Homework 4

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Phys 684

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Problem 1

(a) Letting

$$\vec{R} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}; \quad \vec{\Omega} = \begin{pmatrix} \Omega_0' \\ -\Omega_0'' \\ \delta \end{pmatrix},$$

the evolution of the Bloch vector in the field interaction basis is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\vec{R} = \vec{\Omega} \times \vec{R} \implies \begin{cases} \dot{u} = -\delta v - \Omega_0'' w \\ \dot{v} = \delta u - \Omega_0' w \\ \dot{w} = \Omega_0' v + \Omega_0'' u. \end{cases}$$

Given that $\Omega_0' = \Omega_0$ and $\Omega_0'' = 0$ during the pulse, this simplifies to

$$\begin{aligned} \dot{u} &= -\delta v \\ \dot{v} &= \delta u - \Omega_0 w \\ \dot{w} &= \Omega_0 v. \end{aligned}$$

Then,

$$\begin{aligned} \ddot{v} &= \delta \dot{u} - \Omega_0 \dot{w} \\ &= -\left(\delta^2 + \Omega_0^2\right) v \\ &= -\Omega^2 v \\ \implies \quad v(t) &= A\cos(\Omega t) + B\sin(\Omega t). \end{aligned}$$

Next, the initial condition $\vec{R}(0) = -\hat{w}$ implies $A = 0$, so

$$v(t) = B\sin(\Omega t).$$

Now solving for u,

$$
\begin{aligned}
& & \dot{u} &= -\delta v \\
&\implies & u(t) &= u(0) - \delta \int_0^t \mathrm{d}t' v(t') \\
& & &= \frac{\delta B}{\Omega} \cos(\Omega t')\Big|_0^t \\
& & &= \frac{\delta B}{\Omega} \left(\cos(\Omega t) - 1\right),
\end{aligned}
$$

and then w

$$
\begin{aligned}
& & \dot{w} &= \Omega_0 v \\
&\implies & w(t) &= u(0) + \Omega_0 \int_0^t \mathrm{d}t' v(t') \\
& & &= -1 - \frac{\Omega_0 B}{\Omega} \cos(\Omega t')\Big|_0^t \\
& & &= -1 - \frac{\Omega_0 B}{\Omega} \left(\cos(\Omega t) - 1\right)
\end{aligned}
$$

Using the equation for $\dot{v}$ we can determine the value of B:

$$
\begin{aligned}
& & \dot{v} &= \delta u - \Omega_0 w \\
&\implies & \Omega B \cos(\Omega t) &= \frac{\delta^2 B}{\Omega} \left(\cos(\Omega t) - 1\right) + \Omega_0 \left(1 + \frac{\Omega_0 B}{\Omega} \left(\cos(\Omega t) - 1\right)\right) \\
& & &= \frac{\delta^2 + \Omega_0^2}{\Omega} B \cos(\Omega t) - \frac{\delta^2 + \Omega_0^2}{\Omega} B + \Omega_0 \\
& & &= \Omega B \cos(\Omega t) - \Omega B + \Omega_0 \\
&\implies & \Omega B &= \Omega_0 \\
&\implies & B &= \frac{\Omega_0}{\Omega}.
\end{aligned}
$$

After the pulse ends the precession of the Bloch vector stops, and its final position is given by $\vec{R}(\tau)$. Letting $\theta = \Omega \tau$,

$$
\begin{aligned}
u &= \frac{\Omega_0 \delta}{\Omega^2} \left(\cos\theta - 1\right) \\
v &= \frac{\Omega_0}{\Omega} \sin\theta \\
w &= -\left[1 + \frac{\Omega_0^2}{\Omega^2} \left(\cos\theta - 1\right)\right].
\end{aligned}
$$

Problem 2

See attached Mathematica notebook.

Problem 3

See attached Mathematica notebook.

Problem 4

See attached Mathematica notebook for calculations.

$$H_{\mathrm{d}} = -\frac{\hbar}{2}\Omega_0\sigma_z + \hbar\dot{\theta}\sigma_y$$

$$\begin{aligned}
\dot{\tilde{\rho}}_{\mathrm{d}} &= [H_{\mathrm{d}}, \tilde{\rho}_{\mathrm{d}}] \\
&= \begin{pmatrix} -2\dot{\theta}\,\mathrm{Re}[\tilde{\rho}_{\mathrm{d}12}] & i\Omega_0\rho_{12} - \dot{\theta}\left(\tilde{\rho}_{\mathrm{d}22} - \tilde{\rho}_{\mathrm{d}11}\right) \\ -i\Omega_0\rho_{21} - \dot{\theta}\left(\tilde{\rho}_{\mathrm{d}22} - \tilde{\rho}_{\mathrm{d}11}\right) & 2\dot{\theta}\,\mathrm{Re}[\tilde{\rho}_{\mathrm{d}12}] \end{pmatrix}
\end{aligned}$$

The relaxation terms take the same form as in the previous problem, i.e.

$$\left.\frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}t}\right|_{\mathrm{relaxation}} = -\gamma\left(\sigma_0\tilde{\rho}_{\mathrm{d}} + \tilde{\rho}_{\mathrm{d}}\sigma_0\right) + \gamma_2\sigma_-\tilde{\rho}_{\mathrm{d}}\sigma_+ + 2\Gamma\sigma_0\tilde{\rho}_{\mathrm{d}}\sigma_0.$$

Adding this to the above equation, we get

$$\dot{\tilde{\rho}}_{\mathrm{d}} = \begin{pmatrix} -2\dot{\theta}\,\mathrm{Re}[\tilde{\rho}_{\mathrm{d}12}] + \gamma_2\tilde{\rho}_{\mathrm{d}22} & i\Omega_0\rho_{12} - \dot{\theta}\left(\tilde{\rho}_{\mathrm{d}22} - \tilde{\rho}_{\mathrm{d}11}\right) - \gamma\tilde{\rho}_{\mathrm{d}12} \\ -i\Omega_0\rho_{21} - \dot{\theta}\left(\tilde{\rho}_{\mathrm{d}22} - \tilde{\rho}_{\mathrm{d}11}\right) - \gamma\tilde{\rho}_{\mathrm{d}21} & 2\dot{\theta}\,\mathrm{Re}[\tilde{\rho}_{\mathrm{d}12}] - \gamma_2\rho_{22} \end{pmatrix}$$

Problem 2

In[∘]:=
```
{û, v̂, ŵ} = IdentityMatrix[3];
R⃗0 = -ŵ;
ϕ = ArcTan[Sqrt[Ω^2 - Ω0^2]/Ω0];
```

In[∘]:=
```
(RotationMatrix[-ϕ, v̂].RotationMatrix[θ, û].RotationMatrix[ϕ, v̂].R⃗0 /.
    {Sqrt[Ω^2 - Ω0^2] → δ}) // FullSimplify // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} \frac{\delta\,(-1+\mathrm{Cos}[\theta])\,\Omega_0}{\Omega^2} \\ \frac{\mathrm{Sin}[\theta]\,\Omega_0}{\Omega} \\ -1-\frac{(-1+\mathrm{Cos}[\theta])\,\Omega_0^2}{\Omega^2} \end{pmatrix}$$

Problem 3

In[∘]:=
```
{σx, σy, σz} = Table[PauliMatrix[i], {i, 1, 3}];
{σ+, σ-} = 1/2 (σx ∓ ⅈ σy); σ0 = σ+.σ-;
rho = {{ρ11, ρ12}, {ρ21, ρ22}};
```

In[∘]:=
```
(-γ (σ0.rho + rho.σ0) + γ2 σ-.rho.σ+ + 2 Γ σ0.rho.σ0 // FullSimplify) /.
  {-γ + Γ → -γ2 / 2} // MatrixForm
```

Out[∘]//MatrixForm=

$$\begin{pmatrix} \gamma_2\,\rho_{22} & -\gamma\,\rho_{12} \\ -\gamma\,\rho_{21} & -\gamma_2\,\rho_{22} \end{pmatrix}$$

Problem 4

In[∘]:=
```
Hd = -ℏ/2 Ω0 σz + ℏ θ̇ σy;
```

In[ ]:=
```
rhoDot = 1/(ⅈ ℏ) comm[H_d, rho] // FullSimplify;
rhoDot // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} -\dot{\theta}\,(\rho_{12}+\rho_{21}) & \dot{\theta}\,(\rho_{11}-\rho_{22})+i\,\rho_{12}\,\Omega_0 \\ \dot{\theta}\,(\rho_{11}-\rho_{22})-i\,\rho_{21}\,\Omega_0 & \dot{\theta}\,(\rho_{12}+\rho_{21}) \end{pmatrix}$$

In[ ]:=
```
rhoDot + (-γ (σ_0.rho + rho.σ_0) + γ_2 σ_-.rho.σ_+ + 2 Γ σ_0.rho.σ_0  // FullSimplify) /.
  {-γ + Γ → -γ_2 / 2} // MatrixForm
```

Out[ ]//MatrixForm=

$$\begin{pmatrix} -\dot{\theta}\,(\rho_{12}+\rho_{21})+\gamma_2\,\rho_{22} & -\gamma\,\rho_{12}+\dot{\theta}\,(\rho_{11}-\rho_{22})+i\,\rho_{12}\,\Omega_0 \\ -\gamma\,\rho_{21}+\dot{\theta}\,(\rho_{11}-\rho_{22})-i\,\rho_{21}\,\Omega_0 & \dot{\theta}\,(\rho_{12}+\rho_{21})-\gamma_2\,\rho_{22} \end{pmatrix}$$