Homework 2

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Phys 612

January 25, 2023

Problem 1

The variation of the action is given by

$$
\begin{aligned}
\delta S &= \int_{t_1}^{t_2} \mathrm{d}t\ \left[p_\sigma \delta \dot{q}_\sigma + \dot{q}_\sigma \delta p_\sigma - \frac{\partial H}{\partial q_\sigma}\delta q_\sigma - \frac{\partial H}{\partial p_\sigma}\delta p_\sigma\right] \\
&= \int_{t_1}^{t_2} \mathrm{d}t\ \left[-\dot{p}_\sigma \delta q_\sigma + \dot{q}_\sigma \delta p_\sigma - \frac{\partial H}{\partial q_\sigma}\delta q_\sigma - \frac{\partial H}{\partial p_\sigma}\delta p_\sigma\right] \\
&= \int_{t_1}^{t_2} \mathrm{d}t\ \left[\dot{q}_\sigma \delta p_\sigma - \frac{\partial H}{\partial p_\sigma}\delta p_\sigma - \dot{p}_\sigma \delta q_\sigma - \frac{\partial H}{\partial q_\sigma}\delta q_\sigma\right] \\
&= \int_{t_1}^{t_2} \mathrm{d}t\ \left[\left(\dot{q}_\sigma - \frac{\partial H}{\partial p_\sigma}\right)\delta p_\sigma - \left(\dot{p}_\sigma + \frac{\partial H}{\partial q_\sigma}\right)\delta q_\sigma\right],
\end{aligned}
$$

where the assumption that $\delta q_\sigma(t_1) = \delta q_\sigma(t_2) = 0$ was used to integrate by parts in the second line. Demanding stationary action, we find

$$
\delta S = 0 \implies \begin{matrix} \dot{q}_\sigma - \frac{\partial H}{\partial p_\sigma} = 0 \\ \dot{p}_\sigma + \frac{\partial H}{\partial q_\sigma} = 0 \end{matrix} \implies \begin{matrix} \dot{q}_\sigma = \frac{\partial H}{\partial p_\sigma} \\ \dot{p}_\sigma = -\frac{\partial H}{\partial q_\sigma} \end{matrix}
$$

This is $2n$ conditions for the n coordinates and n momenta.

Problem 2

The Poisson bracket is defined by

$$
\{f, g\} = \sum_i \left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right),
$$

and satisfies the following properties:

$$
\begin{gathered}
\{f, g\} = -\{g, f\}, \\
\{\alpha f + \beta g, h\} = \alpha\{f, h\} + \beta\{g, h\}, \\
\{fg, h\} = f\{g, h\} + g\{f, h\}, \\
\{f, \{g, h\}\} + \{h, \{f, g\}\} + \{g, \{h, f\}\} = 0.
\end{gathered}
$$

i. For a function $F(q,p,t)$ to be a constant of motion, it's time derivative should vanish (subject to the equations of motion). The full time derivative is given by

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}F(q,p,t) &= \frac{\partial F}{\partial q}\frac{\mathrm{d}q}{\mathrm{d}t} + \frac{\partial F}{\partial p}\frac{\mathrm{d}p}{\mathrm{d}t} + \frac{\partial F}{\partial t} \\
&= \frac{\partial F}{\partial q}\frac{\partial H}{\partial q} - \frac{\partial F}{\partial p}\frac{\partial H}{\partial p} + \frac{\partial F}{\partial t} \\
&= \{F,H\} + \frac{\partial F}{\partial t}.
\end{aligned}$$

The condition for being a constant of the motion is then

$$\boxed{\{F,H\} + \frac{\partial F}{\partial t} = 0.}$$

If F does not explicitly depend on time, this is simply

$$\boxed{\{F,H\} = 0.}$$

ii. Since $\{F,G\}$ is just some function of p and q, it's time derivative is given by it's poissonbracket with the hamiltonian:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\{F,G\} &= \{\{F,G\},H\} \\
&= \{G,\{H,F\}\} + \{F,\{G,H\}\} \\
&= \{G,0\} + \{F,0\} \\
&= 0.
\end{aligned}$$

Above, the Jaboci identity was used going from the first to the second line, while the fact that F and G are constants of motion was used to go from the second to the third line.

iii. We have that

$$L_i = \epsilon_{ijk} r_j p_k,$$

and

$$\epsilon_{ijk}\epsilon_{lmk} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}.$$

Thus

- $$\begin{aligned}
\{L_i, L_j\} &= \epsilon_{iab}\epsilon_{jcd}\{r_a p_b, r_c p_d\} \\
&= \epsilon_{iab}\epsilon_{jcd}\left[r_a r_c\{p_b, p_d\} + r_a\{p_b, r_c\}p_d + r_c\{r_a, p_d\}r_b + \{r_a, r_c\}r_a p_b\right] \\
&= \epsilon_{iab}\epsilon_{jcd}\left[r_a p_d\{p_b, r_c\} + r_c p_b\{r_a, p_d\}\right] \\
&= \epsilon_{iab}\epsilon_{jcd}\left[-r_a p_d \delta_{bc} + r_c p_b \delta_{cd}\right] \\
&= -\epsilon_{iab}\epsilon_{jbd} r_a p_d + \epsilon_{iab}\epsilon_{jca} r_c p_b \\
&= -\left(\delta_{id}\delta_{aj} - \delta_{ij}\delta_{ad}\right) r_a p_d + \left(\delta_{bj}\delta_{ic} - \delta_{bc}\delta_{ij}\right) r_c p_b \\
&= -\delta_{id}\delta_{aj} r_a p_d + \delta_{bj}\delta_{ic} r_c p_b \\
&= -r_j p_i + r_i p_j \\
&= \epsilon_{ijk} r_i p_j \\
&= L_k
\end{aligned}$$

- $$\begin{aligned}
\left\{L^2, L_i\right\} &= \left\{L_x^2 + L_y^2 + L_z^2, L_i\right\} \\
&= 2L_x\{L_x, L_i\} + 2L_y\{L_y, L_i\} + 2L_z\{L_z, L_i\} \\
&= -2L_x\epsilon_{ixa}L_a - 2L_y\epsilon_{iyb}L_b - 2L_z\epsilon_{izc}L_c \\
&= -2L_x\left(\delta_{iz}L_y - \delta_{iy}L_z\right) - 2L_y\left(\delta_{ix}L_z - \delta i z L_x\right) - 2L_z\left(\delta_{iy}L_x - \delta_{ix}L_y\right) \\
&= -2\left(\delta_{iz}L_x L_y - \delta_{iy}L_x L_z + \delta_{ix}L_y L_z - \delta_{iz}L_y L_x + \delta_{iy}L_z L_x - \delta_{ix}L_z L_y\right) \\
&= -2\left[\delta_{ix}\left(L_y L_z - L_z L_y\right) + \delta_{iy}\left(L_z L_x - L_x L_z\right) + \delta_{iz}\left(L_x L_y - L_y L_x\right)\right] \\
&= 0
\end{aligned}$$

iv. Nope! The Ls could *not* serve as a set of momenta, because they don't satisfy the cannonical Poisson bracket $\{p_i, p_j\} = 0$.

Problem 3

i. The conjugate momentum is
$$p = \frac{\partial L}{\partial \dot{q}} = m\dot{q}.$$
The hamiltonian is then
$$\begin{aligned}
H &= p\dot{q} - L \\
&= \frac{p^2}{m} - \frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2 \\
&= \boxed{\frac{p^2}{2m} + \frac{1}{2}m\omega^2 q^2}
\end{aligned}$$
Hamilton's equations then give
$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m}$$

$$\dot{p} = -\frac{\partial H}{\partial q} = -m\omega^2 q.$$

The second equation above is a first order ODE for $q(t)$,

$$m\dot{q} + m\omega^2 q = 0,$$

with general solution

$$\boxed{q(t) = \mathrm{Re}[Ae^{i\omega t}]}$$

where A is some complex number with $\mathrm{Re}\, A = q(0)$. The equation for p is then

$$\boxed{p(t) = \mathrm{Re}[im\omega Ae^{i\omega t}]},$$

fixing the imaginary part of A as $\mathrm{Im}\, A = -im\omega p(0)$.
The resulting trajectories in phase space are just circles and ellipses, as seen in Figure 1.

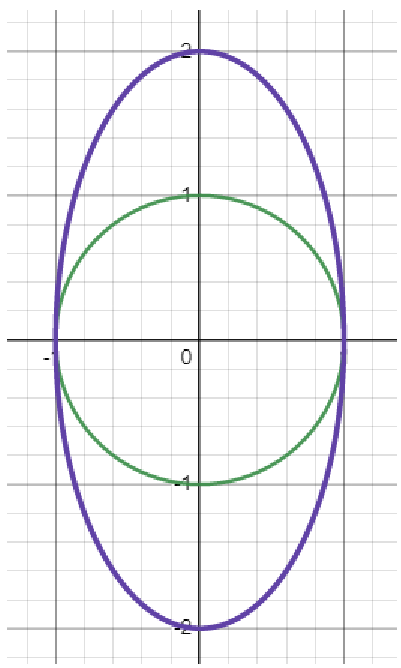

Figure 1: Two sample trajectories in (q, p) space.

ii. Let's go the easy way first:

$$\begin{aligned}
\{q, p\}_{(Q,P)} &= \frac{\partial q}{\partial Q}\frac{\partial p}{\partial P} - \frac{\partial q}{\partial P}\frac{\partial p}{\partial Q} \\
&= \left(\sqrt{\frac{2P}{m\omega}}\cos Q\right)\left(\sqrt{\frac{m\omega}{2P}}\cos Q\right) - \left(\frac{1}{\sqrt{2Pm\omega}}\sin Q\right)\left(-\sqrt{2Pm\omega}\sin Q\right) \\
&= \cos^2 Q + \sin^2 Q \\
&= 1
\end{aligned}$$

Now the other way! When we invert the transformation, we find

$$Q = \tan^{-1}(\frac{m\omega q}{p}); \quad P = \frac{p^2}{2m\omega} + \frac{1}{2}m\omega q^2.$$

Now,

$$\begin{aligned}
\{Q, P\}_{(q,p)} &= \frac{\partial Q}{\partial q}\frac{\partial P}{\partial p} - \frac{\partial Q}{\partial p}\frac{\partial P}{\partial q} \\
&= \left(\frac{m\omega/p}{\left(\frac{m\omega q}{p}\right)^2 + 1}\right)\left(\frac{p}{m\omega}\right) - \left(-\frac{m\omega q}{(m\omega q)^2 + p^2}\right)(m\omega) \\
&= \frac{p^2}{m^2\omega^2 q^2 + p^2} + \frac{m^2\omega^2 q^2}{m^2\omega^2 q^2 + p^2} \\
&= 1
\end{aligned}$$

iii. In the new coordinates, the hamiltonian is simply

$$H = \omega P \cos^2 Q + \omega P \sin^2 Q = \omega P.$$

Hamilton's equations then give

$$\dot{P} = 0; \quad \dot{Q} = \omega,$$

with general solutions

$$P(t) = P_0; \quad Q(t) = \omega t + Q_0$$

The resulting trajectories in phase space are just horizontal lines, as seen in Figure 2.

Figure 2: Two sample trajectories in (Q, P) space.