In[44]:= **Definitions and Assumptions;**

In[45]:=
```
$Assumptions = {M > 0, r > 0, θ ∈ ℝ, t ∈ ℝ, ϕ ∈ ℝ, τ ∈ ℝ};
n = 4;
coord = {t, r, θ, ϕ};
vel = {Dt[t, τ], Dt[r, τ], Dt[θ, τ], Dt[ϕ, τ]};
```

In[49]:=
```
affine :=
  affine = FullSimplify[Table[1/2 * Sum[(inversemetric[[i, s]]) * (D[metric[[s, j]], coord[[k]]] +
          D[metric[[s, k]], coord[[j]]] - D[metric[[j, k]], coord[[s]]]),
       {s, 1, n}], {i, 1, n}, {j , 1, n}, {k, 1, n}]];
```

In[50]:=
```
geodesic := geodesic = FullSimplify[
    Table[-Sum[affine[[i, j, k]] × vel[[j]] × vel[[k]], {j, 1, n}, {k, 1, n}], {i, 1, n}]];
```

In[51]:=
```
listaffine :=
  Table[If[UnsameQ[affine[[i, j, k]], 0], {ToString[Γ[coord[[i]], coord[[j]], coord[[k]]]],
     affine[[i, j, k]]}], {i, 1, n}, {j, 1, n}, {k, 1, j}];
```

In[52]:=
```
listgeodesic := Table[{Dt[vel[[i]], τ], "=", geodesic[[i]]}, {i, 1, n}];
```

In[53]:=

Problem 1;

In[57]:=
```
metric = DiagonalMatrix[{-(1 - 2 M/Sqrt[r^2 + b^2]), (1 - 2 M/Sqrt[r^2 + b^2])^-1, r^2 + b^2, (r^2 + b^2) Sin[θ]^2}];
inversemetric = Simplify[Inverse[metric]];
```

In[59]:=
```
metric // MatrixForm
inversemetric // MatrixForm
```

Out[59]//MatrixForm=

$$\begin{pmatrix} -1+\frac{2M}{\sqrt{b^2+r^2}} & 0 & 0 & 0 \\ 0 & \frac{1}{1-\frac{2M}{\sqrt{b^2+r^2}}} & 0 & 0 \\ 0 & 0 & b^2+r^2 & 0 \\ 0 & 0 & 0 & \left(b^2+r^2\right)\mathrm{Sin}[\theta]^2 \end{pmatrix}$$

Out[60]//MatrixForm=

$$\begin{pmatrix} \frac{\sqrt{b^2+r^2}}{2M-\sqrt{b^2+r^2}} & 0 & 0 & 0 \\ 0 & 1-\frac{2M}{\sqrt{b^2+r^2}} & 0 & 0 \\ 0 & 0 & \frac{1}{b^2+r^2} & 0 \\ 0 & 0 & 0 & \frac{\mathrm{Csc}[\theta]^2}{b^2+r^2} \end{pmatrix}$$

In[61]:=

Problem 2;

In[65]:=
```
TableForm[Partition[DeleteCases[Flatten[listaffine], Null], 2], TableSpacing → {2, 2}]
```

Out[65]//TableForm=

Γ[t, r, t]	$\frac{M\,r}{(b^2+r^2)\left(-2M+\sqrt{b^2+r^2}\right)}$
Γ[r, t, t]	$\frac{M\,r\left(-2M+\sqrt{b^2+r^2}\right)}{(b^2+r^2)^2}$
Γ[r, r, r]	$-\frac{M\,r}{(b^2+r^2)\left(-2M+\sqrt{b^2+r^2}\right)}$
Γ[r, θ, θ]	$r\left(-1+\frac{2M}{\sqrt{b^2+r^2}}\right)$
Γ[r, ϕ, ϕ]	$r\left(-1+\frac{2M}{\sqrt{b^2+r^2}}\right)\text{Sin}[\theta]^2$
Γ[θ, θ, r]	$\frac{r}{b^2+r^2}$
Γ[θ, ϕ, ϕ]	$-\text{Cos}[\theta]\,\text{Sin}[\theta]$
Γ[ϕ, ϕ, r]	$\frac{r}{b^2+r^2}$
Γ[ϕ, ϕ, θ]	$\text{Cot}[\theta]$

In[66]:=
```
TableForm[Partition[DeleteCases[Flatten[listaffine], Null], 2],
   TableSpacing → {2, 2}] /. {b → 0} // FullSimplify
```

Out[66]//TableForm=

Γ[t, r, t]	$\frac{M}{r(-2M+r)}$
Γ[r, t, t]	$\frac{M(-2M+r)}{r^3}$
Γ[r, r, r]	$\frac{M}{2Mr-r^2}$
Γ[r, θ, θ]	$2M-r$
Γ[r, ϕ, ϕ]	$(2M-r)\,\text{Sin}[\theta]^2$
Γ[θ, θ, r]	$\frac{1}{r}$
Γ[θ, ϕ, ϕ]	$-\text{Cos}[\theta]\,\text{Sin}[\theta]$
Γ[ϕ, ϕ, r]	$\frac{1}{r}$
Γ[ϕ, ϕ, θ]	$\text{Cot}[\theta]$

In[67]:=

Problem 3;

In[70]:=
```
TableForm[listgeodesic, TableSpacing → {2, 2}] // FullSimplify // TraditionalForm
```

Out[70]//TraditionalForm=

$$\frac{d^2t}{d\tau^2} = -\frac{2 M r \frac{dr}{d\tau} \frac{dt}{d\tau}}{(b^2+r^2)\left(\sqrt{b^2+r^2}-2M\right)}$$

$$\frac{d^2r}{d\tau^2} = r\left(\frac{\left(\sqrt{b^2+r^2}-2M\right)\left((b^2+r^2)^{3/2}\left(\sin^2(\theta)\left(\frac{d\phi}{d\tau}\right)^2+\left(\frac{d\theta}{d\tau}\right)^2\right)-M\left(\frac{dt}{d\tau}\right)^2\right)}{(b^2+r^2)^2} + \frac{M\left(\frac{dr}{d\tau}\right)^2}{(b^2+r^2)\left(\sqrt{b^2+r^2}-2M\right)}\right)$$

$$\frac{d^2\theta}{d\tau^2} = \sin(\theta)\cos(\theta)\left(\frac{d\phi}{d\tau}\right)^2 - \frac{2r\frac{d\theta}{d\tau}\frac{dr}{d\tau}}{b^2+r^2}$$

$$\frac{d^2\phi}{d\tau^2} = 2\frac{d\phi}{d\tau}\left(-\frac{r\frac{dr}{d\tau}}{b^2+r^2} - \cot(\theta)\frac{d\theta}{d\tau}\right)$$

In[71]:=

Problem 4;

In[74]:=
```
e = -(metric.vel)[[1]];
l = -(metric.vel)[[4]];
```

In[76]:=
```
e // FullSimplify
l // FullSimplify
```

Out[76]=

$$\left(1-\frac{2\,M}{\sqrt{b^2+r^2}}\right) \mathrm{Dt}[t, \tau]$$

Out[77]=

$$-\left(\left(b^2+r^2\right) \mathrm{Dt}[\phi, \tau]\, \mathrm{Sin}[\theta]^2\right)$$

In[78]:=
```
velᵀ.metric.vel // FullSimplify // TraditionalForm
```

Out[78]//TraditionalForm=

$$\left(\frac{2M}{\sqrt{b^2+r^2}}-1\right)\left(\frac{dt}{d\tau}\right)^2 + \frac{\left(\frac{dr}{d\tau}\right)^2}{1-\frac{2M}{\sqrt{b^2+r^2}}} + \left(b^2+r^2\right)\left(\sin^2(\theta)\left(\frac{d\phi}{d\tau}\right)^2+\left(\frac{d\theta}{d\tau}\right)^2\right)$$

Problem 7;

In[87]:=

$$\text{effectivePotential} = \frac{1}{2}\left(\left(1-\frac{2\,M}{\sqrt{r^2+b^2}}\right)\left(1+\frac{L^2}{r^2+b^2}\right)-1\right);$$

In[88]:=
```
conditions = {M → 1, b → 1, L → 5};
```

In[89]:=
```
extrema = FullSimplify[
   Solve[D[effectivePotential /. conditions, r] == 0, r], Assumptions → {L > 0}] // N
```

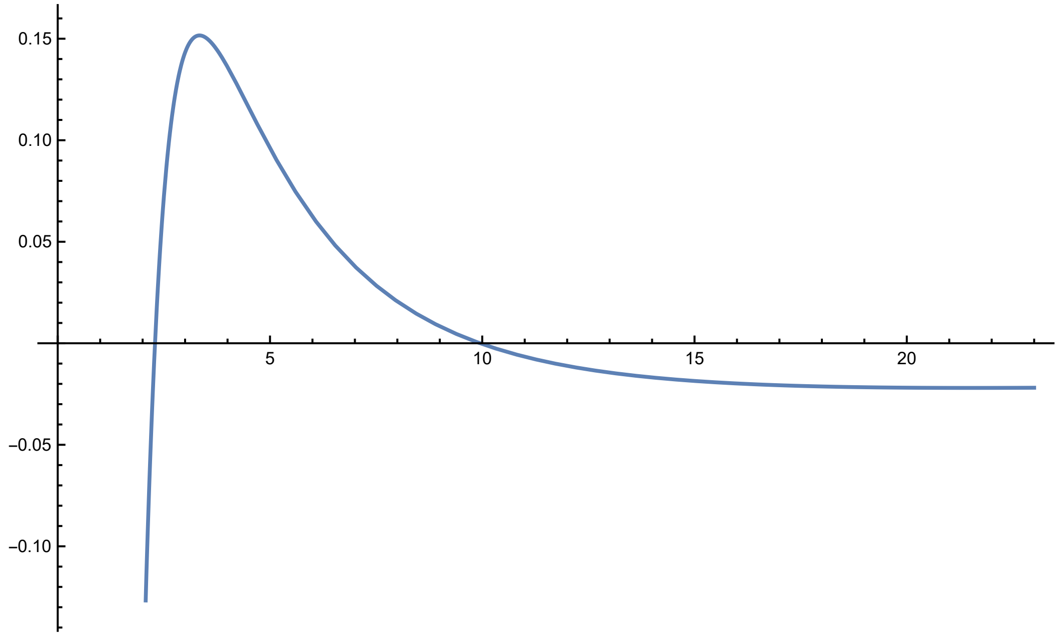
Out[89]=
```
{{r → 3.33962}, {r → 21.4906}}
```

In[90]:=
```
D[D[effectivePotential, r], r] /. extrema /. conditions // N
```

Out[90]=
```
{-0.112016, 0.0000839707}
```

In[91]:=
```
Plot[effectivePotential /. conditions, {r, 0, 23}]
```

Out[91]=



Problem 8;

In[94]:=
```
conditions = {M → 1, b → 3, L → 2};
```

In[95]:=
```
Solve[(effectivePotential /. conditions) == -0.05, r]
```

Solve: Solve was unable to solve the system with inexact coefficients. The answer was obtained by solving a corresponding exact system and numericizing the result.

Out[95]=
```
{{r → 17.7759}}
```

In[96]:=
```
Plot[(effectivePotential /. conditions), {r, 0, 18}]
```

Out[96]=