PYU33C01 Numerical Methods Assignment 4

Completed by Seán Alexander 21364546

Introduction:

This assignment will examine the characteristics of Poisson distributions for various mean values. This was performed under the lens of a scenario in which random darts were distributed across regions on a dartboard where the Poisson distribution was employed to model the probability of finding different numbers of darts in these regions. The distribution was then plotted for different mean values, and their impact on distribution properties was analyzed. Subsequently, simulations of dart-throwing scenarios were conducted and compared to the Poisson distribution with a matching mean. Additionally, the minimum probability threshold at which the simulation approximated the Poisson distribution, with variations based on the number of trials and regions considered were determined.

Plotting Poisson Distributions for Different Mean Values

Generating a random distribution of N darts on a dart board which has been divided into L regions of equal area will result in the associated probability of finding each individual dart in any given region to be L^{-1}. Upon examining such a random distribution such that $L \sim 2N$, one would likely determine that many regions are empty but some regions have a number of darts greater than 1.

The Poisson Distribution is outlined below:

$$P(n = x) = \frac{e^{-\langle n \rangle} \langle n \rangle^n}{n!} \tag{0.1}$$

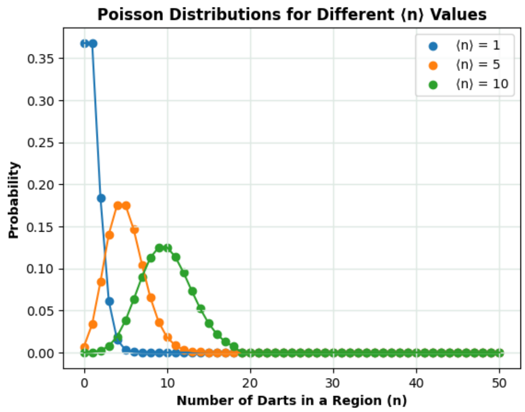
where:

$$\langle n \rangle = \sum_{n=0}^{N} nP(n) \tag{0.2}$$

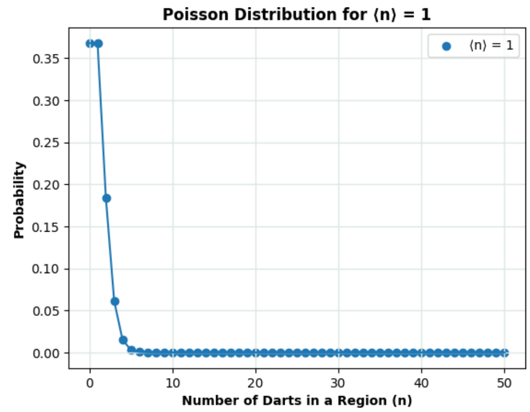
Where: P is the probability that $x = n$, and $\langle n \rangle$ is the mean value of the distribution.

One may model the probability that a given region has a particular number of darts using the Poisson distribution, where x is a given region, n is a given number of darts and $\langle n \rangle$ is the mean number of darts in any given region determined by the the ratio of darts thrown and regions.

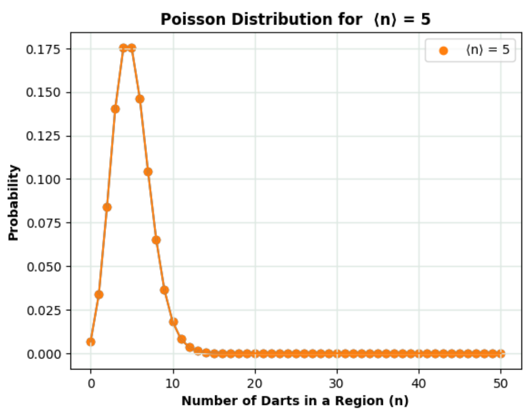
In order to use the Poisson distribution to model such a scenario, the Poisson distribution was plotted for 3 given mean values being: $\langle n \rangle = [1, 5, 10]$ as shown below:



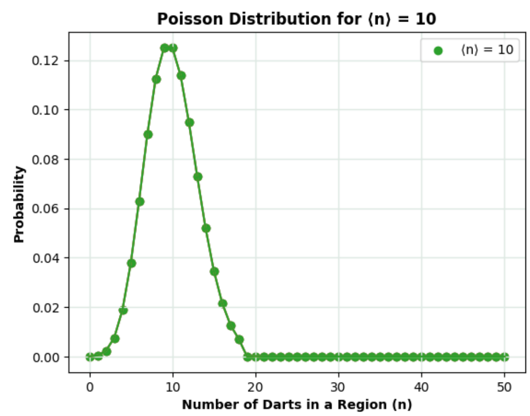
(a) $\langle n \rangle = [1, 5, 10]$



(b) $\langle n \rangle = 1$



(c) $\langle n \rangle = 5$



(d) $\langle n \rangle = 10$

Figure 1: **Poisson Distributions for $\langle \mathbf{n} \rangle = [\mathbf{1}, \mathbf{5}, \mathbf{10}]$**

Poisson Distribution Analysis

In order to examine the effect changing the mean value of the Poisson distribution on the distribution's properties, the following numerical values were determined with a number of darts thrown of $N = 50$:

$$\sum_{n=0}^{N} P(n) \tag{0.3}$$

$$\sum_{n=0}^{N} n \times P(n) \tag{0.4}$$

$$\sum_{n=0}^{N} n^2 \times P(n) \tag{0.5}$$

If the Poisson distributions were correctly normalized, $\sum_{n=0}^{N} P(n)$ would have to equal 1 as the sum of the probabilities of all possibilities is by definition equal to 1. Following from this, numerical values for the variance and standard deviation of the Poisson distributions were using the following methods:

$$\text{variance} = \sum_{n=0}^{N}(n - \langle n \rangle)^2 \cdot \sum_{n=0}^{N} P(n) \tag{0.6}$$

$$\text{Standard Deviation} = \sqrt{\text{variance}} \tag{0.7}$$

These numerical values were tabulated as shown below:

$\langle n \rangle$	$\sum_{n=0}^{N} P(n)$	$\sum_{n=0}^{N} n \times P(n)$	$\sum_{n=0}^{N} n^2 \times P(n)$	Variance	Standard Deviation
1	1.000000	1.000000	2.000000	1.000000	1.000000
5	1.000000	5.000000	30.000000	5.000000	2.236068
10	1.000000	10.000000	110.000000	10.000000	3.162278

Table 1: Poisson Distribution Properties for $\langle n \rangle = [1, 5, 10]$

These calculated numerical values show the distributions are normalised and the variance and therefore the standard deviation increases with the the mean which are both in line with theory. This means that the larger the value of $\langle n \rangle$, the more flat its associated peak becomes.

Throwing Darts and Comparing to Associated Poisson Distribution

In order to evaluate the theoretical model of the dart throwing scenario, random integers were generated and the region they aligned with were tracked to determine the frequency of different numbers of darts in different regions. This was performed $T = 10$ times with $N = 50$ darts over $L = 100$ regions. Each of the trials were plotted overlayed upon each-other on a histogram as shown below:

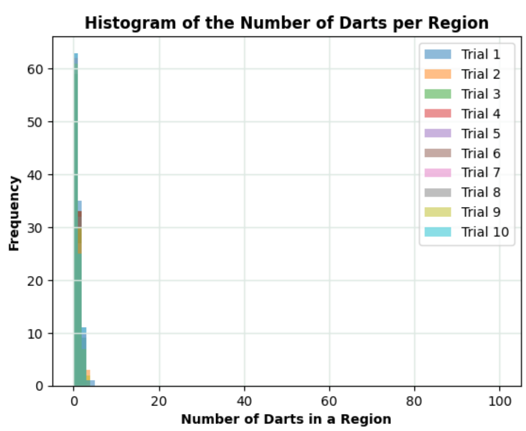


Figure 2: *Histogram of Dart Throwing Scenario* (10 *Trials*)

The summation of these distributions were then normalized using the following expression:

$$P_{sim}(n) = \frac{\sum_{t=0}^{T} H_t}{LT} \tag{0.8}$$

This P_{sim} was then plotted against the Poisson distribution with the same mean ($\langle n \rangle$) as P_{sim} as shown below:

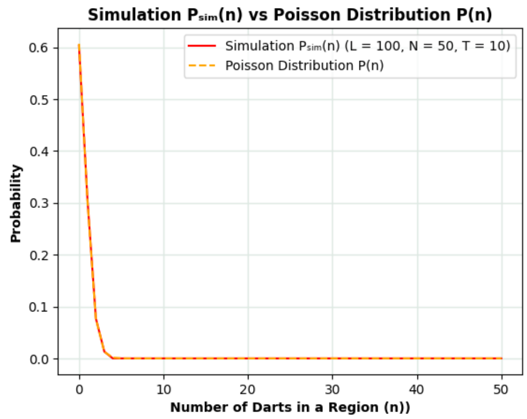
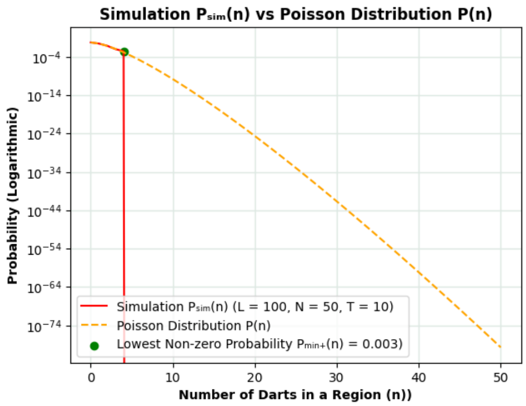


Figure 3: *Simulation $P_{sim}(n)$ vs Poisson Distribution $P(n)$*

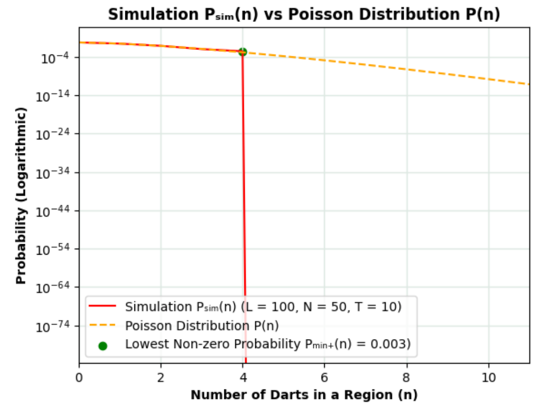
The simulation $P_{sim}(n)$ appears to be in great agreement with the Poisson Distribution $P(n)$ with the same associated mean $\langle n \rangle$ which in this case was approximately $\frac{1}{2}$.

Determining the Minimum Threshold for where Numerical Data Probes The Poisson distribution

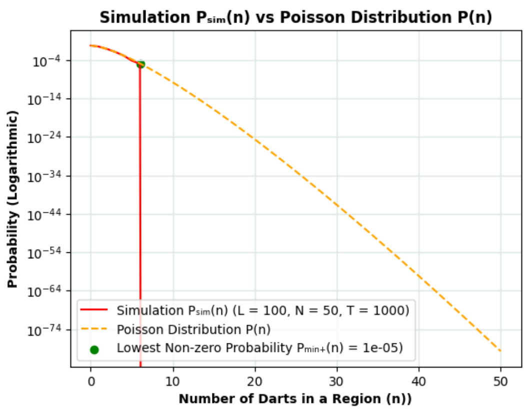
In order to determine the lowest probability value of $P(n)$ the simulation $P_{sim}(n)$ agrees with the theoretical Poisson distribution with the same mean, the previous figure was also plotted with a logarithmic probability scale. This in reveals the smallest non-zero probability as the scale asymptotically approaches $-\infty$. These values were determined and displayed in both an overall and zoomed in graph:



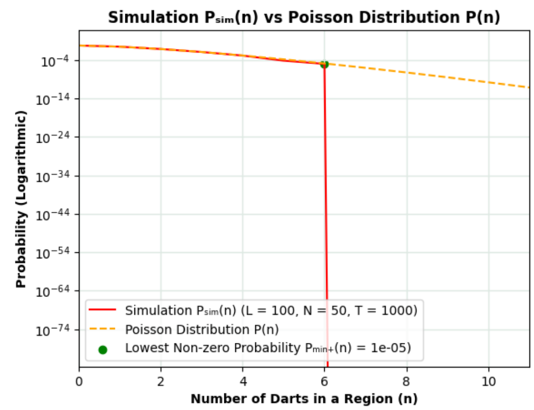
(a) Zoomed Out: $T = 10 \rightarrow P_{min+} = 0.003$



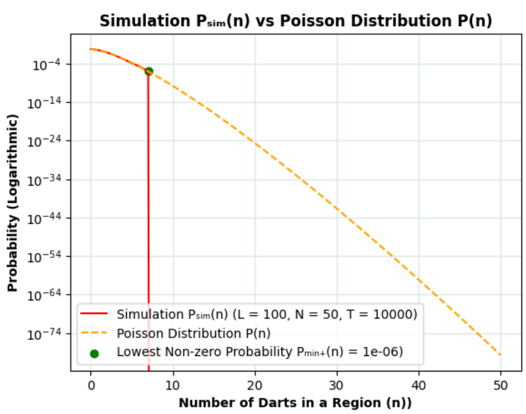
(b) Zoomed In: $T = 10 \rightarrow P_{min+} = 0.003$



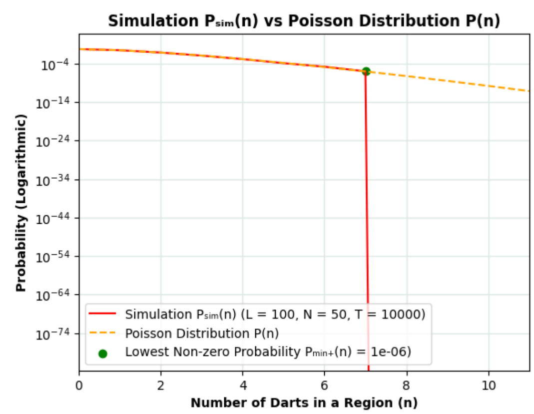
(c) Zoomed Out: $T = 1000 \rightarrow P_{min+} = 10^{-5}$



(d) Zoomed In: $T = 1000 \rightarrow P_{min+} = 10^{-5}$



(e) Zoomed Out: $T = 10000 \rightarrow P_{min+} = 10^{-6}$

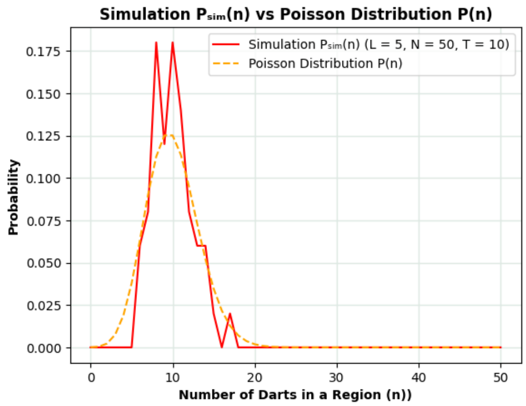


(f) Zoomed In: $T = 10000 \rightarrow P_{min+} = 10^{-6}$

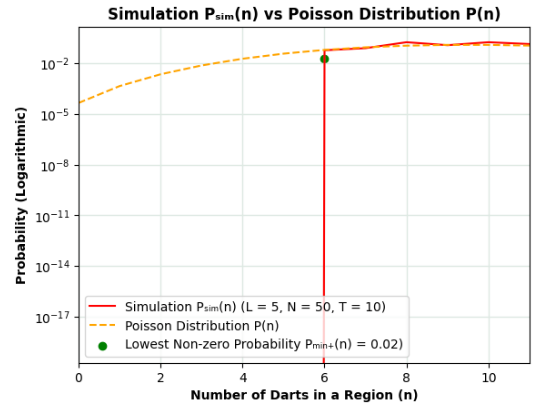
Figure 4: **Probing the Smallest Non-Zero Probability $\mathbf{P_{min+}}$ for** $N = 50$**,** $L = 100$

The lowest non-zero values of probability under the above parameters yielded varied substantially as the code was run but under $T = 1000$ trials, the value was determined to be $\sim 10^{-6}$.

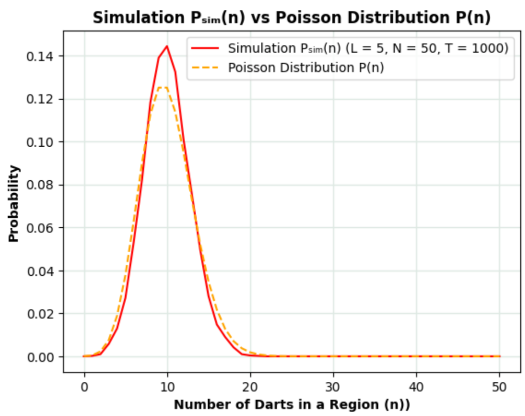
The prior calculations were repeated with $N = 50$ darts over $L = 5$ regions as shown below:



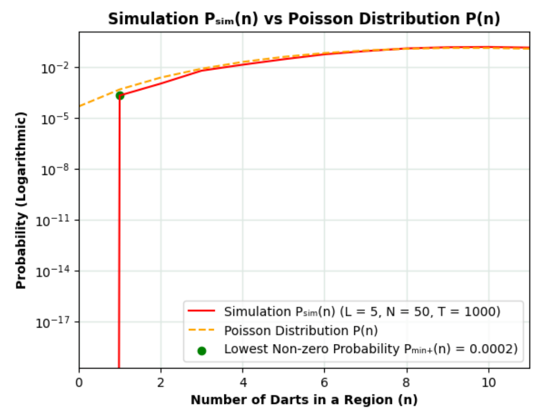
(a) Plot: $T = 10$



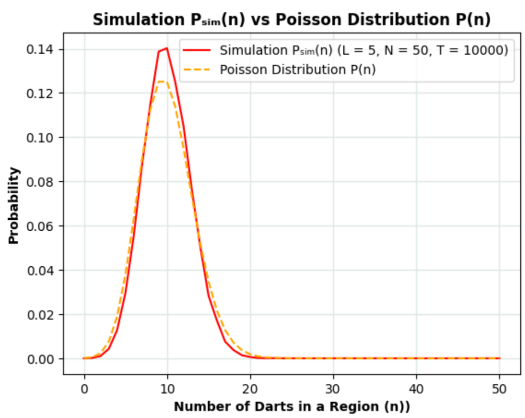
(b) Zoomed In: $T = 10 \rightarrow P_{min+} = 0.02$



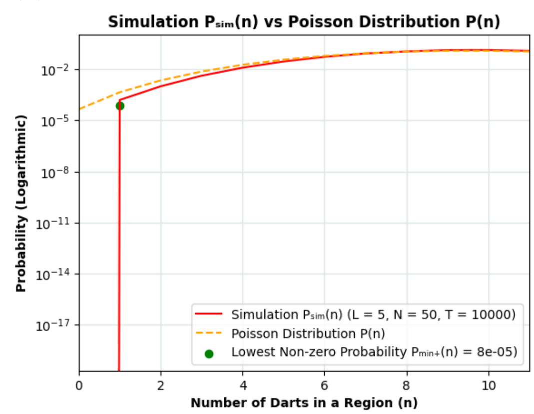
(c) Plot: $T = 1000$



(d) Zoomed In: $T = 1000 \rightarrow P_{min+} = 0.0002$



(e) Plot: $T = 10000$



(f) Zoomed In: $T = 10000 \rightarrow P_{min+} = 8 \times 10^{-5}$

Figure 5: **Probing the Smallest Non-Zero Probability of $\mathbf{P_{min+}}$ for $N = 50$, $L = 5$**

The plots appear to agree far less with the theoretical values when $N = 50$ darts were simulated over just $L = 5$ regions. The lowest non-zero values of probability under the above parameters yielded varied substantially as the code was run but under $T = 1000$ trials, the value was determined to be $\sim 8 \times 10^{-5}$ which is more than an order of magnitude larger than that found for the simulation with twenty times the regions for the same number of trials.

Conclusion

In conclusion, the characteristics of Poisson distributions under the lens of the dart throwing scenario were analysed. The Poisson distribution was used to model the probability of finding different numbers of darts in distinct regions. By plotting the distribution for different mean values, the influence on distribution properties was revealed. Theoretical analysis and simulations of dart-throwing trials were conducted, demonstrating a strong agreement between the simulated results and the Poisson distribution with matching parameters. Furthermore, the assignment examined the minimum probability threshold at which the simulation approximated the Poisson distribution, finding that this threshold varied depending on the number of trials and regions considered.