Classical v. Relativistic Kinematics

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1 Linear Acceleration

Classical

$$\mathcal{L} = \frac{1}{2}m\left|\dot{\vec{r}}\right|^2 + m\vec{r}\cdot\vec{g}$$

Euler-Lagrange Equations

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{\vec{r}}} &= \frac{\partial\mathcal{L}}{\partial r} \\
\implies \qquad \frac{\mathrm{d}}{\mathrm{d}t}\left(m\dot{\vec{r}}\right) &= m\vec{g} \\
\implies \qquad m\ddot{\vec{r}} &= m\vec{g} \\
\implies \qquad \ddot{\vec{r}} &= \vec{g}
\end{aligned}$$

Solutions of Equations of Motion

$$\begin{aligned}
\vec{r}(t) &= \vec{r}_0 + t\dot{\vec{r}}_0 + \frac{1}{2}t^2\vec{g} \\
\dot{\vec{r}}(t) &= \dot{\vec{r}}_0 + t\vec{g} \\
\ddot{\vec{r}}(t) &= \vec{g}
\end{aligned}$$

Hamiltonian

$$\vec{p} = \frac{\partial\mathcal{L}}{\partial\dot{\vec{r}}} = m\dot{\vec{r}} \implies \dot{\vec{r}} = \frac{1}{m}\vec{p}$$

$$H = \dot{\vec{r}}\cdot\vec{p} - \mathcal{L} = \frac{1}{m}\left|\vec{p}\right|^2 - \frac{1}{2m}\left|\vec{p}\right|^2 - m\vec{r}\cdot\vec{g} = \frac{1}{2m}\left|\vec{p}\right|^2 - m\vec{r}\cdot\vec{g}$$

$$\dot{\vec{p}} = -\frac{\partial\mathcal{H}}{\partial\vec{r}} = m\vec{g}$$

Conserved Quantities

Consider orthonormal vectors $\hat{r}_1$, $\hat{r}_2$, $\hat{r}_3$ such that

$$\hat{r}_1 \cdot \vec{g} = \hat{r}_2 \cdot \vec{g} = 0,$$

and

$$\hat{r}_3 = \frac{\vec{g}}{|\vec{g}|}.$$

Then, the quantities

$$\begin{aligned}
p_1 &\coloneqq \hat{r}_1 \cdot \vec{p} \\
p_2 &\coloneqq \hat{r}_2 \cdot \vec{p} \\
L_3 &\coloneqq (\hat{r}_1 \cdot \vec{r})p_2 - (\hat{r}_2 \cdot \vec{r})p_1 \\
&= (\hat{r}_1 \cdot \vec{r})(\hat{r}_2 \cdot \vec{p}) - (\hat{r}_2 \cdot \vec{r})(\hat{r}_1 \cdot \vec{p})
\end{aligned}$$

are conserved.
proof

$$\begin{aligned}
\dot{p}_1 &= \{p_1, H\} \\
&= \frac{\partial p_1}{\partial \vec{r}} \cdot \frac{\partial \mathcal{H}}{\partial \vec{p}} + \frac{\partial p_1}{\partial \vec{p}} \cdot \frac{\partial H}{\partial \vec{r}} \\
&= \vec{0} \cdot \frac{\vec{p}}{m} - m\hat{r}_1 \cdot \vec{g} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\dot{p}_2 &= \{p_2, H\} \\
&= \frac{\partial p_2}{\partial \vec{r}} \cdot \frac{\partial \mathcal{H}}{\partial \vec{p}} + \frac{\partial p_2}{\partial \vec{p}} \cdot \frac{\partial H}{\partial \vec{r}} \\
&= \vec{0} \cdot \frac{\vec{p}}{m} - m\hat{r}_2 \cdot \vec{g} \\
&= 0
\end{aligned}$$

$$\begin{aligned}
\dot{L}_3 &= \{L_3, \mathcal{H}\} \\
&= \frac{\partial L_3}{\partial \vec{r}} \cdot \frac{\partial \mathcal{H}}{\partial \vec{p}} + \frac{\partial L_3}{\partial \vec{p}} \cdot \frac{\partial H}{\partial \vec{r}} \\
&= ((\hat{r}_2 \cdot \vec{p})\hat{r}_1 - (\hat{r}_1 \cdot \vec{p})\hat{r}_2) \cdot \frac{\vec{p}}{m} - m\left((\hat{r}_1 \cdot \vec{r})\hat{r}_2 - (\hat{r}_2 \cdot \vec{r})\hat{r}_1\right) \cdot \vec{g} \\
&= \frac{1}{m}\left[(\hat{r}_2 \cdot \vec{p})(\hat{r}_1 \cdot \vec{p}) - (\hat{r}_1 \cdot \vec{p})(\hat{r}_1 \cdot \vec{p})\right] - m\left[(\hat{r}_1 \cdot \vec{r})(\hat{r}_2 \cdot \vec{g}) - (\hat{r}_2 \cdot \vec{r})(\hat{r}_1 \cdot \vec{g})\right] \\
&= \frac{1}{m}[0] - [0 - 0] \\
&= 0
\end{aligned}$$

Changing Coordinates

Consider coordinates ρ, ϕ and r_3 such that

$$\begin{aligned}\hat{r}_1 \cdot \vec{r} &= \rho\cos\phi,\\ \hat{r}_2 \cdot \vec{r} &= \rho\sin\phi,\\ \hat{r}_3 \cdot \vec{r} &= r_3,\end{aligned}$$

where $\hat{r}_3$ is defined as above. Consider also cylindrical unit vectors $\hat{\rho}$ and $\hat{\phi}$ such that

$$\begin{aligned}\hat{\rho} &= \cos\phi\ \hat{r}_1 + \sin\phi\ \hat{r}_2,\\ \hat{\phi} &= -\sin\phi\ \hat{r}_1 + \cos\phi\ \hat{r}_2.\end{aligned}$$

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\hat{\rho} &= -\sin\phi\dot{\phi}\hat{r}_1 + \cos\phi\dot{\phi}\hat{r}_2\\ &= \dot{\phi}\hat{\phi}\\ \frac{\mathrm{d}}{\mathrm{d}t}\hat{\phi} &= -\cos\phi\dot{\phi}\hat{r}_1 - \sin\phi\dot{\phi}\hat{r}_2\\ &= -\dot{\phi}\hat{\rho}\\ \vec{r} &= \rho\hat{\rho} + \phi\hat{\phi} + r_3\hat{r}_3\\ \dot{\vec{r}} &= \dot{\rho}\hat{\rho} + \rho\dot{\hat{\rho}} + \dot{\phi}\hat{\phi} + \phi\dot{\hat{\phi}} + \dot{r}_3\hat{r}_3\\ &= \dot{\rho}\hat{\rho} + \rho\dot{\phi}\hat{\phi} + \dot{\phi}\hat{\phi} - \phi\dot{\phi}\hat{\rho} + +\dot{r}_3\hat{r}_3\\ &= \left(\dot{\rho} - \phi\dot{\phi}\right)\hat{\rho} + \left(\rho\dot{\phi} + \dot{\phi}\right)\hat{\phi} + \dot{r}_3\hat{r}_3\\ &= \left(\dot{\rho} - \phi\dot{\phi}\right)\hat{\rho} + (\rho+1)\dot{\phi}\hat{\phi} + \dot{r}_3\hat{r}_3\\ \left|\dot{\vec{r}}\right|^2 &= \left(\dot{\rho} - \phi\dot{\phi}\right)^2 + (\rho+1)^2\dot{\phi}^2 + \dot{r}_3^2\\ &= \dot{\rho}^2 - 2\phi\dot{\rho}\dot{\phi} + \phi^2\dot{\phi}^2 + \rho^2\dot{\phi}^2 + 2\rho\dot{\phi}^2 + \dot{\phi}^2 + \dot{r}_3^2\end{aligned}$$

Relativistic

$$\begin{aligned}\mathcal{L} &= -\gamma^{-1}mc^2 + m\vec{r}\cdot\vec{g}\\ &=: -mc^2\sqrt{1 - \frac{\left|\dot{\vec{r}}\right|^2}{c^2}} + m\vec{r}\cdot\vec{g}\end{aligned}$$

Euler-Lagrange Equations

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}} &= \frac{\partial \mathcal{L}}{\partial \vec{r}} \\
\implies \frac{\mathrm{d}}{\mathrm{d}t}\left(-m\gamma\dot{\vec{r}}\right) &= m\vec{g} \\
\implies -m\left(\frac{\mathrm{d}\gamma}{\mathrm{d}t}\dot{\vec{r}} + \gamma\frac{\mathrm{d}\dot{\vec{r}}}{\mathrm{d}t}\right) &= m\vec{g} \\
\implies -\left[\left(\frac{1}{c^2}\gamma^3\dot{\vec{r}}\cdot\ddot{\vec{r}}\right)\dot{\vec{r}} + \gamma\ddot{\vec{r}}\right] &= \vec{g}
\end{aligned}$$

$$\begin{aligned}
\gamma &= \frac{1}{\sqrt{1-\frac{\left|\dot{\vec{r}}\right|^2}{c^2}}} \\
\frac{\partial \gamma}{\partial \dot{\vec{r}}} &= \frac{\gamma^3}{c^2}\dot{\vec{r}} \\
\frac{\mathrm{d}\gamma}{\mathrm{d}t} = \frac{\partial \gamma}{\partial \dot{\vec{r}}}\cdot\frac{\mathrm{d}\dot{\vec{r}}}{\mathrm{d}t} &= \frac{\gamma^3}{c^2}\dot{\vec{r}}\cdot\ddot{\vec{r}} \\
\frac{\partial \gamma^{-1}}{\partial \dot{\vec{r}}} &= -\frac{1}{c^2}\gamma\dot{\vec{r}} \\
\frac{\mathrm{d}\gamma^{-1}}{\mathrm{d}t} &= -\frac{\gamma}{c^2}\dot{\vec{r}}\cdot\ddot{\vec{r}} \\
\frac{\mathrm{d}\gamma^{n}}{\mathrm{d}\dot{\vec{r}}} &= \frac{n\gamma^{n+2}}{c^2}\dot{\vec{r}}
\end{aligned}$$

Hamiltonian

$$
\begin{aligned}
\vec{p} &= \frac{\partial \mathcal{L}}{\partial \dot{\vec{r}}} \\
&= m\gamma\dot{\vec{r}} \\
\implies& \\
\left|\vec{p}\right|^2 &= \frac{m^2\dot{\vec{r}}^2}{1 - \frac{\left|\dot{\vec{r}}\right|^2}{c^2}} \\
\implies& \\
\left|\vec{p}\right|^2 - \frac{1}{c^2}\left|\vec{p}\right|^2\left|\dot{\vec{r}}\right|^2 &= m^2\left|\dot{\vec{r}}\right|^2 \\
\implies& \\
\left|\dot{\vec{r}}\right|^2\left(m^2 + \frac{\left|\vec{p}\right|^2}{c^2}\right) &= \left|\vec{p}\right|^2 \\
\implies& \\
\left|\dot{\vec{r}}\right|^2 = \frac{c^2\left|\vec{p}\right|^2}{m^2c^4 + \left|\vec{p}\right|^2c^2} &=: \frac{c^4\left|\vec{p}\right|^2}{E^2}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{H} &= \dot{\vec{r}} \cdot \vec{p} - \mathcal{L} \\
&= \frac{\left|\vec{p}\right|^2}{m\gamma} + \frac{mc^2}{\gamma} - m\vec{r} \cdot \vec{g} \\
&= \frac{\left|\vec{p}\right|^2 + m^2c^2}{m\gamma} - m\vec{r} \cdot \vec{g}
\end{aligned}
$$

$$
\dot{\vec{p}} = -\frac{\partial \mathcal{H}}{\partial \vec{r}} = m\vec{g}
$$

Solutions of Equations of Motion

$$\begin{aligned}
\vec{p}(t) &= \vec{p}_0 + mt\vec{g} \\
\dot{\vec{r}}(t) &= \frac{\vec{p}(t)}{m\gamma} \\
&= \frac{1}{m}\vec{p}(t)\sqrt{1 - \frac{\left|\dot{\vec{r}}\right|^2}{c^2}} \\
&= \frac{1}{m}\vec{p}(t)\sqrt{1 - \frac{c^2\left|\vec{p}\right|^2}{E^2}} \\
&= (\gamma_0\dot{\vec{r}}_0 + t\vec{g})\sqrt{1 - \frac{c^2\left|m\gamma_0\dot{\vec{r}}_0 + t\vec{g}\right|^2}{m^2c^4 + \left|m\gamma_0\dot{\vec{r}}_0 + t\vec{g}\right|^2 c^2}} \\
&= (\gamma_0\dot{\vec{r}}_0 + t\vec{g})\sqrt{\frac{m^2c^2}{\left|m\gamma_0\dot{\vec{r}}_0 + t\vec{g}\right|^2}} \\
&= mc\frac{(\gamma_0\dot{\vec{r}}_0 + t\vec{g})}{\left|m\gamma_0\dot{\vec{r}}_0 + t\vec{g}\right|}
\end{aligned}$$

Conserved Quantities