Zurek's Density Matrices

Sean Ericson

UO

Group meeting, February 29, 2024

The Total State

$$|\psi_{Z1}\rangle = \sqrt{p}\,|00_A0_B\rangle + \sqrt{q}\,\left|1\tilde{1}_A\tilde{1}_B\right\rangle$$

The Total State

$$|\psi_{Z1}\rangle = \sqrt{p}\,|00_A0_B\rangle + \sqrt{q}\,\left|1\tilde{1}_A\tilde{1}_B\right\rangle$$

where

The Total State

$$|\psi_{Z1}\rangle = \sqrt{p}\,|00_A0_B\rangle + \sqrt{q}\,\left|1\tilde{1}_A\tilde{1}_B\right\rangle$$

where

$$|0_X\rangle := \bigotimes_{i=1}^{N_X} |0\rangle$$

The Total State

$$|\psi_{Z1}\rangle = \sqrt{p}\,|00_A0_B\rangle + \sqrt{q}\,\left|1\tilde{1}_A\tilde{1}_B\right\rangle$$

where

$$|0_X\rangle := \bigotimes_{i=1}^{N_X} |0\rangle$$

$$\left|\tilde{1}_X\right\rangle := \bigotimes_{i=1}^{N_X} \left|\tilde{1}\right\rangle$$

The Total State

$$|\psi_{Z1}\rangle = \sqrt{p}\,|00_A0_B\rangle + \sqrt{q}\,\left|1\tilde{1}_A\tilde{1}_B\right\rangle$$

where

$$|0_X\rangle := \bigotimes_{i=1}^{N_X} |0\rangle$$

$$\left|\tilde{1}_X\right\rangle := \bigotimes_{i=1}^{N_X} \left|\tilde{1}\right\rangle$$

$$\left|\tilde{1}\right\rangle := s\,|0\rangle + c\,|1\rangle$$

The Total State (cont.)

$$\begin{aligned}\rho_{Z1} &= |\psi_{Z1}\rangle\langle\psi_{Z1}| \\ &= p\,|00_A0_B\rangle\langle 00_A0_B| + \sqrt{pq}\,\left|00_A0_B\right\rangle\left\langle 1\tilde{1}_A\tilde{1}_B\right| \\ &\quad + \sqrt{pq}\,\left|1\tilde{1}_A\tilde{1}_B\right\rangle\left\langle 00_A0_B\right| + q\,\left|1\tilde{1}_A\tilde{1}_B\right\rangle\left\langle 1\tilde{1}_A\tilde{1}_B\right|\end{aligned}$$

The System's State

$$\rho_S = \mathsf{Tr}_{AB}[\rho_{Z1}]$$

The System's State

$$\begin{aligned}\rho_S &= \mathrm{Tr}_{AB}[\rho_{Z1}] \\ &= \sum_{\vec{x}\in\{0,1\}^{n_A+n_B}} \left(\mathbb{I}\otimes\langle\vec{x}|\right)\rho_{Z1}\left(\mathbb{I}\otimes|\vec{x}\rangle\right)\end{aligned}$$

The System's State

$$
\begin{aligned}
\rho_S &= \mathrm{Tr}_{AB}[\rho_{Z1}] \\
&= \sum_{\vec{x}\in\{0,1\}^{n_A+n_B}} \left(\mathbb{I}\otimes\langle\vec{x}|\right)\rho_{Z1}\left(\mathbb{I}\otimes|\vec{x}\rangle\right) \\
&= \sum_{\vec{x}} p|\left\langle 0_{AB}|\vec{x}\right\rangle|^2\,|0\rangle\langle 0| + \sqrt{pq}\left\langle\vec{x}\middle|\tilde{1}_{AB}\right\rangle\left\langle 0_{AB}|\vec{x}\right\rangle|0\rangle\langle 1| \\
&\qquad + \sqrt{pq}\left\langle\vec{x}\middle|\tilde{1}_{AB}\right\rangle\left\langle 0_{AB}|\vec{x}\right\rangle|1\rangle\langle 0| + q|\left\langle\tilde{1}_{AB}\middle|\vec{x}\right\rangle|^2\,|1\rangle\langle 1|
\end{aligned}
$$

The System's State (cont.)

Obviously,

$$\sum_{\vec{x}} \langle 0_{AB} | \vec{x} \rangle = \left\langle 0_{AB} \middle| \vec{0} \right\rangle = 1$$

The System's State (cont.)

Obviously,

$$\sum_{\vec{x}} \langle 0_{AB} | \vec{x} \rangle = \left\langle 0_{AB} \middle| \vec{0} \right\rangle = 1$$

but more interesting is

$$\sum_{\vec{x}} | \left\langle \tilde{1}_{AB} \middle| \vec{x} \right\rangle |^2$$

The System's State (cont.)

Obviously,

$$\sum_{\vec{x}} \langle 0_{AB} | \vec{x} \rangle = \left\langle 0_{AB} \middle| \vec{0} \right\rangle = 1$$

but more interesting is

$$\sum_{\vec{x}} | \left\langle \tilde{1}_{AB} \middle| \vec{x} \right\rangle |^2$$

We note that

$$| \left\langle \tilde{1}_{AB} \middle| \vec{x} \right\rangle |^2 = | \left\langle x_1 x_2 \cdots x_N \middle| \bigotimes^N (s |0\rangle + c |1\rangle) \right\rangle |^2$$

The System's State (cont.)

Obviously,

$$\sum_{\vec{x}} \langle 0_{AB} | \vec{x} \rangle = \left\langle 0_{AB} \middle| \vec{0} \right\rangle = 1$$

but more interesting is

$$\sum_{\vec{x}} | \left\langle \tilde{1}_{AB} \middle| \vec{x} \right\rangle |^2$$

We note that

$$| \left\langle \tilde{1}_{AB} \middle| \vec{x} \right\rangle |^2 = | \left\langle x_1 x_2 \cdots x_N \middle| \bigotimes^N (s\,|0\rangle + c\,|1\rangle) \right\rangle |^2$$
$$= s^{N-\mathcal{H}(\vec{x})} c^{\mathcal{H}(\vec{x})}$$

Obviously,

$$\sum_{\vec{x}} \langle 0_{AB} | \vec{x} \rangle = \left\langle 0_{AB} \middle| \vec{0} \right\rangle = 1$$

but more interesting is

$$\sum_{\vec{x}} | \left\langle \tilde{1}_{AB} \middle| \vec{x} \right\rangle |^2$$

We note that

$$| \left\langle \tilde{1}_{AB} \middle| \vec{x} \right\rangle |^2 = | \left\langle x_1 x_2 \cdots x_N \middle| \bigotimes^N (s |0\rangle + c |1\rangle) \right\rangle |^2$$
$$= s^{N - \mathcal{H}(\vec{x})} c^{\mathcal{H}(\vec{x})}$$

where $\mathcal{H}(\vec{x})$ is the Hamming weight of $\vec{x}$ (the number of 1s in it).

The System's State (cont.)

So,

The System's State (cont.)

So,

$$\sum_{\vec{x}} | \left\langle \tilde{1}_{AB} \middle| \vec{x} \right\rangle |^2 = \sum_{i=1}^{N} |s^{N-i} c^i|^2$$

The System's State (cont.)

So,

$$\sum_{\vec{x}} |\left\langle \tilde{1}_{AB} \middle| \vec{x} \right\rangle|^2 = \sum_{i=1}^{N} |s^{N-i} c^i|^2$$
$$= \sum_{i=1}^{N} (s^2)^{N-i} (c^2)^i$$

The System's State (cont.)

So,

$$\sum_{\vec{x}} | \left\langle \tilde{1}_{AB} \middle| \vec{x} \right\rangle |^2 = \sum_{i=1}^{N} |s^{N-i} c^i|^2$$
$$= \sum_{i=1}^{N} (s^2)^{N-i} (c^2)^i$$
$$= (s^2 + c^2)^N$$

The System's State (cont.)

So,

$$\begin{aligned}
\sum_{\vec{x}} |\left\langle \tilde{1}_{AB} \middle| \vec{x} \right\rangle|^2 &= \sum_{i=1}^{N} |s^{N-i} c^i|^2 \\
&= \sum_{i=1}^{N} (s^2)^{N-i} (c^2)^i \\
&= (s^2 + c^2)^N \\
&= 1^N \\
&= 1
\end{aligned}$$

The System's State (cont.)

Then, finally, the system's state is

The System's State (cont.)

Then, finally, the system's state is

$$\rho_S = \sum_{\vec{x}} p|\langle 0_{AB}|\vec{x}\rangle|^2 |0\rangle\langle 0| + \sqrt{pq}\left\langle \vec{x}\middle|\tilde{1}_{AB}\right\rangle \langle 0_{AB}|\vec{x}\rangle |0\rangle\langle 1| + \sqrt{pq}\left\langle \vec{x}\middle|\tilde{1}_{AB}\right\rangle \langle 0_{AB}|\vec{x}\rangle |1\rangle\langle 0| + q|\left\langle \tilde{1}_{AB}\middle|\vec{x}\right\rangle|^2 |1\rangle\langle 1|$$

The System's State (cont.)

Then, finally, the system's state is

$$\rho_S = \sum_{\vec{x}} p|\langle 0_{AB}|\vec{x}\rangle|^2 |0\rangle\langle 0| + \sqrt{pq}\left\langle \vec{x}\middle|\tilde{1}_{AB}\right\rangle \langle 0_{AB}|\vec{x}\rangle |0\rangle\langle 1|$$
$$+ \sqrt{pq}\left\langle \vec{x}\middle|\tilde{1}_{AB}\right\rangle \langle 0_{AB}|\vec{x}\rangle |1\rangle\langle 0| + q|\left\langle \tilde{1}_{AB}\middle|\vec{x}\right\rangle|^2 |1\rangle\langle 1|$$
$$= p\,|0\rangle\langle 0| + s^N\sqrt{pq}\,|0\rangle\langle 1| + s^N\sqrt{pq}\,|1\rangle\langle 0| + q\,|1\rangle\langle 1|$$

The System's State (cont.)

Then, finally, the system's state is

$$
\begin{aligned}
\rho_S &= \sum_{\vec{x}} p|\langle 0_{AB}|\vec{x}\rangle|^2 |0\rangle\langle 0| + \sqrt{pq}\left\langle \vec{x}\middle|\tilde{1}_{AB}\right\rangle \langle 0_{AB}|\vec{x}\rangle |0\rangle\langle 1| \\
&\qquad + \sqrt{pq}\left\langle \vec{x}\middle|\tilde{1}_{AB}\right\rangle \langle 0_{AB}|\vec{x}\rangle |1\rangle\langle 0| + q|\left\langle \tilde{1}_{AB}\middle|\vec{x}\right\rangle|^2 |1\rangle\langle 1| \\
&= p\,|0\rangle\langle 0| + s^N\sqrt{pq}\,|0\rangle\langle 1| + s^N\sqrt{pq}\,|1\rangle\langle 0| + q\,|1\rangle\langle 1| \\
&= \begin{pmatrix} p & s^N\sqrt{pq} \\ s^N\sqrt{pq} & q \end{pmatrix}
\end{aligned}
$$

The System and Alice's (or Bob's) Combined State

Calculating the combined state of the system and one of the observers carries through almost identically:

The System and Alice's (or Bob's) Combined State

Calculating the combined state of the system and one of the observers carries through almost identically:

$$\rho_{SA} = \mathrm{Tr}_B[\rho_{Z1}]$$

The System and Alice's (or Bob's) Combined State

Calculating the combined state of the system and one of the observers carries through almost identically:

$$\begin{aligned}\rho_{SA} &= \mathrm{Tr}_B[\rho_{Z1}] \\ &= \sum_{\vec{x}\in\{0,1\}^{N_B+1}} p|\langle 0_B|\vec{x}\rangle|^2 |00_A\rangle\langle 00_A| + q|\langle \tilde{1}_B|\vec{x}\rangle|^2 \left|1\tilde{1}_A\right\rangle\left\langle 1\tilde{1}_A\right| \\ &\qquad + \sqrt{pq}\,\langle 0_B|\vec{x}\rangle \left\langle \tilde{1}_B|\vec{x}\right\rangle \left[|00_A\rangle\left\langle 1\tilde{1}_a\right| + \left|1\tilde{1}_a\right\rangle\langle 00_A|\right]\end{aligned}$$

The System and Alice's (or Bob's) Combined State

Calculating the combined state of the system and one of the observers carries through almost identically:

$$\begin{aligned}
\rho_{SA} &= \mathrm{Tr}_B[\rho_{Z1}] \\
&= \sum_{\vec{x}\in\{0,1\}^{N_B+1}} p|\langle 0_B|\vec{x}\rangle|^2 \, |00_A\rangle\langle 00_A| + q|\langle \tilde{1}_B|\vec{x}\rangle|^2 \, |1\tilde{1}_A\rangle\langle 1\tilde{1}_A| \\
&\qquad + \sqrt{pq}\, \langle 0_B|\vec{x}\rangle \langle \tilde{1}_B|\vec{x}\rangle \left[|00_A\rangle\langle 1\tilde{1}_a| + |1\tilde{1}_a\rangle\langle 00_A|\right] \\
&= p\,|00_A\rangle\langle 00_A| + q\,|1\tilde{1}_A\rangle\langle 1\tilde{1}_A| + \sqrt{pq}\left[|00_A\rangle\langle 1\tilde{1}_A| + |1\tilde{1}_A\rangle\langle 00_A|\right]
\end{aligned}$$

The System and Alice's (or Bob's) Combined State

Calculating the combined state of the system and one of the observers carries through almost identically:

$$\begin{aligned}
\rho_{SA} &= \mathrm{Tr}_B[\rho_{Z1}] \\
&= \sum_{\vec{x}\in\{0,1\}^{N_B+1}} p|\langle 0_B|\vec{x}\rangle|^2 \,|00_A\rangle\!\langle 00_A| + q|\langle \tilde{1}_B|\vec{x}\rangle|^2 \,|1\tilde{1}_A\rangle\!\langle 1\tilde{1}_A| \\
&\qquad + \sqrt{pq}\,\langle 0_B|\vec{x}\rangle\,\langle \tilde{1}_B|\vec{x}\rangle\,\big[|00_A\rangle\!\langle 1\tilde{1}_a| + |1\tilde{1}_a\rangle\!\langle 00_A|\big] \\
&= p\,|00_A\rangle\!\langle 00_A| + q\,|1\tilde{1}_A\rangle\!\langle 1\tilde{1}_A| + \sqrt{pq}\,\big[|00_A\rangle\!\langle 1\tilde{1}_A| + |1\tilde{1}_A\rangle\!\langle 00_A|\big] \\
&= \begin{pmatrix} p & s^{N_B}\sqrt{pq} \\ s^{N_B}\sqrt{pq} & q \end{pmatrix}
\end{aligned}$$

The System and Alice's (or Bob's) Combined State

Calculating the combined state of the system and one of the observers carries through almost identically:

$$\begin{aligned}
\rho_{SA} &= \mathrm{Tr}_B[\rho_{Z1}] \\
&= \sum_{\vec{x}\in\{0,1\}^{N_B+1}} p|\left\langle 0_B \middle| \vec{x} \right\rangle|^2 \left|00_A\middle\rangle\middle\langle 00_A\right| + q|\left\langle \tilde{1}_B \middle| \vec{x} \right\rangle|^2 \left|1\tilde{1}_A\middle\rangle\middle\langle 1\tilde{1}_A\right| \\
&\qquad + \sqrt{pq}\left\langle 0_B \middle| \vec{x} \right\rangle \left\langle \tilde{1}_B \middle| \vec{x} \right\rangle \left[\left|00_A\middle\rangle\middle\langle 1\tilde{1}_a\right| + \left|1\tilde{1}_a\middle\rangle\middle\langle 00_A\right|\right] \\
&= p\left|00_A\middle\rangle\middle\langle 00_A\right| + q\left|1\tilde{1}_A\middle\rangle\middle\langle 1\tilde{1}_A\right| + \sqrt{pq}\left[\left|00_A\middle\rangle\middle\langle 1\tilde{1}_A\right| + \left|1\tilde{1}_A\middle\rangle\middle\langle 00_A\right|\right] \\
&= \begin{pmatrix} p & s^{N_B}\sqrt{pq} \\ s^{N_B}\sqrt{pq} & q \end{pmatrix}
\end{aligned}$$

Note that the state has a similar form to that of the system alone, but only when written in the $\{|00\rangle, |0\tilde{1}\rangle\}$ basis.

Alice or Bob's State

Here's where the paper looses me.

$$\begin{aligned}
\rho_B &= \mathrm{Tr}_{SA}[\rho_{Z1}] \\
&= \sum_{\vec{x}\in\{0,1\}^{N_A+1}} p|\left\langle 00_A|\vec{x}\right\rangle|^2 \left|0_B\right\rangle\!\left\langle 0_B\right| + q|\left\langle 1\tilde{1}_A|\vec{x}\right\rangle|^2 \left|\tilde{1}_B\right\rangle\!\left\langle\tilde{1}_B\right| \\
&\qquad + \sqrt{pq}\left\langle 00_A|\vec{x}\right\rangle\left\langle 1\tilde{1}_A|\vec{x}\right\rangle\left[\left|0_B\right\rangle\!\left\langle\tilde{1}_B\right| + \left|\tilde{1}_B\right\rangle\!\left\langle 0_B\right|\right] \\
&= p\left|0_B\right\rangle\!\left\langle 0_B\right| + q\left|\tilde{1}_B\right\rangle\!\left\langle\tilde{1}_B\right| \\
&= \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix}
\end{aligned}$$

in the $\{\left|0_B\right\rangle, \left|\tilde{1}_B\right\rangle\}$ basis (*not* orthonormal).

This is not (superficially)

$$\begin{pmatrix} p & s^{N_A}\sqrt{pq} \\ s^{N_A}\sqrt{pq} & q \end{pmatrix}$$

Are They the Same State?

Is there as basis in which ρ_{SA} has the form that Zurek proposes?

Are They the Same State?

Is there as basis in which ρ_{SA} has the form that Zurek proposes?

Eigenvalues

$$\lambda_{\pm} = \frac{1}{2}\left(1 \pm 2\sqrt{p^2 + pqs^2}\right)$$