ST221 Introduction to Statistics

Dr Rafael Moral

Dept of Mathematics and Statistics
Maynooth University

rafael.deandrademoral@mu.ie

2 Discrete Random Variables

A variable whose value is determined in some way by chance is called a random variable.

- Number of people waiting at an ATM.
- Number of malfunctioning components in a batch of 100.
- Height and weight of a randomly selected person.

Types of random variables

- A continuous random variable takes values anywhere on the real line $\mathbb{R}$ or on a subset of $\mathbb{R}$.
- A discrete random variable takes values on the integers or on a subset of the integers.

Examples

- $X =$ weight of a randomly chosen person: any value $(0, \infty)$.
- $X =$ number of people waiting at an ATM: 0 , 1, 2, 3....

For now we will focus on discrete random variables.

2.1 The probability mass function and cumulative distribution function

Example

Toss a coin 3 times.

$$\Omega = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}.$$

There are 8 equally likely outcomes. Let X = the number of heads observed. Then X is a discrete random variable with possible values: 0, 1, 2, 3.

What is the probability mass function of X?

Probability mass function

In general, a probability mass function for a discrete random variable X is a function, p, such that

$$p(x) = P(X = x)$$

Note

- $p(x) \geq 0 \qquad \forall\ x$
- $\sum_x p(x) = 1$

Example: Back to the tossing of three coins experiment.
X = number of heads in 3 tosses.

Show that p(x) is a valid probability mass function.

Cumulative distribution function

The cumulative distribution function of a random variable is a function F such that

$$F(x) = P(X \leq x) = \sum_{k \leq x} p(k)$$

Example: Back to the tossing of three coins experiment.
X = number of heads in 3 tosses.

Give the cumulative distribution function for X.

Example

Let X denote the number of accidents at a busy intersection in a week. Based on historical information, we know that:

x	0	1	2	3	4
$P(X = x)$	0.4	0.3	0.15	0.1	0.05

Questions:

1. Is this a valid pmf?

2. Graph p.

3. Find F.

4. Graph F.

2.2 Expectation and Variance

Definition: The expected value of a discrete random variable is defined as

$$E[X] = \sum_x xP(X = x)$$

Note: E[X] is often called the mean of the random variable and is denoted by μ.

Example: Back to the accidents example. Compute the expected value of X.

Lemma: For some continuous function g,

$$E[g(X)] = \sum_x g(x)p(x)$$

Definition: The variance of a random variable X is

$$Var(X) = E[(X - \mu)^2]$$

Notes

1. The variance of X is denote by σ^2.
2. The variance is the expected squared distance of X from μ.
3. An equivalent formula is

$$\begin{aligned} Var(X) &= E[X^2] - \mu^2 \\ &= E[X^2] - (E[X])^2 \end{aligned}$$

Definition: The standard deviation of a random variable X is

$$\sigma = \sqrt{Var(X)}$$

Example: Back to the accidents example.

Find $\text{Var}(X)$.

Example

Suppose X takes values -1, 0, 1 each with probability $\frac{1}{3}$.

1. Find $\mathrm{E}[X]$.

2. Find $\mathrm{Var}(X)$.

Some notes on expected values

- Discrete random variables: $E[X] = \sum_x xp(x)$.
    - probability weighted sum of possible values.
- Continuous random variables: $E[X] = \int xf(x)dx$.
    - $f(x)$ will be defined later.
    - probability weighted integral of possible values.
- Expected value $\neq$ most probably value and is not necessarily a value that the random variable can take or that it typically takes. It is the long run average over many independent repetitions.
- Probability can be considered as the long run relative frequency.

Types of discrete distributions

For the remainder of this section we will look at 'special' types of discrete distributions. These are the:

- Bernoulli distribution
- Binomial distribution
- Poisson distribution

2.3 The Bernoulli Distribution

A 'Bernoulli trial' or 'Bernoulli random variable' is where there are just two possible outcomes which we denote either a 'success' or a 'failure'.

Example
Toss a coin:
Head = success = 1, Tail = failure = 0.

Properties of the Bernoulli distribution

Probability mass function

$$p(x) = \begin{cases} p & \text{if } x = 1 \\ q & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

where $q = 1 - p$.

Expected value

$$\text{E}[X] = 0 \times P[X = 0] + 1 \times P[X = 1] = p$$

Variance

$$\text{E}[X^2] = 0^2 \times P[X = 0] + 1^2 \times P[X = 1] = p$$

$$\text{Var}(X) = \text{E}[X^2] \text{ - } (E[X])^2 = p - p^2 = p(1 - p) = pq$$

Example

If in the rolling of a fair die, the event of obtaining a 4 or 6 is called a success and 1, 2, 3, 5 a failure, what is the $\mathrm{E}[X]$ and $\mathrm{Var}(X)$?

2.4 The Binomial Distribution

If n Bernoulli trials all with probability p are performed independently, then X, the number of successes out of the n trials is said to be a binomial random variable with parameters n and p.

Examples

1. Toss a coin 10 times. On each toss the probability of getting a tail is 0.5.
   Let $X = \#$ tails obtained.
   We write $X \sim \text{Binomial}(n = 10, p = 0.5)$.

2. A multiple choice test has 100 questions. Each question has four possible answers. A student does not know anything about the subject and so the probability of a correct answer is 0.25.
   We write $X \sim \text{Binomial}(n = 100, p = 0.25)$.

Assumptions for a binomial random variable

- At each trial there are two possible outcomes: 'success' or 'failure'.
- Trials are independent.
- The probability (p) of success at each trial is constant.
- There is a fixed number (n) of identical trials.

Properties of the binomial distribution

If $X \sim \text{Binomial}(n, p)$, the possible outcomes are 0, 1, 2, ..., n.

Probability mass function

$$P(X = x) = \binom{n}{x} p^x (1-p)^{n-x}$$

Expected value

$$\mathrm{E}[X] = np$$

Variance

$$\mathrm{Var}(X) = np(1-p) = npq$$

Example

Let X = no. of tails in 5 tosses of a coin.

1. What distribution does X have?
2. What values can X take?
3. Find the probability mass function of X.
4. Find the $\mathrm{E}[X]$.
5. What is the $\mathrm{P}(X \leq 2)$?

Example

Suppose that 5% of the Irish population are colour blind. Let $X = \#$ colour blind people in a random sample of 100 people.

1. What is the probability that the sample has no colour blind people?

2. What is the probability that the sample has one colour blind person?

3. What is the probability that the sample has two or more colour blind people?

4. Find the $\mathrm{E}[X]$ and $\mathrm{Var}(X)$.

Statistical Tables

2.5 The Poisson Distribution

A probability model for count data.

Examples

1. Number of plankton in a litre of water.
2. Number of calls per hour to a helpline.
3. Number of earthquakes in a year.

Properties of the Poisson distribution

If X is a Poisson random variable it takes values 0, 1, 2, 3,.... We say that $X \sim$ Poisson(λ).

Probability mass function

$$P(X = x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

Expected value

$$\text{E}[X] = \lambda$$

Variance

$$\text{Var}(X) = \lambda$$

Example

The number of calls per hour to a computer helpline is approximated by a Poisson random variable with mean 3. We say that $X \sim \text{Poisson}(\lambda = 3)$.

1 Find

(a) $P(X = 0)$

(b) $P(X = 1)$

(c) $P(X = 2)$

(d) $P(X = 3)$

Example contd.

The number of calls per hour to a computer helpline is approximated by a Poisson random variable with mean 3. We say that $X \sim \text{Poisson}(\lambda = 3)$.

2 What is the probability of getting no calls in a two hour period?

3 What is the probability of getting one call in a two hour period?

Example

On average 6.7 patients arrive in a doctor's office in 1 hour. Arrivals follow a Poisson distribution.

1. What is the probability of at most 3 patients arriving in the next hour?

2. What is the probability of exactly 5 people arriving in the next 90 minutes?

Example

Suppose that the occurrence of earthquakes in a particular region of California follows a Poisson distribution with a rate of 7 per year on average.

1. What is the probability of no earthquakes in one year?

2. What is the probability that in exactly 1 of the next 8 years no earthquakes will occur?

Summary