

Homework 3

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Problem 1

The hamiltonian in the field interaction representation is (neglecting tildes)

$$H = \frac{\hbar}{2} \begin{pmatrix} -\delta(t) & \Omega_0^*(t) \\ \Omega_0(t) & \delta(t) \end{pmatrix},$$

and the equations of motion for the density matrix are

$$\begin{aligned} \dot{\rho} &= \frac{1}{i\hbar} [H, \rho] \\ &= \begin{pmatrix} -\text{Re}[\rho_{12}\Omega_0(t)] & i\rho_{12}\delta(t) - \frac{i}{2}(\rho_{22} - \rho_{11})\Omega_0^*(t) \\ -i\rho_{21}\delta(t) + \frac{i}{2}(\rho_{22} - \rho_{11})\Omega_0(t) & \text{Re}[\rho_{12}\Omega_0(t)] \end{pmatrix} \end{aligned}$$

(See attached Mathematica printout for calculations.)

Problem 2

The Bloch vector is just the vector of expectation values of the Pauli operators, so

$$\begin{aligned} \frac{d}{dt} \vec{B} &= \frac{d}{dt} \text{Tr}[\rho \vec{\sigma}] \\ &= \begin{pmatrix} \text{Tr}[\dot{\rho} \sigma_x] \\ \text{Tr}[\dot{\rho} \sigma_y] \\ \text{Tr}[\dot{\rho} \sigma_z] \end{pmatrix} \\ &= \begin{pmatrix} -\text{Im}[\Omega_0(t)](\rho_{22} - \rho_{11}) - 2\text{Im}[\rho_{12}]\delta(t) \\ \text{Re}[\Omega_0(t)](\rho_{22} - \rho_{11}) - 2\text{Re}[\rho_{12}]\delta(t) \\ -2\text{Im}[\rho_{12}\Omega_0(t)] \end{pmatrix} \end{aligned}$$

(See attached Mathematica printout for calculations.)

Problem 3 (Berman 3.8)

From equation 3.31 in Berman,

$$\begin{aligned}\dot{\rho}_{12} &= i(\omega_0 + i\gamma)\rho_{12}(t) - \frac{i}{2}\Omega_0^*(t)e^{i\omega t}(\rho_{22}(t) - \rho_{11}(t)) \\ \dot{\rho}_{22} &= -2\gamma\rho_{22}(t) + 2\text{Im}[\Omega_0(t)e^{-i\omega t}\rho_{12}(t)] \\ \rho_{22} &= \frac{|\Omega_0|^2/2}{2\gamma^2 + |\Omega|^2} \left[1 - \left(\cos(\lambda t) + \frac{3\gamma}{2\lambda} \sin(\lambda t) \right) e^{-3\gamma t/2} \right] \\ \frac{|\Omega_0|^2/2}{2\gamma^2 + |\Omega_0|^2} &= \frac{1}{2} \left(1 + 2 \left(\frac{\gamma}{|\Omega_0|} \right)^2 \right)^{-1} \\ &\approx \frac{1}{2} \left(1 - 2 \frac{\gamma^2}{|\Omega_0|^2} \right)\end{aligned}$$

$$\begin{aligned}\lambda &= \sqrt{|\Omega_0|^2 - \gamma^2/4} \\ &= |\Omega_0| \sqrt{1 - \left(\frac{\gamma}{2|\Omega_0|} \right)^2} \\ &\approx |\Omega_0| \left(1 - \frac{\gamma^2}{4|\Omega_0|^2} \right) \\ \frac{1}{\lambda} &\approx \frac{1}{|\Omega_0|} \left(1 + \frac{\gamma^2}{4|\Omega_0|^2} \right)\end{aligned}$$

so

$$\begin{aligned}\rho_{22} &\approx \frac{1}{2} \left(1 - 2 \frac{\gamma^2}{|\Omega_0|^2} \right) \left[1 - \left(\cos \left(|\Omega_0| \left(1 - \frac{\gamma^2}{4|\Omega_0|^2} \right) t \right) + \frac{3}{2} \frac{\gamma}{|\Omega_0|} \left(1 + \frac{\gamma^2}{4|\Omega_0|^2} \right) \sin \left(|\Omega_0| \left(1 - \frac{\gamma^2}{4|\Omega_0|^2} \right) t \right) \right) e^{-3\gamma t/2} \right] \\ &\approx \frac{1}{2} \left[1 - \left(\cos(|\Omega_0|t) + \frac{3}{2} \frac{\gamma}{|\Omega_0|} \sin(|\Omega_0|t) \right) e^{-3\gamma t/2} \right] \\ &\approx \frac{1}{2} (1 - e^{-3\gamma t/2} \cos(|\Omega_0|t))\end{aligned}$$

This describes exponentially damped oscillation of the population between the upper and lower states (i.e. damped oscillation of the z -component of the Bloch vector), which asymptotically approaches $\rho_{11} = \rho_{22} = 1/2$ (i.e. the z -component of the Bloch vector goes to 0).

Problem 4 (Berman 3.10)

Parameterizing the state as

$$|\psi\rangle = \cos \frac{\theta}{2} |1\rangle + \sin \frac{\theta}{2} e^{i\phi} |2\rangle,$$

the Bloch vector is

$$\begin{aligned}\vec{B} &= \begin{pmatrix} \langle \psi | \sigma_x | \psi \rangle \\ \langle \psi | \sigma_y | \psi \rangle \\ \langle \psi | \sigma_z | \psi \rangle \end{pmatrix} \\ &= \begin{pmatrix} \cos \phi \sin \theta \\ \sin \phi \sin \theta \\ \cos \theta \end{pmatrix}.\end{aligned}$$

(See attached Mathematica printout for calculations.)

Problem 5 (Berman 3.7)

In the absence of relaxation the Bloch vector has constant unit length (assuming proper normalization). For constant $\vec{\Omega}$,

$$\frac{d}{dt} \vec{B} = \vec{\Omega} \times \vec{B},$$

so, letting θ be the angle between $\vec{\Omega}$ and \vec{B} ,

$$\begin{aligned}\frac{d}{dt} \left(|\vec{B}| |\vec{\Omega}| \cos \theta \right) &= |\vec{\Omega}| \frac{d}{dt} (\cos \theta) \\ &= \frac{d}{dt} (\vec{\Omega} \cdot \vec{B}) \\ &= \vec{\Omega} \cdot \left(\frac{d}{dt} \vec{B} \right) \\ &= \vec{\Omega} \cdot (\vec{\Omega} \times \vec{B}) \\ &= 0\end{aligned}$$

If $\vec{\Omega}$ is a function of time (with rate of change negligible compared to its magnitude), then

$$\begin{aligned}\frac{d}{dt} \cos \theta &= \frac{d}{dt} \frac{\vec{\Omega}(t) \cdot \vec{B}(t)}{|\vec{\Omega}(t)|} \\ &= \left(\frac{d}{dt} \Omega^{-1}(t) \right) (\vec{\Omega}(t) \cdot \vec{B}(t)) + \frac{\left(\frac{d}{dt} \vec{\Omega}(t) \right) \cdot \vec{B}(t) + \vec{\Omega}(t) \cdot \left(\frac{d}{dt} \vec{B}(t) \right)}{\Omega} \\ &= \frac{\dot{\vec{\Omega}}(t)}{\Omega(t)} \cdot \vec{B} - \frac{\dot{\Omega}(t)}{\Omega^2(t)} \\ &\approx 0.\end{aligned}$$

```
In[4]:= Symbolize[Ωθ];
Symbolize[σx]; Symbolize[σy]; Symbolize[σz];
Symbolize[σ+]; Symbolize[σ-]; Symbolize[σθ];
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```
In[7]:= $Assumptions = {ω ∈ ℝ, t ∈ ℝ, θ ∈ ℝ, ϕ ∈ ℝ};
σx = PauliMatrix[1]; σy = PauliMatrix[2]; σz = PauliMatrix[3];
σ+ =  $\frac{1}{2} (\sigma_x - i \sigma_y)$ ; σ- =  $\frac{1}{2} (\sigma_x + i \sigma_y)$ ; σθ = σ+.σ-;
Comm[A_, B_] := A.B - B.A;
CleanUp[x_] := TraditionalForm[MatrixForm[FullSimplify[x]]];
```

Problem 1

```
In[12]:= HFI[t_] =  $\frac{\hbar}{2} (-\delta[t] \sigma_z + \text{Re}[\Omega_\theta[t]] \sigma_x + \text{Im}[\Omega_\theta[t]] \sigma_y)$ ;
rho = {{ρ11, ρ12}, {ρ21, ρ22}};
ρ̇ =  $\frac{1}{i \hbar} \text{Comm}[H_{FI}[t], \text{rho}]$ ;
ρ̇ // CleanUp
```

```
Out[15]//TraditionalForm=
```

$$\begin{pmatrix} -\frac{1}{2} i (\rho_{21} \Omega_0(t)^* - \rho_{12} \Omega_0(t)) & \frac{1}{2} i ((\rho_{11} - \rho_{22}) \Omega_0(t)^* + 2 \rho_{12} \delta(t)) \\ -\frac{1}{2} i (2 \rho_{21} \delta(t) + (\rho_{11} - \rho_{22}) \Omega_0(t)) & \frac{1}{2} i (\rho_{21} \Omega_0(t)^* - \rho_{12} \Omega_0(t)) \end{pmatrix}$$

Problem 2

```
In[16]:= {{Tr[ρ̇.σx], Tr[ρ̇.σy], Tr[ρ̇.σz]]}^T // CleanUp
```

```
Out[16]//TraditionalForm=
```

$$\begin{pmatrix} (\rho_{11} - \rho_{22}) \text{Im}(\Omega_0(t)) + i (\rho_{12} - \rho_{21}) \delta(t) \\ (\rho_{22} - \rho_{11}) \text{Re}(\Omega_0(t)) - (\rho_{12} + \rho_{21}) \delta(t) \\ -i (\rho_{21} \Omega_0(t)^* - \rho_{12} \Omega_0(t)) \end{pmatrix}$$

Problem 3

```
In[17]:= rho[t_] = {{a[t], b[t]}, {c[t], d[t]}};
H[t_] =  $\frac{\hbar}{2} (-\omega_\theta \sigma_z + \Omega_\theta e^{i \omega t} \sigma_+ + (\Omega_\theta e^{i \omega t} \sigma_+)^{\dagger})$ ;
rhoDot[t_] =
 $\frac{1}{i \hbar} (\text{Comm}[H[t], \text{rho}[t]] - i \hbar \gamma (\sigma_\theta.\text{rho}[t] + \text{rho}[t].\sigma_\theta) + 2 i \hbar \gamma \sigma_-.\text{rho}[t].\sigma_+)$ ;
rhoDot[t] // CleanUp
```

```
In[21]:= DSolve[{a'[t] == (rhoDot[t][[1, 1]] /. {d[t] -> 1 - a[t], c[t] -> b[t]*}),
  b'[t] == (rhoDot[t][[1, 2]] /. {d[t] -> 1 - a[t], c[t] -> b[t]*}),
  a[0] == 1, c[0] == 0} // FullSimplify, {a, b}, t] // Cleanup
```

Problem 4

```
In[23]:=  $\psi = \{\{\text{Cos}[\theta / 2]\}, \{e^{i\phi} \text{Sin}[\theta / 2]\}\};$ 
  (( $\psi^\dagger$ .#. $\psi$ )[[1]]) & /@ { $\sigma_x$ ,  $\sigma_y$ ,  $\sigma_z$ } // Cleanup
```

Out[24]//TraditionalForm=

$$\begin{pmatrix} \sin(\theta) \cos(\phi) \\ \sin(\theta) \sin(\phi) \\ \cos(\theta) \end{pmatrix}$$

Problem 5

```
In[25]:=
```