Homework 5

Sean Ericson
Phys 610

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Problem 1

(a) The components of the Riemann curvature tensor were calculated using Mathematica code adapted from Hartle (see Appendix).

$$
\begin{aligned}
R^{\theta}_{\phi\phi\theta} &= -\sin^2\theta \\
R^{\phi}_{\theta\phi\theta} &= 1
\end{aligned}
$$

(b) The integral over the whole two-sphere is given by

$$
\begin{aligned}
\int \sqrt{r_0^4 \sin^2\theta}\frac{2}{r_0^2}\mathrm{d}\phi\mathrm{d}\theta &= 2\int_0^{2\pi}\mathrm{d}\phi\int_0^{\pi}\mathrm{d}\theta\sin\theta \\
&= 8\pi,
\end{aligned}
$$

which is independent of r_0.

Problem 2

(a) The symmetries

$$
\begin{aligned}
R_{\mu\nu\rho\sigma} &= -R_{\mu\nu\sigma\rho} \\
&= -R_{\nu\mu\rho\sigma} \\
&= R_{\rho\sigma\mu\nu}
\end{aligned}
$$

Show that the Riemann tensor can be thought of as a symmetric rank-2 tensor of antisymmetric rank-2 tensors. An antisymmetric rank-2 tensor has

$$
d' = T_{d-1} = \frac{1}{2}d(d-1)
$$

independent components, where T_n is the n^{th} triangular number. A symmetric rank-2 tensor, meanwhile, has

$$
N = T_{d'} = \frac{1}{2}d'(d'+1)
$$

independent components. Combining these, we get

$$N = \frac{1}{2}\left(\frac{1}{2}d(d-1)\right)\left(\frac{1}{2}d(d-1)+1\right).$$

However, we have overcounted. There is an additional symmetry: $R_{\mu\nu\rho\sigma}$ has no totally antisymmetric component:

$$R_{[\mu\nu\rho\sigma]} = 0.$$

A totally antisymmetric rank-4 tensor has $\binom{d}{4}$ components. Subtracting this gives

$$\begin{aligned} N &= \frac{1}{2}\left(\frac{1}{2}d(d-1)\right)\left(\frac{1}{2}d(d-1)+1\right) - \frac{1}{24}d(d-1)(d-2)(d-3) \\ &= \frac{1}{12}d^2(d^2-1) \end{aligned}$$

(b) In two dimensions, the Riemann tensor has only one independent component. Therefore, to get the right index symmetries, it must be that

$$R_{abcd} = f(R)\left(g_{ac}g_{db} - g_{ad}g_{bc}\right)$$

for some function of the curvature scalar. To determine the form of this function, we can contract twice with the metric:

$$\begin{aligned} R &= g^{ac}g^{db}R_{abcd} \\ &= f(R)g^{ac}g^{db}R\left(g_{ac}g_{db} - g_{ad}g_{bc}\right) \\ &= f(R)(\delta^a_a\delta^b_b - \delta^c_c) \\ &= f(R)(4-2) \\ \implies \quad f(R) &= \frac{R}{2}. \end{aligned}$$

Hence, in 2 dimensions,

$$R_{abcd} = \frac{R}{2}\left(g_{ac}g_{db} - g_{ad}g_{bc}\right) =$$

(c) Noting that $R = 2/r_0^2$ (calculated in the attached Mathematica notebook),

$$\begin{aligned} R_{\theta\phi\phi\theta} &= \frac{1}{2}\frac{2}{r^2}(r^4\sin^2\theta) = r^2\sin^2\theta \\ R_{\phi\theta\phi\theta} &= \frac{1}{2}\frac{2}{r^2}r^2 = 1 \end{aligned}$$

Problem 3

(a) Under the transformation

$$t \to t + f(r,t),$$

we have that

$$\begin{aligned}
\mathrm{d}t &\to \mathrm{d}t + \frac{\partial f}{\partial r}\mathrm{d}r + \frac{\partial f}{\partial t}\mathrm{d}t \\
&= \left(1+\dot{f}\right)\mathrm{d}t + f'\mathrm{d}r \\
\implies \quad \mathrm{d}t^2 &\to (1+\dot{f})^2\mathrm{d}t^2 + 2(1+\dot{f})\mathrm{d}t\mathrm{d}r + f'^2\mathrm{d}r.
\end{aligned}$$

Thus, we can pick f such that

$$2(1+\dot{f}) = -2B(r,t) \implies \frac{\partial f}{\partial t} = -(1+B(r,t))$$

to cancel the $\mathrm{d}r\mathrm{d}t$ term.

(b)

$$\begin{aligned}
G_{\hat{t}\hat{r}} &= e^{-(v+\lambda)/2}\dot{\lambda}/r \\
&= 0 \\
\implies \quad \dot{\lambda} &= 0
\end{aligned}$$

Considering $G_{\hat{r}\hat{r}}$, we see that

$$\begin{aligned}
G_{\hat{r}\hat{r}} &= \frac{e^{-\lambda}}{r^2}\left(1 - e^{\lambda} + rv'\right) \\
&= 0 \\
\implies \quad rv' &= 1 - e^{\lambda},
\end{aligned}$$

so we can see that the r dependence of v comes entirely from λ. Now, taking the derivative with respect to time, we find

$$\begin{aligned}
\frac{\partial}{\partial t}(rv') &= \frac{\partial}{\partial t}\left(1 - e^{\lambda}\right) \\
&= -\dot{\lambda}e^{\lambda} \\
&= 0 \\
\implies \quad \partial_t\partial_r v &= 0 \\
\implies \quad v &= \lambda(r) + h(t)
\end{aligned}$$

for some $h(t)$.

(c) Identifying

$$e^v = \left(1 - \frac{2GM}{r}\right); \quad e^{\lambda} = \left(1 - \frac{2GM}{r}\right)^{-1}$$

we recover the Schwarzschild metric. Since this is just a coordinate transformation from the completely general spherically symmetric metric, it must be that the Schwarzschild geometry is the most general asymptotically flat, spherically symmetric solution to the Einstein equation.

Appendix

In[180]:=

```
$Assumptions = {θ ∈ ℝ, r > 0, ϕ ∈ ℝ, τ ∈ ℝ};
n = 2;
coord = {θ, ϕ};
vel = { Dt[θ, τ], Dt[ϕ, τ]};
```

In[184]:=

```
affine :=
  affine = FullSimplify[Table[1/2 * Sum[(inversemetric[[i, s]]) * (D[metric[[s, j]], coord[[k]]] +
          D[metric[[s, k]], coord[[j]]] - D[metric[[j, k]], coord[[s]]]),
       {s, 1, n}], {i, 1, n}, {j , 1, n}, {k, 1, n}]];
```

In[185]:=

```
riemann :=
  riemann = Simplify[Table[D[affine[[i, j, l]], coord[[k]]] - D[affine[[i, j, k]], coord[[l]]] +
      Sum[affine[[s, j, l]] × affine[[i, k, s]] - affine[[s, j, k]] × affine[[i, l, s]], {s, 1, n}],
     {i, 1, n}, {j, 1, n}, {k, 1, n}, {l, 1, n}]];
```

In[186]:=

```
ricci :=
  ricci = Simplify[Table[Sum[riemann[[i, j, i, l]], {i, 1, n}], {j, 1, n}, {l, 1, n}]];
```

In[187]:=

```
Rscalar = Simplify[Sum[inversemetric[[i, j]] × ricci[[i, j]], {i, 1, n}, {j, 1, n}]];
```

In[188]:=

```
listaffine :=
  Table[If[UnsameQ[affine[[i, j, k]], 0], {ToString[Γ[coord[[i]], coord[[j]], coord[[k]]]],
     affine[[i, j, k]]}], {i, 1, n}, {j, 1, n}, {k, 1, j}];
```

In[189]:=

```
listriemann := Table[If[UnsameQ[riemann[[i, j, k, l]], 0],
    {ToString[R[coord[[i]], coord[[j]], coord[[k]], coord[[l]]]], riemann[[i, j, k, l]]}],
   {i, 1, n}, {j, 1, n}, {k, 1, n}, {l, 1, k - 1}];
```

In[190]:=

```
listricci := Table[If[UnsameQ[ricci[[j, l]], 0],
    {ToString[R[coord[[j]], coord[[l]]]], ricci[[j, l]]}], {j, 1, n}, {l, 1, j}];
```

In[191]:=

```
metric = DiagonalMatrix[{r^2, r^2 Sin[θ]^2}];
inversemetric = Simplify[Inverse[metric]];
```

In[193]:=

```
metric // MatrixForm
inversemetric // MatrixForm
```

Out[193]//MatrixForm=

$$\begin{pmatrix} r^2 & 0 \\ 0 & r^2 \operatorname{Sin}[\theta]^2 \end{pmatrix}$$

Out[194]//MatrixForm=

$$\begin{pmatrix} \frac{1}{r^2} & 0 \\ 0 & \frac{\operatorname{Csc}[\theta]^2}{r^2} \end{pmatrix}$$

In[195]:=

```
TableForm[Partition[DeleteCases[Flatten[listaffine], Null], 2], TableSpacing → {2, 2}]
```

Out[195]//TableForm=

$\Gamma[\theta, \phi, \phi]$	$-\text{Cos}[\theta]\,\text{Sin}[\theta]$
$\Gamma[\phi, \phi, \theta]$	$\text{Cot}[\theta]$

In[196]:=

```
TableForm[Partition[DeleteCases[Flatten[listriemann], Null], 2], TableSpacing → {2, 2}]
```

Out[196]//TableForm=

$R[\theta, \phi, \phi, \theta]$	$-\text{Sin}[\theta]^2$
$R[\phi, \theta, \phi, \theta]$	1

In[197]:=

```
TableForm[Partition[DeleteCases[Flatten[listricci], Null], 2], TableSpacing → {2, 2}]
```

Out[197]//TableForm=

$R[\theta, \theta]$	1
$R[\phi, \phi]$	$\text{Sin}[\theta]^2$

In[198]:=

```
Rscalar
```

Out[198]=

$$\frac{2}{r^2}$$

In[199]:=

$$\text{Integrate}\left[\sqrt{\text{Det[metric]}}\ \text{Rscalar},\ \{\theta, 0, \pi\},\ \{\phi, 0, 2\pi\}\right]$$

Out[199]=

8π