$\mathscr{T}$: The Chronological "Operator"

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UO

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What is $\mathscr{T}$? A few different names...

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The Chronological Operator

> "...reorders its argument such that the times are in increasing order from right to left." -Steck, QM

The Question: What is it?

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- and $\theta(t)$ is the Heaviside step function.

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- For our purposes, we can more generally just consider operators to be functions between vector spaces.

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  - Even though it *looks* like one!

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- The Chronological Operator is just a special case of the "path-ordering" operator, and so is a meta-operator.

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but

$$\begin{aligned}\mathscr{T}\{\sigma_3(t_3)\sigma_4(t_4)\} &= \sigma_4(t_4)\sigma_3(t_3) = B \\ &= \mathscr{T}\{A\} = A\end{aligned}$$

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