Homework 5

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Phys 684

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Problem 1

We start with the Maxwell wave equation

$$\left(\vec{\nabla}^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\vec{E}(\vec{r},t) = \mu_0 \frac{\partial^2}{\partial t^2}\vec{P}(\vec{r},t),$$

where

$$\vec{E}(\vec{r},t) = \vec{E}_+(z,t) + \vec{E}_-(z,t); \quad \vec{E}_\pm(z,t) = \frac{1}{2}\hat{x}E_0(z,t)e^{\mp i\alpha(z,t)},$$
$$\vec{P}(\vec{r},t) = \vec{P}_+(z,t) + \vec{P}_-(z,t) = \frac{1}{2}\hat{x}\left(P_0(z,t)e^{-i\alpha(z,t)} + \text{c.c}\right),$$
$$\alpha(z,t) = \omega t - kz - \phi(z,t),$$

and $E_0(z,t) \in \mathbb{R}$ while $P_0(z,t) \in \mathbb{C}$. Noticing that

$$\begin{aligned}
2\frac{\partial}{\partial z}|E_\pm| &= \left(E_0' \mp iE_0\alpha'\right)e^{\mp i\alpha}, \\
2\frac{\partial}{\partial t}|E_\pm| &= \left(\dot{E}_0 \mp iE_0\dot{\alpha}\right)e^{\mp i\alpha}, \\
2\frac{\partial}{\partial t}|P_\pm| &= \left(\dot{P}_0 \mp P_0\dot{\alpha}\right)e^{\mp i\alpha} \\
2\frac{\partial^2}{\partial t^2}|P_\pm| &= \left(\ddot{P}_0 \mp \dot{P}_0\dot{\alpha} \mp P_0\ddot{\alpha} \mp i\dot{P}_0\dot{\alpha} + iP_0\dot{\alpha}^2\right)e^{\mp i\alpha} \\
\alpha' &= -k - \phi' \\
\dot{\alpha} &= \omega - \dot{\phi},
\end{aligned}$$

we factor the differential operator in the wave equation as

$$\partial_z^2 - \frac{1}{c^2}\partial_t^2 = \left(\partial_z + \frac{1}{c}\partial_t\right)\left(\partial_z - \frac{1}{c}\partial_t\right),$$

and begin applying it to the $\pm$ components of the fields:

$$\begin{aligned}
2\left(\partial_z - \frac{1}{c}\partial_t\right)|E_\pm| &= \left[E_0' \mp iE_0\alpha' - \frac{1}{c}\left(\dot{E}_0 \mp iE_0\dot{\alpha}\right)\right]e^{\mp i\alpha} \\
&= \left[\mp iE_0\left(\alpha' - \frac{1}{c}\dot{\alpha}\right) + E_0' - \frac{1}{c}\dot{E}_0\right]e^{\mp i\alpha} \\
&= \left[\pm iE_0(2k + \phi' - \frac{1}{c}\dot{\phi}) + E_0' - \frac{1}{c}\dot{E}_0\right]e^{\mp i\alpha}.
\end{aligned}$$

Now, in the slowly varying amplitude and phase approximation, we neglect terms proportional to $\ddot{E}_0$, E_0'', $\dot{P}_0$, and $\ddot{P}_0$. So, as we apply the second half of the differential operator, let's drop the terms that will produce terms that we'll neglect anyway:

$$\begin{aligned}
2\left(\partial_z + \frac{1}{c}\partial_t\right)\left(\partial_z - \frac{1}{c}\partial_t\right)|E_\pm| &\approx \pm 2ik\left(\partial_z + \frac{1}{c}\partial_t\right)\left[E_0 e^{\mp i\alpha}\right] \\
&= \pm 2ik\left[E_0' + \frac{1}{c}\dot{E}_0 \pm i\left(\phi' + \frac{1}{c}\dot{\phi}\right)\right]e^{\mp i\alpha}
\end{aligned}$$

The other side of the wave equation is approximately

$$\frac{\partial^2}{\partial t^2}P_0 \approx -\omega^2 P_0$$

Equating the real and imaginary parts of each side of then gives the desired results.

Problem 2

Starting with

$$\begin{aligned}
\dot{\tilde{\rho}}_{21} &= -(\gamma + i\delta)\tilde{\rho}_{21} + i\frac{\Omega_0}{2}(\rho_{22} - \rho_{11}) \\
\dot{\rho}_{22} &= -\gamma_2\rho_{22} + \mathrm{Re}[i\Omega_0^*\tilde{\rho}_{21}],
\end{aligned}$$

we first drop the tildes, then let $\rho_{21} = \frac{1}{2}(u - iv)$, and $\Omega_0 = \Omega_0' + i\Omega_0''$. Plugging these in, we get

$$\begin{aligned}
\frac{1}{2}(\dot{u} - i\dot{v}) &= -\frac{1}{2}(\gamma + i\delta)(u - iv) + \frac{i}{2}(\Omega_0' + i\Omega_0'')(2\rho_{22} - 1) \\
\dot{\rho}_{22} &= -\gamma_2\rho_{22} + \frac{1}{2}(\Omega_0'' u + \Omega_0' v)
\end{aligned}$$

$$\Longrightarrow$$

$$\begin{aligned}
\dot{u} &= -\gamma u - \delta v - 2\Omega_0''\rho_{22} + \Omega_0'' \\
\dot{v} &= -\delta u + \gamma v + 2\Omega_0'\rho_{22} - \Omega_0' \\
\dot{\rho}_{22} &= \frac{\Omega_0''}{2}u + \frac{\Omega_0'}{2}v - \gamma_2\rho_{22}
\end{aligned}$$

In the steady state, this is

$$\left.\begin{aligned}
&-\gamma u - \delta v - 2\Omega_0''\rho_{22} = -\Omega_0'' \\
&-\delta u + \gamma v + 2\Omega_0'\rho_{22} = \Omega_0' \\
&\tfrac{\Omega_0''}{2}u + \tfrac{\Omega_0'}{2}v - \gamma_2\rho_{22} = 0
\end{aligned}\right\} \implies \begin{pmatrix} -\gamma & -\delta & 2\Omega_0'' \\ -\delta & +\gamma & 2\Omega_0' \\ \frac{\Omega_0''}{2} & \frac{\Omega_0'}{2} & -\gamma_2 \end{pmatrix} \begin{pmatrix} u \\ v \\ \rho_{22} \end{pmatrix} = \begin{pmatrix} -\Omega_0'' \\ \Omega_0 \\ 0 \end{pmatrix}$$

Inverting and solving, we find

$$\begin{pmatrix} u \\ v \\ \rho_{22} \end{pmatrix} = \frac{1}{\gamma^2 + \delta^2 + \frac{\gamma}{\gamma_2}\left|\Omega_0\right|^2} \begin{pmatrix} \gamma\Omega_0'' - \delta\Omega_0' \\ \delta\Omega_0'' + \gamma\Omega_0' \\ \frac{\gamma\left|\Omega_0\right|^2}{2\gamma_2} \end{pmatrix}.$$

Or, in terms of just density matrix components,

$$\tilde{\rho}_{21} = \frac{-\frac{1}{2}(\delta + i\gamma)\Omega_0}{\gamma^2 + \delta^2 + \frac{\gamma}{\gamma_2}\left|\Omega_0\right|^2}$$

$$\rho_{22} = \frac{\gamma\left|\Omega_0\right|^2}{2\gamma_2}\frac{1}{\gamma^2 + \delta^2 + \frac{\gamma}{\gamma_2}\left|\Omega_0\right|^2}$$

Problem 3

In lecture, we arrived at

$$\chi(\omega) \approx \frac{N}{V}\frac{\mu^2}{\epsilon_0\hbar}\frac{\rho_{22}^{(0)} - \rho_{11}^{(0)}}{\delta - i\gamma},$$

where we used the first order approximation

$$\rho_{21}^{(1)} \approx \frac{\Omega_0}{2}\frac{\rho_{22}^{(0)} - \rho_{11}^{(0)}}{\delta - i\gamma}.$$

If I use the full solution that I got in the previous problem, however, I get an imaginary component of χ that is

$$\chi'' = \frac{N}{V}\frac{\mu^2}{\epsilon_0\hbar}\frac{\gamma}{\gamma^2 + \delta^2 + \frac{\gamma}{\gamma_2}\left|\Omega_0\right|^2},$$

which is the same up to that extra term in the denominator. That extra term seems to mess up the simplification when we set $\delta = 0$ ($\implies \omega = \omega_0$), $\gamma = \gamma_2/2$, and

$$\gamma_2 = \frac{\mu^2\omega_0^3}{3\pi\epsilon_0\hbar c^3}.$$

If we ignore it, then everything goes through exactly as it did in lecture. When we compare

$$\alpha = \frac{\omega}{c}\chi'' = \frac{N}{V}\sigma,$$

we get

$$\sigma = \frac{c^2}{\omega_0^2} 6\pi = \frac{3}{2\pi}\lambda_0^2$$

Given

$$\frac{N}{V} = 3 \times 10^9 \ \frac{\text{atoms}}{\text{cm}^3}$$
$$\gamma_2 = 2\pi \times 10 \text{ MHz}$$
$$\lambda_0 = 600 \text{ nm}$$

we have that

$$\alpha = \frac{N}{V}\frac{3}{2\pi}\lambda_0^2$$

(a)

$$n + 1 + \frac{1}{2}\chi'$$

Problem 4

(a) The rate equation approximation seems to be valid in the first and third cases. It's clearly not valid in the second case, as the coherence is almost exactly 90° out of phase with the population

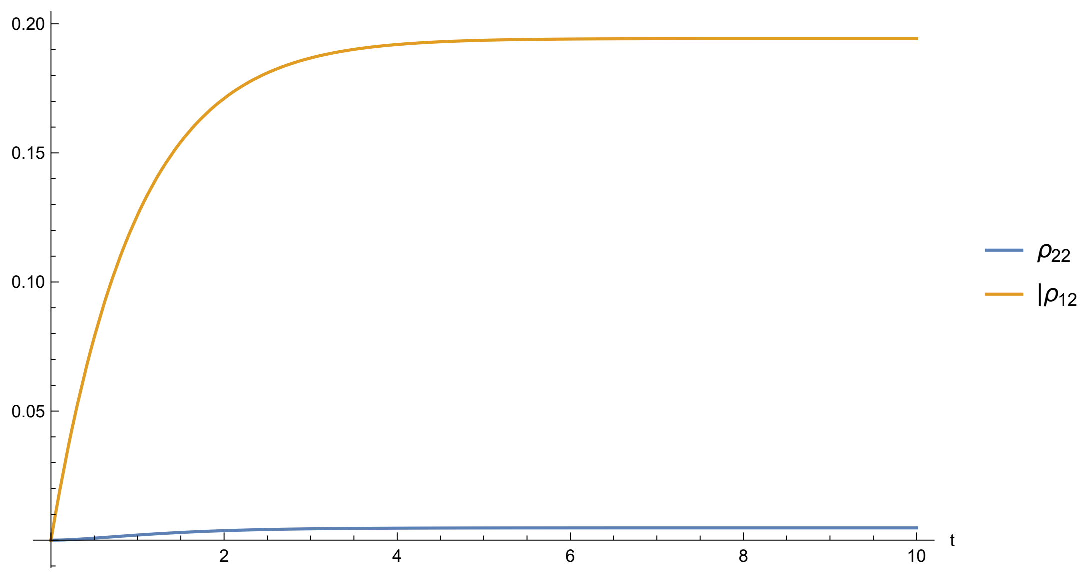
Problem 4 (Berman 4.3)

In[122]:=
```
SolveAndPlot[γ_, χ_, δ_] :=
 ({{uSoln, vSoln, wSoln}} = NDSolve[{u'[t] == -γ u[t] - δ v[t],
     v'[t] == δ u[t] - γ v[t] - (2 χ) w[t], w'[t] == - (2 γ) (w[t] + 1) + χ v[t],
     u[0] == 0, v[0] == 0, w[0] == -1}, {u, v, w}, {t, 0, 10}];
  Plot[{1/2 (w[x] + 1) /. wSoln, Sqrt[u[x]^2 + v[x]^2] /. uSoln /. vSoln}, {x, 0, 10},
   PlotRange → All, PlotLegends → {"ρ22", "|ρ12|"}, AxesLabel → {"t", ""}]);
```

$\gamma = 1;\ \Omega_0 = 0.2,\ \delta = 0.2$

In[123]:=
```
SolveAndPlot[1, 0.1, 0.2]
```

Out[123]=

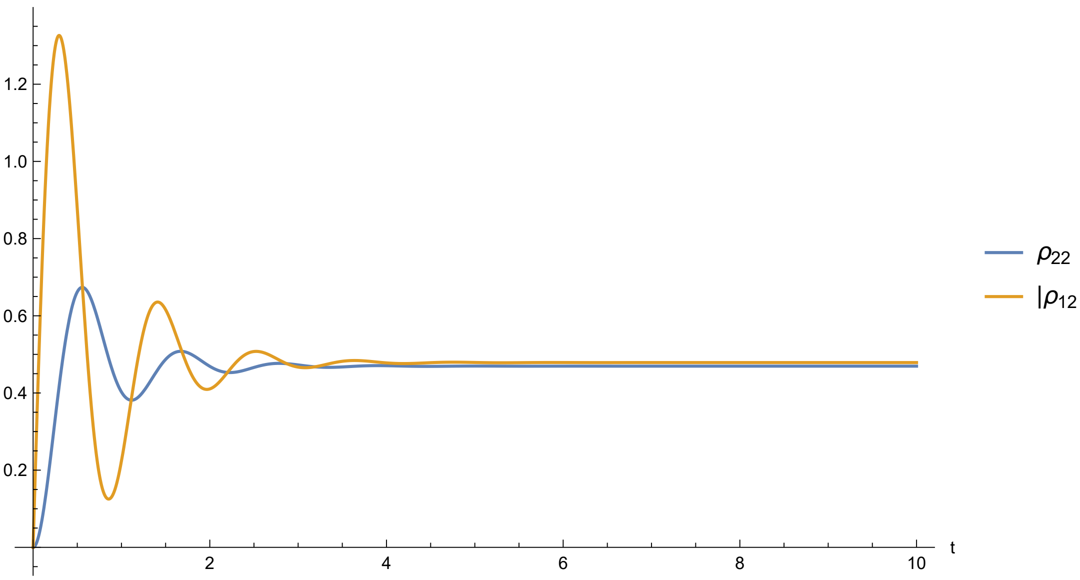


$\gamma = 1;\ \Omega_0 = 8,\ \delta = 0.2$

In[124]:=

```
SolveAndPlot[1, 4, 0.2]
```

Out[124]=



$\gamma = 1;\ \Omega_0 = 8,\ \delta = 50$

In[125]:=

```
SolveAndPlot[1, 4, 50]
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Out[125]=