Construction and Evaluation of a Chaotic Water Wheel

Sean Cummins
Dr Michael Cawley
Department of Experimental Physics
seancummins16@gmail.com

The aim of this project is the construction and evaluation of a Malkus-Howard-Lorenz water wheel. We started by constructing an unideal 4-bucket water wheel and attempted to match the motion observed with a computer simulation. From this we moved on to the Malkus-Howard-Lorenz chaotic water wheel whose governing equations are a subset of the Lorenz system. We constructed a 45-bucket water wheel using a bicycle wheel and syringes. We integrated the Lorenz system to predict the apparatus's motion. Furthermore we numerically generated a map of the type of motion expected for a particular choice of parameter values using Lyapunov exponents. Again we sought to match the type of motion generated by a particular choice of experimental parameter values set on the apparatus with the type of motion predicted by the parameter map.

Background

"Chaos: When the present determines the future, but the approximate present does not approximately determine the future." - Edward Norten Lorenz

Chaos theory is a rich and relatively new area of physics whose origins date back to the 1960's and the work of Edward Norten Lorenz. Today it is present in all areas of science.

"Chaos is aperiodic long-term behaviour in a deterministic system that exhibit sensitive dependence on initial conditions." [1]

This means if two systems are set up such that they differ in initial conditions by a tiny amount, the resultant trajectories will quickly diverge such that (after a short amount of time) neither motion will resemble each other.

The most famous chaotic system on which this project was based is the Lorenz system [2]. The set of equations was discovered by Edward Norten Lorenz and published in the landmark paper *"Deterministic Nonperiodic Flow"* in 1963.

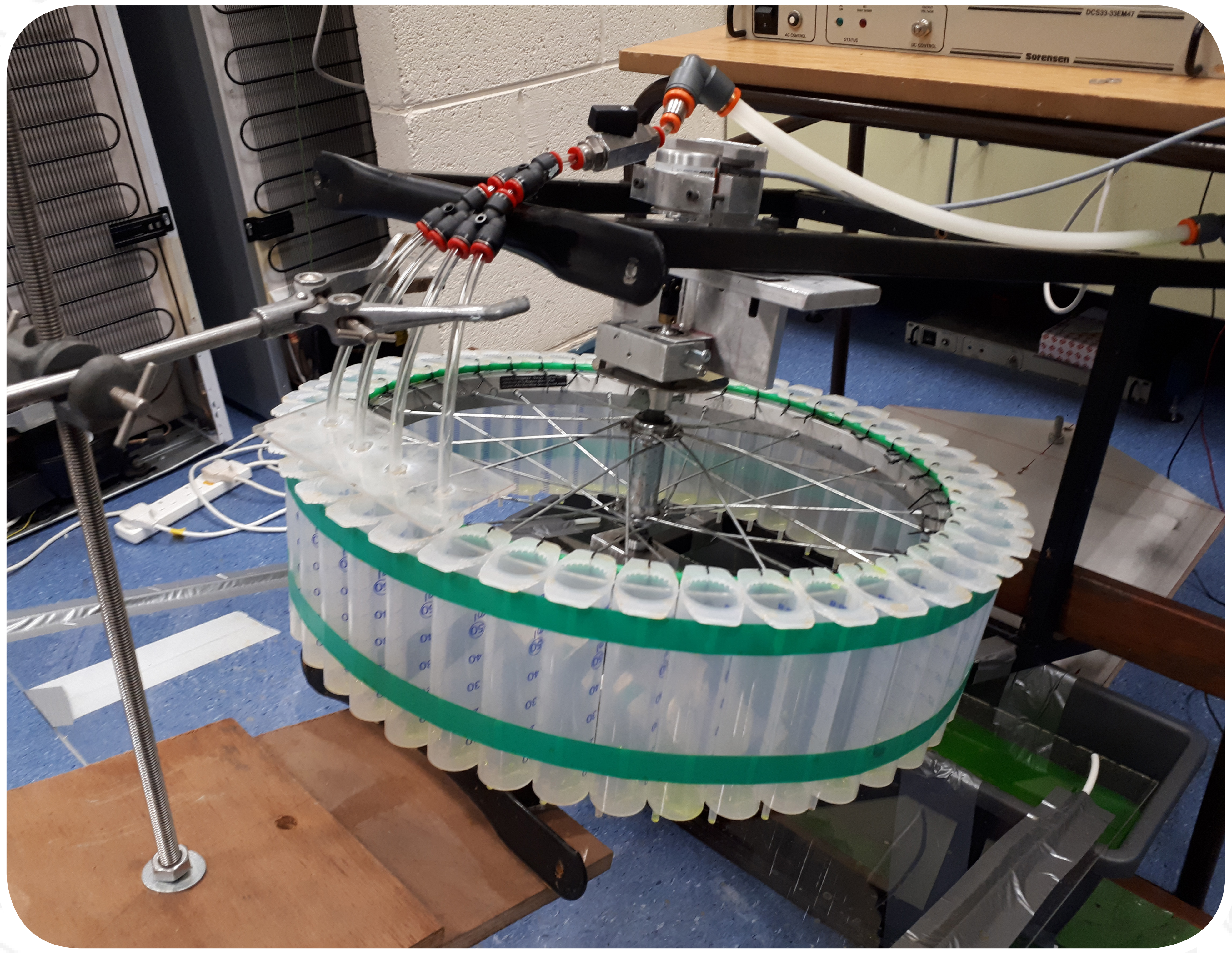
Lorenz Equations	Malkus Water Wheel Equations
$\dot{x}=\sigma(y-x)$ $\dot{y}=\rho x-y-xz$ $\dot{z}=xy-bz$	$\dot{\omega}=\sigma(\omega+y)$ $\dot{y}=\rho x-y-xz$ $\dot{z}=xy-bz$
x: Fluid Motion	ω: Angular velocity of wheel
y: Horizontal Temperature	y: Horizontal component of center of mass
z: Vertical Temperature	z: Vertical component of center of mass
σ: Prandtl Number ρ: Rayleigh Number	$\sigma=\frac{1}{K}\frac{\nu+Q_{tot}R^2}{I_{tot}}$ $\quad \rho=\frac{Q_{tot}}{K^2}\frac{Rg\sin\alpha}{\nu+Q_{tot}R^2}$
b: Aspect ratio of the rolls	K: Outflow
	ν: Magnetic braking coefficient
	Q_{tot}: Inflow
	R: Radius of wheel
	I_{tot}: Total inertia of wheel and water
	g: Acceleration due to gravity
	α: Tilt angle
	$b=1$ (Subset of Lorenz system)

The Lorenz equations in their original context describe a simple climate model (a convection cell). Discovering that these equations were incredibly sensitive initial conditions, Lorenz came to the conclusion that accurate prediction of the weather past a few days was practically impossible.

In the 1970s it was discovered by Malkus, Howard and Lorenz that there existed a mechanical analogue to these equations in the form of a special type of water wheel called a Malkus-Howard-Lorenz water wheel [3] [4].

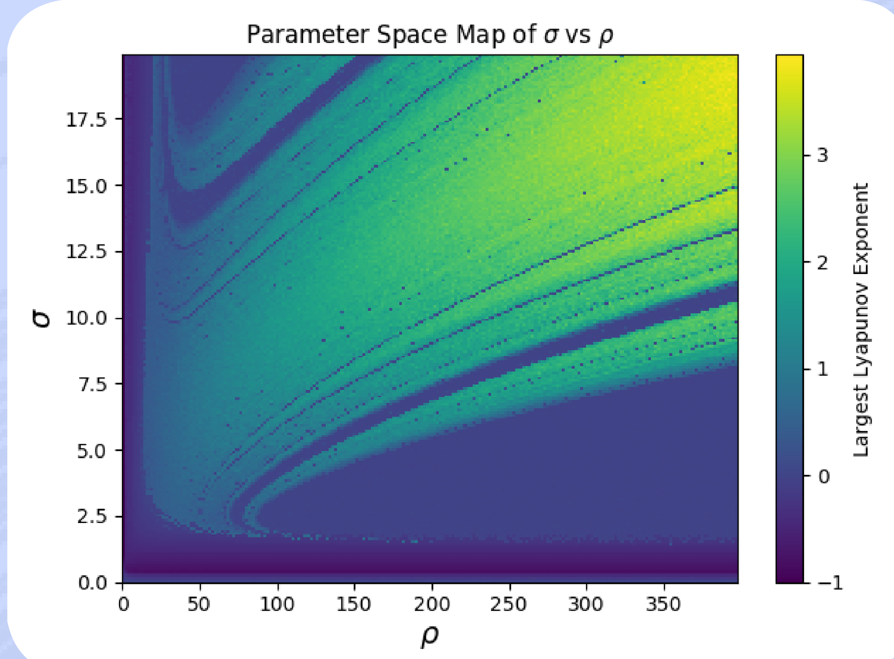
As seen in the above table, the underlying structures of the water wheel's equations are nearly identical to the Lorenz system and are in fact a subset.

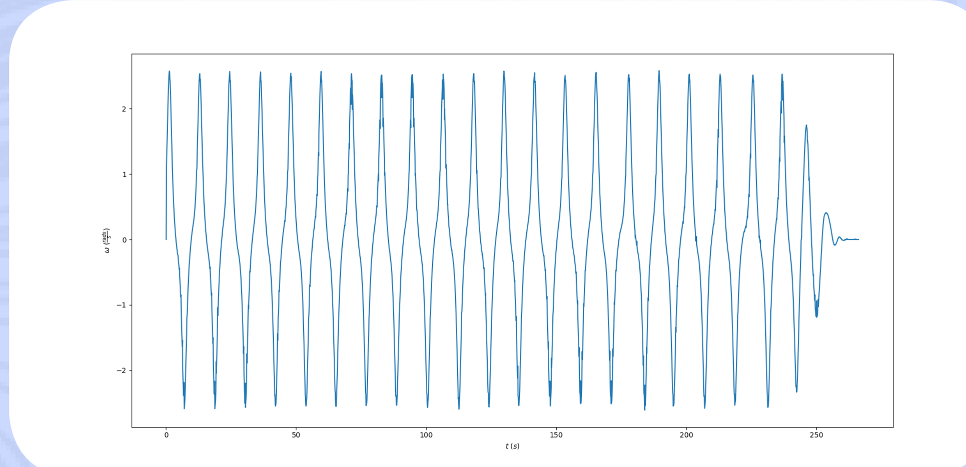
The aim of this project was to construct such a water wheel and record its rotating motion for varies settings of the wheel's parameters. We would then compare the resultant motion with a computer simulation to see if our wheel could be described by the Malkus water wheel equations.



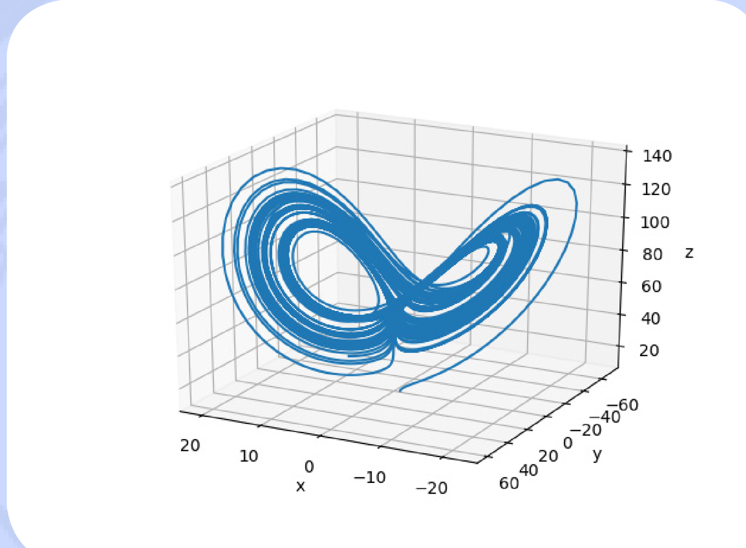
(Above) 45-Bucket Water Wheel. The syringes collect and leak water from inflow tubes. The aluminum disc in the center acts as a magnetic brake. The whole system is tilted at an angle with run off water flowing into a trough to be circulated back into syringes. The green colour of the water is from a florescent dye added to make the water content in each syringe easier to see.

Results

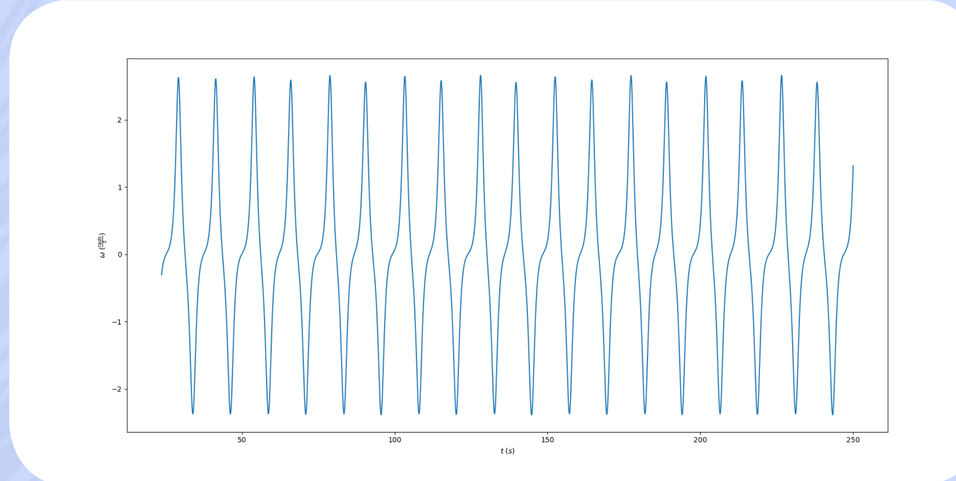




(Above) Plot of data corresponding to periodic motion



(Above) Computer simulation of Malkus water wheel equations



(Above) Simulation corresponding to periodic motion for same parameter values as data plot

Video/Website

Use a QR scanner app on your smartphone to access the following content:

Video of the 45-bucket water wheel undergoing periodic motion

Website for further content on the project

(Above) 4-Bucket Water Wheel
It proved to be a much more complex system to model then the 45-bucket water wheel as the total mass of the system did not approach a constant.

Equipment

- Water wheel: Made from a 20cm radius BMX bike wheel attached to a stool frame
- Buckets: Either vitamin containers for 4-bucket water wheel or syringes for 45-bucket water wheel
- Pump: To circulate water around water wheel
- Rotary Encoder: To record angular rotation of water wheel with electrical pulses
- Quadrature Decoder: To count encoder pulses
- Measurement Computing (MSC) USB 1208 LS: To collect decoder counts and output to computer
- Computer: Interpret decoder counts as angular rotation and save measurements
- Magnetic Brake: Adds a frictional force that slows down and changes motion of wheel

Lyapunov Exponents

Depending on the experimental settings, the motion of the water wheel is stable, periodic or chaotic.

"Can one predict in advance (given the experimental settings) what type of motion will occur?"

The answer (perhaps surprisingly) is yes! [5]

One can compute the Largest Lyapunov Characteristic Exponent (LLCE) [5] for the system for the given experimental settings.

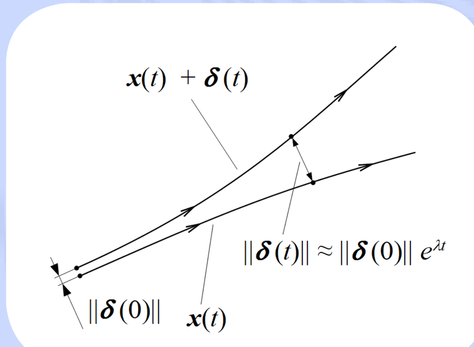
$$\|\delta(t)\| \approx \|\delta(0)\| e^{\lambda t}$$

where λ is the LLCE.

Lyapunov exponents are numbers which characterize how nearby trajectories in the phase space evolve over time.

Encoded in the number is information relating to how strongly stable, periodic or chaotic the system is.

We calculated the LLCE for a 200×200 grid of values for ρ and σ, (b=1).



Conclusions

Construction

Both the 4-bucket water wheel and the 45-bucket water wheel were successfully constructed. The buckets used for the 4-bucket water wheel could be improved but the syringes used for the 45-bucket water wheel were successful. The rotary encoder-decoder system required to measure the angular position of the wheel was correctly implemented and worked perfectly, as did the MSC USB interface. The pump systems implemented also succeeded but the pump system corresponding to the 45 bucket water wheel worked better. The magnetic braking system was implemented but proved to not be a success. It is hoped that further work can be done to improve this important aspect of the system in the future.

Evaluation

The 4 bucket water wheel had both successes and failures. The successes were in the simulations and animations. The failures were in being unable to control accurately most of the experimental parameters. The resultant motion was almost always *chaotic*. The 45-bucket water wheel was more of a success but was built too late to preform many tests. It also suffered from being difficult to measure certain experimental parameters accurately, most noticeably Q_{tot} and ν. However both *stable* and *periodic* motion was found even if the calculated values of ρ and σ for such motions were inaccurate.

It is too early to say if our 45-bucket water wheel truly obeys the Malkus water wheel equations. Further tests must be conducted and accurate measurements of its experimental parameters must be made. However a lot was accomplished in just the 6 weeks allocated for this project. The Malkus-Howard-Lorenz water wheel is an incredibly complex system and it is hoped that further investigations can be made in the future.

References

[1] Steven H Strogatz, *Nonlinear Dynamics and Chaos,* Westview Press; Ist edition (January 19, 2001)
[2] Edward N Lorenz, *Deterministic Nonperiodic Flow,* Journal of Atmospheric Sciences Volume 20, 130
[3] Lucas Illing et al, *Experiments with a Malkus-Lorenz water wheel: Chaos and Synchronization*, Am. J. Phys. 80, 192(2012); DOI: 10.1119/1.3680533
[4] Leslie E. Matson, *The Malkus-Lorenz water wheel revisited,* Am. J. Phys. 75 (12), December 2007; DOI: 10.1119/1.2785209
[5]Macro Sandri, *Numerical Calculation of Lyapunov Exponents,* The Mathematical Journal 84, 1996
https://commons.wikimedia.org/wiki/File:Convection_cells.svg - Convection cell image
https://commons.wikimedia.org/wiki/File:Orbital_instability_(Lyapunov_exponent).png - Lyapunov image

Acknowledgments

This project would not have been possible without the help, support and expertise of a number of key individuals, offices and departments. I would like to extend my deep sense of gratitude to the Experiential Learning office and the SPUR program of Maynooth University as well as the entirety of the Experimental Physics department for giving me this opportunity to participate in this undergraduate research project. I would to thank in particular Dr Aisling Flynn, Experiential Learning Officer for running and coordinating the SPUR program. Finally and most importantly, I would like to thank my mentor Dr Michael Cawley for choosing me for this project and supporting me throughout the project with insight and guidance.

Maynooth University
National University of Ireland Maynooth