Homework 2

Sean Ericson
Phys 622

January 25, 2023

0.2.4 **Functional derivative**

(a) (i)

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int \mathrm{d}y\ \left[\phi(y)+\epsilon\delta(y-x)-\phi(y)\right] = \int \mathrm{d}y\ \delta(y-x) = \boxed{1}$$

(ii)

$$\begin{aligned}
&\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int \mathrm{d}y\ \left[\phi^2(y)+2\epsilon\delta(y-x)+\epsilon^2\delta^2(y-x)-\phi^2(y)\right] \\
&= \int \mathrm{d}y\ 2\phi(y)\delta(y-x) \\
&= \boxed{2\phi(x)}
\end{aligned}$$

(iii)

$$\begin{aligned}
&\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int \mathrm{d}y\ \left[f(\phi(y)+\epsilon\delta(y-x))g(\phi(y)+\epsilon\delta(y-x)) - f(\phi(y))g(\phi(y))\right] \\
&= \lim_{\epsilon\to 0}\frac{1}{\epsilon}\int \mathrm{d}y\ \left[\left(f(\phi(y))+\epsilon\delta(y-x)\frac{\mathrm{d}f}{\mathrm{d}\phi}|_{\phi(y)}\right)\left(g(\phi(y))+\epsilon\delta(y-x)\frac{\mathrm{d}g}{\mathrm{d}\phi}|_{\phi(y)}\right) - f(\phi(y))g(\phi(y))\right] \\
&= \int \mathrm{d}y\ \left[\delta(y-x)f(\phi(y))\frac{\mathrm{d}g}{\mathrm{d}\phi}|_{\phi(y)} + \delta(y-x)g(\phi(y))\frac{\mathrm{d}f}{\mathrm{d}\phi}|_{\phi(y)}\right] \\
&= \boxed{f(\phi(x))\frac{\mathrm{d}g}{\mathrm{d}\phi}|_{\phi(x)} + g(\phi(x))\frac{\mathrm{d}f}{\mathrm{d}\phi}|_{\phi(x)}}
\end{aligned}$$

(iv)

$$\begin{aligned}
&\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int \mathrm{d}y\ \left[\left(\frac{\mathrm{d}}{\mathrm{d}y}\left(\phi(y)+\epsilon\delta(y-x)\right)\right)^2 - \left(\frac{\mathrm{d}\phi}{\mathrm{d}y}\right)^2\right] \\
&= \int \mathrm{d}y\ 2\phi'(y)\delta'(y-x) \\
&= \boxed{-2\phi''(x)}
\end{aligned}$$

(v)

$$\begin{aligned}
&\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int \mathrm{d}y\ \left[V(\phi'(y)-\epsilon\delta'(y-x))-V(\phi'(y))\right]\\
&=\int \mathrm{d}y\ \frac{\mathrm{d}V}{\mathrm{d}\phi'}|_{\phi'(y)}\delta'(y-x)\\
&=\boxed{-\frac{\mathrm{d}}{\mathrm{d}x}\frac{\mathrm{d}V}{\mathrm{d}\phi'}|_{\phi'(x)}}
\end{aligned}$$

(b) We have that

$$\begin{aligned}
\delta S &= \int \mathrm{d}^4x\ \delta\mathscr{L}\\
&=\int \mathrm{d}^4x\ \frac{\partial\mathscr{L}}{\partial\phi}\delta\phi+\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\delta(\partial_\mu\phi)\\
&=\int \mathrm{d}^4x\ \left(\frac{\partial\mathscr{L}}{\partial\phi}-\partial_\mu\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\right)\delta\phi.
\end{aligned}$$

Thus,

$$\delta S=0 \implies \boxed{\frac{\partial\mathscr{L}}{\partial\phi}-\partial_\mu\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}=0.}$$

Or, using the above definition of the functional derivative,

$$\begin{aligned}
&\frac{\delta}{\delta\phi(x)}\int \mathrm{d}^4y\ \mathscr{L}\\
&=\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int \mathrm{d}^4y\ \mathscr{L}(\phi(y)+\epsilon\delta^4(y-x),\partial_\mu\phi(y)+\epsilon\partial_\mu\delta^4(y-x))-\mathscr{L}(\phi(y),\partial_\mu\phi(y))\\
&=\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int \mathrm{d}^4y\ \mathscr{L}(\phi,\partial_\mu\phi)+\epsilon\frac{\partial\mathscr{L}}{\partial\phi}\delta^4(y-x)+\epsilon\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\partial_\mu\delta^4(y-x)-\mathscr{L}(\phi,\partial_\mu\phi)\\
&=\int \mathrm{d}^4y\ \frac{\partial\mathscr{L}}{\partial\phi}\delta^4(y-x)+\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\partial_\mu\delta^4(y-x)\\
&=\int \mathrm{d}^4y\ \frac{\partial\mathscr{L}}{\partial\phi}\delta^4(y-x)-\partial_\mu\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\delta^4(y-x)\\
&=\frac{\partial\mathscr{L}}{\partial\phi(x)}-\partial_\mu\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi(x))}.
\end{aligned}$$

Thus,

$$\delta S=0 \implies \boxed{\frac{\partial\mathscr{L}}{\partial\phi(x)}-\partial_\mu\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi(x))}=0.}$$

0.2.5 **Massive scalar field**

(a) Given the derivatives

$$\frac{\partial\mathscr{L}}{\partial\phi}=-\frac{m^2}{2}\phi^*;\quad \frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}=\frac{1}{2}\partial^\mu\phi^*,$$

$$\frac{\partial \mathscr{L}}{\partial \phi^*} = -\frac{m^2}{2}\phi; \quad \frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi^*)} = \frac{1}{2}\partial^\mu \phi,$$

the Euler-Lagrange equations are simply

$$\boxed{-m^2\phi^* = \partial^2\phi^*; \quad -m^2\phi = \partial^2\phi}$$

(b) Nope, this lagrangian is not invariant under a local $U(1)$ transformation:

$$\begin{aligned}
\mathscr{L} \to & \frac{1}{2}\left(\partial_\mu\left(\phi e^{i\Lambda}\right)\right)\left(\partial^\mu\left(\phi^* e^{-i\Lambda}\right)\right) - \frac{m^2}{2}\left(\phi e^{i\Lambda}\right)\left(\phi^* e^{-i\Lambda}\right) \\
&= \frac{1}{2}\left(\left(\partial_\mu\phi\right)e^{i\Lambda} + \left(\partial_\mu e^{i\Lambda}\right)\phi\right)\left(\left(\partial^\mu\phi^*\right)e^{-i\Lambda} + \left(\partial^\mu e^{-i\Lambda}\right)\phi^*\right) - \frac{m^2}{2}|\phi|^2 \\
&= \frac{1}{2}\left[\partial_\mu\phi\partial^\mu\phi^* - i\left(\partial_\mu\phi\right)\left(\partial^\mu\Lambda\right)\phi^* + i\left(\partial^\mu\phi^*\right)\left(\partial_\mu\Lambda\right) + \partial_\mu\Lambda\partial^\mu\Lambda\right] - \frac{m^2}{2}|\phi|^2 \\
&\neq \frac{1}{2}\partial_\mu\phi\partial^\mu\phi^* - \frac{m^2}{2}|\phi|^2
\end{aligned}$$

0.2.6 Particle in homogeneous E and B fields

(a) The force (and hence acceleration) is given by

$$\vec{F} = e\left(\vec{E} + \frac{\vec{v}}{c}\times\vec{B}\right) = e\begin{pmatrix} \frac{B}{c}\dot{y} \\ E_y - \frac{B}{c}\dot{x} \\ E_z \end{pmatrix}$$

$$\implies \begin{pmatrix} \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{pmatrix} = \frac{e}{m}\begin{pmatrix} \frac{B}{c}\dot{y} \\ E_y - \frac{B}{c}\dot{x} \\ E_z \end{pmatrix} = \begin{pmatrix} \omega\dot{y} \\ \frac{e}{m}E_y - \omega\dot{x} \\ \frac{e}{m}E_z \end{pmatrix}$$

Looking at just the z components, we see that the motion in that direction is uncoupled from the other two:

$$\ddot{z} = \frac{e}{m}E_z$$

(b) From the above expression for the acceleration, we can see that

$$\ddot{\xi} = \ddot{x} + i\ddot{y} = \omega\dot{y} + i\left(\frac{e}{m}E_y - \omega\dot{x}\right) = -i\omega\dot{\xi} + i\frac{e}{m}E_y$$

giving a inhomogeneous ODE for ξ of the form

$$\ddot{\xi} + i\omega\dot{\xi} - \frac{e}{m}E_y = 0,$$

with solution

$$\dot{\xi}(t) = Ae^{i\omega t} + \frac{c}{B}E_y \implies \xi(t) = A'e^{i\omega t} + \frac{c}{B}E_y t + B$$

where A, A', and B are complex constants defined by the initial conditions.

(c) We have that

$$\langle \vec{v} \rangle = c\vec{E} \times \vec{B}/B^2 = c\frac{E_y}{B}\hat{x}.$$

Is this really the time-averaged velocity? Let's calculate it:

$$\begin{aligned}
\langle v_x \rangle &= \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \mathrm{d}t\,\mathrm{Re}\Big\{\dot{\xi}\Big\} \\
&= \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \mathrm{d}t\,\left[A\cos(\omega t) + c\frac{E_y}{B}\right] \\
&= \frac{\omega}{2\pi}\frac{2\pi}{\omega}c\frac{E_y}{B} \\
&= c\frac{E_y}{B} \\
\langle v_y \rangle &= \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \mathrm{d}t\,\mathrm{Im}\Big\{\dot{\xi}\Big\} \\
&\propto \frac{\omega}{2\pi}\int_0^{2\pi/\omega} \sin(\omega t) \\
&= 0
\end{aligned}$$

It is! the time-averaged velocity in the $x-y$ plane is in the x-direction, with magnitude cE_y/B. If we rearrange this slightly, we see

$$\frac{\langle \vec{v} \rangle}{c} = \frac{E_y}{B},$$

so

$$\frac{E_y}{B} \ll 1 \implies \frac{|\vec{v}|}{c} \ll 1$$

and hence the non-relativisitc approximation is valid.

(d) The three qualitatively different trajectories depend on the value of E_y relative to B, as seen in Figure 1.

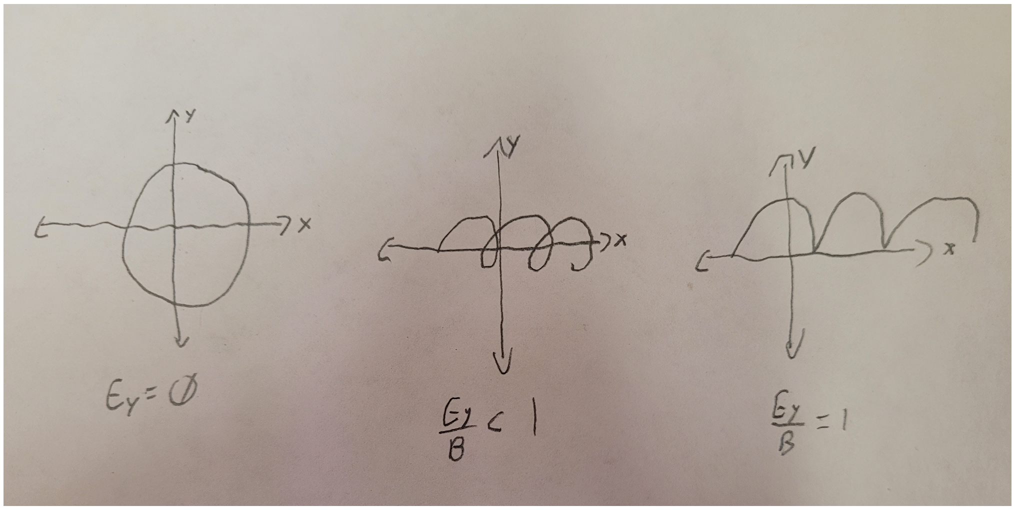

Figure 1: The three qualitatively differnt trajectories.

0.2.7 **Harmonic oscillator coupled to a magnetic field**

The force is given by

$$\begin{aligned}\vec{F} &= -k\vec{r} + \frac{e}{c}\vec{v} \times \vec{B} \\ &= -m\omega_0^2\vec{r} + \frac{eB}{c}\dot{y}\hat{x} + \frac{eB}{c}\dot{x}\hat{y} \\ &= \begin{pmatrix} -m\omega_0^2 x + m\omega_1\dot{y} \\ -m\omega_0^2 y - m\omega_1\dot{x} \\ -m\omega_0^2 z \end{pmatrix}\end{aligned}$$

where

$$\omega_1 = \frac{eB}{mc}.$$

As in the last problem, we can see that motion in the z-direction (parallel to $\vec{B}$) decouples from motion in the $x - y$ plane, as

$$F_z = -m\omega_0^2 z.$$

We can also see that motion is this direction is just that of a 1-dimensional harmonic oscillator with frequency ω_0.
Also like the last problem, we can define

$$\xi = x + iy$$

to help us analyze motion in the $x - y$ plane (the plane perpendicular to $\vec{B}$). From the force above, we can see that the acceleration is

$$\begin{pmatrix} \ddot{x} \\ \ddot{y} \end{pmatrix} = \begin{pmatrix} \omega_1\dot{y} - \omega_0^2 x \\ -\omega_1\dot{x} - \omega_0^2 y \end{pmatrix}.$$

In terms of ξ, this is

$$\ddot{\xi} = -i\omega_1\dot{\xi} - \omega_0^2\xi \implies \ddot{\xi} + i\omega_1\dot{\xi} + \omega_0^2\xi = 0.$$

This is another second-order ODE for ξ, with general solution

$$\xi(t) = \mathrm{Re}[Ae^{i\tilde{\omega}t}],$$

where A is some complex constant determined by the initial conditions, and

$$\tilde{\omega} = \frac{1}{2}\left(\sqrt{\omega_1^2 + 4\omega_0^2} - \omega_1\right)$$

is the frequency of oscillation in the plane.