Homework 3

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Problem 1

$$\left\langle \frac{1}{2}, m_J'; I, m_I' \middle| I_z J_z \middle| \frac{1}{2}, m_J; I, m_I \right\rangle = \hbar^2 m_I m_J \delta_{m_J', m_J} \delta_{m_I', m_I}$$

$$\left\langle \frac{1}{2}, m_J'; I, m_I' \middle| I_+ J_- \middle| \frac{1}{2}, m_J; I, m_I \right\rangle = \hbar^2 \sqrt{(I + m_I + 1)(I - m_I)} \delta_{m_J', -\frac{1}{2}} \delta_{m_J, \frac{1}{2}} \delta_{m_I', m_I + 1}$$

$$\left\langle \frac{1}{2}, m_J'; I, m_I' \middle| I_- J_+ \middle| \frac{1}{2}, m_J; I, m_I \right\rangle = \hbar^2 \sqrt{(I - m_I + 1)(I + m_I)} \delta_{m_J', \frac{1}{2}} \delta_{m_J, -\frac{1}{2}} \delta_{m_I', m_I - 1}$$

$$\left\langle \frac{1}{2}, m_J'; I, m_I' \middle| g_J J_z + g_I I_z \middle| \frac{1}{2} m_J; I m_I \right\rangle = \hbar \left(g_J m_J \delta_{m_J', m_J} + g_I m_I \delta_{m_I', m_I} \right)$$

$$\left\langle \frac{1}{2}, \frac{1}{2}; I, m_I' \middle| H_{\text{hfs}} + H_{\text{hfs}}^{(\text{B})} \middle| \frac{1}{2}, \frac{1}{2}; I, m_I \right\rangle = A_{\text{hfs}} \frac{m_I}{2} \delta_{m_I', m_I} + \mu_B (\frac{g_J}{2} + g_I m_I \delta_{m_I', m_I}) B$$

$$\left\langle \frac{1}{2}, \frac{1}{2}; I, m_I' \middle| H_{\text{hfs}} + H_{\text{hfs}}^{(\text{B})} \middle| \frac{1}{2}, \frac{-1}{2}; I, m_I \right\rangle = A_{\text{hfs}} \sqrt{(I - m_I + 1)(I + m_I)} \delta_{m_I', m_I - 1}$$

$$\left\langle \frac{1}{2}, \frac{-1}{2}; I, m_I' \middle| H_{\text{hfs}} + H_{\text{hfs}}^{(\text{B})} \middle| \frac{1}{2}, \frac{1}{2}; I, m_I \right\rangle = A_{\text{hfs}} \sqrt{(I + m_I + 1)(I - m_I)} \delta_{m_I', m_I + 1}$$

$$\left\langle \frac{1}{2}, \frac{-1}{2}; I, m_I' \middle| H_{\text{hfs}} + H_{\text{hfs}}^{(\text{B})} \middle| \frac{1}{2}, \frac{-1}{2}; I, m_I \right\rangle = -A_{\text{hfs}} \frac{m_I}{2} \delta_{m_I, m_I'} + \mu_B (-\frac{g_J}{2} + g_I m_I \delta_{m_I', m_I})$$

Problem 2

$$
\begin{aligned}
\partial_t U(t,t_0) &= \partial_t U_0(t,t_0) - \frac{i}{\hbar}\partial_t \int_{t_0}^{t} \mathrm{d}t_1 U_0(t,t_1)V(t_1)U(t_1,t_0) \\
&= -\frac{i}{\hbar}H_0 U_0(t,t_0) - \frac{i}{\hbar}\left[U_0(t,t)V(t)U(t,t_0) - \frac{i}{\hbar}\int_{t_0}^{t} \mathrm{d}t_1\; H_0 U_0(t,t_1)V(t_1)U(t_1,t_0)\right] \\
&= -\frac{i}{\hbar}\left[H_0 U_0(t,t_0) + V(t)U(t,t_0) - \frac{i}{\hbar}H_0 \int_{t_0}^{t} \mathrm{d}t_1\; U_0(t,t_1)V(t_1)U(t_1,t_0)\right] \\
&= -\frac{i}{\hbar}\left[H_0\left(U_0(t,t_0) - \frac{i}{\hbar}\int_{t_0}^{t} \mathrm{d}t_1\; U_0(t,t_1)V(t_1)U(t_1,t)\right) + V(t)U(t,t_0)\right] \\
&= -\frac{i}{\hbar}\left[H_0 U(t,t_0) + V(t)U(t,t_0)\right] \\
&= -\frac{i}{\hbar}\left[H_0 + V(t)\right]U(t,t_0)
\end{aligned}
$$

Problem 3

(a)

$$
\begin{aligned}
G^+(x,x_0;E) &= \langle x|\frac{1}{E - p^2/2m + i0^+}|x_0\rangle \\
&= \int_{-\infty}^{\infty} \mathrm{d}p\; \langle x|\frac{1}{E - p^2/2m + i0^+}|p\rangle\,\langle p|x_0\rangle \\
&= \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty} \mathrm{d}p\; \frac{e^{ip(x-x_0)/\hbar}}{E - p^2/2m + i0^+}
\end{aligned}
$$

Let

$$
r = x - x_0, \quad z = \frac{pr}{\hbar}, \quad z_E^2 = \frac{2mr^2E}{\hbar^2}
$$

Then

$$
\begin{aligned}
G^+(x,x_0;E) &= \frac{mr}{\pi\hbar}\int_{-\infty}^{\infty} \mathrm{d}z\; \frac{e^{iz}}{[z - (z_E + i0^+)]\,[z + (z_E - i0^+)]} \\
&= (-2\pi i)\frac{mr}{\pi\hbar}\frac{e^{iz_E}}{-2z_E} \\
&= \frac{im}{z_E\hbar}e^{iz_E} \\
&= \frac{im}{z_E\hbar}e^{ip_E r/\hbar}
\end{aligned}
$$

Problem 4

$$
\begin{aligned}
\tilde{K}^{(3)}_{\mathbf{fi}} &= \frac{i}{\hbar^3}\sum_{jk}\int_0^t \mathrm{d}t_3 \int_0^{t_3}\mathrm{d}t_2 \int_0^{t_2}\mathrm{d}t_3\, V_{\mathbf{f}j}V_{jk}V_{k\mathbf{i}}e^{iE_{\mathbf{f}j}t_3/\hbar}e^{iE_{jk}t_2/\hbar}e^{iE_{k\mathbf{i}}t_1/\hbar} \\
&= \frac{i}{\hbar^3}\sum_{jk} V_{\mathbf{f}j}V_{jk}V_{k\mathbf{i}}\int_0^t \mathrm{d}t_3\, e^{iE_{\mathbf{f}j}t_3/\hbar}\int_0^{t_3}\mathrm{d}t_2 \int_0^{t_2}\mathrm{d}t_1\, e^{iE_{jk}t_2/\hbar}e^{iE_{k\mathbf{i}}t_1/\hbar} \\
&= \frac{i}{\hbar^3}\sum_{jk} V_{\mathbf{f}j}V_{jk}V_{k\mathbf{i}}\int_0^t \mathrm{d}t_3\, e^{iE_{\mathbf{f}j}t_3/\hbar}\int_0^{t_3}\mathrm{d}t_2 \int_0^{t_2}\mathrm{d}t_1\, e^{iE_k(t_1-t_2)/\hbar}e^{iE_jt_2/\hbar}e^{iE_{\mathbf{i}}t_1/\hbar} \\
&= \frac{1}{2\pi\hbar^3}\sum_{jk} V_{\mathbf{f}j}V_{jk}V_{k\mathbf{i}}\int_0^t \mathrm{d}t_3\, e^{iE_{\mathbf{f}j}t_3/\hbar}\int_0^{t_3}\mathrm{d}t_2 \int_0^{t_3}\mathrm{d}t_1 \int_{-\infty}^{\infty}\mathrm{d}E\, \frac{e^{i(E_j-E)t_2/\hbar}e^{i(E-E_i)t_1/\hbar}}{E-E_k+i0^+} \\
&= \frac{2\pi}{\hbar}\sum_{jk} V_{\mathbf{f}j}V_{jk}V_{k\mathbf{i}}\int_0^t \mathrm{d}t_3\, e^{iE_{\mathbf{f}j}t_3/\hbar}\int_{-\infty}^{\infty}\mathrm{d}E\, \frac{\delta_t(E_j-E)\delta_t(E_i-E)e^{i(E_j-E)t_3/2\hbar}e^{i(E-E_i)t_3/2\hbar}}{E-E_k+i0^+} \\
&\approx \frac{2\pi}{\hbar}\sum_{jk}\frac{V_{\mathbf{f}j}V_{jk}V_{k\mathbf{i}}}{E_i-E_k+i0^+}\int_0^t \mathrm{d}t_3\, e^{iE_{\mathbf{f}j}t_3/\hbar}e^{iE_{j\mathbf{i}}t_3/2\hbar}\int_{-\infty}^{\infty}\mathrm{d}E\,\delta_t(E_j-E)\delta_t(E_i-E) \\
&= \frac{2\pi}{\hbar}\sum_{jk}\frac{V_{\mathbf{f}j}V_{jk}V_{k\mathbf{i}}}{E_{\mathbf{i}}-E_k+i0^+}\int_0^t \mathrm{d}t_3\, e^{iE_{\mathbf{f}j}t_3/\hbar}e^{iE_{j\mathbf{i}}t_3/2\hbar}\delta_t(E_{\mathbf{i}j}) \\
&= \frac{1}{\hbar^2}\sum_{jk}\frac{V_{\mathbf{f}j}V_{jk}V_{k\mathbf{i}}}{E_{\mathbf{i}}-E_k+i0^+}\int_0^t \mathrm{d}t_3 \int_0^{t_3}\mathrm{d}t_3'\, e^{iE_{\mathbf{f}j}t_3/\hbar}e^{iE_{j\mathbf{i}}t_3'/\hbar} \\
&= \frac{1}{2\pi i\hbar^2}\sum_{jk}\frac{V_{\mathbf{f}j}V_{jk}V_{k\mathbf{i}}}{E_{\mathbf{i}}-E_k+i0^+}\int_0^t \mathrm{d}t_3 \int_0^{t}\mathrm{d}t_3' \int_{-\infty}^{\infty}\mathrm{d}E\, \frac{e^{i(E_{\mathbf{f}}-E)t_3/\hbar}e^{i(E-E_{\mathbf{i}})t_3'/\hbar}}{E-E_j+i0^+} \\
&= -2\pi i\sum_{jk}\frac{V_{\mathbf{f}j}V_{jk}V_{k\mathbf{i}}}{E_{\mathbf{i}}-E_k+i0^+}\int_{-\infty}^{\infty}\mathrm{d}E\, \frac{\delta_t(E_{\mathbf{f}}-E)\delta_t(E-E_{\mathbf{i}})e^{i(E_{\mathbf{f}}-E)t/2\hbar}e^{i(E-E_{\mathbf{i}})t/2\hbar}}{E-E_j+i0^+} \\
&\approx -2\pi i\sum_{jk}\frac{V_{\mathbf{f}j}V_{jk}V_{k\mathbf{i}}}{(E_{\mathbf{i}}-E_k+i0^+)(E_{\mathbf{i}}-E_j+i0^+)}e^{iE_{\mathbf{fi}}t/2\hbar}\int_{-\infty}^{\infty}\mathrm{d}E\,\delta_t(E_{\mathbf{f}}-E)\delta_t(E-E_{\mathbf{i}}) \\
&= -2\pi i\sum_{jk}\frac{V_{\mathbf{f}j}V_{jk}V_{k\mathbf{i}}}{(E_{\mathbf{i}}-E_k+i0^+)(E_{\mathbf{i}}-E_j+i0^+)}e^{iE_{\mathbf{fi}}t/2\hbar}\delta_t(E_{\mathbf{fi}})
\end{aligned}
$$

Problem 5

$$
\sin^2(\Omega t/2) = \left(\frac{\Omega t}{2}\right)^2 - \frac{1}{3}\left(\frac{\Omega t}{2}\right)^4 + \cdots
$$

We already know that the first order term in the perturbation series gives

$$
\left(\frac{\Omega t}{2}\right)^2
$$

The restriction of the Hilbert space to $\{|i\rangle, |f\rangle\}$ kills any terms in the perturbation series with an odd number of perturbation matrix elements. The next nonzero term gives

$$\tilde{K}^{(3)}_{\mathbf{fi}} = \frac{i}{\hbar^3} \left| V_{\mathbf{fi}} \right|^2 \mathscr{T} \frac{1}{3!} [e]$$

Problem 6

(a)

$$\begin{aligned}
\tilde{K}^{(2)}_{\mathbf{fi}} &= \langle \mathbf{f} | - \frac{1}{2\hbar^2} \int_0^t \mathrm{d}t_2 \int_0^{t_2} \mathrm{d}t_1 \, e^{iH_0 t_2/\hbar} V_0 e^{-i\omega t_2} e^{-iH_0 t_2/\hbar} e^{iH_0 t_1/\hbar} V_0 e^{-i\omega t_1/\hbar} e^{-iH_0 t_1/\hbar} | \mathbf{i} \rangle \\
&\quad + \langle \mathbf{f} | - \frac{1}{2\hbar^2} \int_0^t \mathrm{d}t_2 \int_0^{t_2} \mathrm{d}t_1 \, e^{iH_0 t_2/\hbar} V_0^\dagger e^{i\omega t_2} e^{-iH_0 t_2/\hbar} e^{iH_0 t_1/\hbar} V_0^\dagger e^{i\omega t_1/\hbar} e^{-iH_0 t_1/\hbar} | \mathbf{i} \rangle \\
&= -\frac{1}{2\hbar^2} \sum_k \int_0^t \mathrm{d}t_2 \int_0^{t_2} \mathrm{d}t_1 \, (V_0)_{\mathbf{f}k} e^{i(E_{\mathbf{f}k} - \hbar\omega)t_2} (V_0)_{k\mathbf{i}} e^{i(E_{k\mathbf{i}} - \hbar\omega)t_1} + (V_0^\dagger)_{\mathbf{f}k} e^{i(E_{\mathbf{f}k} + \hbar\omega)t_2} (V_0^\dagger)_{k\mathbf{i}} e^{i(E_{k\mathbf{i}} + \hbar\omega)t_1}
\end{aligned}$$

(b)

$$\tilde{K}^{(2)}_{\mathbf{fi}} = -2\pi i \sum_k \left[\left(\frac{(V_0)_{\mathbf{f}k} (V_0)_{k\mathbf{i}}}{E_{\mathbf{i}k} - \hbar\omega + i0^+} e^{i(E_{\mathbf{fi}} - \hbar\omega)} \delta_t(E_{\mathbf{fi}} - \hbar\omega) \right) + \left(\frac{(V_0)_{\mathbf{f}k} (V_0)_{k\mathbf{i}}}{E_{\mathbf{i}k} + \hbar\omega + i0^+} e^{i(E_{\mathbf{fi}} + \hbar\omega)} \delta_t(E_{\mathbf{fi}} + \hbar\omega) \right) \right]$$

(c) The term on the left will be the dominant term, as the difference $E_{\mathbf{i}k} - \hbar\omega$ in the denominator will be extremely small.