Homework 8

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Problem 1

(a) The Hamiltonian for the Lambda system in the Schrödinger representation is given by

$$\begin{aligned} H &= H_0 + V \\ &= -\hbar \begin{pmatrix} \omega_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \omega_0' \end{pmatrix} + \frac{\hbar}{2} \begin{pmatrix} 0 & \Omega_0^* e^{i\omega t} & 0 \\ \Omega_0 e^{-i\omega t} & 0 & \Omega_0' e^{-i\omega' t} \\ 0 & \Omega_0'^* e^{i\omega' t} & 0 \end{pmatrix} \\ &= \hbar \begin{pmatrix} -\omega_0 & \Omega_0^* e^{i\omega t} & 0 \\ \Omega_0 e^{-i\omega t} & 0 & \Omega_0' e^{-i\omega' t} \\ 0 & \Omega_0'^* e^{i\omega' t} & -\omega_0' \end{pmatrix}. \end{aligned}$$

Time evolution follows from the Schrödinger equation

$$\begin{aligned} \partial_t \left|\psi\right\rangle &= -\frac{i}{\hbar} H \left|\psi\right\rangle \\ \implies & \begin{cases} \dot{c}_1 = & i\omega_0 c_1 - \frac{i}{2}\Omega_0^* e^{i\omega t} \\ \dot{c}_2 = & -\frac{i}{2}\Omega_0 e^{-i\omega t} c_1 - \frac{i}{2}\Omega_0' e^{i\omega' t} c_3 \\ \dot{c}_3 = & -\frac{i}{2}\Omega_0'^* e^{-i\omega' t} c_2 i\omega_0' c_3 \end{cases} \end{aligned}$$

To go to the field interaction representation, we factor out the applied field via

$$c_1 \to \tilde{c}_1 = c_1 e^{-i\omega t}; \quad c_2 \to \tilde{c}_2 = c_2; \quad c_3 \to \tilde{c}_3 = c_3 e^{-i\omega' t}.$$

Then,

$$\begin{aligned}
\dot{\tilde{c}}_1 &= \left(-i\omega c_1 + \dot{c}_1\right) e^{-i\omega t} \\
&= \left(-i\omega c_1 + i\omega_0 c_1 - \frac{i}{2}\Omega_0^* e^{i\omega t}\right) e^{-i\omega t} \\
&= i\delta\tilde{c}_1 - \frac{i}{2}\Omega_0^*\tilde{c}_2 \\
\dot{\tilde{c}}_2 &= -\frac{i}{2}\left(\Omega_0\tilde{c}_1 + \Omega_0'\tilde{c}_3\right) \\
\dot{\tilde{c}}_3 &= \left(i\omega' c_3 + \dot{c}_3\right) e^{i\omega' t} \\
&= \left(-i\omega' c_3 + -\frac{i}{2}\Omega_0'^* e^{-i\omega' t} c_2 + i\omega_0' c_3\right) e^{i\omega' t} \\
&= -\frac{i}{2}\Omega_0'^*\tilde{c}_2 + i\delta'\tilde{c}_3.
\end{aligned}$$

Putting this together, we have

$$\begin{aligned}
\partial_t|\tilde{\psi}\rangle &= i\begin{pmatrix} \delta\tilde{c}_1 - \frac{1}{2}\Omega_0^*\tilde{c}_2 \\ -\frac{1}{2}\Omega_0\tilde{c}_1 - \frac{1}{2}\Omega_0'\tilde{c}_3 \\ -\frac{1}{2}\Omega_0'^*\tilde{c}_2 + \delta'\tilde{c}_3 \end{pmatrix} \\
&= -\frac{i}{\hbar}\tilde{H}|\tilde{\psi}\rangle \\
\implies \qquad \tilde{H} &= \hbar\begin{pmatrix} -2\delta & \frac{\Omega_0^*}{2} & 0 \\ \frac{\Omega_0}{2} & 0 & \frac{\Omega_0'}{2} \\ 0 & \frac{\Omega_0'^*}{2} & -2\delta' \end{pmatrix}.
\end{aligned}$$

(b) Given that

$$\begin{aligned}
|D\rangle &= \frac{1}{\Omega}\left(\Omega_0'\,|1\rangle + \Omega_0\,|3\rangle\right), \\
|B\rangle &= \frac{1}{\Omega}\left(\Omega_0^*\,|1\rangle - \Omega_0'^*\,|3\rangle\right),
\end{aligned}$$

where $\Omega = \sqrt{|\Omega_0|^2 + |\Omega_0'|^2}$. The transformation between these bases is then (dropping tildes)

$$\begin{pmatrix} c_d \\ c_2 \\ c_B \end{pmatrix} = \begin{pmatrix} \frac{\Omega_0'}{\Omega} & 0 & \frac{\Omega_0}{\Omega} \\ 0 & 1 & 0 \\ \frac{\Omega_0^*}{\Omega} & 0 & -\Omega_0'^* \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} =: U\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.$$

Now we can transform the Hamiltonian into this basis (assuming real Rabi frequencies):

$$\begin{aligned}
H_{D2B} &= UHU^\dagger \\
&= \hbar\begin{pmatrix} -2\delta & \frac{\Omega_0\Omega_0'}{\Omega^2} \end{pmatrix}
\end{aligned}$$

(c)

Problem 2

Problem 3

Problem 4

Problem 5 (Berman 9.1-2)