Homework 1

Sean Ericson
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0.2.1 **The brachistochrone problem**

(a) Let
$$\tilde{g} = g \sin \alpha.$$
Conservation of energy implies
$$\frac{1}{2} m v^2 = -m \tilde{g} y \implies \boxed{v = \sqrt{-2\tilde{g} y}}$$

(b) The passage time is given by
$$\begin{aligned} T &= \int \frac{\mathrm{d}s}{v} \\ &= \int_{x_1}^{x_2} \sqrt{\frac{\mathrm{d}x^2 + \mathrm{d}y^2}{-2\tilde{g} y}} \\ &= \int_{x_1}^{x_2} \sqrt{\frac{\mathrm{d}x^2 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\mathrm{d}x\right)^2}{-2\tilde{g} y}} \\ &= \int_{x_1}^{x_2} \mathrm{d}x \sqrt{\frac{1 + (y')^2}{-2\tilde{g} y}}, \end{aligned}$$
where we identify
$$L(y, y') = \sqrt{\frac{1 + (y')^2}{-2\tilde{g} y}}.$$
Now, using the derivative
$$\frac{\partial L}{\partial y'} = \frac{y'}{1 + (y')^2} \sqrt{\frac{1 + (y')^2}{-2\tilde{g} y}},$$

we can write out Jaboci's integral:

$$
\begin{aligned}
C_1 &= y'\frac{\partial L}{\partial y'} - L \\
&= \frac{(y')^2}{1+(y')^2}\sqrt{\frac{1+(y')^2}{-2\tilde{g}y}} - \sqrt{\frac{1+(y')^2}{-2\tilde{g}y}} \\
&= \frac{-1}{1+(y')^2}\sqrt{\frac{1+(y')^2}{-2\tilde{g}y}} \\
&= \frac{-1}{\sqrt{-2\tilde{g}y(1+(y')^2)}}.
\end{aligned}
$$

This equation represents an ODE for $y(x)$.

(c) Let $y' = \cot t$ (note that this implies $\mathrm{d}x/\mathrm{d}y = \tan(t)$). Plugging this into the above ODE gives

$$
-2\tilde{g}y\csc^2(t) = \frac{1}{C_1^2} \implies \boxed{y(t) = -\frac{\sin^2(t)}{2C_1^2\tilde{g}}},
$$

with time derivative

$$
\frac{\mathrm{d}y}{\mathrm{d}t} = -\frac{\sin(t)\cos(t)}{2C_1^2\tilde{g}}
$$

Now,

$$
\begin{aligned}
\frac{\mathrm{d}x}{\mathrm{d}t} &= \frac{\mathrm{d}x}{\mathrm{d}y}\frac{\mathrm{d}y}{\mathrm{d}t} = -\tan(t)\frac{\sin(t)\cos(t)}{2C_1^2\tilde{g}} = -\frac{\sin^2(t)}{C_1^2\tilde{g}} \\
&\implies \boxed{x(t) = \frac{\sin(t)\cos(t) - t}{2C_1^2\tilde{g}} + C_2}
\end{aligned}
$$

Hey, that's a parameterization of a cycloid!

(d) Let $t_1 = 0$. Then $T = t_2$ and we have that

$$
T = \cot^{-1}(y_2')
$$

(e) Let P_1 be the origin. The length of the shortest path between P_1 and P_2 (a straight line) is therefore

$$
D = \sqrt{x_2^2 + y_2^2}
$$

(f) π is the ration of a circle's diameter to its circumference. (Sorry, I didn't get part e finished, so this is the only ratio I can offer)

0.2.2 **Dido's problem**

The full Lagriangian is

$$L = \frac{1}{2}(\dot{x}y - \dot{y}x) + \lambda\sqrt{\dot{x}^2 + \dot{y}^2}.$$

We can start by taking some derivatives:

$$\begin{aligned}
\frac{\partial L}{\partial x} &= -\frac{1}{2}\dot{y} \\
\frac{\partial L}{\partial y} &= \frac{1}{2}\dot{x} \\
\frac{\partial L}{\partial \dot{x}} &= \frac{1}{2}y + \lambda\frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}} \\
\frac{\partial L}{\partial \dot{y}} &= -\frac{1}{2}x + \lambda\frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}
\end{aligned}$$

The Euler-Lagrange equation for the x-coordinate gives

$$\begin{aligned}
-\frac{1}{2}\dot{y} &= \frac{1}{2}\dot{y} + \lambda\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right) \\
\implies -\frac{\mathrm{d}}{\mathrm{d}t}y &= \lambda\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right) \\
\implies y - y_0 &= -\frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2}}
\end{aligned}$$

Similarly, the Euler-Lagrange equation for the y-coordinate gives

$$\begin{aligned}
\frac{1}{2}\dot{x} &= -\frac{1}{2}\dot{x} + \lambda\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right) \\
\implies \frac{\mathrm{d}}{\mathrm{d}t}x &= \lambda\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}\right) \\
\implies x - x_0 &= \frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2}}
\end{aligned}$$

Squaring and adding the previous two results, we see that

$$(x - x_0)^2 + (y - y_0)^2 = \lambda^2,$$

which is the equation for a circle centered at (x_0, y_0) with radius λ. Let the origin be at point O, with the x-axis pointing along the line $\overline{OP}$, as depicted in Figure 1. By symmetry, we can see that

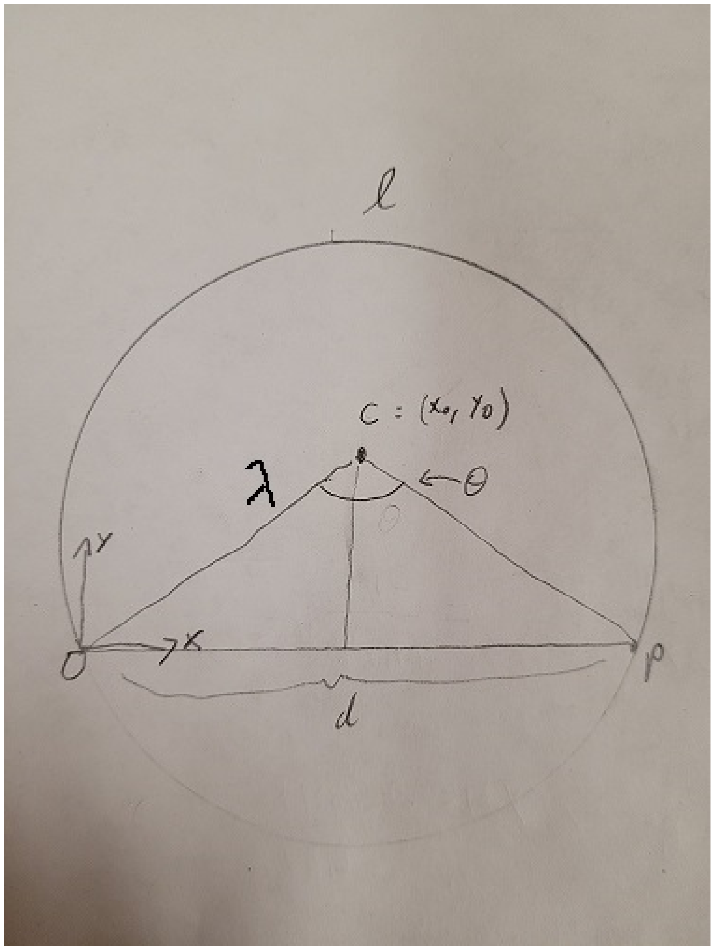

Figure 1: An illustration of the relevent quantities.

$$x_0 = \frac{d}{2},$$

leaving y_0 and λ to be determined as a functions of d and l. From Figure 1, we can see that

$$y_0^2 + \left(\frac{d}{2}\right)^2 = \lambda^2,$$

$$l = (2\pi - \theta)\lambda,$$

and

$$\sin(\theta/2) = \frac{d}{2\lambda}.$$

These three equations implicitly define y_0 and λ as functions of d and l, but they cannot be solved analytically.

0.2.3 **Geodesics on the 2-sphere**

We want to minimize

$$\int \mathrm{d}t \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$$

subject to the constraint

$$x^2 + y^2 + z^2 = R^2$$

We start by taking some derivatives. Let

$$L_1 = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$$

then

$$\frac{\partial L_1}{\partial \dot{x}} = \frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}$$

$$\frac{\partial L_1}{\partial \dot{y}} = \frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}$$

$$\frac{\partial L_1}{\partial \dot{z}} = \frac{\dot{z}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}}$$

The Euler-Lagrange equations then give

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\dot{x}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} = 2\lambda x$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\dot{y}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} = 2\lambda y$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\dot{z}}{\sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}} = 2\lambda z$$