PYU33C01 Numerical Methods Assignment 3

Completed by Seán Alexander 21364546

Introduction:

This assignment will examine the use of numerical solutions in solving ordinary differential equations computationally. Different methods of numerical solutions will be compared.

There are some ordinary differential equations which are impossible to solve with elementary functions. Such as:

$$f(x,t) = \frac{dx}{dt} = (1+t)x + 13t + t^2 \tag{0.1}$$

In order to solve this ODE by the integrating factor method one invariably must solve an integral which includes exponential terms with powers that are polynomials of degree 2. This is impossible without the use of non-elementary functions.

The solution found using Mathematica is displayed below and the constant C was found using the starting conditions outlined later.

$$x(t) = e^{t+\frac{t^2}{2}}\,C + e^{t+\frac{t^2}{2}}\left(-e^{-\frac{1}{2}t(2+t)}(-4+t) + 3\sqrt{2\pi e}\,\times \operatorname{erf}\left(\frac{1+t}{\sqrt{2}}\right)\right) \tag{0.2}$$

The solution shown above requires use of the error function (erf) where erf(*) is defined below:

$$\operatorname{erf}(t) = \frac{2}{\sqrt{\pi}}\int_0^t e^{-k^2}\,dt \tag{0.3}$$

The solution for $x(t)$ obtained above while not entirely without error when computed will be taken as the analytic solution of the ODE when comparing types of numerical solution.

Plotting the Direction Field of $f(x,t)$

One may visualize the solutions for of the ODE using a direction field. The vector field with each vector being of slope v/u where $v = \frac{dx}{dt}$ and $u = 1$ was then normalized such that is was of length unity; where u and v were divided by the magnitude $\sqrt{u^2+v^2}$.

The python command meshgrid(), and quiver() from the numpy and matplotlib.pyplot libraries respectively were used to visualise this directional field.

Using the methods outlined above, the directional field for t ranging from 0 to 5 and x from -3 to 3 was produced at a 25×25 grid. This directional field is shown below.

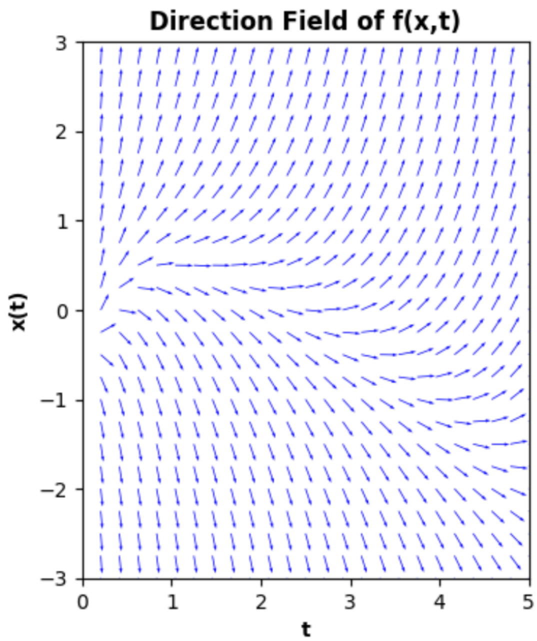

Figure 1: *Direction Field of $f(x,t)$*

Numerical Solution Methods

The initial conditions taken for this exercise were $x(0) = 0.0655$ as this is a condition close to the critical point $x(t_c) = 0.065923...$ that being where $f(x,t) = 0$. This is due to the fact that solutions found below t_c (such as 0.0655) will tend towards minus infinity while solutions above t_c will tend towards plus infinity. This asymptotic behaviour is useful in determining distinctions between different numerical solutions as minor errors can increase dramatically as t increases.

The simple Euler method was the first numerical ODE solution method utilised. This method is characterised by taking the zeroth order approximation of the Taylor expansion:

$$x(t + \Delta t) = x(t) + \left.\frac{dx}{dt}\right|_t \Delta t + \frac{1}{2}\left.\frac{d^2x}{dt^2}\right|_t (\Delta t)^2 + \ldots \tag{0.4}$$

which results in the $(i+1)^{th}$ iteration of the solution of $x(t)$ being:

$$x_{i+1} = x_i + f_i \Delta t \tag{0.5}$$

The improved Euler method behaves similarly but uses the trapezoid rule to better approximate the behaviour of $f(x,t)$ as it is being evaluated to find the next iteration:

$$x_{i+1} = x_i + \left(f(x_i, t_i) + f\left(x_i + f(x_i, t_i)\Delta t, t_{i+1}\right)\right)\frac{\Delta t}{2} \tag{0.6}$$

Finally, the fourth order Runge-Kutta method was employed which is characterised by:

$$x_{i+1} = x_i + \frac{\Delta t}{6}\left(f(x_1, t_1) + 2f(x_2, t_2) + 2f(x_3, t_3) + f(x_4, t_4)\right) \tag{0.7}$$

Where:

$$x_1' = x_i \quad , \quad t_1' = t_i \tag{0.8}$$

$$x_2 = x_i + \frac{1}{2} f(x_1, t_1)\Delta t \quad , \quad t_2 = t_i + \frac{\Delta t}{2} \tag{0.9}$$

$$x_3 = x_i + \frac{1}{2} f(x_2, t_2)\Delta t \quad , \quad t_3 = t_i + \frac{\Delta t}{2} \tag{0.10}$$

$$x_4 = x_i + f(x_3, t_3)\Delta t \quad , \quad t_4 = t_i + \Delta t \tag{0.11}$$

Numerical Solution Behaviours

The simple Euler method was applied with step size 0.04. This was plotted and overlayed on the directional field found earlier. What is immediately noticeable is that the directional field appears to be tangential to the solution at the points the solution intersects. The simple Euler method tends towards positive infinity whilst the analytical solution determined in terms of the error function tends towards negative infinity. This is due to the fact that the simple Euler method is a crude approximation and thus behaves particularly incorrectly near the critical point $t = t_c$.

The improved Euler method and the Runge-Kutta method were then applied. The improved Euler method also behaved similar to the simple Euler method but it appears to attempt to curve down towards negative infinity before ultimately tending toward positive infinity ultimately ass seen on the error graphs below. The Runge-Kutta method aligned faithfully with little relative error until the solution met $x(t) = 0$ and earned a relative error of ~ 0.7. It no longer maintained its almost non-zero relative error afterwards.

The methods were then repeated with a step size of 0.02. This resulted in the greatest change for the simple Euler method which was most susceptible to a change in step size as it maintained a low slope at the critical point before tending towards infinity and both on the high and low scales maintained a lower relative error when compared to the solution derived in terms of the error function.

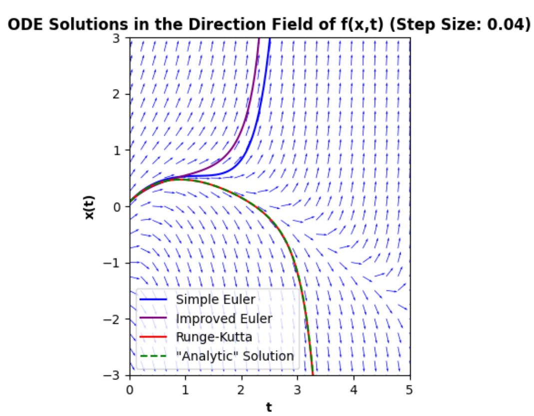

Figure 2: *ODE Numerical Solutions with Step Size 0.04*

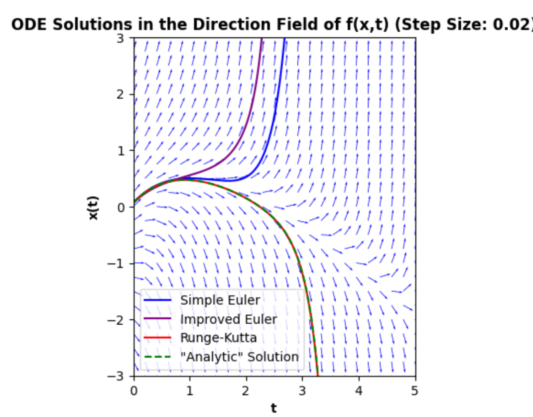

Figure 3: *ODE Numerical Solutions with Step Size 0.02*

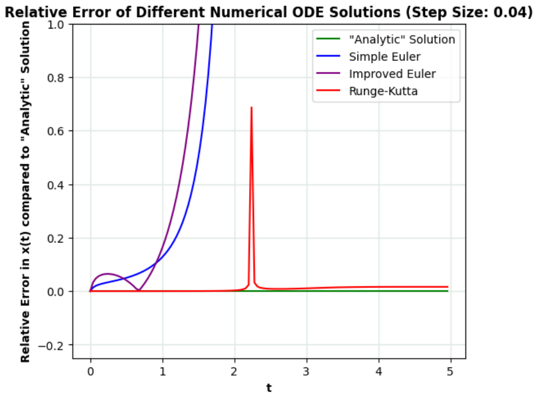

Figure 4: *Relative Error for ODE Numerical Solutions with Step Size 0.04*

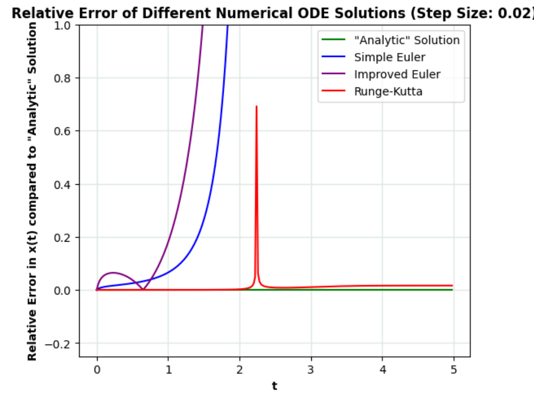

Figure 5: *Relative Error for ODE Numerical Solutions with Step Size 0.02*

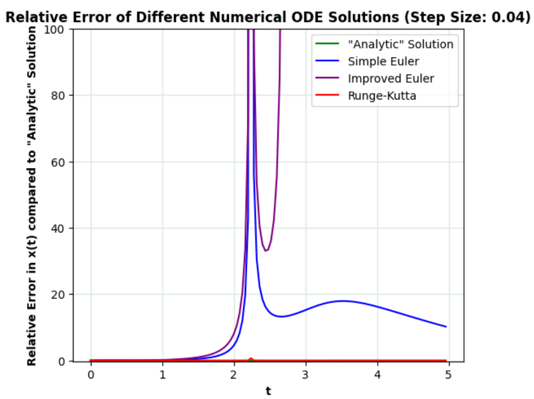

Figure 6: *Larger Scale Relative Error for ODE Numerical Solutions with Step Size 0.04*

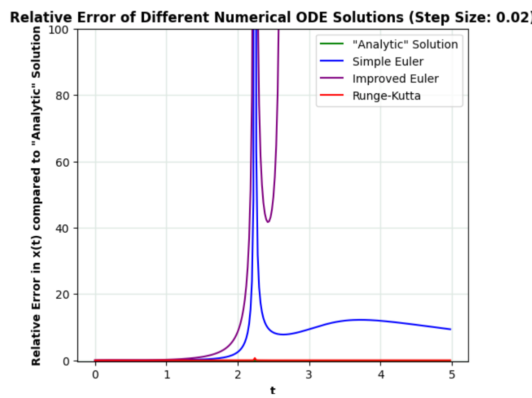

Figure 7: *Larger Scale Relative Error for ODE Numerical Solutions with Step Size 0.02*

Conclusion

Overall, the Runge-Kutta method gave the best approximation of the of the ODE solution. The difference between the different numerical methods applied are to do with the accuracy of the integration schemes as the process outlined is equivalent to determining: $x_{i+1} = x_i + \int_{t_i}^{t_{i+1}} f(x,t)\,dt$ and each method employed improves the method of approximation used in determining the value of the integral term used to determine the $(i+1)^{th}$ value of x. This determines the ultimate path that the entire solution for $x(t)$ takes. This explains how to Runge-Kutta method performs the best as it gives the best approximation of the integral term.It also aligned well with the analytic solution found in terms of the non-elementary error function.