogonal and intersect at the point $O(0,0)$.
- Every point is denoted by P, $P(x_0, y_0)$, $P = (x_0, y_0)$.
- The lines of the form $x = x_0$, are parallel to y-axis, and in particular the y-axis itself has the equation $y\text{-axis}: \; x = 0$. As a set it can be expressed as $y\text{-axis} = \{(0, y) : y \in \mathbb{R}\}$.

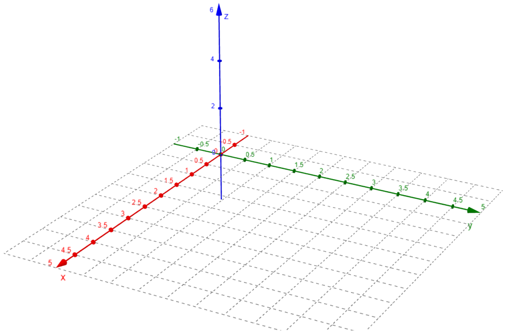
  Similarly the lines $y = y_0$ are parallel to x-axis; horizontal. Then $x\text{-axis}: \; y = 0$ and $x\text{-axis} = \{(x, 0) : x \in \mathbb{R}\}$.
- The Euclidean plane consists of 4 quadrants. We enumerate from 1 to 4 starting from top-right to bottom-right. In the next example B is in the 2nd and C in the 3rd quadrant, while O lies in all of them. Confirm the coordinates and equation of horizontal and vertical lines 🏠.

More points


y
x
l1: x = 2
l2: y = 3
A = (2, 3)
G = (0, 2)
B = (-3, 1)
F = (-4, 0)
E = (2, 0)
C = (-2, -1)
H = (0, -2)
D = (3, -2)
–5
–4
–3
–2
–1
0
1
2
3
4
5
4
3
2
1
–1
–2
–3
–4

The Euclidean space $\mathbb{R}^3$

Our geometric intuition of the space around us can be expressed by the **Euclidean space** $\mathbb{R}^3 = \{(x, y, z) : x, y, z \in \mathbb{R}\}$.
This time we need 3 axes; x,y and z. We may represent them in the page as follows



- The x-axis contains all points with $y = 0$ and $z = 0$

x-axis $:= \{(x, 0, 0) : x \in \mathbb{R}\}$ i.e. its (system of) equation(s) is

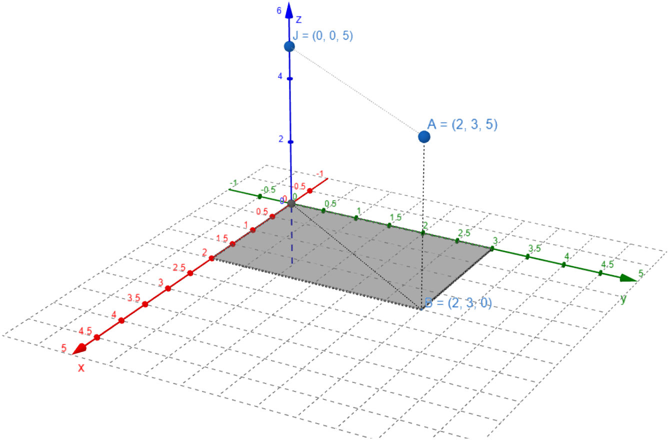
$$x\text{-axis} : \begin{cases} y = 0, \\ z = 0 \end{cases} \tag{2}$$

Similarly for the other two axes .

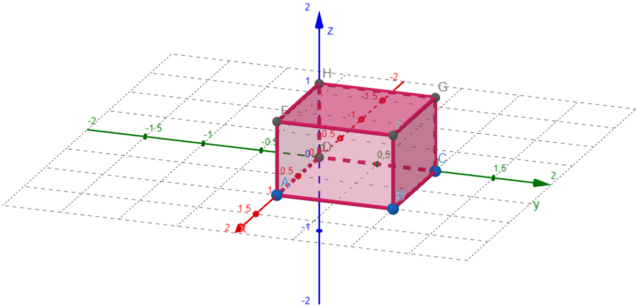
- The xy-plane contains all points with $z = 0$; $\boxed{xy-plane : z=0}$.

xy-plane $:= \{(x, y, 0) : x, y \in \mathbb{R}\}$. What about the other two coordinate planes ?

- The three planes divide $\mathbb{R}^3$ in 3 octants. The first of them is oct1 $:= \{(x, y, z) \in \mathbb{R}^3 : x, y, z \geq 0\}$.
- Note above that with one equation like $z = 0$ on $\mathbb{R}^3$, we take a plane. A surface of dimension 2 that we will say later. For taking a *line*, like the axes, we needed *two* consistent equations like that! We will further elaborate on this in the sequel.
- We represent a point $P(x, y, z)$ by (i) finding its projection $Q(x, y, 0)$ at the xy-plane and (ii) its projection $J(0, 0, z)$ in the z axis.


z
J = (0, 0, 5)
A = (2, 3, 5)
B = (2, 3, 0)
x
y

For another example consider the following cube of the first octant with edge 1.



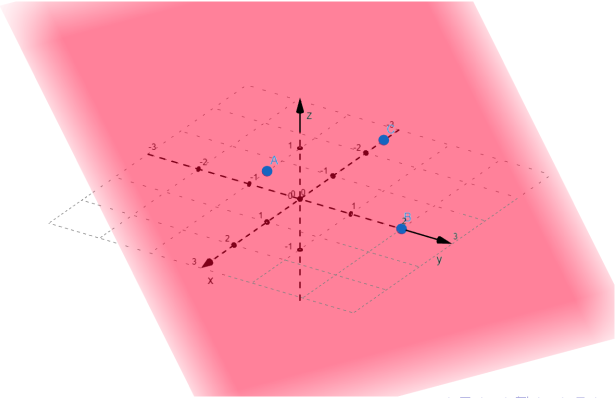
Here $A(1,0,0)$, $B(1,1,0)$, $G(0,1,1)$ are three of its vertices; the other 5? 🏠.
The six faces are lying at the planes $x=0$, $x=1$ (back, front), $y=0$, $y=1$ (left, right) and $z=0$, $z=1$ (down, up).

General form of planes on $\mathbb{R}^3$

Every **plane** has an analytic equation of the form

$$\alpha x + \beta y + \gamma z + \delta = 0. \quad (\alpha, \beta, \gamma) \neq (0,0,0). \tag{3}$$

Below we can see the plane $x + 2y + 3z = 4$ and three points $A(1,0,1)$, $B(0,2,0)$ and $C(-1,1,1)$ on it.



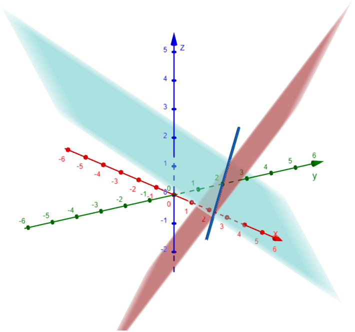
Lines as planes' intersections

Two planes of the form

$$\alpha x + \beta y + \gamma z + \delta_i = 0. \quad \delta_1 \neq \delta_2. \tag{4}$$

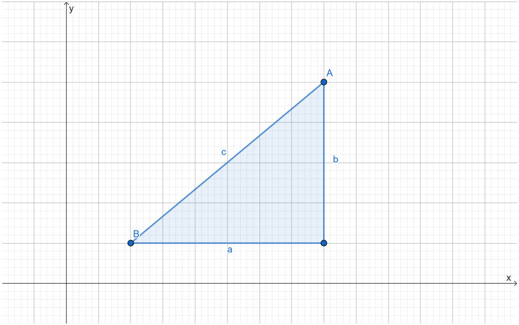
are **parallel**. Otherwise two non-parallel planes intersect defining a line. In the next figure we can see the line

$$\ell : x + 2y + 3z = 4, \; x + y = z + 2. \tag{5}$$



Distance

The **distance** $d(A, B) = (AB) = |AB|$ between two points on $\mathbb{R}^2$, or $\mathbb{R}^3$ is defined by the Pythagorean Theorem; $|AB| = c = \sqrt{a^2 + b^2}$;



Let $A(x, y, z)$, $B(x', y', z')$. Their distance equals

$$|AB| := \sqrt{(x - x')^2 + (y - y')^2 + (z - z')^2}. \tag{6}$$

• For example the distance between the points $A(1,2,3)$ and $B(-3,14,6)$ of $\mathbb{R}^3$ is

$$|AB| = \sqrt{(1-(-3))^2 + (2-14)^2 + (3-6)^2} \tag{7}$$
$$= \sqrt{16+144+9} = \sqrt{169} = 13. \tag{8}$$

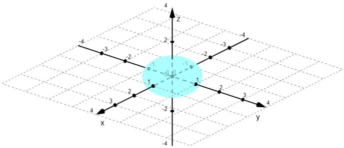
• A **sphere** is the set of points that are all at the same distance $r > 0$ from a given point C. The point C is the *center* of the sphere, and r is its *radius*.
• The interior points are of distance less than r from the centre and the exterior of distance more than r from the centre. The above definition immediately leads to the following.

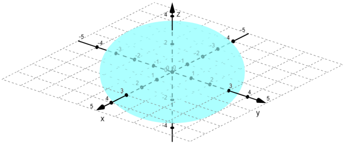
Spheres

The **sphere** of radius $r > 0$ centred at $C(x_0, y_0, z_0)$ has the equation

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2 \tag{9}$$

Below we can see the spheres centred at the origin of radii 1 (left) and 3 (right)





The sets of points $P(x, y, z)$ with

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 < r^2 \quad (> r^2) \tag{10}$$

consist of the interior (exterior) of the above sphere.

Vectors

• The following notion coming from Algebra will be of high importance in our study.

• As a **vector** of $\mathbb{R}^2$ (or $\mathbb{R}^3$) we refer to every directional linear segment; $\mathbf{u} := \vec{u} := \overrightarrow{AB} := \overline{AB}$. We refer to the point A as its start and the point B as its end and the size or length of the vector, is exactly the size of the corresponding linear segment, the distance between A and B; $|\mathbf{u}| := |\overrightarrow{AB}| := |AB|$.

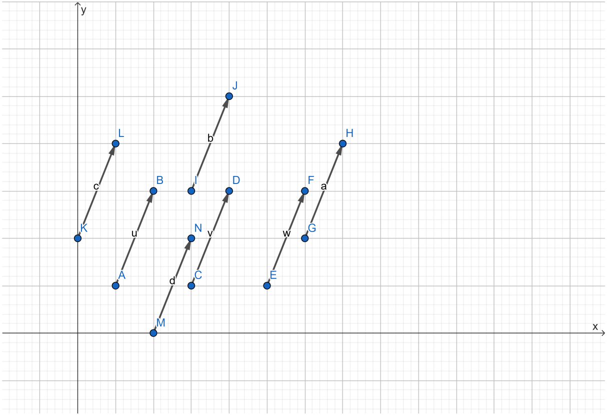
• Two vectors are *equal* when they have the same direction and size. This means that we can move (but not rotate) a vector, while keeping it invariant.

• Velocity, acceleration, and force are typical examples of vectors.

• A vector $\mathbf{u} = (u_1, u_2, u_3)$ on $\mathbb{R}^3$ (similarly $\mathbb{R}^2$) represent the change between the start and end points by u_1 (positive/negative) in the first coordinate, u_2 in the second and u_3 in the third. In other words the vector $\overrightarrow{AB}$ with start at the point $A(a_1, a_2, a_3)$ and end at $B(b_1, b_2, b_3)$ is

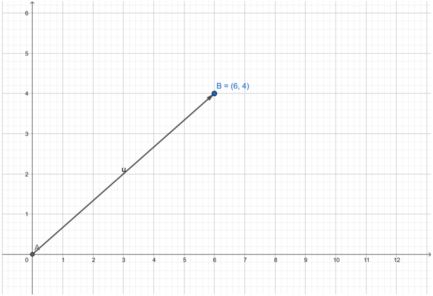
$$\overrightarrow{AB} = (b_1 - a_1, b_2 - a_2, b_3 - a_3). \tag{11}$$

The vector $\mathbf{u} = (1, 2)$ on the plane:



Identification between points and vectors

Every vector $\mathbf{u} = (x, y)$ (say on $\mathbb{R}^2$, but similarly for $\mathbb{R}^3$) can be moved so that to has its start on the origin $O(0, 0)$ and its end at the point $B(x, y)$, i.e. $\mathbf{u} = \overrightarrow{OB}$.
It will be very often convenient to **identify** every point $B(x, y)$ with the vector $\overrightarrow{OB} = (x, y) = \mathbf{u}$. Several times we further identify the letters used; point $P(x, y)$ and vector $\mathbf{p} = (x, y)$, without extra emphasis.



$\mathbb{R}^n$

Let $n \in \mathbb{N}$. The Euclidean space of dimension n is defined as the set of all points/vectors with n real coordinates:

$$\mathbb{R}^n := \{\mathbf{x} = (x_1, \ldots, x_n) : x_i \in \mathbb{R},\ i = 1, \ldots, n\}. \tag{12}$$

• In particular $\mathbb{R}^3$, $\mathbb{R}^2$ and $\mathbb{R}^1$ are referred as the Euclidean space, plane and line respectively.
• We commonly use (x, y) and (x, y, z) for the points/vectors of $\mathbb{R}^2$ and $\mathbb{R}^3$ respectively. Whenever we are on $\mathbb{R}^2$ or $\mathbb{R}^3$ the coordinates' correspondence is $x_1 = x$, $x_2 = y$ and $x_3 = z$.
• The vector $\mathbf{0} := \vec{0} := (0, \ldots, 0)$ is referred as the zero vector of $\mathbb{R}^n$ and plays the role of multi-dimensional analogous of the zero real number.

Vector operations; recap from Linear algebra

Consider $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$ and $\lambda \in \mathbb{R}$.

- The vectors $\mathbf{x}$ and $\mathbf{y}$ are equal, when all of their corresponding coordinates are equal to each other;

$$\mathbf{x} = \mathbf{y} \Leftrightarrow x_i = y_i, \text{ for every } i = 1, \ldots, n. \tag{13}$$

- Their sum $\mathbf{x} + \mathbf{y}$ and their difference $\mathbf{x} - \mathbf{y}$ are the vectors of $\mathbb{R}^n$ given below:

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, \ldots, x_n + y_n) \quad \mathbf{x} - \mathbf{y} = (x_1 - y_1, \ldots, x_n - y_n). \tag{14}$$

- The opposite vector of $\mathbf{x}$ is given by

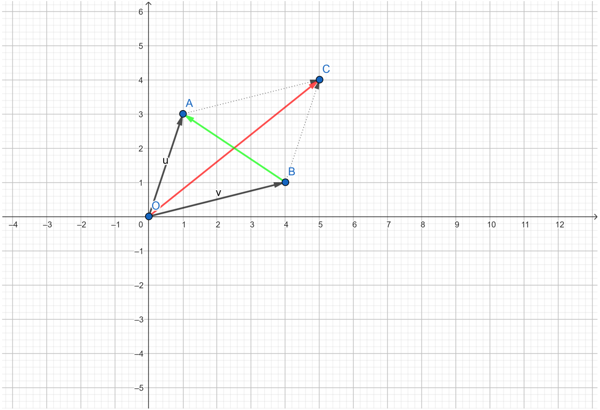
$$-\mathbf{x} := (-x_1, \ldots, -x_n). \tag{15}$$

- The scalar product between λ and $\mathbf{x}$ is the vector:

$$\lambda \mathbf{x} := (\lambda x_1, \ldots, \lambda x_n). \tag{16}$$

Visually: sum; Parallelogram rule

Let sum and difference of two vectors are obtained by the parallelogram low. Below $\mathbf{u}+\mathbf{v}=\overrightarrow{OA}+\overrightarrow{OB}=\overrightarrow{OA}+\overrightarrow{AC}=\overrightarrow{OC}$ is in red and $\mathbf{u}-\mathbf{v}$ is in green



Vector Properties

Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbb{R}^n$ and $\lambda, \mu \in \mathbb{R}$. Recall the next fundamental properties:

- $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$.
- $\mathbf{x} + \mathbf{0} = \mathbf{0} + \mathbf{x} = \mathbf{x}$.
- $(\mathbf{x} + \mathbf{y}) + \mathbf{z} = \mathbf{x} + (\mathbf{y} + \mathbf{z}) =: \mathbf{x} + \mathbf{y} + \mathbf{z}$.
- $\lambda(\mathbf{x} + \mathbf{y}) = \lambda\mathbf{x} + \lambda\mathbf{y}$.
- $(\lambda + \mu)\mathbf{x} = \lambda\mathbf{x} + \mu\mathbf{x}$.
- $\lambda(\mu\mathbf{x}) = (\lambda\mu)\mathbf{x}$.
- $1\mathbf{x} = \mathbf{x}$
- $0\mathbf{x} = \mathbf{0}$.
- $\lambda\mathbf{0} = \mathbf{0}$.

Vector equation of a line

• So far we have seen analytic equations of lines on $\mathbb{R}^2$ and $\mathbb{R}^3$ (as intersections of planes). Vectors can be employed for obtaining further expressions of line equations.

• A line passing through the point $P(p_1, p_2, p_3)$ with direction $\mathbf{d} = (d_1, d_2, d_3) \neq \mathbf{0}$ has the following **vector equation**:

$$\mathbf{x} = \mathbf{p} + t\mathbf{d}, \text{ for every } t \in \mathbb{R}. \tag{17}$$

(recall that $\mathbf{p} = \overrightarrow{OP} = (p_1, p_2, p_3)$) Which may represented coordinate-wise as

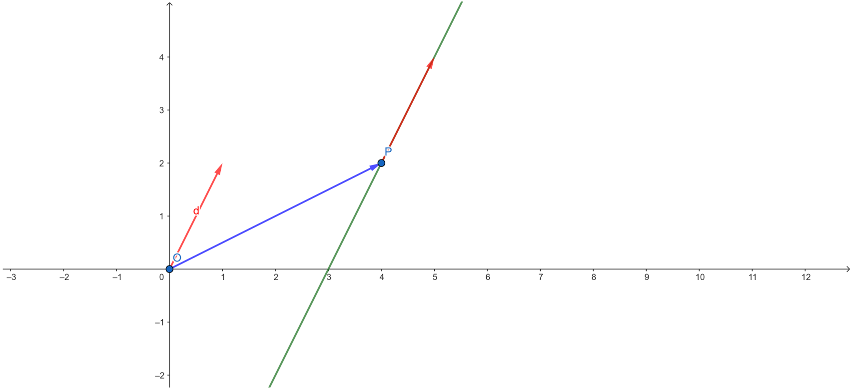
$$x_1 = p_1 + td_1, \; x_2 = p_2 + td_2, \; x_3 = p_3 + td_3, \quad t \in \mathbb{R} \tag{18}$$

• The equation of a line defined by two different points $P(p_1, p_2, p_3)$, $Q(q_1, q_2, q_3)$ can be obtained as above with $\mathbf{d} := \overrightarrow{PQ}$.

• All the above hold exactly in the same way for lines on $\mathbb{R}^n$, for every n.

Vector equation of a line: illustration

Let us draw the line passing from $P(4,2)$ with direction $\mathbf{d} = (1,2)$:



Example:

For example we will determine the vector equation of the line of $\mathbb{R}^3$ passing through the points $P(2,5,-1)$ and $Q(2,3,0)$.

The directional vector is

$$\mathbf{d} := \overrightarrow{PQ} = (2-2, 3-5, 0-(-1)) = (0,-2,1). \tag{19}$$

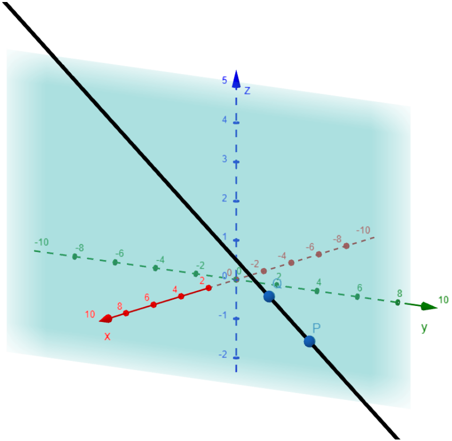
Then every vector $\mathbf{x}$ in the line of P, Q satisfies

$$\mathbf{x} = \mathbf{p} + t\mathbf{d} = (2,5,-1) + t(0,-2,1) = (2, 5-2t, -1+t), \quad t \in \mathbb{R}. \tag{20}$$

$$\text{or} \quad x_1 = 2,\ x_2 = 5-2t,\ x_3 = -1+t, \quad t \in \mathbb{R}. \tag{21}$$

- For $t=0$ we take the point P. For $t=1$ we have the point Q.
- The line defined by P and Q lies on the plane $x_1 = 2$. Therefore it is perpendicular to the x-axis (recall $x_1 = x$) and/or never intersects the yz-plane.

Let's plot it



Recall the dot product, norm and angle

Let $\mathbf{x} = (x_1, \ldots, x_n), \mathbf{y} = (y_1, \ldots, y_n) \in \mathbb{R}^n$.
(i) The **dot product** $\mathbf{x} \cdot \mathbf{y}$ is the **real number**

$$\mathbf{x} \cdot \mathbf{y} := x_1 y_1 + \cdots + x_n y_n =: \sum_{i=1}^{n} x_i y_i. \tag{22}$$

(ii) The **norm** $\|\mathbf{x}\|$ (or the length) of the vector $\mathbf{x}$;

$$\|\mathbf{x}\| := \sqrt{x \cdot x} = \sqrt{x_1^2 + \cdots + x_n^2} = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}. \tag{23}$$

(iii) If $\mathbf{x}, \mathbf{y} \neq \mathbf{0}$, the angle $\theta := (\widehat{\mathbf{x}, \mathbf{y}})$ is defined as the unique number of $[0, \pi]$ such that

$$\cos(\theta) = \frac{\mathbf{x} \cdot \mathbf{y}}{\|\mathbf{x}\| \|\mathbf{y}\|}. \tag{24}$$

Recall the fundamental properties of dot product and norm

- $\mathbf{x} \cdot \mathbf{y} = \mathbf{y} \cdot \mathbf{x}$.
- $\mathbf{x} \cdot (\mathbf{y} + \mathbf{z}) = \mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z}$.
- $(\lambda \mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot (\lambda \mathbf{y}) = \lambda(\mathbf{x} \cdot \mathbf{y})$.
- $\mathbf{0} \cdot \mathbf{x} = 0$.
- $\|\mathbf{x}\| \geq 0$ and $\|\mathbf{x}\| = 0 \Leftrightarrow \mathbf{x} = \mathbf{0}$. [Positivity]
- $\|\lambda \mathbf{x}\| = |\lambda| \|\mathbf{x}\|$. [Homogeneity]
- $\|\mathbf{x} + \mathbf{y}\| \leq \|\mathbf{x}\| + \|\mathbf{y}\|$. [Triangle inequality]
- $|x \cdot y| \leq \|x\| \|y\|$. [Cauchy-Schwarz inequality]
- $\cos(\theta) = 1 \Leftrightarrow \mathbf{x} = \lambda \mathbf{y}$, for some $\lambda > 0$ (parallel and of the same direction); i.e. $\theta = 0$. Similarly $\cos(\theta) = -1 \Leftrightarrow \mathbf{x} = \lambda \mathbf{y}$, for some $\lambda < 0$ (parallel and of the opposite direction); i.e. $\theta = \pi$.
- Two vectors are called **orthogonal** when $\mathbf{x} \cdot \mathbf{y} = 0$. The zero vector is orthogonal to every vector and when $\mathbf{x}, \mathbf{y} \neq \mathbf{0}$, the angle of two vectors is $\pi/2$ if and only if $\mathbf{x} \cdot \mathbf{y} = 0$.
- A vector $\mathbf{v} \in \mathbb{R}^n$ is called a **unit vector** when $\|\mathbf{v}\| = 1$. For every non-zero vector $\mathbf{u}$, the vector $\mathbf{v} := \mathbf{u}/\|\mathbf{u}\|$ is a unit vector of the same direction.

Example

Let $A(0,3,4)$, $B(0,-4,3)$ and $C(1,0,-1)$ and $\mathbf{a},\mathbf{b},\mathbf{c}$ the corresponding vectors. (i) Find the cosine of the angle $\widehat{BOC}$, (ii) show that $\widehat{AOB} = \pi/2$.

(i) Observe that $\widehat{BOC} = \widehat{(\mathbf{b},\mathbf{c})}$. Note also that $\|\mathbf{b}\| = 5$ and $\|\mathbf{c}\| = \sqrt{2}$. We find their dot product

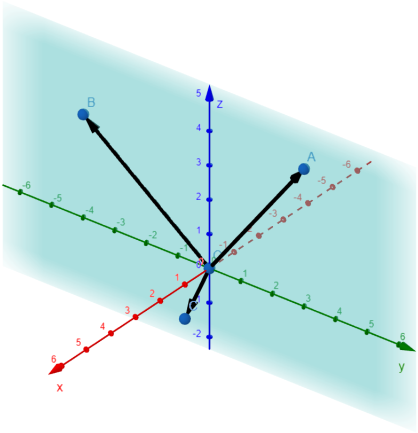
$$\mathbf{b}\cdot\mathbf{c} = 0\cdot 1 + (-4)\cdot 0 + 3\cdot(-1) = -3, \tag{25}$$

thus

$$\cos(\widehat{BOC}) = \cos(\widehat{\mathbf{b},\mathbf{c}}) = \frac{\mathbf{b}\cdot\mathbf{c}}{\|\mathbf{b}\|\|\mathbf{c}\|} = \frac{-3}{5\sqrt{2}}. \tag{26}$$

(ii) As before $\mathbf{a}\cdot\mathbf{b} = 0 - 12 + 12 = 0$, so the vectors are orthogonal; their angle is $\pi/2$.

Let us see the points, the vectors and the yz-plane: $x = 0$, because A and B lie there!



Standard unit vectors

• The standard unit vectors of $\mathbb{R}^n$ are the vectors $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$, where

$$\mathbf{e}_1 = (1, 0, \dots, 0),\ \mathbf{e}_2 = (0, 1, 0, \dots, 0), \dots, \mathbf{e}_n = (0, 0, \dots, 0, 1), \tag{27}$$

($\mathbf{e}_i$ has all coordinates 0, but i-th coordinate which is 1).
• These are *orthogonal* to each other; $\mathbf{e}_i \cdot \mathbf{e}_j = 0,\ \forall i \neq j$.
• Let $\mathbf{v} = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$. Then we can express it as a *unique linear combination* of $\mathbf{e}_1, \dots, \mathbf{e}_n$:

$$\mathbf{v} = v_1\mathbf{e}_1 + \cdots + v_n\mathbf{e}_n. \tag{28}$$

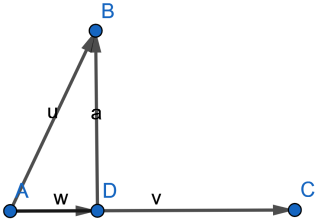
• Geometrically the $\mathbf{e}_i$ vector is the unit vector parallel to the x_i-axis in the positive direction.
• In the literature, when $n = 3$, the vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ of $\mathbb{R}^3$ may appear as $\mathbf{i}, \mathbf{j}, \mathbf{k}$ respectively.

Orthogonal projections

Let $\mathbf{u}, \mathbf{v} \neq \mathbf{0}$. Then **orthogonal projection of u on v** is defined as the vector

$$\mathrm{Proj}_{\mathbf{v}}(\mathbf{u}) := \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v}. \tag{29}$$

$$\text{Then:} \quad \mathrm{Proj}_{\mathbf{v}}(\mathbf{u}) \parallel \mathbf{v}, \quad \mathrm{Proj}_{\mathbf{v}}(\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{v}, \quad (\mathbf{u} - \mathrm{Proj}_{\mathbf{v}}(\mathbf{u})) \cdot \mathbf{v} = 0. \tag{30}$$



In the figure $\mathbf{u} = \overrightarrow{AB}$, $\mathbf{v} = \overrightarrow{AC}$, then $\mathbf{w} = \mathrm{Proj}_{\mathbf{v}}(\mathbf{u}) = \overrightarrow{AD}$ and $\mathbf{a} = \mathbf{u} - \mathrm{Proj}_{\mathbf{v}}(\mathbf{u}) = \overrightarrow{DB}$.

The cross product on $\mathbb{R}^3$

• We restrict on $\mathbb{R}^3$ and let $\mathbf{e}_1 = (1,0,0)$, $\mathbf{e}_2 = (0,1,0)$ and $\mathbf{e}_3 = (0,0,1)$ the standard unit vectors. Let $\mathbf{a} = (a_1, a_2, a_3), \mathbf{b} = (b_1, b_2, b_3) \in \mathbb{R}^3$. The **cross product $\mathbf{a} \times \mathbf{b} \in \mathbb{R}^3$** is the vector defined as

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad \text{(expand the first row of the determinant)} \tag{31}$$

$$= (a_2 b_3 - a_3 b_2)\mathbf{e}_1 + (a_3 b_1 - a_1 b_3)\mathbf{e}_2 + (a_1 b_2 - a_2 b_1)\mathbf{e}_3. \tag{32}$$

• Let $\mathbf{a}, \mathbf{b}, \mathbf{c} \in \mathbb{R}^3$. The real number $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is referred as their **triple product**. It turns out that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} =: \det[\mathbf{a}, \mathbf{b}, \mathbf{c}]. \tag{33}$$

Interpretation and properties

- $\mathbf{0} \times \mathbf{b} = \mathbf{a} \times \mathbf{0} = \mathbf{0}$, for every $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$.
- More generally, when $\mathbf{a}$ and $\mathbf{b}$ are parallel (i.e. $\mathbf{a} = \lambda \mathbf{b}$, for some $\lambda \in \mathbb{R}$), then $\mathbf{a} \times \mathbf{b} = \mathbf{0}$. For instance $\mathbf{a} \times \mathbf{a} = \mathbf{0}$.
- The cross product $\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.
- When $\mathbf{a}$ and $\mathbf{b}$ are not parallel, then $\mathbf{a} \times \mathbf{b}$ is orthogonal to every vector in the plane defined by $\mathbf{a}$ and $\mathbf{b}$.
- $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$ (anticommutativity).
- $\mathbf{e}_1 \times \mathbf{e}_2 = \mathbf{e}_3$, $\mathbf{e}_2 \times \mathbf{e}_3 = \mathbf{e}_1$ and $\mathbf{e}_3 \times \mathbf{e}_1 = \mathbf{e}_2$.
- But $\mathbf{e}_1 \times \mathbf{e}_3 = -\mathbf{e}_2$, etc.
- $\|\mathbf{a} \times \mathbf{b}\| = \|\mathbf{a}\| \|\mathbf{b}\| |\sin(\theta)|$, where $\theta = (\widehat{a, b})$ and therefore it is equal with the area of the parallelogram defined by $\mathbf{a}$ and $\mathbf{b}$.
- $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$.
- $(\lambda \mathbf{a}) \times \mathbf{b} = \mathbf{a} \times (\lambda \mathbf{b}) = \lambda(\mathbf{a} \times \mathbf{b})$.

- **Jacobi identity**:

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) + \mathbf{b} \times (\mathbf{c} \times \mathbf{a}) + c \times (a \times b) = \mathbf{0}. \tag{34}$$

- Clearly when any two of $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are parallel, $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = 0$.
- When there is not any couple of parallel vectors between $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, these are defining a **parallelepiped** having as edges the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. This solid is expressed by the set of the following linear combinations:

$$\mathrm{Par}(\mathbf{a}, \mathbf{b}, \mathbf{c}) = \left\{ \mathbf{u} = \lambda \mathbf{a} + \mu \mathbf{b} + \nu \mathbf{c} : 0 \leq \lambda, \mu, \nu \leq 1 \right\}. \tag{35}$$

The volume of the parallelepiped is the absolute value of the triple product, or of the determinant of the three vectors!

$$\mathrm{Vol}(\mathrm{Par}(\mathbf{a}, \mathbf{b}, \mathbf{c})) = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})| = |\det[\mathbf{a}, \mathbf{b}, \mathbf{c}]| \tag{36}$$

Example

Let $\mathbf{a} = (0,3,4)$, $\mathbf{b} = (0,-4,3)$ and $\mathbf{c} = (1,0,-1)$ as before. Find (i) the orthogonal projection of $\mathbf{a}$ on $\mathbf{c}$, (ii) two vectors $\mathbf{a}_1$ and $\mathbf{a}_2$ such that $\mathbf{a} = \mathbf{a}_1 + \mathbf{a}_2$ and $\mathbf{a}_1$ being parallel with $\mathbf{c}$ and $\mathbf{a}_2$ orthogonal with $\mathbf{c}$, (iii) the cross product of $\mathbf{b}$ and $\mathbf{c}$, (iv) the area of the parallelogram defined by $\mathbf{b}$ and $\mathbf{c}$ (vi) the volume of the parallelepiped defined by $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$.

(i) For the orthogonal projection $\mathbf{a}_1 = \mathrm{Proj}_{\mathbf{c}}(\mathbf{a})$, we take $\mathbf{a} \cdot \mathbf{c} = 0 \cdot 1 + 3 \cdot 0 + 4 \cdot (-1) = -4$. Then

$$\mathbf{a}_1 = \mathrm{Proj}_{\mathbf{c}}(\mathbf{a}) = \frac{\mathbf{a} \cdot \mathbf{c}}{\|\mathbf{c}\|^2}\mathbf{c} = \frac{-4}{2}\mathbf{c} = -2\mathbf{c} = (-2,0,2). \tag{37}$$

(ii) The vector $\mathbf{a}_2$ which is orthogonal with $\mathbf{c}$, is then

$$\mathbf{a}_2 = \mathbf{a} - \mathrm{Proj}_{\mathbf{c}}(\mathbf{a}) = (0,3,4) - (-2,0,2) = (2,3,2). \tag{38}$$

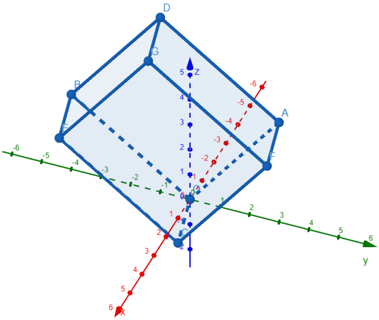
(iii) For the cross product

$$\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -4 & 3 \\ 1 & 0 & -1 \end{vmatrix} = \mathbf{i}(-4)(-1) - \mathbf{j}(-3 \cdot 1) + \mathbf{k}(-(-4) \cdot 1) = (4,3,4). \tag{39}$$

(iv) Area of parallelogram by **b** and **c**$=\|\mathbf{b}\times\mathbf{c}\| = \sqrt{16+9+16} = \sqrt{41}$.
(v) Triple product:

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = (0,3,4)\cdot(4,3,4) = 9+16 = 25. \tag{40}$$

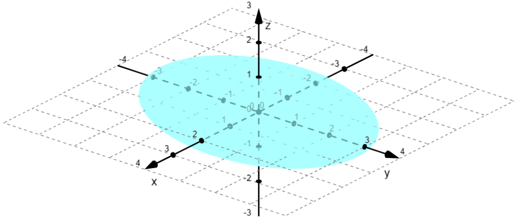
(vi) Volume of parallelepiped$=|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})| = 25$. Note that this solid has edges O, A, B, C and the end points of the vectors $\mathbf{a}+\mathbf{b}$, $\mathbf{b}+\mathbf{c}$, $\mathbf{c}+\mathbf{a}$ and $\mathbf{a}+\mathbf{b}+\mathbf{c}$.



Ellipsoid

The **ellipsoid** (centred at O) with half-axes lengths $\alpha, \beta, \gamma > 0$ is given by

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = 1. \tag{41}$$

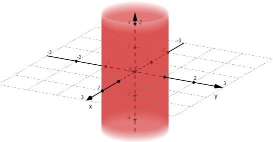


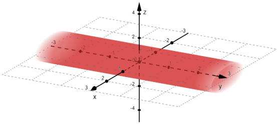
Above we have $\alpha = 2$, $\beta = 3$ and $\gamma = 1$.

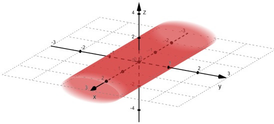
Cylinder

Below we can see the **cylinders** with equations

$$K_z : x^2 + y^2 = 1, \quad K_y : x^2 + z^2 = 1, \quad K_x : y^2 + z^2 = 1. \tag{42}$$



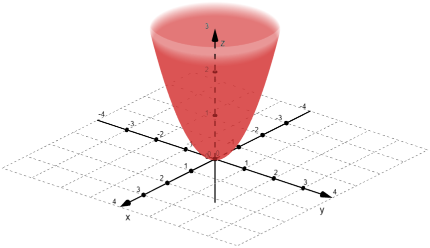


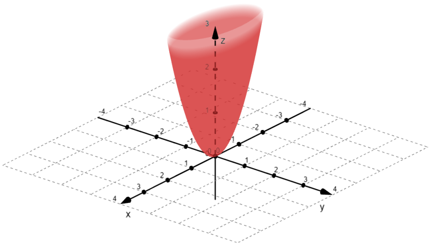


Paraboloid

Below we can see the **paraboloids** with equations

$$P : z = x^2 + y^2 \text{ (standard)}, \quad P_e : z = x^2 + 4y^2 \text{ (elliptic paraboloid)} \tag{43}$$

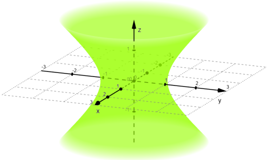


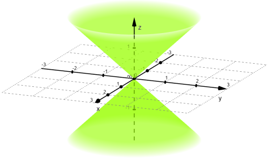


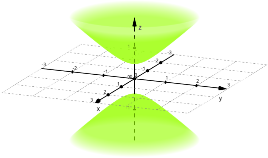
Hyperboloid

Below we can see the **hyperboloids** with equations

$$H_p : x^2 + y^2 - z^2 = 1, \quad H_0 : x^2 + y^2 - z^2 = 0, \quad H_n : x^2 + y^2 - z^2 = -1. \quad (44)$$







the H_0 is mostly referred as a double cone.

Cone

The standard **cone** is given by the equation

$$z = \sqrt{x^2 + y^2}. \tag{45}$$