Exercise Set 9

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Phys 633

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Monday

Exercise 1

$$
\begin{aligned}
(x_{j+1} - x_j)^2 &= (\bar{x}_{j+1} + \delta_{j+1} - (\bar{x}_j + \delta_j))^2 \\
&= (\delta_{j+1} - \delta_j + x_0 + \bar{v}(t_{j+1} - t_0) - x_0 - \bar{v}(t_j - t_0))^2 \\
&= (\delta_{j+1} - \delta_j + \bar{v}(t_{j+1} - t_j))^2 \\
&= (\delta_{j+1} - \delta_j + \bar{v}\delta t)^2 \\
&= (\delta_{j+1} - \delta_j)^2 + 2(\delta_{j+1} - \delta_j)\bar{v}\delta t + \bar{v}^2 \delta t^2
\end{aligned}
$$

Exercise 2

$$
\begin{aligned}
\prod_{j=0}^{N-1} \exp\left[\frac{im\bar{v}^2\delta t}{2\hbar}\right] &= \exp\left[\sum_{j=0}^{N-1} \frac{im\bar{v}^2\delta t}{2\hbar}\right] \\
&= \exp\left[N\frac{im\bar{v}^2\delta t}{2\hbar}\right] \\
&= \exp\left[\frac{im(x-x_0)^2}{2\hbar(t-t_0)}\right]
\end{aligned}
$$

Exercise 3

$$
\begin{aligned}
\det_n &= \begin{vmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{vmatrix}_n \\
&= 2\begin{vmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{vmatrix}_{n-1} + \begin{vmatrix} -1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{vmatrix}_{n-1} \\
&= 2\begin{vmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{vmatrix}_{n-1} - \begin{vmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{vmatrix}_{n-2} \\
&= 2\det_{n-1} - \det_{n-2}
\end{aligned}
$$

Exercise 4

First, assume that, for some j

$$\det_j = j+1, \quad \det_{j+1} = j+2$$

Then

$$
\begin{aligned}
\det_{j+2} &= 2\det_{j+1} - \det_j \\
&= 2(j+2) - (j+1) \\
&= j+3
\end{aligned}
$$

Now, given that

$$\det_1 = \begin{vmatrix} 2 \end{vmatrix} = 2, \quad \det_2 = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 4 - 1 = 3$$

$$\det_3 = 2 \cdot 3 - 2 = 4 = 3 + 1$$

we have that

$$\det_n = n + 1 \quad n \in \mathbb{N}$$

Tuesday

Exercise 1

$$\frac{\partial}{\partial x_k} x_j^n = n x_j^{n-1} \delta_{jk}$$

$$\leftrightarrow$$

$$\frac{\delta}{\delta x(t)} \int_{t_1}^{t_2} \mathrm{d}t'\; x^n(t') = n x^{n-1}(t) \delta(t-t')$$

Exercise 2

Expanding the field commutator into the mode functions gives

$$\left[\hat{\psi}(\vec{r}), \hat{\psi}(\vec{r}')\right] = \sum_{jj'} \left[\hat{a}_j(t), \hat{a}^\dagger_{j'}(t)\right] \phi(\vec{r}) \phi^*(\vec{r}')$$

A particual mode-operator commutator pair can be projected out by multiplying by $\phi^*_k(\vec{r})\phi_{k'}(\vec{r}')$ and integrating:

$$\begin{aligned}
\int \mathrm{d}^3r \mathrm{d}^3r' \left[\hat{\psi}(\vec{r}), \hat{\psi}(\vec{r}')\right] \phi^*_k(\vec{r})\phi_{k'}(\vec{r}') &= \int \mathrm{d}^3r \mathrm{d}^3r' \sum_{jj'} \left[\hat{a}_j(t), \hat{a}^\dagger_{j'}(t)\right] \phi(\vec{r}) \phi^*(\vec{r}') \phi^*_k(\vec{r})\phi_{k'}(\vec{r}') \\
&= \sum_{jj'} \left[\hat{a}_j(t), \hat{a}^\dagger_{j'}(t)\right] \delta_{jk}\delta_{j'k'} \\
&= \left[\hat{a}_k(t), \hat{a}^\dagger_{k'}(t)\right]
\end{aligned}$$

Since the field commutator evaulates as

$$\left[\hat{\psi}(\vec{r}), \hat{\psi}(\vec{r}')\right] = \delta^3(\vec{r} - \vec{r}')$$

we also have that

$$\begin{aligned}
\int \mathrm{d}^3r \mathrm{d}^3r' \left[\hat{\psi}(\vec{r}), \hat{\psi}(\vec{r}')\right] \phi^*_k(\vec{r})\phi_{k'}(\vec{r}') &= \int \mathrm{d}^3r \mathrm{d}^3r' \delta^3(\vec{r} - \vec{r}') \phi^*_k(\vec{r})\phi_{k'}(\vec{r}') \\
&= \delta_{kk'}
\end{aligned}$$

Thus

$$\left[\hat{a}_k(t), \hat{a}^\dagger_{k'}\right] = \delta_{kk'}$$

Exercise 3

The initial joint state is given by

$$\left(\frac{|1_L 0_R\rangle + |0_L 1_R\rangle}{\sqrt{2}}\right) |{\uparrow_A \uparrow_B}\rangle = \frac{|1_L 0_R\rangle |{\uparrow_A \uparrow_B}\rangle + |0_L 1_R\rangle |{\uparrow_A \uparrow_B}\rangle}{\sqrt{2}}.$$

After the interaction (and discarding the mode states) we have the state

$$\frac{|{\downarrow_A}{\uparrow_B}\rangle + |{\uparrow_A}{\downarrow_B}\rangle}{\sqrt{2}},$$

in which the spins of the particals are clearly entangled. Assuming the modes are physically separted, and the spin-flipping interactions between the modes and the particals are local, the entanglment present in the particals' spins must have come from the modes. Hence, mode state

$$\frac{|1_L 0_R\rangle + |0_L 1_R\rangle}{\sqrt{2}}$$

must be entangled.