Homework 1

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Phys 611

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Problem 1

(a) To start with, we need to parameterize a path along the surface of a sphere (along with it's derivative w.r.t. the parameter ϕ):

$$\vec{r}(\phi) = \begin{pmatrix} r\sin\theta(\phi)\cos\phi \\ r\sin\theta(\phi)\sin\phi \\ r\cos\theta(\phi) \end{pmatrix}$$

$$\dot{\vec{r}} = r\begin{pmatrix} \cos\theta\cos\phi\dot{\theta} - \sin\theta\sin\phi \\ \cos\theta\sin\phi\dot{\theta} + \sin\theta\cos\phi \\ -\sin\theta\dot{\theta} \end{pmatrix}$$

The arc length is then given by

$$S = \int_{\phi_0}^{\phi_1} \mathrm{d}\phi\, |\dot{\vec{r}}|$$

Now,

$$\begin{aligned} \left|\dot{\vec{r}}\right|^2 &= \cos^2\theta\cos^2\phi\dot{\theta}^2 - 2\cos\theta\sin\theta\cos\phi\sin\phi\dot{\theta} \\ &\quad + \sin^2\theta\sin^2\phi + \cos^2\theta\sin^\phi\dot{\theta}^2 + 2\cos\theta\sin\theta\cos\phi\sin\phi\dot{\theta} + \sin^2\theta\cos^2\phi + \sin^2\theta\dot{\theta}^2 \\ &= \cos^2\theta\cos^2\phi\dot{\theta}^2 + \cos^2\theta\sin^2\phi\dot{\theta}^2 + \sin^2\theta\sin^2\phi + \sin^2\theta\cos^2\phi \\ &= \cos^2\theta\dot{\theta}^2 + \sin^2\theta \end{aligned}$$

so the arc length is then

$$S = \int_{\phi_0}^{\phi_1} \mathrm{d}\phi\, \sqrt{\cos^2\theta\dot{\theta}^2 + \sin^2\theta}$$

where θ and $\dot{\theta}$ are both functions of ϕ.

(b) b

(c) c

(d) d

Problem 2

(a) Extending the derivation from class, now for $\mathscr{L}(y, \dot{y}, \ddot{y})$,

$$\delta \mathscr{L} = \left(\frac{\partial \mathscr{L}}{\partial x}\right) \delta x + \left(\frac{\partial \mathscr{L}}{\partial \dot{x}}\right) \delta \dot{x} + \left(\frac{\partial \mathscr{L}}{\partial \ddot{x}}\right) \delta \ddot{x}$$

Integrating by parts once on the second term and twice on the third term yields

$$\delta S = \int_{x_0}^{x_1} \mathrm{d}x \, \left(\frac{\partial \mathscr{L}}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial \mathscr{L}}{\partial \dot{x}} + \frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial \mathscr{L}}{\partial \ddot{x}}\right) \delta x$$

Now we can use the stationary action principle to derive the modified Euler-Lagrange equation

$$\delta S = 0 \implies \boxed{\frac{\partial \mathscr{L}}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\frac{\partial \mathscr{L}}{\partial \dot{y}} + \frac{\mathrm{d}^2}{\mathrm{d}y^2}\frac{\partial \mathscr{L}}{\partial \ddot{x}} = 0}$$

(b) Now to apply the modified EL equation. The Lagrangian is given by

$$\mathscr{L}(y, \dot{y}, \ddot{y}) = \ddot{y}^2$$

Now some derivatives!

$$\frac{\partial \mathscr{L}}{\partial y} = 0; \quad \frac{\partial \mathscr{L}}{\partial \dot{y}} = 0; \quad \frac{\partial \mathscr{L}}{\partial \ddot{y}} = 2\ddot{y}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}x^2}\frac{\partial \mathscr{L}}{\partial \ddot{y}} = 2\ddddot{y}$$

The Euler-Lagrange equation then gives

$$\ddddot{y} = 0$$

(c) The general solution to the above equation is

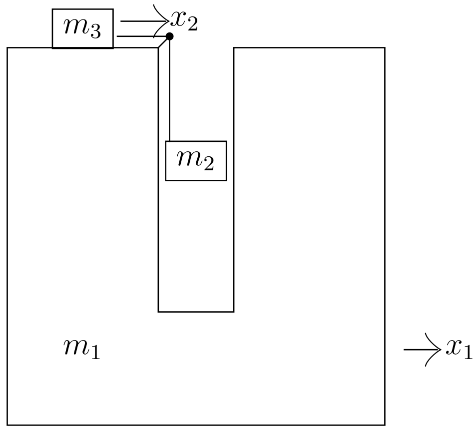
$$y(x) = ax^3 + bx^2 + cx + d$$

Applying the boundary conditions $y(0) = 0$ and $y'(0) = 0$ gives

$$c = d = 0$$

Further, applying the boundary conditions $y(L) = h$ and $y'(L) = m$ gives

Problem 3



(a) a

(b) b

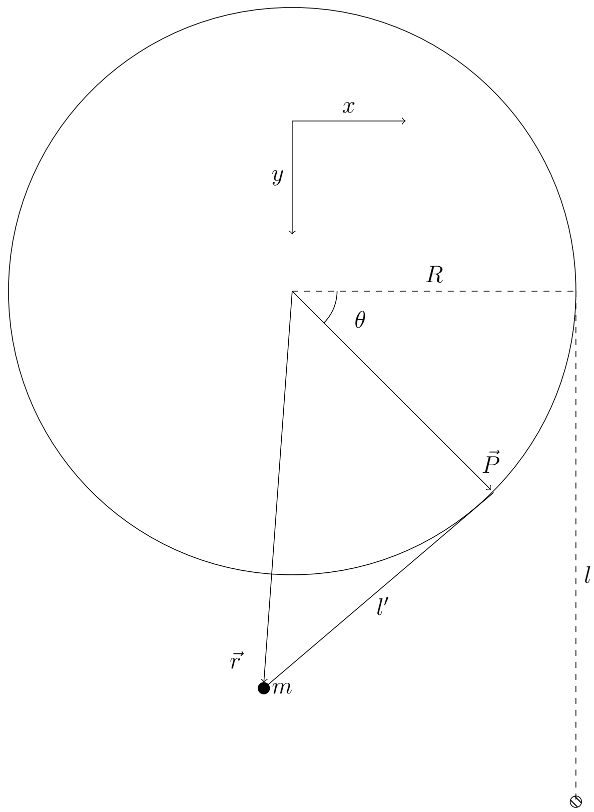
Problem 4

(a) a

(b) b

(c) c

Problem 5



(a) First, some needed quantities/vectors

$$l' = l - R\theta$$

$$\vec{r} - \vec{p} = \begin{pmatrix} -l' \sin\theta \\ l' \cos\theta \end{pmatrix}$$

$$\implies \vec{r} = \begin{pmatrix} -l' \sin\theta + R\cos\theta \\ l' \cos\theta + R\sin\theta \end{pmatrix} = \begin{pmatrix} -(l - R\theta)\sin\theta + R\cos\theta \\ (l - R\theta)\cos\theta + R\sin\theta \end{pmatrix}$$

The kinetic and potential energies are given by

$$\begin{aligned} T &= \frac{1}{2} m \left|\dot{\vec{r}}\right|^2 \\ &= \frac{1}{2} m \left| \begin{matrix} R\sin\theta\dot{\theta} - (l - R\theta)\cos\theta\dot{\theta} - R\sin\theta\dot{\theta} \\ -R\cos\theta\dot{\theta} - (l - R\theta)\sin\theta\dot{\theta} + R\cos\theta\dot{\theta} \end{matrix} \right|^2 \\ &= \frac{1}{2} m \left| \begin{matrix} (R\theta - l)\cos\theta\dot{\theta} \\ (R\theta - l)\sin\theta\dot{\theta} \end{matrix} \right|^2 \\ &= \frac{1}{2} m \dot{\theta}^2 (R\theta - l)^2 \end{aligned}$$

$$\begin{aligned} V &= -mg(\vec{r} \cdot \hat{y}) \\ &= -mg\left((l - R\theta)\cos\theta + R\sin\theta\right) \end{aligned}$$

Combining these, we get the Lagrangian

$$\boxed{\mathscr{L} = \frac{1}{2}m\dot{\theta}^2 + mg(l - R\theta)\cos\theta + mgR\sin\theta}$$

(b) Let's take some derivatives!

$$\begin{aligned}\frac{\partial \mathscr{L}}{\partial \theta} &= m\dot{\theta}^2(R\theta - l) - mgR\cos\theta - mg(l - R\theta)\sin\theta + mgR\cos\theta \\ &= m\dot{\theta}^2(R\theta - l) - mg(l - R\theta)\sin\theta\end{aligned}$$

$$\begin{aligned}\frac{\partial \mathscr{L}}{\partial \dot{\theta}} &= m\dot{\theta}(R\theta - l)^2 \\ \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathscr{L}}{\partial \dot{\theta}} &= m\ddot{\theta}(R\theta - l)^2 + 2m\dot{\theta}(R\theta - l)\end{aligned}$$

The Euler-Lagrange equation is now

$$m\dot{\theta}(R\theta - l)^2 = m\ddot{\theta}(R\theta - l)^2 + 2m\dot{\theta}(R\theta - l)$$

(c) c