Homework 7

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Phys 632

March 1, 2022

Problem 1

(a) Let $\hat{\alpha} = \hat{z}$ and $\hat{\beta} = \cos\theta\hat{z} + \sin\theta\hat{x}$. Then

$$\sigma_{\hat{\beta}} = \cos\theta\sigma_z + \sin\theta\sigma_x = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$$

and

$$\langle + | \sigma_{\hat{\beta}} | + \rangle = \cos\theta$$

(b) When $\hat{\beta}'$ is aligned with $\hat{\alpha}$ the integral evaluates to 1. When $\hat{\beta}'$ is anti-aligned with $\hat{\alpha}$ ($\theta' = \pi$), the integral evaluates to -1. The value of the integral varies lineraly with the angle θ' between 1 and -1 over the range $\theta' = 0$ to $\theta' = \pi$. Thus,

$$\left\langle \sigma(\hat{\beta}, \lambda) \right\rangle = 1 - \frac{2\theta'}{\pi}.$$

To pick $\hat{\beta}'$ (θ') in order to reproduce the results of quantum mechanics, we set

$$\theta' = \frac{\pi}{2}(1 - \cos\theta)$$

Problem 2

(a) As we showed in class, $C(\hat{\alpha}, \hat{\beta}) = -\hat{\alpha} \cdot \hat{\beta}$ for the Bell pair $\frac{1}{\sqrt{2}}(|1, 0\rangle - |0, 1\rangle)$. The first three pairs are 45° appart, while the the last is 135°. Therefore,

$$\mathscr{C} = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 2\sqrt{2}$$

(b) Given

$$\vec{\sigma}_{\hat{\alpha}} = \hat{\alpha} \cdot \vec{\sigma}; \qquad \hat{\gamma} = a\hat{\alpha} + b\hat{\beta} \quad \left(\left|a\right|^2 + \left|b\right|^2 = 1\right),$$

we see

$$\sigma_{\hat{\gamma}} = \hat{\gamma} \cdot \vec{\sigma} = \left(a\hat{\alpha} + b\hat{\beta}\right) \cdot \vec{\sigma} = a\sigma_{\hat{\alpha}} + b\sigma_{\hat{\beta}}.$$

We are therefore justified in writing

$$\sigma_{\hat{\pm}} = \frac{1}{\sqrt{2}} \left(\sigma_{\hat{x}} \pm \sigma_{\hat{y}} \right).$$

Now

$$\begin{aligned}
\sigma^{(1)}_{\hat{x}} \sigma^{(2)}_{\hat{+}} &= \frac{1}{\sqrt{2}} \left(\sigma^{(1)}_{\hat{x}} \sigma^{(2)}_{\hat{x}} + \sigma^{(1)}_{\hat{x}} \sigma^{(2)}_{\hat{y}} \right) \\
\sigma^{(1)}_{\hat{x}} \sigma^{(2)}_{\hat{-}} &= \frac{1}{\sqrt{2}} \left(\sigma^{(1)}_{\hat{x}} \sigma^{(2)}_{\hat{x}} - \sigma^{(1)}_{\hat{x}} \sigma^{(2)}_{\hat{y}} \right) \\
\sigma^{(1)}_{\hat{y}} \sigma^{(2)}_{\hat{+}} &= \frac{1}{\sqrt{2}} \left(\sigma^{(1)}_{\hat{y}} \sigma^{(2)}_{\hat{x}} + \sigma^{(1)}_{\hat{y}} \sigma^{(2)}_{\hat{y}} \right) \\
\sigma^{(1)}_{\hat{y}} \sigma^{(2)}_{\hat{-}} &= \frac{1}{\sqrt{2}} \left(\sigma^{(1)}_{\hat{y}} \sigma^{(2)}_{\hat{x}} - \sigma^{(1)}_{\hat{y}} \sigma^{(2)}_{\hat{y}} \right)
\end{aligned}$$

adding the first three terms and subtracting the last gives

$$\sqrt{2} \left(\sigma^{(1)}_{\hat{x}} \sigma^{(2)}_{\hat{x}} + \sigma^{(1)}_{\hat{y}} \sigma^{(2)}_{\hat{y}} \right)$$

The total correlation combination is easily calcualted as

$$2\sqrt{2} C(\hat{\alpha}, \hat{\alpha}) = -2\sqrt{2}$$

(c) Consider the two cases

$$A_x = 1, A_y = -1, B_+ = 1, B_- = -1 \implies 2$$
$$A_x = 1, A_y = -1, B_+ = 1, B_- = 1 \implies -2$$

Since the other two possible cases are equal by symmetry, we can conclude that any experimental run must result in ± 2.

(d) $|\mathscr{C}| \leq \langle | \pm 2 | \rangle = 2$

Problem 3

$$\begin{aligned}
\partial_t \operatorname{Tr}\left[\rho^2\right] &= \partial_t \sum_{\alpha} \langle \alpha | \rho \rho | \alpha \rangle \\
&= \sum_{\alpha} \langle \alpha | \dot{\rho} \rho + \rho \dot{\rho} | \alpha \rangle \\
&= -\frac{i}{\hbar} \sum_{\alpha} \langle \alpha | [H, \rho] \rho + \rho [H, \rho] | \alpha \rangle \\
&= -\frac{i}{\hbar} \sum_{\alpha} \langle \alpha | H \rho - \rho H \rho + \rho H \rho - H \rho | \alpha \rangle \\
&= 0
\end{aligned}$$

Problem 4

(a) If $\vec{r} = \begin{pmatrix} a & b & c \end{pmatrix}^\intercal$, then

$$\begin{aligned}
\frac{1}{2}\left[\mathcal{I} + \vec{r}\cdot\vec{\sigma}\right] &= \frac{1}{2}\left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} c & 0 \\ 0 & -c \end{pmatrix} + \begin{pmatrix} 0 & a \\ a & 0 \end{pmatrix} + \begin{pmatrix} 0 & -ib \\ ib & 0 \end{pmatrix}\right] \\
&= \frac{1}{2}\begin{pmatrix} 1+c & a-ib \\ a+ib & 1-c \end{pmatrix}
\end{aligned}$$

(b)

$$\begin{aligned}
\mathrm{Tr}[\rho\sigma_x] &= \frac{1}{2}\,\mathrm{Tr}\left[\begin{pmatrix} 1+c & a-ib \\ a+ib & 1-c \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right] = \frac{1}{2}(a - ib + a + ib) = a \\
\mathrm{Tr}[\rho\sigma_y] &= \frac{1}{2}\,\mathrm{Tr}\left[\frac{1}{2}\begin{pmatrix} 1+c & a-ib \\ a+ib & 1-c \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\right] = \frac{1}{2}(ia + b - ia + b) = b \\
\mathrm{Tr}[\rho\sigma_z] &= \frac{1}{2}\,\mathrm{Tr}\left[\frac{1}{2}\begin{pmatrix} 1+c & a-ib \\ a+ib & 1-c \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\right] = \frac{1}{2}(1 + c - 1 + c) = c
\end{aligned}$$

(c) In the "I Know Nothing" state,

$$\langle\sigma_x\rangle = \langle\sigma_y\rangle = \langle\sigma_z\rangle = 0$$

Thus

$$\vec{r} = \vec{0}$$