Homework 4

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Phys 684

October 25, 2024

Problem 1

(a) Letting

$$\vec{R} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}; \quad \vec{\Omega} = \begin{pmatrix} \Omega_0' \\ -\Omega_0'' \\ \delta \end{pmatrix},$$

the evolution of the Bloch vector in the field interaction basis is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{R} = \vec{\Omega} \times \vec{R} \implies \begin{cases} \dot{u} = -\delta v - \Omega_0'' w \\ \dot{v} = \delta u - \Omega_0' w \\ \dot{w} = \Omega_0' v + \Omega_0'' u. \end{cases}$$

Given that $\Omega_0' = \Omega_0$ and $\Omega_0'' = 0$ during the pulse, this simplifies to

$$\begin{aligned} \dot{u} &= -\delta v \\ \dot{v} &= \delta u - \Omega_0 w \\ \dot{w} &= \Omega_0 v. \end{aligned}$$

Then,

$$\begin{aligned} \ddot{v} &= \delta \dot{u} - \Omega_0 \dot{w} \\ &= -\left(\delta^2 + \Omega_0^2\right) v \\ &= -\Omega^2 v \\ \implies \quad v(t) &= A\cos(\Omega t) + B\sin(\Omega t). \end{aligned}$$

Next, the initial condition $\vec{R}(0) = -\hat{w}$ implies $A = 0$, so

$$v(t) = B\sin(\Omega t).$$

Now solving for u,

$$
\begin{aligned}
&& \dot{u} &= -\delta v \\
\implies && u(t) &= u(0) - \delta \int_0^t \mathrm{d}t' v(t') \\
&& &= \frac{\delta B}{\Omega} \cos(\Omega t')\Big|_0^t \\
&& &= \frac{\delta B}{\Omega} \left(\cos(\Omega t) - 1\right),
\end{aligned}
$$

and then w

$$
\begin{aligned}
&& \dot{w} &= \Omega_0 v \\
\implies && w(t) &= u(0) + \Omega_0 \int_0^t \mathrm{d}t' v(t') \\
&& &= -1 - \frac{\Omega_0 B}{\Omega} \cos(\Omega t')\Big|_0^t \\
&& &= -1 - \frac{\Omega_0 B}{\Omega} \left(\cos(\Omega t) - 1\right)
\end{aligned}
$$

Using the equation for $\dot{v}$ we can determine the value of B:

$$
\begin{aligned}
&& \dot{v} &= \delta u - \Omega_0 w \\
\implies && \Omega B \cos(\Omega t) &= \frac{\delta^2 B}{\Omega} \left(\cos(\Omega t) - 1\right) + \Omega_0 \left(1 + \frac{\Omega_0 B}{\Omega} \left(\cos(\Omega t) - 1\right)\right) \\
&& &= \frac{\delta^2 + \Omega_0^2}{\Omega} B \cos(\Omega t) - \frac{\delta^2 + \Omega_0^2}{\Omega} B + \Omega_0 \\
&& &= \Omega B \cos(\Omega t) - \Omega B + \Omega_0 \\
\implies && \Omega B &= \Omega_0 \\
\implies && B &= \frac{\Omega_0}{\Omega}.
\end{aligned}
$$

After the pulse ends the precession of the Bloch vector stops, and its final position is given by $\vec{R}(\tau)$. Letting $\theta = \Omega \tau$,

$$
\begin{aligned}
u &= \frac{\Omega_0 \delta}{\Omega^2} \left(\cos\theta - 1\right) \\
v &= \frac{\Omega_0}{\Omega} \sin\theta \\
w &= -\left[1 + \frac{\Omega_0^2}{\Omega^2} \left(\cos\theta - 1\right)\right].
\end{aligned}
$$

Problem 2

See attached Mathematica notebook.

Problem 3

See attached Mathematica notebook.

Problem 4

See attached Mathematica notebook for calculations.

$$H_{\mathrm{d}} = -\frac{\hbar}{2}\Omega_0\sigma_z + \hbar\dot{\theta}\sigma_y$$

$$\begin{aligned}\dot{\tilde{\rho}}_{\mathrm{d}} &= [H_{\mathrm{d}}, \tilde{\rho}_{\mathrm{d}}] \\ &= \begin{pmatrix} -\dot{\theta}\,\mathrm{Re}[\tilde{\rho}_{d12}] & i\Omega_0\rho_{12} - \dot{\theta}\,(\tilde{\rho}_{d22} - \tilde{\rho}_{d11}) \\ -i\Omega_0\rho_{21} - \dot{\theta}\,(\tilde{\rho}_{d22} - \tilde{\rho}_{d11}) & \dot{\theta}\,\mathrm{Re}[\tilde{\rho}_{d12}] \end{pmatrix}\end{aligned}$$

Problem 2

In[11]:= $\{\hat{u}, \hat{v}, \hat{w}\}$ = IdentityMatrix[3];

$\vec{R}_0 = -\hat{w}$;

ϕ = ArcTan$\left[\frac{\sqrt{\Omega^2 - \Omega_0{}^2}}{\Omega_0}\right]$;

In[14]:= (RotationMatrix[$-\phi$, $\hat{v}$].RotationMatrix[θ, $\hat{u}$].RotationMatrix[ϕ, $\hat{v}$].$\vec{R}_0$ /. $\left\{\sqrt{\Omega^2 - \Omega_0{}^2} \to \delta\right\}$) // FullSimplify // MatrixForm

Out[14]//MatrixForm=

$$\begin{pmatrix} \frac{\delta\,(-1+\text{Cos}[\theta])\,\Omega_0}{\Omega^2} \\ \frac{\text{Sin}[\theta]\,\Omega_0}{\Omega} \\ -1 - \frac{(-1+\text{Cos}[\theta])\,\Omega_0^2}{\Omega^2} \end{pmatrix}$$

Problem 3

In[15]:= $\{\sigma_x, \sigma_y, \sigma_z\}$ = Table[PauliMatrix[i], {i, 1, 3}];

$\{\sigma_+, \sigma_-\} = \frac{1}{2}(\sigma_x \mp i\,\sigma_y)$; $\sigma_0 = \sigma_+.\sigma_-$;

rho = $\{\{\rho_{11}, \rho_{12}\}, \{\rho_{21}, \rho_{22}\}\}$;

In[18]:= $(-\gamma\,(\sigma_0.\text{rho} + \text{rho}.\sigma_0) + \gamma_2\,\sigma_-.\text{rho}.\sigma_+ + 2\,\Gamma\,\sigma_0.\text{rho}.\sigma_0$ // FullSimplify) /. $\{-\gamma + \Gamma \to -\gamma_2 / 2\}$ // MatrixForm

Out[18]//MatrixForm=

$$\begin{pmatrix} \gamma_2\,\rho_{22} & -\gamma\,\rho_{12} \\ -\gamma\,\rho_{21} & -\gamma_2\,\rho_{22} \end{pmatrix}$$

Problem 4

In[19]:= $H_d = \frac{-\hbar}{2}\,\Omega_0\,\sigma_z + \hbar\,\dot{\theta}\,\sigma_y$;

In[20]:= $\frac{1}{i\,\hbar}$ comm[H_d, rho] // FullSimplify // MatrixForm

Out[20]//MatrixForm=

$$\begin{pmatrix} -\dot{\theta}\,(\rho_{12} + \rho_{21}) & \dot{\theta}\,(\rho_{11} - \rho_{22}) + i\,\rho_{12}\,\Omega_0 \\ \dot{\theta}\,(\rho_{11} - \rho_{22}) - i\,\rho_{21}\,\Omega_0 & \dot{\theta}\,(\rho_{12} + \rho_{21}) \end{pmatrix}$$

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