

Homework 5

Sean Ericson
Phys 684

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Problem 1

We start with the Maxwell wave equation

$$\left(\vec{\nabla}^2 - \frac{1}{c^2} \frac{\partial^2}{\partial t^2}\right) \vec{E}(\vec{r}, t) = \mu_0 \frac{\partial^2}{\partial t^2} \vec{P}(\vec{r}, t),$$

where

$$\vec{E}(\vec{r}, t) = \vec{E}_+(z, t) + \vec{E}_-(z, t); \quad \vec{E}_{\pm}(z, t) = \frac{1}{2} \hat{x} E_0(z, t) e^{\mp i \alpha(z, t)},$$

$$\vec{P}(\vec{r}, t) = \vec{P}_+(z, t) + \vec{P}_-(z, t) = \frac{1}{2} \hat{x} (P_0(z, t) e^{-i \alpha(z, t)} + \text{c.c.}),$$

$$\alpha(z, t) = \omega t - kz - \phi(z, t),$$

and $E_0(z, t) \in \mathbb{R}$ while $P_0(z, t) \in \mathbb{C}$. Noticing that

$$2 \frac{\partial}{\partial z} |E_{\pm}| = (E'_0 \mp i E_0 \alpha') e^{\mp i \alpha},$$

$$2 \frac{\partial}{\partial t} |E_{\pm}| = (\dot{E}_0 \mp i E_0 \dot{\alpha}) e^{\mp i \alpha},$$

$$2 \frac{\partial}{\partial t} |P_{\pm}| = (\dot{P}_0 \mp P_0 \dot{\alpha}) e^{\mp i \alpha}$$

$$2 \frac{\partial^2}{\partial t^2} |P_{\pm}| = (\ddot{P}_0 \mp \dot{P}_0 \dot{\alpha} \mp P_0 \ddot{\alpha} \mp i \dot{P}_0 \dot{\alpha} + i P_0 \dot{\alpha}^2) e^{\mp i \alpha}$$

$$\alpha' = -k - \phi'$$

$$\dot{\alpha} = \omega - \dot{\phi},$$

we factor the differential operator in the wave equation as

$$\partial_z^2 - \frac{1}{c^2} \partial_t^2 = \left(\partial_z + \frac{1}{c} \partial_t\right) \left(\partial_z - \frac{1}{c} \partial_t\right),$$

and begin applying it to the \pm components of the fields:

$$\begin{aligned}
2 \left(\partial_z - \frac{1}{c} \partial_t \right) |E_{\pm}| &= \left[E'_0 \mp i E_0 \alpha' - \frac{1}{c} \left(\dot{E}_0 \mp i E_0 \dot{\alpha} \right) \right] e^{\mp i \alpha} \\
&= \left[\mp i E_0 \left(\alpha' - \frac{1}{c} \dot{\alpha} \right) + E'_0 - \frac{1}{c} \dot{E}_0 \right] e^{\mp i \alpha} \\
&= \left[\pm i E_0 \left(2k + \phi' - \frac{1}{c} \dot{\phi} \right) + E'_0 - \frac{1}{c} \dot{E}_0 \right] e^{\mp i \alpha}.
\end{aligned}$$

Now, in the slowly varying amplitude and phase approximation, we neglect terms proportional to \ddot{E}_0 , E''_0 , \dot{P}_0 , and \ddot{P}_0 . So, as we apply the second half of the differential operator, let's drop the terms that will produce terms that we'll neglect anyway:

$$\begin{aligned}
2 \left(\partial_z + \frac{1}{c} \partial_t \right) \left(\partial_z - \frac{1}{c} \partial_t \right) |E_{\pm}| &\approx \pm i \left(\partial_z + \frac{1}{c} \partial_t \right) \left[E_0 \left(2k + \phi' - \frac{1}{c} \dot{\phi} \right) e^{\mp i \alpha} \right] \\
&= \pm i \left[\left(E'_0 + \frac{1}{c} \dot{E}_0 \right) \left(2k + \phi' - \frac{1}{c} \dot{\phi} \right) \right]
\end{aligned}$$

Problem 2

Starting with

$$\begin{aligned}
\dot{\tilde{\rho}}_{21} &= -(\gamma + i\delta) \tilde{\rho}_{21} + i \frac{\Omega_0}{2} (\rho_{22} - \rho_{11}) \\
\dot{\rho}_{22} &= -\gamma_2 \rho_{22} + \text{Re}[i \Omega_0^* \tilde{\rho}_{21}],
\end{aligned}$$

we first drop the tildes, then let $\rho_{21} = \frac{1}{2}(u - iv)$, and $\Omega_0 = \Omega'_0 + i\Omega''_0$. Plugging these in, we get

$$\begin{aligned}
\frac{1}{2}(\dot{u} - i\dot{v}) &= -\frac{1}{2}(\gamma + i\delta)(u - iv) + \frac{i}{2}(\Omega'_0 + i\Omega''_0)(2\rho_{22} - 1) \\
\dot{\rho}_{22} &= -\gamma_2 \rho_{22} + \frac{1}{2}(\Omega''_0 u + \Omega'_0 v) \\
\implies \\
\dot{u} &= -\gamma u - \delta v - 2\Omega''_0 \rho_{22} + \Omega''_0 \\
\dot{v} &= -\delta u + \gamma v + 2\Omega'_0 \rho_{22} - \Omega'_0 \\
\dot{\rho}_{22} &= \frac{\Omega''_0}{2} u + \frac{\Omega'_0}{2} v - \gamma_2 \rho_{22}
\end{aligned}$$

In the steady state, this is

$$\left. \begin{aligned} -\gamma u - \delta v - 2\Omega''_0 \rho_{22} &= -\Omega''_0 \\ -\delta u + \gamma v + 2\Omega'_0 \rho_{22} &= \Omega'_0 \\ \frac{\Omega''_0}{2} u + \frac{\Omega'_0}{2} v - \gamma_2 \rho_{22} &= 0 \end{aligned} \right\} \implies \begin{pmatrix} -\gamma & -\delta & 2\Omega''_0 \\ -\delta & +\gamma & 2\Omega'_0 \\ \frac{\Omega''_0}{2} & \frac{\Omega'_0}{2} & -\gamma_2 \end{pmatrix} \begin{pmatrix} u \\ v \\ \rho_{22} \end{pmatrix} = \begin{pmatrix} -\Omega''_0 \\ \Omega'_0 \\ 0 \end{pmatrix}$$

Inverting and solving, we find

$$\begin{pmatrix} u \\ v \\ \rho_{22} \end{pmatrix} = \frac{1}{\gamma^2 + \delta^2 + \frac{\gamma}{\gamma_2} |\Omega_0|^2} \begin{pmatrix} \gamma \Omega_0'' - \delta \Omega_0' \\ \delta \Omega_0'' + \gamma \Omega_0' \\ \frac{\gamma |\Omega_0|^2}{2\gamma_2} \end{pmatrix}.$$

Or, in terms of just density matrix components,

$$\begin{aligned} \tilde{\rho}_{21} &= \frac{-\frac{1}{2}(\delta + i\gamma)\Omega_0}{\gamma^2 + \delta^2 + \frac{\gamma}{\gamma_2} |\Omega_0|^2} \\ \rho_{22} &= \frac{\gamma |\Omega_0|^2}{2\gamma_2} \frac{1}{\gamma^2 + \delta^2 + \frac{\gamma}{\gamma_2} |\Omega_0|^2} \end{aligned}$$

Problem 3

(a)

Problem 4

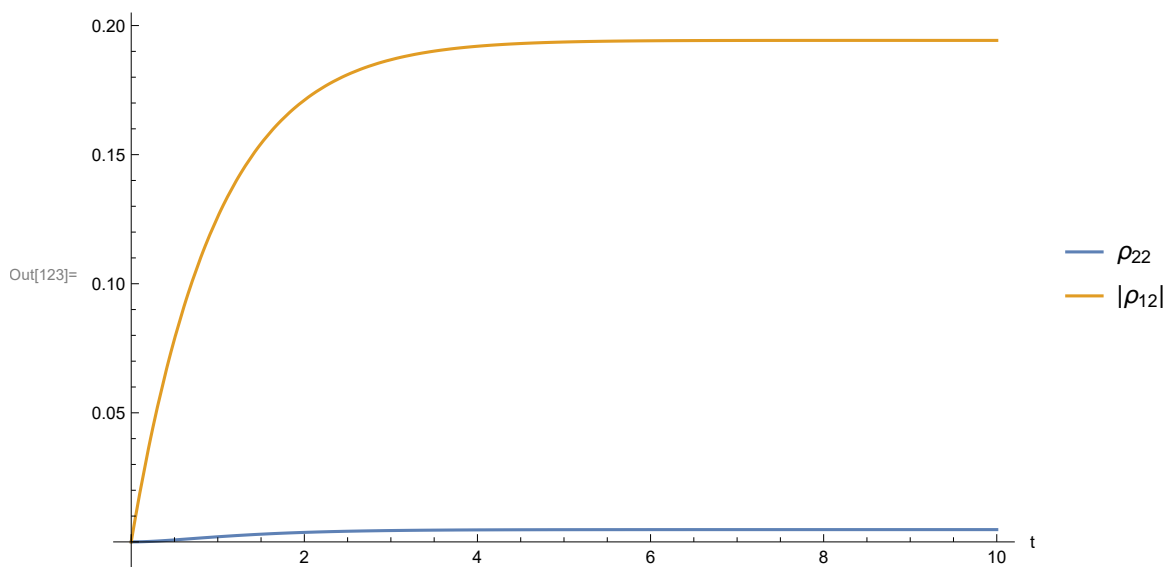
(a)

Problem 4 (Berman 4.3)

```
In[122]:= SolveAndPlot[ $\gamma$ _,  $\chi$ _,  $\delta$ _] :=
  ({{uSoln, vSoln, wSoln}} = NDSolve[{u'[t] == - $\gamma$  u[t] -  $\delta$  v[t],
    v'[t] ==  $\delta$  u[t] -  $\gamma$  v[t] - (2  $\chi$ ) w[t], w'[t] == -(2  $\gamma$ ) (w[t] + 1) +  $\chi$  v[t],
    u[0] == 0, v[0] == 0, w[0] == -1}, {u, v, w}, {t, 0, 10}];
  Plot[ $\left\{\frac{1}{2} (w[x] + 1) /. wSoln, \sqrt{u[x]^2 + v[x]^2} /. uSoln /. vSoln\right\}$ , {x, 0, 10},
    PlotRange -> All, PlotLegends -> {" $\rho_{22}$ ", " $|\rho_{12}|$ "}, AxesLabel -> {"t", ""}]);
```

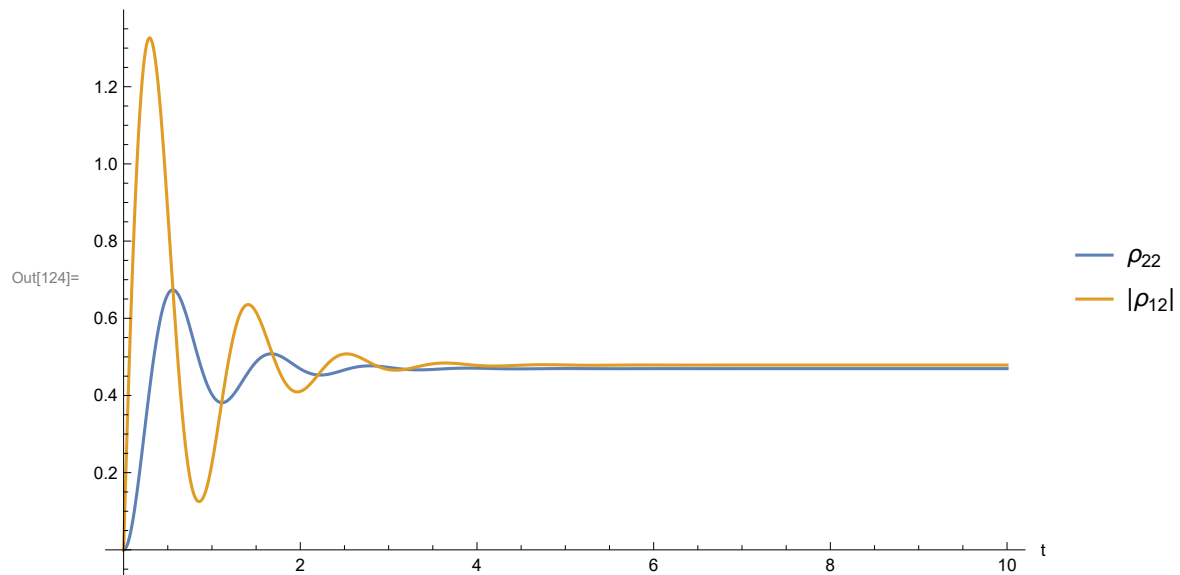
$\gamma = 1$; $\Omega_0 = 0.2$, $\delta = 0.2$

```
In[123]:= SolveAndPlot[1, 0.1, 0.2]
```



$$\gamma = 1; \Omega_0 = 8, \delta = 0.2$$

In[124]:= **SolveAndPlot**[1, 4, 0.2]



$$\gamma = 1; \Omega_0 = 8, \delta = 50$$

In[125]:= **SolveAndPlot**[1, 4, 50]

