Homework 1

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Problem 1

Let's begin with the purified state, $|\psi_{\text{CM}}\rangle$ (where "C" stands for "contestant" and "M" stands for "Monty")

$$|\psi_{\text{CM}}\rangle = \frac{1}{\sqrt{3}} \left(|1\rangle |1\rangle + |2\rangle |2\rangle + |3\rangle |3\rangle \right)$$

The contestant's state represents their state of belief about which door the car is behind. Monty's state represents his state of belief about which door he will open to reveal a goat. The contestant's selection of door 1 induces the transformation

$$\begin{aligned}
|\psi_{\text{CM}}\rangle \rightarrow & \frac{1}{\sqrt{3}} \left[|1\rangle \left(\frac{|2\rangle + |3\rangle}{\sqrt{2}} \right) + |2\rangle |3\rangle + |3\rangle |2\rangle \right] \\
&= \frac{1}{\sqrt{6}} |1\rangle |2\rangle + \frac{1}{\sqrt{6}} |1\rangle |3\rangle + \frac{1}{\sqrt{3}} |2\rangle |3\rangle + \frac{1}{\sqrt{3}} |3\rangle |2\rangle \\
&= \left(\frac{1}{\sqrt{6}} |1\rangle + \frac{1}{\sqrt{3}} |2\rangle \right) |2\rangle + \left(\frac{1}{\sqrt{6}} |1\rangle + \frac{1}{\sqrt{3}} |2\rangle \right) |3\rangle
\end{aligned}$$

Now, a projective measurment on Monty's state (i.e., Monty choosing to open door 2 or 3) will collapse the state one of

$$\left(\sqrt{\frac{1}{3}} |1\rangle + \sqrt{\frac{2}{3}} |2\rangle \right) |2\rangle$$

$$\left(\sqrt{\frac{1}{3}} |1\rangle + \sqrt{\frac{2}{3}} |2\rangle \right) |3\rangle$$

with equal probability. In either case, the contestant's part of the state becomes

$$|\psi_{\text{C}}\rangle = \sqrt{\frac{1}{3}} |1\rangle + \sqrt{\frac{2}{3}} |2\rangle$$

indicating that with probability 2/3 the car is behind door 2 and the contestant should switch.

Problem 2

$$\begin{aligned} x &\approx x_0 + \lambda x_1 + \lambda^2 x_2 + \cdots \\ &= 12.002383785691716 \end{aligned}$$

(a)

$$\begin{aligned} x &= (x_0 + \lambda x_1)^3 \\ &= x_0^3 + 3\lambda x_0^2 x_1 + 3\lambda^2 x_0 x_1^2 + \lambda^3 x_1^3 \\ &= x_0^3 + \lambda \end{aligned}$$

Matching terms proportional to λ gives

$$\lambda = 3\lambda x_0^2 x_1 \implies 3x_0^2 x_1 = 1 \implies x_1 = \frac{1}{3x_0^2}$$

The current approximation for x is therefore

$$x \approx x_0 + \lambda x_1 = 12 + \frac{1.03}{3 \times 12^2} = 12.002384\overline{259}$$

(b) Keeping terms only proportional to λ^2 we have

$$\begin{aligned} (x_0 + \lambda x_1 + \lambda^2 x_2)^3 &\to 3x_0^2 x_2 \lambda^2 + 3x_0 x_1^2 \lambda^2 \\ &= (3x_0^2 x_2 + 3x_0 x_1^2)\lambda^2 \end{aligned}$$

matching terms proportional to λ^2 gives

$$3x_0^2 x_2 + 3x_0 x_1^2 = 0 \implies x_2 = -\frac{x_0 x_1^2}{x_0^2} = -\frac{x_1^2}{x_0} = -\frac{1}{9x_0^5}$$

The current approximation for x is therefore

$$x \approx x_0 + \lambda x_1 + \lambda^2 x_2 = 12 + \frac{1.03}{3 \times 12^2} - \frac{1.03^2}{9 \times 12^5} = 12.00238473298$$

Problem 3

Δ_4 is a sum over all sets of 5 nonnegative indicies that add to 3:

$$\begin{aligned} \Delta_4 =& (01110\cdot) + (10011\cdot) + (10101\cdot) + (11001\cdot) + \\ & (00120\cdot) + (00210\cdot) + (01020\cdot) + (01200\cdot) + (02100\cdot) + (02010\cdot) + (10002\cdot) + (20001\cdot) \\ & (00030\cdot) + (00300\cdot) + (03000\cdot) \end{aligned}$$

Where only terms in which the first and last indicies do not cancel under trace have been kept. Some further simplification is given by

$$(10002\cdot) \to (3000)$$

$$(2001\cdot) \to (3000)$$
$$(00030\cdot), (03000\cdot), (00300\cdot) \to -(3000)$$
$$(00120\cdot) \to -(0011\cdot 1) \to -(10011\cdot)$$
$$(00210\cdot) \to -(001\cdot 11) \to -(11001\cdot)$$
$$(01020\cdot) \to -(0101\cdot 1) \to -(10101\cdot)$$
$$(01200\cdot) \to -(011\cdot 10) \to -(10011\cdot)$$
$$(02100\cdot) \to -(01\cdot 110) \to -(11001\cdot)$$
$$(02010\cdot) \to -(01\cdot 101) \to -(10101\cdot)$$

Giving

$$\begin{aligned}\Delta_4 &= (01110\cdot) - (10011\cdot) - (11001\cdot) - (10101\cdot) - (3000)\\ &= -(0111) - (2001) - (2100) - (2010) - (3000)\end{aligned}$$

Now let's handle the trace of each term individually

$$\begin{aligned}\mathrm{Tr}[-(0111)] &= \mathrm{Tr}\left[P_0 V \sum_{\alpha\neq 0}\frac{|\alpha\rangle\langle\alpha|}{E_{0\alpha}} V \sum_{\beta\neq 0}\frac{|\beta\rangle\langle\beta|}{E_{0\beta}} V \sum_{\gamma\neq 0}\frac{|\gamma\rangle\langle\gamma|}{E_{0\gamma}} V\right]\\ &= \mathrm{Tr}\left[P_0 V \sum_{\alpha\neq 0}\frac{|\alpha\rangle\langle\alpha|}{E_{0\alpha}} V \sum_{\beta\neq 0}\frac{|\beta\rangle\langle\beta|}{E_{0\beta}} V \sum_{\gamma\neq 0}\frac{|\gamma\rangle\langle\gamma|}{E_{0\gamma}} V P_0\right]\\ &= \mathrm{Tr}\left[P_0 \sum_{\alpha,\beta,\gamma\neq 0}\frac{V_{0\alpha}V_{\alpha\beta}V_{\beta\gamma}V_{\gamma 0}}{E_{0\alpha}E_{0\beta}E_{0\gamma}}\right]\\ &= \sum_{\alpha,\beta,\gamma\neq 0}\frac{V_{0\alpha}V_{\alpha\beta}V_{\beta\gamma}V_{\gamma 0}}{E_{0\alpha}E_{0\beta}E_{0\gamma}}\end{aligned}$$

$$\begin{aligned}-\mathrm{Tr}[(2010)] &= -\mathrm{Tr}\left[\sum_{\alpha\neq 0}\frac{|\alpha\rangle\langle\alpha|}{E_{0\alpha}^2} V P_0 V \sum_{\beta\neq 0}\frac{|\beta\rangle\langle\beta|}{E_{0\beta}} V P_0 V\right]\\ &= -\mathrm{Tr}\left[P_0 V \sum_{\alpha\neq 0}\frac{|\alpha\rangle\langle\alpha|}{E_{0\alpha}^2} V P_0 V \sum_{\beta\neq 0}\frac{|\beta\rangle\langle\beta|}{E_{0\beta}} V P_0\right]\\ &= -\mathrm{Tr}\left[P_0 \sum_{\alpha,\beta\neq 0}\frac{V_{0\alpha}V_{\alpha 0}V_{0\beta}V_{\beta 0}}{E_{0\alpha}^2 E_{0\beta}}\right]\\ &= -\sum_{\alpha,\beta\neq 0}\frac{\left|V_{0\alpha}\right|^2\left|V_{0\beta}\right|^2}{E_{0\alpha}^2 E_{0\beta}}\end{aligned}$$

$$
\begin{aligned}
-\operatorname{Tr}[(2001)] &= -\operatorname{Tr}\left[\sum_{\alpha\neq 0}\frac{|\alpha\rangle\langle\alpha|}{E_{0\alpha}^2}VP_0VP_0V\sum_{\beta\neq 0}\frac{|\beta\rangle\langle\beta|}{E_{0\beta}}V\right] \\
&= -\operatorname{Tr}\left[P_0V\sum_{\beta\neq 0}\frac{|\beta\rangle\langle\beta|}{E_{0\beta}}V\sum_{\alpha\neq 0}\frac{|\alpha\rangle\langle\alpha|}{E_{0\alpha}^2}VP_0VP_0\right] \\
&= -\operatorname{Tr}\left[P_0\sum_{\alpha,\beta\neq 0}\frac{V_{0\beta}V_{\beta\alpha}V_{\alpha 0}}{E_{0\alpha}^2E_{0\beta}}V_{00}\right] \\
&= -V_{00}\sum_{\alpha,\beta\neq 0}\frac{V_{0\beta}V_{\beta\alpha}V_{\alpha 0}}{E_{0\alpha}^2E_{0\beta}} \\
&= -V_{00}\sum_{\alpha,\beta\neq 0}\frac{V_{0\alpha}V_{\alpha\beta}V_{\beta 0}}{E_{0\alpha}E_{0\beta}^2}
\end{aligned}
$$

$$
\begin{aligned}
-\operatorname{Tr}[(2100)] &= -\operatorname{Tr}\left[\sum_{\alpha\neq 0}\frac{|\alpha\rangle\langle\alpha|}{E_{0\alpha}^2}V\sum_{\beta\neq 0}\frac{|\beta\rangle\langle\beta|}{E_{0\beta}}VP_0VP_0V\right] \\
&= -\operatorname{Tr}\left[P_0V\sum_{\alpha\neq 0}\frac{|\alpha\rangle\langle\alpha|}{E_{0\alpha}^2}V\sum_{\beta\neq 0}\frac{|\beta\rangle\langle\beta|}{E_{0\beta}}VP_0VP_0\right] \\
&= -\operatorname{Tr}\left[P_0\sum_{\alpha,\beta\neq 0}\frac{V_{0\alpha}V_{\alpha\beta}V_{\beta 0}}{E_{0\alpha}^2E_{0\beta}}V_{00}\right] \\
&= -V_{00}\sum_{\alpha,\beta\neq 0}\frac{V_{0\alpha}V_{\alpha\beta}V_{\beta 0}}{E_{0\alpha}^2E_{0\beta}}
\end{aligned}
$$

$$
\begin{aligned}
-\operatorname{Tr}[(3000)] &= \operatorname{Tr}\left[\sum_{\alpha\neq 0}\frac{|\alpha\rangle\langle\alpha|}{E_{0\alpha}^3}VP_0VP_0VP_0V\right] \\
&= \operatorname{Tr}\left[P_0V\sum_{\alpha\neq 0}\frac{|\alpha\rangle\langle\alpha|}{E_{0\alpha}^3}VP_0VP_0VP_0\right] \\
&= \operatorname{Tr}\left[P_0\sum_{\alpha\neq 0}\frac{V_{0\alpha}V_{\alpha 0}}{E_{0\alpha}^3}V_{00}V_{00}\right] \\
&= V_{00}^2\sum_{\alpha\neq 0}\frac{\left|V_{0\alpha}\right|^2}{E_{0\alpha}^3}
\end{aligned}
$$

Putting it all together we have

$$
\operatorname{Tr}[\Delta_4] = \sum_{\alpha,\beta,\gamma\neq 0}\frac{V_{0\alpha}V_{\alpha\beta}V_{\beta\gamma}V_{\gamma 0}}{E_{0\alpha}E_{0\beta}E_{0\gamma}} - \sum_{\alpha,\beta\neq 0}\frac{\left|V_{0\alpha}\right|^2\left|V_{0\beta}\right|^2}{E_{0\alpha}^2E_{0\beta}} - V_{00}\sum_{\alpha,\beta\neq 0}\left[\frac{V_{0\alpha}V_{\alpha\beta}V_{\beta 0}}{E_{0\alpha}E_{0\beta}^2} + \frac{V_{0\alpha}V_{\alpha\beta}V_{\beta 0}}{E_{0\alpha}^2E_{0\beta}}\right] + V_{00}^2\sum_{\alpha\neq 0}\frac{\left|V_{0\alpha}\right|^2}{E_{0\alpha}^3}
$$

In the case that $\mathrm{Tr}[\Delta_1] = \mathrm{Tr}[\Delta_2] = \mathrm{Tr}[\Delta_3] = 0$ this reduces to just the first term:

$$\mathrm{Tr}[\Delta_4] = \sum_{\alpha,\beta,\gamma\neq 0} \frac{V_{0\alpha}V_{\alpha\beta}V_{\beta\gamma}V_{\gamma 0}}{E_{0\alpha}E_{0\beta}E_{0\gamma}}$$

Problem 4

$$\begin{aligned}
[\vec{r}\cdot\vec{p}, H] &= \left[\vec{r}\cdot\vec{p}, \frac{p^2}{2m}\right] + [\vec{r}\cdot\vec{p}, V(r)] \\
&= \frac{1}{2m}\left[\vec{r}, p^2\right]\cdot\vec{p} + \vec{r}\cdot[\vec{p}, V(r)] \\
&= \frac{1}{2m}(2i\hbar\vec{p})\cdot\vec{p} - i\hbar\vec{r}\cdot\nabla V(r) \\
&= i\hbar\left(\frac{p^2}{m} - \vec{r}\cdot\nabla V(r)\right) \\
&= i\hbar\left(2T - \vec{r}\cdot\nabla V\right)
\end{aligned}$$

Now,

$$\langle[\vec{r}\cdot\vec{p}, H]\rangle = 0 \implies 2\langle T\rangle = \langle\vec{r}\cdot\nabla V\rangle$$

but

$$\langle\vec{r}\cdot\nabla V\rangle = \left\langle\vec{r}\cdot n\lambda r^{n-1}\hat{r}\right\rangle = \langle n\lambda r^n\rangle = n\langle V\rangle$$

so

$$2\langle T\rangle = n\langle V\rangle$$

Problem 5

(a) For the hydrogen atom we have $n = -1$, so

$$\begin{aligned}
\langle T\rangle = -\frac{1}{2}\langle v\rangle &\implies E_n = \langle T\rangle_n + \langle V\rangle_n = \frac{1}{2}\langle V\rangle_n \\
&\implies -\frac{\alpha^2\mu c^2}{2n^2} = \frac{\hbar c\alpha}{2}\left\langle r^{-1}\right\rangle \\
&\implies \left\langle r^{-1}\right\rangle = \frac{\alpha\mu c}{\hbar n^2} = \frac{1}{a_0 n^2}
\end{aligned}$$

(b) Given $H = \frac{p_r^2}{2m_e} + \frac{\hbar^2 L(L+1)}{2m_e r^2} - \frac{\hbar c\alpha}{r}$, $E_n = -\frac{\alpha^2\mu c^2}{2n^2}$,

$$\partial_\alpha E = -\frac{\alpha\mu c^2}{n^2}$$

$$\langle\partial_\alpha H\rangle = \left\langle -\frac{\hbar c}{r}\right\rangle = -\hbar c\left\langle r^{-1}\right\rangle$$

$$\partial_\alpha E = \langle\partial_\alpha H\rangle \implies \left\langle r^{-1}\right\rangle = \frac{\alpha\mu c}{\hbar n^2} = \frac{1}{a_0 n^2}$$

(c)

$$\partial_L E = \partial_L \left(-\frac{\alpha^2 \mu c^2}{2(L+k)^2} \right) = \frac{\alpha^2 \mu c^2}{(L+k)^3} = \frac{\alpha^2 \mu c^2}{n^3}$$

$$\langle \partial_L H \rangle = \left\langle \partial_L \left(\frac{p_r^2}{2\mu} + \frac{\hbar^2 L(L+1)}{2\mu r^2} - \frac{\hbar c \alpha}{r} \right) \right\rangle = \frac{\hbar^2 (L+1/2)}{\mu} \left\langle r^{-2} \right\rangle$$

$$\partial_L E = \langle \partial_L H \rangle \implies \left\langle r^{-2} \right\rangle = \frac{\alpha^2 \mu^2 c^2}{\hbar^2 (L+1/2) n^3} = \frac{1}{a_0^2 (L+1/2) n^3}$$

(d)

$$\begin{aligned} [H, p_r] &= \frac{\hbar^2 L(L+1)}{2\mu} \left[r^{-2}, p_r \right] - \hbar \alpha c \left[r^{-1}, p_r \right] \\ &= -\frac{i\hbar^3 L(L+1)}{\mu} r^{-3} + i\hbar^2 \alpha c r^{-2} \end{aligned}$$

Since the expectation value of any operator with the Hamiltonian vanishes with respect to any of the Hamiltonian's eigenstates,

$$\langle [H, p_r] \rangle = 0 \implies i\hbar^2 \alpha c \left\langle r^{-2} \right\rangle = \frac{i\hbar^3 L(L+1)}{\mu} \left\langle r^{-3} \right\rangle$$

$$\implies \left\langle r^{-3} \right\rangle = \frac{\alpha \mu c}{\hbar L(L+1)} \left\langle r^{-2} \right\rangle = \frac{1}{a_0 L(L+1)} \left\langle r^{-2} \right\rangle$$

Substituting in the value for $\langle r^{-2} \rangle$ gives

$$\left\langle r^{-3} \right\rangle = \frac{1}{L(L+1/2)(L+1) a_0^3 n^3}$$