Homework 1

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Problem 1

We can get the electric field amplitude from the intensity, as

$$I = \frac{P}{A} = \frac{1}{2}c\epsilon_0 E_0^2 \implies E_0 = \sqrt{\frac{2P}{c\epsilon_0 A}} \approx 868 \text{ V/m}.$$

A rough but simple estimate for the dipole moment is just $ea_0 \approx 2.5$ Debye. The Rabi frequency is then

$$\Omega_0 = \frac{\mu E_0}{\hbar} \approx 70 \text{ MHz}$$

See the end of the document for a printout of the Mathematica notebook used for these calculations.

Problem 2

Under the rotating wave approximation, we neglect the counter-rotating term and get as our differential equation (neglecting bars on the 'c's)

$$\dot{c}_1 = -\frac{1}{2}i\Omega_0 e^{i\delta t} c_2$$

$$\dot{c}_2 = -\frac{1}{2}i\Omega_0 e^{-i\delta t} c_1.$$

The Rabi frequency is directly proportional to the applied electric field. In the weak-field limit, we can perturbatively expand the amplitudes as

$$c_i = c_i^{(0)} + \Omega_0 c_i^{(1)} + \Omega_0^2 c_i^{(2)} + \cdots.$$

To zero-th order, the amplitudes are given by the initial conditions $c_1^{(0)} = c_1(0) = 1$, and $c_2^{(0)} = c_2(0) = 0$. Now we go to first order and plug into the differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(c_1^{(0)} + \Omega_0 c_1^{(1)}\right) = -\frac{i}{2}\Omega_0 e^{i\delta t}\left(c_2^0 + \Omega_0 c_2^{(1)}\right)$$

$$\implies \qquad \Omega_0 \dot{c}_1^{(1)} = -\frac{i}{2}\Omega_0^2 e^{i\delta t} c_2^{(1)}$$

Matching terms proportional to Ω_0 gives

$$\dot{c}_1^{(1)} = 0$$

$$\implies \quad c_1^{(1)} = c_1^{(1)}(0) = 0.$$

For c_2, we find

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(c_2^{(0)} + \Omega_0 c_2^{(1)}\right) = -\frac{i}{2}\Omega_0 e^{-i\delta t}\left(c_1^{(0)} + \Omega_0 c_1^{(1)}\right)$$

$$\implies \quad \Omega_0 \dot{c}_2^{(1)} = -\frac{i}{2}\Omega_0 e^{-i\delta t} + O(\Omega^2)$$

$$\implies \quad c_2^{(1)} = -\frac{i}{2}\int_0^t \mathrm{d}t'\, e^{-i\delta t'}$$

$$= \frac{1}{2\delta}\left(e^{-i\delta t} - 1\right).$$

So, to first order we have that

$$\left.\begin{array}{l} c_1 \approx 1 \\ c_2 \approx \frac{\Omega_0}{2\delta}\left(e^{-i\delta t} - 1\right) \end{array}\right\} \implies \begin{array}{l} \left|c_1\right|^2 \approx 1 \\ \left|c_2\right|^2 \approx \frac{\Omega_0^2}{\delta^2}\sin^2\frac{\delta t}{2} \end{array}$$

Repeating the process for second order (matching terms proportional to Ω_0^2),

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(1 + \Omega_0^2 c_1^{(2)}\right) = -\frac{i}{2}\Omega_0 e^{i\delta t}\frac{\Omega_0}{2\delta}\left(e^{-i\delta} - 1\right)$$

$$\implies \quad \dot{c}_1^{(2)} = -\frac{i}{4\delta}\left(1 - e^{i\delta t}\right)$$

$$\implies \quad c_1^{(2)} = -\frac{i}{4\delta}\int_0^t \mathrm{d}t'\left(1 - e^{i\delta t}\right)$$

$$= -\frac{i}{4\delta}\left[t - \frac{1}{i\delta}\left(e^{i\delta t} - 1\right)\right]$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\Omega_0}{2\delta}\left(e^{-i\delta t} - 1\right) + \Omega_0^2 c_2^{(2)}\right) = -\frac{i}{2}\Omega_0 e^{-i\delta t}\left(1 + \Omega_0^2 c_1^{(2)}\right)$$

$$\implies \quad \dot{c}_2^{(2)} = 0$$

$$\implies \quad c_2^{(2)} = 0$$

So, to second order we have

$$c_1 \approx 1 - \frac{\Omega_0^2}{4\delta}\left[i\delta t - \left(e^{i\delta t} - 1\right)\right]$$

$$c_2 \approx \frac{\Omega_0}{2\delta}\left(e^{-i\delta t} - 1\right)$$

$$\implies \begin{cases} |c_1|^2 \approx 1 - \frac{\Omega_0^2}{\delta^2} \sin^2 \frac{\delta t}{2} + \frac{\Omega_0^4}{8\delta^4} \left(1 + \frac{t^2}{2\delta^2} - \cos \delta t - 2t \sin \delta t\right) \\ |c_2|^2 \approx \frac{\Omega_0^2}{\delta^2} \sin^2 \frac{\delta t}{2} \end{cases}$$

For third order I'll just rely on the Mathematica code and skip the details:

$$c_1 \approx 1 - \frac{\Omega_0^2}{4\delta} \left[i\delta t - \left(e^{i\delta t} - 1\right)\right]$$

$$c_2 \approx \frac{\Omega_0}{2\delta} \left(e^{-i\delta t} - 1\right) + \frac{\Omega_0^3}{\delta^3} \left(2 - i\delta t - e^{-i\delta t} \left(2 + i\delta t\right)\right)$$

$$\implies \begin{cases} |c_1|^2 \approx 1 - \frac{\Omega_0^2}{\delta^2} \sin^2 \frac{\delta t}{2} + \frac{\Omega_0^4}{8\delta^4} \left(1 + \frac{t^2}{2\delta^2} - \cos \delta t - 2t \sin \delta t\right) \\ |c_2|^2 \approx \frac{\Omega_0^2}{\delta^2} \sin^2 \frac{\delta t}{2} + \frac{\Omega_0^4}{2\delta^4} \left(\delta t \sin \frac{\delta t}{2} \cos \frac{\delta t}{2} - 2 \sin^2 \frac{\delta t}{2}\right) + \frac{\Omega_0^6}{16\delta^6} \left((\delta t)^2 \cos^2 \frac{\delta t}{2} + 4 \sin^2 \frac{\delta t}{2} - 4\delta t \cos \frac{\delta t}{2} \sin \frac{\delta t}{2}\right) \end{cases}$$

Problem 3

(a) We consider a two-level atom with states denoted $|1\rangle$, $|2\rangle$ and corresponding energies $E_1 = \hbar\omega_1 = -\omega_0/2$ and $E_2 = \hbar\omega_2 = \omega_0/2$. The atom interacts with a linearly polarized optical field described by

$$\vec{E} = \mathrm{Re}[E_0 e^{-i\omega t}] \, \hat{z}.$$

The interaction between the atom and the field is given to lowest order by the electric dipole interaction

$$V = -\vec{\mu} \cdot \vec{E} = ez|E_0| \cos(\omega t - \phi).$$

Assuming the two atomic states have opposite parity, the diagonal interaction matrix elements vanish:

$$V_{11} \propto \langle 1|z|1\rangle = 0 = V_{22}.$$

By a choice of phase for the wavefunction, we can take the off-diagonal elements to be real (and hence equal):

$$V_{12} = e|E_0|z_{12} \cos(\omega t - \phi)$$

The hamiltonian for the combined system is then

$$\begin{aligned} H &= H_0 + V \\ &= \frac{\hbar\omega_0}{2} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \hbar\Omega_0 \cos(\omega t - \phi) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= -\frac{\hbar\omega_0}{2}\sigma_z + \hbar\Omega_0 \cos(\omega t - \phi)\sigma_x, \end{aligned}$$

where we define the Rabi frequency $\Omega_0 = \frac{ez_{12}E_0}{\hbar}$.

In the Schrödinger representation, the state of the atom is described by

$$|\psi(t)\rangle_{\mathrm{S}} = c_1(t) |1\rangle + c_2(t) |2\rangle \quad \left(|c_1|^2 + |c_2|^2 = 1\right).$$

In the absence of the external field, the state would evolve as

$$|\psi(t)\rangle_{\mathrm{S}} = c_1(0)e^{-i\omega_1 t} |1\rangle + c_2(0)e^{-i\omega_2 t} |2\rangle.$$

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In the interaction representation, we factor out this free phase evolution by writing

$$|\psi(t)\rangle_{\mathrm{I}} = \bar{c}_1(t)e^{-i\omega_1 t}\,|1\rangle + \bar{c}_2(t)e^{-i\omega_2 t}\,|2\rangle,$$

that is, we make the (time-dependent) unitary transformation

$$|\psi(t)\rangle_{\mathrm{S}} \to |\psi(t)\rangle_{\mathrm{I}} = U(t)\,|\psi(t)\rangle_{\mathrm{S}},$$

where

$$U(t) = \begin{pmatrix} e^{-i\omega_1 t} & 0 \\ 0 & e^{-i\omega_2 t} \end{pmatrix}.$$

We get the effective interaction hamiltonian by making the inverse transformation on V:

$$V_{\mathrm{I}} = U^\dagger V U = \hbar\Omega_0 \cos(\omega t)\begin{pmatrix} 0 & e^{-i\omega_0 t} \\ e^{i\omega_0 t} & 0 \end{pmatrix},$$

where the phase ϕ has been absorbed into the (complex) Rabi frequency.

(b) To make the rotating wave approximation, we re-write the effective hamiltonian as

$$V_{\mathrm{I}} = \frac{1}{2}\hbar\Omega_0\left(e^{i\omega t} + e^{-i\omega t}\right)e^{i\omega_0 t}\sigma_+ + \mathrm{h.c.}$$
$$= \frac{1}{2}\hbar\Omega_0\left(e^{i(\omega+\omega_0)t} + e^{-i(\omega-\omega_0)t}\right)\sigma_+ + \mathrm{h.c.}$$

where σ_+ is the raising operator $|2\rangle\langle 1|$. Making the RWA amounts to neglecting the counter-rotating term (i.e. the term with $\omega_0 + \omega$), leaving

$$V_{\mathrm{I}} \approx \frac{1}{2}\hbar\Omega_0 e^{i\delta t}\sigma_+ + \mathrm{h.c.},$$

where we have switched to using the detuning $\delta = \omega_0 - \omega$.

(c) The Bloch Siegert shift effectively decreases $\omega_2 - \omega_1 = \omega_0$, and therefore becomes relevant when the level spacing is already small, such as in magnetic field interactions.

Problem 4

Making the simplifying assumptions of a constant Rabi frequency and zero detuning, we have that

$$\left.\begin{array}{l} \dot{c}_1 = -\frac{i}{2}\Omega_0 c_2 - \frac{\gamma_1}{2}c_1 \\ \dot{c}_2 = -\frac{i}{2}\Omega_0 c_1 - \frac{\gamma_2}{2} \end{array}\right\} \implies \begin{pmatrix} \dot{c}_1 \\ \dot{c}_2 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} \gamma_1 & i\Omega_0 \\ i\Omega_0 & \gamma_2 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix},$$

or, more simply,

$$\dot{\psi} = M\psi; \qquad M = -\frac{1}{2}\begin{pmatrix} \gamma_1 & i\Omega_0 \\ i\Omega_0 & \gamma_2 \end{pmatrix}.$$

For $c_1(0) = 1$, $c_2(0) = 0$, the solution to this first order differential equation is

$$\psi(t) = e^{Mt}\psi(0)$$

$$= \frac{1}{2\chi}\left(\begin{matrix} e^{-(\chi+\gamma_1+\gamma_2)t}\left[\chi + \gamma_1 - \gamma_2 + e^{\chi t/2}\left(\chi - \gamma_1 + \gamma_2\right)\right] \\ -2i\Omega_0 e^{-(\gamma_1+\gamma_2+\chi)/4}\left(e^{\chi t/2} - 1\right) \end{matrix} \right)$$

where

$$\chi = \sqrt{(\gamma_1 - \gamma_2)^2 - 4\Omega_0^2}.$$

In particular,

$$c_2(t) = -i\frac{\Omega_0}{\chi}e^{-(\chi+\gamma_1+\gamma_2)t/4}\left(e^{\chi t/2} - 1\right)$$

$$\implies \quad \left|c_2(t)\right|^2 = \frac{\Omega_0^2}{\chi^2}\frac{1 + e^{\chi t} - 2e^{\chi t/2}}{e^{(\chi+\gamma_1+\gamma_2)t/2}}.$$

Note that if $\Omega_0^2 > (\gamma_1 - \gamma_2)^2$, χ will be imaginary and the population will oscillate *and* decay. If $\gamma_1 = \gamma_2 = \gamma$, this reduces to

$$c_1 = e^{\gamma t/2}\cos\frac{\Omega_0 t}{2}$$

$$c_2 = -ie^{\gamma t/2}\sin\frac{\Omega_0 t}{2},$$

with

$$\left|c_2\right|^2 = e^{-\gamma t}\sin^2\frac{\Omega_0 t}{2}$$

Problem 1

In[1]:= `P = 1 mW; A = 1 mm²;`

In[2]:= `μ = e a₀; UnitConvert[μ, "Debyes"]`

Out[2]= `2.541746473 D`

In[3]:= $E_0 = \sqrt{\dfrac{2\,P}{c\,\varepsilon_0\,A}}$; `UnitConvert[E₀, "V/m"]`

Out[3]= `868.021098 V/m`

In[4]:= $\Omega = \dfrac{\mu\,E_0}{\hbar}$; `UnitConvert[Ω, "MHz"]`

Out[4]= `69.7855727 MHz`

Problem 2

In[5]:=
```
Clear[Ω];
c1 = {1, 0, 0, 0};
c2 = {0, 0, 0, 0};
```

In[8]:=
```
For[j = 2, j < 5, j++,

   c1⟦j⟧ = Integrate[-ⅈ/2 ⅇ^(ⅈ δ t) c2⟦j - 1⟧, {t, 0, t}] // FullSimplify;

   c2⟦j⟧ = Integrate[-ⅈ/2 ⅇ^(-ⅈ δ t) c1⟦j - 1⟧, {t, 0, t}] // FullSimplify;

]
```

In[9]:=
```
combine[order_, coeff_] = Sum[Indexed[coeff, j] Ω₀^(j-1), {j, 1, order + 1}];
sqrMag[z_] := Collect[FullSimplify[ComplexExpand[Abs[z]²]], Ω₀];
showOrder[order_] := StringForm["Order ``: \n
  c₁ ~ `` ⇒ |c₁|² ~ `` \n
  c₂ ~ `` ⇒ |c₂|² ~ ``\n\n",
    order, combine[order, c1], sqrMag[combine[order, c1]],
    combine[order, c2], sqrMag[combine[order, c2]]];
```

In[12]:= `Column[Table[showOrder[i], {i, 0, 3}]]`

Out[12]=

Order 0:

$c_1 \sim 1 \Rightarrow |c_1|^2 \sim 1$

$c_2 \sim 0 \Rightarrow |c_2|^2 \sim 0$

Order 1:

$c_1 \sim 1 \Rightarrow |c_1|^2 \sim 1$

$c_2 \sim \frac{\left(-1+e^{-i\,t\,\delta}\right)\Omega_0}{2\,\delta} \Rightarrow |c_2|^2 \sim \frac{\text{Sin}\left[\frac{t\,\delta}{2}\right]^2 \Omega_0^2}{\delta^2}$

Order 2:

$c_1 \sim 1 + \frac{\left(-1+e^{i\,t\,\delta}-i\,t\,\delta\right)\Omega_0^2}{4\,\delta^2} \Rightarrow |c_1|^2 \sim 1 + \frac{(-1+\text{Cos}[t\,\delta])\,\Omega_0^2}{2\,\delta^2} + \frac{\left(2+t^2\,\delta^2-2\,\text{Cos}[t\,\delta]-2\,t\,\delta\,\text{Sin}[t\,\delta]\right)\Omega_0^4}{16\,\delta^4}$

$c_2 \sim \frac{\left(-1+e^{-i\,t\,\delta}\right)\Omega_0}{2\,\delta} \Rightarrow |c_2|^2 \sim \frac{\text{Sin}\left[\frac{t\,\delta}{2}\right]^2 \Omega_0^2}{\delta^2}$

Order 3:

$c_1 \sim 1 + \frac{\left(-1+e^{i\,t\,\delta}-i\,t\,\delta\right)\Omega_0^2}{4\,\delta^2} \Rightarrow |c_1|^2 \sim 1 + \frac{(-1+\text{Cos}[t\,\delta])\,\Omega_0^2}{2\,\delta^2} + \frac{\left(2+t^2\,\delta^2-2\,\text{Cos}[t\,\delta]-2\,t\,\delta\,\text{Sin}[t\,\delta]\right)\Omega_0^4}{16\,\delta^4}$

$c_2 \sim \frac{\left(-1+e^{-i\,t\,\delta}\right)\Omega_0}{2\,\delta} + \frac{\left(2-i\,t\,\delta+e^{-i\,t\,\delta}\,(-2-i\,t\,\delta)\right)\Omega_0^3}{8\,\delta^3} \Rightarrow |c_2|^2 \sim$

$\frac{\text{Sin}\left[\frac{t\,\delta}{2}\right]^2 \Omega_0^2}{\delta^2} + \frac{\left(8\,t\,\delta^3\,\text{Cos}\left[\frac{t\,\delta}{2}\right]\,\text{Sin}\left[\frac{t\,\delta}{2}\right]-16\,\delta^2\,\text{Sin}\left[\frac{t\,\delta}{2}\right]^2\right)\Omega_0^4}{16\,\delta^6} + \frac{\left(t^2\,\delta^2\,\text{Cos}\left[\frac{t\,\delta}{2}\right]^2-4\,t\,\delta\,\text{Cos}\left[\frac{t\,\delta}{2}\right]\,\text{Sin}\left[\frac{t\,\delta}{2}\right]+4\,\text{Sin}\left[\frac{t\,\delta}{2}\right]^2\right)\Omega_0^6}{16\,\delta^6}$

Problem 3

In[13]:= `σ_x = PauliMatrix[1]; σ_z = PauliMatrix[3];`

In[14]:= $H_0 = -\frac{\hbar\,\omega_0}{2}\,\sigma_z;$

$V = \hbar\Omega_0\,\text{Cos}[\omega\,t - \phi]\,\sigma_x;$

$U = \text{MatrixExp}[-i\,H_0\,t\,/\,\hbar];$

In[17]:= $V_I = U^\dagger.V.U$ // ComplexExpand // TrigToExp // FullSimplify;
V_I // MatrixForm

Out[18]//MatrixForm=

$$\begin{pmatrix} 0 & e^{-i\,t\,\omega_0}\,\text{Cos}[\phi - t\,\omega]\,\hbar\Omega_0 \\ e^{i\,t\,\omega_0}\,\text{Cos}[\phi - t\,\omega]\,\hbar\Omega_0 & 0 \end{pmatrix}$$

Problem 4

In[19]:= $M = \dfrac{-1}{2}\,\{\{\gamma_1,\ i\,\Omega_0\},\ \{i\,\Omega_0,\ \gamma_2\}\};$

$\psi 0 = \{\{1\},\ \{0\}\};$

$\psi[t\_] = \{a1[t],\ a2[t]\};$

In[22]:= soln = MatrixExp[M t].$\psi 0$ // FullSimplify;

subs1 = $\left\{\sqrt{(\gamma_1 - \gamma_2)^2 - 4\,\Omega_0{}^2} \to \chi\right\};$

subs2 = $\left\{\gamma_1 \to \dfrac{1}{2}\,(\gamma^+ + \Delta\gamma),\ \gamma_2 \to \dfrac{1}{2}\,(\gamma^+ - \Delta\gamma)\right\};$

soln = (Numerator[soln] /. subs1) / (Denominator[soln] /. subs1);

soln =
   FullSimplify[Numerator[soln] /. subs2] / FullSimplify[Denominator[soln] /. subs2];

StringForm$\left[\text{"c}_1 = \texttt{`` } \backslash\text{nc}_2 = \texttt{`` } \backslash\text{n}|c_2|^2 = \texttt{``"},\ \text{soln}[[1]][[1]]\ \text{// FullSimplify},\right.$
  $\left.\text{soln}[[2]][[1]]\ \text{// FullSimplify},\ \text{sqrMag}[\text{soln}[[2]][[1]]]\ \text{// FullSimplify}\right]$

Out[27]=

$$c_1 = \frac{e^{-\frac{1}{4}t\,(\chi+\gamma^+)}\left(\Delta\gamma + \chi + e^{\frac{t\chi}{2}}\,(-\Delta\gamma + \chi)\right)}{2\,\chi}$$

$$c_2 = -\frac{i\,e^{-\frac{1}{4}t\,(\chi+\gamma^+)}\left(-1 + e^{\frac{t\chi}{2}}\right)\Omega_0}{\chi}$$

$$|c_2|^2 = \frac{\left(1 + e^{t\chi} - 2\,\sqrt{e^{t\chi}}\right)\Omega_0^2}{\sqrt{e^{t\,(\chi+\gamma^+)}}\,\chi^2}$$

In[28]:= soln = soln /. $\{\Delta\gamma \to 0,\ \gamma^+ \to 2\,\gamma,\ \chi \to 2\,i\,\Omega_0\};$

StringForm$\left[\text{"c}_1 = \texttt{`` } \backslash\text{nc}_2 = \texttt{`` } \backslash\text{n}|c_2|^2 = \texttt{``"},\ \text{soln}[[1]][[1]]\ \text{// FullSimplify},\right.$
  $\left.\text{soln}[[2]][[1]]\ \text{// FullSimplify},\ \text{sqrMag}[\text{soln}[[2]][[1]]]\ \text{// FullSimplify}\right]$

Out[29]=

$$c_1 = e^{-\frac{t\gamma}{2}}\,\text{Cos}\left[\frac{t\,\Omega_0}{2}\right]$$

$$c_2 = -i\,e^{-\frac{t\gamma}{2}}\,\text{Sin}\left[\frac{t\,\Omega_0}{2}\right]$$

$$|c_2|^2 = \frac{\text{Sin}\left[\frac{t\,\Omega_0}{2}\right]^2}{\sqrt{e^{2\,t\,\gamma}}}$$