```
In[19]:= H1 = {α, β}ᵀ;
         H2 = {γ, δ}ᵀ;
         ϵ = {{0, 1}, {-1, 0}};
         V = m1^2 H1†.H1 + m2^2 H2†.H2 - b (H1†.H2 + H2†.H1) +
             (g1^2 + g2^2)/8 (H1†.H1 - H2†.H2)^2 + g1^2/2 (H2ᵀ.ϵ.H1) (H2†.ϵ.H1*);
         V // TraditionalForm
```

Out[23]//TraditionalForm=

$$-b\,(\gamma\,\alpha^* + \alpha\,\gamma^* + \delta\,\beta^* + \beta\,\delta^*) + \frac{1}{8}\left(g_1^2 + g_2^2\right)(\alpha\,\alpha^* + \beta\,\beta^* - \gamma\,\gamma^* - \delta\,\delta^*)^2 + \frac{1}{2}\,g_1^2\,(\beta\,\gamma - \alpha\,\delta)(\beta^*\,\gamma^* - \alpha^*\,\delta^*) + m_1^2\,(\alpha\,\alpha^* + \beta\,\beta^*) + m_2^2\,(\gamma\,\gamma^* + \delta\,\delta^*)$$

---

a)

```
In[24]:= sub = {α → 0, β → v/√2, γ → y, δ → z};

In[25]:= (D[V /. α* → a, α] /. a → α*) /. sub // FullSimplify // TraditionalForm
```

Out[25]//TraditionalForm=

$$-\frac{1}{4}\,y^*\left(4\,b + \sqrt{2}\,g_1^2\,v\,z\right)$$

---

b)

```
In[26]:= (V /. sub) /. y → 0 // FullSimplify // TraditionalForm
```

Out[26]//TraditionalForm=

$$-\sqrt{2}\,b\,v\,\mathrm{Re}(z) + \frac{1}{32}\left(g_1^2 + g_2^2\right)\left(v^2 - 2\,z\,z^*\right)^2 + m_2^2\,z\,z^* + \frac{1}{2}\,m_1^2\,v^2$$

---

c)

```
In[27]:= sub2 = {β → (v1 + h1 + ⅈ A1)/√2, δ → (v2 + h2 + ⅈ A2)/√2};
```

In[28]:=
```
(V /. sub2) // TraditionalForm
```

Out[28]//TraditionalForm=

$$-b\left(\frac{1}{2}\,(i\,A_1+h_1+v_1)\,((h_2+v_2)^*-i\,(A_2)^*)+\frac{1}{2}\,(i\,A_2+h_2+v_2)\,((h_1+v_1)^*-i\,(A_1)^*)+\gamma\,\alpha^*+\alpha\,\gamma^*\right)+$$
$$\frac{1}{2}\,g_1^2\left(\frac{\gamma\,(i\,A_1+h_1+v_1)}{\sqrt{2}}-\frac{\alpha\,(i\,A_2+h_2+v_2)}{\sqrt{2}}\right)\left(\frac{\gamma^*\,((h_1+v_1)^*-i\,(A_1)^*)}{\sqrt{2}}-\frac{\alpha^*\,((h_2+v_2)^*-i\,(A_2)^*)}{\sqrt{2}}\right)+$$
$$\frac{1}{8}\,(g_1^2+g_2^2)\left(\frac{1}{2}\,(i\,A_1+h_1+v_1)\,((h_1+v_1)^*-i\,(A_1)^*)-\frac{1}{2}\,(i\,A_2+h_2+v_2)\,((h_2+v_2)^*-i\,(A_2)^*)+\alpha\,\alpha^*-\gamma\,\gamma^*\right)^2+$$
$$m_1^2\left(\alpha\,\alpha^*+\frac{1}{2}\,(i\,A_1+h_1+v_1)\,((h_1+v_1)^*-i\,(A_1)^*)\right)+m_2^2\left(\gamma\,\gamma^*+\frac{1}{2}\,(i\,A_2+h_2+v_2)\,((h_2+v_2)^*-i\,(A_2)^*)\right)$$

In[29]:=
```
eq1 = D[(V // FullSimplify) /. sub2, h₁] /.
    {α → 0, γ → 0, A₁ → 0, h₁ → 0, A₂ → 0, h₂ → 0, m₁ → √M₁} // FullSimplify
eq2 = D[(V // FullSimplify) /. sub2 // FullSimplify, h₂] /.
    {α → 0, γ → 0, A₁ → 0, h₁ → 0, A₂ → 0, h₂ → 0, m₂ → √M₂} // FullSimplify
```

Out[29]=

$$M_1\,v_1-\frac{1}{2}\,b\,(1+v_1)\,v_2+\frac{1}{8}\,(g_1^2+g_2^2)\,v_1\,(v_1^2-v_2^2)$$

Out[30]=

$$-b\,v_1+M_2\,v_2+\frac{1}{8}\,(g_1^2+g_2^2)\,v_2\,(-v_1^2+v_2^2)$$

In[31]:=
```
Solve[{eq1 == 0}, {M₁}] // Quiet // FullSimplify
Solve[eq2 == 0, M₂] // Quiet // FullSimplify
```

Out[31]=

$$\left\{\left\{M_1\to\boxed{\frac{1}{8}\left(4\,b\,v_2+\frac{4\,b\,v_2}{v_1}+(g_1^2+g_2^2)\,(-v_1^2+v_2^2)\right)\ \text{if}\ v_1\neq 0}\right\}\right\}$$

Out[32]=

$$\left\{\left\{M_2\to\boxed{\frac{1}{8}\,(g_1^2+g_2^2)\,v_1^2+\frac{b\,v_1}{v_2}-\frac{1}{8}\,(g_1^2+g_2^2)\,v_2^2\ \text{if}\ v_2\neq 0}\right\}\right\}$$

d)

In[33]:=
```
V2 = V /. sub2;
V2 // TraditionalForm
MM =
  {{D[D[V2, α]*, α]*, D[D[V2, α]*, γ]*}, {D[D[V2, γ]*, α]*, D[D[V2, γ]*, γ]*}} /.
    {α → 0, γ → 0, h1 → 0, h2 → 0, A1 → 0, A2 → 0} // FullSimplify;
MM // MatrixForm
```

Out[34]//TraditionalForm=

$$-b\left(\frac{1}{2}(i A_1+h_1+v_1)((h_2+v_2)^*-i(A_2)^*)+\frac{1}{2}(i A_2+h_2+v_2)((h_1+v_1)^*-i(A_1)^*)+\gamma\alpha^*+\alpha\gamma^*\right)+$$
$$\frac{1}{2}g_1^2\left(\frac{\gamma(i A_1+h_1+v_1)}{\sqrt{2}}-\frac{\alpha(i A_2+h_2+v_2)}{\sqrt{2}}\right)\left(\frac{\gamma^*((h_1+v_1)^*-i(A_1)^*)}{\sqrt{2}}-\frac{\alpha^*((h_2+v_2)^*-i(A_2)^*)}{\sqrt{2}}\right)+$$
$$\frac{1}{8}(g_1^2+g_2^2)\left(\frac{1}{2}(i A_1+h_1+v_1)((h_1+v_1)^*-i(A_1)^*)-\frac{1}{2}(i A_2+h_2+v_2)((h_2+v_2)^*-i(A_2)^*)+\alpha\alpha^*-\gamma\gamma^*\right)^2+$$
$$m_1^2\left(\alpha\alpha^*+\frac{1}{2}(i A_1+h_1+v_1)((h_1+v_1)^*-i(A_1)^*)\right)+m_2^2\left(\gamma\gamma^*+\frac{1}{2}(i A_2+h_2+v_2)((h_2+v_2)^*-i(A_2)^*)\right)$$

Out[36]//MatrixForm=

$$\begin{pmatrix} m_1^2+\frac{1}{8}\left(g_2^2\left(v_1^2-v_2^2\right)+g_1^2\left(v_1^2+v_2^2\right)\right) & -b-\frac{1}{4}g_1^2 v_1 v_2 \\ -b-\frac{1}{4}g_1^2 v_1 v_2 & m_2^2+\frac{1}{8}\left(g_2^2\left(-v_1^2+v_2^2\right)+g_1^2\left(v_1^2+v_2^2\right)\right) \end{pmatrix}$$

In[37]:=
```
Eigenvalues[MM] // FullSimplify // FullSimplify
```

Out[37]=

$$\left\{\frac{1}{8}\left(4m_1^2+4m_2^2+g_1^2\left(v_1^2+v_2^2\right)-\sqrt{64b^2+\left(4m_1^2-4m_2^2+g_2^2v_1^2\right)^2+32b\,g_1^2v_1v_2-2\left(4g_2^2\left(m_1^2-m_2^2\right)+\left(-2g_1^4+g_2^4\right)v_1^2\right)v_2^2+g_2^4v_2^4}\right),\right.$$
$$\left.\frac{1}{8}\left(4m_1^2+4m_2^2+g_1^2\left(v_1^2+v_2^2\right)+\sqrt{64b^2+\left(4m_1^2-4m_2^2+g_2^2v_1^2\right)^2+32b\,g_1^2v_1v_2-2\left(4g_2^2\left(m_1^2-m_2^2\right)+\left(-2g_1^4+g_2^4\right)v_1^2\right)v_2^2+g_2^4v_2^4}\right)\right\}$$

e)

In[38]:=
```
V2 = FullSimplify[V2];
V2 // TraditionalForm
```

Out[39]//TraditionalForm=

$$\frac{1}{32}\Big(-32b(A_1A_2+\gamma\alpha^*+\alpha\gamma^*+(h_1+v_1)(h_2+v_2))+$$
$$8g_1^2(-\alpha A_2+A_1\gamma+i(\alpha(h_2+v_2)-\gamma(h_1+v_1)))(\gamma^*(A_1+i(h_1+v_1))-\alpha^*(A_2+i(h_2+v_2)))+$$
$$(g_1^2+g_2^2)\left(A_1^2-A_2^2+2\alpha\alpha^*-2\gamma\gamma^*+(h_1-h_2+v_1-v_2)(h_1+h_2+v_1+v_2)\right)^2+$$
$$16m_1^2\left(A_1^2+2\alpha\alpha^*+(h_1+v_1)^2\right)+16m_2^2\left(A_2^2+2\gamma\gamma^*+(h_2+v_2)^2\right)\Big)$$

In[40]:=
```
MM2 = {{D[V2, A₁, A₁], D[V2, A₁, A₂]}, {D[V2, A₂, A₁], D[V2, A₂, A₂]}} /.
    {α → 0, γ → 0, h₁ → 0, h₂ → 0, A₁ → 0, A₂ → 0} // FullSimplify;
MM2 // MatrixForm
```

Out[41]//MatrixForm=

$$\begin{pmatrix} m_1^2 + \frac{1}{8}\left(g_1^2 + g_2^2\right)\left(v_1^2 - v_2^2\right) & -b \\ -b & m_2^2 - \frac{1}{8}\left(g_1^2 + g_2^2\right)\left(v_1^2 - v_2^2\right) \end{pmatrix}$$

In[42]:=
```
Eigenvalues[MM2] // FullSimplify
```

Out[42]=

$$\left\{\frac{1}{8}\left(4 m_1^2 + 4 m_2^2 - \sqrt{64 b^2 + \left(4 m_1^2 - 4 m_2^2 + \left(g_1^2 + g_2^2\right)\left(v_1^2 - v_2^2\right)\right)^2}\right),\right.$$
$$\left.\frac{1}{8}\left(4 m_1^2 + 4 m_2^2 + \sqrt{64 b^2 + \left(4 m_1^2 - 4 m_2^2 + \left(g_1^2 + g_2^2\right)\left(v_1^2 - v_2^2\right)\right)^2}\right)\right\}$$

---

f)

In[43]:=
```
MM3 = {{D[V2, h₁, h₁], D[V2, h₁, h₂]}, {D[V2, h₂, h₁], D[V2, h₂, h₂]}} /.
    {α → 0, γ → 0, h₁ → 0, h₂ → 0, A₁ → 0, A₂ → 0} // FullSimplify;
MM3 // MatrixForm
```

Out[44]//MatrixForm=

$$\begin{pmatrix} m_1^2 + \frac{1}{8}\left(g_1^2 + g_2^2\right)\left(3 v_1^2 - v_2^2\right) & -b - \frac{1}{4}\left(g_1^2 + g_2^2\right) v_1 v_2 \\ -b - \frac{1}{4}\left(g_1^2 + g_2^2\right) v_1 v_2 & m_2^2 - \frac{1}{8}\left(g_1^2 + g_2^2\right)\left(v_1^2 - 3 v_2^2\right) \end{pmatrix}$$

In[45]:=
```
Eigenvalues[MM3] // FullSimplify
```

Out[45]=

$$\left\{\frac{1}{8}\left(4 m_1^2 + 4 m_2^2 + \left(g_1^2 + g_2^2\right)\left(v_1^2 + v_2^2\right) - 2\sqrt{\left(16 b^2 + \left(2 m_1^2 - 2 m_2^2 + \left(g_1^2 + g_2^2\right) v_1^2\right)^2 + 8 b\left(g_1^2 + g_2^2\right) v_1 v_2 - \left(g_1^2 + g_2^2\right)\left(4 m_1^2 - 4 m_2^2 + \left(g_1^2 + g_2^2\right) v_1^2\right) v_2^2 + \left(g_1^2 + g_2^2\right)^2 v_2^4\right)}\right),\right.$$
$$\left.\frac{1}{8}\left(4 m_1^2 + 4 m_2^2 + \left(g_1^2 + g_2^2\right)\left(v_1^2 + v_2^2\right) + 2\sqrt{\left(16 b^2 + \left(2 m_1^2 - 2 m_2^2 + \left(g_1^2 + g_2^2\right) v_1^2\right)^2 + 8 b\left(g_1^2 + g_2^2\right) v_1 v_2 - \left(g_1^2 + g_2^2\right)\left(4 m_1^2 - 4 m_2^2 + \left(g_1^2 + g_2^2\right) v_1^2\right) v_2^2 + \left(g_1^2 + g_2^2\right)^2 v_2^4\right)}\right)\right\}$$