Homework 1

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Phys 633

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Problem 1

Let's begin with the purified state, $|\psi_{\text{CM}}\rangle$ (where "C" stands for "contestant" and "M" stands for "Monty")

$$|\psi_{\text{CM}}\rangle = \frac{1}{\sqrt{3}}\left(|1\rangle|1\rangle + |2\rangle|2\rangle + |3\rangle|3\rangle\right)$$

The contestant's state represents their state of belief about which door the car is behind. Monty's state represents his state of belief about which door he will open to reveal a goat. The contestant's selection of door 1 induces the transformation

$$\begin{aligned}
|\psi_{\text{CM}}\rangle \to & \frac{1}{\sqrt{3}}\left[|1\rangle\left(\frac{|2\rangle + |3\rangle}{\sqrt{2}}\right) + |2\rangle|3\rangle + |3\rangle|2\rangle\right] \\
&= \frac{1}{\sqrt{6}}|1\rangle|2\rangle + \frac{1}{\sqrt{6}}|1\rangle|3\rangle + \frac{1}{\sqrt{3}}|2\rangle|3\rangle + \frac{1}{\sqrt{3}}|3\rangle|2\rangle \\
&= \left(\frac{1}{\sqrt{6}}|1\rangle + \frac{1}{\sqrt{3}}|2\rangle\right)|2\rangle + \left(\frac{1}{\sqrt{6}}|1\rangle + \frac{1}{\sqrt{3}}|2\rangle\right)|3\rangle
\end{aligned}$$

Now, a projective measurment on Monty's state (i.e., Monty choosing to open door 2 or 3) will collapse the state one of

$$\left(\sqrt{\frac{1}{3}}|1\rangle + \sqrt{\frac{2}{3}}|2\rangle\right)|2\rangle$$

$$\left(\sqrt{\frac{1}{3}}|1\rangle + \sqrt{\frac{2}{3}}|2\rangle\right)|3\rangle$$

with equal probability. In either case, the contestant's part of the state becomes

$$|\psi_{\text{C}}\rangle = \sqrt{\frac{1}{3}}|1\rangle + \sqrt{\frac{2}{3}}|2\rangle$$

indicating that with probability 2/3 the car is behind door 2 and the contestant should switch.

Problem 2

$$\begin{aligned} x &\approx x_0 + \lambda x_1 + \lambda^2 x_2 + \cdots \\ &= 12.002383785691716 \end{aligned}$$

(a)

$$\begin{aligned} x &= (x_0 + \lambda x_1)^3 \\ &= x_0^3 + 3\lambda x_0^2 x_1 + 3\lambda^2 x_0 x_1^2 + \lambda^3 x_1^3 \\ &= x_0^3 + \lambda \end{aligned}$$

Matching terms proportional to λ gives

$$\lambda = 3\lambda x_0^2 x_1 \implies 3x_0^2 x_1 = 1 \implies x_1 = \frac{1}{3x_0^2}$$

The current approximation for x is therefore

$$x \approx x_0 + \lambda x_1 = 12 + \frac{1.03}{3 \times 12^2} = 12.002384\overline{259}$$

(b) Keeping terms only proportional to λ^2 we have

$$\begin{aligned} (x_0 + \lambda x_1 + \lambda^2 x_2)^3 &\to 3x_0^2 x_2 \lambda^2 + 3x_0 x_1^2 \lambda^2 \\ &= (3x_0^2 x_2 + 3x_0 x_1^2)\lambda^2 \end{aligned}$$

matching terms proportional to λ^2 gives

$$3x_0^2 x_2 + 3x_0 x_1^2 = 0 \implies x_2 = -\frac{x_0 x_1^2}{x_0^2} = -\frac{x_1^2}{x_0} = -\frac{1}{9x_0^5}$$

The current approximation for x is therefore

$$x \approx x_0 + \lambda x_1 + \lambda^2 x_2 = 12 + \frac{1.03}{3 \times 12^2} - \frac{1.03^2}{9 \times 12^5} = 12.00238473298$$

Problem 3

Δ_4 is a sum over all sets of 5 nonnegative indicies that add to 3:

$$\begin{aligned} \Delta_4 = &(01110\cdot) + (10011\cdot) + (10101\cdot) + (11001\cdot) + \\ &(00120\cdot) + (00210\cdot) + (01020\cdot) + (01200\cdot) + (02100\cdot) + (02010\cdot) + (10002\cdot) + (20001\cdot) \\ &(00030\cdot) + (00300\cdot) + (03000\cdot) \end{aligned}$$

Where only terms in which the first and last indicies do not cancel under trace have been kept. Some further simplification is given by

$$(10002\cdot) \to (3000)$$

$$(2001\cdot) \to (3000)$$

$$(00030\cdot), (03000\cdot), (00300\cdot) \to -(3000)$$

$$(00120\cdot) \to -(0011\cdot 1) \to -(10011\cdot)$$

$$(00210\cdot) \to -(001\cdot 11) \to -(11001\cdot)$$

$$(01020\cdot) \to -(0101\cdot 1) \to -(10101\cdot)$$

$$(01200\cdot) \to -(011\cdot 10) \to -(10011\cdot)$$

$$(02100\cdot) \to -(01\cdot 110) \to -(11001\cdot)$$

$$(02010\cdot) \to -(01\cdot 101) \to -(10101\cdot)$$

Giving

$$\begin{aligned}\Delta_4 &= (01110\cdot) - (10011\cdot) - (11001\cdot) - (10101\cdot) - (3000)\\ &= -(0111) - (2001) - (2100) - (2010) - (3000)\end{aligned}$$

Now let's handle the trace of each term individually

$$\begin{aligned}\mathrm{Tr}[-(0111)] &= \mathrm{Tr}\left[P_0 V \sum_{\alpha\neq 0} \frac{|\alpha\rangle\langle\alpha|}{E_{0\alpha}} V \sum_{\beta\neq 0} \frac{|\beta\rangle\langle\beta|}{E_{0\beta}} V \sum_{\gamma\neq 0} \frac{|\gamma\rangle\langle\gamma|}{E_{0\gamma}} V\right]\\ &= \mathrm{Tr}\left[P_0 V \sum_{\alpha\neq 0} \frac{|\alpha\rangle\langle\alpha|}{E_{0\alpha}} V \sum_{\beta\neq 0} \frac{|\beta\rangle\langle\beta|}{E_{0\beta}} V \sum_{\gamma\neq 0} \frac{|\gamma\rangle\langle\gamma|}{E_{0\gamma}} V P_0\right]\\ &= \mathrm{Tr}\left[P_0 \sum_{\alpha,\beta,\gamma\neq 0} \frac{V_{0\alpha}V_{\alpha\beta}V_{\beta\gamma}V_{\gamma 0}}{E_{0\alpha}E_{0\beta}E_{0\gamma}}\right]\\ &= \sum_{\alpha,\beta,\gamma\neq 0} \frac{V_{0\alpha}V_{\alpha\beta}V_{\beta\gamma}V_{\gamma 0}}{E_{0\alpha}E_{0\beta}E_{0\gamma}}\end{aligned}$$

$$\begin{aligned}-\mathrm{Tr}[(2010)] &= -\mathrm{Tr}\left[\sum_{\alpha\neq 0} \frac{|\alpha\rangle\langle\alpha|}{E_{0\alpha}^2} V P_0 V \sum_{\beta\neq 0} \frac{|\beta\rangle\langle\beta|}{E_{0\beta}} V P_0 V\right]\\ &= -\mathrm{Tr}\left[P_0 V \sum_{\alpha\neq 0} \frac{|\alpha\rangle\langle\alpha|}{E_{0\alpha}^2} V P_0 V \sum_{\beta\neq 0} \frac{|\beta\rangle\langle\beta|}{E_{0\beta}} V P_0\right]\\ &= -\mathrm{Tr}\left[P_0 \sum_{\alpha,\beta\neq 0} \frac{V_{0\alpha}V_{\alpha 0}V_{0\beta}V_{\beta 0}}{E_{0\alpha}^2 E_{0\beta}}\right]\\ &= -\sum_{\alpha,\beta\neq 0} \frac{\left|V_{0\alpha}\right|^2\left|V_{0\beta}\right|^2}{E_{0\alpha}^2 E_{0\beta}}\end{aligned}$$

$$
\begin{aligned}
-\operatorname{Tr}[(2001)] &= -\operatorname{Tr}\left[\sum_{\alpha\neq 0}\frac{|\alpha\rangle\langle\alpha|}{E_{0\alpha}^2}VP_0VP_0V\sum_{\beta\neq 0}\frac{|\beta\rangle\langle\beta|}{E_{0\beta}}V\right]\\
&= -\operatorname{Tr}\left[P_0V\sum_{\beta\neq 0}\frac{|\beta\rangle\langle\beta|}{E_{0\beta}}V\sum_{\alpha\neq 0}\frac{|\alpha\rangle\langle\alpha|}{E_{0\alpha}^2}VP_0VP_0\right]\\
&= -\operatorname{Tr}\left[P_0\sum_{\alpha,\beta\neq 0}\frac{V_{0\beta}V_{\beta\alpha}V_{\alpha 0}}{E_{0\alpha}^2E_{0\beta}}V_{00}\right]\\
&= -V_{00}\sum_{\alpha,\beta\neq 0}\frac{V_{0\beta}V_{\beta\alpha}V_{\alpha 0}}{E_{0\alpha}^2E_{0\beta}}\\
&= -V_{00}\sum_{\alpha,\beta\neq 0}\frac{V_{0\alpha}V_{\alpha\beta}V_{\beta 0}}{E_{0\alpha}E_{0\beta}^2}
\end{aligned}
$$

$$
\begin{aligned}
-\operatorname{Tr}[(2100)] &= -\operatorname{Tr}\left[\sum_{\alpha\neq 0}\frac{|\alpha\rangle\langle\alpha|}{E_{0\alpha}^2}V\sum_{\beta\neq 0}\frac{|\beta\rangle\langle\beta|}{E_{0\beta}}VP_0VP_0V\right]\\
&= -\operatorname{Tr}\left[P_0V\sum_{\alpha\neq 0}\frac{|\alpha\rangle\langle\alpha|}{E_{0\alpha}^2}V\sum_{\beta\neq 0}\frac{|\beta\rangle\langle\beta|}{E_{0\beta}}VP_0VP_0\right]\\
&= -\operatorname{Tr}\left[P_0\sum_{\alpha,\beta\neq 0}\frac{V_{0\alpha}V_{\alpha\beta}V_{\beta 0}}{E_{0\alpha}^2E_{0\beta}}V_{00}\right]\\
&= -V_{00}\sum_{\alpha,\beta\neq 0}\frac{V_{0\alpha}V_{\alpha\beta}V_{\beta 0}}{E_{0\alpha}^2E_{0\beta}}
\end{aligned}
$$

$$
\begin{aligned}
-\operatorname{Tr}[(3000)] &= \operatorname{Tr}\left[\sum_{\alpha\neq 0}\frac{|\alpha\rangle\langle\alpha|}{E_{0\alpha}^3}VP_0VP_0VP_0V\right]\\
&= \operatorname{Tr}\left[P_0V\sum_{\alpha\neq 0}\frac{|\alpha\rangle\langle\alpha|}{E_{0\alpha}^3}VP_0VP_0VP_0\right]\\
&= \operatorname{Tr}\left[P_0\sum_{\alpha\neq 0}\frac{V_{0\alpha}V_{\alpha 0}}{E_{0\alpha}^3}V_{00}V_{00}\right]\\
&= V_{00}^2\sum_{\alpha\neq 0}\frac{\left|V_{0\alpha}\right|^2}{E_{0\alpha}^3}
\end{aligned}
$$

Putting it all together we have

$$
\operatorname{Tr}[\Delta_4] = \sum_{\alpha,\beta,\gamma\neq 0}\frac{V_{0\alpha}V_{\alpha\beta}V_{\beta\gamma}V_{\gamma 0}}{E_{0\alpha}E_{0\beta}E_{0\gamma}} - \sum_{\alpha,\beta\neq 0}\frac{\left|V_{0\alpha}\right|^2\left|V_{0\beta}\right|^2}{E_{0\alpha}^2E_{0\beta}} - V_{00}\sum_{\alpha,\beta\neq 0}\left[\frac{V_{0\alpha}V_{\alpha\beta}V_{\beta 0}}{E_{0\alpha}E_{0\beta}^2}+\frac{V_{0\alpha}V_{\alpha\beta}V_{\beta 0}}{E_{0\alpha}^2E_{0\beta}}\right] + V_{00}^2\sum_{\alpha\neq 0}\frac{\left|V_{0\alpha}\right|^2}{E_{0\alpha}^3}
$$

In the case that $\mathrm{Tr}[\Delta_1] = \mathrm{Tr}[\Delta_2] = \mathrm{Tr}[\Delta_3] = 0$ this reduces to just the first term:

$$\mathrm{Tr}[\Delta_4] = \sum_{\alpha,\beta,\gamma\neq 0} \frac{V_{0\alpha}V_{\alpha\beta}V_{\beta\gamma}V_{\gamma 0}}{E_{0\alpha}E_{0\beta}E_{0\gamma}}$$

Problem 4

$$\begin{aligned}
[\vec{r}\cdot\vec{p}, H] &= \left[\vec{r}\cdot\vec{p}, \frac{p^2}{2m}\right] + [\vec{r}\cdot\vec{p}, V(r)] \\
&= \frac{1}{2m}\left[\vec{r}, p^2\right]\cdot\vec{p} + \vec{r}\cdot[\vec{p}, V(r)] \\
&= \frac{1}{2m}(2i\hbar\vec{p})\cdot\vec{p} - i\hbar\vec{r}\cdot\nabla V(r) \\
&= i\hbar\left(\frac{p^2}{m} - \vec{r}\cdot\nabla V(r)\right) \\
&= i\hbar\,(2T - \vec{r}\cdot\nabla V)
\end{aligned}$$

Now,

$$\langle[\vec{r}\cdot\vec{p}, H]\rangle = 0 \implies 2\langle T\rangle = \langle\vec{r}\cdot\nabla V\rangle$$

but

$$\langle\vec{r}\cdot\nabla V\rangle = \left\langle\vec{r}\cdot n\lambda r^{n-1}\hat{r}\right\rangle = \langle n\lambda r^n\rangle = n\langle V\rangle$$

so

$$2\langle T\rangle = n\langle V\rangle$$

Problem 5

(a) For the hydrogen atom we have $n = -1$, so

$$\langle T\rangle = -\frac{1}{2}\langle v\rangle \implies E_n = \langle T\rangle_n + \langle V\rangle_n = \frac{1}{2}\langle V\rangle_n$$

$$\begin{aligned}
&\implies -\frac{\alpha^2\mu c^2}{2n^2} = \frac{\hbar c\alpha}{2}\left\langle r^{-1}\right\rangle \\
&\implies \left\langle r^{-1}\right\rangle = \frac{\alpha\mu c}{\hbar n^2} = \frac{1}{a_0 n^2}
\end{aligned}$$

(b) Given $H = \frac{p_r^2}{2m_e} + \frac{\hbar^2 L(L+1)}{2m_e r^2} - \frac{\hbar c\alpha}{r}$, $E_n = -\frac{\alpha^2\mu c^2}{2n^2}$,

$$\partial_\alpha E = -\frac{\alpha\mu c^2}{n^2}$$

$$\langle\partial_\alpha H\rangle = \left\langle-\frac{\hbar c}{r}\right\rangle = -\hbar c\left\langle r^{-1}\right\rangle$$

$$\partial_\alpha E = \langle\partial_\alpha H\rangle \implies \left\langle r^{-1}\right\rangle = \frac{\alpha\mu c}{\hbar n^2} = \frac{1}{a_0 n^2}$$

(c)

$$\partial_L E = \partial_L\left(-\frac{\alpha^2\mu c^2}{2(L+k)^2}\right) = \frac{\alpha^2\mu c^2}{(L+k)^3} = \frac{\alpha^2\mu c^2}{n^3}$$

$$\langle \partial_L H\rangle = \left\langle \partial_L\left(\frac{p_r^2}{2\mu} + \frac{\hbar^2 L(L+1)}{2\mu r^2} - \frac{\hbar c\alpha}{r}\right)\right\rangle = \frac{\hbar^2(L+1/2)}{\mu}\left\langle r^{-2}\right\rangle$$

$$\partial_L E = \langle \partial_L H\rangle \implies \left\langle r^{-2}\right\rangle = \frac{\alpha^2\mu^2c^2}{\hbar^2(L+1/2)n^3} = \frac{1}{a_0^2(L+1/2)n^3}$$

(d)

$$\begin{aligned}[H,p_r] &= \frac{\hbar^2 L(L+1)}{2\mu}\left[r^{-2},p_r\right] - \hbar\alpha c\left[r^{-1},p_r\right] \\ &= -\frac{i\hbar^3 L(L+1)}{\mu}r^{-3} + i\hbar^2\alpha c r^{-2}\end{aligned}$$

Since the expectation value of any operator with the Hamiltonian vanishes with respect to any of the Hamiltonian's eigenstates,

$$\langle [H,p_r]\rangle = 0 \implies i\hbar^2\alpha c\left\langle r^{-2}\right\rangle = \frac{i\hbar^3 L(L+1)}{\mu}\left\langle r^{-3}\right\rangle$$

$$\implies \left\langle r^{-3}\right\rangle = \frac{\alpha\mu c}{\hbar L(L+1)}\left\langle r^{-2}\right\rangle = \frac{1}{a_0 L(L+1)}\left\langle r^{-2}\right\rangle$$

Substituting in the value for $\langle r^{-2}\rangle$ gives

$$\left\langle r^{-3}\right\rangle = \frac{1}{L(L+1/2)(L+1)a_0^3 n^3}$$

Homework 2

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Problem 1

Problem 2

Problem 3

Problem 4

Problem 5

Homework 3

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Problem 1

$$\left\langle \frac{1}{2}, m_J'; I, m_I' \middle| I_z J_z \middle| \frac{1}{2}, m_J; I, m_I \right\rangle = \hbar^2 m_I m_J \delta_{m_J', m_J} \delta_{m_I', m_I}$$
$$\left\langle \frac{1}{2}, m_J'; I, m_I' \middle| I_+ J_- \middle| \frac{1}{2}, m_J; I, m_I \right\rangle = \hbar^2 \sqrt{(I + m_I + 1)(I - m_I)} \delta_{m_J', -\frac{1}{2}} \delta_{m_J, \frac{1}{2}} \delta_{m_I', m_I+1}$$
$$\left\langle \frac{1}{2}, m_J'; I, m_I' \middle| I_- J_+ \middle| \frac{1}{2}, m_J; I, m_I \right\rangle = \hbar^2 \sqrt{(I - m_I + 1)(I + m_I)} \delta_{m_J', \frac{1}{2}} \delta_{m_J, -\frac{1}{2}} \delta_{m_I', m_I-1}$$
$$\left\langle \frac{1}{2}, m_J'; I, m_I' \middle| g_J J_z + g_I I_z \middle| \frac{1}{2} m_J; I m_I \right\rangle = \hbar \left(g_J m_J \delta_{m_J', m_J} + g_I m_I \delta_{m_I', m_I} \right)$$

$$\left\langle \frac{1}{2}, \frac{1}{2}; I, m_I' \middle| H_{\text{hfs}} + H_{\text{hfs}}^{(\text{B})} \middle| \frac{1}{2}, \frac{1}{2}; I, m_I \right\rangle = A_{\text{hfs}} \frac{m_I}{2} \delta_{m_I', m_I} + \mu_B \left(\frac{g_J}{2} + g_I m_I \delta_{m_I', m_I}\right) B$$
$$\left\langle \frac{1}{2}, \frac{1}{2}; I, m_I' \middle| H_{\text{hfs}} + H_{\text{hfs}}^{(\text{B})} \middle| \frac{1}{2}, \frac{-1}{2}; I, m_I \right\rangle = A_{\text{hfs}} \sqrt{(I - m_I + 1)(I + m_I)} \delta_{m_I', m_I-1}$$
$$\left\langle \frac{1}{2}, \frac{-1}{2}; I, m_I' \middle| H_{\text{hfs}} + H_{\text{hfs}}^{(\text{B})} \middle| \frac{1}{2}, \frac{1}{2}; I, m_I \right\rangle = A_{\text{hfs}} \sqrt{(I + m_I + 1)(I - m_I)} \delta_{m_I', m_I+1}$$
$$\left\langle \frac{1}{2}, \frac{-1}{2}; I, m_I' \middle| H_{\text{hfs}} + H_{\text{hfs}}^{(\text{B})} \middle| \frac{1}{2}, \frac{-1}{2}; I, m_I \right\rangle = -A_{\text{hfs}} \frac{m_I}{2} \delta_{m_I, m_I'} + \mu_B \left(-\frac{g_J}{2} + g_I m_I \delta_{m_I', m_I}\right)$$

Problem 2

$$\begin{aligned}
\partial_t U(t,t_0) &= \partial_t U_0(t,t_0) - \frac{i}{\hbar}\partial_t \int_{t_0}^{t} \mathrm{d}t_1 U_0(t,t_1)V(t_1)U(t_1,t_0) \\
&= -\frac{i}{\hbar}H_0 U_0(t,t_0) - \frac{i}{\hbar}\left[U_0(t,t)V(t)U(t,t_0) - \frac{i}{\hbar}\int_{t_0}^{t}\mathrm{d}t_1\; H_0 U_0(t,t_1)V(t_1)U(t_1,t_0)\right] \\
&= -\frac{i}{\hbar}\left[H_0 U_0(t,t_0) + V(t)U(t,t_0) - \frac{i}{\hbar}H_0\int_{t_0}^{t}\mathrm{d}t_1\; U_0(t,t_1)V(t_1)U(t_1,t_0)\right] \\
&= -\frac{i}{\hbar}\left[H_0\left(U_0(t,t_0) - \frac{i}{\hbar}\int_{t_0}^{t}\mathrm{d}t_1\; U_0(t,t_1)V(t_1)U(t_1,t)\right) + V(t)U(t,t_0)\right] \\
&= -\frac{i}{\hbar}\left[H_0 U(t,t_0) + V(t)U(t,t_0)\right] \\
&= -\frac{i}{\hbar}\left[H_0 + V(t)\right]U(t,t_0)
\end{aligned}$$

Problem 3

(a)

$$\begin{aligned}
G^{+}(x,x_0;E) &= \langle x|\frac{1}{E - p^2/2m + i0^{+}}|x_0\rangle \\
&= \int_{-\infty}^{\infty}\mathrm{d}p\;\langle x|\frac{1}{E - p^2/2m + i0^{+}}|p\rangle\langle p|x_0\rangle \\
&= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\mathrm{d}p\;\frac{e^{ip(x-x_0)/\hbar}}{E - p^2/2m + i0^{+}}
\end{aligned}$$

Let

$$r = x - x_0, \quad z = \frac{pr}{\hbar}, \quad z_E^2 = \frac{2mr^2E}{\hbar^2}$$

Then

$$\begin{aligned}
G^{+}(x,x_0;E) &= \frac{mr}{\pi\hbar}\int_{-\infty}^{\infty}\mathrm{d}z\;\frac{e^{iz}}{[z - (z_E + i0^{+})][z + (z_E - i0^{+})]} \\
&= (-2\pi i)\frac{mr}{\pi\hbar}\frac{e^{iz_E}}{-2z_E} \\
&= \frac{im}{z_E\hbar}e^{iz_E} \\
&= \frac{im}{z_E\hbar}e^{ip_E r/\hbar}
\end{aligned}$$

Problem 4

$$
\begin{aligned}
\tilde{K}^{(3)}_{\mathbf{fi}} &= \frac{i}{\hbar^3}\sum_{jk}\int_0^t \mathrm{d}t_3 \int_0^{t_3}\mathrm{d}t_2 \int_0^{t_2}\mathrm{d}t_3\, V_{\mathbf{f}j}V_{jk}V_{k\mathbf{i}}e^{iE_{\mathbf{f}j}t_3/\hbar}e^{iE_{jk}t_2/\hbar}e^{iE_{k\mathbf{i}}t_1/\hbar} \\
&= \frac{i}{\hbar^3}\sum_{jk}V_{\mathbf{f}j}V_{jk}V_{k\mathbf{i}}\int_0^t \mathrm{d}t_3\, e^{iE_{\mathbf{f}j}t_3/\hbar}\int_0^{t_3}\mathrm{d}t_2 \int_0^{t_2}\mathrm{d}t_1\, e^{iE_{jk}t_2/\hbar}e^{iE_{k\mathbf{i}}t_1/\hbar} \\
&= \frac{i}{\hbar^3}\sum_{jk}V_{\mathbf{f}j}V_{jk}V_{k\mathbf{i}}\int_0^t \mathrm{d}t_3\, e^{iE_{\mathbf{f}j}t_3/\hbar}\int_0^{t_3}\mathrm{d}t_2 \int_0^{t_2}\mathrm{d}t_1\, e^{iE_k(t_1-t_2)/\hbar}e^{iE_jt_2/\hbar}e^{iE_{\mathbf{i}}t_1/\hbar} \\
&= \frac{1}{2\pi\hbar^3}\sum_{jk}V_{\mathbf{f}j}V_{jk}V_{k\mathbf{i}}\int_0^t \mathrm{d}t_3\, e^{iE_{\mathbf{f}j}t_3/\hbar}\int_0^{t_3}\mathrm{d}t_2 \int_0^{t_3}\mathrm{d}t_1 \int_{-\infty}^{\infty}\mathrm{d}E\,\frac{e^{i(E_j-E)t_2/\hbar}e^{i(E-E_i)t_1/\hbar}}{E-E_k+i0^+} \\
&= \frac{2\pi}{\hbar}\sum_{jk}V_{\mathbf{f}j}V_{jk}V_{k\mathbf{i}}\int_0^t \mathrm{d}t_3\, e^{iE_{\mathbf{f}j}t_3/\hbar}\int_{-\infty}^{\infty}\mathrm{d}E\,\frac{\delta_t(E_j-E)\delta_t(E_i-E)e^{i(E_j-E)t_3/2\hbar}e^{i(E-E_i)t_3/2\hbar}}{E-E_k+i0^+} \\
&\approx \frac{2\pi}{\hbar}\sum_{jk}\frac{V_{\mathbf{f}j}V_{jk}V_{k\mathbf{i}}}{E_i-E_k+i0^+}\int_0^t \mathrm{d}t_3\, e^{iE_{\mathbf{f}j}t_3/\hbar}e^{iE_{j\mathbf{i}}t_3/2\hbar}\int_{-\infty}^{\infty}\mathrm{d}E\,\delta_t(E_j-E)\delta_t(E_i-E) \\
&= \frac{2\pi}{\hbar}\sum_{jk}\frac{V_{\mathbf{f}j}V_{jk}V_{k\mathbf{i}}}{E_{\mathbf{i}}-E_k+i0^+}\int_0^t \mathrm{d}t_3\, e^{iE_{\mathbf{f}j}t_3/\hbar}e^{iE_{j\mathbf{i}}t_3/2\hbar}\delta_t(E_{\mathbf{i}j}) \\
&= \frac{1}{\hbar^2}\sum_{jk}\frac{V_{\mathbf{f}j}V_{jk}V_{k\mathbf{i}}}{E_{\mathbf{i}}-E_k+i0^+}\int_0^t \mathrm{d}t_3 \int_0^{t_3}\mathrm{d}t_3'\, e^{iE_{\mathbf{f}j}t_3/\hbar}e^{iE_{j\mathbf{i}}t_3'/\hbar} \\
&= \frac{1}{2\pi i\hbar^2}\sum_{jk}\frac{V_{\mathbf{f}j}V_{jk}V_{k\mathbf{i}}}{E_{\mathbf{i}}-E_k+i0^+}\int_0^t \mathrm{d}t_3 \int_0^{t}\mathrm{d}t_3' \int_{-\infty}^{\infty}\mathrm{d}E\,\frac{e^{i(E_{\mathbf{f}}-E)t_3/\hbar}e^{i(E-E_{\mathbf{i}})t_3'/\hbar}}{E-E_j+i0^+} \\
&= -2\pi i\sum_{jk}\frac{V_{\mathbf{f}j}V_{jk}V_{k\mathbf{i}}}{E_{\mathbf{i}}-E_k+i0^+}\int_{-\infty}^{\infty}\mathrm{d}E\,\frac{\delta_t(E_{\mathbf{f}}-E)\delta_t(E-E_{\mathbf{i}})e^{i(E_{\mathbf{f}}-E)t/2\hbar}e^{i(E-E_{\mathbf{i}})t/2\hbar}}{E-E_j+i0^+} \\
&\approx -2\pi i\sum_{jk}\frac{V_{\mathbf{f}j}V_{jk}V_{k\mathbf{i}}}{(E_{\mathbf{i}}-E_k+i0^+)(E_{\mathbf{i}}-E_j+i0^+)}e^{iE_{\mathbf{fi}}t/2\hbar}\int_{-\infty}^{\infty}\mathrm{d}E\,\delta_t(E_{\mathbf{f}}-E)\delta_t(E-E_{\mathbf{i}}) \\
&= -2\pi i\sum_{jk}\frac{V_{\mathbf{f}j}V_{jk}V_{k\mathbf{i}}}{(E_{\mathbf{i}}-E_k+i0^+)(E_{\mathbf{i}}-E_j+i0^+)}e^{iE_{\mathbf{fi}}t/2\hbar}\delta_t(E_{\mathbf{fi}})
\end{aligned}
$$

Problem 5

$$\sin^2(\Omega t/2) = \left(\frac{\Omega t}{2}\right)^2 - \frac{1}{3}\left(\frac{\Omega t}{2}\right)^4 + \cdots$$

We already know that the first order term in the perturbation series gives

$$\left(\frac{\Omega t}{2}\right)^2$$

The restriction of the Hilbert space to $\{|i\rangle, |f\rangle\}$ kills any terms in the perturbation series with an odd number of perturbation matrix elements. The next nonzero term gives

$$\tilde{K}^{(3)}_{\mathbf{fi}} = \frac{i}{\hbar^3}\left|V_{\mathbf{fi}}\right|^2 \mathscr{T}\frac{1}{3!}\,[e]$$

Problem 6

(a)

$$\begin{aligned}
\tilde{K}^{(2)}_{\mathbf{fi}} &= \langle\mathbf{f}|-\frac{1}{2\hbar^2}\int_0^t \mathrm{d}t_2 \int_0^{t_2} \mathrm{d}t_1\; e^{iH_0t_2/\hbar}V_0e^{-i\omega t_2}e^{-iH_0t_2/\hbar}e^{iH_0t_1/\hbar}V_0e^{-i\omega t_1/\hbar}e^{-iH_0t_1/\hbar}|\mathbf{i}\rangle \\
&\quad + \langle\mathbf{f}|-\frac{1}{2\hbar^2}\int_0^t \mathrm{d}t_2 \int_0^{t_2} \mathrm{d}t_1\; e^{iH_0t_2/\hbar}V_0^\dagger e^{i\omega t_2}e^{-iH_0t_2/\hbar}e^{iH_0t_1/\hbar}V_0^\dagger e^{i\omega t_1/\hbar}e^{-iH_0t_1/\hbar}|\mathbf{i}\rangle \\
&= -\frac{1}{2\hbar^2}\sum_k \int_0^t \mathrm{d}t_2 \int_0^{t_2} \mathrm{d}t_1\; (V_0)_{\mathbf{f}k}e^{i(E_{\mathbf{f}k}-\hbar\omega)t_2}(V_0)_{k\mathbf{i}}e^{i(E_{k\mathbf{i}}-\hbar\omega)t_1} + (V_0^\dagger)_{\mathbf{f}k}e^{i(E_{\mathbf{f}k}+\hbar\omega)t_2}(V_0^\dagger)_{k\mathbf{i}}e^{i(E_{k\mathbf{i}}+\hbar\omega)t_1}
\end{aligned}$$

(b)

$$\tilde{K}^{(2)}_{\mathbf{fi}} = -2\pi i\sum_k\left[\left(\frac{(V_0)_{\mathbf{f}k}(V_0)_{k\mathbf{i}}}{E_{\mathbf{i}k}-\hbar\omega+i0^+}e^{i(E_{\mathbf{fi}}-\hbar\omega)}\delta_t(E_{\mathbf{fi}}-\hbar\omega)\right)+\left(\frac{(V_0)_{\mathbf{f}k}(V_0)_{k\mathbf{i}}}{E_{\mathbf{i}k}+\hbar\omega+i0^+}e^{i(E_{\mathbf{fi}}+\hbar\omega)}\delta_t(E_{\mathbf{fi}}+\hbar\omega)\right)\right]$$

(c) The term on the left will be the dominant term, as the difference $E_{\mathbf{i}k}-\hbar\omega$ in the denominator will be extremely small.

Homework 4

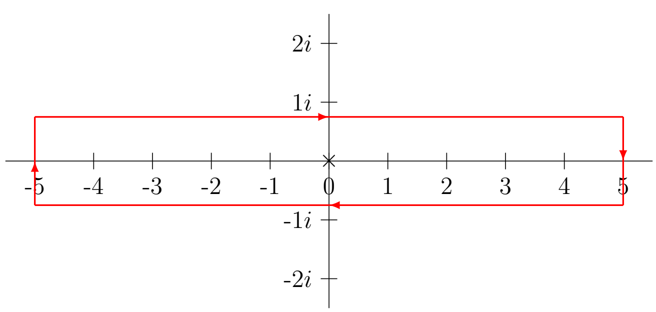
Sean Ericson
Phys 633

May 24, 2022

Problem 1

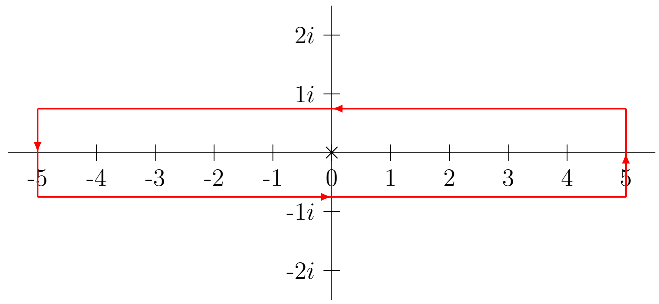
$$\begin{aligned}
\int_{-\infty}^{\infty} \mathrm{d}x \, \frac{f(x)}{x+i0^+} - \mathscr{P}\int_{-\infty}^{\infty} \mathrm{d}x \, \frac{1}{x} &= \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{1}{x+i0^+} - \frac{1/2}{1+i0^+} - \frac{1/2}{x-i0^+} \\
&= \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}x \, \left[\frac{1}{x+i0^+} - \frac{1}{x-i0^+}\right] \\
&= \frac{1}{2}(-2\pi i f(0)) \\
&= -\pi i f(0)
\end{aligned}$$

Where the following contour was used:



$$\begin{aligned}
\int_{-\infty}^{\infty} \mathrm{d}x \, \frac{f(x)}{x-i0^+} - \mathscr{P}\int_{-\infty}^{\infty} \mathrm{d}x \, \frac{1}{x} &= \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{1}{x-i0^+} - \frac{1/2}{1+i0^+} - \frac{1/2}{x-i0^+} \\
&= \frac{1}{2}\int_{-\infty}^{\infty} \mathrm{d}x \, \left[\frac{1}{x-i0^+} - \frac{1}{x+i0^+}\right] \\
&= \frac{1}{2}(2\pi i f(0)) \\
&= \pi i f(0)
\end{aligned}$$

Where the following contour integral was used:



Thus

$$\int_{-\infty}^{\infty} \mathrm{d}x \, \frac{f(x)}{x \pm i0^+} - \mathscr{P} \int_{-\infty}^{\infty} \mathrm{d}x \, \frac{f(x)}{x} = \mp i\pi f(0)$$

$$\Longrightarrow$$

$$\frac{1}{x \pm i0^+} = \mathscr{P}\frac{1}{x} \mp i\pi\delta(x)$$

Problem 2

Given that

$$\overleftrightarrow{\chi}(\omega) - \overleftrightarrow{\chi}_0 = \frac{1}{\pi i} \fint_{-\infty}^{\infty} \mathrm{d}\omega' \, \frac{\overleftrightarrow{\chi}(\omega') - \overleftrightarrow{\chi}_0}{\omega' - \omega}$$

we have

$$\begin{aligned} \frac{1}{\pi} \fint_{-\infty}^{\infty} \mathrm{d}\omega' \, \frac{\overleftrightarrow{\zeta}(\omega') - \overleftrightarrow{\chi}_0}{\omega' - \omega} &= \frac{1}{\pi} \fint_{-\infty}^{\infty} \mathrm{d}\omega' \, \frac{\overleftrightarrow{\chi}(\omega') - \overleftrightarrow{\chi}_0}{\omega' - \omega} + \frac{i}{\pi} \fint_{-\infty}^{\infty} \mathrm{d}\omega' \, \frac{\overleftrightarrow{\sigma}(\omega')}{\omega'(\omega' - \omega)} \\ &= i(\overleftrightarrow{\chi}(\omega) - \overleftrightarrow{\chi}_0) - \frac{\overleftrightarrow{\sigma}(\omega)}{\omega} + \frac{\overleftrightarrow{\sigma}_0}{\omega} \\ &= i\left(\overleftrightarrow{\zeta}(\omega) - \overleftrightarrow{\chi}_0 - \frac{i\overleftrightarrow{\sigma}_0}{\omega}\right) \end{aligned}$$

Therefore

$$\overleftrightarrow{\zeta}(\omega) - \overleftrightarrow{\chi}_0 = \frac{1}{\pi i} \fint_{-\infty}^{\infty} \mathrm{d}\omega' \, \frac{\overleftrightarrow{\zeta}(\omega') - \overleftrightarrow{\chi}_0}{\omega' - \omega} + \frac{i\overleftrightarrow{\sigma}_0}{\omega}$$

while splitting up the real and imaginary parts gives

$$\begin{aligned} \mathrm{Re}[\overleftrightarrow{\zeta}(\omega) - \overleftrightarrow{\chi}_0] &= \quad \frac{1}{\pi} \fint_{-\infty}^{\infty} \mathrm{d}\omega' \, \frac{\mathrm{Im}[\overleftrightarrow{\zeta}(\omega) - \overleftrightarrow{\chi}_0]}{\omega' - \omega} \\ \mathrm{Im}[\overleftrightarrow{\zeta}(\omega) - \overleftrightarrow{\chi}_0] &= -\frac{1}{\pi} \fint_{-\infty}^{\infty} \mathrm{d}\omega' \, \frac{\mathrm{Re}[\overleftrightarrow{\zeta}(\omega) - \overleftrightarrow{\chi}_0]}{\omega' - \omega} + \frac{\overleftrightarrow{\sigma}_0}{\omega} \end{aligned}$$

Problem 3

Firstly,

$$\overset{\leftrightarrow}{\chi}(-\omega^*) = \frac{i}{\hbar}\int_0^\infty \mathrm{d}\tau \ \langle [\tilde{x}_\alpha(\tau), \tilde{x}_\beta(0)] \rangle \, e^{-i\omega^*\tau}$$

$$\overset{\leftrightarrow}{\chi}^*(\omega) = -\frac{i}{\hbar}\int_0^\infty \mathrm{d}\tau \ \langle [\tilde{x}_\beta(0), \tilde{x}_\alpha(\tau)] \rangle \, e^{-i\omega^*\tau} = \frac{i}{\hbar}\int_0^\infty \mathrm{d}\tau \ \langle [\tilde{x}_\alpha(\tau), \tilde{x}_\beta(0)] \rangle \, e^{-i\omega^*\tau}$$

so we have

$$\overset{\leftrightarrow}{\chi}(-\omega^*) = \overset{\leftrightarrow}{\chi}^*(\omega)$$

This means, for a real frequency ω, $\mathrm{Re}[\overset{\leftrightarrow}{\chi}(\omega)]$ is an even function, while $\mathrm{Im}[\overset{\leftrightarrow}{\chi}(\omega)]$ is odd. Now, noting that $\overset{\leftrightarrow}{\chi}_0$ must be real,

$$\begin{aligned}
\mathrm{Re}[\overset{\leftrightarrow}{\chi}(\omega)] &= \overset{\leftrightarrow}{\chi}_0 + \frac{1}{\pi}\fint_{-\infty}^{\infty} \mathrm{d}\omega' \, \frac{\mathrm{Im}[\overset{\leftrightarrow}{\chi}(\omega')]}{\omega' - \omega} \\
&= \overset{\leftrightarrow}{\chi}_0 + \frac{1}{\pi}\fint_{-\infty}^{\infty} \mathrm{d}\omega' \, \frac{\mathrm{Im}[\overset{\leftrightarrow}{\chi}(\omega')]}{\omega'^2 - \omega^2}(\omega' + \omega) \\
&= \overset{\leftrightarrow}{\chi}_0 + \frac{1}{\pi}\fint_{-\infty}^{\infty} \mathrm{d}\omega' \, \frac{\omega' \, \mathrm{Im}[\overset{\leftrightarrow}{\chi}(\omega')]}{\omega'^2 - \omega^2} + \frac{\omega}{\pi}\fint_{-\infty}^{\infty} \mathrm{d}\omega' \, \frac{\mathrm{Im}[\overset{\leftrightarrow}{\chi}(\omega')]}{\omega'^2 - \omega^2} \\
&= \overset{\leftrightarrow}{\chi}_0 + \frac{2}{\pi}\fint_{0}^{\infty} \mathrm{d}\omega' \, \frac{\omega' \, \mathrm{Im}[\overset{\leftrightarrow}{\chi}(\omega')]}{\omega'^2 - \omega^2}
\end{aligned}$$

Where, in the second to last line, the second integral vanishes due to $\mathrm{Im}[\overset{\leftrightarrow}{\chi}(\omega')]$ being odd, while the evenness of $\mathrm{Re}[\overset{\leftrightarrow}{\chi}(\omega')]$ allows the domain of the first integral to be restricted to the positive reals (with an additional factor of 2).
Similarly,

$$\begin{aligned}
\mathrm{Im}[\overset{\leftrightarrow}{\chi}(\omega)] &= -\frac{1}{\pi}\fint_{-\infty}^{\infty} \mathrm{d}\omega' \, \frac{\mathrm{Re}[\overset{\leftrightarrow}{\chi}(\omega')] - \overset{\leftrightarrow}{\chi}_0}{\omega' - \omega} \\
&= -\frac{1}{\pi}\fint_{-\infty}^{\infty} \mathrm{d}\omega' \, \frac{\mathrm{Re}[\overset{\leftrightarrow}{\chi}(\omega')] - \overset{\leftrightarrow}{\chi}_0}{\omega'^2 - \omega^2}(\omega' + \omega) \\
&= -\frac{1}{\pi}\fint_{-\infty}^{\infty} \mathrm{d}\omega' \, \frac{\omega' \left(\mathrm{Re}[\overset{\leftrightarrow}{\chi}(\omega')] + \overset{\leftrightarrow}{\chi}_0\right)}{\omega'^2 - \omega^2} - \frac{\omega}{\pi}\fint_{-\infty}^{\infty} \mathrm{d}\omega' \, \frac{\mathrm{Re}[\overset{\leftrightarrow}{\chi}(\omega')] - \overset{\leftrightarrow}{\chi}_0}{\omega'^2 - \omega^2} \\
&= -\frac{2\omega}{\pi}\fint_{0}^{\infty} \mathrm{d}\omega' \, \frac{\mathrm{Re}[\overset{\leftrightarrow}{\chi}(\omega')] - \overset{\leftrightarrow}{\chi}_0}{\omega'^2 - \omega^2}
\end{aligned}$$

Problem 4

(a) By the definition of the Green's function,

$$\left(\frac{\partial^2}{\partial t^2} + \gamma\frac{\partial}{\partial t} + \omega_0^2\right) g(t, t') = \delta(t - t').$$

Taking the Fourier transform of the above equation (with respect to t) yields

$$\left(-\omega^2 - i\gamma\omega + \omega_0^2\right) g(\omega, t') = \int_{-\infty}^{\infty} \mathrm{d}t\, \delta(t-t')e^{i\omega t} = e^{i\omega t'}$$
$$\Longrightarrow$$
$$g(\omega, t') = \frac{e^{i\omega t'}}{-\omega^2 - i\gamma\omega + \omega_0^2} \tag{1}$$

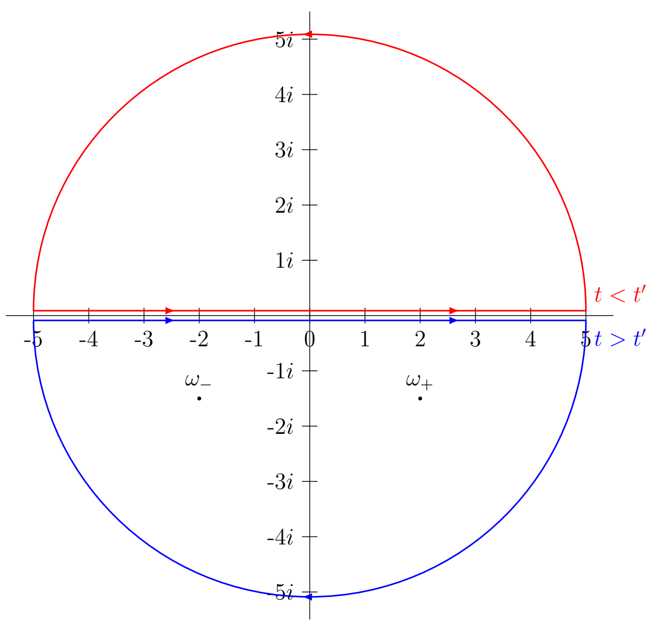
Inverting the Fourier transform gives

$$\begin{aligned} g(t,t') &= -\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\, \frac{e^{i\omega t'}e^{-i\omega t}}{\omega^2 + i\gamma\omega - \omega_0^2} \\ &= -\frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\, \frac{e^{-i\omega(t-t')}}{(\omega-\omega_+)(\omega-\omega_-)} \end{aligned}$$

where

$$\omega_\pm = -\frac{i\gamma}{2} \pm \tilde{\omega}; \quad \tilde{\omega} = \sqrt{\omega_0^2 - \frac{\gamma^2}{4}}$$

This integral can be calculated using Cauchy's residue theorem by considering the contours below:



Only the lower contour encloses the poles, thus

$$g(t,t') = i\Theta(t-t')\left[\frac{e^{-i\omega_+(t-t')}}{\omega_+ - \omega_-} + \frac{e^{-i\omega_-(t-t')}}{\omega_- - \omega_+}\right] = \boxed{\Theta(t-t')\frac{e^{-\gamma(t-t')/2}}{\tilde{\omega}}\sin[\tilde{\omega}(t-t')]}$$

(b) Let

$$\mathscr{L}x(t) = f(t); \quad \mathscr{L} = \frac{\partial^2}{\partial t^2} + \gamma\frac{\partial}{\partial t} + \omega_0^2$$

Then

$$\begin{aligned} f(t) &= \int_{-\infty}^{\infty} \mathrm{d}t'\ \delta(t-t')f(t') \\ &= \int_{-\infty}^{\infty} \mathrm{d}t'\ [\mathscr{L}g(t-t')]\, f(t') \\ &= \mathscr{L}\int_{-\infty}^{\infty} \mathrm{d}t'\ g(t-t')f(t') \end{aligned}$$

$$\implies x(t) = \int_{-\infty}^{\infty} \mathrm{d}t'\ g(t-t')f(t')$$

showing that the general solution $x(t)$ for a given forcing function $f(t)$ is simply the convolution of the forcing and Green function $(g * f)(x)$. Using the Green function calculated above,

$$\begin{aligned} x(t) &= \int_{-\infty}^{\infty} \mathrm{d}t'\ \Theta(t-t')\frac{e^{-\gamma(t-t')/2}}{\tilde{\omega}} \sin[\tilde{\omega}(t-t')]f(t') \\ &= \frac{1}{\tilde{\omega}}\int_{0}^{\infty} \mathrm{d}t'\ e^{-\gamma(t-t')/2} \sin[\tilde{\omega}(t-t')]f(t') \end{aligned}$$

(c) See (1)

(d) By the convolution theorem,

$$x(t) = (g * f)(t) \implies \tilde{x}(\omega) = \tilde{g}(\omega)\tilde{f}(\omega)$$

Applying the inverse Fourier transform then gives

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\ \tilde{x}(\omega)e^{-i\omega t} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}\omega\ \tilde{g}(\omega)\tilde{f}(\omega)e^{-i\omega t}$$

Problem 5

(a)

$$n(E) = 2(\pi k_E^2)\frac{A}{(2\pi)^2} = \frac{E^2 A}{2\pi\hbar^2 c^2}$$

$$\implies \rho(E) = \frac{EA}{\pi\hbar^2 c^2}$$

$$\implies \rho(E_{\mathrm{f}}) = \frac{\omega_{\mathrm{eg}} A}{\pi\hbar c^2}$$

$$
\begin{aligned}
\Gamma_{\mathrm{i}\to\mathcal{F}} &= \frac{2\pi}{\hbar}\left|V_{\mathbf{fi}}\right|^2\rho(E_{\mathrm{f}}) \\
&= \frac{2\pi}{\hbar}\left|\hat{\epsilon}\cdot\vec{d}_{\mathrm{eg}}\right|^2\frac{\hbar\omega}{2\epsilon_0 AL}\frac{\omega_{\mathrm{eg}}A}{\pi\hbar c^2} \\
&= \frac{2\pi}{\hbar}\frac{e^2\left|\vec{r}_{\mathrm{eg}}\right|^2}{2}\frac{\hbar\omega}{2\epsilon_0 AL}\frac{\omega_{\mathrm{eg}}A}{\pi\hbar c^2} \\
&\approx \frac{\omega_{\mathrm{eg}}^2 e^2\left|\vec{r}_{\mathrm{eg}}\right|^2}{2\epsilon_0\hbar c^2 L}
\end{aligned}
$$

(b)

$$
\begin{aligned}
n(E) = 2(2k_E)\frac{L}{2\pi} &= \frac{2EL}{\pi\hbar c} \\
\implies \rho(E) &= \frac{2L}{\pi\hbar c} \\
\implies \rho(E_{\mathrm{f}}) &= \frac{2L}{\pi\hbar c}
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_{\mathrm{i}\to\mathcal{F}} &= \frac{2\pi}{\hbar}\left|V_{\mathbf{fi}}\right|^2\rho(E_{\mathrm{f}}) \\
&= \frac{2\pi}{\hbar}\left|\hat{\epsilon}\cdot\vec{d}_{\mathrm{eg}}\right|^2\frac{\hbar\omega}{2\epsilon_0 AL}\frac{2L}{\pi\hbar c} \\
&= \frac{2\pi}{\hbar}e^2\left|\vec{r}_{\mathrm{eg}}\right|^2\frac{\hbar\omega}{2\epsilon_0 AL}\frac{2L}{\pi\hbar c} \\
&\approx \frac{2\omega_{\mathrm{eg}}e^2\left|r_{\mathrm{eg}}\right|^2}{\epsilon_0\hbar c A}
\end{aligned}
$$

Homework 5

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Phys 633

June 4, 2022

Problem 1

(a)

$$\begin{aligned}
\mathscr{E}(\mathbf{r}) &= \int_{-\infty}^{\infty} \mathrm{d}T\, \varphi(\mathbf{r}, T) e^{ik^2 T} \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}T \int_{-\infty}^{\infty} \mathrm{d}E\, \tilde{\varphi}(\mathbf{r}, E) e^{-iET} e^{ik^2 T} \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{d}E \int_{-\infty}^{\infty} \mathrm{d}T\, \tilde{\varphi}(\mathbf{r}, E) e^{-i(E-k^2)T} \\
&= \int_{-\infty}^{\infty} \mathrm{d}E\, \tilde{\varphi}(\mathbf{r}, E) \delta(E - k^2) \\
&= \tilde{\varphi}(\mathbf{r}, k^2)
\end{aligned}$$

The $\tilde{\varphi}$ are solutions for particualar values of energy, and our desired solution is the $\tilde{\varphi}$ with energy k^2 as expected.

(b) Given that

$$\hbar = 1, \quad m = 1/2$$

we have that

$$L(x, \dot{x}) = \frac{\dot{x}}{4} - V(x) = \frac{\dot{x}^2}{4} - k^2(1 - n^2(\mathbf{r}))$$

Thus

$$\begin{aligned}
K(x, T; x_0, 0) &= \int \mathrm{D}x \exp\left[i \int_0^T \mathrm{d}\tau\, L(x, \dot{x}) \right] \\
&= \int \mathrm{D}x \exp\left[i \int_0^T \mathrm{d}\tau\, \left(\frac{\dot{x}^2}{4} - V(x) \right) \right] \\
&= \int \mathrm{D}x \exp\left[i \int_0^T \mathrm{d}\tau\, \left(\frac{\dot{x}^2}{4} - k^2(1 - n^2(\mathbf{r})) \right) \right]
\end{aligned}$$

(c) We have that

$$\begin{aligned}G^{+}(\mathbf{r},\mathbf{r}';\tau) &= K(\mathbf{r},\mathbf{r}';\tau)\Theta(\tau) \\ &= \Theta(\tau)\int \mathrm{D}x \exp\left[i\int_0^T \mathrm{d}\tau\ \left(\frac{\dot{x}^2}{4} - V(x)\right)\right]\end{aligned}$$

Therefore

$$\begin{aligned}G^{+}(\mathbf{r},\mathbf{r}';E) &= \frac{1}{i\hbar}\int_0^\infty \mathrm{d}\tau\ e^{i(E+i0^+)\tau/\hbar} G^{+}(\mathbf{r},\mathbf{r}',\tau) \\ &= -i\int_0^\infty \mathrm{d}\tau\ e^{i(E+i0^+)\tau/\hbar}\int \mathrm{D}x \exp\left[i\int_0^T \mathrm{d}\tau\ \left(\frac{\dot{x}^2}{4} - V(x)\right)\right]\end{aligned}$$

where $\hbar = 1$ was used. Finally,

$$G^{+}(\mathbf{r},\mathbf{r}';k) = -i\int_0^\infty \mathrm{d}T\ e^{ik^2T}\int \mathrm{D}x \exp\left[i\int_0^T \mathrm{d}\tau\ \frac{\dot{x}^2}{4} - k^2(1-n^2(\mathbf{r}))\right]$$

where the convergence helper $i0^+$ is understood to be present if necessary.

(d)

$$\begin{aligned}S_{\text{reduced}}[x] &= \int_0^s \mathrm{d}s'\ \sqrt{k^2 - V(x)} \\ &= \int_0^s \mathrm{d}s'\ \sqrt{k^2 - k^2(1-n^2(x))} \\ &= k\int_0^s \mathrm{d}s'\ n(x)\end{aligned}$$

Given that

$$l[\mathbf{r}] = \int_0^d \mathrm{d}s\ n(x)$$

we have that

$$\delta l = 0 \iff \delta S_{\text{reduced}} = 0$$

Problem 2

(a) Firstly,

$$\begin{aligned}|\Psi(\mathbf{r}_1,\cdots,\mathbf{r}_N)\rangle &= \frac{1}{\sqrt{N!}}\hat{\psi}^\dagger(\mathbf{r}_1)\cdots\hat{\psi}^\dagger(\mathbf{r}_N)\,|0\rangle \\ &= \frac{1}{\sqrt{N!}}\left(\sum_{j_1}\hat{a}^\dagger_{j_1}(t)\phi^*_{j_1}(\mathbf{r}_1)\right)\cdots\left(\sum_{j_N}\hat{a}^\dagger_{j_N}(t)\phi^*_{j_N}(\mathbf{r}_1)\right)|0\rangle\end{aligned}$$

Now,

$$
\begin{aligned}
\hat{N}\left|\Psi(\mathbf{r}_1,\cdots,\mathbf{r}_N)\right\rangle &= \left(\sum_{j_0} \hat{a}^\dagger_{j_0}\hat{a}_{j_0}\right)\left|\Psi(\mathbf{r}_1,\cdots,\mathbf{r}_N)\right\rangle \\
&= \frac{1}{\sqrt{N!}}\left(\sum_{j_0} \hat{a}^\dagger_{j_0}\hat{a}_{j_0}\right)\left(\sum_{j_1} \hat{a}^\dagger_{j_1}(t)\phi^*_{j_1}(\mathbf{r}_1)\right)\cdots\left(\sum_{j_N} \hat{a}^\dagger_{j_N}(t)\phi^*_{j_N}(\mathbf{r}_1)\right)\left|0\right\rangle \\
&= \frac{1}{\sqrt{N!}}\sum_{j_0,\cdots,j_N} \hat{a}^\dagger_{j_0}\hat{a}_{j_0}\hat{a}^\dagger_{j_1}\hat{a}^\dagger_{j_2}\cdots\hat{a}^\dagger_{j_N}\left|0\right\rangle\phi^*_{j_1}\cdots\phi^*_{j_N} \\
&= \frac{1}{\sqrt{N!}}\sum_{j_0,\cdots,j_N} \hat{a}^\dagger_{j_0}\left(\hat{a}^\dagger_{j_1}\hat{a}_{j_0}+\delta_{j_0j_1}\right)\hat{a}^\dagger_{j_2}\cdots\hat{a}^\dagger_{j_N}\left|0\right\rangle\phi^*_{j_1}\cdots\phi^*_{j_N} \\
&= \frac{1}{\sqrt{N!}}\sum_{j_0,\cdots,j_N} \hat{a}^\dagger_{j_0}\hat{a}^\dagger_{j_1}\hat{a}_{j_0}\hat{a}^\dagger_{j_2}\cdots\hat{a}^\dagger_{j_N}\left|0\right\rangle\phi^*_{j_1}\cdots\phi^*_{j_N} + \left|\Psi\right\rangle \\
\cdots& \\
&= \frac{1}{\sqrt{N!}}\sum_{j_0,\cdots,j_N} \hat{a}^\dagger_{j_0}\hat{a}^\dagger_{j_1}\cdots\hat{a}^\dagger_{j_N}\hat{a}_{j_0}\left|0\right\rangle + N\left|\Psi\right\rangle \\
&= 0 + N\left|\Psi\right\rangle \\
&= N\left|\Psi\right\rangle
\end{aligned}
$$

Notice that commuting the j_0 annihilation operator over one position yields a copy of $\left|\Psi\right\rangle$. After moving it over N times, the j_0 annihilation operator will be next to the vaccum state, killing that term and leaving us with N coppies of $\left|\Psi\right\rangle$. Therefore,

$$\hat{N}\left|\Psi\right\rangle = N\left|\Psi\right\rangle$$

(b) The same argument carries over to the fermionic case with one slight change. Commuting the annihilation operator over one position will yeild a term like

$$\delta_{j_0j_i} - \hat{a}^\dagger_{j_i}\hat{a}_{j_0}$$

In the bosonic case, at the end of the argument we're effective left with

$$\hat{N}\left|\Psi\right\rangle = 0 + N\left|\Psi\right\rangle = N\left|\Psi\right\rangle$$

In the fermionic case, we're instead left with

$$\hat{N}\left|\Psi\right\rangle = N\left|\Psi\right\rangle - 0 = N\left|\Psi\right\rangle$$

Problem 3

(a) Let

$$\psi_i := \hat{\psi}(\mathbf{r}_i); \quad \psi^\dagger_i := \hat{\psi}^\dagger(\mathbf{r}_i)$$

$$\psi_{i'} \coloneqq \hat{\psi}(\mathbf{r}'_i); \quad \psi^\dagger_{i'} \coloneqq \hat{\psi}^\dagger(\mathbf{r}'_i)$$

$$|\Psi\rangle \coloneqq \frac{1}{\sqrt{N!}}\psi^\dagger_1 \cdots \psi^\dagger_N \,|0\rangle$$

$$|\Psi'\rangle \coloneqq \frac{1}{\sqrt{N!}}\psi^\dagger_{1'} \cdots \psi^\dagger_{N'} \,|0\rangle$$

Let's work out the first few cases explicitly:
$N = 1$

$$\begin{aligned}
(1!)\,\langle\Psi'|\Psi\rangle_1 &= \langle 0|\psi_{1'}\psi^\dagger_1|0\rangle \\
&= \delta^3(\mathbf{r}_1 - \mathbf{r}'_1)\,\langle 0|0\rangle + \langle 0|\psi^\dagger_1\psi_{1'}|0\rangle \\
&= \delta^3(\mathbf{r}_1 - \mathbf{r}'_1)
\end{aligned}$$

$N = 2$

$$\begin{aligned}
(2!)\,\langle\Psi'|\Psi\rangle_2 &= \langle 0|\psi_{2'}\psi_{1'}\psi^\dagger_1\psi^\dagger_2|0\rangle \\
&= \delta^3(\mathbf{r}_1 - \mathbf{r}'_1)\,\langle 0|\psi_{2'}\psi^\dagger_2|0\rangle + \langle 0|\psi_{2'}\psi^\dagger_1\psi_{1'}\psi^\dagger_2|0\rangle \\
&= \delta^3(\mathbf{r}_1 - \mathbf{r}'_1)\delta^3(\mathbf{r}_2 - \mathbf{r}'_2) + \delta^3(\mathbf{r}_2 - \mathbf{r}'_1)\,\langle 0|\psi_{2'}\psi^\dagger_1|0\rangle + \langle 0|\psi_{2'}\psi^\dagger_1\psi^\dagger_2\psi_{1'}|0\rangle \\
&= \delta^3(\mathbf{r}_1 - \mathbf{r}'_1)\delta^3(\mathbf{r}_2 - \mathbf{r}'_2) + \delta^3(\mathbf{r}_1 - \mathbf{r}'_2)\delta^3(\mathbf{r}_2 - \mathbf{r}'_1)
\end{aligned}$$

$N = 3$

$$\begin{aligned}
(3!)\,\langle\Psi'|\Psi\rangle_3 &= \langle 0|\psi_{3'}\psi_{2'}\psi_{1'}\psi^\dagger_1\psi^\dagger_2\psi^\dagger_3|0\rangle \\
&= \delta^3(\mathbf{r}_1 - \mathbf{r}'_1)\,\langle 0|\psi_{3'}\psi_{2'}\psi^\dagger_2\psi^\dagger_3|0\rangle + \delta^3(\mathbf{r}_2 - \mathbf{r}'_1)\,\langle 0|\psi_{3'}\psi_{2'}\psi^\dagger_1\psi^\dagger_3|0\rangle + \delta^3(\mathbf{r}_3 - \mathbf{r}'_1)\,\langle 0|\psi_{3'}\psi_{2'}\psi^\dagger_1\psi^\dagger_2|0 \\
&= \delta^3(\mathbf{r}_1 - \mathbf{r}'_1)\delta^3(\mathbf{r}_2 - \mathbf{r}'_2)\delta^3(\mathbf{r}_3 - \mathbf{r}'_3) + \delta^3(\mathbf{r}_1 - \mathbf{r}'_1)\delta^3(\mathbf{r}_2 - \mathbf{r}'_3)\delta^3(\mathbf{r}_3 - \mathbf{r}'_2) \\
&\quad + \delta^3(\mathbf{r}_1 - \mathbf{r}'_2)\delta^3(\mathbf{r}_2 - \mathbf{r}'_1)\delta^3(\mathbf{r}_3 - \mathbf{r}'_3) + \delta^3(\mathbf{r}_1 - \mathbf{r}'_3)\delta^3(\mathbf{r}_2 - \mathbf{r}'_1)\delta^3(\mathbf{r}_3 - \mathbf{r}'_2) \\
&\quad + \delta^3(\mathbf{r}_1 - \mathbf{r}'_2)\delta^3(\mathbf{r}_3 - \mathbf{r}'_1)\delta^3(\mathbf{r}_2 - \mathbf{r}'_3) + \delta^3(\mathbf{r}_1 - \mathbf{r}'_3)\delta^3(\mathbf{r}_2 - \mathbf{r}'_2)\delta^3(\mathbf{r}_3 - \mathbf{r}'_1)
\end{aligned}$$

For each N, we first use the commutation relation to move $\psi_{1'}$ over all the way to the right, generating N terms. Each of these terms contains a delta function and a matrix element of the form $(N-1)$, up to a re-labeling of the indicies. Continuing inductively,

$$\langle\Psi'|\Psi\rangle = \frac{1}{N!}\sum_{\sigma\in P(N)} \delta^3(\mathbf{r}_1 - \mathbf{r}'_{\sigma_1})\cdots\delta^3(\mathbf{r}_N - \mathbf{r}'_{\sigma_N})$$

(b) For a fermionic state, nearly the same argument applies, but now the anticommutation relation introduces a minus sign everytime we commute. A commutation corresponds to a transpostion of two of the indicies, so the even permutations will get an even number of minus signs, while the odd permutations will get an odd number of minus signs. The net result is therefore

$$\langle\Psi'|\Psi\rangle = \frac{1}{N!}\sum_{\sigma\in P(N)} \epsilon_\sigma\delta^3(\mathbf{r}_1 - \mathbf{r}'_{\sigma_1})\cdots\delta^3(\mathbf{r}_N - \mathbf{r}'_{\sigma_N})$$