Mathematics of Physical Quantities and Units:

Dimensional Analysis and The Π Theorem

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UO

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Scalars, Physical Quantities, and Dimensions

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$\ker[d] = \mathrm{im}[i]$.

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Splittings (i.e. Choosing Base Units)

A linear map $U : \mathcal{D} \to \mathcal{P}$ such that $d(U(D)) = D \quad \forall D \in \mathcal{D}$ is called a *linear splitting*.

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χ as given is a bijection, hence $\mathcal{S} \times \mathcal{D} \simeq \mathcal{P}$.

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Example: $\quad lwh - V = 0$

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Consider a $G(\mathcal{E})$-invariant $F \subseteq \langle E_1 \rangle, \cdots, \langle E_m \rangle, \langle D_1 \rangle, \cdots, \langle D_n \rangle$ where the $\{E_i\}$ are linearly independent and $D_j = \sum k_{ji} E_i$.

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Then, there exists $\Phi \subseteq \mathcal{S}^n$ such that

$$(p_1, \cdots, p_m, q_1, \cdots, q_n) \in F$$

$$\implies (q_1 - \sum k_{1i} p_i, \cdots, q_n - \sum k_{ni} p_i) \in \Phi$$

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Then, applying $G(\mathcal{E})$-invariance of F implies

$$(U_0(E_1), \cdots, U_0(E_m), q_1 + U_0(D_1) - U(D_1), \cdots, q_n + U_0(D_n) - U(D_n)) \in F$$

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$$U(D_i) = U(\sum k_{ij} E_j) = \sum k_{ji} U(E_j) = \sum k_{ij} p_j \square$$

An (Obvious) Proposition

We say a function $f : \langle E_1 \rangle \times \cdots \times \langle E_m \rangle \times \langle D_1 \rangle \cdots \langle D_n \rangle \to \langle C \rangle$ is $G(\mathcal{E})$-invariant if, for all $\theta \in G(\mathcal{E})$,

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f is clearly not invariant under this map, a contradiction $\square$.

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Then,

1) There exists k_{ij} such that $C = \sum k_{n+1,i} E_i$
2) There exists $\phi : \mathcal{S}^n \to \mathcal{S}$ such that for all $(p_1, \cdots, q_n) \in \langle E_1 \rangle \times \cdots \times \langle D_n \rangle$,

$$f(\vec{p}, \vec{q}) = \phi(q_1 - \sum k_{1j} p_j, \cdots, q_n - \sum k_{nj} p_j) + \sum k_{n+1,j} p_j$$

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ϕ is then defined by

$$\Phi = (\lambda_1, \cdots, \lambda_n, \phi(\lambda_1, \cdots, \lambda_n))$$

Example: The Harmonic Oscillator

Consider the relation

$$F(m, \hbar, \omega; x, p, H) = H - \frac{p^2}{2m} - \frac{1}{2}m\omega^2 x^2 = 0$$

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$$[m] = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \; [\hbar] = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}, \; [\omega] = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

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$$[m] = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ [\hbar] = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix},\ [\omega] = \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}.$$

These vectors span the space:

$$|\mathcal{M}| = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & -1 & -1 \end{vmatrix} = -2 \neq 0$$

We can then solve for the dimensions of the variables in terms of the dimensions of the parameters:

Example: The Harmonic Oscillator (cont.)

We can then solve for the dimensions of the variables in terms of the dimensions of the parameters:

$$[x] = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \quad \mathcal{M}^{-1}[x] = \begin{pmatrix} -1/2 \\ 1/2 \\ -1/2 \end{pmatrix} \implies [x] = \left[\sqrt{\frac{\hbar}{m\omega}}\right]$$

$$[p] = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}; \quad \mathcal{M}^{-1}[p] = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \end{pmatrix} \implies [p] = \left[\sqrt{m\hbar\omega}\right]$$

$$[H] = \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}; \quad \mathcal{M}^{-1}[H] = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \implies [H] = [\hbar\omega]$$

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Another issue: many quantities are best described not by elements of $\mathbb{R}$, but rather by elements of $\mathbb{R}$-*torsors*.

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If X is a *topological group*, we say that X is a *topological space* and that the action is *continuous*.

If X is a *Lie group*, we say that X is a *smooth manifold* and that the action is *smooth*.

Bonus: *Torsors*

"An affine space is like a vector space that forgot its origin; a torsor is like a group that forgot its identity".

Technical definition: a torsor (or *principle homogenous space*) for a group G with identity element e is a set X on which G acts such that

1) $\forall x \in X,\ ex = x$.
2) $\forall x, y \in X, \exists!\ g \in G$ such that $gx = y$.

If X is a *topological group*, we say that X is a *topological space* and that the action is *continuous*.

If X is a *Lie group*, we say that X is a *smooth manifold* and that the action is *smooth*.

If X is a *algebraic group*, we say that X is a *algebraic variety* and that the action is *regular*.

Torsors cont.

Examples

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- Energies ($\mathbb{R}$-torsor)

Torsors cont.

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- Energies ($\mathbb{R}$-torsor)
- Phases ($U(1)$-torsor)

Torsors cont.

Examples

- Energies ($\mathbb{R}$-torsor)
- Phases ($U(1)$-torsor)
- Dates ($\mathbb{Z}$-torsor)

One More Thing...

What's up with "Natural units" ($c = \hbar = 1$)??

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How to express this with the vector space notation above?

Oh! Don't Forget about the 2019 SI Redefinition!

The quantities $\Delta\nu_{\mathrm{Cs}}, c, \hbar, e$, and k_B span the dimensional space $\{\text{mass}, \text{length}, \text{time}, \text{temperature}, \text{charge}\}$.

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All SI base units are defined as a fixed linear transformation of these quantities.