Problem 50.5

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The Problem Statement

(a) Prove the useful identities

$$\langle p|\,\gamma^{\mu}|k] = [k|\gamma^{\mu}\,|p\rangle \tag{50.38}$$

$$\langle p|\,\gamma^{\mu}|k]^{*} = \langle k|\,\gamma^{\mu}|p] \tag{50.39}$$

$$\langle p|\,\gamma^{\mu}|p] = 2p^{\mu} \tag{50.40}$$

$$\langle p|\gamma^{\mu}|k\rangle = 0 \tag{50.41}$$

$$[p|\gamma^{\mu}|k] = 0 \tag{50.42}$$

The Problem Statement

(b) Extend the last two identities of part (a): show that the product of an odd number of gamma matrices sandwiched between either $\langle p|$ and $|k\rangle$ or $[p|$ and $|k]$ vanishes. Also show that the product of an even number of gamma matrices betweeh either $\langle p|$ and $|k]$ or $[p|$ and $|k\rangle$ vanishes.

The Problem Statement

(c) Prove the Fierz identities,

$$-\frac{1}{2}\langle p|\gamma_\mu|q]\gamma^\mu = |q]\langle p| + |p\rangle[q|, \tag{50.43}$$

$$-\frac{1}{2}[p|\gamma_\mu|q\rangle\gamma^\mu = |q\rangle[p| + |p]\langle q|\,. \tag{50.44}$$

Now take the matrix element of eq. (50.44) between $\langle r|$ and $|s]$ to get another useful form of the Fierz identity,

$$[p|\gamma^\mu|q\rangle\langle r|\gamma_\mu|s] = 2[ps]\langle qr\rangle\,. \tag{50.45}$$

Some Important Definitions and Relations

Our twister bras and kets are defined by

$$|p] = u_-(\vec{p}) = \begin{pmatrix} \phi_a \\ 0 \end{pmatrix} \qquad [p| = \bar{u}_+(\vec{p}) = \begin{pmatrix} \phi^a & 0 \end{pmatrix}$$

$$|p\rangle = u_+(\vec{p}) = \begin{pmatrix} 0 \\ \phi^{*\dot{a}} \end{pmatrix} \qquad \langle p| = \bar{u}_-(\vec{p}) = \begin{pmatrix} 0 & \phi^*_{\dot{a}} \end{pmatrix}$$

Some Important Definitions and Relations

They obey the inner product relations

$$\langle k| \, |p\rangle := \langle kp\rangle = -\langle pk\rangle$$
$$[k||p] := [kp] = -[pk]$$
$$\langle k| \, p] = [k \, |p\rangle = 0$$
$$[kp]^* = \langle pk\rangle$$

where

$$\langle pk\rangle = \phi^*_{\dot{a}}\kappa^{*\dot{a}}, \quad [pk] = \phi^a\kappa_a$$

Some Important Definitions and Relations

The twistors are related to the 4-momentum by

$$p_{a\dot{a}} := p_\mu \sigma^\mu_{a\dot{a}} = -\phi_a \phi^*_{\dot{a}},$$

from which we can easily derive the useful relation

$$-\not{p} = |p\rangle [p| + |p] \langle p| .$$

Solution: 50.5 (a)

$\langle p| \gamma^\mu |k] = [k|\gamma^\mu |p\rangle$

First, let

$$q_\nu = -\delta_\nu^\mu \implies -\not{q} = \delta_\nu^\mu \gamma^\nu = \gamma^\mu.$$

Now,

$$\begin{aligned}
\langle p| - \not{q}|k] &= \langle p| \left(|q\rangle [q| + |q] \langle q|\right) |k] \\
&= \langle pq\rangle [qk] + \langle p| q][q|k\rangle \\
&= \langle pq\rangle [qk] \\
[k| - \not{q} |p\rangle &= [k| \left(|q\rangle [q| + |q] \langle q|\right) |p\rangle \\
&= [k|q\rangle [q|p\rangle + [kq] \langle qp\rangle \\
&= \langle qp\rangle [kq] \\
&= \langle pq\rangle [qk] \qquad \square
\end{aligned}$$

Solution: 50.5 (a)

$\langle p|\gamma^\mu|k]^* = \langle k|\gamma^\mu|p]$

Define q_ν as before, then

$$
\begin{aligned}
\langle p| - \not{q}|k]^* &= (\langle pq\rangle [qk])^* \\
&= [qp]\langle kq\rangle \\
[k| - \not{q}|p] &= \langle kq\rangle [qp] \\
&= [qp]\langle kq\rangle \qquad \square
\end{aligned}
$$

Solution: 50.5 (a)

$\langle p| \gamma^\mu |p] = 2p^\mu$

Define q_ν as before, then

$$
\begin{aligned}
\langle p| - \not{q} |p] &= \langle pq \rangle [qp] \\
&= \phi^*_{\dot{a}} \kappa^{*\dot{a}} \phi_a \kappa^a \\
&= \phi_a \phi^*_{\dot{a}} \kappa^{*\dot{a}} \kappa^a \\
&= p_{a\dot{a}} q^{\dot{a}a} \\
&= p_\mu q_\nu \sigma^\mu_{a\dot{a}} \bar{\sigma}^{\nu \dot{a}a} \\
&= -2 p_\mu q_\nu g^{\mu\nu} \\
&= -2 p^\nu q_\nu \\
&= 2 p^\nu \delta^\mu_\nu \\
&= 2p^\mu \qquad \square
\end{aligned}
$$

Solution: 50.5 (a)

$\langle p|\gamma^\mu|k\rangle = 0$, $[p|\gamma^\mu|k] = 0$

Define q_ν as before, then

$$\begin{aligned}
\langle p|-\not{q}|k\rangle &= \langle p|\left(|q\rangle\,[q| + |q]\,\langle q|\right)|k\rangle \\
&= \langle pq\rangle\,[q\,|k\rangle + \langle p|\,q]\,\langle qk\rangle \\
&= 0 \qquad \square \\
[p| - \not{q}|k] &= [p|\left(|q\rangle\,[q| + |q]\,\langle q|\right)|k] \\
&= [p\,|q\rangle\,[qk] + [pq]\,\langle q|\,k] \\
&= 0 \qquad \square
\end{aligned}$$

Solution: 50.5 (b)

First, note that

$$
\begin{aligned}
-\not{p}_1 &= |p_1\rangle\,[p_1| + |p_1]\,\langle p_1| \\
(-\not{p}_1)(-\not{p}_2) &= \left(|p_1\rangle\,[p_1| + |p_1]\,\langle p_1|\right)\left(|p_2\rangle\,[p_2| + |p_2]\,\langle p_2|\right) \\
&= [p_1 p_2]\,|p_1\rangle\langle p_2| + \langle p_1 p_2\rangle\,|p_1][p_2| \\
(-\not{p}_1)(-\not{p}_2)(-\not{p}_3) &= (-\not{p}_1)(-\not{p}_2)\left(|p_3\rangle\,[p_3| + |p_3]\,\langle p_3|\right) \\
&= [p_1 p_2]\,\langle p_2 p_3\rangle\,\langle p_1|\,[p_3| + \langle p_1 p_2\rangle\,[p_2 p_3]|p_1]\,\langle p_3|
\end{aligned}
$$

Solution: 50.5 (b)

Let's refer to outer products of the form

$$|p\rangle\langle q|\,, \quad |p][q|$$

as *homogeneous*, and outer products of the form

$$|p\rangle\,[q|, \quad |p]\,\langle q|$$

as *heterogeneous*.

Analogous definitions can be made for inner products and matrix elements.

Solution: 50.5 (b)

The following pattern is clear from the previous slide:

- The product of an *odd* number of γ is given by *heterogeneous* outer products
- The product of an *even* number of γ is given by *homogeneous* outer products

Solution: 50.5 (b)

Therefore, we can conlude that

- *Homogeneous* matrix elements of an *odd* number of γ vanish. i.e.

$$\langle p| \prod_{i=1}^{2n+1} \gamma^{\mu_i} |k\rangle = [p| \prod_{i=1}^{2n+1} \gamma^{\mu_i} |k] = 0$$

- Similarly, *heterogeneous* matrix elements of an *even* number of γ vanish. i.e.

$$\langle p| \prod_{i=1}^{2n} \gamma^{\mu_i} |k] = [p| \prod_{i=1}^{2n} \gamma^{\mu_i} |k\rangle = 0$$

Solution: 50.5 (c)

$-\frac{1}{2}\langle p|\gamma_\mu|q]\gamma^\mu = |q]\langle p| + |p\rangle[q|$

$$\langle p|\gamma_\mu|q]\gamma^\mu = \begin{pmatrix} 0 & \phi^*_{\dot a} \end{pmatrix}\begin{pmatrix} 0 & \sigma_{\mu a\dot c} \\ \bar\sigma_\mu^{\dot a c} & 0 \end{pmatrix}\begin{pmatrix} \kappa_c \\ 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^\mu_{b\dot d} \\ \bar\sigma^{\mu\dot b d} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \phi^*_{\dot a}\kappa_c\bar\sigma_\mu^{\dot a c}\sigma^\mu_{b\dot d} \\ \phi^*_{\dot a}\kappa_c\bar\sigma_\mu^{\dot a c}\bar\sigma^{\mu\dot b d} & \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -2\phi^*_{\dot a}\kappa_b \\ -2\phi^{*\dot b}\kappa^b & \end{pmatrix}$$

$$|q]\langle p| + |p\rangle[q| = \begin{pmatrix} \kappa_b \\ 0 \end{pmatrix}\begin{pmatrix} 0 & \phi^*_{\dot a} \end{pmatrix} + \begin{pmatrix} 0 & \phi^{*\dot b} \end{pmatrix}\begin{pmatrix} \kappa^b \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \phi^*_{\dot a}\kappa_b \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \phi^{*\dot b}\kappa^b & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \phi^*_{\dot a}\kappa_b \\ \phi^{*\dot b}\kappa^b & 0 \end{pmatrix} \qquad \square$$

Solution: 50.5 (c)

$-\frac{1}{2}[p|\gamma_\mu\,|q\rangle\,\gamma^\mu = |q\rangle\,[p| + |p]\,\langle q|$

$$
\begin{aligned}
[p|\gamma_\mu\,|q\rangle &= \begin{pmatrix}\phi^a & 0\end{pmatrix}\begin{pmatrix}0 & \sigma_{\mu a\dot{c}}\\ \bar{\sigma}_\mu^{\dot{a}c} & 0\end{pmatrix}\begin{pmatrix}0\\ \kappa^{*\dot{a}}\end{pmatrix}\begin{pmatrix}0 & \sigma^\mu_{b\dot{d}}\\ \bar{\sigma}^{\mu\dot{b}d} & 0\end{pmatrix}\\
&= \begin{pmatrix}0 & \phi^a\kappa^{*\dot{a}}\sigma_{\mu a\dot{a}}\sigma^\mu_{b\dot{d}}\\ \phi^a\kappa^{*\dot{a}}\sigma_{\mu a\dot{a}}\sigma^{\mu\dot{b}d} & 0\end{pmatrix}\\
&= \begin{pmatrix}0 & -2\phi_b\kappa^*_{\dot{b}}\\ -2\phi^a\kappa^{*\dot{a}} & 0\end{pmatrix}\\
|q\rangle\,[p| + |p]\,\langle q| &= \begin{pmatrix}0\\ \kappa^{*\dot{a}}\end{pmatrix}\begin{pmatrix}\phi^a & 0\end{pmatrix} + \begin{pmatrix}\phi_b\\ 0\end{pmatrix}\begin{pmatrix}0 & \kappa^*_{\dot{b}}\end{pmatrix}\\
&= \begin{pmatrix}0 & 0\\ \phi^a\kappa^{*\dot{a}} & 0\end{pmatrix} + \begin{pmatrix}0 & \phi_b\kappa^*_{\dot{b}}\end{pmatrix}\\
&= \begin{pmatrix}0 & \phi_b\kappa^*_{\dot{b}}\\ \phi^a\kappa^{*\dot{a}} & 0\end{pmatrix} \qquad \square
\end{aligned}
$$

Solution: 50.5 (c)

$[p|\gamma^{\mu}\,|q\rangle\,\langle r|\,\gamma_m u|s] = 2[ps]\,\langle qr\rangle$

$$\begin{aligned}
\langle r|\left(-\frac{1}{2}[p|\gamma_{\mu}\,|q\rangle\,\gamma^{\mu}\right)|s] &= \frac{1}{2}[p|\gamma_{\mu}\,|q\rangle\,\langle r|\,\gamma^{\mu}|s] \\
&= -\,\langle r|\,(|q\rangle\,[p| + |p]\,\langle q|)\,|s] \\
&= -\,\langle rq\rangle\,[ps] \\
&= [ps]\,\langle qr\rangle \qquad \square
\end{aligned}$$