Homework 7

Sean Ericson
Phys 663

June 2, 2024

Problem 1

i) For a radiation dominated flat universe, we have that

$$\begin{aligned} a(t) &= \left(\frac{t}{t_0}\right)^{1/2} \\ \implies \qquad t &= t_0 a^2 \\ \implies \qquad \mathrm{d}t &= 2t_0 a \mathrm{d}a. \end{aligned}$$

Then,

$$\begin{aligned} H(t) &= \frac{1}{2}t^{-1} \\ \implies \qquad H_0 &= \frac{1}{2}t_0^{-1} \\ \implies \qquad t_0 &= \frac{1}{2}H_0^{-1}, \end{aligned}$$

and

$$\begin{aligned} d_{\max}(t) &= a(t)\int_0^{t_0} \frac{\mathrm{d}t'}{a(t')} \\ &= 2t_0 a(t) \int_0^1 \mathrm{d}a \\ &= 2t_0 a(t) \\ &= H_0^{-1} a(t). \end{aligned}$$

Evaluating this at the current epoch gives a cosmological horizon of

$$\boxed{d_{\max}(t_0) = H_0^{-1}}$$

ii) For a matter dominated flat universe, we have that

$$
\begin{aligned}
a(t) &= \left(\frac{t}{t_0}\right)^{2/3} \\
\implies \qquad t &= t_0 a^{3/2} \\
\implies \qquad \mathrm{d}t &= \frac{3}{2} t_0 a^{1/2} \mathrm{d}a.
\end{aligned}
$$

Then,

$$
\begin{aligned}
H(t) &= \frac{2}{3} t^{-1} \\
\implies \qquad H_0 &= \frac{2}{3} t_0^{-1} \\
\implies \qquad t_0 &= \frac{2}{3} H_0^{-1},
\end{aligned}
$$

and

$$
\begin{aligned}
d_{\max}(t) &= a(t) \int_0^{t_0} \frac{\mathrm{d}t'}{a(t')} \\
&= \frac{3}{2} t_0 \int_0^1 \frac{\mathrm{d}a}{a^{1/2}} \\
&= 3t_0 \\
&= 2H_0^{-1}.
\end{aligned}
$$

Evaluating this at the current epoch gives a cosmological horizon of

$$
\boxed{d_{\max}(t_0) = 2H_0^{-1}}
$$

iii) For a cosmological constant dominated flat universe, we have that

$$
\begin{aligned}
a(t) &= e^{H_0 t} \\
\implies \qquad t &= H_0^{-1} \ln(a) \\
\implies \qquad \mathrm{d}t &= \frac{\mathrm{d}a}{H_0 a}.
\end{aligned}
$$

Clearly there is no finite value of the time coordinate t such that $a(t) = 0$. Spencer said to take $t_{\text{beg}} = 0$ in this case, but I'll just leave it as t_{beg}

$$
\begin{aligned}
d_{\max}(t) &= a(t) \int_{t_{\text{beg}}}^{t_0} \frac{\mathrm{d}t'}{a(t')} \\
&= a(t) \int_{t_{\text{beg}}}^{t_0} \mathrm{d}t' e^{-H_0 t'} \\
&= -a(t) H_0^{-1} \left(a^{-1}(t_0) - a^{-1}(t_{\text{beg}})\right)
\end{aligned}
$$

Evaluating this at the current epoch gives a cosmological horizon of

$$\boxed{d_{\max}(t_0) = H_0^{-1}\left(a^{-1}(t_{\text{beg}}) - 1\right)}$$

Problem 2

Similarly to part ii) of Problem 1,

$$\begin{aligned} d(t_0) &= \int_{t_{\text{emit}}}^{t_0} \frac{\mathrm{d}t'}{a(t')} \\ &= H_0^{-1} \int_{a(t_{\text{emit}})}^{1} \frac{\mathrm{d}a}{a^{1/2}} \\ &= H_0^{-1} \int_0^z \frac{\mathrm{d}z}{(1+z)^{(3/2)}} \\ &= 2H_0^{-1}\left(1 - (1+z)^{-1/2}\right) \\ \implies \qquad H_0 d &= z - \frac{3}{4}z^2 + \frac{5}{8}z^3 + \cdots, \end{aligned}$$

so it looks like the quadratic coefficient is $-3/4$.

Problem 3

(a) A static solution should have $\dot{a} = \ddot{a} = 0$. For the second Friedman equation this implies

$$\sum_i (\rho_i + 3p_i) = \sum_i (1 + 3w_i)\rho_i = 0.$$

For normal matter $w = 0$, so we can write the above condition as

$$\rho_{\text{matter}} + (1 + 3w_{\text{other}})\rho_{\text{other}} = 0.$$

Assuming positive energy densities, this implies

$$w_{\text{other}} = -\frac{1}{3}\left(\frac{\rho_{\text{matter}}}{\rho_{\text{other}}} + 1\right) < -\frac{1}{3}$$

(b) Let $\zeta = 8\pi G/3$, and assume positive energy densities. Then

$$\left(\frac{\dot{a}}{a}\right)^2 = 0 = \zeta \sum_i \rho_i - \frac{k}{a^2}$$

$$\implies k = \zeta a^2 \sum_i \rho_i > 0$$

(c) We can rearrange the Friedman equation as

$$\dot{a}^2 - \zeta a^2 \sum_i \rho_i + k = 0,$$

where as in class we can interpret the first term as a "kinetic energy" and the remaining terms as a "potential energy". The potential energy term is an inverted parabola, so clearly the equilibrium is unstable.