Homework 8

Sean Ericson
Phys 684

November 25, 2024

Problem 1

(a) The Hamiltonian for the Lambda system in the Schrödinger representation is given by

$$\begin{aligned} H &= H_0 + V \\ &= -\hbar \begin{pmatrix} \omega_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \omega_0' \end{pmatrix} + \frac{\hbar}{2} \begin{pmatrix} 0 & \Omega_0^* e^{i\omega t} & 0 \\ \Omega_0 e^{-i\omega t} & 0 & \Omega_0' e^{-i\omega' t} \\ 0 & \Omega_0'^* e^{i\omega' t} & 0 \end{pmatrix} \\ &= \hbar \begin{pmatrix} -\omega_0 & \Omega_0^* e^{i\omega t} & 0 \\ \Omega_0 e^{-i\omega t} & 0 & \Omega_0' e^{-i\omega' t} \\ 0 & \Omega_0'^* e^{i\omega' t} & -\omega_0' \end{pmatrix}. \end{aligned}$$

Time evolution follows from the Schrödinger equation

$$\begin{aligned} \partial_t \left|\psi\right\rangle &= -\frac{i}{\hbar} H \left|\psi\right\rangle \\ \implies & \begin{cases} \dot{c}_1 = & i\omega_0 c_1 - \frac{i}{2}\Omega_0^* e^{i\omega t} \\ \dot{c}_2 = & -\frac{i}{2}\Omega_0 e^{-i\omega t} c_1 - \frac{i}{2}\Omega_0' e^{i\omega' t} c_3 \\ \dot{c}_3 = & -\frac{i}{2}\Omega_0'^* e^{-i\omega' t} c_2 i\omega_0' c_3 \end{cases} \end{aligned}$$

To go to the field interaction representation, we factor out the applied field via

$$c_1 \to \tilde{c}_1 = c_1 e^{-i\omega t}; \quad c_2 \to \tilde{c}_2 = c_2; \quad c_3 \to \tilde{c}_3 = c_3 e^{-i\omega' t}.$$

Then,

$$
\begin{aligned}
\dot{\tilde{c}}_1 &= \left(-i\omega c_1 + \dot{c}_1\right) e^{-i\omega t} \\
&= \left(-i\omega c_1 + i\omega_0 c_1 - \frac{i}{2}\Omega_0^* e^{i\omega t}\right) e^{-i\omega t} \\
&= i\delta \tilde{c}_1 - \frac{i}{2}\Omega_0^* \tilde{c}_2 \\
\dot{\tilde{c}}_2 &= -\frac{i}{2}\left(\Omega_0 \tilde{c}_1 + \Omega_0' \tilde{c}_3\right) \\
\dot{\tilde{c}}_3 &= \left(i\omega' c_3 + \dot{c}_3\right) e^{i\omega' t} \\
&= \left(-i\omega' c_3 + -\frac{i}{2}\Omega_0'^* e^{-i\omega' t} c_2 + i\omega_0' c_3\right) e^{i\omega' t} \\
&= -\frac{i}{2}\Omega_0'^* \tilde{c}_2 + i\delta' \tilde{c}_3.
\end{aligned}
$$

Putting this together, we have

$$
\begin{aligned}
\partial_t |\tilde{\psi}\rangle &= i \begin{pmatrix} \delta \tilde{c}_1 - \frac{1}{2}\Omega_0^* \tilde{c}_2 \\ -\frac{1}{2}\Omega_0 \tilde{c}_1 - \frac{1}{2}\Omega_0' \tilde{c}_3 \\ -\frac{1}{2}\Omega_0'^* \tilde{c}_2 + \delta' \tilde{c}_3 \end{pmatrix} \\
&= -\frac{i}{\hbar}\tilde{H}|\tilde{\psi}\rangle \\
\implies \qquad \tilde{H} &= \hbar \begin{pmatrix} -2\delta & \frac{\Omega_0^*}{2} & 0 \\ \frac{\Omega_0}{2} & 0 & \frac{\Omega_0'}{2} \\ 0 & \frac{\Omega_0'^*}{2} & -2\delta' \end{pmatrix}.
\end{aligned}
$$

(b) Given that

$$
\begin{aligned}
|D\rangle &= \frac{1}{\Omega}\left(\Omega_0' |1\rangle + \Omega_0 |3\rangle\right), \\
|B\rangle &= \frac{1}{\Omega}\left(\Omega_0^* |1\rangle - \Omega_0'^* |3\rangle\right),
\end{aligned}
$$

where $\Omega = \sqrt{|\Omega_0|^2 + |\Omega_0'|^2}$, we solve for the $|1\rangle$ and $|3\rangle$ states to get

$$
\begin{aligned}
|1\rangle &= \frac{1}{\Omega}\left(\Omega_0'^* |D\rangle + \Omega_0 |B\rangle\right) \\
|3\rangle &= \frac{1}{\Omega}\left(\Omega_0^* |D\rangle - \Omega_0' |B\rangle\right)
\end{aligned}
$$

(c)

Problem 2

Problem 3

Problem 4

Problem 5 (Berman 9.1-2)