Problem 3

In[1]:=
```
<< Notation`
```

In[14]:=
```
$Assumptions = {hϕ ∈ ℝ, hψ ∈ ℝ, vϕ ∈ ℝ, vψ ∈ ℝ, ϕ1 ∈ ℝ,
    ϕ2 ∈ ℝ, ϕ3 ∈ ℝ, W1 ∈ ℝ, W2 ∈ ℝ, W3 ∈ ℝ, Z ∈ ℝ, B ∈ ℝ, Y ∈ ℝ, g1 > 0, g2 > 0};
```

$\$Assumptions = \{h_\phi \in \mathbb{R}, h_\psi \in \mathbb{R}, v_\phi \in \mathbb{R}, v_\psi \in \mathbb{R}, \phi_1 \in \mathbb{R}, \phi_2 \in \mathbb{R}, \phi_3 \in \mathbb{R}, W^1 \in \mathbb{R}, W^2 \in \mathbb{R}, W^3 \in \mathbb{R}, Z \in \mathbb{R}, B \in \mathbb{R}, Y \in \mathbb{R}, g_1 > 0, g_2 > 0\};$

In[15]:=
```
reorderSymbols[expr_, symbols_List] := With[{s = symbols},
    HoldForm[Evaluate[expr /. Thread[s → Sort@s]] /. Thread[Sort@s → s]];
order[expr_] :=
  reorderSymbols[expr, {g1, g2, vϕ, vψ, Y, hϕ, hψ, W1, W2, W3, W+, W-, B, Z, A}]
```

$order[expr\_] := reorderSymbols[expr, \{g_1, g_2, v_\phi, v_\psi, Y, h_\phi, h_\psi, W^1, W^2, W^3, W^+, W^-, B, Z, A\}]$

a)

In[17]:=

$\phi = \{\phi_1, \phi_2, \phi_3\}^T;$

$W = \{W^1, W^2, W^3\}^T;$

$T = \left\{\frac{1}{\sqrt{2}} \{\{0, 1, 0\}, \{1, 0, 1\}, \{0, 1, 0\}\}, \frac{i}{\sqrt{2}} \{\{0, -1, 0\}, \{1, 0, -1\}, \{0, 1, 0\}\}, \{\{1, 0, 0\}, \{0, 0, 0\}, \{0, 0, -1\}\}\right\};$

$\theta_w = \text{ArcTan}\left[\frac{g_2}{g_1}\right];$

$s_w = \text{Sin}[\theta_w]$ // FullSimplify;

$c_w = \text{Cos}[\theta_w]$ // FullSimplify;

StringForm["s_w = ``, $c_w$ = ``", s_w, c_w]

Out[23]=

$$s_w = \frac{g_2}{\sqrt{g_1^2 + g_2^2}}, \quad c_w = \frac{g_1}{\sqrt{g_1^2 + g_2^2}}$$

In[24]:=
```
Row[Table[StringForm["T`` = ``  ", a, T[[a]] // MatrixForm], {a, 1, 3}]]
```

Out[24]=

$$T^1 = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & 0 \end{pmatrix} \quad T^2 = \begin{pmatrix} 0 & -\frac{i}{\sqrt{2}} & 0 \\ \frac{i}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} \\ 0 & \frac{i}{\sqrt{2}} & 0 \end{pmatrix} \quad T^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

In[25]:=
```
StringForm["WaT^a = ``", Sum[W[[a]] × T[[a]], {a, 1, 3}] // MatrixForm]
```

Out[25]=

$$W_a T^a = \begin{pmatrix} W^3 & \frac{W^1}{\sqrt{2}} - \frac{i W^2}{\sqrt{2}} & 0 \\ \frac{W^1}{\sqrt{2}} + \frac{i W^2}{\sqrt{2}} & 0 & \frac{W^1}{\sqrt{2}} - \frac{i W^2}{\sqrt{2}} \\ 0 & \frac{W^1}{\sqrt{2}} + \frac{i W^2}{\sqrt{2}} & -W^3 \end{pmatrix}$$

In[26]:=
```
Unprotect[D];
D = i g1 Sum[W[[a]] × T[[a]], {a, 1, 3}] + i g2 Y B IdentityMatrix[3];
StringForm["Dμ ⊃ ``", D // MatrixForm]
```

Out[28]=

$$D_\mu \supset \begin{pmatrix} i g_1 W^3 + i B g_2 Y & i g_1 \left(\frac{W^1}{\sqrt{2}} - \frac{i W^2}{\sqrt{2}}\right) & 0 \\ i g_1 \left(\frac{W^1}{\sqrt{2}} + \frac{i W^2}{\sqrt{2}}\right) & i B g_2 Y & i g_1 \left(\frac{W^1}{\sqrt{2}} - \frac{i W^2}{\sqrt{2}}\right) \\ 0 & i g_1 \left(\frac{W^1}{\sqrt{2}} + \frac{i W^2}{\sqrt{2}}\right) & -i g_1 W^3 + i B g_2 Y \end{pmatrix}$$

In[29]:=
```
Dϕ = D.ϕ;
StringForm["Dμϕ ⊃ ``", Dϕ // MatrixForm]
```

Out[30]=

$$D_\mu \phi \supset \begin{pmatrix} \left(i g_1 W^3 + i B g_2 Y\right) \phi_1 + i g_1 \left(\frac{W^1}{\sqrt{2}} - \frac{i W^2}{\sqrt{2}}\right) \phi_2 \\ i g_1 \left(\frac{W^1}{\sqrt{2}} + \frac{i W^2}{\sqrt{2}}\right) \phi_1 + i B g_2 Y \phi_2 + i g_1 \left(\frac{W^1}{\sqrt{2}} - \frac{i W^2}{\sqrt{2}}\right) \phi_3 \\ i g_1 \left(\frac{W^1}{\sqrt{2}} + \frac{i W^2}{\sqrt{2}}\right) \phi_2 + \left(-i g_1 W^3 + i B g_2 Y\right) \phi_3 \end{pmatrix}$$

In[31]:=
```
L = Dϕ†.Dϕ // FullSimplify;
StringForm["L ⊃ ``", L // order]
```

Out[32]=

$$L \supset$$

$$2 g_1 g_2 Y B \left(\sqrt{2} W^1 \phi_2 + W^3 (\phi_1 - \phi_3)\right) (\phi_1 + \phi_3) + g_2^2 Y^2 B^2 \left(\phi_1^2 + \phi_2^2 + \phi_3^2\right) + \frac{1}{2} g_1^2 \left(\left(W^2\right)^2 \left(2 \phi_2^2 + (\phi_1 - \phi_3)^2\right) + 2 \sqrt{2} W^1 W^3 \phi_2 (\phi_1 - \phi_3) + 2 \left(W^3\right)^2 \left(\phi_1^2 + \phi_3^2\right) + \left(W^1\right)^2 \left(2 \phi_2^2 + (\phi_1 + \phi_3)^2\right)\right)$$

In[33]:=
```
L = L /. {W^1 → (1/(√2)) (W+ + W-), W^2 → (i/(√2)) (W+ - W-),
      W^3 → cw Z + sw A, B → cw A - sw Z} // FullSimplify // FullSimplify;
StringForm["L ⊃ ``", L // order];
StringForm["mW^2 = ``", L /. {hϕ → 0, A → 0, Z → 0, W- → 1, W+ → 1} // order];
StringForm["mZ^2 = ``", L /. {hϕ → 0, A → 0, Z → 1, W- → 0, W+ → 0} // order];
StringForm["mA^2 = ``", L /. {hϕ → 0, A → 1, Z → 0, W- → 0, W+ → 0} // order];
```

In[38]:=
```
L₁ = L /. {Y → 1, ϕ₁ → 0, ϕ₂ → 0, ϕ₃ → (v_ϕ + h_ϕ) / √2} // FullSimplify;
(m_W^2)_{Y=1} = L₁ /. {h_ϕ → 0, A → 0, Z → 0, W⁻ → 1, W⁺ → 1};
(m_Z^2)_{Y=1} = 2 L₁ /. {h_ϕ → 0, A → 0, Z → 1, W⁻ → 0, W⁺ → 0};
(m_A^2)_{Y=1} = 2 L₁ /. {h_ϕ → 0, A → 1, Z → 0, W⁻ → 0, W⁺ → 0} ;
StringForm["L|_{Y=1} ⊃ ``", L1 // order]
StringForm["m_W^2 = ``", (m_W^2)_{Y=1} // order]
StringForm["m_Z^2 = ``", (m_Z^2)_{Y=1} // order]
StringForm["m_A^2 = ``", (m_A^2)_{Y=1} // order]
```

Out[42]=

$L|_{Y=1} \supset L1$

Out[43]=

$m_W{}^2 = \frac{1}{2} g_1^2 v_\phi^2$

Out[44]=

$m_Z{}^2 = \left(g_1^2 + g_2^2\right) v_\phi^2$

Out[45]=

$m_A{}^2 = 0$

In[46]:=
```
L₀ = L /. {Y → 0, ϕ₁ → 0, ϕ₂ → (v_ϕ + h_ϕ) / √2 , ϕ₃ → 0} // FullSimplify;
(m_W^2)_{Y=0} = L₀ /. {h_ϕ → 0, A → 0, Z → 0, W⁻ → 1, W⁺ → 1};
(m_Z^2)_{Y=0} = 2 L₀ /. {h_ϕ → 0, A → 0, Z → 1, W⁻ → 0, W⁺ → 0};
(m_A^2)_{Y=0} = 2 L₀ /. {h_ϕ → 0, A → 1, Z → 0, W⁻ → 0, W⁺ → 0} ;
StringForm["L|_{Y=0} ⊃ ``", L0 // order]
StringForm["m_W^2 = ``", (m_W^2)_{Y=0} // order]
StringForm["m_Z^2 = ``", (m_Z^2)_{Y=0} // order]
StringForm["m_A^2 = ``", (m_A^2)_{Y=0} // order]
```

Out[50]=

$L|_{Y=0} \supset L0$

Out[51]=

$m_W{}^2 = g_1^2 v_\phi^2$

Out[52]=

$m_Z{}^2 = 0$

Out[53]=

$m_A{}^2 = 0$

In[54]:=
```
D' = D /. {W¹ → 0, W² → 0, W³ → c_w Z + s_w A, B → c_w A - s_w Z} // FullSimplify ;
D' // MatrixForm
```

Out[55]//MatrixForm=

$$\begin{pmatrix} \frac{i \left(A g_1 g_2 (1+Y) + \left(g_1^2 - g_2^2 Y\right) Z\right)}{\sqrt{g_1^2+g_2^2}} & 0 & 0 \\ 0 & \frac{i g_2 Y (A g_1 - g_2 Z)}{\sqrt{g_1^2+g_2^2}} & 0 \\ 0 & 0 & \frac{i \left(A g_1 g_2 (-1+Y) - \left(g_1^2 + g_2^2 Y\right) Z\right)}{\sqrt{g_1^2+g_2^2}} \end{pmatrix}$$

In[56]:=
```
chargeTerm = Coefficient[D', A]  // FullSimplify;
chargeTerm // MatrixForm
```

Out[57]//MatrixForm=

$$\begin{pmatrix} \frac{i\, g_1 g_2 (1+Y)}{\sqrt{g_1^2+g_2^2}} & 0 & 0 \\ 0 & \frac{i\, g_1 g_2 Y}{\sqrt{g_1^2+g_2^2}} & 0 \\ 0 & 0 & \frac{i\, g_1 g_2 (-1+Y)}{\sqrt{g_1^2+g_2^2}} \end{pmatrix}$$

In[58]:=
```
chargeTerm == i g1 sw (T[[3]] + Y IdentityMatrix[3]) // FullSimplify
```

Out[58]=

True

b)

In[59]:=

$$\rho_1 = \frac{\left(m_W{}^2\right)_{Y=1}}{\left(m_Z{}^2\right)_{Y=1} c_W{}^2} \text{ // FullSimplify;}$$

$$\rho_0 = \frac{\left(m_W{}^2\right)_{Y=0}}{\left(m_Z{}^2\right)_{Y=0} c_W{}^2} \text{ // FullSimplify;}$$

```
StringForm["ρ1 = ``", ρ1]
StringForm["ρ0 = ``", ρ0]
```

Power: Infinite expression $\frac{1}{0}$ encountered.

Out[61]=

$\rho_1 = \frac{1}{2}$

Out[62]=

ρ_0 = ComplexInfinity

In[63]:=

$\psi = \left\{0,\ (v_\psi + h_\psi) \big/ \sqrt{2},\ 0\right\}^T;$

$L = (D.\psi)^\dagger.(D.\psi)$ /. $Y \to 0$;

$L = L + (D\phi)^\dagger.D\phi$ /. $\left\{Y \to 1,\ \phi_1 \to 0,\ \phi_2 \to 0,\ \phi_3 \to (v_\phi + h_\phi) \big/ \sqrt{2}\right\}$ // FullSimplify;

In[66]:=
```
L = L /. {W^1 → 1/√2 (W^+ + W^-), W^2 → i/√2 (W^+ - W^-), W^3 → c_w Z + s_w A, B → c_w A - s_w Z} //
    FullSimplify;
m_W^2 = L /. {h_ϕ → 0, h_ψ → 0, A → 0, Z → 0, W^- → 1, W^+ → 1} // FullSimplify;
m_Z^2 = 2 L /. {h_ϕ → 0, h_ψ → 0, A → 0, Z → 1, W^- → 0, W^+ → 0} // FullSimplify;
m_A^2 = 2 L /. {h_ϕ → 0, h_ψ → 0, A → 1, Z → 0, W^- → 0, W^+ → 0} // FullSimplify;
StringForm["m_A^2 = ``", m_A^2 // order]
StringForm["m_W^2 = ``", m_W^2 // order]
StringForm["m_Z^2 = ``", m_Z^2 // order]
```

Out[70]=

$m_A{}^2 = 0$

Out[71]=

$m_W{}^2 = \frac{1}{2} g_1^2 \left(v_\phi^2 + 2 v_\psi^2\right)$

Out[72]=

$m_Z{}^2 = \left(g_1^2 + g_2^2\right) v_\phi^2$

In[73]:=
```
ρ = m_W^2/(m_Z^2 c_w^2) // FullSimplify;
StringForm["ρ = ``", ρ]
```

Out[74]=

$\rho = \frac{1}{2} + \frac{v_\psi^2}{v_\phi^2}$