# Homework 6

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## Problem 1

In terms of the $c_\pm$ amplitudes, the expectation values of the Pauli operators are

$$\langle \sigma_x \rangle = c_+^* c_- + \text{c.c.}$$
$$\langle \sigma_y \rangle = -i c_+^* c_- + \text{c.c.}$$
$$\langle \sigma_y \rangle = c_+^* c_+ + c_-^* c_-$$

The equations of motion for the amplitudes are given by

$$\dot{c}_+ = -\frac{i}{2}\left(\Delta c_+ - \Omega c_-\right)$$
$$\dot{c}_- = -\frac{i}{2}\left(\Omega^* c_+ - \Delta c_-\right)$$

Combining these, we can see that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\langle\sigma_x\rangle &= \dot{c}_+^* c_- + c_+^* \dot{c}_- + \text{c.c.} \\
&= \frac{i}{2}\left(\Delta c_+^* - \Omega^* c_-^*\right) c_- - \frac{i}{2} c_+^* \left(\Omega^* c_+ - \Delta c_-\right) + \text{c.c.} \\
&= -\frac{i}{2}\left(\Omega^* \left|c_+\right|^2 - \Omega^* \left|c_-\right|^2 - 2\Delta c_+^* c_-\right) + \text{c.c.} \\
&= -\Delta\langle\sigma_y\rangle - \mathrm{Im}[\sigma]\langle\sigma_z\rangle \\
\frac{\mathrm{d}}{\mathrm{d}t}\langle\sigma_y\rangle &= -i\dot{c}_+^* c_- - i c_+^* \dot{c}_- + \text{c.c.} \\
&= -i\left[\frac{i}{2}\left(\Delta c_+^* - \Omega^* c_-^*\right)\right] - i c_+^* \left[\frac{-i}{2}\left(\Omega^* c_+ - \Delta c_-\right)\right] \\
&= \frac{1}{2}\left(\Omega^* \left|c_+\right|^2 - \Omega^* \left|c_-\right|^2 - 2\Delta c_+^* c_-\right) + \text{c.c.} \\
&= \Delta\langle\sigma_x\rangle - \mathrm{Re}[\Omega]\langle\sigma_z\rangle \\
\frac{\mathrm{d}}{\mathrm{d}t}\langle\sigma_z\rangle &= c_+^* \dot{c}_+ - c_-^* \dot{c}_- + \text{c.c.} \\
&= -\frac{i}{2}\left(\Delta\left|c_+\right|^2 + \Omega c_+^* c_- - \Omega^* c_-^* c_+ - \Delta\left|c_-\right|^2\right) + \text{c.c.} \\
&= \mathrm{Re}[\Omega]\langle\sigma_y\rangle + \mathrm{Im}[\Omega]\langle\sigma_x\rangle
\end{aligned}
$$

This is equivalent to

$$
\vec{P} \times \langle\vec{\sigma}\rangle = \begin{pmatrix} \mathrm{Re}[\Omega] \\ -\mathrm{Im}[\Omega] \\ \Delta \end{pmatrix} \times \begin{pmatrix} \langle\sigma_x\rangle \\ \langle\sigma_y\rangle \\ \langle\sigma_z\rangle \end{pmatrix} = \begin{pmatrix} -\Delta\langle\sigma_y\rangle - \mathrm{Im}[\Omega]\langle\sigma_z\rangle \\ \Delta\langle\sigma_x\rangle - \mathrm{Re}[\Omega]\langle\sigma_z\rangle \\ \mathrm{Im}[\Omega]\langle\sigma_x\rangle + \mathrm{Re}[\Omega]\langle\sigma_y\rangle \end{pmatrix}
$$

# Problem 2

Given

$$
H = -\vec{\mu}_S \cdot \vec{B} = \frac{g_S \mu_B}{\hbar} \vec{S} \cdot \vec{B},
$$

The equation of motion for the operator $S_\alpha$ is given by

$$
\begin{aligned}
\dot{S}_\alpha &= -\frac{i}{\hbar}[S_\alpha, H] \\
&= -\frac{i g_S \mu_B}{\hbar^2}[S_\alpha, S_\beta B_\beta] \\
&= -\frac{i g_S \mu_B}{\hbar^2}[S_\alpha, S_\beta] B_\beta \\
&= -\frac{i g_S \mu_B}{\hbar^2}\epsilon_{\alpha\beta\gamma} S_\gamma B_\beta \\
&= \frac{i g_S \mu_B}{\hbar^2}\epsilon_{\alpha\beta\gamma} S_\beta B_\gamma \\
&= \vec{\mu}_S \times \vec{B}
\end{aligned}
$$

# Problem 3

Orient the coordiante system such that $\hat{\alpha}$ points in the $\hat{z}$ direction. Then,

$$\begin{aligned}
e^{-i\vec{\alpha}\cdot\vec{S}/\hbar} &= e^{-i\alpha S_z/\hbar} \\
&= e^{-i\alpha\sigma_z/2} \\
&= \mathbb{I} - i\frac{\alpha}{2}\sigma_z + \frac{1}{2}\left(\frac{i}{2}\alpha\sigma_z\right)^2 + \frac{1}{6}\left(\frac{i}{2}\alpha\sigma_z\right)^3 + \dots \\
&= \mathbb{I} - i\frac{\alpha}{2}\sigma_z - \frac{1}{2}\left(\frac{\alpha}{2}\right)^2\mathbb{I} - \frac{1}{6}\left(\frac{\alpha}{2}\right)^3\sigma_z + \dots \\
&= \left(\mathbb{I} - \frac{1}{2}\left(\frac{\alpha}{2}\right)^2 + \dots\right) - i\left(\frac{\alpha}{2}\sigma_z + \frac{1}{6}\left(\frac{\alpha}{2}\right)^3 + \dots\right) \\
&= \cos\left(\frac{\alpha}{2}\right)\mathbb{I} - i\sin\left(\frac{\alpha}{2}\right)\sigma_z.
\end{aligned}$$

Given that the coordinate orientation was arbitrary, we have

$$e^{-i\vec{\alpha}\cdot\vec{S}} = \cos\left(\frac{\alpha}{2}\right)\mathbb{I} - i\sin\left(\frac{\alpha}{2}\right)(\hat{\alpha}\cdot\vec{\sigma}).$$

# Problem 4

(a) The relevant rotation operators in the standard basis are

$$\begin{aligned}
R_x(\pi) &= \cos\left(\frac{\pi}{2}\right)\mathbb{I} - i\sin\left(\frac{\pi}{2}\right)\sigma_x = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \\
R_x(\frac{\pi}{2}) &= \cos\left(\frac{\pi}{4}\right)\mathbb{I} - i\sin\left(\frac{\pi}{4}\right)\sigma_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \\
R_y(-\pi) &= \cos\left(\frac{\pi}{2}\right)\mathbb{I} + i\sin\left(\frac{\pi}{2}\right)\sigma_y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\end{aligned}$$

Now,

$$R_x(\pi)\left|+\right\rangle = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -i\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -i\left|-\right\rangle.$$

While

$$\begin{aligned}
R_x(\frac{\pi}{2})R_y(-\pi)R_x(\frac{\pi}{2})\left|+\right\rangle &= \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= -\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= -\left|-\right\rangle
\end{aligned}$$

The final states are equivalent up to a $\frac{\pi}{2}$ difference in phase.

(b) Taking into account the error $\epsilon$, the rotation operators are

$$R_x(\pi+\epsilon) = \begin{pmatrix} -\epsilon & -i\left(1-\frac{\epsilon^2}{2}\right) \\ -i\left(1-\frac{\epsilon^2}{2}\right) & -\epsilon \end{pmatrix}$$

$$R_x(\frac{\pi}{2}+\epsilon) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1-\epsilon-\frac{\epsilon^2}{2} & -i\left(1+\epsilon-\frac{\epsilon^2}{2}\right) \\ -i\left(1+\epsilon-\frac{\epsilon^2}{2}\right) & 1-\epsilon-\frac{\epsilon^2}{2} \end{pmatrix}$$

$$R_y(-\pi+\epsilon) = \begin{pmatrix} \epsilon & 1-\frac{\epsilon^2}{2} \\ \frac{\epsilon^2}{2}-1 & \epsilon \end{pmatrix}$$

The single $R_x(\pi+\epsilon)$ rotation gives

$$R_x(\pi+\epsilon)\left|+\right\rangle = -\epsilon\left|+\right\rangle - i(1-\frac{\epsilon^2}{2})\left|-\right\rangle$$

With error

$$\langle +|R_x(\pi+\epsilon)|+\rangle = -\epsilon$$

The composite rotation is given by (to second-order in $\epsilon$)

$$R_x(\pi/2+\epsilon)R_y(-\pi+\epsilon)R_x(\pi/2+\epsilon) = \begin{pmatrix} -2\epsilon^2 & 1-i\epsilon-\frac{\epsilon^2}{2} \\ \frac{\epsilon^2}{2}-i\epsilon-1 & -2\epsilon^2 \end{pmatrix}$$

With error

$$\langle +|R_x(\pi/2+\epsilon)R_y(-\pi+\epsilon)R_x(\pi/2+\epsilon)|+\rangle = -2\epsilon^2$$

Thus, the error of the single rotation is of order $\epsilon$, while the composite rotation's error is of order $\epsilon^2$.