All About Spinors

...on *flat* spacetime

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UO

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Some Philisophical Motivation

Special and General Covariance

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- Bargmann ('54): reps up to sign are *exactly* the true reps of the universal cover

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- In fact, $\widehat{ISO}(3,1)^+ \cong ISL(2,\mathbb{C})$

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Spinors and Null Vectors

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Representations and Hilbert Space

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Summary

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