Homework 8

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Problem 1

$$\Gamma = cT^n; \quad N_{\text{int}} = \int_t^\infty \mathrm{d}t' \Gamma(t')$$

For a radiation dominated universe

$$a(t) = \sqrt{\frac{t}{t_0}} \implies H = \frac{1}{2t}, \quad H_0 = \frac{1}{2t_0}$$

$$\begin{aligned} T &\propto \frac{1}{a} \\ \implies \quad \Gamma &= c\left(\frac{t}{t_0}\right)^{n/2} \\ \implies \quad N_{\text{int}} &= c\int_t^\infty \mathrm{d}t' \left(\frac{t}{t_0}\right)^{n/2} \\ &= \frac{2c}{n-2}\frac{t^{-(n-2)/2}}{(2H_0)^{n/2}} \quad (n > 2) \end{aligned}$$

Now,

$$\begin{aligned} N_{\text{int}}(t_d) &= 1 \\ \implies \quad \frac{2c}{n-2}\frac{t_d^{-(n-2)/2}}{(2H_0)^{n/2}} &= 1 \\ \implies \quad t_d &= \frac{1}{2}\left(\frac{n-2}{c}\right)^{\frac{-2}{n-2}} H_0^{\frac{-n}{n-2}}. \end{aligned}$$

Finally,

$$\begin{aligned} \frac{\Gamma(t)}{H(t)} &= c\frac{t^{(2-n)/2}}{t_0^{-n/2}} \\ \implies \quad \frac{\Gamma(t_d)}{H(t_d)} &= \frac{n-2}{2}, \end{aligned}$$

which is greater than 1 for $n > 4$

Problem 2

i) The entropy density is given by

$$\begin{aligned}
s_0 &= \frac{2\pi^2}{45} g_{*s} T_\gamma^3 \\
&= \frac{2\pi^2}{45}\left(2 + \frac{7}{8}\times 3\times 2\left(\frac{T_\nu}{T_\gamma}\right)^3\right) T_\gamma^3 \\
&= \frac{2\pi^2}{45}\left(2 + \frac{21}{4}\frac{4}{11}\right)(2.73\text{ K})^3 \\
&= 39.4\text{ K}^3 \\
&\approx 4\times 10^{-38}\text{ GeV}^3
\end{aligned}$$

The critical density is

$$\rho_c = \frac{3H_0^2}{8\pi G} = 5\times 10^{-6}\text{ GeVcm}^{-3}$$

the dark matter density is then

$$\rho_{\text{DM}} = \Omega_{\text{DM}}\rho_c \approx 1.3\times 10^{-6}\text{GeVcm}^{-3}$$

The number density, n_{DM} is given by

$$n_{\text{DM}} = Y_{\text{DM}} s_0.$$

Using

$$Y_{\text{DM}} \sim 0.2\frac{g}{g_{*s}} \approx 0.007$$

I get

$$n_{\text{DM}} \approx 3\times 10^{-40}\text{ GeV}^3$$

ii)

$$\begin{aligned}
\Omega_{\text{DM}} \approx 0.25 = \frac{\rho_{\text{DM}}}{\rho_c} &= \frac{m_{\text{DM}} n_{\text{DM}}}{\rho_c} \\
\implies m_{\text{DM}} &= \frac{\Omega_{\text{DM}}\rho_c}{n_{\text{DM}}} \approx 37\text{ eV}
\end{aligned}$$

This is wayyyy below the weak scale. But I'm sure my yield calculation was nonsense =P.

Problem 3

Given that

$$\begin{aligned}
\rho_{\text{DM}} &= 0.3\text{ GeVcm}^{-3}, \\
R &= 20\text{ kpc}, \\
\langle\sigma v\rangle &= 3\times 10^{-26}\text{ cm}^3\text{s}^{-1}, \\
m_{\text{DM}} &= 100\text{ GeV}
\end{aligned}$$

we have that

$$\begin{aligned}\Gamma &\approx n\,\langle\sigma v\rangle \\ &= \frac{\rho_{\rm DM}}{m_{\rm DM}}\,\langle\sigma v\rangle \\ &\approx 1\times 10^{-28}\ {\rm s}^{-1}\end{aligned}$$

The total number of dark matter particles within the given radius is

$$N = \frac{4}{3}\pi R^3\frac{\rho_{\rm DM}}{m_{\rm DM}} \approx 10^{66}$$

The current rate of annihilations in the galaxy is then $\sim 10^{38}$/sec. Given that the time constant $(1/\Gamma)$ is about 10 orders of magnitude larger than the current age of the universe, it seems we are not presently at great risk of galactic dark matter depletion. Using

$$\rho_{\rm DM} = \Omega_{\rm DM}\rho_c = \frac{3\Omega_{\rm DM}H_0^2}{8\pi G} \approx 1.2\times 10^{-6}\ {\rm GeVcm}^{-3}$$

as the universal dark matter density, we find a universal dark matter annihilation rate of

$$\Gamma \approx 10^{-34}\ {\rm s}^{-1}$$