Homework 2

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Phys 684

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Problem 1 (Berman 2.9)

In the adiabatic approximation, the dressed-state amplitudes satisfy (eq. 2.146 in the textbook)

$$\begin{aligned} c_{d_1}(t) &= e^{\frac{i}{2}\xi(t)}c_{d_1}(t_0) \\ c_{d_2}(t) &= e^{-\frac{i}{2}\xi(t)}c_{d_2}(t_0), \end{aligned}$$

where

$$\xi(t) = \int_{t_0}^{t} \mathrm{d}t'\Omega(t').$$

In this case, $t_0 = -\infty$ and

$$\Omega_0(t) = \Omega_0 e^{-\left(\frac{t}{T}\right)^2},$$

so we can integrate that to get

$$\xi(t) = \frac{\sqrt{\pi}}{2}\Omega_0 T\left(1 + \operatorname{erf}\left(\frac{t}{T}\right)\right).$$

We transform between the dressed-states and the field-interaction basis states via

$$\begin{pmatrix}\tilde{c}_1 \\ \tilde{c}_2\end{pmatrix} = \begin{pmatrix} c_\theta & s_\theta \\ -s_\theta & c_\theta \end{pmatrix}\begin{pmatrix} c_{d_1} \\ c_{d_2}\end{pmatrix}; \quad \begin{pmatrix} c_{d_1} \\ c_{d_2}\end{pmatrix} = \begin{pmatrix} c_\theta & -s_\theta \\ s_\theta & c_\theta \end{pmatrix}\begin{pmatrix}\tilde{c}_1 \\ \tilde{c}_2\end{pmatrix},$$

where

$$c_\theta = \sqrt{\frac{1}{2}\left(1 + \frac{\delta}{\Omega(t)}\right)}; \quad s_\theta = \sqrt{\frac{1}{2}\left(1 - \frac{\delta}{\Omega(t)}\right)},$$

and $\Omega(t) = \sqrt{\delta^2 + \Omega_0^2(t)}$. Given that $\tilde{c}_1(-\infty) = 1$ and $\tilde{c}_2(-\infty) = 0$, we find that

$$\begin{aligned} c_{d_1}(-\infty) &= c_\theta(-\infty) = 1 \\ c_{d_2}(-\infty) &= s_\theta(-\infty) = 0 \end{aligned}$$

Putting everything together, we find

$$\begin{aligned}
\begin{pmatrix} \tilde{c}_1(t) \\ \tilde{c}_2(t) \end{pmatrix} &= \begin{pmatrix} c_\theta & s_\theta \\ -s_\theta & c_\theta \end{pmatrix} \begin{pmatrix} c_{d_1}(t) \\ c_{d_2}(t) \end{pmatrix} \\
&= \begin{pmatrix} c_\theta(t) c_{d_1}(t) + s_\theta(t) c_{d_2}(t) \\ -s_\theta c_{d_1}(t) + c_\theta c_{d_2}(t) \end{pmatrix} \\
&= \begin{pmatrix} c_\theta(t) \exp\left[\frac{i}{2}\xi(t)\right] \\ -s_\theta(t) \exp\left[\frac{i}{2}\xi(t)\right] \end{pmatrix}
\end{aligned}$$

Problem 2 (Berman 2.17)

Problem 3

(a) Given that

$$H_0 = \hbar \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix},$$

we have that

$$\begin{aligned}
\exp(iH_0 t/\hbar) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + it \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix} + \frac{(it)^2}{2} \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix}^2 + \frac{(it)^3}{3!} \begin{pmatrix} \omega_1 & 0 \\ 0 & \omega_2 \end{pmatrix}^3 + \cdots \\
&= \begin{pmatrix} 1 + i\omega_1 t + \frac{1}{2}(i\omega_1 t)^2 + \frac{1}{3!}(i\omega_1 t)^3 + \cdots & 0 \\ 0 & 1 + i\omega_2 t + \frac{1}{2}(i\omega_2 t)^2 + \frac{1}{3!}(i\omega_2 t)^3 + \cdots \end{pmatrix} \\
&= \begin{pmatrix} \exp(i\omega_1 t) & 0 \\ 0 & \exp(i\omega_2 t) \end{pmatrix}.
\end{aligned}$$

(b) In the interaction representation, we factor out this free phase evolution by writing

$$|\psi(t)\rangle_{\mathrm{I}} = \bar{c}_1(t) e^{-i\omega_1 t} |1\rangle + \bar{c}_2(t) e^{-i\omega_2 t} |2\rangle,$$

that is, we make the (time-dependent) unitary transformation

$$|\psi(t)\rangle_{\mathrm{S}} \to |\psi(t)\rangle_{\mathrm{I}} = U(t) |\psi(t)\rangle_{\mathrm{S}},$$

where

$$U(t) = \begin{pmatrix} e^{-i\omega_1 t} & 0 \\ 0 & e^{-i\omega_2 t} \end{pmatrix}.$$

We get the effective interaction hamiltonian by making the inverse transformation on V:

$$V_{\mathrm{I}} = U^\dagger V U = \hbar\Omega_0 \cos(\omega t) \begin{pmatrix} 0 & e^{-i\omega_0 t} \\ e^{i\omega_0 t} & 0 \end{pmatrix},$$

where the phase ϕ has been absorbed into the (complex) Rabi frequency.

Problem 4

We seek to find the eigenvectors of

$$\tilde{H} = \frac{\hbar}{2}\begin{pmatrix} -\delta & \Omega_0 \\ \Omega_0 & \delta \end{pmatrix}.$$

Firstly, the eigenvalues of a 2x2 matrix are given by

$$\frac{1}{2}\left(T \pm \sqrt{T^2 - 4D}\right),$$

where T is the trace, and D is the determinant. In this case,

$$T = 0; \quad D = -\delta^2 - \Omega_0^2 =: -\frac{\hbar^2\Omega^2}{4},$$

so the eigenvalues are simply $\pm\frac{1}{2}\hbar\Omega$. Let the + eigenvector be of the form

$$\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix},$$

i.e.

$$\frac{\hbar}{2}\begin{pmatrix} -\delta & \Omega_0 \\ \Omega_0 & \delta \end{pmatrix}\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} = \frac{\hbar\Omega}{2}\begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}.$$

Taking the first entry, we find

$$\begin{aligned} & -\delta\sin\theta + \Omega_0\cos\theta = \Omega\sin\theta \\ \implies \qquad & (\Omega + \delta)\sin\theta = \Omega_0\cos\theta \\ \implies \qquad & \tan\theta = \frac{\Omega_0}{\Omega + \delta} \\ \implies \qquad & \theta = \operatorname{atan}\frac{\Omega_0}{\Omega + \delta}. \end{aligned}$$

Now, using the identity

$$\tan(2\operatorname{atan}(x)) = \frac{-2x}{x^2 - 1},$$

we find

$$\begin{aligned} \tan(2\theta) &= \frac{-2\left(\frac{\Omega_0}{\Omega+\delta}\right)}{\left(\frac{\Omega_0}{\Omega+\delta}\right)^2 - 1} \\ &= \frac{\Omega_0}{\delta}. \end{aligned}$$

The eigenvectors are then

$$\begin{aligned} |+\rangle &= \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} \\ |-\rangle &= \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix}, \end{aligned}$$

with θ given implicitly above.

Problem 4

The general solution is given by

$$\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} = \begin{pmatrix} e^{-i\delta t/2}\left[\left[\cos\frac{\Omega t}{2} + i\frac{\delta}{\Omega}\sin\frac{\Omega t}{2}\right]c_1(0) - i\frac{\Omega_0^*}{\Omega}\sin\frac{\Omega t}{2}c_2(0)\right] \\ e^{i\delta t/2}\left[-i\frac{\Omega_0}{\Omega}\sin\frac{\Omega t}{2}c_1(0) + \left[\cos\frac{\Omega t}{2} - i\frac{\delta}{\Omega}\sin\frac{\Omega t}{2}\right]c_2(0)\right] \end{pmatrix}.$$

In the $|\delta| \gg \Omega_0$ limit, we have that

$$\Omega = \sqrt{\Omega_0^2 + \delta^2} \approx \delta(1 + \frac{\Omega_0^2}{\delta^2})$$

(a) With the initial conditions $c_1(0) = 1$ and $c_2(0) = 0$, we have

$$\begin{pmatrix} c_1(t) \\ c_2(t) \end{pmatrix} = \begin{pmatrix} e^{-i\delta t/2}\left(\cos\frac{\Omega t}{2} + i\frac{\delta}{\Omega}\sin\frac{\Omega t}{2}\right) \\ -i\frac{\Omega_0}{\Omega}e^{i\delta t/2}\sin\frac{\Omega t}{2} \end{pmatrix}$$

Focusing on c_1, we find

$$\begin{aligned} c_1(t) &= e^{-i\delta t/2}\left(\cos\frac{\Omega t}{2} + i\frac{\delta}{\Omega}\sin\frac{\Omega t}{2}\right) \\ &= \frac{1}{2}e^{-i\delta t/2}\left[\left(e^{i\Omega t/2} + e^{-i\Omega t/2}\right) + \frac{\delta}{\Omega}\left(e^{i\Omega t/2} - e^{-i\Omega t/2}\right)\right] \\ &= \frac{1}{2\Omega}e^{-i(\delta+\Omega)t/2}\left[(\Omega - \delta) + (\Omega + \delta)e^{i\Omega t}\right] \\ &= \frac{\Omega + \delta}{2\Omega}e^{-i(\delta-\Omega)t/2} + O(\frac{\Omega_0^2}{\delta^2}). \end{aligned}$$

Now, since this is in the interaction representation, we have that

$$\begin{aligned} |\psi(t)\rangle &= c_1(t)e^{-i\omega_1 t}|1\rangle \\ &\propto e^{-i(\omega_1 + (\delta-\Omega)/2)t}|1\rangle, \end{aligned}$$

i.e., the $|1\rangle$ state has an apparent energy shift of

$$\begin{aligned} \Delta &= \frac{1}{2}(\delta - \Omega) \\ &\approx \frac{1}{2}\left(\delta - \delta(1 - \frac{\Omega_0^2}{2\delta^2})\right) \\ &= \frac{\Omega_0^2}{4\delta} \end{aligned}$$

In[5]:=
```
Symbolize[δ₀]; Symbolize[Ω₀]; Symbolize[T₀]; Symbolize[H_I];
Symbolize[ψ_I]; Symbolize[c₁]; Symbolize[c₂];
Symbolize[s_θ]; Symbolize[c_θ];
$Assumptions = {T > 0};
```

Problem 1

In[9]:=
$$\Omega_0[t\_] = A\, e^{-\left(\frac{t}{T}\right)^2};$$
$$R[t\_] = \sqrt{\delta^2 + (\Omega_0[t])^2} \text{ // FullSimplify;}$$

In[11]:= Integrate[Ω_0[t], {t, -∞, t}]

Out[11]=
$$\frac{1}{2} A \sqrt{\pi}\, T \left(1 + \text{Erf}\left[\frac{t}{T}\right]\right)$$

In[12]:=
$$c_\theta[t\_] = \sqrt{\frac{1}{2}\left(1 + \frac{\delta}{R[t]}\right)} \text{ // FullSimplify;}$$
$$s_\theta[t\_] = \sqrt{\frac{1}{2}\left(1 - \frac{\delta}{R[t]}\right)} \text{ // FullSimplify;}$$

In[15]:= s_θ[-∞] // FullSimplify

Out[15]=
$$\frac{\sqrt{1 - \frac{\delta}{\sqrt{\delta^2}}}}{\sqrt{2}}$$

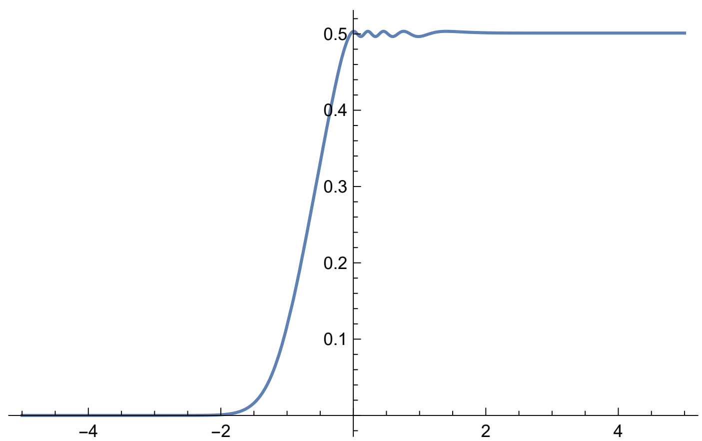
Problem 2

In[ ]:=
$$\delta[t\_] = \delta_0 \left(1 - e^{\frac{t}{T}}\right)^3 \text{HeavisideTheta}[-t];$$
$$H_I[t\_] = \frac{1}{2}\left\{\left\{0,\ \Omega_0[t]\, e^{-i\,\delta[t]}\right\},\ \left\{\Omega_0[t]\, e^{i\,\delta[t]},\ 0\right\}\right\};$$
$$\psi_I[t\_] = \{\{c_1[t]\}, \{c_2[t]\}\};$$

In[•]:=
```
T0 = 5;
T = 1;
δ0 = 30;
A = 30;
soln1 =
  NDSolve[{ⅈ D[ψI[t], t] == HI[t].ψI[t], ψI[-T0] == {{1}, {0}}}, c1, {t, -T0, T0}];
soln2 = NDSolve[{ⅈ D[ψI[t], t] == HI[t].ψI[t], ψI[-T0] == {{1}, {0}}}, c2, {t, -T0, T0}];
soln12 = Union[soln1, soln2];
```

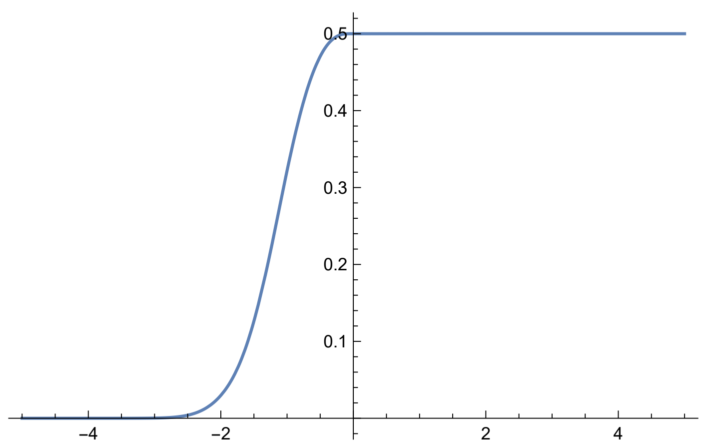
In[•]:=
```
Plot[Evaluate[Abs[c2[x]]^2 /. soln2], {x, -T0, T0}, PlotRange → All]
```

Out[•]=



In[•]:=
```
Plot[Evaluate[Evaluate[Abs[c1[x] Conjugate[c2[x]]] /. soln1] /. soln2],
  {x, -T0, T0}, PlotRange → All]
```

Out[•]=



Problem 3

In[•]:=
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a
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Out[•]=

a

Problem 4

In[∘]:= a

Out[∘]=

a

Problem 5

In[∘]:=
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"Ω = √(δ² + A²);"
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$$c_1 = e^{-i\,\delta\,t/2}\left(\text{Cos}\left[\frac{\Omega\,t}{2}\right] + \frac{i\,\delta}{\Omega}\,\text{Sin}\left[\frac{\Omega\,t}{2}\right]\right);$$
$$c_2 = -\frac{i\,A}{\Omega}\,e^{i\,\delta\,t/2}\,\text{Sin}\left[\frac{\Omega\,t}{2}\right];$$

Out[∘]=

$$\Omega = \sqrt{\delta^2 + A^2};$$

In[∘]:= c_1 // TrigToExp // Simplify

Out[∘]=

$$\frac{e^{-\frac{1}{2}\,i\,t\,\delta}\,e^{-\frac{1}{2}\,i\,t\,\Omega}\left(\left(-1 + e^{i\,t\,\Omega}\right)\delta + \left(1 + e^{i\,t\,\Omega}\right)\Omega\right)}{2\,\Omega}$$

In[∘]:= c_2 // TrigToExp // Simplify

Out[∘]=

$$-\frac{A\,e^{\frac{1}{2}\,i\,t\,(\delta-\Omega)}\left(-1 + e^{i\,t\,\Omega}\right)}{2\,\Omega}$$

In[∘]:=
$$\frac{\left(\left(-1 + e^{i\,t\,\Omega}\right)\delta + \left(1 + e^{i\,t\,\Omega}\right)\Omega\right)}{2\,\Omega} \text{ // FullSimplify}$$

Out[∘]=

$$\frac{-\delta + \Omega + e^{i\,t\,\Omega}\,(\delta + \Omega)}{2\,\Omega}$$