Homework 1

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Problem 1

Let

$$E_s = \hbar\omega_x, \quad t_s = \frac{1}{\omega_x}, \quad x_s = \sqrt{\frac{\hbar}{m\omega_x}}, \quad p_s = \sqrt{m\hbar\omega_x}.$$

$$\tilde{H} = \frac{H}{E_s}, \quad \tilde{x} = \frac{x}{x_s}, \quad \tilde{y} = \frac{y}{x_s} \quad \tilde{p}_{x(y)} = \frac{p_{x(y)}}{p_s}, \quad \tilde{\omega}_y = \frac{\omega_y}{\omega_x}$$

The Hamiltonian,

$$H = \frac{1}{2m}\left(p_x^2 + p_y^2\right) + \frac{m}{2}\left(\omega_x^2 x^2 + \omega_y^2 y^2\right),$$

can then be rewritten as

$$\tilde{H}E_s = \frac{1}{2m}\left((\tilde{p}_x p_s)^2 + (\tilde{p}_y p_s)^2\right) + \frac{m}{2}\left(\omega_x^2(\tilde{x}x_s)^2 + (\tilde{\omega}_y\omega_x)^2(\tilde{y}x_s)^2\right).$$

Dropping tildes and substituting in values, this reduces to

$$H = \frac{1}{2}\left(p_x^2 + p_y^2 + x^2 + \omega_y^2 y^2\right)$$

Problem 2

In scaled variables,

$$\begin{aligned}
\partial_t \langle x \rangle &= -i \langle [x, H] \rangle \\
&= -\frac{i}{2} \left\langle \left[x, p^2 \right] \right\rangle \\
&= \langle p \rangle
\end{aligned}$$

$$\begin{aligned}
\partial_t \langle p \rangle &= -i \langle [p, H] \rangle \\
&= -\frac{i}{2} \left\langle \left[p, x^2 \right] \right\rangle \\
&= - \langle x \rangle
\end{aligned}$$

$$\begin{aligned}
\partial_t V_x &= \partial_t \left(\left\langle x^2 \right\rangle - \langle x \rangle^2 \right) \\
&= -i \left\langle \left[x^2, H \right] \right\rangle - 2 \langle x \rangle \partial_t \langle x \rangle \\
&= -\frac{i}{2} \left\langle \left[x^2, p^2 \right] \right\rangle - 2 \langle x \rangle \langle p \rangle \\
&= 2 \left\langle [x, p]_+ \right\rangle - 2 \langle x \rangle \langle p \rangle \\
&= 2 C_{xp}
\end{aligned}$$

$$\begin{aligned}
\partial_t V_p &= \partial_t \left(\left\langle p^2 \right\rangle - \langle p \rangle^2 \right) \\
&= -i \left\langle \left[p^2, H \right] \right\rangle - 2 \langle p \rangle \partial_t \langle p \rangle \\
&= -\frac{i}{2} \left\langle \left[p^2, x^2 \right] \right\rangle + 2 \langle x \rangle \langle p \rangle \\
&= -2 \left\langle [x, p]_+ \right\rangle + 2 \langle x \rangle \langle p \rangle \\
&= -2 C_{xp}
\end{aligned}$$

$$\begin{aligned}
\partial_t C_{xp} &= \partial_t \left(\left\langle [x, p]_+ \right\rangle - \langle x \rangle \langle p \rangle \right) \\
&= -i \langle [xp + px, H] \rangle - \langle p \rangle^2 + \langle x \rangle^2 \\
&= -\frac{i}{2} \left\langle \left[xp, x^2 \right] + \left[xp, p^2 \right] + \left[px, x^2 \right] + \left[px, p^2 \right] \right\rangle - \langle p \rangle^2 + \langle x \rangle^2 \\
&= \left\langle p^2 \right\rangle - \left\langle x^2 \right\rangle - \langle p \rangle^2 + \langle x \rangle^2 \\
&= V_p - V_x
\end{aligned}$$

Restoring units,

$$\begin{aligned}
\partial_t \langle x \rangle &= \frac{1}{m} \langle p \rangle \\
\partial_t \langle p \rangle &= -m\omega^2 \langle x \rangle \\
\partial_t V_x &= \frac{2}{m} C_{xp} \\
\partial_t V_P &= -m\omega^2 C_{xp} \\
\partial_t C_{xp} &= \frac{1}{m} V_p - m\omega^2 V_x
\end{aligned}$$

Problem 3

From the result of Homework 3 Problem 3 from last term,

$$V_x(t) = \sigma^2(t) = \sigma^2 + \frac{\hbar^2 t^2}{4m^2\sigma^2}$$

$$\implies \dot{V}_x(t) = \frac{\hbar^2 t}{2m^2\sigma^2}$$

Using the equations from problem 2 above (with $\omega = 0$) we see that

$$\partial_t V_p = 0 \implies V_p = c$$

for some constant c. This implies that the time dependence of C_{xp} has the simple form

$$C_{xp}(t) = \frac{c}{m} t + k$$

for another constant k, which in turn implies

$$V_x(t) = V_x(0) + \frac{c}{m^2} t$$

The substituting in the initial conditions gives

$$V_x(t) = \sigma^2 + \frac{\hbar^2 t^2}{4m^2\sigma^2}$$

Problem 4

Let

$$U_g = V_x V_p - C_{xp}^2$$

Then

$$\begin{aligned}
\partial_t U_g &= \dot{V}_x V_p + V_x \dot{V}_p - 2C_{xp}\dot{C}_{xp} \\
&= 2C_{xp}V_p - 2C_{xp}V_x - 2C_{xp}(V_p - V_x) \\
&= 2C_{xp}(V_p - V_x - V_p + V_x) \\
&= 0
\end{aligned}$$