ST221 Introduction to Statistics

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Example

Toss a coin, get a H or T. Since H and T are equally likely, we write

$$P(H) = P(T) = 1/2$$

1.1 Experiments, sample spaces and events

Formal terminology:

- **Experiment**: procedure where the result cannot be known in advance.

  E.g. Toss a coin, observe the result.
- **Sample space**: Set of all outcomes, denoted by Ω.

  E.g. $\Omega = \{H, T\}$
- **Event**: Any subset of Ω.

  E.g. Event that we get T.

Example

Experiment: roll a die.

1. List the outcomes in the sample space.
2. List the outcomes that makeup the following events:

$$\begin{aligned} A &= \text{event an even number is obtained,} \\ B &= \text{event that a number greater than 4 is obtained.} \end{aligned}$$

Example: Write out the sample space for the following experiments:

1. Toss a coin twice, observe the result.

2. Toss a coin twice, count the number of heads.

3. Toss a coin twice, observe whether the two tosses are the same.

Note:

- Sample spaces may be discrete or continuous.
- A sample space is discrete if it is finite or countable i.e. the elements can be listed even if it is an infinite list.
- In a continuous sample space there are no 'gaps' between the elements and the elements cannot be listed.

1.2 Complements, unions and intersections

Definitions:

- **Complement**: A^c = event that A does not occur.
- **Union**: $A \cup B$ = event that A and/or B occurs.
- **Intersection**: $A \cap B$ = event that both A and B occur.

Venn diagram examples

Example: Back to the roll a die example. Recap:

$$\begin{aligned} \text{A} &= \text{event an even number is obtained,} \\ \text{B} &= \text{event that a number greater than 4 is obtained.} \end{aligned}$$

List the outcomes in:

1. A^c
2. $A \cup B$
3. $A \cap B$

Note: For events A, B and C, the following hold:

- commutative laws:

$$A \cup B = B \cup A$$
$$A \cap B = B \cap A$$

- associative laws:

$$(A \cup B) \cup C = A \cup (B \cup C)$$
$$(A \cap B) \cap C = A \cap (B \cap C)$$

- distributive laws:

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$
$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

1.3 Probability when all outcomes are equally likely

When all outcomes in Ω are equally likely:

$$P(\text{event}) = \frac{\#\text{ of outcomes in the event}}{\#\text{ of outcomes in }\Omega}$$

Example: Back to the roll a die example. Recap:

A = event an even number is obtained,

B = event that a number greater than 4 is obtained.

Find:

1. $P(A)$
2. $P(B)$
3. $P(A \cup B)$
4. $P(A \cap B)$
5. $P(B^c)$

Example

3 letters addressed to Tom, Dick and Harry are randomly placed in three envelopes addressed to Tom, Dick and Harry. List the outcomes in the sample space and find the probability that everyone gets the right letter.

Answer:

If lowercase denotes the letter and order is the envelope:
$\Omega = \{\text{tdh,thd, hdt, htd, dht, dth}\}$

Sample space has 6 equally likely outcomes, so $P(\text{tdh}) = 1/6$.

Example contd.

Find the probability that:

1. Tom gets the right letter
2. Exactly one person gets the right letter
3. Someone gets the right letter
4. No one gets the right letter

Example

A family has three children. Assume that the probability of a boy and of a girl is 0.5.

1. List the outcomes in the sample space.
2. What is the probability that all three children are girls?
3. What is the probability that they have two girls and one boy?
4. What is the probability of at least one girl?

1.4 Probability function

Example: Suppose in a college we can break down the student population into the following groups:

	female	male
arts	30%	30%
not arts	25%	15%

Experiment: One student is randomly selected. Let:

F = event that the selected student is female

A = event that the selected student studies arts.

1.4 Probability function

Example contd.

	female	male
arts	30%	30%
not arts	25%	15%

Find:

1. $P(F)$
2. $P(F^c)$
3. $P(A)$
4. $P(A^c)$
5. $P(F \cap A)$
6. $P(F^c \cap A)$
7. $P(F \cup A)$

What is P, the probability function?

In an applied sense, this is a measure of the likelihood or chance that something will happen, e.g. that a (random) event will occur.

When dealing with experiments that are random and well-defined, one can consider probability to be the relative frequency 'in the long run' of outcomes. E.g. Tossing a coin - repeat the experiment infinitely many times.

Axiomatic definition: there are rules that tell us what is a valid probability function.

Axioms of probability

1. For any event A, $0 \leq P(A) \leq 1$.

   i.e. probability must be between 0 and 1.

2. Let $a =$ outcome, $A =$ event. Then

$$P(A) = \sum_{a \in A} P(a).$$

   i.e. probability of event A is the sum of the probabilities of all outcomes in A.

3. $$P(\Omega) = \sum_{a \in \Omega} P(a) = 1$$

   i.e. the sum of the probabilities of all outcomes must be 1.

Example: Suppose there are 4 possible outcomes to an experiment,

i.e. $\Omega = \{a_1, a_2, a_3, a_4\}$.

We are told $P(a_1) = P(a_2) = 0.2$, $P(a_3) = 0.5$.

What is $P(a_4)$?

Example: Suppose there are 4 possible outcomes to an experiment,

i.e. $\Omega = \{a_1, a_2, a_3, a_4\}$.

We are told $P(a_1) = P(a_2) = 0.2$, $P(a_3) = P(a_4) = 0.5$.

These are **invalid** assignments!

1.5 Probability for complement, union and intersection

All outcomes belong to A or A^c. Therefore:

$$P(A^c) = P(\Omega) - P(A) = 1 - P(A).$$

$A \cup B$ is composed of outcomes in A or B or both. $P(A) + P(B)$ counts the events that are in both A and B twice. So:

$$P(A \cup B) = P(A) + P(B) - P(A \cap B).$$

Q: $P(A^c \cap B^c)$=?

Example: A house needs to be re-roofed for spring. To do this, a dry windless day is needed. The probability of getting a dry day is 0.7, a windy day is 0.4 and a wet and windy day is 0.2. On a randomly selected day, what is the probability that the house can be re-roofed?

Answer:

Experiment: weather on a randomly selected day.

Let D denote the event that the randomly selected day is dry and W that it is windy.

1.6 Sampling problems

Sampling can happen with or without replacement.

Sampling without replacement

Example
A box has 5 tickets labelled 1, 2, 3, 4, 5. Suppose that 2 tickets are sampled without replacement.
The outcomes are:

	(1,2)	(1,3)	(1,4)	(1,5)
(2,1)		(2,3)	(2,4)	(2,5)
(3,1)	(3,2)		(3,4)	(3,5)
(4,1)	(4,2)	(4,3)		(4,5)
(5,1)	(5,2)	(5,3)	(5,4)	

There are 20 equally likely outcomes.

Example contd.

1. What is the probability of a 3 followed by a 4?

2. What is the probability of a 3, 4 in any order?

If we do not keep track of the order in which the tickets are sampled there are 10 unordered outcomes, each of which are equally likely.

In general, if a box has n tickets and a sample of size k is taken without replacement, then there are

$$C_k^n = \binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)...(n-k+1)}{k!}$$

equally likely unordered outcomes and

$$P_k^n = \binom{n}{k} k! = \frac{n!}{(n-k)!} = n(n-1)...(n-k+1)$$

equally likely ordered outcomes.

Sampling with replacement

Example

A box has 5 tickets labelled 1, 2, 3, 4, 5. Suppose that 2 tickets are sampled with replacement.

The outcomes are:

$$\begin{array}{ccccc}
(1,1) & (1,2) & (1,3) & (1,4) & (1,5) \\
(2,1) & (2,2) & (2,3) & (2,4) & (2,5) \\
(3,1) & (3,2) & (3,3) & (3,4) & (3,5) \\
(4,1) & (4,2) & (4,3) & (4,4) & (4,5) \\
(5,1) & (5,2) & (5,3) & (5,4) & (5,5)
\end{array}$$

There are 25 possible ordered outcomes each of which are equally likely.

Example contd.

1. What is the probability of a 3 followed by a 4?

2. What is the probability of a 3, 4 in any order?

If we do not keep track of the order in which the tickets are sampled there are 15 unordered outcomes, each of which are NOT equally likely.

In general, if a box has n tickets and a sample of size k is taken with replacement, then there are

$$n^k$$

equally likely ordered outcomes and there are

$$\binom{n+k-1}{k}$$

unordered outcomes which are not equally likely.

Example

Suppose a box has 100 chips of which 10 are defective. If 4 chips are selected at random from the box:

1. What is the probability that the sample has no defectives?
2. What is the probability that there are exactly 2 defectives in the sample?

Answer
We can think of the chips as being numbered 1,...,100 where 1,...,10 are defective and 11,...,100 are okay.

Example

Suppose 5 cards are dealt at random from a well-shuffled deck of 52 cards. Note: The hand can be thought of as a random sample of size 5 from 52 with no replacement. Compute the following probabilities.

1. P(4 aces)

2. P(3 aces and 2 kings)

3. P(4 of a kind)

Example

Suppose you buy a lotto ticket. Assume 6 numbers drawn without replacement from 1,...,45.

1. What is the probability of winning the jackpot?
2. What is the probability of winning if you buy two tickets?
3. What is the probability of matching 5 of the 6 numbers?

Example

The Birthday Problem. In a room of 30 students, what is the probability that at least two people share a birthday?

1. How many possibilities are there for the birthdays for the 30 students?

2. How many possibilities are there if no two students share a birthday?

3. P(no matching birthday)?

4. P(at least two people matching)?

Additional notes on permutations

1. As we know already, the number of k-element permutations of a set containing n objects is given by

$$P_k^n = \frac{n!}{(n-k)!}$$

If $k = n$ this becomes

$$P_n^n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$$

2. The number of distinguishable permutations of n objects of r different types, where n_1 are alike, n_2 are alike,..., n_r are alike and $n = n_1 + ... + n_r$ is

$$\frac{n!}{n_1!n_2!...n_r!}$$

Example

In how many ways can we paint 11 offices so that four of them will be painted green, three yellow, two white and the remaining two pink?

1.7 Independent events

Definition: Two events A and B are **independent** if

$$P(A \cap B) = P(A)P(B).$$

Example: A card is drawn at random from a 52 card deck. Let:

A = event card is an ace,

B = event card is a diamond.

Are A and B independent?

Answer:

Example: A box has 3 red and 2 black balls. Two are drawn at random without replacement. Let:

A = event 1st ball is red,

B = event 2nd ball is red.

Are A and B independent?

Answer:

Suppose balls are numbered 1 to 5. The first three are red.

We list ordered outcomes since the events A and B distinguish first and second draw.

Example contd.

There are $5 \times 4 = 20$ equally likely ordered outcomes.

$$\begin{aligned}\Omega = \{ \quad & (1,2),(1,3),(1,4),(1,5),\\ & (2,1),(2,3),(2,4),(2,5),\\ & (3,1),(3,2),(3,4),(3,5),\\ & (4,1),(4,2),(4,3),(4,5),\\ & (5,1),(5,2),(5,3),(5,4)\}\end{aligned}$$

Example: If the two balls are drawn with replacement, there are $5 \times 5 = 25$ equally likely ordered outcomes.

1.8 Conditional probability

Definition: Given two events A and B, the **conditional probability** of A given B is defined as

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

Therefore:

$$\begin{aligned} P(A \cap B) &= P(A|B)P(B) \\ &= P(B|A)P(A). \end{aligned}$$

Example: Back to the student example

	female	male
arts	30%	30%
not arts	25%	15%

A random student is selected. Let:

F = event that the selected student is female.

A = event that the selected student studies arts.

Then:

$P(F) = 0.55$

$P(A) = 0.6$

$P(F \cap A) = 0.3$

Probability tree diagram

Questions: A student is chosen at random.

1. If the student studies arts, what is the probability that they are female?
2. If the student is female, what is the probability that they study arts?
3. If the student is female, what is the probability that they don't study arts?
4. If the student is male, what is the probability that they study arts?

Definition: Two events A and B are **independent** if $P(A|B) = P(A)$.

i.e. knowing B has occurred does not change the probability of A.

Back to the student example: Are the events F and A independent?

Note: If A and B are independent, then A^c and B are independent and A^c and B^c are independent.

Proof:

If A and B are independent then:

$$\begin{aligned} P(A|B) &= P(A) \\ 1 - P(A|B) &= 1 - P(A) \\ P(A^c|B) &= P(A^c) \end{aligned}$$

$\Rightarrow A^c$ and B are independent.

Then:

$$\begin{aligned} P(B|A^c) &= P(B) \\ 1 - P(B|A^c) &= 1 - P(B) \\ P(B^c|A^c) &= P(B^c) \end{aligned}$$

$\Rightarrow A^c$ and B^c are independent.

The Multiplicative Rule
If A and B are independent, then

$$P(A \cap B) = P(A)P(B)$$

The General Multiplicative Rule
In general,

$$P(A \cap B) = P(A)P(B|A)$$

or

$$P(A \cap B) = P(B)P(A|B)$$

This general rule does not require A and B to be independent. But remember that if A and B are independent, then $P(B|A) = P(B)$ and $P(A|B) = P(A)$.

1.9 Mutually exclusive events

Definition: Two events A and B are **mutually exclusive**, or disjoint, if $A \cap B = \emptyset$.

Venn diagram:

Does this mean A and B are independent?

Definition: A **partition** of the sample space Ω is a collection of mutually exclusive events whose union is Ω.

Any set or event can be partitioned.

Definition: If $B_1, ..., B_k$ form a partition of Ω, then $A \cap B_1, ..., A \cap B_k$ form a partition of A.

The partition theorem: Let $B_1, ..., B_k$ form a partition of Ω. Then for any event A:

$$P(A) = \sum_{i=1}^{k} P(A \cap B_i)$$

Example: In a survey:

- 13% of respondents had children and had a large car
- 12% of respondents did not have children and had a large car

Find the probability that a respondent picked at random had a large car.

Answer:

Example

In a game show, there are 3 doors, behind two doors are goats and behind the other door is a car.

You as the contestant pick and door and announce your choice but the door remains closed. The host (who knows where the car is) opens one of the remaining doors to reveal a goat. You can stick with your original choice or switch to the other closed door.

Question: Should you stick or switch?

Example

Travelling by train with sister – without tickets!

Caught by inspector

Inspector has powers to administer on-the-spot punishment.

- 9 chocolates; 3 with deadly poison
- Must select and eat one

Should you go first or second?

1.10 Bayes Theorem

From the definition of conditional probability, we can derive:

$$\begin{aligned} P(A|B) &= \frac{P(A \cap B)}{P(B)} \\ &= \frac{P(B|A)P(A)}{P(B)} \\ &= \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|A^c)P(A^c)} \end{aligned}$$

This result is known as the **Bayes theorem**.

Example: Suppose in a certain population, 50% are male and 50% are female. 10% of males are colourblind as are 1% of females. What is the probability that a randomly selected colourblind person from this population is male?

Answer:
Let:

$$M = \text{event randomly selected person is male}$$

$$C = \text{event randomly selected person is colourblind.}$$

Example:

- A disease infects 1 out of 1000 people.
- There is a test (not perfect, if a person has the disease, the test comes back positive 99% of the time, but gives 2% of false positives).
- You tested positive.
- What are the chances you actually have the disease?

Answer:
Let:

$$D = \text{event you have the disease}$$

$$T = \text{event you tested positive}$$

Example

- Travelling by train with sister – without tickets!
- Caught by inspector
- Inspector has powers to administer on-the-spot punishment:

- 9 chocolates, 3 with deadly poison
- You must select and eat one
- Should you go **first** or **second**?

Answer: