Homework 4

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Phys 633

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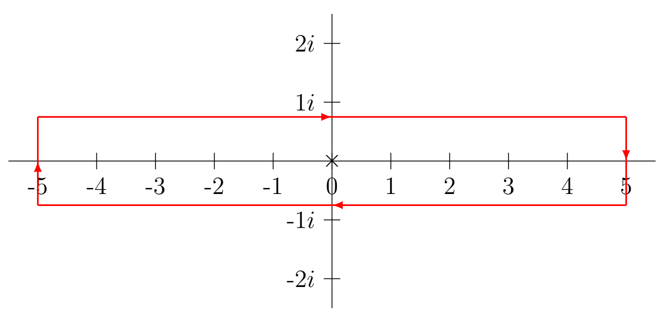
Problem 1

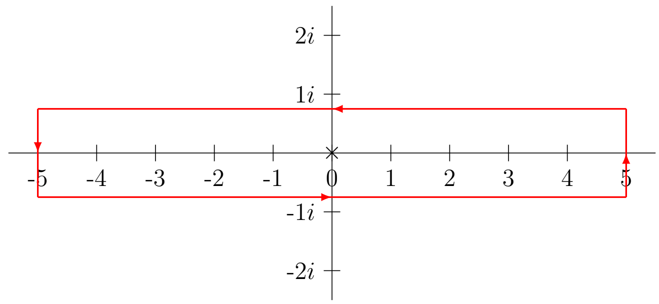
$$\begin{aligned}
\int_{-\infty}^{\infty} \mathrm{d}x \, \frac{f(x)}{x + i0^+} - \mathscr{P} \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{1}{x} &= \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{1}{x + i0^+} - \frac{1/2}{1 + i0^+} - \frac{1/2}{x - i0^+} \\
&= \frac{1}{2} \int_{-\infty}^{\infty} \mathrm{d}x \, \left[\frac{1}{x + i0^+} - \frac{1}{x - i0^+} \right] \\
&= \frac{1}{2}(-2\pi i f(0)) \\
&= -\pi i f(0)
\end{aligned}$$

Where the following contour was used:



$$\begin{aligned}
\int_{-\infty}^{\infty} \mathrm{d}x \, \frac{f(x)}{x - i0^+} - \mathscr{P} \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{1}{x} &= \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{1}{x - i0^+} - \frac{1/2}{1 + i0^+} - \frac{1/2}{x - i0^+} \\
&= \frac{1}{2} \int_{-\infty}^{\infty} \mathrm{d}x \, \left[\frac{1}{x - i0^+} - \frac{1}{x + i0^+} \right] \\
&= \frac{1}{2}(2\pi i f(0)) \\
&= \pi i f(0)
\end{aligned}$$

Where the following contour integral was used:



Thus

$$\int_{-\infty}^{\infty} \mathrm{d}x \, \frac{f(x)}{x \pm i0^+} - \mathscr{P} \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{f(x)}{x} = \mp i\pi f(0)$$
$$\implies$$
$$\frac{1}{x \pm i0^+} = \mathscr{P}\frac{1}{x} \mp i\pi\delta(x)$$

Problem 2

Given that

$$\overleftrightarrow{\chi}(\omega) - \overleftrightarrow{\chi}_0 = \frac{1}{\pi i}\fint_{-\infty}^{\infty} \mathrm{d}\omega' \, \frac{\overleftrightarrow{\chi}(\omega') - \overleftrightarrow{\chi}_0}{\omega' - \omega}$$

we have

$$\begin{aligned} \frac{1}{\pi}\fint_{-\infty}^{\infty} \mathrm{d}\omega' \, \frac{\overleftrightarrow{\zeta}(\omega') - \overleftrightarrow{\chi}_0}{\omega' - \omega} &= \frac{1}{\pi}\fint_{-\infty}^{\infty} \mathrm{d}\omega' \, \frac{\overleftrightarrow{\chi}(\omega') - \overleftrightarrow{\chi}_0}{\omega' - \omega} + \frac{i}{\pi}\fint_{-\infty}^{\infty} \mathrm{d}\omega' \, \frac{\overleftrightarrow{\sigma}(\omega')}{\omega'(\omega' - \omega)} \\ &= i(\overleftrightarrow{\chi}(\omega) - \overleftrightarrow{\chi}_0) - \frac{\overleftrightarrow{\sigma}(\omega)}{\omega} + \frac{\overleftrightarrow{\sigma}_0}{\omega} \\ &= i\left(\overleftrightarrow{\zeta}(\omega) - \overleftrightarrow{\chi}_0 - \frac{i\overleftrightarrow{\sigma}_0}{\omega}\right) \end{aligned}$$

Therefore

$$\overleftrightarrow{\zeta}(\omega) - \overleftrightarrow{\chi}_0 = \frac{1}{\pi i}\fint_{-\infty}^{\infty} \mathrm{d}\omega' \, \frac{\overleftrightarrow{\zeta}(\omega') - \overleftrightarrow{\chi}_0}{\omega' - \omega} + \frac{i\overleftrightarrow{\sigma}_0}{\omega}$$

while splitting up the real and imaginary parts gives

$$\begin{aligned} \mathrm{Re}[\overleftrightarrow{\zeta}(\omega) - \overleftrightarrow{\chi}_0] &= \quad \frac{1}{\pi}\fint_{-\infty}^{\infty} \mathrm{d}\omega' \, \frac{\mathrm{Im}[\overleftrightarrow{\zeta}(\omega) - \overleftrightarrow{\chi}_0]}{\omega' - \omega} \\ \mathrm{Im}[\overleftrightarrow{\zeta}(\omega) - \overleftrightarrow{\chi}_0] &= -\frac{1}{\pi}\fint_{-\infty}^{\infty} \mathrm{d}\omega' \, \frac{\mathrm{Re}[\overleftrightarrow{\zeta}(\omega) - \overleftrightarrow{\chi}_0]}{\omega' - \omega} + \frac{\overleftrightarrow{\sigma}_0}{\omega} \end{aligned}$$

Problem 3

Firstly,

$$\overleftrightarrow{\chi}(-\omega^*) = \frac{i}{\hbar}\int_0^\infty \mathrm{d}\tau \ \langle[\tilde{x}_\alpha(\tau), \tilde{x}_\beta(0)]\rangle \, e^{-i\omega^*\tau}$$

$$\overleftrightarrow{\chi}^*(\omega) = -\frac{i}{\hbar}\int_0^\infty \mathrm{d}\tau \ \langle[\tilde{x}_\beta(0), \tilde{x}_\alpha(\tau)]\rangle \, e^{-i\omega^*\tau} = \frac{i}{\hbar}\int_0^\infty \mathrm{d}\tau \ \langle[\tilde{x}_\alpha(\tau), \tilde{x}_\beta(0)]\rangle \, e^{-i\omega^*\tau}$$

so we have

$$\overleftrightarrow{\chi}(-\omega^*) = \overleftrightarrow{\chi}^*(\omega)$$

This means, for a real frequency ω, $\mathrm{Re}[\overleftrightarrow{\chi}(\omega)]$ is an even function, while $\mathrm{Im}[\overleftrightarrow{\chi}(\omega)]$ is odd. Now, noting that $\overleftrightarrow{\chi}_0$ must be real,

$$\begin{aligned}
\mathrm{Re}[\overleftrightarrow{\chi}(\omega)] &= \overleftrightarrow{\chi}_0 + \frac{1}{\pi}⨍_{-\infty}^{\infty} \mathrm{d}\omega' \, \frac{\mathrm{Im}[\overleftrightarrow{\chi}(\omega')]}{\omega' - \omega} \\
&= \overleftrightarrow{\chi}_0 + \frac{1}{\pi}⨍_{-\infty}^{\infty} \mathrm{d}\omega' \, \frac{\mathrm{Im}[\overleftrightarrow{\chi}(\omega')]}{\omega'^2 - \omega^2}(\omega' + \omega) \\
&= \overleftrightarrow{\chi}_0 + \frac{1}{\pi}⨍_{-\infty}^{\infty} \mathrm{d}\omega' \, \frac{\omega' \, \mathrm{Im}[\overleftrightarrow{\chi}(\omega')]}{\omega'^2 - \omega^2} + \frac{\omega}{\pi}⨍_{-\infty}^{\infty} \mathrm{d}\omega' \, \frac{\mathrm{Im}[\overleftrightarrow{\chi}(\omega')]}{\omega'^2 - \omega^2} \\
&= \overleftrightarrow{\chi}_0 + \frac{2}{\pi}⨍_{0}^{\infty} \mathrm{d}\omega' \, \frac{\omega' \, \mathrm{Im}[\overleftrightarrow{\chi}(\omega')]}{\omega'^2 - \omega^2}
\end{aligned}$$

Where, in the second to last line, the second integral vanishes due to $\mathrm{Im}[\overleftrightarrow{\chi}(\omega')]$ being odd, while the evenness of $\mathrm{Re}[\overleftrightarrow{\chi}(\omega')]$ allows the domain of the first integral to be restricted to the positive reals (with an additional factor of 2).
Similarly,

$$\begin{aligned}
\mathrm{Im}[\overleftrightarrow{\chi}(\omega)] &= -\frac{1}{\pi}⨍_{-\infty}^{\infty} \mathrm{d}\omega' \, \frac{\mathrm{Re}[\overleftrightarrow{\chi}(\omega')] - \overleftrightarrow{\chi}_0}{\omega' - \omega} \\
&= -\frac{1}{\pi}⨍_{-\infty}^{\infty} \mathrm{d}\omega' \, \frac{\mathrm{Re}[\overleftrightarrow{\chi}(\omega')] - \overleftrightarrow{\chi}_0}{\omega'^2 - \omega^2}(\omega' + \omega) \\
&= -\frac{1}{\pi}⨍_{-\infty}^{\infty} \mathrm{d}\omega' \, \frac{\omega'\left(\mathrm{Re}[\overleftrightarrow{\chi}(\omega')] + \overleftrightarrow{\chi}_0\right)}{\omega'^2 - \omega^2} - \frac{\omega}{\pi}⨍_{-\infty}^{\infty} \mathrm{d}\omega' \, \frac{\mathrm{Re}[\overleftrightarrow{\chi}(\omega')] - \overleftrightarrow{\chi}_0}{\omega'^2 - \omega^2} \\
&= -\frac{2\omega}{\pi}⨍_{0}^{\infty} \mathrm{d}\omega' \, \frac{\mathrm{Re}[\overleftrightarrow{\chi}(\omega')] - \overleftrightarrow{\chi}_0}{\omega'^2 - \omega^2}
\end{aligned}$$

Problem 4

(a) By the definition of the Green's function,

$$\left(\frac{\partial^2}{\partial t^2} + \gamma\frac{\partial}{\partial t} + \omega_0^2\right) g(t, t') = \delta(t - t').$$

Taking the Fourier transform of the above equation (with respect to t) yields

$$\left(-\omega^2 - i\gamma\omega + \omega_0^2\right) g(\omega, t') = \int_{-\infty}^{\infty} \mathrm{d}t\, \delta(t-t') e^{i\omega t} = e^{i\omega t'}$$
$$\Longrightarrow$$
$$g(\omega, t') = \frac{e^{i\omega t'}}{-\omega^2 - i\gamma\omega + \omega_0^2} \tag{1}$$

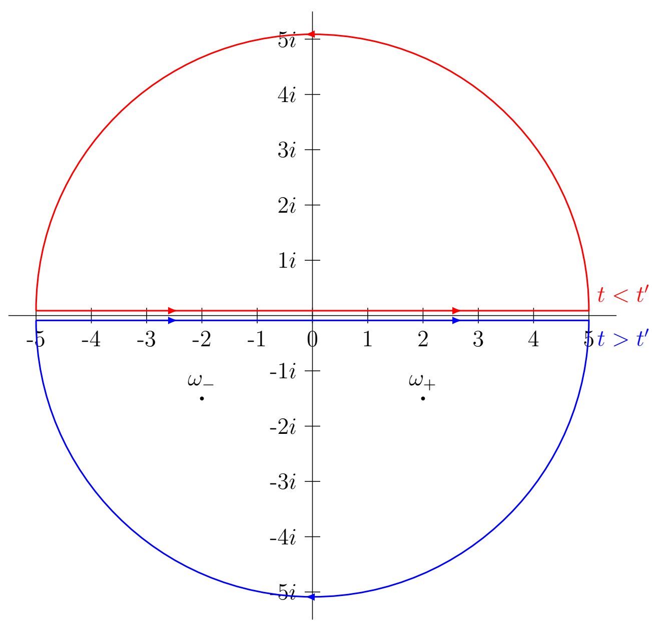
Inverting the Fourier transform gives

$$\begin{aligned} g(t,t') &= -\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\, \frac{e^{i\omega t'} e^{-i\omega t}}{\omega^2 + i\gamma\omega - \omega_0^2} \\ &= -\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\, \frac{e^{-i\omega(t-t')}}{(\omega-\omega_+)(\omega-\omega_-)} \end{aligned}$$

where

$$\omega_\pm = -\frac{i\gamma}{2} \pm \tilde{\omega}; \quad \tilde{\omega} = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$$

This integral can be calculated using Cauchy's residue theorem by considering the contours below:



Only the lower contour encloses the poles, thus

$$g(t,t') = i\Theta(t-t')\left[\frac{e^{-i\omega_+(t-t')}}{\omega_+ - \omega_-} + \frac{e^{-i\omega_-(t-t')}}{\omega_- - \omega_+}\right] = \boxed{\Theta(t-t')\frac{e^{-\gamma(t-t')/2}}{\tilde{\omega}}\sin[\tilde{\omega}(t-t')]}$$

(b) Let

$$\mathscr{L}x(t) = f(t); \quad \mathscr{L} = \frac{\partial^2}{\partial t^2} + \gamma\frac{\partial}{\partial t} + \omega_0^2$$

Then

$$\begin{aligned}
f(t) &= \int_{-\infty}^{\infty} \mathrm{d}t' \; \delta(t-t') f(t') \\
&= \int_{-\infty}^{\infty} \mathrm{d}t' \; [\mathscr{L} g(t-t')] \, f(t') \\
&= \mathscr{L} \int_{-\infty}^{\infty} \mathrm{d}t' \; g(t-t') f(t')
\end{aligned}$$

$$\implies x(t) = \int_{-\infty}^{\infty} \mathrm{d}t' \; g(t-t') f(t')$$

showing that the general solution $x(t)$ for a given forcing function $f(t)$ is simply the convolution of the forcing and Green function $(g * f)(x)$. Using the Green function calculated above,

$$\begin{aligned}
x(t) &= \int_{-\infty}^{\infty} \mathrm{d}t' \; \Theta(t-t') \frac{e^{-\gamma(t-t')/2}}{\tilde{\omega}} \sin[\tilde{\omega}(t-t')] f(t') \\
&= \frac{1}{\tilde{\omega}} \int_{0}^{\infty} \mathrm{d}t' \; e^{-\gamma(t-t')/2} \sin[\tilde{\omega}(t-t')] f(t')
\end{aligned}$$

(c) See (1)

(d) By the convolution theorem,

$$x(t) = (g * f)(t) \implies \tilde{x}(\omega) = \tilde{g}(\omega) \tilde{f}(\omega)$$

Applying the inverse Fourier transform then gives

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\omega \; \tilde{x}(\omega) e^{-i\omega t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}\omega \; \tilde{g}(\omega) \tilde{f}(\omega) e^{-i\omega t}$$

Problem 5

(a)

$$n(E) = 2(\pi k_E^2) \frac{A}{(2\pi)^2} = \frac{E^2 A}{2\pi \hbar^2 c^2}$$

$$\implies \rho(E) = \frac{EA}{\pi \hbar^2 c^2}$$

$$\implies \rho(E_{\mathrm{f}}) = \frac{\omega_{\mathrm{eg}} A}{\pi \hbar c^2}$$

$$
\begin{aligned}
\Gamma_{\mathrm{i}\to\mathcal{F}} &= \frac{2\pi}{\hbar}\left|V_{\mathbf{fi}}\right|^2\rho(E_{\mathrm{f}}) \\
&= \frac{2\pi}{\hbar}\left|\hat{\epsilon}\cdot\vec{d}_{\mathrm{eg}}\right|^2\frac{\hbar\omega}{2\epsilon_0 AL}\frac{\omega_{\mathrm{eg}}A}{\pi\hbar c^2} \\
&= \frac{2\pi}{\hbar}\frac{e^2\left|\vec{r}_{\mathrm{eg}}\right|^2}{2}\frac{\hbar\omega}{2\epsilon_0 AL}\frac{\omega_{\mathrm{eg}}A}{\pi\hbar c^2} \\
&\approx \frac{\omega_{\mathrm{eg}}^2 e^2\left|\vec{r}_{\mathrm{eg}}\right|^2}{2\epsilon_0\hbar c^2 L}
\end{aligned}
$$

(b)

$$
\begin{aligned}
n(E) = 2(2k_E)\frac{L}{2\pi} &= \frac{2EL}{\pi\hbar c} \\
\implies \rho(E) &= \frac{2L}{\pi\hbar c} \\
\implies \rho(E_{\mathrm{f}}) &= \frac{2L}{\pi\hbar c}
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_{\mathrm{i}\to\mathcal{F}} &= \frac{2\pi}{\hbar}\left|V_{\mathbf{fi}}\right|^2\rho(E_{\mathrm{f}}) \\
&= \frac{2\pi}{\hbar}\left|\hat{\epsilon}\cdot\vec{d}_{\mathrm{eg}}\right|^2\frac{\hbar\omega}{2\epsilon_0 AL}\frac{2L}{\pi\hbar c} \\
&= \frac{2\pi}{\hbar}e^2\left|\vec{r}_{\mathrm{eg}}\right|^2\frac{\hbar\omega}{2\epsilon_0 AL}\frac{2L}{\pi\hbar c} \\
&\approx \frac{2\omega_{\mathrm{eg}}e^2\left|r_{\mathrm{eg}}\right|^2}{\epsilon_0\hbar cA}
\end{aligned}
$$