Functions of several variables

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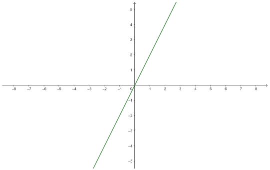
Contents

Functions $f : \mathbb{R}^n \to \mathbb{R}$

• A **function**, $f : \mathbb{R}^n \supset D \to \mathbb{R}$ is a rule which maps each element $\mathbf{x} \in D$ to a unique number $f(\mathbf{x}) \in \mathbb{R}$. The set $D = D_f$ is referred as the **domain** of f. The set $C_f := \{(\mathbf{x}, f(\mathbf{x}) : \mathbf{x} \in D\} \subset \mathbb{R}^{n+1}$ is called the **graph** of f.
• We will denote by $z = f(x, y)$ the rule of a function of two variables (x, y) and by $w = f(x, y, z)$ the case of three variables (x, y, z).
• **Example 1:** Let us find the domain of the following functions on $\mathbb{R}^2$:

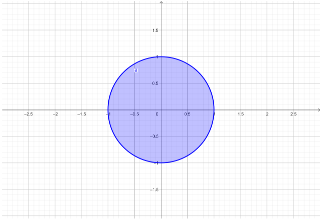
$$f(x,y) = \frac{x}{2x-y}, \quad g(x,y) = \sqrt{1-x^2-y^2}, \quad h(x,y) = \ln(y-x^2). \qquad (1)$$

We must restrict $\mathbb{R}^2$ based on the formulae of the three functions.
• D_f : Here it must be $2x - y \neq 0$. Therefore the domain of f contains all $\mathbb{R}^2$ except the points in the line $2x - y = 0 \Leftrightarrow y = 2x$ below



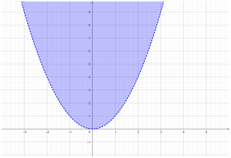
i.e. $D_f = \{(x, y) \in \mathbb{R}^2 : y \neq 2x\}$.

• D_g : Here it must be $1 - x^2 - y^2 \geq 0 \Leftrightarrow x^2 + y^2 \leq 1$. Therefore the domain of g contains all $\mathbb{R}^2$ inside and on the circumference of the circle $x^2 + y^2 = 1$.



i.e. $D_g = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$

• D_h : Here it must be $y - x^2 > 0 \Leftrightarrow y > x^2$. Therefore the domain of h contains all $\mathbb{R}^2$ over the parabola $y = x^2$.



i.e. $D_h = \{(x, y) \in \mathbb{R}^2 : y > x^2\}$.

Values

For finding the value $f(x, y)$ of a function $f : \mathbb{R}^2 \to \mathbb{R}$, we just replace a given x and y in its formula. In the Example 1,

$$f(2, -4) = \frac{2}{2 \cdot 2 - (-4)} = \frac{2}{8} = \frac{1}{4}. \tag{2}$$

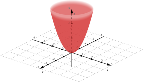
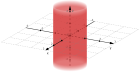
$$g\left(\frac{-1}{2}, \frac{1}{\sqrt{2}}\right) = \sqrt{1 - \left(\frac{-1}{2}\right)^2 - \left(\frac{1}{\sqrt{2}}\right)^2} = \sqrt{1 - \frac{1}{4} - \frac{1}{2}} = \frac{1}{2} \tag{3}$$

$$h(-3, 10) = \ln(10 - (-3)^2) = \ln(10 - 9) = \ln 1 = 0. \tag{4}$$

Example 2:

Find/describe the domains of the functions on $\mathbb{R}^3$:

$$f(x,y,z) = \frac{x}{y-z},\ g(x,y,z) = \sqrt{z - x^2 - y^2},\ h(x,y,z) = \ln(x^2 + y^2 - 1). \tag{5}$$

D_f is entire $\mathbb{R}^3$ without the plane $y = z$; $D_f = \{(x,y,z) \in \mathbb{R}^3 : y \neq z\}$.
D_g contains all points with $z \geq x^2 + y^2$. Which is the points higher than or on the paraboloid $z = x^2 + y^2$; *inside* according our intuition  .
$D_h = \{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 > 1\}$ the *exterior* of the cylinder $K_z : x^2 + y^2 = 1$  .

Level sets

Let $f : \mathbb{R}^n \supset D \to \mathbb{R}$ and $c \in \mathbb{R}$. The set

$$L_c(f) := \{\mathbf{x} \in D : f(\mathbf{x}) = c\} \tag{6}$$

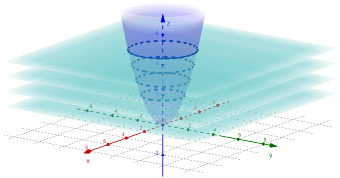
is referred as the level set of value c of f. For $n = 2$, $L_c(f)$ is called a level curve and for $n = 3$ a level surface.
In the case of $n = 2$, the level curve $L_c(f)$ consists of the projection in $\mathbb{R}^2$ of the intersection of the graph C_f with the horizontal plane $z = c$.

Example 3:

Find/describe the level curves of the function

$$f(x,y) = x^2 + y^2. \tag{7}$$

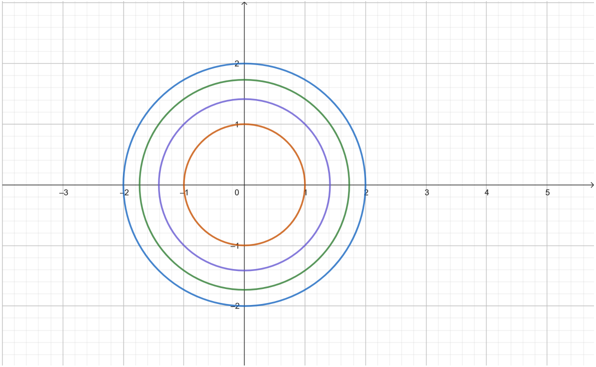
C_f is a paraboloid and when intersecting with $z = c$, $c > 0$ we get circles.



here $c \in \{1, 2, 3, 4\}$.

Indeed, let $c > 0$ and set $f(x,y) = c \Leftrightarrow x^2 + y^2 = \sqrt{c}^2$, which is a circle with center $O(0,0)$ and radius $\sqrt{c}$. For $c = 0$, $L_0(f) = \{O\}$, while for $c < 0$, $L_c(f) = \emptyset$.

The level curves $L_c(f)$ of the paraboloid $f(x, y) = x^2 + y^2$, for $c \in \{1, 2, 3, 4\}$ are as follows:



Why do we need level curves?

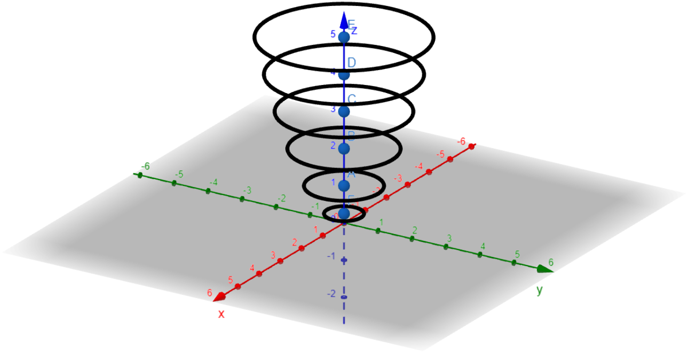
- Let $f : \mathbb{R}^2 \to \mathbb{R}$. Then its graph is a surface $C_f = \{(x, y, z) \in \mathbb{R}^3 : z = f(x, y)\}$. We need to determine its points. As we saw in the beginning of this module, for clarifying the location of a point $P(x, y, z)$, we must have its projection $Q(x, y, 0)$ at the xy-plane and its projection $J(0, 0, z)$ at z-axis.

- When fixing $z = c$, the level curve $L_c(f)$ contains exactly the points (x, y) such that $f(x, y) = z = c$, that is the projections at xy-plane of the points of the graph that are in high $z = c$; i.e. projected at $J(0, 0, c) \in z$-axis.

- By doing so for every $c \in \mathbb{R}$, we are building the graph of f.

We are applying this technique for building the paraboloid $z = f(x, y) = x^2 + y^2$ of our previous example:

The points of the paraboloid with $z = c$, $c \in \{1/4, 1, 2, 3, 4, 5\}$:

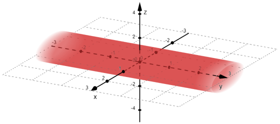


Example 4:

Find/describe the level surfaces of the function

$$f(x,y,z) = x^2 + z^2, \quad (x,y,z) \in \mathbb{R}^3. \tag{8}$$

Let $c > 0$ and set $f(x,y,z) = c \Leftrightarrow x^2 + z^2 = \sqrt{c}^2$, which is a cylinder with

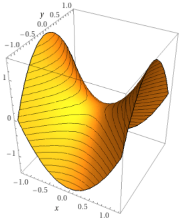


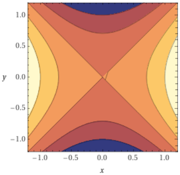
radius $\sqrt{c}$ revoluted around y-axis .
For $c = 0$, $L_0(f) = \{O\}$, while for $c < 0$, $L_c(f) = \emptyset$.

Example 5:

Plot the level curves of the function $f(x, y) = x^2 - y^2$.

Let $c > 0$ and set $f(x, y) = c \Leftrightarrow x^2 - y^2 = \sqrt{c}^2 \Leftrightarrow \frac{x^2}{\sqrt{c}^2} - \frac{y^2}{\sqrt{c}^2} = 1$, which is a hyperbola. Similarly when $c < 0$ we get the hyperbola $\frac{y^2}{\sqrt{c}^2} - \frac{x^2}{\sqrt{c}^2} = 1$.
When $c = 0$ we get $x^2 = y^2 \Leftrightarrow y = \pm x$, which are the two orthogonal lines $y = x$ and $y = -x$.





The surface $z = x^2 - y^2$ is called a hyperbolic paraboloid because of its level curves.

Limits: Definition

A fundamental notion in Calculus is the one of the limit of a function.

Definition

Let $f : \mathbb{R}^n \supset D \to \mathbb{R}$ and let $\mathbf{x}_0 \in \mathbb{R}^n$ such that $B(\mathbf{x}_0, \delta)^* := \{\mathbf{x} \in \mathbb{R}^n : 0 < \|\mathbf{x} - \mathbf{x}_0\| < \delta\} \subset D$, for some $\delta > 0$. When while the independent variable $\mathbf{x}$ is approaching $\mathbf{x}_0$, then its values $f(\mathbf{x})$ approach a number $L \in \mathbb{R}$, we say that the **limit of f at $\mathbf{x}_0$ is L** and we write

$$\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = L. \tag{9}$$

• Other wording is: f *tends to* L, when $\mathbf{x}$ tends to $\mathbf{x}_0$;

$$f(\mathbf{x}) \to L, \quad \text{when } \mathbf{x} \to \mathbf{x}_0. \tag{10}$$

"Tends" is often replaced by "approaches" or "converges".
• When $f(\mathbf{x})$ becomes larger (smaller) than any positive (negative) real number $M > 0$, while $\mathbf{x} \to \mathbf{x}_0$, we say that it tends to ∞ $(-\infty)$ and we write

$$\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x}) = +\infty \quad (-\infty). \tag{11}$$

Limits: Remarks and properties

• It is not assumed that f is defined at $\mathbf{x}_0$ itself, rather than just *around it*; it must be $B(\mathbf{x}_0, \delta)^* \subset D_f$, for some $\delta > 0$.
• If the $\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x})$ exists, then it is *unique*.
• If $\lim_{\mathbf{x} \to \mathbf{x}_0} f_i(\mathbf{x}) = L_i$, $i = 1, 2$, then

$$\lim_{\mathbf{x} \to \mathbf{x}_0} (f_1(\mathbf{x}) + f_2(\mathbf{x})) = L_1 + L_2, \quad \text{similarly for } -, \cdot, : \tag{12}$$

and the same for scalar multiples, powers, roots, composition, under the well-definiteness of every operation applied.
• If $\lim_{\mathbf{x} \to \mathbf{x}_0} f(\mathbf{x})$ exists and it's real, then f is *bounded* in a neighbourhood of $\mathbf{x}_0$; there exists $K > 0$ such that for some $\delta > 0$:

$$|f(\mathbf{x})| < K, \quad \text{for every } \mathbf{x} \in D_f \cap B(\mathbf{x}_0, \delta) = \{\mathbf{x} \in D_f : \|\mathbf{x} - \mathbf{x}_0\| < \delta\}. \tag{13}$$

When the limit is $\pm\infty$, the f is unbounded in a neighbourhood of $\mathbf{x}_0$; for every $K > 0$, there exists $\delta > 0$:

$$|f(\mathbf{x})| > K, \quad \text{for every } \mathbf{x} \in D_f \cap B(\mathbf{x}_0, \delta)^*. \tag{14}$$

Limits: Remarks and properties

• When $\mathbf{x}_0 \in D_f$ and $\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = f(\mathbf{x}_0)$, we say that f is **continuous at $\mathbf{x}_0$**. When f is continuous in every point of its domain, then it is referred simply as **continuous**.

• Polynomial, trigonometric, exponential, logarithmic functions, are continuous (in their domains) and then so do the several operations between them in the corresponding domains.

• When $\mathbf{x} \to \mathbf{x}_0$, then $x_i \to x_{0i}$, for every $i = 1, \ldots, n$ and $\|\mathbf{x} - \mathbf{x}_0\| \to 0$.

• When

$$g_1(\mathbf{x}) \leq f(\mathbf{x}) \leq g_2(\mathbf{x}), \quad \text{around } \mathbf{x}_0 \text{ and } \lim_{\mathbf{x}\to\mathbf{x}_0} g_i(\mathbf{x}) = L \in \mathbb{R},\ i = 1, 2, \tag{15}$$

then the limit of f at $\mathbf{x}_0$ exists and it is equal with L;

$$\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = L. \tag{16}$$

• Finally (applying the above) we have the equivalence

$$\lim_{\mathbf{x}\to\mathbf{x}_0} f(\mathbf{x}) = 0 \Leftrightarrow \lim_{\mathbf{x}\to\mathbf{x}_0} |f(\mathbf{x})| = 0. \tag{17}$$

These properties will be crucial for finding limits.

Example 6:

Find the limit of the function

$$f(x,y,z) = x^2 - e^y + \ln(z-x) \quad \text{at } (1,0,2). \tag{18}$$

Things are simple here, as the function is continuous, thanks to the above discussions

$$\lim_{(x,y,z)\to(1,0,2)} f(x,y,z) = f(1,0,2) = 1^2 - e^0 + \ln(2-1) = 1 - 1 + 0 = 0. \tag{19}$$

Example 7:

Show that the limit

$$\lim_{(x,y)\to(0,0)} \left(\frac{x^3}{x^2+y^2} + e^{x+y} \right) = 1. \tag{20}$$

S1. We look at the exponential term first which obviously gives

$$\lim_{(x,y)\to(0,0)} e^{x+y} = e^{0+0} = 1. \tag{21}$$

S2. About the quotient of polynomials $\frac{x^3}{x^2+y^2}$, I cannot replace $x = 0$ and $y = 0$, as the function is not defined at the point $(0,0)$. Moreover, when $(x,y) \to (0,0)$ we have

$$\lim_{(x,y)\to(0,0)} x^3 = 0 = \lim_{(x,y)\to(0,0)} (x^2+y^2); \quad \text{limit of the form } \left(\frac{0}{0}\right). \tag{22}$$

S3. Let us bound from above the absolute value of the fraction. We start with an obvious but extremely useful inequality:

$$\boxed{|x| = \sqrt{x^2} \leq \sqrt{x^2+y^2}}, \tag{23}$$

with the right hand term still tending to 0!
We use this inequality to extract

$$0 \leq \left|\frac{x^3}{x^2+y^2}\right| = \frac{|x^3|}{|x^2+y^2|} = \frac{|x|^3}{x^2+y^2} \tag{24}$$

$$\leq \frac{\sqrt{x^2+y^2}^3}{x^2+y^2} = \frac{(x^2+y^2)\sqrt{x^2+y^2}}{x^2+y^2} = \sqrt{x^2+y^2}. \tag{25}$$

But of course

$$\lim_{(x,y)\to(0,0)} 0 = 0 = \lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2}, \tag{26}$$

therefore

$$\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = 0 \Longrightarrow \lim_{(x,y)\to(0,0)} \left(\frac{x^3}{x^2+y^2} + e^{x+y}\right) = 0+1 = 1. \tag{27}$$

Methodology

The above ideas may help us find several limits like in Example 7.
In S1 we clarified where the "easy" parts of the function are tending.
In S2 we detected the interesting (0/0) part; denote here as $f_1(x, y)$.
In S3 we proved that

$$0 \leq |f_1(x, y)| \leq g_2(x, y) \to 0, \tag{28}$$

for some function g_2. This meant that $|f_1(x, y)| \to 0$, therefore $f_1(x, y) \to 0$.
The last combined with the first step, ends the exercise.

As we understand the main trick was the tight bound in (28). Let us see tools that we can use in this direction:

- We have $(x, y) \to (0, 0)$, which means that $x, y \to 0$. Also $x^\alpha, y^\alpha \to 0$, for every $\alpha > 0$.
- Moreover $|x|, |y| \leq \sqrt{x^2 + y^2} \to 0$.
- Other inequalities such as $|sin(x)| \leq |x|$ could be used to confirm the bound (28).
- Maybe some direct algebra may help to avoid the (0/0)-indeterminacy, as we will see in the next

Example 8:

Show that the limit

$$\lim_{(x,y)\to(0,0)} \left(\frac{x^2 - y^2}{\sqrt{x} - \sqrt{y}} - x^2 + 2\cos(y) \right) = 2. \tag{29}$$

S1. We look at the safe term first which obviously gives

$$\lim_{(x,y)\to(0,0)} (-x^2 + 2\cos(xy)) = -0^2 + 2\cos(0 \cdot 0) = 2. \tag{30}$$

S2. About the quotient $f_1(x,y) := \frac{x^2 - y^2}{\sqrt{x} - \sqrt{y}}$, obviously is of the undetermined form $(0/0)$.

S3. Although some elementary algebra can be applied:

$$f_1(x,y) = \frac{x^2 - y^2}{\sqrt{x} - \sqrt{y}} = \frac{(x-y)(x+y)(\sqrt{x}+\sqrt{y})}{(\sqrt{x}-\sqrt{y})(\sqrt{x}+\sqrt{y})} \tag{31}$$

$$= \frac{(x-y)(x+y)(\sqrt{x}+\sqrt{y})}{x-y} = (x+y)(\sqrt{x}+\sqrt{y}), \tag{32}$$

which is not in an undetermined form any more and gives

$$\lim_{(x,y)\to(0,0)} f_1(x,y) = \lim_{(x,y)\to(0,0)} (x+y)(\sqrt{x}+\sqrt{y}) = 0 \tag{33}$$

$$\Longrightarrow \lim_{(x,y)\to(0,0)} \left(\frac{x^2 - y^2}{\sqrt{x} - \sqrt{y}} - x^2 + 2\cos(y) \right) = 2. \tag{34}$$

Example 9:

Show that the function $f : \mathbb{R}^3 \to \mathbb{R}$ with formula

$$f(x,y,z) = \begin{cases} \frac{(x+y)^2 - 2z^2}{\sqrt{x^2+y^2+z^2}}, & (x,y,z) \neq (0,0,0) \\ 1, & (x,y,z) = (0,0,0) \end{cases} \tag{35}$$

is **not continuous** at the origin.

We have $f(0,0,0) = 1$ and we are looking for the limit $\lim_{(x,y,z)\to(0,0,0)} f(x,y,z)$.
We immediately note that it is a limit of an undetermined $(0/0)$ form.
We are trying to get a tight upper bound for $|f(x,y,z)|$.
By the triangle inequality of the absolute value we take

$$|(x+y)^2 - 2z^2| \leq |(x+y)^2| + |2z^2| = (|x+y|)^2 + 2z^2 \tag{36}$$

$$\leq (|x| + |y|)^2 + 2z^2 = x^2 + 2|x||y| + y^2 + 2z^2. \tag{37}$$

Recall now that $|x|, |y| \leq \sqrt{x^2+y^2+z^2}$, thus

$$|x||y| \leq \sqrt{x^2+y^2+z^2}\sqrt{x^2+y^2+z^2} = x^2+y^2+z^2. \tag{38}$$

Also, obviously

$$x^2+y^2+2z^2 \leq 2(x^2+y^2+z^2). \tag{39}$$

All the above together imply

$$0 \leq |f(x,y,z)| = \frac{|(x+y)^2-2z^2|}{\sqrt{x^2+y^2+z^2}} \leq \frac{4(x^2+y^2+z^2)}{\sqrt{x^2+y^2+z^2}} \tag{40}$$

$$= 4\sqrt{x^2+y^2+z^2} \to 0, \tag{41}$$

and therefore $\lim_{(x,y,z)\to(0,0,0)} f(x,y,z) = 0 \neq 1 = f(0,0,0)$ so the function f is not continuous at the origin.

Limits through paths

- Let $f : \mathbb{R} \to \mathbb{R}$ a single variable function and $x_0 \in \mathbb{R}$. The approach of x to x_0 could happen in just two possible ways; from the left ($x \to x_0^-$) and the right ($x \to x_0^+$), which is exactly through the rays $(-\infty, x_0)$ and $(x_0, +\infty)$.

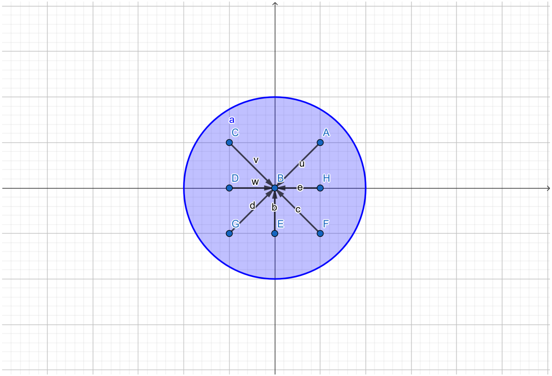
- When $f : \mathbb{R}^n \to \mathbb{R}$ and $\mathbf{x}_0 \in \mathbb{R}^n$ for some $n \geq 2$, this approach can happen in infinity many ways, referred as **paths**.

- If the limit exists, it is unique. Thus if we find two different paths giving different limit values, then the limit does not exist!
- This is our general methodology for proving the non-existence of limits!

- On the other hand, if a large family of paths give a particular limit value, then this is a good indication (of course not a proof!!) that this value has some chances to be the limit of the function.

Paths to the origin

In the next plot we present several paths tending to $O(0,0) \in \mathbb{R}^2$



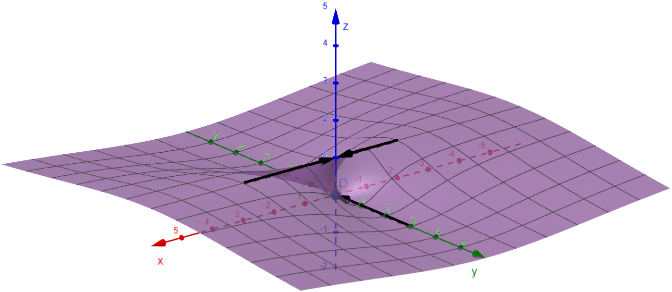
Algebraically the paths can be of the form $x = 0$, $y = 0$,

$$y = \lambda x, \quad \lambda \in \mathbb{R} \tag{42}$$

or another way such as $y = \lambda x^2$ etc. Similarly for any other point $\mathbf{x}_0 \in \mathbb{R}^n$.

Non-existence of a limit

In the next plot we present the behaviour of the function $f(x,y) = x^2/(x^2+y^2)$, when (x,y) tending to $O(0,0) \in \mathbb{R}^2$



Through the y axes, the values of f are $f(0,y) = 0 \to 0$.
Through the x axes, the values are $f(x,0) = 1 \to 1 \neq 0$, and the limit does not exist.

Example 10:

Show that the limit of the function

$$f(x,y) = \frac{2x^2 - y^2}{x^2 + y^2} \tag{43}$$

at the origin **does not exist**.

The simplest possible paths are the axes. We use these here.
(i) Let $P_y:\ x = 0$, the path of y-axis. Then

$$\lim_{(x,y)\to(0,0);x=0} f(x,y) = \lim_{y\to 0} f(0,y) = \lim_{y\to 0} \frac{2\cdot 0^2 - y^2}{0^2 + y^2} = -1. \tag{44}$$

Remark: if the limit exists, then it must be -1. If I find another path giving a different answer, then the limit just doesn't exist.
(ii) Let $P_x:\ y = 0$, which gives $f(x,0) = 2 \to 2$, i.e. the limit does not exist.

Example 11:

Show that the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2 - y^3}{xy} \qquad (45)$$

does not exist.

Let $f(x,y) = \frac{x^2-y^3}{xy}$.
Here the y-axis and x-axis paths are not applicable, because f is not defined on the axes! I must find an alternative.
Then next simplest paths are the lines $P_\lambda : y = \lambda x$, approaching the origin.
Let $\lambda \in \mathbb{R}^*$ and $(x,y) \in P_\lambda$; $y = \lambda x$, then

$$\lim_{(x,y)\to(0,0),y=\lambda x} f(x,y) = \lim_{x\to 0} f(x,\lambda x) = \lim_{x\to 0} \frac{x^2-\lambda^3x^3}{\lambda x^2} \tag{46}$$

$$= \lim_{x\to 0} \frac{1-\lambda^3 x}{\lambda} = \frac{1}{\lambda}. \tag{47}$$

The limit is λ-depended, i.e. depends on the path, which breaks the uniqueness of the limit and therefore the limit does not exist and the claim has been proved.

If I want to see further details, I choose $\lambda_1 = 1$ and $\lambda_2 = 2$. From the above we have that

$$\lim_{(x,y)\to(0,0),y=x} f(x,y) = 1 \neq \frac{1}{2} = \lim_{(x,y)\to(0,0),y=2x} f(x,y), \tag{48}$$

which implies the non-existence of the limit.

Example 12:

Show that the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2+y}{y} \tag{49}$$

does not exist.

Let $f(x,y) = \frac{x^2+y}{y}$. Here the path through the y axis: $x = 0$ gives

$$\lim_{(x,y)\to(0,0);x=0} f(x,y) = \lim_{y\to 0} f(0,y) = 1. \tag{50}$$

Although the x-axis $y = 0$ is not available based on the function's domain.

My next idea is to check the paths $P_\lambda : y = \lambda x$. Let $\lambda \in \mathbb{R}^*$. Then

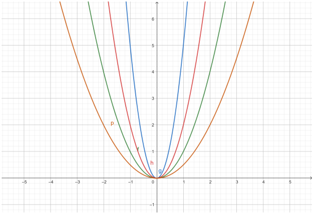
$$\lim_{(x,y)\to(0,0),P_\lambda} f(x,y) = \lim_{x\to 0} f(x,\lambda x) = \lim_{x\to 0} \frac{x^2+\lambda x}{\lambda x} \tag{51}$$

$$= \lim_{x\to 0} \frac{x+\lambda}{\lambda} = 1 \quad \text{again!.} \tag{52}$$

At the moment, there is no answer to our problem, because via all paths studied, f converges to the same limit (here 1).

Although close to start thinking of the limit being 1 (as all path-limits so far), the formula of f inspires me to check the connection between y and x^2 as in the numerator of $f(x,y)$. We will check parabolic paths:

$$\Pi_\lambda : y = \lambda x^2, \quad \lambda \in \mathbb{R}^*. \tag{53}$$



Let $\lambda \in \mathbb{R}^*$, then

$$\lim_{(x,y)\to(0,0);\Pi_\lambda} f(x,y) = \lim_{x\to 0} f(x,\lambda x^2) = \lim_{x\to 0} \frac{x^2+\lambda x^2}{\lambda x^2} = \frac{1+\lambda}{\lambda} \tag{54}$$

i.e. λ-dependent which confirms that the limit does not exist.

Example 13:

Show that the limit

$$\lim_{(x,y)\to(1,2)} \frac{(x+y-3)^2}{(x-1)(y-2)}, \quad \text{does not exist.} \tag{55}$$

Let $f(x,y) = \frac{(x+y-3)^2}{(x-1)(y-2)}$. When $(x,y) \to (1,2)$, I must look for paths tending to this point and so the axes are not eligible.
Checking the vertical line $x = 1$ and the horizontal line $y = 2$, we observe that the function $f(x,y)$ is not defined there, so we must check other paths.
We start thinking of linear paths crossing $(1,2)$.
One of them is given by $P1 : y = 2x$, because when $x \to 1$ then $y \to 2$. Here it holds that

$$\lim_{(x,y)\to(1,2);P_1} f(x,y) = \lim_{x\to 1} f(x,2x) = \lim_{x\to 1} \frac{(x+2x-3)^2}{(x-1)(2x-2)} \tag{56}$$

$$= \lim_{x\to 1} \frac{(3x-3)^2}{(x-1)2(x-1)} = \lim_{x\to 1} \frac{9(x-1)^2}{2(x-1)^2} = \frac{9}{2}. \tag{57}$$

We focus now at the path $P2 : y = x + 1$. Here we take

$$\lim_{(x,y)\to(1,2);P_2} f(x,y) = \lim_{x\to 1} f(x, x+1) = \lim_{x\to 1} \frac{(x+x-2)^2}{(x-1)(x-1)} \quad (58)$$

$$= \lim_{x\to 1} \frac{(2x-2)^2}{(x-1)^2} = \lim_{x\to 1} \frac{4(x-1)^2}{(x-1)^2} = \lim_{x\to 1} 4 = 4 \neq \frac{9}{2}, \quad (59)$$

so the limit does not exist.

Example 14:

Does exist the limit

$$\lim_{(x,y)\to(0,0)} \frac{2x^3 - 4y^5}{x^2 + y^2}? \tag{60}$$

We consider the function $f(x,y) = \frac{2x^3-4y^5}{x^2+y^2}$.
The path on y-axis gives $f(0,y) = -4y^3 \to 0$, when $y \to 0$.
The path on x-axis gives $f(x,0) = 2x \to 0$, when $x \to 0$.
We are looking at the paths $P_\lambda : y = \lambda x$: here

$$f(x,\lambda x) = \frac{2x^3 - 4\lambda^5 x^5}{(1+\lambda^2)x^2} = \frac{2x - 4\lambda^5 x^3}{1+\lambda^2} \to 0, \tag{61}$$

when $x \to 0$.

Naturally we are thinking of a zero limit as through all paths above. Recall that $|a \pm b| \leq |a| + |b|$ and that $|x|, |y| \leq \sqrt{x^2 + y^2}$. Then

$$|2x^3 - 4y^5| \leq 2|x|^3 + 4|y|^5 \leq 2\sqrt{x^2+y^2}^3 + 4\sqrt{x^2+y^2}^5 \tag{62}$$

$$\leq 6\sqrt{x^2+y^2}^3, \tag{63}$$

where for the above we assume that: $q := x^2 + y^2 \leq 1$, which gives $q^5 \leq q^3$. Then

$$0 \leq |f(x,y)| \leq \frac{6\sqrt{x^2+y^2}^3}{x^2+y^2} = 6\sqrt{x^2+y^2} \to 0, \tag{64}$$

which is all we need to obtain that $\lim_{(x,y)\to(0,0)} \frac{2x^3-4y^5}{x^2+y^2} = 0$.