# Homework 5

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Phys 684

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## Problem 1

We start with the Maxwell wave equation

$$\left(\vec{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E}(\vec{r},t) = \mu_0 \frac{\partial^2}{\partial t^2}\vec{P}(\vec{r},t),$$

where

$$\vec{E}(\vec{r},t) = \vec{E}_+(z,t) + \vec{E}_-(z,t); \quad \vec{E}_\pm(z,t) = \frac{1}{2}\hat{x}E_0(z,t)e^{\mp i\alpha(z,t)},$$

$$\vec{P}(\vec{r},t) = \vec{P}_+(z,t) + \vec{P}_-(z,t) = \frac{1}{2}\hat{x}\left(P_0(z,t)e^{-i\alpha(z,t)} + \text{c.c}\right),$$

$$\alpha(z,t) = \omega t - kz - \phi(z,t),$$

and $E_0(z,t) \in \mathbb{R}$ while $P_0(z,t) \in \mathbb{C}$. Next, we note that

$$2\frac{\partial}{\partial z}|E_\pm| = \left(E_0' \mp iE_0\alpha'\right)e^{\mp i\alpha},$$

$$2\frac{\partial}{\partial t}|E_\pm| = \left(\dot{E}_0 \mp iE_0\dot{\alpha}\right)e^{\mp i\alpha},$$

$$2\frac{\partial}{\partial t}|P_\pm| = \left(\dot{P}_0 \mp P_0\dot{\alpha}\right)e^{\mp i\alpha}$$

## Problem 2

Starting with

$$\dot{\tilde{\rho}}_{21} = -\gamma\tilde{\rho}_{21} - i\delta\tilde{\rho}_{21} + i\frac{\Omega_0}{2}\left(\rho_{22} - \rho_{11}\right)$$

$$\dot{\rho}_{22} = -\gamma_2\rho_{22} + \text{Re}[i\Omega_0^*\tilde{\rho}_{21}],$$

we first drop the tildes, then let $\rho_{21} = \frac{1}{2}(u - iv)$, and $\Omega_0 = \Omega_0' + i\Omega_0''$. Plugging these in, we get

$$\begin{aligned}
\frac{1}{2}(\dot{u} - i\dot{v}) &= -\frac{\gamma}{2}(u - iv) - i\frac{\delta}{2}(u - iv) + \frac{i}{2}\left(\Omega_0' + i\Omega_0''\right)(2\rho_{22} - 1) \\
\dot{\rho}_{22} &= -\gamma_2\rho_{22} + \frac{1}{2}\left(\Omega_0'' u + \Omega_0' v\right)
\end{aligned}$$

$$\Longrightarrow$$

$$\begin{aligned}
\dot{u} &= -\gamma u - \delta v - 2\Omega_0''\rho_{22} + \Omega_0'' \\
\dot{v} &= -\delta u + \gamma v + 2\Omega_0'\rho_{22} - \Omega_0' \\
\dot{\rho}_{22} &= \frac{\Omega_0''}{2}u + \frac{\Omega_0'}{2}v - \gamma_2\rho_{22}
\end{aligned}$$

In the steady state, this is

$$\left.\begin{aligned}
-\gamma u - \delta v - 2\Omega_0''\rho_{22} &= -\Omega_0'' \\
-\delta u + \gamma v + 2\Omega_0'\rho_{22} &= \Omega_0' \\
\tfrac{\Omega_0''}{2}u + \tfrac{\Omega_0'}{2}v - \gamma_2\rho_{22} &= 0
\end{aligned}\right\} \Longrightarrow \begin{pmatrix} -\gamma & -\delta & 2\Omega_0'' \\ -\delta & +\gamma & 2\Omega_0' \\ \frac{\Omega_0''}{2} & \frac{\Omega_0'}{2} & -\gamma_2 \end{pmatrix} \begin{pmatrix} u \\ v \\ \rho_{22} \end{pmatrix} = \begin{pmatrix} -\Omega_0'' \\ \Omega_0 \\ 0 \end{pmatrix}$$

Inverting and solving, we find

$$\begin{pmatrix} \dot{u} \\ \dot{v} \\ \dot{\rho}_{22} \end{pmatrix} = \frac{1}{\gamma^2 + \delta^2 + \frac{\gamma}{\gamma_2}\left|\Omega_0\right|^2} \begin{pmatrix} \gamma\Omega_0'' - \delta\Omega_0' \\ \delta\Omega_0'' + \gamma\Omega_0' \\ \frac{\gamma\left|\Omega_0\right|^2}{2\gamma_2} \end{pmatrix}.$$

Or, in terms of just density matrix components,

$$\begin{aligned}
\tilde{\rho}_{21} &= \frac{-\frac{1}{2}(\delta + i\gamma)\Omega_0}{\gamma^2 + \delta^2 + \frac{\gamma}{\gamma_2}\left|\Omega_0\right|^2} \\
\rho_{22} &= \frac{\gamma\left|\Omega_0\right|^2}{2\gamma_2}\frac{1}{\gamma^2 + \delta^2 + \frac{\gamma}{\gamma_2}\left|\Omega_0\right|^2}
\end{aligned}$$

# Problem 3

(a)

# Problem 4

(a)

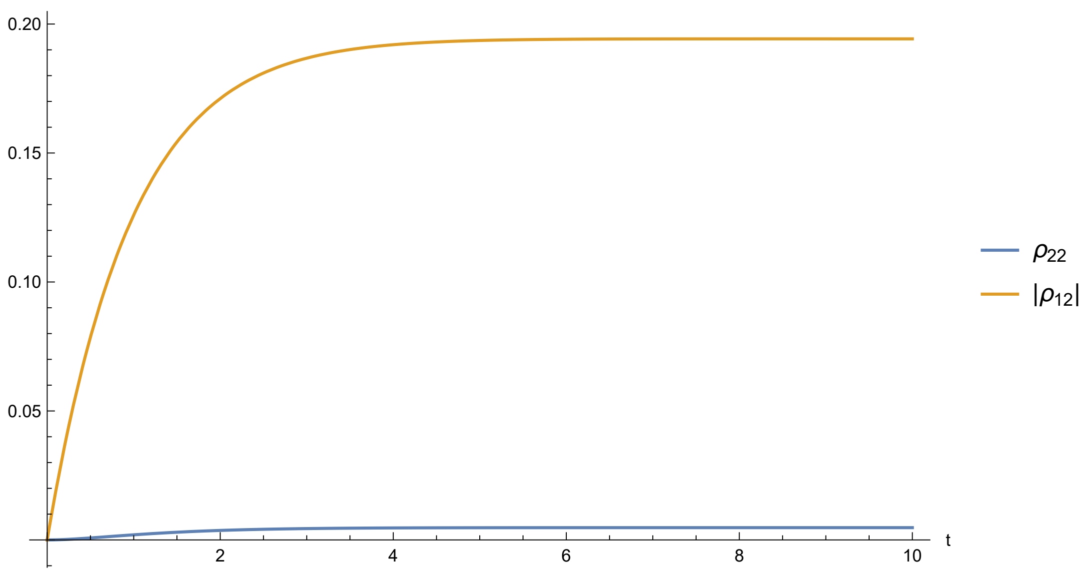
# Problem 4 (Berman 4.3)

In[122]:=
```
SolveAndPlot[γ_, χ_, δ_] :=
 ({{uSoln, vSoln, wSoln}} = NDSolve[{u'[t] == -γ u[t] - δ v[t],
     v'[t] == δ u[t] - γ v[t] - (2 χ) w[t], w'[t] == - (2 γ) (w[t] + 1) + χ v[t],
     u[0] == 0, v[0] == 0, w[0] == -1}, {u, v, w}, {t, 0, 10}];
  Plot[{1/2 (w[x] + 1) /. wSoln, Sqrt[u[x]^2 + v[x]^2] /. uSoln /. vSoln}, {x, 0, 10},
   PlotRange → All, PlotLegends → {"ρ22", "|ρ12|"}, AxesLabel → {"t", ""}]);
```

## $\gamma = 1;\ \Omega_0 = 0.2,\ \delta = 0.2$

In[123]:=
```
SolveAndPlot[1, 0.1, 0.2]
```

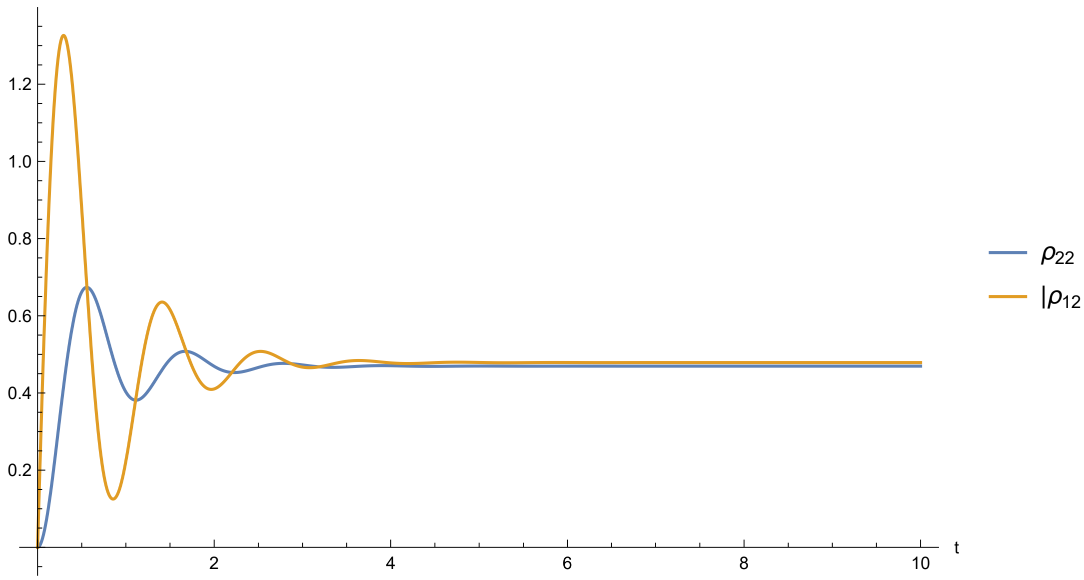
Out[123]=



## $\gamma = 1;\ \Omega_0 = 8,\ \delta = 0.2$

In[124]:=
```
SolveAndPlot[1, 4, 0.2]
```

Out[124]=




## $\gamma = 1;\ \Omega_0 = 8,\ \delta = 50$

In[125]:=
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SolveAndPlot[1, 4, 50]
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Out[125]=