# Homework 3

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## Problem 1

In components, the cross product side of the "bac-cab" rule is

$$\left(\vec{B} \times \vec{C}\right)_r = B_s C_t \epsilon_{str}$$

$$\left(\vec{A} \times \left(\vec{B} \times \vec{C}\right)\right)_p = A_q B_s C_t \epsilon_{str} \epsilon_{qrp}$$

The dot product side is

$$\begin{aligned}
\left(\vec{B}\left(\vec{A} \cdot \vec{C}\right) - \vec{C}\left(\vec{A} \cdot \vec{B}\right)\right)_p &= A_t B_p C_t - A_s B_s C_p \\
&= A_q B_p C_t \delta_{qt} - A_q B_s C_p \delta_{qs} \\
&= A_q B_s C_t \delta_{qt} \delta_{ps} - A_q B_s C_t \delta_{qs} \delta_{pt} \\
&= A_q B_s C_t \left(\delta_{qt} \delta_{ps} - \delta_{qs} \delta_{pt}\right)
\end{aligned}$$

Putting it together,

$$\begin{aligned}
\vec{A} \times \left(\vec{B} \times \vec{C}\right) &= \vec{B}\left(\vec{A} \cdot \vec{C}\right) - \vec{C}\left(\vec{A} \cdot \vec{B}\right) \\
\implies A_q B_s C_t \epsilon_{str} \epsilon_{qrp} &= A_q B_s C_t \left(\delta_{qt} \delta_{ps} - \delta_{qs} \delta_{pt}\right) \\
\implies \epsilon_{str} \epsilon_{qrp} &= \delta_{qt} \delta_{ps} - \delta_{qs} \delta_{pt}
\end{aligned}$$

## Problem 2

Using the ladder operators,

$$L_\pm = L_x \pm i L_y,$$

we can write $L_x$ as

$$L_x = \frac{1}{2}\left(J_+ + J_-\right).$$

The expectation value, $\langle L_x \rangle$, is now immediatly obvious:

$$\langle L_x \rangle = \frac{1}{2}\left(\langle J_+ \rangle + \langle J_- \rangle\right) = 0.$$

For the expectation value of $L_x^2$, we simply note that by symmetry it must have the same value as $L_y^2$. Therefore,

$$L_x^2 = L_y^2 = \frac{1}{2}\left(L^2 - L_z^2\right) = \frac{\hbar^2}{2}(j(j+1) - m^2)$$

Since the first moments are zero, the variance (and hence the uncertainty) are trivial:

$$\sigma_x = \sigma_y = \sqrt{\frac{\hbar^2}{2}(j(j+1) - m^2)}$$

The restriction of $m$ to the range $\{-j, ...j\}$ ensures that $(j(j+1) - m^2) \geq 1$, so

$$\sigma_x \sigma_y = \frac{\hbar^2}{2}(j(j+1) - m^2) \geq \frac{\hbar}{2}$$

# Problem 3

For a radial displacement, $\vec{r} \to \vec{r} + \hat{r}\mathrm{d}r$,

$$f(r + \mathrm{d}r) - f(r) = \mathrm{d}r\hat{r} \cdot \nabla f$$

$$\implies \hat{r} \cdot \nabla f = \frac{f(r + \mathrm{d}r) - f(r)}{\mathrm{d}r} = \partial_r f.$$

For a polar angular displacement, $\vec{r} \to \vec{r} + r\mathrm{d}\theta\hat{\theta}$,

$$f(\theta + \mathrm{d}\theta) - f(\theta) = r\mathrm{d}\theta\hat{\theta}\nabla f$$

$$\implies \hat{\theta} \cdot \nabla f = \frac{1}{r}\frac{f(\theta + \mathrm{d}\theta) - f(\theta)}{\mathrm{d}\theta} = \frac{1}{r}\partial_\theta f.$$

For an azimuthal angular displacement, $\vec{r} \to \vec{r} + r\sin\theta\hat{\phi}$,

$$f(\theta + \mathrm{d}\phi) - f(\phi) = r\sin\theta\mathrm{d}\phi\hat{\phi} \cdot \nabla f$$

$$\implies \hat{\phi} \cdot \nabla f = \frac{1}{r\sin\theta}\partial_\phi f.$$