Homework 4

Sean Ericson
Phys 610

February 13, 2024

Problem 1

(a)

$$\begin{aligned}
&& l \cdot u = 0 \\
\implies && -(1+2\Phi)l^0 u^0 + u^i l_i = 0 \\
\implies && l^0 \frac{\mathrm{d}t}{\mathrm{d}\tau} = (1+2\Phi)^{-1}\frac{\mathrm{d}x^i}{\mathrm{d}\tau} l_i \\
\implies && l^0 = (1+2\Phi)^{-1}\dot{x}^i l_i \\
\implies && l^0 \approx (1-2\Phi)\dot{x}^i l_i \\
\implies && l^0 \approx \dot{x}^i l_i
\end{aligned}$$

(b) Parallel transport of l implies

$$\frac{\mathrm{d}}{\mathrm{d}t} l^\mu + \Gamma^\mu_{\sigma\rho}\dot{x}^\sigma l^\rho = 0$$

Solving for the i^{th} component of l, we find

$$\begin{aligned}
\dot{l}^i &= -\frac{1}{2} g^{ij}\left(\partial_\sigma g_{\rho j} + \partial_\rho g_{\sigma j} - \partial_j g_{\sigma\rho}\right)\dot{x}^\sigma l^\rho \\
&= -\frac{1}{2} g^{jj}\left(\partial_\sigma g_{ii}\dot{x}^\sigma l^i + \partial_\rho g_{ii}\dot{x}^i l^\rho - \partial_i g_{\alpha\alpha}\dot{x}^\alpha l^\alpha\right) \\
&= -\frac{1}{2}(1-2\Phi)^{-1}\left(\partial_\sigma(1-2\Phi)\dot{x}^\sigma l^i + \partial_\rho(1-2\Phi)\dot{x}^i l^\rho - \partial_i g_{\alpha\alpha}\dot{x}^\alpha l^\alpha\right) \\
&= (1-2\Phi)^{-1}\left(\dot{x}^\sigma l^i \Phi_{,\sigma} + \dot{x}^i l^\rho \Phi_{,\rho} + \frac{1}{2}\partial_i g_{\alpha\alpha}\dot{x}^\alpha l^\alpha\right) \\
&= (1-2\Phi)^{-1}\left(\dot{x}^\sigma l^i \Phi_{,\sigma} + \dot{x}^i l^\rho \Phi_{,\rho} - \frac{1}{2}\partial_i(1+2\Phi)\dot{x}^0 l^0 + \frac{1}{2}\partial_i(1-2\Phi)\dot{x}^j l^j\right) \\
&= (1-2\Phi)^{-1}\left(\dot{x}^\sigma l^i \Phi_{,\sigma} + \dot{x}^i l^\rho \Phi_{,\rho} - \Phi_{,i} l^0 - \Phi_{,i}\dot{x}^j l^j\right) \\
&= (1-2\Phi)^{-1}\left(l^i\dot{\Phi} + \dot{x}^j l^j \Phi_{,j} + \dot{x}^i l^0 \dot{\Phi} + \dot{x}^i l^j \Phi, j - 2\dot{x}^j l^j \Phi_{,i}\right) \\
&\approx -2\dot{x}^j l^j \Phi_{,i} + l^i\dot{\Phi} + \dot{x}^i l^k \Phi_{,k} + \dot{x}^m l^i \Phi_{,m},
\end{aligned}$$

(c) ??

(d) ??

(e) ??

Problem 2

??

Problem 3

(a) We begin by demanding that the covariant derivative of a vector transforms like a $(1,1)$ tensor:

$$\begin{aligned}
&& \tilde{\Delta}_\mu \tilde{V}^\nu &= A^\alpha_\mu (A^{-1})^\nu_\beta \Delta_\alpha V^\beta \\
\implies && \tilde{\partial}_\mu \tilde{V}^\nu + \tilde{\Gamma}^\nu_{\mu\gamma}\tilde{V}^\gamma &= A^\alpha_\mu (A^{-1})^\nu_\beta \left(\partial_\alpha V^\beta + \Gamma^\beta_{\alpha\lambda} V^\lambda\right) \\
\implies && \tilde{\partial}_\mu \left((A^{-1})^\nu_\lambda V^\lambda\right) + \tilde{\Gamma}^\nu_{\mu\gamma}(A^{-1})^\gamma_\lambda V^\lambda &= \text{``} \qquad\qquad\qquad \text{''} \\
\implies && (\tilde{\partial}_\mu (A^{-1})^\nu_\lambda) V^\lambda + (A^{-1})^\nu_\lambda (\tilde{\partial}_\mu V^\lambda) + \tilde{\Gamma}^\nu_{\mu\gamma}(A^{-1})^\gamma_\lambda V^\lambda &= \\
\implies && A^\gamma_\mu(\partial_\gamma (A^{-1})^\nu_\lambda) V^\lambda + (A^{-1})^\nu_\lambda A^\alpha_\mu \partial_\alpha V^\lambda + \tilde{\Gamma}^\nu_{\mu\gamma}(A^{-1})^\gamma_\lambda V^\lambda &= \\
\implies && A^\gamma_\mu(\partial_\gamma (A^{-1})^\nu_\lambda) V^\lambda + \tilde{\Gamma}^\nu_{\mu\gamma}(A^{-1})^\gamma_\lambda V^\lambda &= A^\alpha_\mu (A^{-1})^\nu_\beta \Gamma^\beta_{\alpha\lambda} V^\lambda \\
\implies && \tilde{\Gamma}^\nu_{\mu\gamma}(A^{-1})^\gamma_\lambda &= A^\alpha_\mu (A^{-1})^\nu_\beta \Gamma^\beta_{\alpha\lambda} - A^\gamma_\mu \partial_\gamma (A^{-1})^\nu_\lambda \\
\implies && \tilde{\Gamma}^\nu_{\mu\rho} &= A^\lambda_\rho A^\alpha_\mu (A^{-1})^\nu_\beta \Gamma^\beta_{\alpha\lambda} - A^\lambda_\rho A^\alpha_\mu \partial_\alpha (A^{-1})^\nu_\lambda \\
\implies && \tilde{\Gamma}^\nu_{\mu\rho} &= A^\lambda_\rho A^\alpha_\mu \left((A^{-1})^\nu_\beta \Gamma^\beta_{\alpha\lambda} - \partial_\alpha (A^{-1})^\nu_\lambda\right)
\end{aligned}$$

This form agrees with equation 3.10 in Carrol's *Spacetime and Geometry*. To get it into the desired form, we can relabel some indices and focus on the last term:

$$\begin{aligned}
-A^\lambda_\rho A^\alpha_\mu \partial_\alpha (A^{-1})^\mu_\lambda &= \tilde{\partial}_\rho \delta^\mu_\nu - A^\lambda_\nu A^\alpha_\rho \partial_\alpha (A^{-1})^\nu_\lambda \\
&= \tilde{\partial}_\rho \left((A^{-1})^\mu_\tau A^\tau_\nu\right) - A^\lambda_\nu A^\alpha_\rho \partial_\alpha (A^{-1})^\nu_\lambda \\
&= (A^{-1})^\mu_\tau \tilde{\partial}_\rho A^\tau_\nu \\
&= (A^{-1})^\mu_\tau A^\sigma_\rho \partial_\sigma A^\tau_\nu
\end{aligned}$$

Finally, then, we have that

$$\begin{aligned}
\tilde{\Gamma}^\mu_{\rho\nu} &= A^\lambda_\nu A^\alpha_\rho (A^{-1})^\mu_\tau \Gamma^\tau_{\alpha\lambda} - A^\lambda_\nu A^\alpha_\rho \partial_\alpha (A^{-1})^\mu_\lambda \\
&= A^\lambda_\nu A^\sigma_\rho (A^{-1})^\mu_\tau \Gamma^\tau_{\sigma\lambda} + (A^{-1})^\mu_\tau A^\sigma_\rho \partial_\sigma A^\tau_\nu \\
&= A^\sigma_\rho (A^{-1})^\mu_\tau \left(A^\lambda_\nu \Gamma^\tau_{\sigma\lambda} + \partial_\sigma A^\tau_\nu\right)
\end{aligned}$$

(b) Yes, the difference of two connections transforms like a tensor, as the non-tensorial part of the transformations cancel:

$$
\begin{aligned}
\tilde{S}^{\mu}_{\rho\nu} &= (\tilde{\Gamma}_1)^{\mu}_{\rho\nu} - (\tilde{\Gamma}_2)^{\mu}_{\rho\nu} \\
&= A^{\sigma}_{\rho}(A^{-1})^{\mu}_{\tau}A^{\lambda}_{\nu}(\Gamma_1)^{\tau}_{\sigma\lambda} + A^{\sigma}_{\rho}(A^{-1})^{\mu}_{\tau}\partial_{\sigma}A^{\tau}_{\nu} - A^{\sigma}_{\rho}(A^{-1})^{\mu}_{\tau}A^{\lambda}_{\nu}(\Gamma_2)^{\tau}_{\sigma\lambda} - A^{\sigma}_{\rho}(A^{-1})^{\mu}_{\tau}\partial_{\sigma}A^{\tau}_{\nu} \\
&= A^{\sigma}_{\rho}(A^{-1})^{\mu}_{\tau}A^{\lambda}_{\nu}(\Gamma_1)^{\tau}_{\sigma\lambda} - A^{\sigma}_{\rho}(A^{-1})^{\mu}_{\tau}A^{\lambda}_{\nu}(\Gamma_2)^{\tau}_{\sigma\lambda} \\
&= A^{\sigma}_{\rho}(A^{-1})^{\mu}_{\tau}A^{\lambda}_{\nu}S^{\tau}_{\sigma\lambda}
\end{aligned}
$$