Notes

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Bell Superposition

$$
\begin{aligned}
|B_1\rangle &= \left|\Phi^+\right\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \\
|B_2\rangle &= \left|\Phi^-\right\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right) \\
|B_3\rangle &= \left|\Psi^+\right\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right) \\
|B_4\rangle &= \left|\Psi^-\right\rangle = \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right)
\end{aligned}
$$

$$
\mathcal{B} = \begin{pmatrix} B_1 & B_2 & B_3 & B_4 \end{pmatrix}
$$

$$
|\psi(p; B_i, B_j)\rangle \coloneqq \sqrt{1-p}\,|B_i\rangle_{TA}\,|0\rangle_B + \sqrt{p}\,|B_j\rangle_{TA}\,|1\rangle_B
$$

$$
\begin{aligned}
|\psi(p; B_1, B_4)\rangle &= \sqrt{1-p}\left|\Phi^+\right\rangle|0\rangle + \sqrt{p}\left|\Psi^-\right\rangle|1\rangle \\
&= \sqrt{\frac{1-p}{2}}\left(|000\rangle + |110\rangle\right) + \sqrt{\frac{p}{2}}\left(|011\rangle - |101\rangle\right) \\
&= \begin{pmatrix} \sqrt{\frac{1-p}{2}} \\ 0 \\ 0 \\ \sqrt{\frac{p}{2}} \\ 0 \\ -\sqrt{\frac{p}{2}} \\ \sqrt{\frac{1-p}{2}} \\ 0 \end{pmatrix}
\end{aligned}
$$

$$\rho^{1,4} = \begin{pmatrix} \frac{1-p}{2} & 0 & 0 & \frac{1}{2}\sqrt{(1-p)p} & 0 & -\frac{1}{2}\sqrt{(1-p)p} & \frac{1-p}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{2}\sqrt{(1-p)p} & 0 & 0 & \frac{p}{2} & 0 & -\frac{p}{2} & \frac{1}{2}\sqrt{(1-p)p} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{2}\sqrt{(1-p)p} & 0 & 0 & -\frac{p}{2} & 0 & \frac{p}{2} & -\frac{1}{2}\sqrt{(1-p)p} & 0 \\ \frac{1-p}{2} & 0 & 0 & \frac{1}{2}\sqrt{(1-p)p} & 0 & -\frac{1}{2}\sqrt{(1-p)p} & \frac{1-p}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\rho^{1,4}_{TA} = \begin{pmatrix} \frac{1-p}{2} & 0 & 0 & \frac{1-p}{2} \\ 0 & \frac{p}{2} & -\frac{p}{2} & 0 \\ 0 & -\frac{p}{2} & \frac{p}{2} & 0 \\ \frac{1-p}{2} & 0 & 0 & \frac{1-p}{2} \end{pmatrix}$$

$$\rho^{1,4}_{TB} = \begin{pmatrix} \frac{1-p}{2} & 0 & 0 & -\frac{1}{2}\sqrt{(1-p)p} \\ 0 & \frac{p}{2} & \frac{1}{2}\sqrt{(1-p)p} & 0 \\ 0 & \frac{1}{2}\sqrt{(1-p)p} & \frac{1-p}{2} & 0 \\ -\frac{1}{2}\sqrt{(1-p)p} & 0 & 0 & \frac{p}{2} \end{pmatrix}$$

$$\rho^{1,4}_{AB} = \begin{pmatrix} \frac{1-p}{2} & 0 & 0 & \frac{1}{2}\sqrt{(1-p)p} \\ 0 & \frac{p}{2} & -\frac{1}{2}\sqrt{(1-p)p} & 0 \\ 0 & -\frac{1}{2}\sqrt{(1-p)p} & \frac{1-p}{2} & 0 \\ \frac{1}{2}\sqrt{(1-p)p} & 0 & 0 & \frac{p}{2} \end{pmatrix}$$

$$\rho^{1,4}_{T} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$\rho^{1,4}_{A} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

$$\rho^{1,4}_{B} = \begin{pmatrix} 1-p & 0 \\ 0 & p \end{pmatrix}$$

$$
\rho^{1,4}_{T|A} = \begin{pmatrix}
1-p & 0 & 0 & 0 & 0 & 0 & 1-p & 0 \\
0 & 1-p & 0 & 0 & 0 & 0 & 0 & 1-p \\
0 & 0 & p & 0 & -p & 0 & 0 & 0 \\
0 & 0 & 0 & p & 0 & -p & 0 & 0 \\
0 & 0 & -p & 0 & p & 0 & 0 & 0 \\
0 & 0 & 0 & -p & 0 & p & 0 & 0 \\
1-p & 0 & 0 & 0 & 0 & 0 & 1-p & 0 \\
0 & 1-p & 0 & 0 & 0 & 0 & 0 & 1-p
\end{pmatrix}
$$

$$
= \begin{pmatrix}
\begin{pmatrix} 1-p & 0 \\ 0 & 1-p \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1-p & 0 \\ 0 & 1-p \end{pmatrix} \\
\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} & \begin{pmatrix} -p & 0 \\ 0 & -p \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} -p & 0 \\ 0 & -p \end{pmatrix} & \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\
\begin{pmatrix} 1-p & 0 \\ 0 & 1-p \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1-p & 0 \\ 0 & 1-p \end{pmatrix}
\end{pmatrix}
$$

$$
= \begin{pmatrix}
1-p & 0 & 0 & 1-p \\
0 & p & -p & 0 \\
0 & -p & p & 0 \\
1-p & 0 & 0 & 1-p
\end{pmatrix}_{TA} \otimes \mathbb{I}_B
$$

$$
\rho^{1,4}_{T|B} = \begin{pmatrix}
1-p & 0 & 0 & 0 & 0 & -\sqrt{(1-p)p} & 0 & 0 \\
0 & p & 0 & 0 & \sqrt{(1-p)p} & 0 & 0 & 0 \\
0 & 0 & 1-p & 0 & 0 & 0 & 0 & -\sqrt{(1-p)p} \\
0 & 0 & 0 & p & 0 & 0 & \sqrt{(1-p)p} & 0 \\
0 & \sqrt{(1-p)p} & 0 & 0 & 1-p & 0 & 0 & 0 \\
-\sqrt{(1-p)p} & 0 & 0 & 0 & 0 & p & 0 & 0 \\
0 & 0 & 0 & \sqrt{(1-p)p} & 0 & 0 & 1-p & 0 \\
0 & 0 & -\sqrt{(1-p)p} & 0 & 0 & 0 & 0 & p
\end{pmatrix}
$$

$$
= \pi_{132} \begin{pmatrix}
1-p & 0 & 0 & 0 & 0 & 0 & -\sqrt{(1-p)p} & 0 \\
0 & 1-p & 0 & 0 & 0 & 0 & 0 & -\sqrt{(1-p)p} \\
0 & 0 & p & 0 & \sqrt{-((p-1)p)} & 0 & 0 & 0 \\
0 & 0 & 0 & p & 0 & \sqrt{-((p-1)p)} & 0 & 0 \\
0 & 0 & \sqrt{-((p-1)p)} & 0 & 1-p & 0 & 0 & 0 \\
0 & 0 & 0 & \sqrt{-((p-1)p)} & 0 & 1-p & 0 & 0 \\
-\sqrt{(1-p)p} & 0 & 0 & 0 & 0 & 0 & p & 0 \\
0 & -\sqrt{(1-p)p} & 0 & 0 & 0 & 0 & 0 & p
\end{pmatrix} \pi^{\dagger}_{132}
$$

$$
= \pi_{132} \left[\begin{pmatrix}
1-p & 0 & 0 & -\sqrt{(1-p)p} \\
0 & p & \sqrt{-((p-1)p)} & 0 \\
0 & \sqrt{-((p-1)p)} & 1-p & 0 \\
-\sqrt{(1-p)p} & 0 & 0 & p
\end{pmatrix}_{TB} \otimes \mathbb{I}_A \right] \pi^{\dagger}_{132}
$$

$\rho_{TB}^{1,2}$

$\lambda = 0$

$$\sqrt{p}\,|00\rangle - \sqrt{1-p}\,|01\rangle$$
$$\sqrt{p}\,|10\rangle - \sqrt{1-p}\,|11\rangle$$

$\lambda = 1/2$

$$\sqrt{1-p}\,|00\rangle + \sqrt{p}\,|01\rangle$$
$$\sqrt{1-p}\,|10\rangle - \sqrt{p}\,|11\rangle$$

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{p} & \sqrt{1-p} & \sqrt{1-p} & \sqrt{p} \\ -\sqrt{1-p} & \sqrt{p} & \sqrt{p} & -\sqrt{1-p} \\ \sqrt{p} & \sqrt{1-p} & -\sqrt{1-p} & -\sqrt{p} \\ \sqrt{1-p} & -\sqrt{p} & \sqrt{p} & -\sqrt{1-p} \end{pmatrix}$$

$\rho_{TB}^{2,1}$

$\lambda = 0$

$$\sqrt{p}\,|00\rangle - \sqrt{1-p}\,|01\rangle$$
$$\sqrt{p}\,|10\rangle - \sqrt{1-p}\,|11\rangle$$

$\lambda = 1/2$

$$\sqrt{1-p}\,|00\rangle + \sqrt{p}\,|01\rangle$$
$$\sqrt{1-p}\,|10\rangle - \sqrt{p}\,|11\rangle$$

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{p} & \sqrt{1-p} & \sqrt{1-p} & -\sqrt{p} \\ -\sqrt{1-p} & \sqrt{p} & \sqrt{p} & \sqrt{1-p} \\ \sqrt{p} & \sqrt{1-p} & -\sqrt{1-p} & \sqrt{p} \\ \sqrt{1-p} & -\sqrt{p} & \sqrt{p} & \sqrt{1-p} \end{pmatrix}$$

$\rho_{TB}^{1,3}$

$\lambda = 0$

$$\sqrt{p}\,|00\rangle - \sqrt{1-p}\,|01\rangle$$
$$\sqrt{1-p}\,|10\rangle - \sqrt{p}\,|11\rangle$$

$\lambda = 1/2$

$$\sqrt{1-p}\,|00\rangle + \sqrt{p}\,|01\rangle$$
$$\sqrt{p}\,|10\rangle + \sqrt{1-p}\,|11\rangle$$

$\rho_{TB}^{3,1}$

$\lambda = 0$

$$\sqrt{p}\,|00\rangle - \sqrt{1-p}\,|01\rangle$$
$$\sqrt{1-p}\,|10\rangle - \sqrt{p}\,|11\rangle$$

$\lambda = 1/2$

$$\sqrt{1-p}\,|00\rangle + \sqrt{p}\,|01\rangle$$
$$\sqrt{p}\,|10\rangle + \sqrt{1-p}\,|11\rangle$$

$\rho_{TB}^{1,4}$

$\lambda = 0$

$$\sqrt{p}\,|00\rangle + \sqrt{1-p}\,|11\rangle$$
$$\sqrt{1-p}\,|01\rangle - \sqrt{p}\,|10\rangle$$

$\lambda = 1/2$

$$\sqrt{1-p}\,|00\rangle - \sqrt{p}\,|11\rangle$$
$$\sqrt{p}\,|01\rangle + \sqrt{1-p}\,|10\rangle$$

$\rho_{TB}^{4,1}$

$\lambda = 0$

$$\sqrt{p}\,|00\rangle - \sqrt{1-p}\,|11\rangle$$
$$\sqrt{1-p}\,|01\rangle + \sqrt{p}\,|10\rangle$$

$\lambda = 1/2$

$$\sqrt{1-p}\,|00\rangle + \sqrt{p}\,|11\rangle$$
$$\sqrt{p}\,|01\rangle - \sqrt{1-p}\,|10\rangle$$

$\rho_{TB}^{2,3}$

$\lambda = 0$

$$\sqrt{p}\,|00\rangle - \sqrt{1-p}\,|11\rangle$$
$$\sqrt{1-p}\,|01\rangle + \sqrt{p}\,|10\rangle$$

$\lambda = 1/2$

$\rho_{TB}^{2,4}$

$\lambda = 0$

$$\sqrt{p}\,|00\rangle + \sqrt{1-p}\,|11\rangle$$
$$\sqrt{1-p}\,|01\rangle + \sqrt{p}\,|10\rangle$$

$\rho_{TB}^{3,4}$

$\lambda = 0$

$$\sqrt{p}\,|00\rangle - \sqrt{1-p}\,|01\rangle$$
$$\sqrt{p}\,|10\rangle + \sqrt{1-p}\,|11\rangle$$

$\tilde{\rho}_{T|A}^{1,4}$ **Eigensystem**

CONJECTURE $\{i, j, k, l\} = \{1, 2, 3, 4\}$

$$\tilde{\rho}_{T|A}^{i,j} = 2(1-p)\,|B_i\rangle_{TA}\,\langle B_i|_{TA} + 2p\,|B_j\rangle_{TA}\,\langle B_j|_{TA}$$

$$\tilde{\rho}_{T|B}^{i,j} = |B'_k\rangle_{TB}\,\langle B'_k|_{TB} + |B'_l\rangle_{TB}\,\langle B'_l|_{TB}$$

$$\left|B'_{k(l)}\right\rangle := U\left|B_{k(l)}\right\rangle$$

$$U = \frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{1-p}+\sqrt{p} & 0 & 0 & \sqrt{p}-\sqrt{1-p} \\ 0 & \sqrt{1-p}+\sqrt{p} & \sqrt{p}-\sqrt{1-p} & 0 \\ 0 & \sqrt{1-p}-\sqrt{p} & \sqrt{1-p}+\sqrt{p} & 0 \\ \sqrt{1-p}-\sqrt{p} & 0 & 0 & \sqrt{1-p}+\sqrt{p} \end{pmatrix}$$

$\rho_{T|A}^{1,4}$ **Eigensystem**

$\lambda = 0$ subspace (kernel):

$$\frac{1}{\sqrt{2}}\left(|111\rangle - |010\rangle\right)$$
$$\frac{1}{\sqrt{2}}\left(|110\rangle - |000\rangle\right)$$
$$\frac{1}{\sqrt{2}}\left(|011\rangle - |101\rangle\right)$$
$$\frac{1}{\sqrt{2}}\left(|010\rangle - |100\rangle\right)$$

$\lambda = 2p$ subspace:

$$\frac{1}{\sqrt{2}}\left(|101\rangle - |011\rangle\right)$$
$$\frac{1}{\sqrt{2}}\left(|100\rangle - |010\rangle\right)$$

$\lambda = 2(1-p)$ subspace:

$$\frac{1}{\sqrt{2}}\left(|001\rangle + |111\rangle\right)$$
$$\frac{1}{\sqrt{2}}\left(|000\rangle + |110\rangle\right)$$

$\rho_{T|B}$ Eigensystem

$\lambda = 0$ subspace (kernel):

$$\sqrt{1-p}\,|111\rangle + \sqrt{p}\,|010\rangle$$
$$\sqrt{1-p}\,|011\rangle - \sqrt{p}\,|110\rangle$$
$$\sqrt{1-p}\,|101\rangle + \sqrt{p}\,|000\rangle$$
$$\sqrt{1-p}\,|001\rangle - \sqrt{p}\,|100\rangle$$

$\lambda = 1$ subspace:

$$\sqrt{1-p}\,|010\rangle - \sqrt{p}\,|111\rangle$$
$$\sqrt{1-p}\,|110\rangle + \sqrt{p}\,|011\rangle$$
$$\sqrt{1-p}\,|000\rangle - \sqrt{p}\,|101\rangle$$
$$\sqrt{1-p}\,|100\rangle + \sqrt{p}\,|001\rangle$$