Homework 1

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Problem 1

We can get the electric field amplitude from the intensity, as

$$I = \frac{P}{A} = \frac{1}{2} c\epsilon_0 E_0^2 \implies E_0 = \sqrt{\frac{2P}{c\epsilon_0 A}} \approx 868 \text{ V/m}.$$

A rough but simple estimate for the dipole moment is just $ea_0 \approx 2.5$ Debye. The Rabi frequency is then

$$\Omega_0 = \frac{\mu E_0}{\hbar} \approx 70 \text{ MHz}$$

See the end of the document for a printout of the Mathematica notebook used for these calculations.

Problem 2

Under the rotating wave approximation, we neglect the counter-rotating term and get as our differential equation (neglecting bars on the 'c's)

$$\dot{c}_1 = -\frac{1}{2} i\Omega_0 e^{i\delta t} c_2$$
$$\dot{c}_2 = -\frac{1}{2} i\Omega_0 e^{-i\delta t} c_1.$$

The Rabi frequency is directly proportional to the applied electric field. In the weak-field limit, we can perturbatively expand the amplitudes as

$$c_i = c_i^{(0)} + \Omega_0 c_i^{(1)} + \Omega_0^2 c_i^{(2)} + \cdots .$$

To zero-th order, the amplitudes are given by the initial conditions $c_1^{(0)} = c_1(0) = 1$, and $c_2^{(0)} = c_2(0) = 0$. Now we go to first order and plug into the differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(c_1^{(0)} + \Omega_0 c_1^{(1)}\right) = -\frac{i}{2}\Omega_0 e^{i\delta t}\left(c_2^0 + \Omega_0 c_2^{(1)}\right)$$
$$\implies \qquad \Omega_0 \dot{c}_1^{(1)} = -\frac{i}{2}\Omega_0^2 e^{i\delta t} c_2^{(1)}$$

Matching terms proportional to Ω_0 gives

$$\begin{aligned} & \dot{c}_1^{(1)} = 0 \\ \implies \quad & c_1^{(1)} = c_1^{(1)}(0) = 0. \end{aligned}$$

For c_2, we find

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}t}\left(c_2^{(0)} + \Omega_0 c_2^{(1)}\right) = -\frac{i}{2}\Omega_0 e^{-i\delta t}\left(c_1^{(0)} + \Omega_0 c_1^{(1)}\right) \\ \implies \quad & \Omega_0 \dot{c}_2^{(1)} = -\frac{i}{2}\Omega_0 e^{-i\delta t} + O(\Omega^2) \\ \implies \quad & c_2^{(1)} = -\frac{i}{2}\int_0^t \mathrm{d}t' e^{-i\delta t'} \\ & = \frac{1}{2\delta}\left(e^{-i\delta t} - 1\right). \end{aligned}$$

So, to first order we have that

$$\left.\begin{aligned} c_1 &\approx 1 \\ c_2 &\approx \tfrac{\Omega_0}{2\delta}\left(e^{-i\delta t} - 1\right) \end{aligned}\right\} \implies \begin{aligned} |c_1|^2 &\approx 1 \\ |c_2|^2 &\approx \frac{\Omega_0^2}{2\delta^2}\left(1 - \cos\delta t\right) \end{aligned}$$

Repeating the process for second order (matching terms proportional to Ω_0^2),

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}t}\left(1 + \Omega_0^2 c_1^{(2)}\right) = -\frac{i}{2}\Omega_0 e^{i\delta t}\frac{\Omega_0}{2\delta}\left(e^{-i\delta} - 1\right) \\ \implies \quad & \dot{c}_1^{(2)} = -\frac{i}{4\delta}\left(1 - e^{i\delta t}\right) \\ \implies \quad & c_1^{(2)} = -\frac{i}{4\delta}\int_0^t \mathrm{d}t'\left(1 - e^{i\delta t}\right) \\ & = -\frac{i}{4\delta}\left[t - \frac{1}{i\delta}\left(e^{i\delta t} - 1\right)\right] \end{aligned}$$

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\Omega_0}{2\delta}\left(e^{-i\delta t} - 1\right) + \Omega_0^2 c_2^{(2)}\right) = -\frac{i}{2}\Omega_0 e^{-i\delta t}\left(1 + \Omega_0^2 c_1^{(2)}\right) \\ \implies \quad & \dot{c}_2^{(2)} = 0 \\ \implies \quad & c_2^{(2)} = 0 \end{aligned}$$

So, to second order we have

$$\begin{aligned} c_1 &\approx 1 - \frac{i\Omega_0^2}{4\delta}\left[t - \frac{1}{i\delta}\left(e^{i\delta t} - 1\right)\right] \\ c_2 &\approx \frac{\Omega_0}{2\delta}\left(e^{-i\delta t} - 1\right) \end{aligned}$$

$$\implies \begin{cases} \left|c_1\right|^2 \approx 1 - \frac{\Omega_0^2}{2\delta}\left(1-\cos\delta t\right) + \frac{\Omega_0^4}{8\delta^4}\left(1+\frac{t^2}{2\delta^2}-\cos\delta t - 2t\sin\delta t\right) \\ \left|c_2\right|^2 \approx \frac{\Omega_0^2}{2\delta^2}\left(1-\cos\delta t\right) \end{cases}$$

Now, for third order,

$$\begin{aligned} \Omega_0^3 \dot{c}_1^{(3)} &= -\frac{i}{2}\Omega_0 e^{i\delta t}\left(\frac{\Omega_0}{2\delta}\left(e^{-i\delta t}-1\right)\right) \\ \implies \qquad \dot{c}_1^{(3)} &= 0 \\ \implies \qquad \dot{c}_1^{(3)} &= 0 \end{aligned}$$

$$\begin{aligned} \Omega_0^3 \dot{c}_2^{(3)} &= -\frac{i}{2}\Omega_0 e^{-i\delta t}\left(1-\frac{i\Omega_0^2}{4\delta}\left[t-\frac{1}{i\delta}\left(e^{i\delta t}-1\right)\right]\right) \\ \implies \qquad \dot{c}_2^{(3)} &= -\frac{1}{8\delta}\left[t-\frac{1}{i\delta}\left(e^{i\delta t}-1\right)\right] \\ \implies \qquad c_2^{(3)} &= -\frac{1}{16\delta^3}\left[2\left(e^{i\delta t}-1\right)+\delta t(\delta t-2i)\right]. \end{aligned}$$

So, to third order

$$\begin{aligned} c_1 &\approx 1-\frac{i\Omega_0^2}{4\delta}\left[t-\frac{1}{i\delta}\left(e^{i\delta t}-1\right)\right] \\ c_2 &\approx \frac{\Omega_0}{2\delta}\left(e^{-i\delta t}-1\right)-\frac{\Omega_0^3}{16\delta^3}\left[2\left(e^{i\delta t}-1\right)+\delta t(\delta t-2i)\right] \end{aligned}$$

$$\implies \begin{cases} \left|c_1\right|^2 \approx 1 - \frac{\Omega_0^2}{2\delta}\left(1-\cos\delta t\right) + \frac{\Omega_0^4}{8\delta^4}\left(1+\frac{t^2}{2\delta^2}-\cos\delta t - 2t\sin\delta t\right) \\ \left|c_2\right|^2 \approx \frac{\Omega_0^2}{2\delta^2}\left(1-\cos\delta t\right) - \end{cases}$$

Problem 3

(a) We consider a two-level atom with states denoted $|1\rangle$, $|2\rangle$ and corresponding energies $E_1 = \hbar\omega_1 = -\omega_0/2$ and $E_2 = \hbar\omega_2 = \omega_0/2$. The atom interacts with a linearly polarized optical field described by

$$\vec{E} = \mathrm{Re}[E_0 e^{-i\omega t}]\ \hat{z}.$$

The interaction between the atom and the field is given to lowest order by the electric dipole interaction

$$V = -\vec{\mu}\cdot\vec{E} = ez|E_0|\cos(\omega t-\phi).$$

Assuming the two atomic states have opposite parity, the diagonal interaction matrix elements vanish:

$$V_{11} \propto \langle 1|z|1\rangle = 0 = V_{22}.$$

By a choice of phase for the wavefunction, we can take the off-diagonal elements to be real (and hence equal):

$$V_{12} = e|E_0|z_{12}\cos(\omega t-\phi)$$

The hamiltonian for the combined system is then

$$\begin{aligned} H &= H_0 + V \\ &= \frac{\hbar\omega_0}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \hbar\Omega_0 \cos(\omega t - \phi)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= -\frac{\hbar\omega_0}{2}\sigma_z + \hbar\Omega_0\cos(\omega t - \phi)\sigma_x, \end{aligned}$$

where we define the Rabi frequency $\Omega_0 = \frac{ez_{12}E_0}{\hbar}$.

In the Schrödinger representation, the state of the atom is described by

$$|\psi(t)\rangle_{\mathrm{S}} = c_1(t)\,|1\rangle + c_2(t)\,|2\rangle \quad \left(\left|c_1\right|^2 + \left|c_2\right|^2 = 1\right).$$

In the absence of the external field, the state would evolve as

$$|\psi(t)\rangle_{\mathrm{S}} = c_1(0)e^{-i\omega_1 t}\,|1\rangle + c_2(0)e^{-i\omega_2 t}\,|2\rangle\,.$$

In the interaction representation, we factor out this free phase evolution by writing

$$|\psi(t)\rangle_{\mathrm{I}} = \bar{c}_1(t)e^{-i\omega_1 t}\,|1\rangle + \bar{c}_2(t)e^{-i\omega_2 t}\,|2\rangle\,,$$

that is, we make the (time-dependent) unitary transformation

$$|\psi(t)\rangle_{\mathrm{S}} \to |\psi(t)\rangle_{\mathrm{I}} = U(t)\,|\psi(t)\rangle_{\mathrm{S}}\,,$$

where

$$U(t) = \begin{pmatrix} e^{-i\omega_1 t} & 0 \\ 0 & e^{-i\omega_2 t} \end{pmatrix}.$$

We get the effective interaction hamiltonian by making the inverse transformation on V:

$$V_{\mathrm{I}} = U^\dagger V U = \hbar\Omega_0\cos(\omega t)\begin{pmatrix} 0 & e^{-i\omega_0 t} \\ e^{i\omega_0 t} & 0 \end{pmatrix},$$

where the phase ϕ has been absorbed into the (complex) Rabi frequency.

(b) To make the rotating wave approximation, we re-write the effective hamiltonian as

$$\begin{aligned} V_{\mathrm{I}} &= \frac{1}{2}\hbar\Omega_0\left(e^{i\omega t} + e^{-i\omega t}\right)e^{i\omega_0 t}\sigma_+ + \text{h.c.} \\ &= \frac{1}{2}\hbar\Omega_0\left(e^{i(\omega+\omega_0)t} + e^{-i(\omega-\omega_0)t}\right)\sigma_+ + \text{h.c.} \end{aligned}$$

where σ_+ is the raising operator $|2\rangle\langle 1|$. Making the RWA amounts to neglecting the counter-rotating term (i.e. the term with $\omega_0 + \omega$), leaving

$$V_{\mathrm{I}} \approx \frac{1}{2}\hbar\Omega_0 e^{i\delta t}\sigma_+ + \text{h.c.},$$

where we have switched to using the detuning $\delta = \omega_0 - \omega$.

(c) The Bloch Siegert shift effectively decreases $\omega_2 - \omega_1 = \omega_0$, and therefore becomes relevant when the level spacing is already small, such as in magnetic field interactions.

Problem 4

Making the simplifying assumptions of a constant Rabi frequency and zero detuning, we have that

$$\left.\begin{aligned}\dot{c}_1 &= -\tfrac{i}{2}\Omega_0 c_2 - \tfrac{\gamma_1}{2}c_1\\ \dot{c}_2 &= -\tfrac{i}{2}\Omega_0 c_1 - \tfrac{\gamma_2}{2}\end{aligned}\right\} \implies \begin{pmatrix}\dot{c}_1\\ \dot{c}_2\end{pmatrix} = -\frac{1}{2}\begin{pmatrix}\gamma_1 & i\Omega_0\\ i\Omega_0 & \gamma_2\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix},$$

or, more simply,

$$\dot{\psi} = M\psi; \qquad M = -\frac{1}{2}\begin{pmatrix}\gamma_1 & i\Omega_0\\ i\Omega_0 & \gamma_2\end{pmatrix}.$$

For $c_1(0) = 1,\ c_2(0) = 0$, the solution to this first order differential equation is

$$\begin{aligned}\psi(t) &= e^{Mt}\psi(0)\\ &= \frac{1}{2\chi}\begin{pmatrix}e^{-(\gamma_1+\gamma_2+\chi)t}\left[\chi\left(e^{\chi t/2}+1\right)+(\gamma_1+\gamma_2)\left(e^{\chi t/2}-1\right)\right]\\ -2ie^{-(\gamma_1+\gamma_2+\chi)/4}\left(e^{\chi t/2}-1\right)\end{pmatrix}\end{aligned}$$

Problem 1

```
In[1]:= P = 1 mW ; A = 1 mm^2 ;

In[2]:= μ = e a0 ; UnitConvert[μ, "Debyes"]

Out[2]= 2.541746473 D

In[3]:= E0 = Sqrt[(2 P)/(c ε0 A)] ; UnitConvert[E0, "V/m"]

Out[3]= 868.021098 V/m

In[4]:= Ω = (μ E0)/ℏ ; UnitConvert[Ω, "MHz"]

Out[4]= 69.7855727 MHz
```

Problem 2

```
In[5]:= Clear[Ω];
        c1 = {1, 0, 0, 0};
        c2 = {0, 0, 0, 0};

In[8]:= For[j = 2, j < 5, j++,
          c1[[j]] = Integrate[(-ⅈ)/2 ⅇ^(ⅈ δ t) c2[[j - 1]], {t, 0, t}] // FullSimplify ;
          c2[[j]] = Integrate[(-ⅈ)/2 ⅇ^(-ⅈ δ t) c1[[j - 1]], {t, 0, t}] // FullSimplify;
        ]

In[9]:= combine[order_, coeff_] = Sum[Indexed[coeff, j] Ω0^(j-1), {j, 1, order + 1}];
        sqrMag[z_] := FullSimplify[ComplexExpand[Abs[z]^2]];
        showOrder[order_] := StringForm["Order ``: \n
         c1 ~ `` ⇒ |c1|^2 ~ `` \n
         c2 ~ `` ⇒ |c2|^2 ~ ``\n\n",
           order, combine[order, c1], sqrMag[combine[order, c1]],
           combine[order, c2], sqrMag[combine[order, c2]]];
```

In[12]:=
```
Column[Table[showOrder[i], {i, 0, 3}]]
```

Out[12]=

Order 0:

$$c_1 \sim 1 \Rightarrow |c_1|^2 \sim 1$$

$$c_2 \sim 0 \Rightarrow |c_2|^2 \sim 0$$

Order 1:

$$c_1 \sim 1 \Rightarrow |c_1|^2 \sim 1$$

$$c_2 \sim \frac{\left(-1+e^{-i t \delta}\right) \Omega_0}{2 \delta} \Rightarrow |c_2|^2 \sim \frac{\text{Sin}\left[\frac{t \delta}{2}\right]^2 \Omega_0^2}{\delta^2}$$

Order 2:

$$c_1 \sim 1+\frac{\left(-1+e^{i t \delta}-i t \delta\right) \Omega_0^2}{4 \delta^2} \Rightarrow |c_1|^2 \sim 1+\frac{8 \delta^2 (-1+\text{Cos}[t \delta]) \Omega_0^2+\left(2+t^2 \delta^2-2 \text{Cos}[t \delta]-2 t \delta \text{Sin}[t \delta]\right) \Omega_0^4}{16 \delta^4}$$

$$c_2 \sim \frac{\left(-1+e^{-i t \delta}\right) \Omega_0}{2 \delta} \Rightarrow |c_2|^2 \sim \frac{\text{Sin}\left[\frac{t \delta}{2}\right]^2 \Omega_0^2}{\delta^2}$$

Order 3:

$$c_1 \sim 1+\frac{\left(-1+e^{i t \delta}-i t \delta\right) \Omega_0^2}{4 \delta^2} \Rightarrow |c_1|^2 \sim 1+\frac{8 \delta^2 (-1+\text{Cos}[t \delta]) \Omega_0^2+\left(2+t^2 \delta^2-2 \text{Cos}[t \delta]-2 t \delta \text{Sin}[t \delta]\right) \Omega_0^4}{16 \delta^4}$$

$$c_2 \sim \frac{\left(-1+e^{-i t \delta}\right) \Omega_0}{2 \delta}+\frac{\left(2-i t \delta+e^{-i t \delta}(-2-i t \delta)\right) \Omega_0^3}{8 \delta^3} \Rightarrow |c_2|^2 \sim \frac{\left(t \delta \text{Cos}\left[\frac{t \delta}{2}\right] \Omega_0^3-2 \text{Sin}\left[\frac{t \delta}{2}\right] \Omega_0\left(-2 \delta^2+\Omega_0^2\right)\right)^2}{16 \delta^6}$$

Problem 3

In[13]:=
$$\sigma_x = \text{PauliMatrix}[1];\ \sigma_z = \text{PauliMatrix}[3];$$

In[14]:=
$$H_0 = -\frac{\hbar\, \omega_0}{2} \sigma_z;$$
$$V = \hbar \Omega_0 \text{Cos}[\omega t - \phi]\, \sigma_x;$$
$$U = \text{MatrixExp}[-i H_0 t / \hbar];$$

In[17]:=
```
V_I = U†.V.U // ComplexExpand // TrigToExp // FullSimplify;
V_I // MatrixForm
```

Out[18]//MatrixForm=

$$\begin{pmatrix} 0 & e^{-i t \omega_0} \mathrm{Cos}[\phi - t\,\omega]\,\hbar\Omega_0 \\ e^{i t \omega_0} \mathrm{Cos}[\phi - t\,\omega]\,\hbar\Omega_0 & 0 \end{pmatrix}$$

Problem 4

In[19]:=
```
M = -1/2 {{γ1, i Ω0}, {i Ω0, γ2}};
ψ0 = {{1}, {0}};
ψ[t_] = {a1[t], a2[t]};
```

In[22]:=
```
soln = MatrixExp[M t].ψ0 // FullSimplify;
soln = (Numerator[soln] /. {Sqrt[(γ1 - γ2)^2 - 4 Ω0^2] → χ}) /
   (Denominator[soln] /. {Sqrt[(γ1 - γ2)^2 - 4 Ω0^2] → χ});
StringForm["c1 = `` \nc2 = `` \n|c2|^2 = ``", soln[[1]][[1]] // FullSimplify,
 soln[[2]][[1]] // FullSimplify, sqrMag[soln[[2]][[1]]] // FullSimplify]
```

Out[24]=

$$c_1 = \frac{e^{-\frac{1}{4} t (\chi+\gamma_1+\gamma_2)} \left(\chi + \gamma_1 - \gamma_2 + e^{\frac{t\chi}{2}} (\chi - \gamma_1 + \gamma_2)\right)}{2\chi}$$

$$c_2 = -\frac{i\, e^{-\frac{1}{4} t (\chi+\gamma_1+\gamma_2)} \left(-1 + e^{\frac{t\chi}{2}}\right) \Omega_0}{\chi}$$

$$|c_2|^2 = \frac{\left(1 + e^{t\chi} - 2\sqrt{e^{t\chi}}\right) \Omega_0^2}{\sqrt{e^{t(\chi+\gamma_1+\gamma_2)}}\ \chi^2}$$

In[25]:=
```
soln = soln /. {γ1 → γ, γ2 → γ, χ → 2 i Ω0};
StringForm["c1 = `` \nc2 = `` \n|c2|^2 = ``", soln[[1]][[1]] // FullSimplify,
 soln[[2]][[1]] // FullSimplify, sqrMag[soln[[2]][[1]]] // FullSimplify]
```

Out[26]=

$$c_1 = e^{-\frac{t\gamma}{2}} \mathrm{Cos}\left[\frac{t\,\Omega_0}{2}\right]$$

$$c_2 = -i\, e^{-\frac{t\gamma}{2}} \mathrm{Sin}\left[\frac{t\,\Omega_0}{2}\right]$$

$$|c_2|^2 = \frac{\mathrm{Sin}\left[\frac{t\,\Omega_0}{2}\right]^2}{\sqrt{e^{2t\gamma}}}$$