Homework 2

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Phys 684

October 14, 2024

Problem 1 (Berman 2.9)

In the adiabatic approximation, the dressed-state amplitudes satisfy (eq. 2.146 in the textbook)

$$\begin{aligned} c_{d_1}(t) &= e^{\frac{i}{2}\xi(t)} c_{d_1}(t_0) \\ c_{d_2}(t) &= e^{-\frac{i}{2}\xi(t)} c_{d_2}(t_0), \end{aligned}$$

where

$$\xi(t) = \int_{t_0}^{t} \mathrm{d}t' \Omega(t').$$

In this case, $t_0 = -\infty$ and

$$\Omega_0(t) = \Omega_0 e^{-\left(\frac{t}{T}\right)^2},$$

so we can integrate that to get

$$\xi(t) = \frac{\sqrt{\pi}}{2} \Omega_0 T \left(1 + \operatorname{erf}\left(\frac{t}{T}\right)\right).$$

We transform between the dressed-states and the field-interaction basis states via

$$\begin{pmatrix} \tilde{c}_1 \\ \tilde{c}_2 \end{pmatrix} = \begin{pmatrix} c_\theta & s_\theta \\ -s_\theta & c_\theta \end{pmatrix} \begin{pmatrix} c_{d_1} \\ c_{d_2} \end{pmatrix}; \quad \begin{pmatrix} c_{d_1} \\ c_{d_2} \end{pmatrix} = \begin{pmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{pmatrix} \begin{pmatrix} \tilde{c}_1 \\ \tilde{c}_2 \end{pmatrix},$$

where

$$c_\theta = \sqrt{\frac{1}{2}\left(1 + \frac{\delta}{\Omega(t)}\right)}; \quad s_\theta = \sqrt{\frac{1}{2}\left(1 - \frac{\delta}{\Omega(t)}\right)},$$

and $\Omega(t) = \sqrt{\delta^2 + \Omega_0^2(t)}$. Given that $\tilde{c}_1(-\infty) = 1$ and $\tilde{c}_2(-\infty) = 0$, we find that

$$\begin{aligned} c_{d_1}(-\infty) &= c_\theta(-\infty) = 1 \\ c_{d_2}(-\infty) &= s_\theta(-\infty) = 0 \end{aligned}$$

Putting everything together, we find

$$\begin{aligned}\begin{pmatrix}\tilde{c}_1(t)\\ \tilde{c}_2(t)\end{pmatrix} &= \begin{pmatrix}c_\theta & s_\theta\\ -s_\theta & c_\theta\end{pmatrix}\begin{pmatrix}c_{d_1}(t)\\ c_{d_2}(t)\end{pmatrix}\\ &= \begin{pmatrix}c_\theta(t)c_{d_1}(t) + s_\theta(t)c_{d_2}(t)\\ -s_\theta c_{d_1}(t) + c_\theta c_{d_2}(t)\end{pmatrix}\\ &= \begin{pmatrix}c_\theta(t)\exp\left[\frac{i}{2}\xi(t)\right]\\ -s_\theta(t)\exp\left[\frac{i}{2}\xi(t)\right]\end{pmatrix}\end{aligned}$$

Problem 2 (Berman 2.17)

See attached Mathematica print-out.

Problem 3

(a) Given that

$$H_0 = \hbar\begin{pmatrix}\omega_1 & 0\\ 0 & \omega_2\end{pmatrix},$$

we have that

$$\begin{aligned}\exp(iH_0t/\hbar) &= \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + it\begin{pmatrix}\omega_1 & 0\\ 0 & \omega_2\end{pmatrix} + \frac{(it)^2}{2}\begin{pmatrix}\omega_1 & 0\\ 0 & \omega_2\end{pmatrix}^2 + \frac{(it)^3}{3!}\begin{pmatrix}\omega_1 & 0\\ 0 & \omega_2\end{pmatrix}^3 + \cdots\\ &= \begin{pmatrix}1 + i\omega_1 t + \frac{1}{2}(i\omega_1 t)^2 + \frac{1}{3!}(i\omega_1 t)^3 + \cdots & 0\\ 0 & 1 + i\omega_2 t + \frac{1}{2}(i\omega_2 t)^2 + \frac{1}{3!}(i\omega_2 t)^3 + \cdots\end{pmatrix}\\ &= \begin{pmatrix}\exp(i\omega_1 t) & 0\\ 0 & \exp(i\omega_2 t)\end{pmatrix}.\end{aligned}$$

(b) In the interaction representation, we factor out this free phase evolution by writing

$$|\psi(t)\rangle_{\mathrm{I}} = \bar{c}_1(t)e^{-i\omega_1 t}|1\rangle + \bar{c}_2(t)e^{-i\omega_2 t}|2\rangle,$$

that is, we make the (time-dependent) unitary transformation

$$|\psi(t)\rangle_{\mathrm{S}} \to |\psi(t)\rangle_{\mathrm{I}} = U(t)\,|\psi(t)\rangle_{\mathrm{S}},$$

where

$$U(t) = \begin{pmatrix}e^{-i\omega_1 t} & 0\\ 0 & e^{-i\omega_2 t}\end{pmatrix}.$$

We get the effective interaction hamiltonian by making the inverse transformation on V:

$$V_{\mathrm{I}} = U^\dagger V U = \hbar\Omega_0\cos(\omega t)\begin{pmatrix}0 & e^{-i\omega_0 t}\\ e^{i\omega_0 t} & 0\end{pmatrix},$$

where the phase ϕ has been absorbed into the (complex) Rabi frequency.

(c) In the field-interaction representation,

$$|\psi(t)\rangle_{\text{FI}} = \tilde{c}_1(t)e^{i\omega t/2}\,|1\rangle + \tilde{c}_2(t)e^{-i\omega t/2}\,|2\rangle\,.$$

So, in this case,

$$U = \begin{pmatrix} e^{-i\omega t/2} & 0 \\ 0 & e^{i\omega t/2} \end{pmatrix}$$

Problem 4

We seek to find the eigenvectors of

$$\tilde{H} = \frac{\hbar}{2}\begin{pmatrix} -\delta & \Omega_0 \\ \Omega_0 & \delta \end{pmatrix}.$$

Firstly, the eigenvalues of a 2x2 matrix are given by

$$\frac{1}{2}\left(T \pm \sqrt{T^2 - 4D}\right),$$

where T is the trace, and D is the determinant. In this case,

$$T = 0; \quad D = -\delta^2 - \Omega_0^2 =: -\frac{\hbar^2\Omega^2}{4},$$

so the eigenvalues are simply $\pm\frac{1}{2}\hbar\Omega$. Let the + eigenvector be of the form

$$\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix},$$

i.e.

$$\frac{\hbar}{2}\begin{pmatrix} -\delta & \Omega_0 \\ \Omega_0 & \delta \end{pmatrix}\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} = \frac{\hbar\Omega}{2}\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}.$$

Taking the first entry, we find

$$\begin{aligned} -\delta\sin\theta + \Omega_0\cos\theta &= \Omega\sin\theta \\ \implies \qquad (\Omega + \delta)\sin\theta &= \Omega_0\cos\theta \\ \implies \qquad \tan\theta &= \frac{\Omega_0}{\Omega + \delta} \\ \implies \qquad \theta &= \operatorname{atan}\frac{\Omega_0}{\Omega + \delta}. \end{aligned}$$

Now, using the identity

$$\tan(2\operatorname{atan}(x)) = \frac{-2x}{x^2 - 1},$$

we find

$$\begin{aligned} \tan(2\theta) &= \frac{-2\left(\frac{\Omega_0}{\Omega+\delta}\right)}{\left(\frac{\Omega_0}{\Omega+\delta}\right)^2 - 1} \\ &= \frac{\Omega_0}{\delta}. \end{aligned}$$

The eigenvectors are then

$$
\begin{aligned}
|+\rangle &= \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} \\
|-\rangle &= \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix},
\end{aligned}
$$

with θ given implicitly above.

Problem 5

The general solution is given by

$$
\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} = \begin{pmatrix} e^{-i\delta t/2}\left[\left[\cos\frac{\Omega t}{2} + i\frac{\delta}{\Omega}\sin\frac{\Omega t}{2}\right]c_1(0) - i\frac{\Omega_0^*}{\Omega}\sin\frac{\Omega t}{2}c_2(0)\right] \\ e^{i\delta t/2}\left[-i\frac{\Omega_0}{\Omega}\sin\frac{\Omega t}{2}c_1(0) + \left[\cos\frac{\Omega t}{2} - i\frac{\delta}{\Omega}\sin\frac{\Omega t}{2}\right]c_2(0)\right] \end{pmatrix}.
$$

In the $|\delta| \gg \Omega_0$ limit, we have that

$$
\Omega = \sqrt{\Omega_0^2 + \delta^2} \approx \delta(1 + \frac{\Omega_0^2}{\delta^2})
$$

(a) With the initial conditions $c_1(0) = 1$ and $c_2(0) = 0$, we have

$$
\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} = \begin{pmatrix} e^{-i\delta t/2}\left(\cos\frac{\Omega t}{2} + i\frac{\delta}{\Omega}\sin\frac{\Omega t}{2}\right) \\ -i\frac{\Omega_0}{\Omega}e^{i\delta t/2}\sin\frac{\Omega t}{2} \end{pmatrix}
$$

Focusing on c_1, we find

$$
\begin{aligned}
c_1(t) &= e^{-i\delta t/2}\left(\cos\frac{\Omega t}{2} + i\frac{\delta}{\Omega}\sin\frac{\Omega t}{2}\right) \\
&= \frac{1}{2}e^{-i\delta t/2}\left[\left(e^{i\Omega t/2} + e^{-i\Omega t/2}\right) + \frac{\delta}{\Omega}\left(e^{i\Omega t/2} - e^{-i\Omega t/2}\right)\right] \\
&= \frac{1}{2\Omega}e^{-i(\delta+\Omega)t/2}\left[(\Omega - \delta) + (\Omega + \delta)e^{i\Omega t}\right] \\
&= \frac{\Omega + \delta}{2\Omega}e^{-i(\delta-\Omega)t/2} + O(\frac{\Omega_0^2}{\delta^2}).
\end{aligned}
$$

Now, since this is in the interaction representation, we have that

$$
\begin{aligned}
|\psi(t)\rangle &= c_1(t)e^{-i\omega_1 t}|1\rangle \\
&\propto e^{-i(\omega_1 + (\delta-\Omega)/2)t}|1\rangle,
\end{aligned}
$$

i.e., the $|1\rangle$ state has an apparent energy shift of

$$
\begin{aligned}
\Delta &= \frac{1}{2}(\delta - \Omega) \\
&\approx \frac{1}{2}\left(\delta - \delta(1 - \frac{\Omega_0^2}{2\delta^2})\right) \\
&= \frac{\Omega_0^2}{4\delta}
\end{aligned}
$$

(b) With the initial conditions $c_1(0) = 0$ and $c_2(0) = 1$, focusing on c_2 we have

$$\begin{aligned}
c_2(t) &= e^{i\delta t/2}\left[\cos\frac{\Omega t}{2} - i\frac{\delta}{\Omega}\sin\frac{\Omega t}{2}\right] \\
&= \frac{1}{2}e^{i\delta t/2}\left[\left(e^{i\Omega t/2} + e^{-i\Omega t/2}\right) - i\frac{\delta}{\Omega}\left(e^{i\Omega t/2} - e^{-i\Omega t/2}\right)\right] \\
&= \frac{1}{2\Omega}e^{i(\delta-\Omega)t/2}\left[\delta + \Omega - (\delta - \Omega)e^{i\Omega t}\right] \\
&= \frac{\Omega+\delta}{2\Omega}e^{i(\delta-\Omega)t/2} + O(\frac{\Omega_0^2}{\delta^2})
\end{aligned}$$

So,

$$\begin{aligned}
|\psi(t)\rangle &= c_2(t)e^{-i\omega_2 t}\,|2\rangle \\
&\propto e^{-i(\omega_2-(\delta-\Omega)/2)t}\,|2\rangle\,,
\end{aligned}$$

and we see that the shift is the same as for the $|1\rangle$ state, just with opposite sign, i.e. $-\Omega_0^2/4\delta$.

```
In[10]:= Symbolize[δ0]; Symbolize[Ω0]; Symbolize[T0]; Symbolize[HI];
Symbolize[ψI]; Symbolize[c1]; Symbolize[c2];
Symbolize[sθ]; Symbolize[cθ]; Symbolize[ω1]; Symbolize[ω2];
$Assumptions = {T > 0 , t ∈ ℝ, ω1 > 0, ω2 > 0, ω > 0};
```

Problem 1

```
In[∘]:= Ω0[t_] = A ⅇ^(-(t/T)^2);
R[t_] = Sqrt[δ^2 + (Ω0[t])^2] // FullSimplify;

In[∘]:= Integrate[Ω0[t], {t, -∞, t}]
Out[∘]=
```

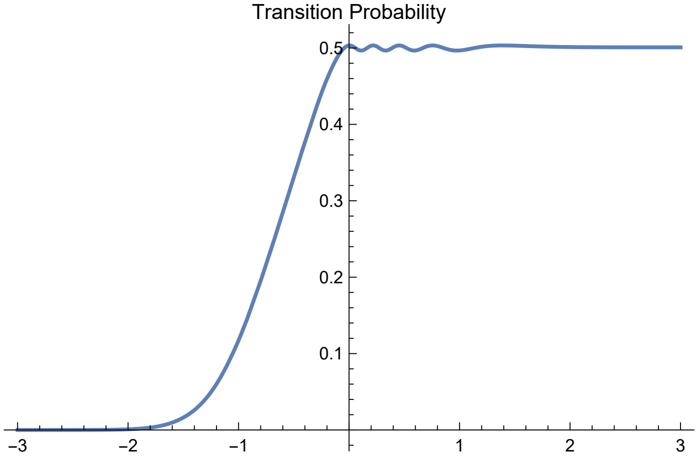
$$\frac{1}{2} A \sqrt{\pi}\, T \left(1 + \mathrm{Erf}\left[\frac{t}{T}\right]\right)$$

Problem 2

```
In[∘]:= δ[t_] = δ0 (1 - ⅇ^(t/T))^3 HeavisideTheta[-t];
HI[t_] = 1/2 {{0, Ω0[t] ⅇ^(-ⅈ δ[t])}, {Ω0[t] ⅇ^(ⅈ δ[t]), 0}};
ψI[t_] = {{c1[t]}, {c2[t]}};
T0 = 3;
T = 1;
δ0 = 30;
A = 30;
soln1 =
  NDSolve[{ⅈ D[ψI[t], t] == HI[t].ψI[t], ψI[-T0] == {{1}, {0}}}, c1, {t, -T0, T0}];
soln2 = NDSolve[{ⅈ D[ψI[t], t] == HI[t].ψI[t], ψI[-T0] == {{1}, {0}}}, c2, {t, -T0, T0}];
```

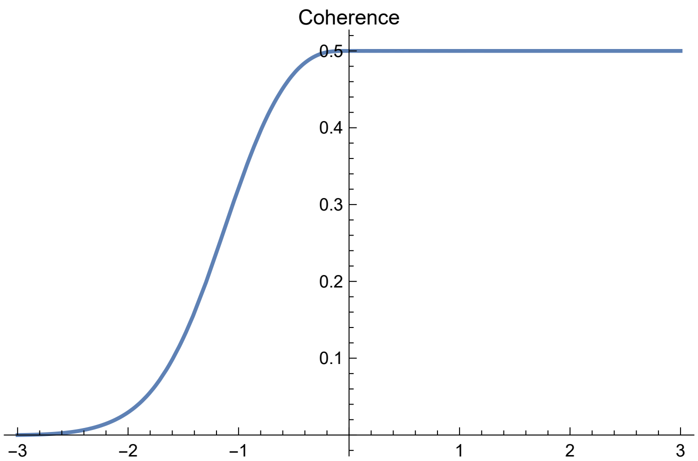
```
In[•]:= Plot[Evaluate[Abs[c2[x]]^2 /. soln2], {x, -T0, T0},
         PlotRange → All, PlotLabel → "Transition Probability"]
```

Out[•]=




```
In[•]:= Plot[Evaluate[Evaluate[Abs[c1[x] Conjugate[c2[x]]] /. soln1] /. soln2],
         {x, -T0, T0}, PlotRange → All, PlotLabel → "Coherence"]
```

Out[•]=



Problem 3

```
In[5]:= U = {{e^(-i ω1 t), 0}, {0, e^(-i ω2 t)}};
        V = {{0, Omega Cos[ω t]}, {Omega Cos[ω t], 0}};

In[7]:= U†.V.U // FullSimplify // MatrixForm
```

Out[7]//MatrixForm=

$$\begin{pmatrix} 0 & e^{i\,t\,(\omega_1-\omega_2)}\ \text{Omega Cos}[t\,\omega] \\ e^{i\,t\,(-\omega_1+\omega_2)}\ \text{Omega Cos}[t\,\omega] & 0 \end{pmatrix}$$

In[16]:= $U = \{\{e^{-i\omega t/2}, 0\}, \{0, e^{i\omega t/2}\}\};$

$U^{\dagger}.V.U$ // FullSimplify // MatrixForm

Out[17]//MatrixForm=

$$\begin{pmatrix} 0 & e^{i t \omega}\,\text{Omega Cos}[t\,\omega] \\ e^{-i t \omega}\,\text{Omega Cos}[t\,\omega] & 0 \end{pmatrix}$$

Problem 4

In[∘]:= $R = \sqrt{\delta^2 + \Omega_0{}^2};$

$$\frac{-2\,\frac{\Omega_0}{R+\delta}}{\left(\frac{\Omega_0}{R+\delta}\right)^2 - 1} \text{ // FullSimplify}$$

Out[∘]=

$$\frac{\Omega_0}{\delta}$$

Problem 5

In[∘]:= "$\Omega = \sqrt{\delta^2 + A^2};$"

$$c_1 = e^{-i\delta t/2}\left(\text{Cos}\left[\frac{\Omega t}{2}\right] + \frac{i\delta}{\Omega}\,\text{Sin}\left[\frac{\Omega t}{2}\right]\right);$$

Out[∘]=

$\Omega = \sqrt{\delta^2 + A^2};$

In[∘]:= c_1 // TrigToExp // Simplify

Out[∘]=

$$\frac{1}{2\,\Omega}\,e^{-\frac{1}{2} i t \delta}\,e^{-\frac{1}{2} i t \Omega}\left(\left(-1 + e^{i t \Omega}\right)\delta + \left(1 + e^{i t \Omega}\right)\Omega\right)$$

In[∘]:=
$$\frac{\left(\left(-1 + e^{i t \Omega}\right)\delta + \left(1 + e^{i t \Omega}\right)\Omega\right)}{2\,\Omega} \text{ // FullSimplify}$$

Out[∘]=

$$\frac{-\delta + \Omega + e^{i t \Omega}\,(\delta + \Omega)}{2\,\Omega}$$

In[∘]:= $$c_2 = e^{i\delta t/2}\left(\text{Cos}\left[\frac{\Omega t}{2}\right] - i\,\frac{\delta}{\Omega}\,\text{Sin}\left[\frac{\Omega t}{2}\right]\right);$$

In[∘]:= c_2 // TrigToExp // FullSimplify

Out[∘]=

$$\frac{e^{\frac{1}{2} i t (\delta - \Omega)}\left(\delta + \Omega + e^{i t \Omega}\,(-\delta + \Omega)\right)}{2\,\Omega}$$