Homework 6

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Problem 1

(a) Symmetrizing under $\epsilon_1 \leftrightarrow \epsilon_2$ and $\epsilon_4 \leftrightarrow \epsilon_3$, we get

$$\mathcal{M} \to a\,(\vec{\epsilon}_1 \cdot \vec{\epsilon}_2)\,(\vec{\epsilon}_3^* \cdot \vec{\epsilon}_4^*) + \frac{b+c}{2}\left[(\vec{\epsilon}_1 \cdot \vec{\epsilon}_3^*)\,(\vec{\epsilon}_2 \cdot \vec{\epsilon}_4^*) + (\vec{\epsilon}_1 \cdot \vec{\epsilon}_4^*)\,(\vec{\epsilon}_2 \cdot \vec{\epsilon}_3^*)\right],$$

which implies $b = c$, and we get two independent terms

$$\begin{aligned}
\mathcal{M}_0 &\propto (\vec{\epsilon}_1 \cdot \vec{\epsilon}_2)\,(\vec{\epsilon}_3^* \cdot \vec{\epsilon}_4^*) \\
\mathcal{M}_> &\propto (\vec{\epsilon}_1 \cdot \vec{\epsilon}_3^*)\,(\vec{\epsilon}_2 \cdot \vec{\epsilon}_4^*) + (\vec{\epsilon}_1 \cdot \vec{\epsilon}_4^*)\,(\vec{\epsilon}_2 \cdot \vec{\epsilon}_3^*)\,.
\end{aligned}$$

(b) In the center of mass frame, the momenta are

$$\begin{aligned}
p_1 &= \frac{m_h}{2}(1,0,0,1) & p_3 &= \frac{m_h}{2}(1,\sin\theta,0,\cos\theta) \\
p_2 &= \frac{m_h}{2}(1,0,0,-1) & p_4 &= \frac{m_h}{2}(1,-\sin\theta,0,-\cos\theta)
\end{aligned}$$

For spin-0 Higgs, by conservation of angular momentum the particles in the in state must have equal helicities (and similarly for the out state), while a spin-2 Higgs places no such restriction. For the matrix element $\mathcal{M}(g_+g_+ \to h \to \gamma_+\gamma_+)$, where the gluons and photons all have helicity $+1$, we have

$$\begin{aligned}
\vec{\epsilon}_1 &= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ i\\ 0\end{pmatrix}, & \vec{\epsilon}_3 &= \frac{1}{\sqrt{2}}\begin{pmatrix}\cos\theta\\ i\\ -\sin\theta\end{pmatrix}, \\
\vec{\epsilon}_2 &= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -i\\ 0\end{pmatrix}, & \vec{\epsilon}_4 &= \frac{1}{\sqrt{2}}\begin{pmatrix}-\cos\theta\\ i\\ \sin\theta\end{pmatrix}.
\end{aligned}$$

The relevant dot products are then

$$\begin{aligned}
\vec{\epsilon}_1 \cdot \vec{\epsilon}_2 &= 1 \\
\vec{\epsilon}_3^* \cdot \vec{\epsilon}_4^* &= -1 \\
\vec{\epsilon}_1 \cdot \vec{\epsilon}_3^* = -\vec{\epsilon}_2 \cdot \vec{\epsilon}_4^* &= \frac{1}{2}(\cos\theta + 1) \\
\vec{\epsilon}_2 \cdot \vec{\epsilon}_3^* = -\vec{\epsilon}_1 \cdot \vec{\epsilon}_4^* &= \frac{1}{2}(\cos\theta - 1)
\end{aligned}$$

Clearly, the first independent matrix element, $\mathcal{M}_0$, does not depend on θ, as is appropriate for a spin-0 Higgs. Meanwhile, in this spin configuration, the second term is θ-dependent but non-vanishing.

$$\mathcal{M}_> \propto (\cos\theta + 1)^2 + (\cos\theta - 1)^2 > 0$$

This is consistent with a spin-2 Higgs with 0 z-component spin projection (i.e. $j = 2,\ m = 0$) For the matrix element $\mathcal{M}(g_+ g_- \to h \to \gamma_+\gamma_-)$, the polarization vectors are given by

$$\vec{\epsilon}_1 = \vec{\epsilon}_2 = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix},$$

$$\vec{\epsilon}_3 = \vec{\epsilon}_4 = \begin{pmatrix} \cos\theta \\ i \\ -\sin\theta \end{pmatrix}.$$

Clearly, the spin-0 component vanishes (both dot products are zero), while

$$M_> \propto \sin^2\theta,$$

so the photons can't come out in the exact same orientation as the gluons came in.

(c) If we have a levi-civita fully contracted with the polarizations,

$$\epsilon_{\mu\nu\rho\sigma}\epsilon_1^\mu \epsilon_2^\nu \epsilon_3^\rho \epsilon_4^\sigma,$$

symmetrization with respect to any two indices will vanish (due to the antisymmetry of the levi-civita), but we need to symmetrize w.r.t $1 \leftrightarrow 2$ and $3 \leftrightarrow 4$.

Problem 2

(a)

$$\begin{aligned} V &= m_1^2 \left|H_1\right|^2 + m_2^2 \left|H_2\right|^2 - b(H_1^\dagger H_2 + H_2^\dagger H_1) + \frac{g^2 + g'^2}{8}\left(\left|H_1\right|^2 - \left|H_2\right|^2\right)^2 + \frac{g^2}{2}\left(H_2^\intercal \epsilon H_1\right)\left(H_2^\dagger \epsilon H_1^*\right) \\ &= m_1^2\left(\left|H_1^+\right|^2 + \left|H_1^-\right|^2\right) + m_2^2\left(\left|H_2^+\right|^2 + \left|H_2^-\right|^2\right) - 2b\,\mathrm{Re}[H_1^{+*}H_2^+ + H_1^{-*}H_2^-] \\ &\quad + \frac{g^2 + g'^2}{8}\left(\left|H_1^+\right|^2 + \left|H_1^-\right|^2 - \left|H_2^+\right|^2 - \left|H_2^-\right|^2\right)^2 + \frac{g^2}{2}\left|H_1^+ H_2^- - H_1^- H_2^+\right|^2 \end{aligned}$$

Given that

$$\frac{\partial V}{\partial H_1^+}\Big|_{H_1 = \langle H_1\rangle, H_2 = \langle H_2\rangle} = -\frac{y^*}{4}\left(4b + \sqrt{2}g^2 vz\right),$$

we can see that $y = 0$ gives a minimum of the potential.

(b)

$$V\big|_{H_1^+\to 0,\ H_2^+\to 0,;H_1^-\to v/\sqrt{2}} =$$

For any complex value of H_2^-, keeping the magnitude fixed while rotating the number towards the positive real axis will decrease the value of the potential. Therefore, at the minimum of the potential H_2^- must lie on the positive real axis.

(c)

$$\begin{aligned}
V = {} & m_1^2\left(\left|H_1^+\right|^2 + \frac{1}{2}\left|v_1 + h_1 + iA_1\right|^2\right) + m_2^2\left(\left|H_2^+\right|^2 + \frac{1}{2}\left|v_2 + h_2 + iA_2\right|^2\right) \\
& - 2b\,\mathrm{Re}[H_1^{+*}H_2^+] - b\left[(v_1+h_1)(v_2+h_2) + A_1A_2\right] \\
& + \frac{g^2+g'^2}{8}\left(\left|H_1^+\right|^2 + \left|v_1+h_1+iA_1\right|^2 - \left|H_2^+\right|^2 - \left|v_2+h_2+iA_2\right|^2\right)^2 \\
& + \frac{g^2}{4}\left|H_1^+\left(v_2+h_2+iA_2\right) - \left(v_1+h_1+iA_1\right)H_2^+\right|^2
\end{aligned}$$

$$\begin{aligned}
\frac{\partial V}{\partial h_1}\Big|_{H_1\to\langle H_1\rangle,\ H_2\to\langle H_2\rangle} &= v_1 m_1^2 - bv_2 + \frac{1}{8}(g^2+g'^2)v_1(v_1^2 - v_2^2) = 0 \\
\implies m_1^2 &= b\frac{v_1}{v_2} - \frac{1}{8}(g^2+g'^2)(v_1^2 - v_2^2)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial V}{\partial h_2}\Big|_{H_1\to\langle H_1\rangle,\ H_2\to\langle H_2\rangle} &= v_2 m_2^2 - bv_1 + \frac{1}{8}(g^2+g'^2)v_2(v_2^2 - v_1^2) = 0 \\
\implies m_2^2 &= b\frac{v_2}{v_1} - \frac{1}{8}(g^2+g'^2)(v_2^2 - v_1^2)
\end{aligned}$$

(d) See Mathematica printout

(e) See Mathematica printout

(f) See Mathematica printout

```
In[19]:= H₁ = {α, β}ᵀ;
H₂ = {γ, δ}ᵀ;
ϵ = {{0, 1}, {-1, 0}};
V = m₁² H₁†.H₁ + m₂² H₂†.H₂ - b (H₁†.H₂ + H₂†.H₁) +
    (g₁² + g₂²)/8 (H₁†.H₁ - H₂†.H₂)² + g₁²/2 (H₂ᵀ.ϵ.H₁) (H₂†.ϵ.H₁*);
V // TraditionalForm
```

Out[23]//TraditionalForm=

$$-b\,(\gamma\,\alpha^* + \alpha\,\gamma^* + \delta\,\beta^* + \beta\,\delta^*) + \frac{1}{8}\left(g_1^2 + g_2^2\right)(\alpha\,\alpha^* + \beta\,\beta^* - \gamma\,\gamma^* - \delta\,\delta^*)^2 + \frac{1}{2}\,g_1^2\,(\beta\,\gamma - \alpha\,\delta)\,(\beta^*\,\gamma^* - \alpha^*\,\delta^*) + m_1^2\,(\alpha\,\alpha^* + \beta\,\beta^*) + m_2^2\,(\gamma\,\gamma^* + \delta\,\delta^*)$$

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a)

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In[24]:= sub = {α → 0, β → v/√2, γ → y, δ → z};
```

```
In[25]:= (D[V /. α* → a, α] /. a → α*) /. sub // FullSimplify // TraditionalForm
```

Out[25]//TraditionalForm=

$$-\frac{1}{4}\,y^*\left(4\,b + \sqrt{2}\,g_1^2\,v\,z\right)$$

---

b)

```
In[26]:= (V /. sub) /. y → 0 // FullSimplify // TraditionalForm
```

Out[26]//TraditionalForm=

$$-\sqrt{2}\,b\,v\,\mathrm{Re}(z) + \frac{1}{32}\left(g_1^2 + g_2^2\right)\left(v^2 - 2\,z\,z^*\right)^2 + m_2^2\,z\,z^* + \frac{1}{2}\,m_1^2\,v^2$$

---

c)

```
In[27]:= sub2 = {β → (v₁ + h₁ + ⅈ A₁)/√2, δ → (v₂ + h₂ + ⅈ A₂)/√2};
```

In[28]:=
```
(V /. sub2) // TraditionalForm
```

Out[28]//TraditionalForm=

$$-b\left(\frac{1}{2}(i A_1+h_1+v_1)((h_2+v_2)^*-i(A_2)^*)+\frac{1}{2}(i A_2+h_2+v_2)((h_1+v_1)^*-i(A_1)^*)+\gamma\alpha^*+\alpha\gamma^*\right)+$$
$$\frac{1}{2}g_1^2\left(\frac{\gamma(i A_1+h_1+v_1)}{\sqrt{2}}-\frac{\alpha(i A_2+h_2+v_2)}{\sqrt{2}}\right)\left(\frac{\gamma^*((h_1+v_1)^*-i(A_1)^*)}{\sqrt{2}}-\frac{\alpha^*((h_2+v_2)^*-i(A_2)^*)}{\sqrt{2}}\right)+$$
$$\frac{1}{8}(g_1^2+g_2^2)\left(\frac{1}{2}(i A_1+h_1+v_1)((h_1+v_1)^*-i(A_1)^*)-\frac{1}{2}(i A_2+h_2+v_2)((h_2+v_2)^*-i(A_2)^*)+\alpha\alpha^*-\gamma\gamma^*\right)^2+$$
$$m_1^2\left(\alpha\alpha^*+\frac{1}{2}(i A_1+h_1+v_1)((h_1+v_1)^*-i(A_1)^*)\right)+m_2^2\left(\gamma\gamma^*+\frac{1}{2}(i A_2+h_2+v_2)((h_2+v_2)^*-i(A_2)^*)\right)$$

In[29]:=
```
eq1 = D[(V // FullSimplify) /. sub2, h1] /.
    {α → 0, γ → 0, A1 → 0, h1 → 0, A2 → 0, h2 → 0, m1 → √M1} // FullSimplify
eq2 = D[(V // FullSimplify) /. sub2 // FullSimplify, h2] /.
    {α → 0, γ → 0, A1 → 0, h1 → 0, A2 → 0, h2 → 0, m2 → √M2} // FullSimplify
```

Out[29]=

$$M_1 v_1-\frac{1}{2}b(1+v_1)v_2+\frac{1}{8}(g_1^2+g_2^2)v_1(v_1^2-v_2^2)$$

Out[30]=

$$-b v_1+M_2 v_2+\frac{1}{8}(g_1^2+g_2^2)v_2(-v_1^2+v_2^2)$$

In[31]:=
```
Solve[{eq1 == 0}, {M1}] // Quiet // FullSimplify
Solve[eq2 == 0, M2] // Quiet // FullSimplify
```

Out[31]=

$$\left\{\left\{M_1\to\boxed{\frac{1}{8}\left(4 b v_2+\frac{4 b v_2}{v_1}+(g_1^2+g_2^2)(-v_1^2+v_2^2)\right)\text{ if }v_1\neq 0}\right\}\right\}$$

Out[32]=

$$\left\{\left\{M_2\to\boxed{\frac{1}{8}(g_1^2+g_2^2)v_1^2+\frac{b v_1}{v_2}-\frac{1}{8}(g_1^2+g_2^2)v_2^2\text{ if }v_2\neq 0}\right\}\right\}$$

d)

In[33]:=
```
V2 = V /. sub2;
V2 // TraditionalForm
MM =
  {{D[D[V2, α]*, α]*, D[D[V2, α]*, γ]*}, {D[D[V2, γ]*, α]*, D[D[V2, γ]*, γ]*}} /.
   {α → 0, γ → 0, h1 → 0, h2 → 0, A1 → 0, A2 → 0} // FullSimplify;
MM // MatrixForm
```

Out[34]//TraditionalForm=

$$-b\left(\frac{1}{2}\,(i\,A_1+h_1+v_1)\,((h_2+v_2)^*-i\,(A_2)^*)+\frac{1}{2}\,(i\,A_2+h_2+v_2)\,((h_1+v_1)^*-i\,(A_1)^*)+\gamma\,\alpha^*+\alpha\,\gamma^*\right)+$$
$$\frac{1}{2}\,g_1^2\left(\frac{\gamma\,(i\,A_1+h_1+v_1)}{\sqrt{2}}-\frac{\alpha\,(i\,A_2+h_2+v_2)}{\sqrt{2}}\right)\left(\frac{\gamma^*\,((h_1+v_1)^*-i\,(A_1)^*)}{\sqrt{2}}-\frac{\alpha^*\,((h_2+v_2)^*-i\,(A_2)^*)}{\sqrt{2}}\right)+$$
$$\frac{1}{8}\,(g_1^2+g_2^2)\left(\frac{1}{2}\,(i\,A_1+h_1+v_1)\,((h_1+v_1)^*-i\,(A_1)^*)-\frac{1}{2}\,(i\,A_2+h_2+v_2)\,((h_2+v_2)^*-i\,(A_2)^*)+\alpha\,\alpha^*-\gamma\,\gamma^*\right)^2+$$
$$m_1^2\left(\alpha\,\alpha^*+\frac{1}{2}\,(i\,A_1+h_1+v_1)\,((h_1+v_1)^*-i\,(A_1)^*)\right)+m_2^2\left(\gamma\,\gamma^*+\frac{1}{2}\,(i\,A_2+h_2+v_2)\,((h_2+v_2)^*-i\,(A_2)^*)\right)$$

Out[36]//MatrixForm=

$$\begin{pmatrix} m_1^2+\frac{1}{8}\left(g_2^2\left(v_1^2-v_2^2\right)+g_1^2\left(v_1^2+v_2^2\right)\right) & -b-\frac{1}{4}\,g_1^2\,v_1\,v_2 \\ -b-\frac{1}{4}\,g_1^2\,v_1\,v_2 & m_2^2+\frac{1}{8}\left(g_2^2\left(-v_1^2+v_2^2\right)+g_1^2\left(v_1^2+v_2^2\right)\right) \end{pmatrix}$$

In[37]:=
```
Eigenvalues[MM] // FullSimplify // FullSimplify
```

Out[37]=

$$\left\{\frac{1}{8}\left(4\,m_1^2+4\,m_2^2+g_1^2\left(v_1^2+v_2^2\right)-\sqrt{64\,b^2+\left(4\,m_1^2-4\,m_2^2+g_2^2\,v_1^2\right)^2+32\,b\,g_1^2\,v_1\,v_2-2\left(4\,g_2^2\left(m_1^2-m_2^2\right)+\left(-2\,g_1^4+g_2^4\right)v_1^2\right)v_2^2+g_2^4\,v_2^4}\right),\right.$$
$$\left.\frac{1}{8}\left(4\,m_1^2+4\,m_2^2+g_1^2\left(v_1^2+v_2^2\right)+\sqrt{64\,b^2+\left(4\,m_1^2-4\,m_2^2+g_2^2\,v_1^2\right)^2+32\,b\,g_1^2\,v_1\,v_2-2\left(4\,g_2^2\left(m_1^2-m_2^2\right)+\left(-2\,g_1^4+g_2^4\right)v_1^2\right)v_2^2+g_2^4\,v_2^4}\right)\right\}$$

e)

In[38]:=
```
V2 = FullSimplify[V2];
V2 // TraditionalForm
```

Out[39]//TraditionalForm=

$$\frac{1}{32}\Big(-32\,b\,(A_1\,A_2+\gamma\,\alpha^*+\alpha\,\gamma^*+(h_1+v_1)\,(h_2+v_2))+$$
$$8\,g_1^2\,(-\alpha\,A_2+A_1\,\gamma+i\,(\alpha\,(h_2+v_2)-\gamma\,(h_1+v_1)))\,(\gamma^*\,(A_1+i\,(h_1+v_1))-\alpha^*\,(A_2+i\,(h_2+v_2)))+$$
$$(g_1^2+g_2^2)\left(A_1^2-A_2^2+2\,\alpha\,\alpha^*-2\,\gamma\,\gamma^*+(h_1-h_2+v_1-v_2)\,(h_1+h_2+v_1+v_2)\right)^2+$$
$$16\,m_1^2\left(A_1^2+2\,\alpha\,\alpha^*+(h_1+v_1)^2\right)+16\,m_2^2\left(A_2^2+2\,\gamma\,\gamma^*+(h_2+v_2)^2\right)\Big)$$

In[40]:=
```
MM2 = {{D[V2, A₁, A₁], D[V2, A₁, A₂]}, {D[V2, A₂, A₁], D[V2, A₂, A₂]}} /.
    {α → 0, γ → 0, h₁ → 0, h₂ → 0, A₁ → 0, A₂ → 0} // FullSimplify;
MM2 // MatrixForm
```

Out[41]//MatrixForm=

$$\begin{pmatrix} m_1^2 + \frac{1}{8}\left(g_1^2+g_2^2\right)\left(v_1^2-v_2^2\right) & -b \\ -b & m_2^2 - \frac{1}{8}\left(g_1^2+g_2^2\right)\left(v_1^2-v_2^2\right) \end{pmatrix}$$

In[42]:=
```
Eigenvalues[MM2] // FullSimplify
```

Out[42]=

$$\left\{\frac{1}{8}\left(4\,m_1^2+4\,m_2^2-\sqrt{64\,b^2+\left(4\,m_1^2-4\,m_2^2+\left(g_1^2+g_2^2\right)\left(v_1^2-v_2^2\right)\right)^2}\right),\right.$$
$$\left.\frac{1}{8}\left(4\,m_1^2+4\,m_2^2+\sqrt{64\,b^2+\left(4\,m_1^2-4\,m_2^2+\left(g_1^2+g_2^2\right)\left(v_1^2-v_2^2\right)\right)^2}\right)\right\}$$

---

f)

In[43]:=
```
MM3 = {{D[V2, h₁, h₁], D[V2, h₁, h₂]}, {D[V2, h₂, h₁], D[V2, h₂, h₂]}} /.
    {α → 0, γ → 0, h₁ → 0, h₂ → 0, A₁ → 0, A₂ → 0} // FullSimplify;
MM3 // MatrixForm
```

Out[44]//MatrixForm=

$$\begin{pmatrix} m_1^2 + \frac{1}{8}\left(g_1^2+g_2^2\right)\left(3\,v_1^2-v_2^2\right) & -b-\frac{1}{4}\left(g_1^2+g_2^2\right)v_1\,v_2 \\ -b-\frac{1}{4}\left(g_1^2+g_2^2\right)v_1\,v_2 & m_2^2 - \frac{1}{8}\left(g_1^2+g_2^2\right)\left(v_1^2-3\,v_2^2\right) \end{pmatrix}$$

In[45]:=
```
Eigenvalues[MM3] // FullSimplify
```

Out[45]=

$$\left\{\frac{1}{8}\left(4\,m_1^2+4\,m_2^2+\left(g_1^2+g_2^2\right)\left(v_1^2+v_2^2\right)-2\sqrt{\left(16\,b^2+\left(2\,m_1^2-2\,m_2^2+\left(g_1^2+g_2^2\right)v_1^2\right)^2+\right.}\right.\right.$$
$$\left.\left.8\,b\left(g_1^2+g_2^2\right)v_1\,v_2-\left(g_1^2+g_2^2\right)\left(4\,m_1^2-4\,m_2^2+\left(g_1^2+g_2^2\right)v_1^2\right)v_2^2+\left(g_1^2+g_2^2\right)^2v_2^4\right)\right),$$
$$\frac{1}{8}\left(4\,m_1^2+4\,m_2^2+\left(g_1^2+g_2^2\right)\left(v_1^2+v_2^2\right)+2\sqrt{\left(16\,b^2+\left(2\,m_1^2-2\,m_2^2+\left(g_1^2+g_2^2\right)v_1^2\right)^2+\right.}\right.$$
$$\left.\left.\left.8\,b\left(g_1^2+g_2^2\right)v_1\,v_2-\left(g_1^2+g_2^2\right)\left(4\,m_1^2-4\,m_2^2+\left(g_1^2+g_2^2\right)v_1^2\right)v_2^2+\left(g_1^2+g_2^2\right)^2v_2^4\right)\right)\right\}$$