Homework 2

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Problem 1

Given the Lagrangian

$$L = \frac{1}{2} m g_{ij}(x) \dot{x}^i \dot{x}^j,$$

and the definition of the christoffel symbols

$$\Gamma^i_{jk} = \frac{1}{2} g^{il} \left(\partial_k g_{lj} + \partial_j g_{lk} - \partial_l g_{jk} \right),$$

we can start by taking derivatives of the Lagrangian:

$$\begin{aligned}
\frac{\partial L}{\partial x^l} &= \frac{1}{2} m \frac{\partial g_{ij}(x)}{\partial x^l} \dot{x}^i \dot{x}^j \\
&= \frac{1}{2} m \partial_l g_{ij} \dot{x}^i \dot{x}^j
\end{aligned}$$

$$\begin{aligned}
\frac{\partial L}{\partial \dot{x}^l} &= \frac{1}{2} m g_{ij} \left(\frac{\partial \dot{x}^i}{\partial \dot{x}^l} \dot{x}^j + \dot{x}^i \frac{\partial \dot{x}^j}{\partial \dot{x}^l} \right) \\
&= \frac{1}{2} m g_{ij} \left(\delta^i{}_l \dot{x}^j + \delta^j{}_l \dot{x}^i \right) \\
&= \frac{1}{2} m \left(g_{lj} \dot{x}^j + g_{il} \dot{x}^i \right) \\
&= m g_{li} \dot{x}^i
\end{aligned}$$

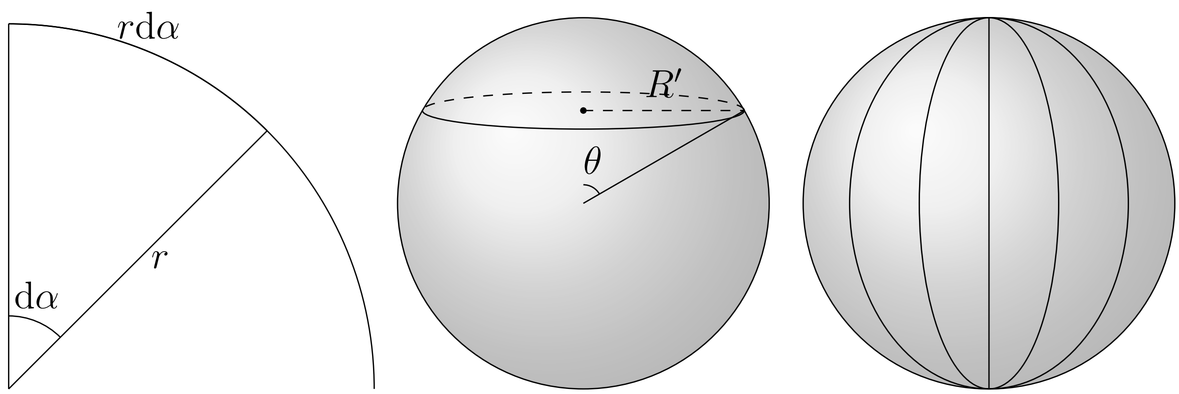
$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{x}^l} &= \frac{\mathrm{d}}{\mathrm{d}t} \left(m g_{li} \dot{x}^i \right) \\
&= m \frac{\mathrm{d} g_{li}}{\mathrm{d}t} \dot{x}^i + m g_{li} \frac{\mathrm{d} \dot{x}^i}{\mathrm{d}t} \\
&= m \frac{\partial g_{li}}{\partial x^j} \frac{\mathrm{d} x^j}{\mathrm{d}t} \dot{x}^i + m g_{li} \ddot{x}^i \\
&= m \partial_j g_{li} \dot{x}^i \dot{x}^j + m g_{li} \ddot{x}^i
\end{aligned}$$

Now we can just plug these in to the Euler-Lagrange equation:

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}^l} &= \frac{\partial L}{\partial x^l} \\
\implies \qquad m\partial_j g_{li}\dot{x}^i\dot{x}^j + m g_{li}\ddot{x}^i &= \frac{1}{2}m\partial_l g_{ij}\dot{x}^i\dot{x}^j \\
\implies \qquad g_{li}\ddot{x}^i &= \left(\frac{1}{2}\partial_l g_{ij} - \partial_j g_{li}\right)\dot{x}^i\dot{x}^j \\
&= -\frac{1}{2}\left(2\partial_j g_{li} - \partial_l g_{ij}\right)\dot{x}^i\dot{x}^j \\
&= -\frac{1}{2}\left(\partial_j g_{li} + \partial_j g_{li} - \partial_l g_{ij}\right)\dot{x}^i\dot{x}^j \\
&= -\frac{1}{2}\left(\partial_j g_{li} + \partial_i g_{lj} - \partial_l g_{ij}\right)\dot{x}^i\dot{x}^j \\
\implies \qquad \ddot{x}^i &= -\frac{1}{2}g^{il}\left(\partial_k g_{lj} + \partial_j g_{lk} - \partial_l g_{jk}\right)\dot{x}^j\dot{x}^k \\
&= -\Gamma^i_{jk}\dot{x}^j\dot{x}^k \\
\implies \qquad \ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k &= 0
\end{aligned}
$$

In the third-to-last line, the index relabeling $i \to j$ and $j \to k$ for contracted indices was used on the right-hand side.

Problem 2



(a) (b) (c)

Figure 1: (a) Arc length for a circle of radius r and angle $\mathrm{d}\alpha$. (b) A constant-θ circle on a 2-sphere. (c) Constant-ϕ circles on a 2-sphere.

(a) As illustrated in Figure 1a, the arc length for a circle of radius r and angle $\mathrm{d}\alpha$ is $r\mathrm{d}\alpha$. On a 2-sphere of radius R, paths of constant θ are circles of radius $R\sin\theta$ (Figure 1b), while paths of constant ϕ are circles of radius R (Figure 1c). The $\hat{\theta}$ and $\hat{\phi}$ directions are orthogonal everywhere on the sphere, so the square of the total displacement is the sum of the squares of the displacements in either direction:

$$\mathrm{d}s^2 = R^2 \mathrm{d}\theta^2 + R^2 \sin^2\theta \mathrm{d}\phi^2$$

(b) The only nonzero christoffel symbols are

$$\begin{aligned}
\Gamma^{\theta}_{\phi\phi} &= \frac{1}{2} g^{\theta l} \left(\partial_\phi g_{l\phi} + \partial_\phi g_{l\phi} - \partial_l g_{\phi\phi}\right) \\
&= -\frac{1}{2} g^{\theta\theta} \partial_\theta g_{\phi\phi} \\
&= -\frac{1}{2} \frac{1}{R^2} \frac{\partial}{\partial\theta} R^2 \sin^2\theta \\
&= -\sin\theta\cos\theta \\
\Gamma^{\phi}_{\theta\phi} &= \frac{1}{2} g^{\phi l} \left(\partial_\phi g_{l\theta} + \partial_\theta g_{l\phi} - \partial_l g_{\theta\phi}\right) \\
&= \frac{1}{2} g^{\phi\phi} \partial_\theta g_{\phi\phi} \\
&= \frac{1}{2} \frac{1}{R^2 \sin^2\theta} \frac{\partial}{\partial\theta} R^2 \sin^2\theta \\
&= \cot\theta \\
&= \Gamma^{\phi}_{\phi\theta}
\end{aligned}$$

Problem 3

(a)

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}s^2} = -\Gamma^i_{jk} \dot{x}^j \dot{x}^k$$

$$\frac{\mathrm{d}^2\theta}{\mathrm{d}s^2} = \sin\theta\cos\theta\dot{\phi}^2$$

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}s^2} = -\left(\Gamma^{\phi}_{\theta\phi} + \Gamma^{\phi}_{\phi\theta}\right)\dot{\theta}\dot{\phi} = -2\cot\theta\dot{\theta}\dot{\phi}$$

(b)

$$\begin{aligned}
& \ddot{\phi} + 2\cot\theta\dot{\theta}\dot{\phi} = 0 \\
\implies\quad & \sin^2\theta\ddot{\phi} + 2\sin\theta\cos\theta\dot{\theta}\dot{\phi} = 0 \\
\implies\quad & \frac{\mathrm{d}}{\mathrm{d}t}\left(\sin^2\theta\dot{\phi}\right) = 0
\end{aligned}$$

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}\right) &= 2\dot{\theta}\ddot{\theta} + 2\sin\theta\cos\theta\dot{\theta}\dot{\phi}^2 + 2\sin^2\theta\dot{\phi}\ddot{\phi} \\
&= 2\sin\theta\cos\theta\dot{\theta}\dot{\phi}^2 + 2\sin\theta\cos\theta\dot{\theta}\dot{\phi}^2 - 4\sin\theta\cos\theta\dot{\theta}\dot{\phi}^2 \\
&= 0
\end{aligned}$$

(c)

$$\tilde{L} = R^2 \sin^2\theta \dot{\phi} \implies \dot{\phi} = \frac{1}{R^2 \sin^2\theta}$$

$$u{\cdot}u = R^2\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) = 1 \implies \dot{\theta}^2 = \frac{1}{R^2}\left(1 - \frac{L^2}{R^2\sin^2\theta}\right) \implies \dot{\theta} = \pm\frac{1}{R}\sqrt{1 - \frac{L^2}{R^2\sin^2\theta}}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}s} = \pm\frac{1}{R}\sqrt{1 - \frac{L^2}{R^2\sin^2\theta}} \implies \int \mathrm{d}s = \pm\int \frac{R\mathrm{d}\theta}{\sqrt{1 - \frac{L^2}{R^2\sin^2\theta}}}$$

(d)

$$z = R\cos\theta \implies \mathrm{d}z = -R\sin\theta\mathrm{d}\theta$$

$$z = R\sin I\cos\psi \implies \mathrm{d}z = -R\sin I\sin\psi$$

$$\implies \psi = \cos^{-1}\frac{\cos\theta}{\sin I}$$

$$\begin{aligned}
s &= \pm\int \frac{R\mathrm{d}\theta}{\sqrt{1 - \frac{L^2}{R^2\sin^2\theta}}} \\
&= \mp\int \frac{R\mathrm{d}z}{\sqrt{R^2\sin^2\theta - L^2}} \\
&= \mp\int \frac{R\mathrm{d}z}{\sqrt{R^2\left(1-\cos^2\theta\right) - L^2}} \\
&= \mp\int \frac{R\mathrm{d}z}{\sqrt{R^2 - L^2 - z^2}} \\
&= \pm\int \frac{R^2\sin I\sin\psi\mathrm{d}\psi}{\sqrt{R^2 - R^2\cos^2 I - R^2\sin^2 I\cos^2\psi}} \\
&= \pm\int \frac{R\sin I\sin\psi\mathrm{d}\psi}{\sqrt{1 - \cos^2 I - \sin^2 I\cos^2\psi}} \\
&= \pm\int \frac{R\sin I\sin\psi\mathrm{d}\psi}{\sqrt{\sin^2 I - \sin^2 I\cos^2\psi}} \\
&= \pm\int \frac{R\sin\psi\mathrm{d}\psi}{\sqrt{1 - \cos^2\psi}} \\
&= \pm\int R\mathrm{d}\psi \\
&= R\psi + C
\end{aligned}$$

$$\implies \qquad \frac{s - C}{R} = \cos^{-1}\frac{\cos\theta}{\sin I}$$

$$\implies \qquad \cos\theta = \sin I\cos\frac{s - C}{R}$$

(e) Project the great circle onto a plane containing the z-axis. The result will be a (generalized) ellipse, hence motion in the z-direction will be sinusoidal. In this picture, z is the z-component of the position on the geodesic in the 3-space in which the 2-sphere is embedded. I and ψ are related to the axis about which the geodesic "rotates".

Problem 4

$$\Delta_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - \Gamma^\lambda_{\rho\mu} g_{\lambda\nu} - \Gamma^\lambda_{\rho\nu} g_{\mu\lambda} = 0 \tag{1}$$

$$\Delta_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho} - \Gamma^\lambda_{\mu\nu} g_{\lambda\rho} - \Gamma^\lambda_{\mu\rho} g_{\nu\lambda} = 0 \tag{2}$$

$$\Delta_\nu g_{\rho\mu} = \partial_\nu g_{\rho\mu} - \Gamma^\lambda_{\nu\rho} g_{\lambda\mu} - \Gamma^\lambda_{\nu\mu} g_{\rho\lambda} = 0 \tag{3}$$

(1) − (2) − (3):

$$\begin{aligned}
\partial_\rho g_{\mu\nu} - \partial_\mu g_{\nu\rho} - \partial_\nu g_{\rho\mu} + 2\Gamma^\lambda_{\mu\nu} g_{\rho\lambda} &= 0 \\
\implies g_{\rho\lambda}\Gamma^\lambda_{\mu\nu} &= \frac{1}{2}\left(\partial_\nu g_{\rho\mu} + \partial_\mu g_{\nu\rho} - \partial_\rho g_{\mu\nu}\right) \\
\implies \Gamma^\rho_{\mu\nu} &= \frac{1}{2} g^{\rho\lambda}\left(\partial_\nu g_{\lambda\mu} + \partial_\mu g_{\nu\lambda} - \partial_\lambda g_{\mu\nu}\right)
\end{aligned}$$