Nash Equilibrium of the LUPI Game

Direct Calculation and Simulation via Genetic Algorithm

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UO

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- If no player selects a unique integer, no one wins.

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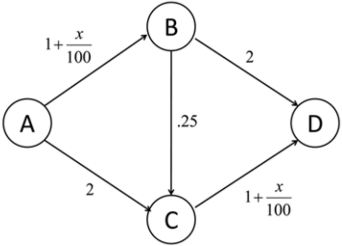
- Often not optimal

Example: Traffic (Braess's Paradox)



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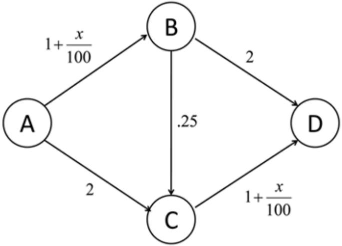

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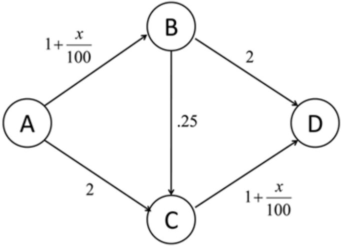

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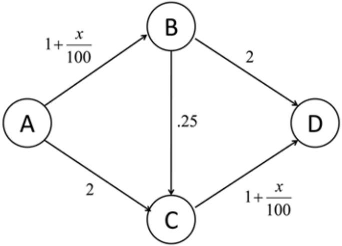

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Optimum: $t_{ABD} = t_{ACD} = 3.5, \quad t_{ABCD} = 3.25$

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- Note that $\text{Coef}[Z_0;\, p_{i_1}^{n_1} p_{i_2}^{n_2} \ldots p_{i_k}^{n_k}]$ is the number of ways that the $(N-1)$ players can pick the number i_1 n_1 times, etc.

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- The term

$$A_1 p_1 = \left.\frac{\mathrm{d}Z_0}{\mathrm{d}p_1}\right|_{p_1=0} p_1$$

contains exactly the cases in question.

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- Clearly,

$$[p_i, p_j] = [D_i, D_j] = [E_i, E_j] = [D_i, E_j] = [p_i, E_j] = 0,$$

and

$$[D_i, p_j] = \delta_{ij}$$

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$$\begin{aligned} Z_1 &:= Z_0 - L_1[Z_0] \\ &= \left(\sum_{i=1}^{N} p_i\right)^{N-1} - (N-1)p_1\left(\sum_{i=2}^{N} p_i\right)^{N-2}. \end{aligned}$$

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- Continuing, cases excluding unique selection of 1 *and* 2 are given by

$$Z_2 = Z_1 - L_2[Z_1].$$

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- Then, we can write the Z_k as

$$Z_k = \left[\prod_{i=1}^{k}(1 - L_i)\right] Z_0$$

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- That is, $c_i = \pi_i E_i[Z_i]$
- The N^{th} player's expected performance is then

$$W(\vec{\pi}; \vec{p}) = \sum_{i=1}^{N} c_i \pi_i$$

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- Under the Nash equilibrium, we have that

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- We therefore have a system of N equations,

$$c_i(\vec{p}) = c_0 \quad 1 \leq i \leq N,$$

with N degrees of freedom ($(N-1)$ in $\vec{p}$, and c_0).

NE LUPI Distributions

- For $N = 3$, the system of equations in analytically solvable, and we find

$$\begin{aligned} c_0 &= 28 - 16\sqrt{3} \\ c_1 &= 2\sqrt{3} - 3 \\ c_2 = c_3 &= 2 - \sqrt{3} \end{aligned}$$

Distributions (cont.)