

96
190

Homework 7

Sean Ericson
Phys 663

June 2, 2024

Problem 1

i) For a radiation dominated flat universe, we have that

$$\begin{aligned} a(t) &= \left(\frac{t}{t_0}\right)^{1/2} \\ \Rightarrow t &= t_0 a^2 \\ \Rightarrow dt &= 2t_0 a da. \end{aligned}$$

ok

Then,

$$\begin{aligned} H(t) &= \frac{1}{2} t^{-1} \\ \Rightarrow H_0 &= \frac{1}{2} t_0^{-1} \\ \Rightarrow t_0 &= \frac{1}{2} H_0^{-1}, \end{aligned}$$

fine

and

$$\begin{aligned} d_{\max}(t) &= a(t) \int_0^{t_0} \frac{dt'}{a(t')} \\ &= 2t_0 a(t) \int_0^1 da \\ &= 2t_0 a(t) \\ &= H_0^{-1} a(t). \end{aligned}$$

✓

Evaluating this at the current epoch gives a cosmological horizon of

$$\boxed{d_{\max}(t_0) = H_0^{-1}}$$

✓

ii) For a matter dominated flat universe, we have that

$$\begin{aligned} a(t) &= \left(\frac{t}{t_0}\right)^{2/3} \\ \Rightarrow t &= t_0 a^{3/2} \\ \Rightarrow dt &= \frac{3}{2} t_0 a^{1/2} da. \end{aligned}$$

from

Then,

$$\begin{aligned} H(t) &= \frac{2}{3} t^{-1} \\ \Rightarrow H_0 &= \frac{2}{3} t_0^{-1} \\ \Rightarrow t_0 &= \frac{2}{3} H_0^{-1}, \end{aligned}$$

and

$$\begin{aligned} d_{\max}(t) &= a(t) \int_0^{t_0} \frac{dt'}{a(t')} \\ &= \frac{3}{2} t_0 \int_0^1 \frac{da}{a^{1/2}} \\ &= 3 t_0 \\ &= 2 H_0^{-1}. \end{aligned}$$

Evaluating this at the current epoch gives a cosmological horizon of

$$d_{\max}(t_0) = 2 H_0^{-1}$$

iii) For a cosmological constant dominated flat universe, we have that

$$\begin{aligned} a(t) &= e^{H_0 t} \\ \Rightarrow t &= H_0^{-1} \ln(a) \\ \Rightarrow dt &= \frac{da}{H_0 a}. \end{aligned}$$

Clearly there is no finite value of the time coordinate t such that $a(t) = 0$. Spencer said to take $t_{\text{beg}} = 0$ in this case, but I'll just leave it as t_{beg}

$$\begin{aligned} d_{\max}(t) &= a(t) \int_{t_{\text{beg}}}^{t_0} \frac{dt'}{a(t')} \\ &= a(t) \int_{t_{\text{beg}}}^{t_0} dt' e^{-H_0 t'} \\ &= -a(t) H_0^{-1} (a^{-1}(t_0) - a^{-1}(t_{\text{beg}})) \end{aligned}$$

Evaluating this at the current epoch gives a cosmological horizon of

$$\frac{26}{30}$$

$$d_{\max}(t_0) = H_0^{-1} (a^{-1}(t_{\text{beg}}) - 1)$$

-4

what happens when $a(t_{\text{beg}}) = 0$

Problem 2

Similarly to part ii) of Problem 1,

$$\begin{aligned} d(t_0) &= \int_{t_{\text{emit}}}^{t_0} \frac{dt'}{a(t')} \\ &= H_0^{-1} \int_{a(t_{\text{emit}})}^1 \frac{da}{a^{1/2}} \\ &= H_0^{-1} \int_0^z \frac{dz}{(1+z)^{3/2}} \\ &= 2H_0^{-1} (1 - (1+z)^{-1/2}) \\ \Rightarrow H_0 d &= z - \frac{3}{4}z^2 + \frac{5}{8}z^3 + \dots, \end{aligned}$$

so it looks like the quadratic coefficient is $-3/4$.

Problem 3

(a) A static solution should have $\dot{a} = \ddot{a} = 0$. For the second Friedman equation this implies

$$\sum_i (\rho_i + 3p_i) = \sum_i (1 + 3w_i)\rho_i = 0.$$

For normal matter $w = 0$, so we can write the above condition as

$$\rho_{\text{matter}} + (1 + 3w_{\text{other}})\rho_{\text{other}} = 0.$$

Assuming positive energy densities, this implies

$$w_{\text{other}} = -\frac{1}{3} \left(\frac{\rho_{\text{matter}}}{\rho_{\text{other}}} + 1 \right) < -\frac{1}{3}$$

(b) Let $\zeta = 8\pi G/3$, and assume positive energy densities. Then

$$\left(\frac{\dot{a}}{a} \right)^2 = 0 = \zeta \sum_i \rho_i - \frac{k}{a^2}$$

$$\Rightarrow k = \zeta a^2 \sum_i \rho_i > 0$$

(c) We can rearrange the Friedman equation as

$$\dot{a}^2 - \zeta a^2 \sum_i \rho_i + k = 0,$$

where as in class we can interpret the first term as a “kinetic energy” and the remaining terms as a “potential energy”. The potential energy term is an inverted parabola, so clearly the equilibrium is unstable. ✓

40
40