Homework 1

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Phys 662

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Problem 1 (Peskin 15.2)

(See attached Mathematica printout for calculations)

(a) Equation (15.40) for the rate of muon decay is

$$\Gamma = \frac{G_F^2 m_\mu^5}{192\pi^3}.$$

The calculation for tau decay rate is identical (assuming we neglect the electron and muon masses). Therefore, substituting the tau mass for the muon's, we find

$$\Gamma(\tau \to \nu_\tau \mu^- \bar{\nu}_\mu) = \Gamma(\tau \to \nu_\tau e^- \bar{\nu}_e) = 4.0476 \times 10^{-13} \text{ GeV}$$

(b) For the given hadronic decay mode, we need to average over the color of the final quarks. The final state is a color singlet, so there's only one color degree of freedom. Therefore, the color average amounts to dividing by 3. Using a value of 0.31 for the strong force at the mass of the tau, we find

$$\Gamma(\tau \to \nu_\tau d\bar{u}) \approx \frac{1}{3}\Gamma_l \left(1 + \frac{\alpha_s(m_\tau)}{\pi}\right) = 1.4823 \times 10^{-13} \text{ GeV}$$

(c) The total rate is given by

$$\Gamma_{\text{total}} = 2\Gamma_l + \Gamma_h = 2.1436 \times 10^{-12} \text{ GeV}.$$

The branching ratio to leptonic modes is

$$\text{BR}(\tau \to \nu_\tau l^- \bar{\nu}_l) = \frac{2\Gamma_l}{\Gamma_{\text{total}}} = 37.76\%,$$

and the lifetime is

$$\tau(\tau) = \frac{\hbar}{\Gamma_{\text{total}}} = 3.0706 \times 10^{-13} \text{ s}.$$

The PDG gives the following values for the branching ration and lifetime:

$$\mathrm{BR}(\tau \to \nu_\tau \mu^- \bar{\nu}_\mu) = (17.3937 \pm 0.0384)\% \tag{1}$$

$$\mathrm{BR}(\tau \to \nu_\tau e^- \bar{\nu}_e) = (17.8175 \pm 0.0399)\% \tag{2}$$

$$\implies \mathrm{BR}(\tau \to \nu_\tau l^- \bar{\nu}_l) = (35.2112 \pm 0.0783)\% \tag{3}$$

$$\tau(\tau) = (2.903 \pm 0.005) \times 10^{-13}\ \mathrm{s} \tag{4}$$

Very close!

Problem 2

The full lagrangian is

$$\mathscr{L} = i\bar{\Psi}\not{D}\Psi - (D_\mu\phi)^\dagger D^\mu\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \lambda\left(|\phi|^2 - \frac{v^2}{2}\right)^2 - y\phi\bar{\Psi}_L\Psi_R - y^*\phi^*\bar{\Psi}_R\Psi_L.$$

Expanding $\phi = \frac{1}{\sqrt{2}}(v + r)$, where we've used the $U(1)$ gauge transformation $\phi \to e^{i\alpha(x)}\phi$ to set the imaginary component of the field to zero. Plugging this into the Yukawa coupling terms and keeping the parts that depend only on the fermion field we find

$$\mathscr{L} \supset -\frac{1}{\sqrt{2}}vy\bar{\Psi}_L\Psi_R - \frac{1}{\sqrt{2}}vy^*\bar{\Psi}_R\Psi_L = -\frac{vy}{\sqrt{2}}\bar{\Psi}\Psi,$$

giving

$$m_\Psi^2 = \frac{vy}{\sqrt{2}}.$$

Similarly, keeping the parts that depend on the real component r of ϕ, we find the fermion interaction with the higgs is

$$\frac{1}{\sqrt{2}}yr\bar{\Psi}_L\Psi_R - \frac{1}{\sqrt{2}}y^*r\bar{\Psi}_R\Psi_L$$

Problem 3

See attached Mathematica sheet for calculations.

(a) $Y = 1$:

$$m_W^2 = \frac{1}{2}g^2v^2$$
$$m_Z^2 = (g^2 + (g')^2)v^2$$
$$m_A^2 = 0$$

I'm gonna guess that we have a $U(1)$ left unbroken in this case.
$Y = 0$:

$$m_W^2 = g^2v^2$$
$$m_Z^2 = 0$$
$$m_Z^2 = 0$$

Perhaps it's a whole $SU(2)$ unbroken in this case...

(b) Using both $Y = 0$ and $Y = 1$, we find

$$\rho = \frac{1}{2} + \frac{v_0^2}{v_1^2}.$$

Then,

$$\rho = 1 \implies v_1^2 = 2v_0$$

Problem 1 (Peskin 15.2)

In[∘]:=
```
<< Units`
<< Notation`
```

In[∘]:=
```
Symbolize[Γ1]; Symbolize[Γh];
```

a)

In[∘]:=
```
StringForm[" G0F = ``", UnitConvert[G0F, "GeV^(-2)"] // ScientificForm]
```

Out[∘]=

$G_F^0 = 1.16638 \times 10^{-5}\ /\mathrm{GeV}^2$

In[∘]:=
```
Γ1 = UnitConvert[((G0F)^2 (tau PARTICLE [mass] c^2)^5) / (192 π^3), "GeV"];
StringForm["Γ1 = ``", Γ1]
```

Out[∘]=

$\Gamma_1 = 4.0476 \times 10^{-13}\ \mathrm{GeV}$

b)

In[∘]:=
```
αs = 0.31;
Γh = 3 Γ1 (1 + αs/π);
StringForm["Γh = ``", Γh]
```

Out[∘]=

$\Gamma_h = 1.33409 \times 10^{-12}\ \mathrm{GeV}$

c)

In[•]:=
```
Γtotal = 2 Γl + Γh;
BR = 2 Γl / Γtotal;
τ = UnitConvert[ħ / Γtotal, "s"];
StringForm["Γtotal = ``", Γtotal]
StringForm["BR(τ → ντ l⁻ ν̄l) = ``", BR]
StringForm["τ = ``", τ]
```

Out[•]=

Γ_{total} = 2.1436×10^{-12} GeV

Out[•]=

BR($\tau \to \nu_\tau l^- \bar{\nu}_l$) = 0.3776414940934138`

Out[•]=

τ = 3.07058×10^{-13} s

Problem 3

In[1]:=
```
<< Notation`
```

In[14]:=
$$\$Assumptions = \{h_\phi \in \mathbb{R},\ h_\psi \in \mathbb{R},\ v_\phi \in \mathbb{R},\ v_\psi \in \mathbb{R},\ \phi_1 \in \mathbb{R},\ \phi_2 \in \mathbb{R},\ \phi_3 \in \mathbb{R},\ W^1 \in \mathbb{R},\ W^2 \in \mathbb{R},\ W^3 \in \mathbb{R},\ Z \in \mathbb{R},\ B \in \mathbb{R},\ Y \in \mathbb{R},\ g_1 > 0,\ g_2 > 0\};$$

In[15]:=
```
reorderSymbols[expr_, symbols_List] := With[{s = symbols},
   HoldForm[Evaluate[expr /. Thread[s → Sort@s]]] /. Thread[Sort@s → s]];
order[expr_] :=
 reorderSymbols[expr, {g₁, g₂, v_ϕ, v_ψ, Y, h_ϕ, h_ψ, W¹, W², W³, W⁺, W⁻, B, Z, A}]
```

a)

In[17]:=
$$\phi = \{\phi_1, \phi_2, \phi_3\}^{\mathsf{T}};$$
$$W = \{W^1, W^2, W^3\}^{\mathsf{T}};$$
$$T = \left\{\frac{1}{\sqrt{2}}\{\{0,1,0\},\{1,0,1\},\{0,1,0\}\},\ \frac{i}{\sqrt{2}}\{\{0,-1,0\},\{1,0,-1\},\{0,1,0\}\},\ \{\{1,0,0\},\{0,0,0\},\{0,0,-1\}\}\right\};$$
$$\theta_w = \text{ArcTan}\left[\frac{g_2}{g_1}\right];$$
$$s_w = \text{Sin}[\theta_w]\ //\ \text{FullSimplify};$$
$$c_w = \text{Cos}[\theta_w]\ //\ \text{FullSimplify};$$
```
StringForm["s_w = ``, c_w = ``", s_w, c_w]
```

Out[23]=
$$s_w = \frac{g_2}{\sqrt{g_1^2 + g_2^2}},\quad c_w = \frac{g_1}{\sqrt{g_1^2 + g_2^2}}$$

In[24]:=
```
Row[Table[StringForm["T`` = ``  ", a, T[[a]] // MatrixForm], {a, 1, 3}]]
```

Out[24]=
$$T^1 = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \quad T^2 = \begin{pmatrix} 0 & -\frac{i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & 0 \end{pmatrix} \quad T^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

```
In[25]:= StringForm["WaTa = ``", Sum[W[[a]] × T[[a]], {a, 1, 3}] // MatrixForm]
```

Out[25]=

$$W_a T^a = \begin{pmatrix} W^3 & \frac{W^1}{\sqrt{2}} - \frac{i W^2}{\sqrt{2}} & 0 \\ \frac{W^1}{\sqrt{2}} + \frac{i W^2}{\sqrt{2}} & 0 & \frac{W^1}{\sqrt{2}} - \frac{i W^2}{\sqrt{2}} \\ 0 & \frac{W^1}{\sqrt{2}} + \frac{i W^2}{\sqrt{2}} & -W^3 \end{pmatrix}$$

```
In[26]:= Unprotect[D];
D = i g1 Sum[W[[a]] × T[[a]], {a, 1, 3}] + i g2 Y B IdentityMatrix[3];
StringForm["Dμ ⊃ ``", D // MatrixForm]
```

Out[28]=

$$D_\mu \supset \begin{pmatrix} i g_1 W^3 + i B g_2 Y & i g_1 \left(\frac{W^1}{\sqrt{2}} - \frac{i W^2}{\sqrt{2}}\right) & 0 \\ i g_1 \left(\frac{W^1}{\sqrt{2}} + \frac{i W^2}{\sqrt{2}}\right) & i B g_2 Y & i g_1 \left(\frac{W^1}{\sqrt{2}} - \frac{i W^2}{\sqrt{2}}\right) \\ 0 & i g_1 \left(\frac{W^1}{\sqrt{2}} + \frac{i W^2}{\sqrt{2}}\right) & -i g_1 W^3 + i B g_2 Y \end{pmatrix}$$

```
In[29]:= Dϕ = D.ϕ;
StringForm["Dμϕ ⊃ ``", Dϕ // MatrixForm]
```

Out[30]=

$$D_\mu \phi \supset \begin{pmatrix} \left(i g_1 W^3 + i B g_2 Y\right) \phi_1 + i g_1 \left(\frac{W^1}{\sqrt{2}} - \frac{i W^2}{\sqrt{2}}\right) \phi_2 \\ i g_1 \left(\frac{W^1}{\sqrt{2}} + \frac{i W^2}{\sqrt{2}}\right) \phi_1 + i B g_2 Y \phi_2 + i g_1 \left(\frac{W^1}{\sqrt{2}} - \frac{i W^2}{\sqrt{2}}\right) \phi_3 \\ i g_1 \left(\frac{W^1}{\sqrt{2}} + \frac{i W^2}{\sqrt{2}}\right) \phi_2 + \left(-i g_1 W^3 + i B g_2 Y\right) \phi_3 \end{pmatrix}$$

```
In[31]:= L = Dϕ†.Dϕ // FullSimplify;
StringForm["L ⊃ ``", L // order]
```

Out[32]=

$L \supset$

$$2 g_1 g_2 Y B \left(\sqrt{2}\, W^1 \phi_2 + W^3 (\phi_1 - \phi_3)\right) (\phi_1 + \phi_3) + g_2^2 Y^2 B^2 \left(\phi_1^2 + \phi_2^2 + \phi_3^2\right) + \frac{1}{2} g_1^2 \left(\left(W^2\right)^2 \left(2 \phi_2^2 + (\phi_1 - \phi_3)^2\right) + 2\sqrt{2}\, W^1 W^3 \phi_2 (\phi_1 - \phi_3) + 2 \left(W^3\right)^2 \left(\phi_1^2 + \phi_3^2\right) + \left(W^1\right)^2 \left(2 \phi_2^2 + (\phi_1 + \phi_3)^2\right)\right)$$

```
In[33]:= L = L /. {W1 → (1/(√2)) (W+ + W-), W2 → (i/(√2)) (W+ - W-),
        W3 → cw Z + sw A, B → cw A - sw Z} // FullSimplify // FullSimplify;
StringForm["L ⊃ ``", L // order];
StringForm["mW2 = ``", L /. {hϕ → 0, A → 0, Z → 0, W- → 1, W+ → 1} // order];
StringForm["mZ2 = ``", L /. {hϕ → 0, A → 0, Z → 1, W- → 0, W+ → 0} // order];
StringForm["mA2 = ``", L /. {hϕ → 0, A → 1, Z → 0, W- → 0, W+ → 0} // order];
```

In[38]:=
```
L₁ = L /. {Y → 1, ϕ₁ → 0, ϕ₂ → 0, ϕ₃ → (v_ϕ + h_ϕ) / √2} // FullSimplify;
(m_W^2)_{Y=1} = L₁ /. {h_ϕ → 0, A → 0, Z → 0, W⁻ → 1, W⁺ → 1};
(m_Z^2)_{Y=1} = 2 L₁ /. {h_ϕ → 0, A → 0, Z → 1, W⁻ → 0, W⁺ → 0};
(m_A^2)_{Y=1} = 2 L₁ /. {h_ϕ → 0, A → 1, Z → 0, W⁻ → 0, W⁺ → 0} ;
StringForm["L|_{Y=1} ⊃ ``", L1 // order]
StringForm["m_W^2 = ``", (m_W^2)_{Y=1} // order]
StringForm["m_Z^2 = ``", (m_Z^2)_{Y=1} // order]
StringForm["m_A^2 = ``", (m_A^2)_{Y=1} // order]
```

Out[42]=

$L|_{Y=1} \supset L1$

Out[43]=

$$m_W{}^2 = \frac{1}{2} g_1^2 v_\phi^2$$

Out[44]=

$$m_Z{}^2 = \left(g_1^2 + g_2^2\right) v_\phi^2$$

Out[45]=

$$m_A{}^2 = 0$$

In[46]:=
```
L₀ = L /. {Y → 0, ϕ₁ → 0, ϕ₂ → (v_ϕ + h_ϕ) / √2 , ϕ₃ → 0} // FullSimplify;
(m_W^2)_{Y=0} = L₀ /. {h_ϕ → 0, A → 0, Z → 0, W⁻ → 1, W⁺ → 1};
(m_Z^2)_{Y=0} = 2 L₀ /. {h_ϕ → 0, A → 0, Z → 1, W⁻ → 0, W⁺ → 0};
(m_A^2)_{Y=0} = 2 L₀ /. {h_ϕ → 0, A → 1, Z → 0, W⁻ → 0, W⁺ → 0} ;
StringForm["L|_{Y=0} ⊃ ``", L0 // order]
StringForm["m_W^2 = ``", (m_W^2)_{Y=0} // order]
StringForm["m_Z^2 = ``", (m_Z^2)_{Y=0} // order]
StringForm["m_A^2 = ``", (m_A^2)_{Y=0} // order]
```

Out[50]=

$L|_{Y=0} \supset L0$

Out[51]=

$$m_W{}^2 = g_1^2 v_\phi^2$$

Out[52]=

$$m_Z{}^2 = 0$$

Out[53]=

$$m_A{}^2 = 0$$

In[54]:=
```
D' = D /. {W¹ → 0, W² → 0, W³ → c_w Z + s_w A, B → c_w A - s_w Z} // FullSimplify ;
D' // MatrixForm
```

Out[55]//MatrixForm=

$$\begin{pmatrix} \frac{i \left(A g_1 g_2 (1+Y) + \left(g_1^2 - g_2^2 Y\right) Z\right)}{\sqrt{g_1^2+g_2^2}} & 0 & 0 \\ 0 & \frac{i g_2 Y (A g_1 - g_2 Z)}{\sqrt{g_1^2+g_2^2}} & 0 \\ 0 & 0 & \frac{i \left(A g_1 g_2 (-1+Y) - \left(g_1^2 + g_2^2 Y\right) Z\right)}{\sqrt{g_1^2+g_2^2}} \end{pmatrix}$$

In[56]:=
```
chargeTerm = Coefficient[D', A]  // FullSimplify;
chargeTerm // MatrixForm
```

Out[57]//MatrixForm=

$$\begin{pmatrix} \frac{i\, g_1\, g_2\, (1+Y)}{\sqrt{g_1^2+g_2^2}} & 0 & 0 \\ 0 & \frac{i\, g_1\, g_2\, Y}{\sqrt{g_1^2+g_2^2}} & 0 \\ 0 & 0 & \frac{i\, g_1\, g_2\, (-1+Y)}{\sqrt{g_1^2+g_2^2}} \end{pmatrix}$$

In[58]:= chargeTerm == $i\, g_1\, s_w$ (T⟦3⟧ + Y IdentityMatrix[3]) // FullSimplify

Out[58]=

True

b)

In[59]:= $\rho_1 = \frac{\left(m_W{}^2\right)_{Y=1}}{\left(m_Z{}^2\right)_{Y=1} c_w{}^2}$ // FullSimplify;

$\rho_0 = \frac{\left(m_W{}^2\right)_{Y=0}}{\left(m_Z{}^2\right)_{Y=0} c_w{}^2}$ // FullSimplify;

StringForm["ρ_1 = ``", ρ_1]

StringForm["ρ_0 = ``", ρ_0]

Power: Infinite expression $\frac{1}{0}$ encountered.

Out[61]=

$\rho_1 = \frac{1}{2}$

Out[62]=

ρ_0 = ComplexInfinity

In[63]:= $\psi = \left\{0,\ (v_\psi + h_\psi) \big/ \sqrt{2},\ 0\right\}^T$;

L = $(D.\psi)^\dagger.(D.\psi)$ /. Y → 0;

L = L + + $(D\phi)^\dagger.D\phi$ /. $\left\{Y \to 1,\ \phi_1 \to 0,\ \phi_2 \to 0,\ \phi_3 \to (v_\phi + h_\phi) \big/ \sqrt{2}\right\}$ // FullSimplify;

In[66]:=
```
L = L /. {W^1 → 1/√2 (W^+ + W^-), W^2 → i/√2 (W^+ - W^-), W^3 → c_w Z + s_w A, B → c_w A - s_w Z} //
    FullSimplify;
m_W^2 = L /. {h_ϕ → 0, h_ψ → 0, A → 0, Z → 0, W^- → 1, W^+ → 1} // FullSimplify;
m_Z^2 = 2 L /. {h_ϕ → 0, h_ψ → 0, A → 0, Z → 1, W^- → 0, W^+ → 0} // FullSimplify;
m_A^2 = 2 L /. {h_ϕ → 0, h_ψ → 0, A → 1, Z → 0, W^- → 0, W^+ → 0} // FullSimplify;
StringForm["m_A^2 = ``", m_A^2 // order]
StringForm["m_W^2 = ``", m_W^2 // order]
StringForm["m_Z^2 = ``", m_Z^2 // order]
```

Out[70]=

$m_A{}^2 = 0$

Out[71]=

$m_W{}^2 = \frac{1}{2} g_1^2 \left(v_\phi^2 + 2 v_\psi^2\right)$

Out[72]=

$m_Z{}^2 = \left(g_1^2 + g_2^2\right) v_\phi^2$

In[73]:=
```
ρ = m_W^2/(m_Z^2 c_w^2) // FullSimplify;
StringForm["ρ = ``", ρ]
```

Out[74]=

$\rho = \frac{1}{2} + \frac{v_\psi^2}{v_\phi^2}$