All About Spinors

...on *flat* spacetime

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UO

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Some Philisophical Motivation

Special and General Covariance

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- Special/General covariance $\to$ Special/General relativity

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- Bargmann ('54): reps up to sign are *exactly* the true reps of the universal cover

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- In fact, $\mathcal{U}(ISO(3,1)^+) \cong ISL(2,\mathbb{C})$

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  - $\to$ use $\epsilon^{A}{}_{B}$, $\epsilon_{C}{}^{D}$ and their conjugates to avoid confusion

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  - Form the *Tensor algebra* $T(\mathfrak{g}) \coloneqq \mathbb{C} \oplus \mathfrak{g} \oplus (\mathfrak{g} \otimes \mathfrak{g}) \cdots$
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  - Fully lifted, $[,]$ obeys Leibniz's law: $T(\mathfrak{g})$ is a *poisson* algebra
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- Physically relevant cases are 1 and 2.1

Representations and Hilbert Space

- Specify representatives for class 1 via

$$()$$

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