Homework 1

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Phys 632

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Problem 1

Let

$$E_s = \hbar\omega_x, \quad t_s = \frac{1}{\omega_x}, \quad x_s = \sqrt{\frac{\hbar}{m\omega_x}}, \quad p_s = \sqrt{m\hbar\omega_x}.$$

$$\tilde{H} = \frac{H}{E_s}, \quad \tilde{x} = \frac{x}{x_s}, \quad \tilde{y} = \frac{y}{x_s} \quad \tilde{p}_{x(y)} = \frac{p_{x(y)}}{p_s}, \quad \tilde{\omega}_y = \frac{\omega_y}{\omega_x}$$

The Hamiltonian,

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \frac{m}{2}\left(\omega_x^2 x^2 + \omega_y^2 y^2\right),$$

can then be rewritten as

$$\tilde{H}E_s = \frac{1}{2m}\left((\tilde{p}_x p_s)^2 + (\tilde{p}_y p_s)^2\right) + \frac{m}{2}\left(\omega_x^2(\tilde{x}x_s)^2 + (\tilde{\omega}_y\omega_x)^2(\tilde{y}x_s)^2\right).$$

Dropping tildes and substituting in values, this reduces to

$$H = \frac{1}{2}\left(p_x^2 + p_y^2 + x^2 + \omega_y^2 y^2\right)$$

Problem 2

In scaled variables,

$$\begin{aligned}
\partial_t \langle x \rangle &= -i \langle [x, H] \rangle \\
&= -\frac{i}{2} \left\langle \left[x, p^2\right] \right\rangle \\
&= \langle p \rangle
\end{aligned}$$

$$\begin{aligned}
\partial_t \langle p \rangle &= -i \langle [p, H] \rangle \\
&= -\frac{i}{2} \left\langle \left[p, x^2\right] \right\rangle \\
&= - \langle x \rangle
\end{aligned}$$

$$\begin{aligned}
\partial_t V_x &= \partial_t \left(\left\langle x^2 \right\rangle - \langle x \rangle^2 \right) \\
&= -i \left\langle \left[x^2, H\right] \right\rangle - 2 \langle x \rangle \partial_t \langle x \rangle \\
&= -\frac{i}{2} \left\langle \left[x^2, p^2\right] \right\rangle - 2 \langle x \rangle \langle p \rangle \\
&= 2 \left\langle [x, p]_+ \right\rangle - 2 \langle x \rangle \langle p \rangle \\
&= 2C_{xp}
\end{aligned}$$

$$\begin{aligned}
\partial_t V_p &= \partial_t \left(\left\langle p^2 \right\rangle - \langle p \rangle^2 \right) \\
&= -i \left\langle \left[p^2, H\right] \right\rangle - 2 \langle p \rangle \partial_t \langle p \rangle \\
&= -\frac{i}{2} \left\langle \left[p^2, x^2\right] \right\rangle + 2 \langle x \rangle \langle p \rangle \\
&= -2 \left\langle [x, p]_+ \right\rangle + 2 \langle x \rangle \langle p \rangle \\
&= -2C_{xp}
\end{aligned}$$

$$\begin{aligned}
\partial_t C_{xp} &= \partial_t \left(\left\langle [x, p]_+ \right\rangle - \langle x \rangle \langle p \rangle \right) \\
&= -i \langle [xp + px, H] \rangle - \langle p \rangle^2 + \langle x \rangle^2 \\
&= -\frac{i}{2} \left\langle \left[xp, x^2\right] + \left[xp, p^2\right] + \left[px, x^2\right] + \left[px, p^2\right] \right\rangle - \langle p \rangle^2 + \langle x \rangle^2 \\
&= \left\langle p^2 \right\rangle - \left\langle x^2 \right\rangle - \langle p \rangle^2 + \langle x \rangle^2 \\
&= V_p - V_x
\end{aligned}$$

Restoring units,

$$\begin{aligned}
\partial_t \langle x \rangle &= \frac{1}{m} \langle p \rangle \\
\partial_t \langle p \rangle &= -m\omega^2 \langle x \rangle \\
\partial_t V_x &= \frac{2}{m} C_{xp} \\
\partial_t V_P &= -m\omega^2 C_{xp} \\
\partial_t C_{xp} &= \frac{1}{m} V_p - m\omega^2 V_x
\end{aligned}$$

Problem 3

From the result of Homework 3 Problem 3 from last term,

$$V_x(t) = \sigma^2(t) = \sigma^2 + \frac{\hbar^2 t^2}{4m^2\sigma^2}$$

$$\implies \dot{V}_x(t) = \frac{\hbar^2 t}{2m^2\sigma^2}$$

Using the equations from problem 2 above (with $\omega = 0$) we see that

$$\partial_t V_p = 0 \implies V_p = c$$

for some constant c. This implies that the time dependence of C_{xp} has the simple form

$$C_{xp}(t) = \frac{c}{m} t + k$$

for another constant k, which in turn implies

$$V_x(t) = V_x(0) + \frac{c}{m^2} t$$

The substituting in the initial conditions gives

$$V_x(t) = \sigma^2 + \frac{\hbar^2 t^2}{4m^2\sigma^2}$$

Problem 4

Let

$$U_g = V_x V_p - C_{xp}^2$$

Then

$$\begin{aligned}
\partial_t U_g &= \dot{V}_x V_p + V_x \dot{V}_p - 2C_{xp}\dot{C}_{xp} \\
&= 2C_{xp}V_p - 2C_{xp}V_x - 2C_{xp}(V_p - V_x) \\
&= 2C_{xp}(V_p - V_x - V_p + V_x) \\
&= 0
\end{aligned}$$

Homework 2

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Problem 1

Let's look at x_1 first. The WKB approximation is

$$\psi_{\text{WKB}}(x) = \begin{cases} \frac{A_<}{\sqrt{p(x)}} \cos\left(\frac{1}{\hbar}\int_x^{x_1} p(x')\mathrm{d}x'\right) + \frac{B_<}{\sqrt{p(x)}} \sin\left(\frac{1}{\hbar}\int_x^{x_1} p(x')\mathrm{d}x'\right) & x < x_1 \\ \frac{A_F}{\sqrt{|p(x)|}} \exp\left[-\frac{1}{\hbar}\int_{x_1}^{x} |p(x')|\mathrm{d}x'\right] + \frac{B_F}{\sqrt{|p(x)|}} \exp\left[\frac{1}{\hbar}\int_{x_1}^{x} |p(x')|\mathrm{d}x'\right] & x > x_1 \end{cases}$$

$$\psi_{\text{patch}}(x) = a\text{Ai}(\nu x) + b\text{Bi}(\nu x), \quad \nu = \left(\frac{2mV'(x_1)}{\hbar^2}\right)^{\frac{1}{3}}.$$

To match them up, let's first consider $x \lesssim x_1$. In this region,

$$p(x) \approx \hbar\sqrt{\nu^3(x_1 - x)}.$$

The WKB solution is thus approximately

$$\psi_{\text{WKB}}(x) \approx \frac{A_<}{\sqrt{\hbar}\nu^{3/4}(x_1-x)^{1/4}} \cos\left(\frac{2}{3}\nu^{3/2}(x_1-x)^{3/2}\right) + \frac{B_<}{\sqrt{\hbar}\nu^{3/4}(x_1-x)^{1/4}} \sin\left(\frac{2}{3}\nu^{3/2}(x_1-x)^{3/2}\right),$$

while the asymptotic form of the patch wavefunction is

$$\psi_{\text{patch}} \approx \frac{a}{\sqrt{\pi}\nu^{1/4}(x_1-x)^{1/4}} \cos\left(\frac{2}{3}\nu^{3/2}(x_1-x)^{3/2} - \frac{\pi}{4}\right) + \frac{b}{\sqrt{\pi}\nu^{1/4}(x_1-x)^{1/4}} \sin\left(\frac{2}{3}\nu^{3/2}(x_1-x)^{3/2} - \frac{\pi}{4}\right).$$

However, given that

$$\cos\left(x - \frac{\pi}{4}\right) = \frac{\sin(x)}{\sqrt{2}} + \frac{\cos(x)}{\sqrt{2}}$$
$$\sin\left(x - \frac{\pi}{4}\right) = \frac{\sin(x)}{\sqrt{2}} - \frac{\cos(x)}{\sqrt{2}},$$

the patch wavefunction can be rewritten as

$$\psi_{\text{patch}} \approx \frac{a-b}{\sqrt{2\pi}\nu^{1/4}(x_1-x)^{1/4}} \cos\left(\frac{2}{3}\nu^{3/2}(x_1-x)^{3/2}\right) + \frac{a+b}{\sqrt{2\pi}\nu^{1/4}(x_1-x)^{1/4}} \sin\left(\frac{2}{3}\nu^{3/2}(x_1-x)^{3/2}\right).$$

The two solutions are equivalent subject to

$$a - b = \sqrt{\frac{2\pi}{\hbar\nu}} A_<, \quad a + b = \sqrt{\frac{2\pi}{\hbar\nu}} B_< \tag{1}$$

Now we can turn to $x \gtrsim x_1$. In this region,

$$|p(x)| \approx \hbar\sqrt{\nu^3(x - x_1)}$$

The WKB solution is thus approximately

$$\psi_{\text{WKB}} \approx \frac{A_F}{\sqrt{\hbar}\nu^{3/4}(x-x_1)^{1/4}} \exp\left[-\frac{2}{3}\nu^{3/2}(x-x_1)^{3/2}\right] + \frac{B_F}{\sqrt{\hbar}\nu^{3/4}(x-x_1)^{1/4}} \exp\left[\frac{2}{3}\nu^{3/2}(x-x_1)^{3/2}\right],$$

while the asymptotic form of the patch wavefunction is

$$\frac{a}{\sqrt{4\pi}\nu^{1/4}(x-x_1)^{1/4}} \exp\left[-\frac{2}{3}\nu^{3/2}(x-x_1)^{3/2}\right] + \frac{b}{\sqrt{\pi}\nu^{1/4}(x-x_1)^{1/4}} \exp\left[\frac{2}{3}\nu^{3/2}(x-x_1)^{3/2}\right]$$

The two solutions are equivalent subjet to

$$a = \sqrt{\frac{4\pi}{\hbar\nu}} A_F, \quad b = \sqrt{\frac{\pi}{\hbar\nu}} B_F. \tag{2}$$

Combining the two conditions for equality gives

$$2A_F - B_F = \sqrt{2} A_<$$

$$2A_F + B_F = \sqrt{2} B_<$$

Problem 2

The approximate wavefucntions around the turning points are

$$\psi_{\text{WKB}} = \begin{cases} \frac{A_<}{\sqrt{p(x)}} \cos\left(\frac{1}{\hbar}\int_x^{x_1} p(x')\mathrm{d}x'\right) + \frac{B_<}{\sqrt{p(x)}} \sin\left(\int_x^{x_1} p(x')\mathrm{d}x'\right) & x \lesssim x_1 \\ \frac{A_F^{(1)}}{\sqrt{|p(x)|}} \exp\left[-\frac{1}{\hbar}\int_{x_1}^{x} |p(x')|\mathrm{d}x'\right] + \frac{B_F^{(1)}}{\sqrt{|p(x)|}} \exp\left[\frac{1}{\hbar}\int_{x_1}^{x} |p(x')|\mathrm{d}x'\right] & x \gtrsim x_1 \\ \frac{A_F^{(2)}}{\sqrt{|p(x)|}} \exp\left[\frac{1}{\hbar}\int_{x}^{x_2} |p(x')|\mathrm{d}x'\right] + \frac{B_F^{(2)}}{\sqrt{|p(x)|}} \exp\left[-\frac{1}{\hbar}\int_{x}^{x_2} |p(x')|\mathrm{d}x'\right] & x \lesssim x_2 \\ \frac{A_>}{\sqrt{p(x)}} \cos\left(\frac{1}{\hbar}\int_{x_2}^{x} p(x')\mathrm{d}x'\right) + \frac{B_>}{\sqrt{p(x)}} \sin\left(\int_{x_2}^{x} p(x')\mathrm{d}x'\right) & x \gtrsim x_2 \end{cases}$$

Requiring the two components within the barrier to be equivalent gives

$$\frac{A_F^{(1)}}{\sqrt{|p(x)|}} \exp\left[-\frac{1}{\hbar}\int_{x_1}^{x} |p(x')|\mathrm{d}x'\right] = \frac{A_F^{(2)}}{\sqrt{|p(x)|}} \exp\left[\frac{1}{\hbar}\int_{x}^{x_2} |p(x')|\mathrm{d}x'\right]$$

$$\implies \frac{A_F^{(1)}}{A_F^{(2)}} = \exp\left[\frac{1}{\hbar}\int_{x_1}^{x} |p(x')|\mathrm{d}x'\right]\exp\left[\frac{1}{\hbar}\int_{x}^{x_2} |p(x')|\mathrm{d}x'\right]$$
$$= \exp\left[\frac{1}{\hbar}\int_{x_1}^{x_2} |p(x')|\mathrm{d}x'\right]$$
$$= \sqrt{T_1}$$

and, similarly,

$$\frac{B_F^{(1)}}{B_F^{(2)}} = \frac{1}{\sqrt{T_1}}$$

Using the result from the last problem we can write the coefficients inside the barrier in terms of the external coefficients:

$$A_F^{(1,2)} = \frac{1}{2\sqrt{2}}(A_{<,>} + B_{<,>})$$

$$B_F^{(1,2)} = \frac{1}{\sqrt{2}}(B_{<,>} - B_{<,>})$$

Combining these, we can write the left-interior coefficients in terms of the right-transmission region coefficients:

$$A_F^{(1)} = \frac{\sqrt{T_1}}{2\sqrt{2}}(A_> + B_>)$$
$$B_F^{(1)} = \frac{1}{\sqrt{2T_1}}(B_> - A_>)$$

We can write the transmission regions as superpositions of left and right going waves:

$$\psi_{\mathrm{WKB}} = \frac{A_< - iB_<}{2}\exp\left[\frac{i}{\hbar}\int_{x}^{x_1} p(x')\mathrm{d}x'\right] + \frac{A_< + iB_<}{2}\exp\left[-\frac{i}{\hbar}\int_{x}^{x_1} p(x')\mathrm{d}x'\right] \quad x \lesssim x_1$$
$$\psi_{\mathrm{WKB}} = \frac{A_> + iB_>}{2}\exp\left[-\frac{i}{\hbar}\int_{x_2}^{x} p(x')\mathrm{d}x'\right] + \frac{A_> - iB_>}{2}\exp\left[\frac{i}{\hbar}\int_{x_2}^{x} p(x')\mathrm{d}x'\right] \quad x \gtrsim x_2$$

To model the action of a wave incident on the barrier from the left side, we set

$$A_< - iB_< = 0$$

The trasnmission probability is then

$$T = |\frac{A_> - iB_>}{A_< + iB_<}|^2$$

Combining with the results above gives

$$T = \frac{T_1}{1 + T_1/4}$$

Problem 3

The approimate tunneling probability is given by

$$\begin{aligned}
T &\approx \exp\left[-\frac{2}{\hbar}\int_{x_1}^{x_2}|p(x')|\mathrm{d}x'\right] \\
&= \exp\left[-\frac{2}{\hbar}\int_{0}^{\frac{V_0-E}{e\mathscr{E}}}\left|\sqrt{2m(E-V_0+e\mathscr{E}x)}\right|\mathrm{d}x'\right] \\
&= \exp\left[-\frac{2}{\hbar}\int_{0}^{\frac{V_0-E}{e\mathscr{E}}}\sqrt{2m(V_0-E-e\mathscr{E}x)}\mathrm{d}x'\right] \\
&= \exp\left[-\frac{2\left(2m(V_0-E)\right)^{3/2}}{3\hbar m e\mathscr{E}}\right]
\end{aligned}$$

If $E = V_0 - W$ then

$$T \approx \exp\left[-\frac{2(2mW)^{3/2}}{3\hbar m e\mathscr{E}}\right]$$

Homework 3

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Problem 1

In components, the cross product side of the "bac-cab" rule is

$$\left(\vec{B} \times \vec{C}\right)_r = B_s C_t \epsilon_{str}$$

$$\left(\vec{A} \times \left(\vec{B} \times \vec{C}\right)\right)_p = A_q B_s C_t \epsilon_{str} \epsilon_{qrp}$$

The dot product side is

$$\begin{aligned}\left(\vec{B}\left(\vec{A} \cdot \vec{C}\right) - \vec{C}\left(\vec{A} \cdot \vec{B}\right)\right)_p &= A_t B_p C_t - A_s B_s C_p \\ &= A_q B_p C_t \delta_{qt} - A_q B_s C_p \delta_{qs} \\ &= A_q B_s C_t \delta_{qt} \delta_{ps} - A_q B_s C_t \delta_{qs} \delta_{pt} \\ &= A_q B_s C_t \left(\delta_{qt} \delta_{ps} - \delta_{qs} \delta_{pt}\right)\end{aligned}$$

Putting it together,

$$\begin{aligned}\vec{A} \times \left(\vec{B} \times \vec{C}\right) &= \vec{B}\left(\vec{A} \cdot \vec{C}\right) - \vec{C}\left(\vec{A} \cdot \vec{B}\right) \\ \implies A_q B_s C_t \epsilon_{str} \epsilon_{qrp} &= A_q B_s C_t \left(\delta_{qt} \delta_{ps} - \delta_{qs} \delta_{pt}\right) \\ \implies \epsilon_{str} \epsilon_{qrp} &= \delta_{qt} \delta_{ps} - \delta_{qs} \delta_{pt}\end{aligned}$$

Problem 2

Using the ladder operators,

$$L_{\pm} = L_x \pm i L_y,$$

we can write L_x as

$$L_x = \frac{1}{2}\left(J_+ + J_-\right).$$

The expectation value, $\langle L_x \rangle$, is now immediatly obvious:

$$\langle L_x \rangle = \frac{1}{2}\left(\langle J_+ \rangle + \langle J_- \rangle\right) = 0.$$

For the expectation value of L_x^2, we simply note that by symmetry it must have the same value as L_y^2. Therefore,

$$L_x^2 = L_y^2 = \frac{1}{2}\left(L^2 - L_z^2\right) = \frac{\hbar^2}{2}(j(j+1) - m^2)$$

Since the first moments are zero, the variance (and hence the uncertainty) are trivial:

$$\sigma_x = \sigma_y = \sqrt{\frac{\hbar^2}{2}(j(j+1) - m^2)}$$

The restriction of m to the range $\{-j, ...j\}$ ensures that $(j(j+1) - m^2) \geq 1$, so

$$\sigma_x \sigma_y = \frac{\hbar^2}{2}(j(j+1) - m^2) \geq \frac{\hbar}{2}$$

Problem 3

For a radial displacement, $\vec{r} \to \vec{r} + \hat{r}\mathrm{d}r$,

$$f(r + \mathrm{d}r) - f(r) = \mathrm{d}r\hat{r} \cdot \nabla f$$

$$\implies \hat{r} \cdot \nabla f = \frac{f(r + \mathrm{d}r) - f(r)}{\mathrm{d}r} = \partial_r f.$$

For a polar angular displacement, $\vec{r} \to \vec{r} + r\mathrm{d}\theta\hat{\theta}$,

$$f(\theta + \mathrm{d}\theta) - f(\theta) = r\mathrm{d}\theta\hat{\theta}\nabla f$$

$$\implies \hat{\theta} \cdot \nabla f = \frac{1}{r}\frac{f(\theta + \mathrm{d}\theta) - f(\theta)}{\mathrm{d}\theta} = \frac{1}{r}\partial_\theta f.$$

For an azimuthal angular displacement, $\vec{r} \to \vec{r} + r\sin\theta\hat{\phi}$,

$$f(\theta + \mathrm{d}\phi) - f(\phi) = r\sin\theta\mathrm{d}\phi\hat{\phi} \cdot \nabla f$$

$$\implies \hat{\phi} \cdot \nabla f = \frac{1}{r\sin\theta}\partial_\phi f.$$

Homework 4

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Problem 1

Firstly,

$$[A, J_x] = [A, J_y] = 0 \implies [A, J_+] = 0$$

Therefore,

$$[A, J_+J_-] = 0$$
$$\implies \left[A, J_x^2 + J_y^2 - \hbar J_z\right] = \hbar[A, J_z] = 0$$

Problem 2

(a) Consider $[J^2, J_\alpha^2]$. We know every individual component commutes with J^2, so

$$\left[J^2, J_\alpha^2\right] = 0 \implies \left[J_x^2, J_\alpha^2\right] + \left[J_y^2, J_\alpha^2\right] + \left[J_z^2, J_\alpha^2\right] = 0$$

When we substitute one of $\{x, y, z\}$ for α, one of the terms above will disappear and another will be "out of order". For example, $\alpha = x$ gives

$$\left[J_y^2, J_x^2\right] + \left[J_z^2, J_x^2\right] = 0 \implies \left[J_z^2, J_x^2\right] = \left[J_x^2, J_y^2\right]$$

Substituting the remaining values of α gives other two requisite equations.

(b) First let

$$J_\pm^2 \left|j, m\right\rangle = c_{j,m}^\pm \left|j, m \pm 2\right\rangle.$$

Then, for $j = 1$,

$$J_z^2(J_+^2 + J_-^2)\left|1, 1\right\rangle = J_z^2 J_-^2 \left|1, 1\right\rangle = \hbar^2 c_{1,1}^- \left|1, -1\right\rangle$$
$$(J_+^2 + J_-^2)J_z^2 \left|1, 1\right\rangle = \hbar^2 J_-^2 \left|1, 1\right\rangle = \hbar^2 c_{1,1}^- \left|1, -1\right\rangle$$
$$J_z^2(J_+^2 + J_-^2)\left|1, -1\right\rangle = J_z^2 J_+^2 \left|1, -1\right\rangle = \hbar^2 c_{1,-1}^+ \left|1, 1\right\rangle$$
$$(J_+^2 + J_-^2)J_z^2 \left|1, -1\right\rangle = \hbar^2 J_+^2 \left|1, -1\right\rangle = \hbar^2 c_{1,-1}^+ \left|1, 1\right\rangle$$
$$J_z^2(J_+^2 + J_-^2)\left|1, 0\right\rangle = (J_+^2 + J_-^2)J_z^2 \left|1, 0\right\rangle = 0$$

Obviously for $j = 1/2$ and $j = 0$ the operator vanishishes similarly to the $|1, 0\rangle$ case. Therefore it is the case that, for $j \in \{0, \frac{1}{2}, 1\}$,

$$\left[J_z^2, J_+^2 + J_-^2\right] = 0$$

Now, since

$$J_+^2 + J_-^2 = 2J_x^2 - 2J_y^2 = 2J^2 - 2J_z^2 - 4J_y^2$$

we see that

$$\left[J_z^2, J_+^2 + J_-^2\right] = 0 \implies \left[J_z^2, 2J^2 - 2J_z^2 - 4J_y^2\right] = -4\left[J_z^2, J_y^2\right] = 0$$

This combined with the result from part (a) give the desired result.

Problem 3

In the case that $l = 1/2$, we *should* have that

$$\Theta_{1/2}^{-1/2}(\theta) = \Theta_{1/2}^{1/2}(\theta) \propto \sqrt{\sin\theta}$$

It should also be the case that

$$L_+\Theta_{1/2}^{-1/2}(\theta)e^{-i\phi/2} \propto \Theta_{1/2}^{1/2}(\theta)e^{i\phi/2}$$

However, applying L_+ to $\Theta_{1/2}^{-1/2}(\theta)e^{-i\phi/2}$, we see that

$$\begin{aligned}
L_+\Theta_{1/2}^{-1/2}(\theta)e^{-i\phi/2} &= \hbar e^{i\phi/2}\left(i\cot\theta\partial_\phi + \partial_\theta\right)\sqrt{\sin\theta}e^{-i\phi/2} \\
&= i\hbar e^{i\phi/2}\cot\theta\partial_\phi\sqrt{\sin\theta}e^{-i\phi/2} + \hbar e^{i\phi/2}\partial_\theta\sqrt{\sin\theta}e^{-i\phi/2} \\
&= \frac{\hbar}{2}e^{i\phi/2}\cot\theta\sqrt{\sin\theta}e^{-i\phi/2} + \frac{\hbar}{2}e^{i\phi/2}\cos\theta\sqrt{\sin\theta}e^{-i\phi/2} \\
&= \frac{\hbar}{2}e^{i\phi/2}\frac{\cos\theta}{\sqrt{\sin\theta}}e^{-i\phi/2}
\end{aligned}$$

Which is *not* proportional to $\sqrt{\sin\theta}$.

Problem 4

$$|11\rangle = \sqrt{\frac{3}{7}}\,|2,2;2-1\rangle - \sqrt{\frac{1}{14}}\,|2,2;1,0\rangle - \sqrt{\frac{1}{14}}\,|2,2;1,0\rangle + \sqrt{\frac{3}{7}}\,|2,2;-1,2\rangle$$

$$\implies P(m_1 = 0 \text{ OR } m_2 = 0) = 2 \times \frac{1}{14} = \frac{1}{7}$$

Problem 5

Let $\alpha = \langle 1, 0; j, 0|j, 0\rangle$. The symmetry relation

$$\langle j_1, m_1; j_2, m_2|j_3, m_3\rangle = (-1)^{j_1+j_2-j_3}\langle j_1, -m_1; j_2 - m_2|j_3, -m_3\rangle$$

implies that $\alpha = -\alpha$, therefore it must be that $\alpha = 0$.

Homework 5

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Problem 1

(a) The rotation matrix for a $\pi/2$ rotation about the y-axis (in the standard basis) is

$$\mathbf{d}^{(1)}(\hat{y}\pi/2) = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & \sqrt{2} & 1 \end{pmatrix}$$

Applying this rotation to $|1,0\rangle$ gives

$$\sum_m |1,m\rangle\, d^{(1)}_{m,0}(\hat{y}\pi/2) = \frac{1}{\sqrt{2}}\left(|1,1\rangle - |1,-1\rangle\right)$$

(b) The rotation matrix for a $-\pi/2$ rotation about the x-axis is

$$\mathbf{d}^{(1)}(-\hat{x}\pi/2) = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{2}i & -1 \\ -\sqrt{2}i & 0 & -\sqrt{2}i \\ -1 & -\sqrt{2}i & 1 \end{pmatrix}$$

Applying this rotation to $|1,1\rangle$ gives

$$\sum_m |1,m\rangle\, d^{(1)}_{m,0}(-\hat{x}\pi/2) = \frac{-i}{\sqrt{2}}\left(|1,1\rangle + |1,-1\rangle\right)$$

Problem 2

(a) The transformation from cartesian to spherical coordinates is given by

$$\begin{aligned} A_1 &= \frac{-1}{\sqrt{2}}A_x - \frac{i}{\sqrt{2}}A_y \\ A_0 &= A_z \\ A_{-1} &= \frac{1}{\sqrt{2}}A_x - \frac{i}{\sqrt{2}}A_y \end{aligned}$$

or, in matrix form,

$$\begin{pmatrix} A_1 \\ A_0 \\ A_{-1} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & -i & 0 \\ 0 & 0 & 1 \\ 1 & -i & 0 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

(b) The cartesian rotation matricies about the y and z axes are given by

$$R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \quad R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The composite passive-rotation operator, $P = R_z(-\gamma)R_y(-\beta)R_z(-\alpha)$, in cartesian coordinates is

$$\begin{pmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & \cos\beta\cos\gamma\sin\alpha + \cos\alpha\sin\alpha & -\cos\gamma\sin\beta \\ -\cos\gamma\sin\alpha - \cos\alpha\cos\beta\sin\gamma & \cos\alpha\cos\gamma - \cos\beta\sin\alpha\sin\gamma & \sin\beta\sin\gamma \\ \cos\alpha\cos\beta & \sin\alpha\sin\beta & \cos\beta \end{pmatrix}$$

Transforming to the spherical basis gives

$$UPU^\dagger = \begin{pmatrix} \frac{1}{2}(\cos\beta + 1)e^{-i\alpha}e^{-i\gamma} & \frac{1}{\sqrt{2}}\sin\beta e^{-i\gamma} & \frac{1}{2}(1-\cos\beta)e^{i\alpha}e^{-i\gamma} \\ -\frac{1}{\sqrt{2}}\sin\beta e^{-i\alpha} & \cos\beta & \frac{1}{\sqrt{2}}\sin\beta e^{i\alpha} \\ \frac{1}{2}(1-\cos\beta)e^{-i\alpha}e^{i\gamma} & -\frac{1}{\sqrt{2}}\sin\beta e^{i\gamma} & \frac{1}{2}(1+\cos\beta)e^{i\alpha}e^{i\gamma} \end{pmatrix}$$

(c) Finally,

$$\left(UPU^\dagger\right)^\intercal = \begin{pmatrix} \frac{1}{2}(\cos\beta + 1)e^{-i\alpha}e^{-i\gamma} & -\frac{1}{\sqrt{2}}\sin\beta e^{-i\alpha} & \frac{1}{2}(1-\cos\beta)e^{-i\alpha}e^{i\gamma} \\ \frac{1}{\sqrt{2}}\sin\beta e^{-i\gamma} & \cos\beta & -\frac{1}{\sqrt{2}}\sin\beta e^{i\gamma} \\ \frac{1}{2}(1-\cos\beta)e^{i\alpha}e^{-i\gamma} & \frac{1}{\sqrt{2}}\sin\beta e^{i\alpha} & \frac{1}{2}(1+\cos\beta)e^{i\alpha}e^{i\gamma} \end{pmatrix})$$

Problem 3

$$
\begin{aligned}
T_1^{(1)} &= \frac{1}{\sqrt{2}}\left(A_1B_0 - A_0B_1\right)\\
&= \frac{1}{2}\left[\left(-A_x - iA_y\right)B_z - A_z\left(-B_x - iB_y\right)\right]\\
&= \frac{1}{2}\left[A_zB_x - A_xB_z + i\left(A_zB_y - A_yB_z\right)\right]\\
&= \frac{-i}{\sqrt{2}}\left[\left(\vec{A}\times\vec{B}\right)_x + i\left(\vec{A}\times\vec{B}\right)_y\right]\\
&= \frac{i}{\sqrt{2}}\left(\vec{A}\times\vec{B}\right)_1\\
T_0^{(1)} &= \frac{1}{\sqrt{2}}\left(A_{-1}B_1 - A_1B_{-1}\right)\\
&= \frac{1}{2^{3/2}}\left[\left(A_x - iA_y\right)\left(-B_x - iB_y\right) - \left(-A_x - iA_y\right)\left(B_x - iB_y\right)\right]\\
&= \frac{i}{\sqrt{2}}\left(A_xB_y - A_yB_x\right)\\
&= \frac{i}{\sqrt{2}}\left(\vec{A}\times\vec{B}\right)_0\\
T_{-1}^{(1)} &= \frac{1}{\sqrt{2}}\left(A_{-1}B_0 - A_0B_{-1}\right)\\
&= \frac{1}{2}\left[\left(A_x - iA_y\right)B_z - A_z\left(B_x - iB_y\right)\right]\\
&= \frac{1}{2}\left[A_xB_z - A_zB_x + i\left(A_zB_y - A_yB_z\right)\right]\\
&= \frac{i}{\sqrt{2}}\left[\left(\vec{A}\times\vec{B}\right)_x - i\left(\vec{A}\times\vec{B}\right)_y\right]\\
&= \frac{i}{\sqrt{2}}\left(\vec{A}\times\vec{B}\right)_{-1}
\end{aligned}
$$

Problem 4

Consider $|\theta_1,\phi_1\rangle$, $|\theta_2,\phi_2\rangle$, $|\theta_1',\phi_1'\rangle$, $|\theta_2',\phi_2'\rangle$ such that

$$|\theta_\alpha',\phi_\alpha'\rangle = R\,|\theta_\alpha,\phi_\alpha\rangle$$

for some rotation R.
Consider now the quantity

$$C_l = \sum_m \langle\theta_2,\phi_2|l,m\rangle\,\langle l,m|\theta_1,\phi_1\rangle = \sum_m \left[Y_l^m(\theta_1,\phi_1)\right]^* Y_l^m(\theta_2,\phi_2)$$

This quantity my be expressed in terms of the primed coordinates as so:

$$C_l' = \sum_{m,m',m''} d^{(l)}_{m',m}\left(d^{(l)}_{m,m''}\right)^* \langle\theta_2,\phi_2|l,m'\rangle\,\langle l,m''|\theta_1,\phi_1\rangle = \sum_m \left[Y_l^m(\theta_1',\phi_1')\right]^* Y_l^m(\theta_2',\phi_2'),$$

where $\mathbf{d}^{(l)}$ is the matrix representation for R in the standard basis. Note that, due to the unitarity of R,

$$\sum_m d^{(l)}_{m',m} d^{(l)}_{m,m''} = \delta_{m',m''}.$$

The tripple sum above therefore reduces to

$$C_l' = \sum_{m'} \langle \theta_2, \phi_2 | l, m' \rangle \langle l, m' | \theta_1, \phi_1 \rangle = C_l$$

and we see that the primed and un-primed quantities are equal.
Next consider the case where (θ_1', ϕ_1') points along the z-axis (i.e. $\theta_1' = 0$), and $\phi_2' = 0$. The value of the spherical harmonic along the z-axis is

$$Y_l^m(0, \phi) = \sqrt{\frac{2l+1}{4\pi}} \delta_{m,0},$$

from the definition of the spherical harmonics. This eliminates all but one of the terms in the quantity above:

$$C_l' = \sqrt{\frac{2l+1}{4\pi}} Y_l^0(\theta_2', 0).$$

However, since this is equal to the rotated quantity, we have

$$Y_l^0(\theta, 0) = \sqrt{\frac{4\pi}{2l+1}} \sum_m \left[Y_l^m(\theta_1, \phi_1)\right]^* Y_l^m(\theta_2, \phi_2)$$

where θ is the angle between the directions given by (θ_1, ϕ_1) and (θ_2, ϕ_2). Letting $\theta_1 = \theta_2$ and $\phi_1 = \phi_2$, we get

$$\sqrt{\frac{2l+1}{4\pi}} = \sqrt{\frac{4\pi}{2l+1}} \sum_m \left|Y_l^m(\theta, \phi)\right|^2$$

$$\implies \sum_m \left|Y_l^m(\theta, \phi)\right|^2 = \frac{2l+1}{4\pi}$$

Homework 6

Sean Ericson
Phys 632

February 22, 2022

Problem 1

In terms of the $c_\pm$ amplitudes, the expectation values of the Pauli operators are

$$\begin{aligned}
\langle \sigma_x \rangle &= c_+^* c_- + \text{c.c.} \\
\langle \sigma_y \rangle &= -i c_+^* c_- + \text{c.c.} \\
\langle \sigma_y \rangle &= c_+^* c_+ + c_-^* c_-
\end{aligned}$$

The equations of motion for the amplitudes are given by

$$\begin{aligned}
\dot{c}_+ &= -\frac{i}{2} \left(\Delta c_+ - \Omega c_- \right) \\
\dot{c}_- &= -\frac{i}{2} \left(\Omega^* c_+ - \Delta c_- \right)
\end{aligned}$$

Combining these, we can see that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\langle\sigma_x\rangle &= \dot{c}_+^* c_- + c_+^*\dot{c}_- + \text{c.c.} \\
&= \frac{i}{2}\left(\Delta c_+^* - \Omega^* c_-^*\right)c_- - \frac{i}{2}c_+^*\left(\Omega^* c_+ - \Delta c_-\right) + \text{c.c.} \\
&= -\frac{i}{2}\left(\Omega^*\left|c_+\right|^2 - \Omega^*\left|c_-\right|^2 - 2\Delta c_+^* c_-\right) + \text{c.c.} \\
&= -\Delta\langle\sigma_y\rangle - \text{Im}[\sigma]\langle\sigma_z\rangle \\
\frac{\mathrm{d}}{\mathrm{d}t}\langle\sigma_y\rangle &= -i\dot{c}_+^* c_- - ic_+^*\dot{c}_- + \text{c.c.} \\
&= -i\left[\frac{i}{2}\left(\Delta c_+^* - \Omega^* c_-^*\right)\right] - ic_+^*\left[\frac{-i}{2}\left(\Omega^* c_+ - \Delta c_-\right)\right] \\
&= \frac{1}{2}\left(\Omega^*\left|c_+\right|^2 - \Omega^*\left|c_-\right|^2 - 2\Delta c_+^* c_-\right) + \text{c.c.} \\
&= \Delta\langle\sigma_x\rangle - \text{Re}[\Omega]\langle\sigma_z\rangle \\
\frac{\mathrm{d}}{\mathrm{d}t}\langle\sigma_z\rangle &= c_+^*\dot{c}_+ - c_-^*\dot{c}_- + \text{c.c.} \\
&= -\frac{i}{2}\left(\Delta\left|c_+\right|^2 + \Omega c_+^* c_- - \Omega^* c_-^* c_+ - \Delta\left|c_-\right|^2\right) + \text{c.c.} \\
&= \text{Re}[\Omega]\langle\sigma_y\rangle + \text{Im}[\Omega]\langle\sigma_x\rangle
\end{aligned}
$$

This is equivalent to

$$
\vec{P}\times\langle\vec{\sigma}\rangle = \begin{pmatrix}\text{Re}[\Omega] \\ -\text{Im}[\Omega] \\ \Delta\end{pmatrix} \times \begin{pmatrix}\langle\sigma_x\rangle \\ \langle\sigma_y\rangle \\ \langle\sigma_z\rangle\end{pmatrix} = \begin{pmatrix}-\Delta\langle\sigma_y\rangle - \text{Im}[\Omega]\langle\sigma_z\rangle \\ \Delta\langle\sigma_x\rangle - \text{Re}[\Omega]\langle\sigma_z\rangle \\ \text{Im}[\Omega]\langle\sigma_x\rangle + \text{Re}[\Omega]\langle\sigma_y\rangle\end{pmatrix}
$$

Problem 2

Given

$$
H = -\vec{\mu}_S\cdot\vec{B} = \frac{g_S\mu_B}{\hbar}\vec{S}\cdot\vec{B},
$$

The equation of motion for the operator S_α is given by

$$
\begin{aligned}
\dot{S}_\alpha &= -\frac{i}{\hbar}[S_\alpha, H] \\
&= -\frac{ig_S\mu_B}{\hbar^2}[S_\alpha, S_\beta B_\beta] \\
&= -\frac{ig_S\mu_B}{\hbar^2}[S_\alpha, S_\beta]B_\beta \\
&= -\frac{ig_S\mu_B}{\hbar^2}\epsilon_{\alpha\beta\gamma}S_\gamma B_\beta \\
&= \frac{ig_S\mu_B}{\hbar^2}\epsilon_{\alpha\beta\gamma}S_\beta B_\gamma \\
&= \vec{\mu}_S\times\vec{B}
\end{aligned}
$$

Problem 3

Orient the coordiante system such that $\hat{\alpha}$ points in the $\hat{z}$ direction. Then,

$$
\begin{aligned}
e^{-i\vec{\alpha}\cdot\vec{S}/\hbar} &= e^{-i\alpha S_z/\hbar} \\
&= e^{-i\alpha\sigma_z/2} \\
&= \mathbb{I} - i\frac{\alpha}{2}\sigma_z + \frac{1}{2}\left(\frac{i}{2}\alpha\sigma_z\right)^2 + \frac{1}{6}\left(\frac{i}{2}\alpha\sigma_z\right)^3 + \dots \\
&= \mathbb{I} - i\frac{\alpha}{2}\sigma_z - \frac{1}{2}\left(\frac{\alpha}{2}\right)^2\mathbb{I} - \frac{1}{6}\left(\frac{\alpha}{2}\right)^3\sigma_z + \dots \\
&= \left(\mathbb{I} - \frac{1}{2}\left(\frac{\alpha}{2}\right)^2 + \dots\right) - i\left(\frac{\alpha}{2}\sigma_z + \frac{1}{6}\left(\frac{\alpha}{2}\right)^3 + \dots\right) \\
&= \cos\left(\frac{\alpha}{2}\right)\mathbb{I} - i\sin\left(\frac{\alpha}{2}\right)\sigma_z.
\end{aligned}
$$

Given that the coordinate orientation was arbitrary, we have

$$
e^{-i\vec{\alpha}\cdot\vec{S}} = \cos\left(\frac{\alpha}{2}\right)\mathbb{I} - i\sin\left(\frac{\alpha}{2}\right)(\hat{\alpha}\cdot\vec{\sigma}).
$$

Problem 4

(a) The relevant rotation operators in the standard basis are

$$
\begin{aligned}
R_x(\pi) &= \cos\left(\frac{\pi}{2}\right)\mathbb{I} - i\sin\left(\frac{\pi}{2}\right)\sigma_x = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \\
R_x(\frac{\pi}{2}) &= \cos\left(\frac{\pi}{4}\right)\mathbb{I} - i\sin\left(\frac{\pi}{4}\right)\sigma_x = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \\
R_y(-\pi) &= \cos\left(\frac{\pi}{2}\right)\mathbb{I} + i\sin\left(\frac{\pi}{2}\right)\sigma_y = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\end{aligned}
$$

Now,

$$
R_x(\pi)\,|+\rangle = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = -i\begin{pmatrix} 0 \\ 1 \end{pmatrix} = -i\,|-\rangle.
$$

While

$$
\begin{aligned}
R_x(\frac{\pi}{2})R_y(-\pi)R_x(\frac{\pi}{2})\,|+\rangle &= \frac{1}{2}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\
&= -\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
&= -\,|-\rangle
\end{aligned}
$$

The final states are equivalent up to a $\frac{\pi}{2}$ difference in phase.

(b) Taking into account the error ϵ, the rotation operators are

$$R_x(\pi+\epsilon) = \begin{pmatrix} -\epsilon & -i\left(1-\frac{\epsilon^2}{2}\right) \\ -i\left(1-\frac{\epsilon^2}{2}\right) & -\epsilon \end{pmatrix}$$
$$R_x(\frac{\pi}{2}+\epsilon) = \frac{1}{\sqrt{2}}\begin{pmatrix} 1-\epsilon-\frac{\epsilon^2}{2} & -i\left(1+\epsilon-\frac{\epsilon^2}{2}\right) \\ -i\left(1+\epsilon-\frac{\epsilon^2}{2}\right) & 1-\epsilon-\frac{\epsilon^2}{2} \end{pmatrix}$$
$$R_y(-\pi+\epsilon) = \begin{pmatrix} \epsilon & 1-\frac{\epsilon^2}{2} \\ \frac{\epsilon^2}{2}-1 & \epsilon \end{pmatrix}$$

The single $R_x(\pi+\epsilon)$ rotation gives

$$R_x(\pi+\epsilon)\left|+\right\rangle = -\epsilon\left|+\right\rangle - i(1-\frac{\epsilon^2}{2})\left|-\right\rangle$$

With error

$$\langle +|R_x(\pi+\epsilon)|+\rangle = -\epsilon$$

The composite rotation is given by (to second-order in ϵ)

$$R_x(\pi/2+\epsilon)R_y(-\pi+\epsilon)R_x(\pi/2+\epsilon) = \begin{pmatrix} -2\epsilon^2 & 1-i\epsilon-\frac{\epsilon^2}{2} \\ \frac{\epsilon^2}{2}-i\epsilon-1 & -2\epsilon^2 \end{pmatrix}$$

With error

$$\langle +|R_x(\pi/2+\epsilon)R_y(-\pi+\epsilon)R_x(\pi/2+\epsilon)|+\rangle = -2\epsilon^2$$

Thus, the error of the single rotation is of order ϵ, while the composite rotation's error is of order ϵ^2.

Homework 7

Sean Ericson
Phys 632

March 1, 2022

Problem 1

(a) Let $\hat{\alpha} = \hat{z}$ and $\hat{\beta} = \cos\theta\hat{z} + \sin\theta\hat{x}$. Then

$$\sigma_{\hat{\beta}} = \cos\theta\sigma_z + \sin\theta\sigma_x = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

and

$$\langle +|\sigma_{\hat{\beta}}|+\rangle = \cos\theta$$

(b) When $\hat{\beta}'$ is aligned with $\hat{\alpha}$ the integral evaluates to 1. When $\hat{\beta}'$ is anti-aligned with $\hat{\alpha}$ ($\theta' = \pi$), the integral evaluates to -1. The value of the integral varies lineraly with the angle θ' between 1 and -1 over the range $\theta' = 0$ to $\theta' = \pi$. Thus,

$$\left\langle \sigma(\hat{\beta}, \lambda) \right\rangle = 1 - \frac{2\theta'}{\pi}.$$

To pick $\hat{\beta}'$ (θ') in order to reproduce the results of quantum mechanics, we set

$$\theta' = \frac{\pi}{2}(1 - \cos\theta)$$

Problem 2

(a) As we showed in class, $C(\hat{\alpha}, \hat{\beta}) = -\hat{\alpha}\cdot\hat{\beta}$ for the Bell pair $\frac{1}{\sqrt{2}}(|1,0\rangle - |0,1\rangle)$. The first three pairs are 45° appart, while the the last is 135°. Therefore,

$$\mathscr{C} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 2\sqrt{2}$$

(b) Given

$$\vec{\sigma}_{\hat{\alpha}} = \hat{\alpha}\cdot\vec{\sigma}; \qquad \hat{\gamma} = a\hat{\alpha} + b\hat{\beta} \quad (\left|a\right|^2 + \left|b\right|^2 = 1),$$

we see

$$\sigma_{\hat{\gamma}} = \hat{\gamma}\cdot\vec{\sigma} = \left(a\hat{\alpha} + b\hat{\beta}\right)\cdot\vec{\sigma} = a\sigma_{\hat{\alpha}} + b\sigma_{\hat{\beta}}.$$

We are therefore justified in writing

$$\sigma_{\hat{\pm}} = \frac{1}{\sqrt{2}}\left(\sigma_{\hat{x}} \pm \sigma_{\hat{y}}\right).$$

Now

$$\begin{aligned}
\sigma^{(1)}_{\hat{x}}\sigma^{(2)}_{\hat{+}} &= \frac{1}{\sqrt{2}}\left(\sigma^{(1)}_{\hat{x}}\sigma^{(2)}_{\hat{x}} + \sigma^{(1)}_{\hat{x}}\sigma^{(2)}_{\hat{y}}\right) \\
\sigma^{(1)}_{\hat{x}}\sigma^{(2)}_{\hat{-}} &= \frac{1}{\sqrt{2}}\left(\sigma^{(1)}_{\hat{x}}\sigma^{(2)}_{\hat{x}} - \sigma^{(1)}_{\hat{x}}\sigma^{(2)}_{\hat{y}}\right) \\
\sigma^{(1)}_{\hat{y}}\sigma^{(2)}_{\hat{+}} &= \frac{1}{\sqrt{2}}\left(\sigma^{(1)}_{\hat{y}}\sigma^{(2)}_{\hat{x}} + \sigma^{(1)}_{\hat{y}}\sigma^{(2)}_{\hat{y}}\right) \\
\sigma^{(1)}_{\hat{y}}\sigma^{(2)}_{\hat{-}} &= \frac{1}{\sqrt{2}}\left(\sigma^{(1)}_{\hat{y}}\sigma^{(2)}_{\hat{x}} - \sigma^{(1)}_{\hat{y}}\sigma^{(2)}_{\hat{y}}\right)
\end{aligned}$$

adding the first three terms and subtracting the last gives

$$\sqrt{2}\left(\sigma^{(1)}_{\hat{x}}\sigma^{(2)}_{\hat{x}} + \sigma^{(1)}_{\hat{y}}\sigma^{(2)}_{\hat{y}}\right)$$

The total correlation combination is easily calcualted as

$$2\sqrt{2}C(\hat{\alpha}, \hat{\alpha}) = -2\sqrt{2}$$

(c) Consider the two cases

$$A_x = 1, A_y = -1, B_+ = 1, B_- = -1 \implies 2$$

$$A_x = 1, A_y = -1, B_+ = 1, B_- = 1 \implies -2$$

Since the other two possible cases are equal by symmetry, we can conclude that any experimental run must result in ± 2.

(d) $|\mathscr{C}| \leq \langle |\pm 2| \rangle = 2$

Problem 3

$$\begin{aligned}
\partial_t \operatorname{Tr}\left[\rho^2\right] &= \partial_t \sum_{\alpha} \langle \alpha | \rho\rho | \alpha \rangle \\
&= \sum_{\alpha} \langle \alpha | \dot{\rho}\rho + \rho\dot{\rho} | \alpha \rangle \\
&= -\frac{i}{\hbar} \sum_{\alpha} \langle \alpha | [H, \rho]\rho + \rho[H, \rho] | \alpha \rangle \\
&= -\frac{i}{\hbar} \sum_{\alpha} \langle \alpha | H\rho - \rho H\rho + \rho H\rho - H\rho | \alpha \rangle \\
&= 0
\end{aligned}$$

Problem 4

(a) If $\vec{r} = \begin{pmatrix} a & b & c \end{pmatrix}^\intercal$, then

$$\begin{aligned}
\frac{1}{2}\left[\mathcal{I} + \vec{r}\cdot\vec{\sigma}\right] &= \frac{1}{2}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix} + \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} + \begin{pmatrix} 0 & -ib \\ ib & 0 \end{pmatrix}\right] \\
&= \frac{1}{2}\begin{pmatrix} 1+c & a-ib \\ a+ib & 1-c \end{pmatrix}
\end{aligned}$$

(b)

$$\begin{aligned}
\mathrm{Tr}[\rho\sigma_x] &= \frac{1}{2}\,\mathrm{Tr}\left[\begin{pmatrix} 1+c & a-ib \\ a+ib & 1-c \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] = \frac{1}{2}(a - ib + a + ib) = a \\
\mathrm{Tr}[\rho\sigma_y] &= \frac{1}{2}\,\mathrm{Tr}\left[\frac{1}{2}\begin{pmatrix} 1+c & a-ib \\ a+ib & 1-c \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right] = \frac{1}{2}(ia + b - ia + b) = b \\
\mathrm{Tr}[\rho\sigma_z] &= \frac{1}{2}\,\mathrm{Tr}\left[\frac{1}{2}\begin{pmatrix} 1+c & a-ib \\ a+ib & 1-c \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] = \frac{1}{2}(1 + c - 1 + c) = c
\end{aligned}$$

(c) In the "I Know Nothing" state,

$$\langle\sigma_x\rangle = \langle\sigma_y\rangle = \langle\sigma_z\rangle = 0$$

Thus

$$\vec{r} = \vec{0}$$

Homework 8

Sean Ericson
Phys 632

March 10, 2022

Problem 1

First, let's identify

$$\mathbf{e} = e^{\beta\hbar\omega}, \quad \coth(\beta\hbar\omega) = \frac{\mathbf{e}^1 + \mathbf{e}^{-1}}{\mathbf{e}^1 - \mathbf{e}^{-1}}, \quad \operatorname{csch}(\beta\hbar\omega) = \frac{2}{\mathbf{e}^1 - \mathbf{e}^{-1}}$$

as well as

$$Z = \sum_n \mathbf{e}^{-(n+1/2)} = \mathbf{e}^{-\frac{1}{2}} \sum_n \mathbf{e}^{-n} = \frac{\mathbf{e}^{-\frac{1}{2}}}{1 - \mathbf{e}^{-1}} = \frac{1}{\mathbf{e}^{\frac{1}{2}} - \mathbf{e}^{-\frac{1}{2}}}$$

Now, neglecting units for the time being,

$$\begin{aligned}
\rho(x, x') &= \langle x|\rho|x'\rangle \\
&= Z^{-1} \sum_n \mathbf{e}^{-(n+1/2)} \langle x|n\rangle \langle n|x'\rangle \\
&= \frac{1}{\sqrt{\pi}} \left(\mathbf{e}^{\frac{1}{2}} - \mathbf{e}^{-\frac{1}{2}}\right) \mathbf{e}^{-\frac{1}{2}} e^{-(x^2+(x')^2)/2} \sum_n \frac{\mathbf{e}^{-n}}{2^n n!} H_n(x) H_n(x')
\end{aligned}$$

Letting $z = \mathbf{e}^{-1}$ and applying Mehler's formula gives

$$\begin{aligned}
\rho(x, x') &= \frac{1}{\sqrt{\pi}} \frac{1 - \mathbf{e}^{-1}}{\sqrt{1 - \mathbf{e}^{-2}}} e^{-(x^2+(x')^2)/2} \exp\left[\frac{2xx'\mathbf{e}^{-1} - (x^2 + (x')^2)\mathbf{e}^{-2}}{1 - \mathbf{e}^{-2}}\right] \\
&= \frac{1}{\sqrt{\pi}} \frac{\mathbf{e}^{\frac{1}{2}}(\mathbf{e}^{\frac{1}{2}} - \mathbf{e}^{-\frac{1}{2}})}{\mathbf{e}^{\frac{1}{2}}\sqrt{\mathbf{e}^1 - \mathbf{e}^{-1}}} \exp\left[\frac{2xx'}{\mathbf{e}^1 - \mathbf{e}^{-1}} - (x^2 + (x')^2)\left(\frac{\mathbf{e}^{-1}}{\mathbf{e}^1 - \mathbf{e}^{-1}} + \frac{1}{2}\right)\right] \\
&= \frac{1}{\sqrt{\pi}} \frac{\mathbf{e}^{\frac{1}{2}} - \mathbf{e}^{-\frac{1}{2}}}{\sqrt{(\mathbf{e}^{\frac{1}{2}} - \mathbf{e}^{-\frac{1}{2}})(\mathbf{e}^{\frac{1}{2}} + \mathbf{e}^{-\frac{1}{2}})}} \exp\left[xx' \operatorname{csch}(\beta\hbar\omega) - \frac{1}{2}(x^2 + (x')^2) \coth(\beta\hbar\omega)\right] \\
&= \frac{1}{\sqrt{\pi \coth(\beta\hbar\omega/2)}} \exp\left[xx' \operatorname{csch}(\beta\hbar\omega) - \frac{1}{2}(x^2 + (x')^2) \coth(\beta\hbar\omega)\right]
\end{aligned}$$

Restoring units,

$$\rho(x,x') = \frac{1}{\sqrt{\pi x_s^2 \coth(\beta\hbar\omega/2)}} \exp\left[\frac{xx'}{x_s^2}\operatorname{csch}(\beta\hbar\omega) - \frac{(x^2+(x')^2))}{2x_s^2}\coth(\beta\hbar\omega)\right]$$

For the position uncertainty, $\rho(x,x)$ is a gaussian, so we can immediately read off the uncertainty as

$$\sigma_x = \sqrt{\frac{1}{2}x_s^2\coth(\beta\hbar\omega/2)} = \sqrt{\frac{\hbar\coth(\beta\hbar\omega/2)}{2m\omega}}$$

As $T\to 0$, this becomes the typical harmonic oscillator ground state position uncertainty,

$$\sigma_x = \sqrt{\frac{\hbar}{2m\omega}}.$$

As $T \to \inf$, $\coth(\beta\hbar\omega/2)\to 2/\beta\hbar\omega$, and the uncertainty approaches

$$\sigma_x = \sqrt{\frac{k_B T}{m\omega^2}}$$

as expexted by the equipartition theorem, $\frac{1}{2}m\omega^2\sigma_x^2 = \frac{1}{2}k_b T$.

Problem 2

Before Alice performs her measurement, the three qubits are in the state

$$|\psi_{\text{TAB}}\rangle = \frac{1}{2}\left[|11\rangle|\psi_{11}\rangle + |10\rangle|\psi_{10}\rangle + |01\rangle|\psi_{01}\rangle - |00\rangle|\psi_{00}\rangle\right]$$

where

$$\begin{aligned}
|\psi_{11}\rangle &= c_0|1\rangle + c_1|0\rangle = \sigma_x|\psi_{10}\rangle \\
|\psi_{10}\rangle &= c_1|1\rangle + c_0|0\rangle = \mathcal{I}|\psi_{10}\rangle \\
|\psi_{01}\rangle &= c_0|1\rangle - c_1|0\rangle = i\sigma_y|\psi_{10}\rangle \\
|\psi_{00}\rangle &= c_1|1\rangle - c_0|0\rangle = \sigma_z|\psi_{10}\rangle
\end{aligned}$$

After Bob hears about Alice's measurement result and performs the necessary unitary operation, his state is

$$\begin{aligned}
\rho_B &= (1-\epsilon)|\psi_{10}\rangle\langle\psi_{10}| + \frac{\epsilon}{3}\left(|\psi_{11}\rangle\langle\psi_{11}| + |\psi_{01}\rangle\langle\psi_{01}| + |\psi_{00}\rangle\langle\psi_{00}|\right) \\
&= (1-\epsilon-\frac{\epsilon}{3})|\psi_{10}\rangle\langle\psi_{10}| + \frac{\epsilon}{3}\left(|\psi_{11}\rangle\langle\psi_{11}| + |\psi_{01}\rangle\langle\psi_{01}| + |\psi_{00}\rangle\langle\psi_{00}| + |\psi_{10}\rangle\langle\psi_{10}|\right) \\
&= (1-\frac{4}{3}\epsilon)|\psi_{10}\rangle\langle\psi_{10}| + \frac{4}{3}\epsilon\mathcal{I}
\end{aligned}$$

With a matrix representation the $\{|1\rangle, |0\rangle\}$ basis of

$$\begin{pmatrix} (1-\frac{4}{3}\epsilon)|c_1|^2 + \frac{4}{3}\epsilon & (1-\frac{4}{3}\epsilon)c_0^* c_1 \\ (1-\frac{4}{3}\epsilon)c_0 c_1^* & (1-\frac{4}{3})\epsilon|c_0|^2 + \frac{4}{3}\epsilon \end{pmatrix}$$

Problem 3

The mean-squared separation is given by

$$\left\langle (x_A - x_B)^2 \right\rangle = \left\langle x^2 \right\rangle_A + \left\langle x^2 \right\rangle_B - 2 \left\langle x \right\rangle_A \left\langle x \right\rangle_B \mp 2 \Big| \left\langle \psi_1 | x | \psi_2 \right\rangle \Big|^2$$
$$= \sigma_x^2(n) + \sigma_x^2(n') \mp 2 \Big| \left\langle n | x | n' \right\rangle \Big|^2$$

where the last term is + for fermions, - for bosons, and 0 for distinguishable particles. The position-variances are simply

$$\sigma_x^2 = \frac{\hbar(n + 1/2)}{m\omega_0}$$

and the overlap is

$$\left\langle n | x | n' \right\rangle = \frac{1}{\sqrt{2}} \left\langle n | a + a^\dagger | n' \right\rangle$$
$$= \frac{1}{\sqrt{2}} \left(\sqrt{n+1}\delta_{n,n'-1} + \sqrt{n}\delta_{n,n'+1} \right)$$

So, for distinguishable particles we have

$$\left\langle (x_A - x_B)^2 \right\rangle = \frac{\hbar(n + n' + 1)}{m\omega_0}$$

For indistinguishable bosons,

$$\left\langle (x_A - x_B)^2 \right\rangle = \frac{\hbar(n + n' + 1)}{m\omega_0} - \frac{1}{\sqrt{2}} \left(\sqrt{n+1}\delta_{n,n'-1} + \sqrt{n}\delta_{n,n'+1} \right)$$

For indistinguishable fermions,

$$\left\langle (x_A - x_B)^2 \right\rangle = \frac{\hbar(n + n' + 1)}{m\omega_0} + \frac{1}{\sqrt{2}} \left(\sqrt{n+1}\delta_{n,n'-1} + \sqrt{n}\delta_{n,n'+1} \right)$$

Problem 4

(a) Parahelium should have the lower energy. Since the electrons' spins are antisymmetric, their spacial degree of freedom must be bosonic, so the can both occupy the lowest energy state.

(b) For the delium atom, the electrons' wavefunctions need not be symmetric or antisymmetric, so their degree of localization should be betwen that of the para/orthohelium cases, and its energy should also be between that of para/orthohelium.

Problem 5

$$\mathrm{Tr}[Q] = p_0 - p_1; \qquad \det[Q] = \left[(p_0 - p_1)^2 - 1\right] \sin^2\theta \cos^2\theta$$

$$\begin{aligned}
q_\pm &= \frac{\mathrm{Tr}[Q]}{2} \pm \sqrt{\left(\frac{\mathrm{Tr}[Q]}{2}\right)^2 - \det[Q]} \\
&= \frac{p_0 - p_0}{2} \pm \sqrt{\left(\frac{p_0 - p_1}{2}\right)^2 - [(p_0 - p_1)^2 - 1] \sin^2\theta \cos^2\theta} \\
&= \frac{1}{2}\left(p_0 - p_1 \pm \sqrt{(p_0 - p_1)^2 - [(p_0 - p_1)^2 - 1] \sin^2(2\theta)}\right) \\
&= \frac{1}{2}\left(p_0 - p_1 \pm \sqrt{(p_0 - p_1)^2 - [(p_0 - p_1)^2 - 1]\left(1 - \cos^2(2\theta)\right)}\right) \\
&= \frac{1}{2}\left(p_0 - p_1 \pm \sqrt{1 - \cos^2(2\theta) - (p_0 - p_1)^2 \cos^2(2\theta)}\right) \\
&= \frac{1}{2}\left(p_0 - p_1 \pm \sqrt{1 + (1 - (p_0 - p_1)^2) \cos^2(2\theta)}\right) \\
&= \frac{1}{2}\left(p_0 - p_1 \pm \sqrt{1 - 4 p_0 p_1 \cos^2 2\theta}\right)
\end{aligned}$$