Homework 4

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Problem 1

(a)

$$\begin{aligned}\gamma &= \frac{1}{\sqrt{1-\tanh^2(\alpha)}}\\ &= \frac{1}{\sqrt{\operatorname{sech}^2(\alpha)}}\\ &= \cosh(\alpha)\end{aligned}$$

$$\Lambda = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} = \begin{pmatrix} \cosh(\alpha) & 0 & 0 & -\sinh(\alpha) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh(\alpha) & 0 & 0 & \cosh(\alpha) \end{pmatrix}$$

(b) For p_3, boosting in the $-\hat{z}$ direction by rapidity α,

$$\begin{aligned} p_3' &= \Lambda p_3 \\ &= p_T \begin{pmatrix} \cosh(\alpha) & 0 & 0 & \sinh(\alpha) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh(\alpha) & 0 & 0 & \cosh(\alpha) \end{pmatrix} \begin{pmatrix} \cosh(y_3) \\ \cos(\phi) \\ \sin(\phi) \\ \sinh(y_3) \end{pmatrix} \\ &= p_T \begin{pmatrix} \cosh(\alpha)\cosh(y_3) + \sinh(\alpha)\sinh(y_3) \\ \cos(\phi) \\ \sin(\phi) \\ \sinh(\alpha)\cosh(y_3) + \cosh(\alpha)\sinh(y_3) \end{pmatrix} \\ &= p_T \begin{pmatrix} \cosh(y_3+\alpha) \\ \cos(\phi) \\ \sin(\phi) \\ \sinh(y_3+\alpha) \end{pmatrix}, \end{aligned}$$

and similarly for p_4. Hence, $y_i \to y_i + \alpha$.

(c) The sum of p_3 and p_4 is

$$p_3 + p_4 = p_T \begin{pmatrix} \cosh(y_3) + \cosh(y_4) \\ 0 \\ 0 \\ \sinh(y_3) + \sinh(y_4) \end{pmatrix}.$$

Considering just the $\hat{t}$ and $\hat{z}$ components, we want the transformation that takes this to the CM Frame, i.e.,

$$\begin{aligned}
\begin{pmatrix} p_T \\ 0 \end{pmatrix} &= p_T \begin{pmatrix} \cosh(\alpha) & -\sinh(\alpha) \\ -\sinh(\alpha) & \cosh(\alpha) \end{pmatrix} \begin{pmatrix} \cosh(y_3) + \cosh(y_4) \\ \sinh(y_3) + \sinh(y_4) \end{pmatrix} \\
&= p_T \begin{pmatrix} \cosh(\alpha)\cosh(y_3) + \cosh(\alpha)\cosh(y_4) - \sinh(\alpha)\sinh(y_3) - \sinh(\alpha)\sinh(y_4) \\ -\sinh(\alpha)\cosh(y_3) - \sinh(\alpha)\cosh(y_4) + \cosh(\alpha)\sinh(y_3) + \cosh(\alpha)\sinh(y_4) \end{pmatrix} \\
&= p_T \begin{pmatrix} \cosh(y_3 - \alpha) + \cosh(y_4 - \alpha) \\ \sinh(y_3 - \alpha) + \sinh(y_4 - \alpha) \end{pmatrix} \\
\implies \quad 0 &= \sinh(y_3 - \alpha) + \sinh(y_4 - \alpha) \\
\implies \quad \alpha &= \frac{1}{2}(y_3 + y_4)
\end{aligned}$$

(d)

$$p_1 + p_2 = \begin{pmatrix} \frac{\sqrt{s}}{2}(x_1 + x_2) \\ \frac{\sqrt{s}}{2}(x_1 - x_2) \end{pmatrix}$$

$$\begin{aligned}
p_1 + p_2 &= p_3 + p_4 \\
\implies \quad \frac{\sqrt{s}}{2}(x_1 + x_2) &= p_T(\cosh(y_3) + \cosh(y_4)) \\
\frac{\sqrt{s}}{2}(x_1 - x_2) &= p_T(\sinh(y_3) + \sinh(y_4)) \\
\implies \quad x_1 = \frac{p_T}{\sqrt{s}}\left(e^{y_3} + e^{y_4}\right),\ & x_2 = \frac{p_T}{\sqrt{s}}\left(e^{-y_3} + e^{-y_4}\right)
\end{aligned}$$

In the last step, substitute the definitions for sinh and cosh in terms of exponentials, then add the two equations to get x_1, and subtract to get x_2.

(e)

$$\begin{aligned}
\hat{s} &= (p_1 + p_2)^2 \\
&= s x_1 x_2 \\
&= p_T^2 \left(e^{y_3} + e^{y_4}\right)\left(e^{-y_3} + e^{-y_4}\right) \\
&= p_T^2 \left(2 + e^{y_3 - y_4} + e^{y_4 - y_3}\right) \\
&= p_T^2 \left(2 + 2\cosh(y_4 - y_3)\right)
\end{aligned}$$

$$\begin{aligned}
(p_3 + p_4)^2 &= p_T^2 \left[(\cosh(y_3) + \cosh(y_4))^2 - (\sinh(y_3) + \sinh(y_4))^2\right] \\
&= p_T^2 \left(2 + 2\cosh(y_4 - y_3)\right)
\end{aligned}$$

Where the last equality was confirmed with WolframAlpha.

(f) I think I could use

$$\hat{t} = (p_1 - p_3)^2 - \frac{1}{2}\hat{s}(1 - \cos\hat{\theta}),$$

but when I plug in the values for $(p_1 - p_3)^2$, and $\hat{s}$ the algebra gets really messy...

Problem 2

(a) The term with the $\hat{t}^2$ denominator corresponds to the leftmost diagram of eq. 13.31 (the t-channel process), while the other two terms correspond to the other two diagrams (the s- and u-channel diagrams).

(b)

(c) Due to the symmetry, the equation should be invariant under $\theta \to -\theta$. Under this transformation, we have that $\hat{t} \leftrightarrow \hat{u}$. Equation 13.26 is clearly invariant under $\hat{t} \leftrightarrow \hat{u}$, so it is consistent with this symmetry.

Problem 3

Seems colinear safe, but not IR safe. $\left|\vec{1}\right\rangle$