Homework 2

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Problem 1

Let's look at x_1 first. The WKB approximation is

$$\psi_{\text{WKB}}(x) = \begin{cases} \frac{A_<}{\sqrt{p(x)}} \cos\left(\frac{1}{\hbar}\int_x^{x_1} p(x')\mathrm{d}x'\right) + \frac{B_<}{\sqrt{p(x)}} \sin\left(\frac{1}{\hbar}\int_x^{x_1} p(x')\mathrm{d}x'\right) & x < x_1 \\ \frac{A_F}{\sqrt{|p(x)|}} \exp\left[-\frac{1}{\hbar}\int_{x_1}^{x} |p(x')|\mathrm{d}x'\right] + \frac{B_F}{\sqrt{|p(x)|}} \exp\left[\frac{1}{\hbar}\int_{x_1}^{x} |p(x')|\mathrm{d}x'\right] & x > x_1 \end{cases}$$

$$\psi_{\text{patch}}(x) = a\mathrm{Ai}(\nu x) + b\mathrm{Bi}(\nu x), \quad \nu = \left(\frac{2mV'(x_1)}{\hbar^2}\right)^{\frac{1}{3}}.$$

To match them up, let's first consider $x \lesssim x_1$. In this region,

$$p(x) \approx \hbar\sqrt{\nu^3(x_1 - x)}.$$

The WKB solution is thus approximately

$$\psi_{\text{WKB}}(x) \approx \frac{A_<}{\sqrt{\hbar}\nu^{3/4}(x_1-x)^{1/4}} \cos\left(\frac{2}{3}\nu^{3/2}(x_1-x)^{3/2}\right) + \frac{B_<}{\sqrt{\hbar}\nu^{3/4}(x_1-x)^{1/4}} \sin\left(\frac{2}{3}\nu^{3/2}(x_1-x)^{3/2}\right),$$

while the asymptotic form of the patch wavefunction is

$$\psi_{\text{patch}} \approx \frac{a}{\sqrt{\pi}\nu^{1/4}(x_1-x)^{1/4}} \cos\left(\frac{2}{3}\nu^{3/2}(x_1-x)^{3/2} - \frac{\pi}{4}\right) + \frac{b}{\sqrt{\pi}\nu^{1/4}(x_1-x)^{1/4}} \sin\left(\frac{2}{3}\nu^{3/2}(x_1-x)^{3/2} - \frac{\pi}{4}\right).$$

However, given that

$$\cos\left(x - \frac{\pi}{4}\right) = \frac{\sin(x)}{\sqrt{2}} + \frac{\cos(x)}{\sqrt{2}}$$
$$\sin\left(x - \frac{\pi}{4}\right) = \frac{\sin(x)}{\sqrt{2}} - \frac{\cos(x)}{\sqrt{2}},$$

the patch wavefunction can be rewritten as

$$\psi_{\text{patch}} \approx \frac{a-b}{\sqrt{2\pi}\nu^{1/4}(x_1-x)^{1/4}} \cos\left(\frac{2}{3}\nu^{3/2}(x_1-x)^{3/2}\right) + \frac{a+b}{\sqrt{2\pi}\nu^{1/4}(x_1-x)^{1/4}} \sin\left(\frac{2}{3}\nu^{3/2}(x_1-x)^{3/2}\right).$$

The two solutions are equivalent subject to

$$a - b = \sqrt{\frac{2\pi}{\hbar\nu}}A_<, \quad a + b = \sqrt{\frac{2\pi}{\hbar\nu}}B_< \tag{1}$$

Now we can turn to $x \gtrsim x_1$. In this region,

$$|p(x)| \approx \hbar\sqrt{\nu^3(x - x_1)}$$

The WKB solution is thus approximately

$$\psi_{\text{WKB}} \approx \frac{A_F}{\sqrt{\hbar}\nu^{3/4}(x - x_1)^{1/4}} \exp\left[-\frac{2}{3}\nu^{3/2}(x - x_1)^{3/2}\right] + \frac{B_F}{\sqrt{\hbar}\nu^{3/4}(x - x_1)^{1/4}} \exp\left[\frac{2}{3}\nu^{3/2}(x - x_1)^{3/2}\right],$$

while the asymptotic form of the patch wavefunction is

$$\frac{a}{\sqrt{4\pi}\nu^{1/4}(x - x_1)^{1/4}} \exp\left[-\frac{2}{3}\nu^{3/2}(x - x_1)^{3/2}\right] + \frac{b}{\sqrt{\pi}\nu^{1/4}(x - x_1)^{1/4}} \exp\left[\frac{2}{3}\nu^{3/2}(x - x_1)^{3/2}\right]$$

The two solutions are equivalent subjet to

$$a = \sqrt{\frac{4\pi}{\hbar\nu}}A_F, \quad b = \sqrt{\frac{\pi}{\hbar\nu}}B_F. \tag{2}$$

Combining the two conditions for equality gives

$$2A_F - B_F = \sqrt{2}A_<$$

$$2A_F + B_F = \sqrt{2}B_<$$

Problem 2

The approximate wavefucntions around the turning points are

$$\psi_{\text{WKB}} = \begin{cases} \frac{A_<}{\sqrt{p(x)}} \cos\left(\frac{1}{\hbar}\int_x^{x_1} p(x')\mathrm{d}x'\right) + \frac{B_<}{\sqrt{p(x)}} \sin\left(\int_x^{x_1} p(x')\mathrm{d}x'\right) & x \lesssim x_1 \\ \frac{A_F^{(1)}}{\sqrt{|p(x)|}} \exp\left[-\frac{1}{\hbar}\int_{x_1}^{x} |p(x')|\mathrm{d}x'\right] + \frac{B_F^{(1)}}{\sqrt{|p(x)|}} \exp\left[\frac{1}{\hbar}\int_{x_1}^{x} |p(x')|\mathrm{d}x'\right] & x \gtrsim x_1 \\ \frac{A_F^{(2)}}{\sqrt{|p(x)|}} \exp\left[\frac{1}{\hbar}\int_x^{x_2} |p(x')|\mathrm{d}x'\right] + \frac{B_F^{(2)}}{\sqrt{|p(x)|}} \exp\left[-\frac{1}{\hbar}\int_x^{x_2} |p(x')|\mathrm{d}x'\right] & x \lesssim x_2 \\ \frac{A_>}{\sqrt{p(x)}} \cos\left(\frac{1}{\hbar}\int_{x_2}^{x} p(x')\mathrm{d}x'\right) + \frac{B_>}{\sqrt{p(x)}} \sin\left(\int_{x_2}^{x} p(x')\mathrm{d}x'\right) & x \gtrsim x_2 \end{cases}$$

Requiring the two components within the barrier to be equivalent gives

$$\frac{A_F^{(1)}}{\sqrt{|p(x)|}} \exp\left[-\frac{1}{\hbar}\int_{x_1}^{x} |p(x')|\mathrm{d}x'\right] = \frac{A_F^{(2)}}{\sqrt{|p(x)|}} \exp\left[\frac{1}{\hbar}\int_x^{x_2} |p(x')|\mathrm{d}x'\right]$$

$$\implies \frac{A_F^{(1)}}{A_F^{(2)}} = \exp\left[\frac{1}{\hbar}\int_{x_1}^{x}|p(x')|\mathrm{d}x'\right]\exp\left[\frac{1}{\hbar}\int_{x}^{x_2}|p(x')|\mathrm{d}x'\right]$$
$$= \exp\left[\frac{1}{\hbar}\int_{x_1}^{x_2}|p(x')|\mathrm{d}x'\right]$$
$$= \sqrt{T_1}$$

and, similarly,

$$\frac{B_F^{(1)}}{B_F^{(2)}} = \frac{1}{\sqrt{T_1}}$$

Using the result from the last problem we can write the coefficients inside the barrier in terms of the external coefficients:

$$A_F^{(1,2)} = \frac{1}{2\sqrt{2}}(A_{<,>} + B_{<,>})$$

$$B_F^{(1,2)} = \frac{1}{\sqrt{2}}(B_{<,>} - B_{<,>})$$

Combining these, we can write the left-interior coefficients in terms of the right-transmission region coefficients:

$$A_F^{(1)} = \frac{\sqrt{T_1}}{2\sqrt{2}}(A_> + B_>)$$
$$B_F^{(1)} = \frac{1}{\sqrt{2T_1}}(B_> - A_>)$$

We can write the transmission regions as superpositions of left and right going waves:

$$\psi_{\mathrm{WKB}} = \frac{A_< - iB_<}{2}\exp\left[\frac{i}{\hbar}\int_{x}^{x_1}p(x')\mathrm{d}x'\right] + \frac{A_< + iB_<}{2}\exp\left[-\frac{i}{\hbar}\int_{x}^{x_1}p(x')\mathrm{d}x'\right] \quad x \lesssim x_1$$
$$\psi_{\mathrm{WKB}} = \frac{A_> + iB_>}{2}\exp\left[-\frac{i}{\hbar}\int_{x_2}^{x}p(x')\mathrm{d}x'\right] + \frac{A_> - iB_>}{2}\exp\left[\frac{i}{\hbar}\int_{x_2}^{x}p(x')\mathrm{d}x'\right] \quad x \gtrsim x_2$$

To model the action of a wave incident on the barrier from the left side, we set

$$A_< - iB_< = 0$$

The trasnmission probability is then

$$T = |\frac{A_> - iB_>}{A_< + iB_<}|^2$$

Combining with the results above gives

$$T = \frac{T_1}{1 + T_1/4}$$

Problem 3

The approimate tunneling probability is given by

$$\begin{aligned}
T &\approx \exp\left[-\frac{2}{\hbar}\int_{x_1}^{x_2}|p(x')|\mathrm{d}x'\right] \\
&= \exp\left[-\frac{2}{\hbar}\int_{0}^{\frac{V_0-E}{e\mathscr{E}}}\left|\sqrt{2m(E-V_0+e\mathscr{E}x)}\right|\mathrm{d}x'\right] \\
&= \exp\left[-\frac{2}{\hbar}\int_{0}^{\frac{V_0-E}{e\mathscr{E}}}\sqrt{2m(V_0-E-e\mathscr{E}x)}\mathrm{d}x'\right] \\
&= \exp\left[-\frac{2\left(2m(V_0-E)\right)^{3/2}}{3\hbar me\mathscr{E}}\right]
\end{aligned}$$

If $E = V_0 - W$ then

$$T \approx \exp\left[-\frac{2(2mW)^{3/2}}{3\hbar me\mathscr{E}}\right]$$