

Homework 2

Sean Ericson
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Problem 1 (Berman 2.9)

In the adiabatic approximation, the dressed-state amplitudes satisfy (eq. 2.146 in the text-book)

$$\begin{aligned}c_{d_1}(t) &= e^{\frac{i}{2}\xi(t)}c_{d_1}(t_0) \\c_{d_2}(t) &= e^{-\frac{i}{2}\xi(t)}c_{d_2}(t_0),\end{aligned}$$

where

$$\xi(t) = \int_{t_0}^t dt' \Omega(t').$$

In this case, $t_0 = -\infty$ and

$$\Omega_0(t) = \Omega_0 e^{-\left(\frac{t}{T}\right)^2},$$

so we can integrate that to get

$$\xi(t) = \frac{\sqrt{\pi}}{2} \Omega_0 T \left(1 + \operatorname{erf} \left(\frac{t}{T} \right) \right).$$

We transform between the dressed-states and the field-interaction basis states via

$$\begin{pmatrix} \tilde{c}_1 \\ \tilde{c}_2 \end{pmatrix} = \begin{pmatrix} c_\theta & s_\theta \\ -s_\theta & c_\theta \end{pmatrix} \begin{pmatrix} c_{d_1} \\ c_{d_2} \end{pmatrix}; \quad \begin{pmatrix} c_{d_1} \\ c_{d_2} \end{pmatrix} = \begin{pmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{pmatrix} \begin{pmatrix} \tilde{c}_1 \\ \tilde{c}_2 \end{pmatrix},$$

where

$$c_\theta = \sqrt{\frac{1}{2} \left(1 + \frac{\delta}{\Omega(t)} \right)}; \quad s_\theta = \sqrt{\frac{1}{2} \left(1 - \frac{\delta}{\Omega(t)} \right)},$$

and $\Omega(t) = \sqrt{\delta^2 + \Omega_0^2(t)}$. Given that $\tilde{c}_1(-\infty) = 1$ and $\tilde{c}_2(-\infty) = 0$, we find that

$$\begin{aligned}c_{d_1}(-\infty) &= c_\theta(-\infty) = 1 \\c_{d_2}(-\infty) &= s_\theta(-\infty) = 0\end{aligned}$$

Putting everything together, we find

$$\begin{aligned}
\begin{pmatrix} \tilde{c}_1(t) \\ \tilde{c}_2(t) \end{pmatrix} &= \begin{pmatrix} c_\theta & s_\theta \\ -s_\theta & c_\theta \end{pmatrix} \begin{pmatrix} c_{d_1}(t) \\ c_{d_2}(t) \end{pmatrix} \\
&= \begin{pmatrix} c_\theta(t)c_{d_1}(t) + s_\theta(t)c_{d_2}(t) \\ -s_\theta(t)c_{d_1}(t) + c_\theta(t)c_{d_2}(t) \end{pmatrix} \\
&= \begin{pmatrix} c_\theta(t) \exp\left[\frac{i}{2}\xi(t)\right] \\ -s_\theta(t) \exp\left[\frac{i}{2}\xi(t)\right] \end{pmatrix}
\end{aligned}$$

Problem 2 (Berman 2.17)

See attached Mathematica print-out.

Problem 3

(a) Given that

$$H_0 = \hbar \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix},$$

we have that

$$\begin{aligned}
\exp(iH_0t/\hbar) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + it \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix} + \frac{(it)^2}{2} \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix}^2 + \frac{(it)^3}{3!} \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix}^3 + \dots \\
&= \begin{pmatrix} 1 + i\omega_1t + \frac{1}{2}(i\omega_1t)^2 + \frac{1}{3!}(i\omega_1t)^3 + \dots & 0 \\ 0 & 1 + i\omega_2t + \frac{1}{2}(i\omega_2t)^2 + \frac{1}{3!}(i\omega_2t)^3 + \dots \end{pmatrix} \\
&= \begin{pmatrix} \exp(i\omega_1t) & 0 \\ 0 & \exp(i\omega_2t) \end{pmatrix}.
\end{aligned}$$

(b) In the interaction representation, we factor out this free phase evolution by writing

$$|\psi(t)\rangle_I = \bar{c}_1(t)e^{-i\omega_1t}|1\rangle + \bar{c}_2(t)e^{-i\omega_2t}|2\rangle,$$

that is, we make the (time-dependent) unitary transformation

$$|\psi(t)\rangle_S \rightarrow |\psi(t)\rangle_I = U(t) |\psi(t)\rangle_S,$$

where

$$U(t) = e^{iH_0t/\hbar} = \begin{pmatrix} e^{i\omega_1t} & 0 \\ 0 & e^{i\omega_2t} \end{pmatrix}.$$

Time evolution in the interaction representation is given by

$$\begin{aligned}
|\dot{\psi}\rangle_I &= \frac{iH_0}{\hbar} e^{iH_0t/\hbar} |\psi\rangle_S + e^{iH_0t/\hbar} |\dot{\psi}\rangle_S \\
&= -\frac{i}{\hbar} e^{iH_0t/\hbar} (H - H_0) |\psi\rangle_S \\
&= -\frac{i}{\hbar} V_I |\psi\rangle_I,
\end{aligned}$$

where

$$\begin{aligned}
V_I &= e^{iH_0 t/\hbar} V e^{-iH_0 t/\hbar} \\
&= \hbar \begin{pmatrix} e^{i\omega_1 t} & 0 \\ 0 & e^{i\omega_2 t} \end{pmatrix} \begin{pmatrix} 0 & \Omega_0 \cos(\omega t) \\ \Omega_0 \cos(\omega t) & 0 \end{pmatrix} \begin{pmatrix} e^{-i\omega_1 t} & 0 \\ 0 & e^{-i\omega_2 t} \end{pmatrix} \\
&= \hbar \Omega_0 \cos(\omega t) \begin{pmatrix} 0 & e^{-i\omega_0 t} \\ e^{i\omega_0 t} & 0 \end{pmatrix}
\end{aligned}$$

We get the effective interaction hamiltonian by making the inverse transformation on V :

$$V_I = U^\dagger V U = \hbar \Omega_0 \cos(\omega t) \begin{pmatrix} 0 & e^{-i\omega_0 t} \\ e^{i\omega_0 t} & 0 \end{pmatrix},$$

where the phase ϕ has been absorbed into the (complex) Rabi frequency.

(c) In the field-interaction representation,

$$|\psi(t)\rangle_{\text{FI}} = \tilde{c}_1(t) e^{i\omega t/2} |1\rangle + \tilde{c}_2(t) e^{-i\omega t/2} |2\rangle.$$

So, in this case,

$$U = \begin{pmatrix} e^{-i\omega t/2} & 0 \\ 0 & e^{i\omega t/2} \end{pmatrix} = e^{-i\omega \sigma_z t/2}$$

and

$$\begin{aligned}
|\dot{\psi}\rangle_{\text{FI}} &= -\frac{i\omega}{2} \sigma_z e^{-i\omega \sigma_z t/2} |\psi\rangle_{\text{S}} + e^{i\omega \sigma_z t/2} |\dot{\psi}\rangle_{\text{S}} \\
&= -\frac{i\omega}{2} \sigma_z e^{-i\omega \sigma_z t/2} |\psi\rangle_{\text{S}} - \frac{i}{\hbar} e^{-i\omega \sigma_z t/2} H |\psi\rangle_{\text{S}} \\
&= \frac{i}{\hbar} \sigma_z |\psi\rangle_{\text{FI}} - \frac{i}{\hbar} e^{-i\omega \sigma_z t/2} H e^{i\omega \sigma_z t/2} |\psi\rangle_{\text{FI}} \\
&= -\frac{i}{\hbar} H_{\text{FI}} |\psi\rangle_{\text{FI}},
\end{aligned}$$

where

$$H_{\text{FI}} = \frac{\hbar\omega}{2} \sigma_z + e^{-i\omega \sigma_z t} H e^{i\omega \sigma_z t}$$

Problem 4

We seek to find the eigenvectors of

$$\tilde{H} = \frac{\hbar}{2} \begin{pmatrix} -\delta & \Omega_0 \\ \Omega_0 & \delta \end{pmatrix}.$$

Firstly, the eigenvalues of a 2x2 matrix are given by

$$\frac{1}{2} \left(T \pm \sqrt{T^2 - 4D} \right),$$

where T is the trace, and D is the determinant. In this case,

$$T = 0; \quad D = -\delta^2 - \Omega_0^2 =: -\frac{\hbar^2 \Omega^2}{4},$$

so the eigenvalues are simply $\pm \frac{1}{2} \hbar \Omega$. Let the $+$ eigenvector be of the form

$$\begin{pmatrix} \sin \theta \\ \cos \theta \end{pmatrix},$$

i.e.

$$\frac{\hbar}{2} \begin{pmatrix} -\delta & \Omega_0 \\ \Omega_0 & \delta \end{pmatrix} \begin{pmatrix} \sin \theta \\ \cos \theta \end{pmatrix} = \frac{\hbar \Omega}{2} \begin{pmatrix} \sin \theta \\ \cos \theta \end{pmatrix}.$$

Taking the first entry, we find

$$\begin{aligned} & -\delta \sin \theta + \Omega_0 \cos \theta = \Omega \sin \theta \\ \implies & (\Omega + \delta) \sin \theta = \Omega_0 \cos \theta \\ \implies & \tan \theta = \frac{\Omega_0}{\Omega + \delta} \\ \implies & \theta = \text{atan} \frac{\Omega_0}{\Omega + \delta}. \end{aligned}$$

Now, using the identity

$$\tan(2 \text{atan}(x)) = \frac{-2x}{x^2 - 1},$$

we find

$$\begin{aligned} \tan(2\theta) &= \frac{-2 \left(\frac{\Omega_0}{\Omega + \delta} \right)}{\left(\frac{\Omega_0}{\Omega + \delta} \right)^2 - 1} \\ &= \frac{\Omega_0}{\delta}. \end{aligned}$$

The eigenvectors are then

$$\begin{aligned} |+\rangle &= \begin{pmatrix} \sin \theta \\ \cos \theta \end{pmatrix} \\ |-\rangle &= \begin{pmatrix} \cos \theta \\ -\sin \theta \end{pmatrix}, \end{aligned}$$

with θ given implicitly above.

Problem 5

The general solution is given by

$$\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} = \begin{pmatrix} e^{-i\delta t/2} \left[\cos \frac{\Omega t}{2} + i \frac{\delta}{\Omega} \sin \frac{\Omega t}{2} \right] c_1(0) - i \frac{\Omega_0^*}{\Omega} \sin \frac{\Omega t}{2} c_2(0) \\ e^{i\delta t/2} \left[-i \frac{\Omega_0}{\Omega} \sin \frac{\Omega t}{2} c_1(0) + \left[\cos \frac{\Omega t}{2} - i \frac{\delta}{\Omega} \sin \frac{\Omega t}{2} \right] c_2(0) \right] \end{pmatrix}.$$

In the $|\delta| \gg \Omega_0$ limit, we have that

$$\Omega = \sqrt{\Omega_0^2 + \delta^2} \approx \delta(1 + \frac{\Omega_0^2}{\delta^2})$$

(a) With the initial conditions $c_1(0) = 1$ and $c_2(0) = 0$, we have

$$\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} = \begin{pmatrix} e^{-i\delta t/2} \left(\cos \frac{\Omega t}{2} + i \frac{\delta}{\Omega} \sin \frac{\Omega t}{2} \right) \\ -i \frac{\Omega_0}{\Omega} e^{i\delta t/2} \sin \frac{\Omega t}{2} \end{pmatrix}$$

Focusing on c_1 , we find

$$\begin{aligned} c_1(t) &= e^{-i\delta t/2} \left(\cos \frac{\Omega t}{2} + i \frac{\delta}{\Omega} \sin \frac{\Omega t}{2} \right) \\ &= \frac{1}{2} e^{-i\delta t/2} \left[(e^{i\Omega t/2} + e^{-i\Omega t/2}) + \frac{\delta}{\Omega} (e^{i\Omega t/2} - e^{-i\Omega t/2}) \right] \\ &= \frac{1}{2\Omega} e^{-i(\delta+\Omega)t/2} [(\Omega - \delta) + (\Omega + \delta)e^{i\Omega t}] \\ &= \frac{\Omega + \delta}{2\Omega} e^{-i(\delta-\Omega)t/2} + O\left(\frac{\Omega_0^2}{\delta^2}\right). \end{aligned}$$

Now, since this is in the interaction representation, we have that

$$\begin{aligned} |\psi(t)\rangle &= c_1(t) e^{-i\omega_1 t} |1\rangle \\ &\propto e^{-i(\omega_1 + (\delta - \Omega)/2)t} |1\rangle, \end{aligned}$$

i.e., the $|1\rangle$ state has an apparent energy shift of

$$\begin{aligned} \Delta &= \frac{1}{2}(\delta - \Omega) \\ &\approx \frac{1}{2} \left(\delta - \delta(1 - \frac{\Omega_0^2}{2\delta^2}) \right) \\ &= \frac{\Omega_0^2}{4\delta} \end{aligned}$$

(b) With the initial conditions $c_1(0) = 0$ and $c_2(0) = 1$, focusing on c_2 we have

$$\begin{aligned} c_2(t) &= e^{i\delta t/2} \left[\cos \frac{\Omega t}{2} - i \frac{\delta}{\Omega} \sin \frac{\Omega t}{2} \right] \\ &= \frac{1}{2} e^{i\delta t/2} \left[(e^{i\Omega t/2} + e^{-i\Omega t/2}) - i \frac{\delta}{\Omega} (e^{i\Omega t/2} - e^{-i\Omega t/2}) \right] \\ &= \frac{1}{2\Omega} e^{i(\delta-\Omega)t/2} [\delta + \Omega - (\delta - \Omega)e^{i\Omega t}] \\ &= \frac{\Omega + \delta}{2\Omega} e^{i(\delta-\Omega)t/2} + O\left(\frac{\Omega_0^2}{\delta^2}\right) \end{aligned}$$

So,

$$\begin{aligned} |\psi(t)\rangle &= c_2(t) e^{-i\omega_2 t} |2\rangle \\ &\propto e^{-i(\omega_2 - (\delta - \Omega)/2)t} |2\rangle, \end{aligned}$$

and we see that the shift is the same as for the $|1\rangle$ state, just with opposite sign, i.e. $-\Omega_0^2/4\delta$.

```
In[10]:= Symbolize[ $\delta_\theta$ ]; Symbolize[ $\Omega_\theta$ ]; Symbolize[ $T_\theta$ ]; Symbolize[ $H_I$ ];
Symbolize[ $\psi_I$ ]; Symbolize[ $c_1$ ]; Symbolize[ $c_2$ ];
Symbolize[ $s_\theta$ ]; Symbolize[ $c_\theta$ ]; Symbolize[ $\omega_1$ ]; Symbolize[ $\omega_2$ ];
$Assumptions = {T > 0, t ∈ ℝ,  $\omega_1$  > 0,  $\omega_2$  > 0,  $\omega$  > 0};
```

Problem 1

```
In[ ]:=  $\Omega_\theta[t_] = A e^{-\left(\frac{t}{T}\right)^2}$ ;
R[t_] =  $\sqrt{\delta^2 + (\Omega_\theta[t])^2}$  // FullSimplify;

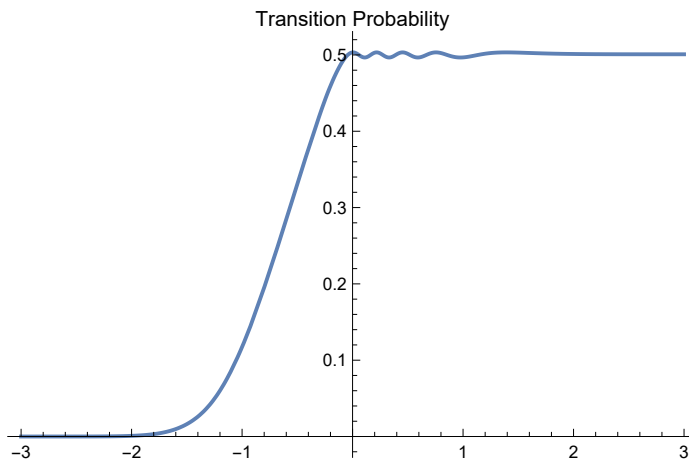
In[ ]:= Integrate[ $\Omega_\theta[t]$ , {t, -∞, t}]
Out[ ]:=  $\frac{1}{2} A \sqrt{\pi} T \left(1 + \operatorname{Erf}\left[\frac{t}{T}\right]\right)$ 
```

Problem 2

```
In[ ]:=  $\delta[t_] = \delta_\theta \left(1 - e^{\frac{t}{T}}\right)^3 \operatorname{HeavisideTheta}[-t]$ ;
 $H_I[t_] = \frac{1}{2} \left\{ \{0, \Omega_\theta[t] e^{-\frac{1}{2} \delta[t]}\}, \{\Omega_\theta[t] e^{\frac{1}{2} \delta[t]}, 0\} \right\}$ ;
 $\psi_I[t_] = \{\{c_1[t]\}, \{c_2[t]\}\}$ ;
 $T_\theta = 3$ ;
 $T = 1$ ;
 $\delta_\theta = 30$ ;
 $A = 30$ ;
soln1 =
NDSolve[{ $\frac{1}{2} D[\psi_I[t], t] = H_I[t] \cdot \psi_I[t]$ ,  $\psi_I[-T_\theta] = \{\{1\}, \{0\}\}$ }, c1, {t, -T_\theta, T_\theta}];
soln2 = NDSolve[{ $\frac{1}{2} D[\psi_I[t], t] = H_I[t] \cdot \psi_I[t]$ ,  $\psi_I[-T_\theta] = \{\{1\}, \{0\}\}$ }, c2, {t, -T_\theta, T_\theta}];
```

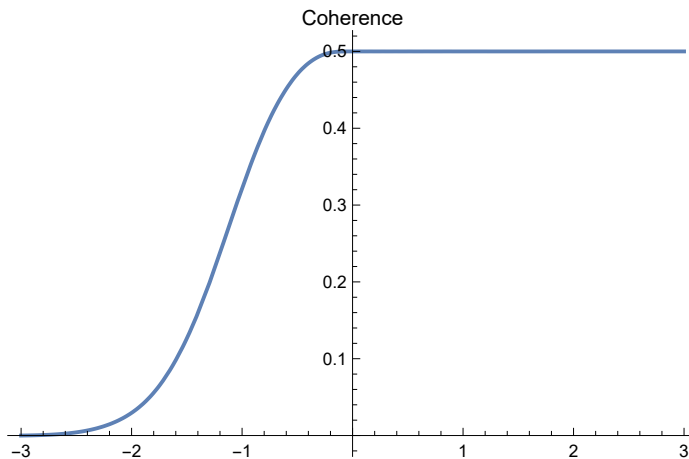
```
In[6]:= Plot[Evaluate[Abs[c2[x]]^2 /. soln2], {x, -T0, T0},
  PlotRange -> All, PlotLabel -> "Transition Probability"]
```

```
Out[6]=
```



```
In[7]:= Plot[Evaluate[Evaluate[Abs[c1[x] Conjugate[c2[x]]] /. soln1] /. soln2],
  {x, -T0, T0}, PlotRange -> All, PlotLabel -> "Coherence"]
```

```
Out[7]=
```



Problem 3

```
In[5]:= U = {{e^{-i \omega_1 t}, 0}, {0, e^{-i \omega_2 t}}};
  V = {{0, Omega Cos[\omega t]}, {Omega Cos[\omega t], 0}};
```

```
In[7]:= U^\dagger.V.U // FullSimplify // MatrixForm
```

```
Out[7]//MatrixForm=
```

$$\begin{pmatrix} 0 & e^{i t (\omega_1 - \omega_2)} \Omega \cos[t \omega] \\ e^{i t (-\omega_1 + \omega_2)} \Omega \cos[t \omega] & 0 \end{pmatrix}$$

```
In[16]:= U = {{e^{-i \omega t/2}, 0}, {0, e^{i \omega t/2}}};
          U†.V.U // FullSimplify // MatrixForm

Out[17]//MatrixForm=

$$\begin{pmatrix} 0 & e^{i t \omega} \Omega \cos[t \omega] \\ e^{-i t \omega} \Omega \cos[t \omega] & 0 \end{pmatrix}$$

```

Problem 4

```
In[*]:= R = \sqrt{\delta^2 + \Omega^2};
          \frac{-2 \frac{\Omega_0}{R+\delta}}{\left(\frac{\Omega_0}{R+\delta}\right)^2 - 1} // FullSimplify

Out[*]=

$$\frac{\Omega_0}{\delta}$$

```

Problem 5

```
In[*]:= "\Omega = \sqrt{\delta^2 + A^2};"
          c_1 = e^{-i \delta t/2} \left( \cos\left[\frac{\Omega t}{2}\right] + \frac{i \delta}{\Omega} \sin\left[\frac{\Omega t}{2}\right] \right);

Out[*]=

$$\Omega = \sqrt{\delta^2 + A^2};$$


In[*]:= c_1 // TrigToExp // Simplify

Out[*]=

$$\frac{1}{2 \Omega} e^{-\frac{1}{2} i t \delta} e^{-\frac{1}{2} i t \Omega} \left( (-1 + e^{i t \Omega}) \delta + (1 + e^{i t \Omega}) \Omega \right)$$


In[*]:= \frac{\left( (-1 + e^{i t \Omega}) \delta + (1 + e^{i t \Omega}) \Omega \right)}{2 \Omega} // FullSimplify

Out[*]=

$$\frac{-\delta + \Omega + e^{i t \Omega} (\delta + \Omega)}{2 \Omega}$$


In[*]:= c_2 = e^{i \delta t/2} \left( \cos\left[\frac{\Omega t}{2}\right] - \frac{i \delta}{\Omega} \sin\left[\frac{\Omega t}{2}\right] \right);

In[*]:= c_2 // TrigToExp // FullSimplify

Out[*]=

$$\frac{e^{\frac{1}{2} i t (\delta - \Omega)} (\delta + \Omega + e^{i t \Omega} (-\delta + \Omega))}{2 \Omega}$$

```