$[Q, R] = 0 \not\Longrightarrow \sigma_Q \sigma_R = 0$

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Phys 631

October 13, 2023

Problem

Consider two observables Q and R such that $[Q, R] = 0$. The uncertainty principle states that $\sigma_Q \sigma_R > 0$. Is it *always* the case that $\sigma_Q \sigma_R = 0$?

Solution

Consider an orthonormal set of states $|\psi_i\rangle$ that span the Hilbert space

$$\sum_i |\psi_i\rangle\langle\psi_i| = \mathbb{I},$$

and are eigenvectors for both operators:

$$Q = \sum_i q_i |\psi_i\rangle\langle\psi_i| ; \quad R = \sum_i r_i |\psi_i\rangle\langle\psi_i| .$$

For an arbitrary state

$$|\psi\rangle = \sum_i a_i |\psi_i\rangle ,$$

the variance is given by

$$\sigma_Q^2 = \langle\psi|Q^2|\psi\rangle - \langle\psi|Q|\psi\rangle^2 .$$

Evaluating the first term, we have

$$\begin{aligned}
\langle\psi|Q^2|\psi\rangle &= \left(\sum_i a_i^* \langle\psi_i|\right)\left(\sum_j q_j |\psi_j\rangle\langle\psi_j|\right)^2\left(\sum_k a_k |\psi_k\rangle\right) \\
&= \sum_{i,j,j',k} a_i^* a_k q_j q_{j'} \langle\psi_i|\psi_j\rangle \langle\psi_j|\psi_{j'}\rangle \langle\psi_{j'}|\psi_k\rangle \\
&= \sum_{i,j,j',k} a_i^* a_k q_j q_{j'} \delta_{ij} \delta j j' \delta_{j'k} \\
&= \sum_i |a_i|^2 q_i^2 .
\end{aligned}$$

The second terms, meanwhile, is

$$\begin{aligned}
\langle \psi | Q | \psi \rangle^2 &= \left[\left(\sum_i a_i^* \langle \psi_i | \right) \left(\sum_j q_j | \psi_j \rangle\langle \psi_j | \right) \left(\sum_k a_k | \psi_k \rangle \right) \right]^2 \\
&= \left[\sum_{i,j,k} a_i^* a_k q_j \langle \psi_i | \psi_j \rangle \langle \psi_j | \psi_k \rangle \right]^2 \\
&= \left[\sum_{i,j,k} a_i^* a_k q_j \delta_{ij} \delta_{jk} \right]^2 \\
&= \left[\sum_i |a_i|^2 q_i \right]^2 .
\end{aligned}$$

Now,

$$\sigma_Q \sigma_R = \sqrt{\sum_i |a_i|^2 q_i^2 - \left(\sum_i |a_i|^2 q_i \right)^2} \sqrt{\sum_i |a_i|^2 r_i^2 - \left(\sum_i |a_i|^2 r_i \right)^2} .$$

Clearly, $\sigma_Q \sigma_R = 0$ if and only if one of the two factors above is zero. That is,

$$\boxed{\sum_i |a_i|^2 q_i^2 = \left(\sum_i |a_i| q_i \right)^2 \quad \text{OR} \quad \sum_i |a_i|^2 r_i^2 = \left(\sum_i |a_i| r_i \right)^2 \implies \sigma_Q \sigma_R = 0}$$

$$\sum_i |a_i|^2 q_i^2 = \left(\sum_i |a_i| q_i \right)^2$$

$$\implies$$

$$\sum_{i \neq j} |a_i a_j| q_i q_j = 0$$