All About Spinors

...on *flat* spacetime

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UO

Theory meeting, June 27, 2024

Special and General Covariance

Some Philisophical Motivation

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- Special/General covariance $\to$ Special/General relativity

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- Bargmann ('54): reps up to sign are *exactly* the true reps of the universal cover

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- Fundamental group of $ISO(3,1)^+$ is $\mathbb{Z}_2 \rightarrow$ double cover
- In fact, $\mathcal{U}(ISO(3,1)^+) \cong ISL(2,\mathbb{C})$

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  - $\rightarrow$ use $\epsilon^{A}{}_{B}$, $\epsilon_{C}{}^{D}$ and their conjugates to avoid confusion

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- Given a Lie algebra $\mathfrak{g}$, the *universal enveloping algebra* $\mathscr{U}(\mathfrak{g})$ is the unique unital associative algebra whose representations correspond exactly to the representations of $\mathfrak{g}$
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  - Form the *Tensor algebra* $T(\mathfrak{g}) \coloneqq \mathbb{C} \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \cdots$
  - Recursively lift $[,]$ from $\mathfrak{g}$ to $T(\mathfrak{g})$
  - Fully lifted, $[,]$ obeys Leibniz's law: $T(\mathfrak{g})$ is a *poisson* algebra
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- Physically relevant cases are 1 and 2.1

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- For $s = 1/2$, the pair $(\phi^A, \xi_{A'})$ is known as a *Dirac* spinor, and the above equations are just the Dirac equation

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  where ξ and ζ are the auxiliary fields corresponding to ϕ and ψ, respectively
- Define an inner product by integrating the normal component of the particle current of a Cauchy surface Σ:

$$\langle\phi,\psi\rangle := \int_\Sigma j^{AA'} n_{AA'}\mathrm{d}V$$

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Summary

- The natural action of $ISL(2, \mathbb{C})$ on spinoral tensor fields $\phi^{A_1 \dots A_n}$ gives rise to all known physical fields
  - For even n (bosons), the representations are true reps of the Poincaré group (spinors not actually required)
  - For odd n (fermions), the reps are only reps up to sign, and spinors are necessary to describe them

References

[1] Wikipedia.

[2] R. M. Wald. *General Relativity*. The University of Chicago Press, 1984.