Midterm Rewrite

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Problem 1

Consider, without loss of generality, $J_z = -\frac{i}{\hbar}[J_x, J_y]$. Due to the linearity and commutational invariance of the trace,

$$\begin{aligned}
\mathrm{tr}[J_z] &= -\frac{i}{\hbar}\,\mathrm{tr}[J_x J_y - J_y J_x] \\
&= -\frac{i}{\hbar}\left(\mathrm{tr}[J_x J_y] - \mathrm{tr}[J_y J_x]\right) \\
&= -\frac{i}{\hbar}\left(\mathrm{tr}[J_x J_y] - \mathrm{tr}[J_x J_y]\right) \\
&= 0
\end{aligned}$$

Problem 2

Using the saddle-point approximation with $f(x,t) = xt - e^t$, we have

$$\begin{aligned}
\frac{\partial f}{\partial t} &= x - e^t \\
\frac{\partial^2 f}{\partial t^2} &= -e^t
\end{aligned}$$

Therefore,

$$t_0 = \ln(x); \quad f_0 = f(x, t_0) = x\ln(x) - x; \quad f_0'' = \frac{\partial^2 f}{\partial t^2}|_{t=t_0} = -x$$

and applying the saddle-point approximation gives

$$\begin{aligned}
I(x) &\approx e^{f_0}\sqrt{\frac{2\pi}{|f_0''|}} \\
&\approx e^{x\ln(x)-x}\sqrt{\frac{2\pi}{x}} \\
&\approx \sqrt{2\pi}e^{-x}x^{x-\frac{1}{2}}
\end{aligned}$$

Problem 3

(a) No, $\vec{r}$ and $\vec{p}$ are vector operators, but their components each commute amongst themselves.

(b)

$$\vec{V}\cdot\vec{J}-\vec{J}\cdot\vec{V}=[V_\alpha,J_\alpha]=i\hbar\epsilon_{\alpha\alpha\beta}V_\beta=0 \implies \vec{V}\cdot\vec{J}=\vec{J}\cdot\vec{V}$$

(c)

$$\begin{aligned}\left[J_\alpha,\vec{V}\cdot\vec{J}\right] &= [J_\alpha,V_\beta J_\beta] \\ &= V_\beta[J_\alpha,J_\beta]+[J_\alpha,V_\beta]J_\beta \\ &= i\hbar\left(\epsilon_{\alpha\beta\gamma}V_\beta J_\gamma+\epsilon_{\alpha\beta\gamma}V_\gamma J_\beta\right) \\ &= 0\end{aligned}$$

where the last equality comes from the antisymmetry of the Levi-Civita symbol.

(d) Firstly,

$$\begin{aligned}\left[J^2,V_\beta\right] &= J_\alpha[J_\alpha,V_\beta]+[J_\alpha,V_\beta]J_\alpha \\ &= i\hbar\epsilon_{\alpha\beta\gamma}[J_\alpha,V_\gamma]_+\end{aligned}$$

Now,

$$\begin{aligned}\left[J^2,\left[J^2,V_\beta\right]\right] &= J_\sigma[J_\sigma,V_\beta]+\left[J_\sigma,\left[J^2,V_\beta\right]\right]J_\sigma \\ &= i\hbar\epsilon_{\alpha\beta\gamma}\left[J_\sigma,\left[J_\sigma,[J_\alpha,V_\gamma]_+\right]\right]_+ \\ &= i\hbar\epsilon_{\alpha\beta\gamma}[J_\sigma,[J_\sigma,J_\alpha V_\gamma+V_\gamma J_\alpha]]_+\end{aligned}$$

The inner commutator gives

$$\begin{aligned}[J_\sigma,J_\alpha V_\gamma+V_\gamma J_\alpha] &= J_\alpha[J_\sigma,V_\gamma]+[J_\sigma,J_\alpha]V_\gamma+V_\gamma[J_\sigma,J_\alpha]+[J_\sigma,V_\gamma]J_\alpha \\ &= i\hbar\epsilon_{\sigma\gamma\delta}J_\alpha V_\delta+i\hbar\epsilon_{\sigma\alpha\delta}J_\delta V_\gamma+i\hbar\epsilon_{\sigma\gamma\delta}V_\delta J_\alpha \\ &= i\hbar\epsilon_{\sigma\gamma\delta}[J_\alpha,V_\delta]_+ + i\hbar\epsilon_{\sigma\alpha\delta}[J_\delta,V_\gamma]_+\end{aligned}$$

The double commutator therefore becomes

$$\begin{aligned}\left[J^2,\left[J^2,V_\beta\right]\right] &= -\hbar^2\epsilon_{\alpha\beta\gamma}\left[J_\alpha,\epsilon_{\sigma\gamma\delta}[J_\alpha,V_\delta]_+ + \epsilon_{\sigma\alpha\delta}[J_\delta,V_\gamma]_+\right]_+ \\ &= \hbar^2\epsilon_{\alpha\beta\gamma}\left[J_\sigma,\epsilon_{\sigma\alpha\delta}[J_\gamma,V_\delta]_+ - \epsilon_{\sigma\alpha\delta}[J_\delta,V_\gamma]_+\right]_+ \\ &= \hbar^2\epsilon_{\alpha\beta\gamma}\epsilon_{\alpha\delta\sigma}\left[J_\sigma,[J_\gamma,V_\delta]_+ - [J_\delta,V_\gamma]_+\right]_+\end{aligned}$$

Then, using

$$\epsilon_{\mu\alpha\beta}\epsilon_{\mu\sigma\tau}=\delta_{\alpha\sigma}\delta_{\beta\tau}-\delta_{\alpha\tau}\delta_{\beta\sigma},$$

we get

$$\begin{aligned}\left[J^2,\left[J^2,V_\beta\right]\right] &= \hbar^2(\delta_{\beta\delta}\delta_{\gamma\sigma}-\delta_{\beta\sigma}\delta_{\gamma\delta})\left[J_\sigma,[J_\gamma,V_\delta]_+ - [J_\delta,V_\gamma]_+\right] \\ &= \hbar^2\left[J_\gamma,[J_\gamma,V_\beta]_+ - [J_\beta,V_\gamma]_+\right]_+ - \hbar^2\left[J_\beta,[J_\gamma,V_\gamma]_+ - [J_\gamma,V_\gamma]_+\right]_+ \\ &= \hbar^2\left[J_\gamma,[J_\gamma,V_\beta]_+ - [J_\beta,V_\gamma]_+\right]_+\end{aligned}$$

The first term of the outer commutator in the above equation gives

$$\begin{aligned}
\big[J_\gamma, [J_\gamma, V_\beta]_+\big]_+ &= J_\gamma J_\gamma V_\beta + J_\gamma V_\beta J_\gamma + J_\gamma V_\beta J_\gamma + V_\beta J_\gamma J_\gamma \\
&= 2J_\gamma J_\gamma V_\beta + 2V_\beta J_\gamma J_\gamma + J_\gamma [V_\beta, J_\gamma] + [J_\gamma, V_\beta] J_\gamma \\
&= 2\big[J^2, V_\beta\big]_+ - i\hbar\epsilon_{\gamma\beta\sigma} J_\gamma V_\sigma + i\hbar\epsilon_{\gamma\beta\sigma} V_\sigma J_\gamma \\
&= 2\big[J^2, V_\beta\big]_+ + \hbar^2 \epsilon_{\gamma\beta\sigma}\epsilon_{\gamma\sigma\tau} V_\tau \\
&= 2\big[J^2, V_\beta\big]_+ - 2\hbar^2 V_\beta
\end{aligned}$$

and the second term gives

$$\begin{aligned}
\big[J_\gamma, [J_\beta, V_\gamma]_+\big]_+ &= J_\gamma J_\beta V_\gamma + J_\gamma V_\gamma J_\beta + J_\beta V_\gamma J_\gamma + V_\gamma J_\beta J_\gamma \\
&= 4J_\beta \vec{V} \cdot \vec{J} + [J_\gamma, J_\beta] V_\gamma + V_\gamma [J_\beta, J_\gamma] \\
&= 4J_\beta \vec{V} \cdot \vec{J} + i\hbar\epsilon_{\gamma\beta\sigma} [J_\sigma, V_\gamma] \\
&= 4J_\beta \vec{V} \cdot \vec{J} - \hbar^2 \epsilon_{\gamma\beta\sigma}\epsilon_{\sigma\gamma\tau} V_\tau \\
&= 4J_\beta \vec{V} \cdot \vec{J} - 2\hbar^2 V_\beta
\end{aligned}$$

Finally,

$$\begin{aligned}
\big[J^2, \big[J^2, V_\beta\big]\big] &= 2\hbar^2 \big[J^2, V_\beta\big]_+ - 4\hbar^2 J_\beta \vec{V} \cdot \vec{J} \\
\implies \Big[J^2, \Big[J^2, \vec{V}\Big]\Big] &= 2\hbar^2 \Big[J^2, \vec{V}\Big]_+ - 4\hbar^2 (\vec{V} \cdot \vec{J}) \vec{J}
\end{aligned}$$

Problem 4

(a) For $0 \leq x \leq L$, the potential can be written

$$V(x) = V_0 \left(\frac{x}{L} - 1 \right),$$

and the turning point is given by

$$V(x_2) = V_0 \left(\frac{x_2}{L} - 1 \right) = -|E| \implies x_2 = L \left(1 - \frac{|E|}{V_0} \right)$$

The WKB quantization condtion tells us that

$$\begin{aligned}
(n + 1/2)\pi\hbar &= 2 \int_0^{x_2} \sqrt{2m(E - V(x))} \mathrm{d}x \\
&= \sqrt{32m} \int_0^{x_2} \sqrt{V_0 - |E| - V_0 \frac{x}{L}} \mathrm{d}x \\
&= \frac{\sqrt{128m}L}{3V_0} (V_0 - |E|)^{3/2}
\end{aligned}$$

Thus

$$E_n = \left(\frac{3V_0 (n + 1/2)\pi\hbar}{\sqrt{32}mL} \right) - V_0$$

(b) The energy of the highest bound state is as close to 0 as possible without being positive. Setting $E_n = 0$ gives

$$V_0 = \left(\frac{3V_0(n_{\max} + 1/2)\pi\hbar}{\sqrt{32}mL} \right) \implies n_{\max} \approx \frac{sqrt32mV_0L}{3\pi\hbar} - \frac{1}{2}$$

(c) Setting $n_{\max} = 0$ gives

$$\frac{\sqrt{32mV_0}L}{3\pi\hbar} = \frac{1}{2} \implies V_0L \geq \frac{9\pi^2\hbar^2}{128m}$$