Homework 6

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Phys 662

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Problem 1

(a) See the attached Mathematica notebook for the calculations.

Q (GeV)	α_s
4.18	0.2121
2	0.2675
1.28	0.3179

(b) The evolution of $m_f(Q)$ is given by

$$\begin{aligned} & \frac{\mathrm{d}}{\mathrm{d}\log Q} m_f(Q) = -\frac{2}{\pi}\alpha_s(Q) m_f(Q) \\ \implies \quad & \frac{\mathrm{d}}{\mathrm{d}Q} m_f(Q) = -\frac{a m_f(Q)}{Q\,(1 + b\ln(Q/Q_0))} \\ \implies \quad & m_f(Q) = m_f(Q_0)\left(\frac{1}{1 + b\ln(Q/Q_0)}\right)^{a/b}. \end{aligned}$$

Substituting in the values $a = (2/\pi)\alpha_s(Q_0)$ and $b = b_0\alpha_s(Q_0)/2\pi$, we get

$$\frac{m_f(Q)}{m_f(Q_0)} = \left(\frac{\alpha_s(Q)}{\alpha_s(Q_0)}\right)^{4/b_0}$$

(c) We find a value of 1.1784 GeV for the mass of the charm quark at $Q = 2$ GeV.

flavor (GeV)	m_f (GeV)
u	0.0020
d	0.0042
s	0.0860
c	1.0577

(d) The calculated masses for the four lightest quarks at $Q = m_b$ are listed in the table above.

Problem 2

(a) See attached Mathematica notebook.

(b) Consider the operator

$$df_\pi^4 \operatorname{Tr}\big[Q_L \Sigma Q_R \Sigma^\dagger + \text{h.c.}\big],$$

where d is a dimensionless constant. This operator is $SU(3)_L \times SU(3)_R$ symmetric:

$$\begin{aligned}
\operatorname{Tr}\big[Q_L \Sigma Q_R \Sigma^\dagger\big] &\to \operatorname{Tr}\Big[U_L Q_L U_L^\dagger U_L \Sigma U_R^\dagger U_R Q_R U_R^\dagger U_R \Sigma^\dagger U_L^\dagger\Big] \\
&= \operatorname{Tr}\Big[U_L Q_L \Sigma Q_R \Sigma^\dagger U_L^\dagger\Big] \\
&= \operatorname{Tr}\big[Q_L \Sigma Q_R \Sigma^\dagger\big].
\end{aligned}$$

In the last equality, the cyclic property of the trace in used. This operator results in mass corrections

$$\delta m_{\pi^+}^2 = \delta m_{K^+}^2 = 4 d f_\pi^2$$

(c) We have

$$\begin{aligned}
\frac{c}{f_\pi^{2n-4}} \operatorname{Tr}\big[D_{\mu_1} \cdots D_{\mu_n} \Sigma D^{\mu_1} \cdots D^{\mu_n} \Sigma^\dagger\big] &= f_\pi^2 \Lambda^2 \operatorname{Tr}\left[\frac{D_{\mu_1}}{\Lambda} \cdots \frac{D_{\mu_n}}{\Lambda} \Sigma \frac{D^{\mu_1}}{\Lambda} \cdots \frac{D^{\mu_n}}{\Lambda} \Sigma^\dagger\right] \\
&= f_\pi^2 \Lambda^{2-2n} \operatorname{Tr}\big[D_{\mu_1} \cdots D_{\mu_n} \Sigma D^{\mu_1} \cdots D^{\mu_n} \Sigma^\dagger\big] \\
\implies \qquad c &= \left(\frac{f_\pi}{\Lambda}\right)^{2n-2} \\
&= \frac{1}{(4\pi)^{2n-2}}
\end{aligned}$$

We can get the four-point vertex by expanding Σ to first order:

$$\Sigma \approx 1 + i\Pi.$$

The terms that are fourth-order in Π are of the forms

$$Tr[D^n \Sigma D^n \Sigma^\dagger] \supset \lambda_1 D^n \Pi^3 D^n \Pi + \lambda_2 D^n \Pi^3 D^n \Pi + \lambda_3 D^n \Pi^2 D^n \Pi^2,$$

and the matrix element is roughly

$$\mathcal{M} \sim \frac{c}{f_\pi^{2n-2}} (p \cdot p)^{n-2} \sim \left(\frac{E_{\text{CM}}}{4\pi f_\pi}\right)^{2n-4}.$$

We can see that this is independent of n when $E_{\text{CM}} \approx 4\pi f_\pi = \Lambda$.

Problem 1 (Peskin 14.2)

In[1]:=
```
<< Notation`
```

In[2]:=
```
Symbolize[αs]; Symbolize[αs0]; Symbolize[Q0]; Symbolize[b0]; Symbolize[nf];
Symbolize[m0]; Symbolize[mf]; Symbolize[mu]; Symbolize[md];
Symbolize[ms]; Symbolize[mc]; Symbolize[mt]; Symbolize[mb];
```

In[4]:=
```
$Assumptions = {Q > 0};
```

In[5]:=
```
mu = 0.0022;
md = 0.0047;
ms = 0.096;
mc = 1.28;
mb = 4.18;
mt = 164;
```

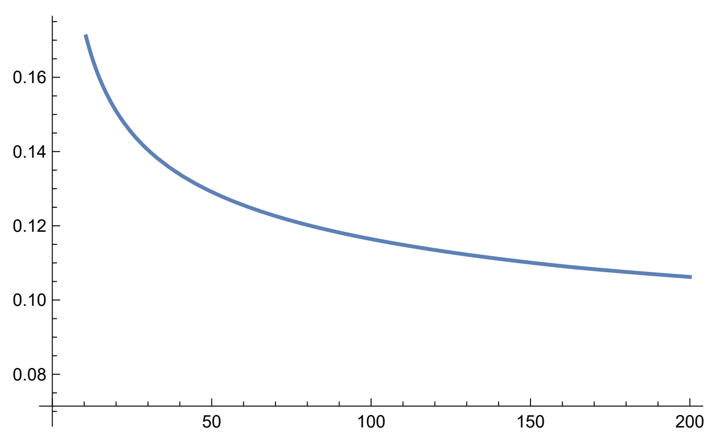
In[11]:=
```
b0 = 11 - 2/3 nf;

f[q_, q0_, α0_] := α0 / (1 + (b0 α0 / (2 π)) Log[q / q0]);

αs[Q_] := Piecewise[{{f[Q, mt - 1, αs[mt - 1]] /. nf → 6, mt < Q},
    {f[Q, 91, 0.118] /. nf → 5, mb ≤ Q < mt},
    {f[Q, mb , αs[mb]] /. nf → 4, mc ≤ Q < mb}, {f[Q, mc, αs[mc]] /. nf → 3,
     ms ≤ Q < mc}, {f[Q, ms, αs[ms]] /. nf → 2, md ≤ Q < ms},
    {f[Q, md, αs[md]] /. nf → 1, mu ≤ Q < md}, {f[Q, mu, αs[mu]] /. nf → 0, Q < mu} }];
```

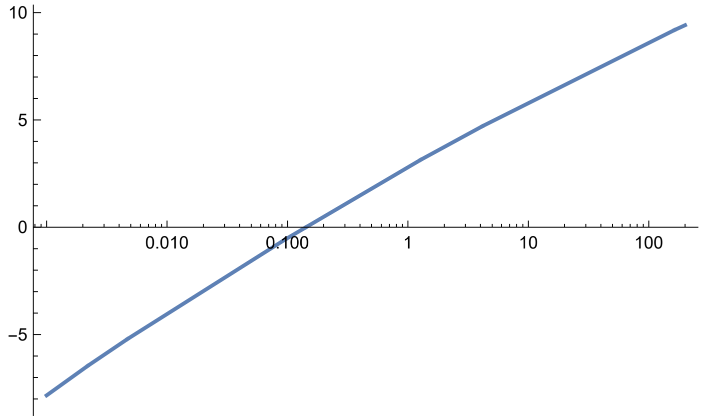
In[14]:=
```
Plot[αs[Q], {Q, 0.001, 200}]
```

Out[14]=



In[15]:= $\text{LogLinearPlot}\left[\frac{1}{\alpha_s[Q]}, \{Q, 0.001, 200\}\right]$

Out[15]=



a)

In[16]:= $\alpha_s[m_b]$

Out[16]=

0.212056

In[17]:= $\alpha_s[2]$

Out[17]=

0.26752

In[18]:= $\alpha_s[m_c]$

Out[18]=

0.317851

b)

$$m[Q\_, m0\_, Q0\_] := m0 \left(\frac{\alpha_s[Q]}{\alpha_s[Q0]}\right)^{\frac{4}{b_0}}$$

c)

In[32]:=

$m[2, m_c, m_c] \;/.\; n_f \to 4$

Out[32]=

1.17835

d)

In[38]:=
```
nf = 5;
m[mb, mu, 2]
m[mb, md, 2]
m[mb, ms, 2]
m[mb, mc, 1.28]
```

Out[39]=
0.00197001

Out[40]=
0.00420866

Out[41]=
0.0859641

Out[42]=
1.05772

Problem 2

In[1]:=
```
<< Notation`
```

In[2]:=

$$\text{Symbolize}[\pi^0];\ \text{Symbolize}[\pi^+];\ \text{Symbolize}[\pi^-];$$
$$\text{Symbolize}[K^0];\ \text{Symbolize}[K^+];\ \text{Symbolize}[K^-];\ \text{Symbolize}[\overline{K}^0];$$
$$\text{Symbolize}[\eta];\ \text{Symbolize}[f_\pi];\ \text{Symbolize}[\Sigma^+];$$
$$\text{Symbolize}[m_u];\ \text{Symbolize}[m_d];\ \text{Symbolize}[m_s];$$
$$\text{Symbolize}[Q_L];\ \text{Symbolize}[Q_R];\quad \text{Symbolize}[L_2];$$

In[7]:=

$$\$\text{Assumptions} = \{m_u \in \mathbb{R},\ m_d \in \mathbb{R},\ m_s \in \mathbb{R},\ f_\pi \in \mathbb{R}\};$$

In[8]:=

$$\Pi = \left\{\left\{\pi^0 + \frac{1}{\sqrt{3}}\eta,\ \sqrt{2}\,\pi^+,\ \sqrt{2}\,K^+\right\},\right.$$
$$\left.\left\{\sqrt{2}\,\pi^-,\ -\pi^0 + \frac{1}{\sqrt{3}}\eta,\ \sqrt{2}\,K^0\right\},\ \left\{\sqrt{2}\,K^-,\ \sqrt{2}\,\overline{K}^0,\ \frac{-2}{\sqrt{3}}\eta\right\}\right\};$$

$$M = \text{DiagonalMatrix}[\{m_u,\ m_d,\ m_s\}];$$
$$\text{MatrixForm} /@ \{\Pi,\ M\} \mathbin{//} \text{Row}$$

Out[10]=

$$\begin{pmatrix} \frac{\eta}{\sqrt{3}} + \pi^0 & \sqrt{2}\,\pi^+ & \sqrt{2}\,K^+ \\ \sqrt{2}\,\pi^- & \frac{\eta}{\sqrt{3}} - \pi^0 & \sqrt{2}\,K^0 \\ \sqrt{2}\,K^- & \sqrt{2}\,\overline{K}^0 & -\frac{2\eta}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{pmatrix}$$

In[11]:=

$$n = 2;$$
$$\Sigma = \sum_{i=0}^{n} \frac{1}{i\,!}\left(\frac{i}{f_\pi}\right)^i \text{MatrixPower}[\Pi,\ i];$$
$$\Sigma^+ = \sum_{i=0}^{n} \frac{1}{i\,!}\left(\frac{-i}{f_\pi}\right)^i \text{MatrixPower}[\Pi,\ i];$$
$$\text{MatrixForm} /@ \{\Sigma,\ \Sigma^+\} \mathbin{//} \text{Row}$$

Out[14]=

$$\left(\begin{matrix}
1 + \frac{i\left(\frac{\eta}{\sqrt{3}}+\pi^0\right)}{f_\pi} - \frac{2K^-K^+ + \left(\frac{\eta}{\sqrt{3}}+\pi^0\right)^2 + 2\pi^-\pi^+}{2f_\pi^2} & \frac{i\sqrt{2}\,\pi^+}{f_\pi} - \frac{2\overline{K}^0K^+ + \sqrt{2}\left(\frac{\eta}{\sqrt{3}}-\pi^0\right)\pi^+ + \sqrt{2}\left(\frac{\eta}{\sqrt{3}}+\pi^0\right)\pi^+}{2f_\pi^2} & \frac{i\sqrt{2}\,K^+}{f_\pi} - \frac{-2\sqrt{\frac{2}{3}}K^+\eta + \sqrt{\cdot}}{} \\
\frac{i\sqrt{2}\,\pi^-}{f_\pi} - \frac{2K^0K^- + \sqrt{2}\left(\frac{\eta}{\sqrt{3}}-\pi^0\right)\pi^- + \sqrt{2}\left(\frac{\eta}{\sqrt{3}}+\pi^0\right)\pi^-}{2f_\pi^2} & 1 + \frac{i\left(\frac{\eta}{\sqrt{3}}-\pi^0\right)}{f_\pi} - \frac{2\overline{K}^0K^0 + \left(\frac{\eta}{\sqrt{3}}-\pi^0\right)^2 + 2\pi^-\pi^+}{2f_\pi^2} & \frac{i\sqrt{2}\,K^0}{f_\pi} - \frac{-2\sqrt{\frac{2}{3}}K^0\eta + \sqrt{\cdot}}{} \\
\frac{i\sqrt{2}\,K^-}{f_\pi} - \frac{-2\sqrt{\frac{2}{3}}K^-\eta + \sqrt{2}\,K^-\left(\frac{\eta}{\sqrt{3}}+\pi^0\right) + 2\overline{K}^0\pi^-}{2f_\pi^2} & \frac{i\sqrt{2}\,\overline{K}^0}{f_\pi} - \frac{-2\sqrt{\frac{2}{3}}\overline{K}^0\eta + \sqrt{2}\,\overline{K}^0\left(\frac{\eta}{\sqrt{3}}-\pi^0\right) + 2K^-\pi^+}{2f_\pi^2} & 1 - \frac{2i\eta}{\sqrt{3}\,f_\pi} - \frac{2\overline{K}^0 \cdots}{}
\end{matrix}\right.$$

a)

In[15]:=
```
L = c fπ^3 Tr[M.Σ + M†.Σ+] // FullSimplify
```

Out[15]=

$$\frac{1}{3} c f_\pi \left(-6 \overline{K}^0 K^0 (m_d + m_s) - 6 K^- K^+ (m_s + m_u) + 6 f_\pi^2 (m_d + m_s + m_u) - (m_d + 4 m_s + m_u) \eta^2 + 2\sqrt{3} (m_d - m_u) \eta \pi^0 - 3 (m_d + m_u) \left(\pi^0\right)^2 - 6 (m_d + m_u) \pi^- \pi^+\right)$$

In[16]:=
```
-Coefficient[L, K+ K-]
```

Out[16]=

$$2 c f_\pi (m_s + m_u)$$

In[17]:=
```
-Coefficient[L, K0 K̄0]
```

Out[17]=

$$2 c f_\pi (m_d + m_s)$$

In[18]:=
```
-2 Coefficient[L, η^2] // Simplify
```

Out[18]=

$$\frac{2}{3} c f_\pi (m_d + 4 m_s + m_u)$$

b)

In[19]:=
```
QL = DiagonalMatrix[{2/3, -1/3, -1/3}];
QR = QL;
```

In[21]:=
```
L2 = d fπ^3 Tr[QL.Σ.QR.Σ+ + Σ.QR.Σ+.QL] // FullSimplify
```

Out[21]=

$$\frac{1}{54 f_\pi} d \left(7 \eta^4 + 4\sqrt{3} \eta^3 \pi^0 + 6 \eta^2 \left(4 \overline{K}^0 K^0 + 4 K^- K^+ + 5 \left(\pi^0\right)^2 - 2 \pi^- \pi^+\right) + 12\sqrt{3} \eta \left(\left(\pi^0\right)^3 - \sqrt{2} \left(\overline{K}^0 K^+ \pi^- + K^0 K^- \pi^+\right) + 2 \pi^0 (2 K^- K^+ + \pi^- \pi^+)\right) + 3 \left(24 f_\pi^4 + 8 (\overline{K}^0)^2 \left(K^0\right)^2 + 5 \left(\pi^0\right)^4 - 12\sqrt{2} \pi^0 \left(\overline{K}^0 K^+ \pi^- + K^0 K^- \pi^+\right) - 72 f_\pi^2 (K^- K^+ + \pi^- \pi^+) + 4 \left(\pi^0\right)^2 \left(2 \overline{K}^0 K^0 + 2 K^- K^+ + 5 \pi^- \pi^+\right) + 4 (K^- K^+ + \pi^- \pi^+) \left(-2 \overline{K}^0 K^0 + 5 K^- K^+ + 5 \pi^- \pi^+\right)\right)\right)$$

In[22]:=
```
-Coefficient[L2, K+ K-] /. {π0 → 0, π+ → 0, π- → 0, K0 → 0, K̄0 → 0, η → 0}
```

Out[22]=

$$4 d f_\pi$$

In[23]:=
```
-Coefficient[L2, π+ π-] /. {π0 → 0, K+ → 0, K- → 0, K0 → 0, K̄0 → 0, η → 0}
```

Out[23]=

$$4 d f_\pi$$

In[24]:=
```
-Coefficient[L₂, K⁰ K̄⁰] /. {π⁰ → 0, π⁺ → 0, π⁻ → 0, K⁺ → 0, K⁻ → 0, η → 0}
```
Out[24]=

0

In[25]:=
```
-Coefficient[L₂, π⁰ π⁰] /. {π⁺ → 0, π⁻ → 0, K⁺ → 0, K⁻ → 0, K⁰ → 0, K̄⁰ → 0, η → 0}
```
Out[25]=

0

In[26]:=
```
-2 Coefficient[L₂, η²] /. {π⁰ → 0, π⁺ → 0, π⁻ → 0, K⁺ → 0, K⁻ → 0, K⁰ → 0, K̄⁰ → 0}
```
Out[26]=

0