Homework 3

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Phys 684

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Problem 1

The hamiltonian in the field interaction representation is (neglecting tildes)

$$H = \frac{\hbar}{2}\begin{pmatrix} -\delta(t) & \Omega_0^*(t) \\ \Omega_0(t) & \delta(t) \end{pmatrix},$$

and the equations of motion for the density matrix are

$$\begin{aligned}\dot{\rho} &= \frac{1}{i\hbar}[H, \rho] \\ &= \begin{pmatrix} -\operatorname{Re}[\rho_{12}\Omega_0(t)] & i\rho_{12}\delta(t) - \frac{i}{2}(\rho_{22} - \rho_{11})\Omega_0^*(t) \\ -i\rho_{21}\delta(t) + \frac{i}{2}(\rho_{22} - \rho_{11})\Omega_0(t) & \operatorname{Re}[\rho_{12}\Omega_0(t)] \end{pmatrix}\end{aligned}$$

(See attached Mathematica printout for calculations.)

Problem 2

The Bloch vector is just the vector of expectation values of the Pauli operators, so

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\vec{B} &= \frac{\mathrm{d}}{\mathrm{d}t}\operatorname{Tr}[\rho\vec{\sigma}] \\ &= \begin{pmatrix} \operatorname{Tr}[\dot{\rho}\sigma_x] \\ \operatorname{Tr}[\dot{\rho}\sigma_y] \\ \operatorname{Tr}[\dot{\rho}\sigma_z] \end{pmatrix} \\ &= \begin{pmatrix} -\operatorname{Im}[\Omega_0(t)](\rho_{22} - \rho_{11}) - 2\operatorname{Im}[\rho_{12}]\delta(t) \\ \operatorname{Re}[\Omega_0(t)]\,(\rho_{22} - \rho_{11}) - 2\operatorname{Re}[\rho_{12}]\delta(t) \\ -2\operatorname{Im}[\rho_{12}\Omega_0(t)] \end{pmatrix}\end{aligned}$$

(See attached Mathematica printout for calculations.)

Problem 3 (Berman 3.8)

From equation 3.31 in Berman,

$$\begin{aligned}
\dot{\rho}_{12} &= i(\omega_0 + i\gamma)\rho_{12}(t) - \frac{i}{2}\Omega_0^*(t)e^{i\omega t}\left(\rho_{22}(t) - \rho_{11}(t)\right) \\
&= (i\omega_0 - \gamma)\rho_{12}(t) - \frac{i}{2}\Omega_0 e^{i\omega t}(2\rho_{22} - 1) \\
\dot{\rho}_{22} &= -2\gamma\rho_{22}(t) + 2\,\mathrm{Im}[\Omega_0(t)e^{-i\omega t}\rho_{12}(t)]
\end{aligned}$$

$$\rho_{22} = \frac{\left|\Omega_0\right|^2/2}{2\gamma^2 + \left|\Omega\right|^2}\left[1 - \left(\cos(\lambda t) + \frac{3\gamma}{2\lambda}\sin(\lambda t)\right)e^{-3\gamma t/2}\right]$$

$$\begin{aligned}
\frac{\left|\Omega_0\right|^2/2}{2\gamma^2 + \left|\Omega_0\right|^2} &= \frac{1}{2}\left(1 + 2\left(\frac{\gamma}{|\Omega_0|}\right)^2\right)^{-1} \\
&\approx \frac{1}{2}\left(1 - 2\frac{\gamma^2}{\left|\Omega_0\right|^2}\right)
\end{aligned}$$

$$\begin{aligned}
\lambda &= \sqrt{\left|\Omega_0\right|^2 - \gamma^2/4} \\
&= |\Omega_0|\sqrt{1 - \left(\frac{\gamma}{2|\Omega_0|}\right)^2} \\
&\approx |\Omega_0|\left(1 - \frac{\gamma^2}{4\left|\Omega_0\right|^2}\right) \\
\frac{1}{\lambda} &\approx \frac{1}{|\Omega_0|}\left(1 + \frac{\gamma^2}{4\left|\Omega_0\right|^2}\right)
\end{aligned}$$

so

$$\begin{aligned}
\rho_{22} &\approx \frac{1}{2}\left(1 - 2\frac{\gamma^2}{\left|\Omega_0\right|^2}\right)\left[1 - \left(\cos\left(|\Omega_0|\left(1 - \frac{\gamma^2}{4\left|\Omega_0\right|^2}\right)t\right) + \frac{3}{2}\frac{\gamma}{|\Omega_0|}\left(1 + \frac{\gamma^2}{4\left|\Omega_0\right|^2}\right)\sin\left(|\Omega_0|\left(1 - \frac{\gamma^2}{4\left|\Omega_0\right|^2}\right)t\right)\right)e^{-3\gamma t/2}\right] \\
&\approx \frac{1}{2}\left[1 - \left(\cos(|\Omega_0|t) + \frac{3}{2}\frac{\gamma}{|\Omega_0|}\sin(|\Omega_0|t)\right)e^{-3\gamma t/2}\right] \\
&\approx \frac{1}{2}\left(1 - e^{-3\gamma t/2}\cos(|\Omega_0|t)\right)
\end{aligned}$$

This describes exponentially damped oscillation of the population between the upper and lower states (i.e. damped oscillation of the z-component of the Bloch vector), which asymptotically approaches $\rho_{11} = \rho_{22} = 1/2$ (i.e. the z-component of the Bloch vector goes to 0).

Problem 4 (Berman 3.10)

Parameterizing the state as

$$|\psi\rangle = \cos\frac{\theta}{2}|1\rangle + \sin\frac{\theta}{2}e^{i\phi}|2\rangle\,,$$

the Bloch vector is

$$\begin{aligned}\vec{B} &= \begin{pmatrix}\langle\psi|\sigma_x|\psi\rangle\\ \langle\psi|\sigma_y|\psi\rangle\\ \langle\psi|\sigma_z|\psi\rangle\end{pmatrix}\\ &= \begin{pmatrix}\cos\phi\sin\theta\\ \sin\phi\sin\theta\\ \cos\theta\end{pmatrix}.\end{aligned}$$

(See attached Mathematica printout for calculations.)

Problem 5 (Berman 3.7)

In the absence of relaxation the Bloch vector has constant unit length (assuming proper normalization). For constant $\vec{\Omega}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{B} = \vec{\Omega}\times\vec{B},$$

so, letting θ be the angle between $\vec{\Omega}$ and $\vec{B}$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\left|\vec{B}\right|\left|\vec{\Omega}\right|\cos\theta\right) &= \left|\vec{\Omega}\right|\frac{\mathrm{d}}{\mathrm{d}t}(\cos\theta)\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\vec{\Omega}\cdot\vec{B}\right)\\ &= \vec{\Omega}\cdot\left(\frac{\mathrm{d}}{\mathrm{d}t}\vec{B}\right)\\ &= \vec{\Omega}\cdot\left(\vec{\Omega}\times\vec{B}\right)\\ &= 0\end{aligned}$$

If $\vec{\Omega}$ is a function of time (with rate of change negligible compared to its magnitude), then

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\cos\theta &= \frac{\mathrm{d}}{\mathrm{d}t}\frac{\vec{\Omega}(t)\cdot\vec{B}(t)}{\left|\vec{\Omega}(t)\right|}\\ &= \left(\frac{\mathrm{d}}{\mathrm{d}t}\Omega^{-1}(t)\right)\left(\vec{\Omega}(t)\cdot\vec{B}(t)\right) + \frac{\left(\frac{\mathrm{d}}{\mathrm{d}t}\vec{\Omega}(t)\right)\cdot\vec{B}(t) + \vec{\Omega}(t)\cdot\left(\frac{\mathrm{d}}{\mathrm{d}t}\vec{B}(t)\right)}{\Omega}\\ &= \frac{\dot{\vec{\Omega}}(t)}{\Omega(t)}\cdot\vec{B} - \frac{\dot{\Omega}(t)}{\Omega^2(t)}\\ &\approx 0.\end{aligned}$$

In[4]:=
```
Symbolize[Ω0];
Symbolize[σx]; Symbolize[σy]; Symbolize[σz];
Symbolize[σ+]; Symbolize[σ-]; Symbolize[σ0];
```

In[7]:=
```
$Assumptions = {ω ∈ ℝ, t ∈ ℝ, θ ∈ ℝ, ϕ ∈ ℝ};
σx = PauliMatrix[1]; σy = PauliMatrix[2]; σz = PauliMatrix[3];
σ+ = 1/2 (σx - ⅈ σy); σ- = 1/2 (σx + ⅈ σy); σ0 = σ+.σ-;
Comm[A_, B_] := A.B - B.A;
CleanUp[x_] := TraditionalForm[MatrixForm[FullSimplify[x]]];
```

Problem 1

In[12]:=
```
HFI[t_] = ℏ/2 (-δ[t] σz + Re[Ω0[t]] σx + Im[Ω0[t]] σy);
rho = {{ρ11, ρ12}, {ρ21, ρ22}};
ρ̇ = 1/(ⅈ ℏ) Comm[HFI[t], rho];
ρ̇ // CleanUp
```

Out[15]//TraditionalForm=

$$\begin{pmatrix} -\frac{1}{2}\, i\,(\rho_{21}\,\Omega_0(t)^* - \rho_{12}\,\Omega_0(t)) & \frac{1}{2}\, i\,((\rho_{11}-\rho_{22})\,\Omega_0(t)^* + 2\,\rho_{12}\,\delta(t)) \\ -\frac{1}{2}\, i\,(2\,\rho_{21}\,\delta(t) + (\rho_{11}-\rho_{22})\,\Omega_0(t)) & \frac{1}{2}\, i\,(\rho_{21}\,\Omega_0(t)^* - \rho_{12}\,\Omega_0(t)) \end{pmatrix}$$

Problem 2

In[16]:=
```
{{Tr[ρ̇.σx], Tr[ρ̇.σy], Tr[ρ̇.σz]}}ᵀ // CleanUp
```

Out[16]//TraditionalForm=

$$\begin{pmatrix} (\rho_{11}-\rho_{22})\,\mathrm{Im}(\Omega_0(t)) + i\,(\rho_{12}-\rho_{21})\,\delta(t) \\ (\rho_{22}-\rho_{11})\,\mathrm{Re}(\Omega_0(t)) - (\rho_{12}+\rho_{21})\,\delta(t) \\ -i\,(\rho_{21}\,\Omega_0(t)^* - \rho_{12}\,\Omega_0(t)) \end{pmatrix}$$

Problem 3

In[17]:=
```
rho[t_] = {{a[t], b[t]}, {c[t], d[t]}};
H[t_] = ℏ/2 (-ω0 σz + Ω0 ⅇ^(ⅈ ω t) σ+ + (Ω0 ⅇ^(ⅈ ω t) σ+)†);
rhoDot[t_] =
  1/(ⅈ ℏ) (Comm[H[t], rho[t]] - ⅈ ℏ γ (σ0.rho[t] + rho[t].σ0) + 2 ⅈ ℏ γ σ-.rho[t].σ+);
rhoDot[t] // CleanUp
```

In[21]:=
```
DSolve[{a'[t] == (rhoDot[t]〚1, 1〛 /. {d[t] → 1 - a[t], c[t] → b[t]*}),
    b'[t] == (rhoDot[t]〚1, 2〛 /. {d[t] → 1 - a[t], c[t] → b[t]*}),
    a[0] == 1, c[0] == 0} // FullSimplify, {a, b}, t] // CleanUp
```

Problem 4

In[23]:=
```
ψ = {{Cos[θ / 2]}, {ⅇ^(ⅈ ϕ) Sin[θ / 2]}};
((ψ†.#.ψ)〚1〛) & /@ {σx, σy, σz} // CleanUp
```

Out[24]//TraditionalForm=

$$\begin{pmatrix} \sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) \\ \cos(\theta) \end{pmatrix}$$

Problem 5

In[25]:=