Homework 6

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Problem 1

$$r(\phi) = \frac{a(1-e^2)}{1+e\cos\phi}$$

$$\begin{aligned}
x &= r\cos\phi \\
&= \frac{a(1-e^2)\cos\phi}{1+e\cos\phi} \\
y &= r\sin\phi \\
&= \frac{a(1-e^2)\sin\phi}{1+e\cos\phi}
\end{aligned}$$

$$b = a\sqrt{1-e^2}$$

$$T^{00} = M\delta(z)\left[\delta(x-x_1)\delta(y-y_1) + \delta(x-x_2)\delta(y-y_2)\right]$$

$$\begin{aligned}
I_{xx} &= \int \mathrm{d}^3x\; x^2 T^{00} \\
&= M(x_1^2 + x_2^2) \\
I_{yy} &= M(y_1^2 + y_2^2) \\
I_{xy} &= M(x_1y_1 + x_2y_2)
\end{aligned}$$

Problem 2

(a) First, contracting with the metric we can see that

$$\begin{aligned}
& 0 = (G_{\mu\nu} + \Lambda g_{\mu\nu}) \\
\implies \quad & 0 = g^{\mu\nu}\left(R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu}\right) \\
& \quad = R - \frac{1}{2}4R + 4\Lambda \\
\implies \quad & R = 4\Lambda.
\end{aligned}$$

Then,

$$
\begin{aligned}
0 &= R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} \\
&= R_{\mu\nu} - \Lambda g_{\mu\nu}
\end{aligned}
$$

Assuming the general form for the line element (as in Carrol)

$$
\mathrm{d}s^2 = -e^{2\alpha(r)}\mathrm{d}t^2 + e^{2\beta(r)}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right),
$$

the non-zero components of the Ricci tensor are

$$
\begin{aligned}
R_{tt} &= e^{2(\alpha-\beta)}\left(\frac{2}{r}\alpha' + \alpha'^2 + \alpha''\right) \\
R_{rr} &= \frac{2}{r}\beta' + \alpha'\beta' - \alpha'^2 - \alpha'' \\
R_{\theta\theta} &= e^{-2\beta}\left(r\beta' - r\alpha'\right) + 1 \\
R_{\phi\phi} &= \sin^2\theta\left(e^{-2\beta}\left(\left(r\beta' - r\alpha'\right) - 1\right) + 1 + r^2\right).
\end{aligned}
$$

The Einstein equation then gives

$$
\begin{aligned}
0 &= e^{2(\alpha-\beta)}\left(\frac{2}{r}\alpha' + \alpha'^2 + \alpha''\right) - \Lambda e^{2\alpha} \\
0 &= \frac{2}{r}\beta' + \alpha'\beta' - \alpha'^2 - \alpha'' + \Lambda e^{2\beta} \\
0 &= e^{-2\beta}\left(\left(r\beta' - r\alpha'\right) - 1\right) + 1 + \Lambda r^2 \\
0 &= \sin^2\theta\left(e^{-2\beta}\left(\left(r\beta' - r\alpha'\right) - 1\right) + 1 + \Lambda r^2\right).
\end{aligned}
$$

Multiplying the first equation by $e^{-2\alpha}$, the second by $e^{-2\beta}$, then adding gives

$$
\begin{aligned}
0 &= e^{-2\beta}\left(\frac{2}{r}\alpha' + \alpha'^2 + \alpha''\right) - \Lambda + e^{-2\beta}\left(\frac{2}{r}\beta' + \alpha'\beta' - \alpha'^2 - \alpha''\right) + \Lambda \\
&= \frac{2}{r}\left(\alpha' + \beta'\right) \\
\implies \quad \alpha' &= -\beta' \\
\implies \quad \alpha(r) &= -\beta(r) + c.
\end{aligned}
$$

However, we can set $c = 0$ by taking $t \to e^{-c}t$. The third of the Einstein equations is now

$$
\begin{aligned}
0 &= e^{-2\beta}\left(\left(r\beta' - r\alpha'\right) - 1\right) + 1 + \Lambda r^2 \\
&= e^{2\alpha}\left(-2r\alpha' - 1\right) + 1 + \Lambda r^2 \\
&= -\partial_r\left(re^{2\alpha}\right) + 1 + \Lambda r^2 \\
\implies \quad \partial_r\left(re^{2\alpha}\right) &= 1 + \Lambda r^2 \\
\implies \quad re^{2\alpha} &= r + \frac{1}{3}\Lambda r^3 + \alpha_0 \\
\implies \quad e^{2\alpha} &= 1 + \frac{1}{3}\Lambda r^2 + \frac{\alpha_0}{r},
\end{aligned}
$$

where α_0 is a yet to be determined constant. Plugging this back into the line element, we have

$$\mathrm{d}s^2 = -\left(1 + \frac{\alpha_0}{r} + \frac{1}{3}\Lambda r^2\right)\mathrm{d}t^2 + \left(1 + \frac{\alpha_0}{r} + \frac{1}{3}\Lambda r^2\right)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2\right).$$

Clearly this reduces to the Schwarzschild metric when $\Lambda \to 0$ if we choose $\alpha_0 = -2GM$.

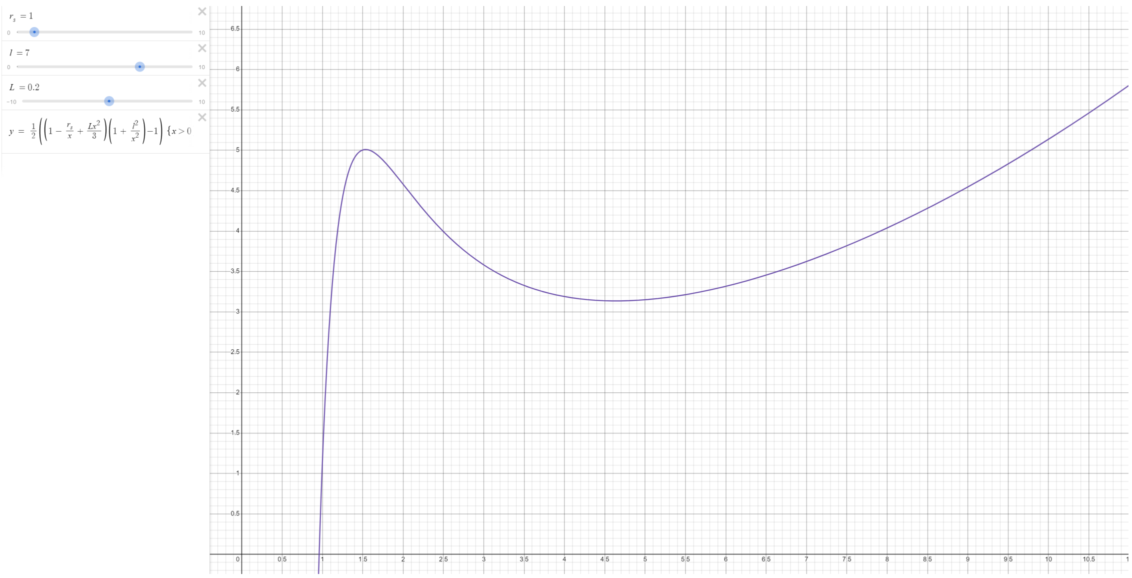

Figure 1: $V_{\text{eff}}(r)$ with $r_s = 1$, $l = 7$, $\Lambda = 0.2$.

(b) We have the conserved quantities

$$e := -\xi_t^\mu \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = \left(1 - \frac{r_s}{r} + \frac{1}{3}\Lambda r^2\right)\frac{\mathrm{d}t}{\mathrm{d}\tau}$$
$$l := -\xi_\phi^\mu \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} = -r^2\sin^2\theta\frac{\mathrm{d}\phi}{\mathrm{d}\tau}$$

Normalization of the four-velocity gives

$$\begin{aligned}
\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}\frac{\mathrm{d}x_\mu}{\mathrm{d}\tau} &= -1 \\
&= -\left(1 + \frac{r_s}{r} + \frac{1}{3}\Lambda r^2\right)\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 + \left(1 + \frac{\alpha_0}{r} + \frac{1}{3}\Lambda r^2\right)^{-1}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + r^2\left(\frac{\mathrm{d}\phi}{\mathrm{d}\tau}\right)^2 \\
&= -e^2\left(1 + \frac{r_s}{r} + \frac{1}{3}\Lambda r^2\right)^{-1} + \left(1 + \frac{\alpha_0}{r} + \frac{1}{3}\Lambda r^2\right)^{-1}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + \frac{l^2}{r^2} \\
\implies \qquad e^2 &= 1 + \frac{r_s}{r} + \frac{1}{3}\Lambda r^2 + \left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + \frac{l^2}{r^2}\left(1 + \frac{r_s}{r} + \frac{1}{3}\Lambda r^2\right) \\
&= \left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + \left(1 + \frac{r_s}{r} + \frac{1}{3}\Lambda r^2\right)\left(1 + \frac{l^2}{r^2}\right)
\end{aligned}$$

$$\implies \quad \mathcal{E} := \frac{1}{2}(e^2 - 1) = \frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 + V_{\text{eff}}(r)$$

where

$$
\begin{aligned}
V_{\text{eff}}(r) &= \frac{1}{2}\left(\left(1 - \frac{r_s}{r} + \frac{1}{3}\Lambda r^2\right)\left(1 + \frac{l^2}{r^2}\right) - 1\right) \\
&= \frac{1}{6}\Lambda(r^2 + l^2) - \frac{r_s}{2r} + \frac{l^2}{2r^2} - \frac{r_s l^2}{2r^3}
\end{aligned}
$$

Problem 3

(a) Calculation done in Mathematica, see Appendix.

(b) The transformation is given by

$$
h_{\mu\nu} \to h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_n u \xi_\mu
$$

To kill off the cs, we need

$$
\begin{aligned}
2\partial_0 \xi_0 &= -c\sin(k(x-t)), \\
2\partial_1 \xi_1 &= c\sin(k(x-t)), \\
\partial_0 \xi_i &= -\partial_i \xi_0 = 0, \\
\partial_i \xi_j &= -\partial_j \xi_i = 0 \quad (i, j = y, z).
\end{aligned}
$$

Integrating, we find that

$$
\xi_\mu = \begin{pmatrix} -1 \\ -1 \\ 0 \\ 0 \end{pmatrix} \frac{c}{2k}\cos(k(x-t)) + b_\mu
$$

where b_μ is any constant vector. Under this transformation a is unchanged.

Appendix

```
$Assumptions = {t ∈ ℝ, x ∈ ℝ, y ∈ ℝ, z ∈ ℝ, c > 0, a > 0, k > 0};
```

In[10]:=
```
n = 4;
coord = {t, x, y, z};
η = DiagonalMatrix[{-1, 1, 1, 1}];
h = Sin[k (x - t)] × DiagonalMatrix[{c, -c, a, -a}];
```

In[5]:=
```
η // MatrixForm // TraditionalForm
h // MatrixForm // TraditionalForm
```

Out[5]//TraditionalForm=

$$\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Out[6]//TraditionalForm=

$$\begin{pmatrix} c\sin(k(x-t)) & 0 & 0 & 0 \\ 0 & -c\sin(k(x-t)) & 0 & 0 \\ 0 & 0 & a\sin(k(x-t)) & 0 \\ 0 & 0 & 0 & -a\sin(k(x-t)) \end{pmatrix}$$

In[14]:=
```
linearRicci = Simplify[
   Table[(Sum[D[(η.h)[[s, i]], {coord[[s]], 1}, {coord[[j]], 1}] + D[(η.h)[[s, j]], {coord[[s]], 1},
          {coord[[i]], 1}], {s, 1, n }] - D[Tr[η.h], {coord[[i]], 1}, {coord[[j]], 1}] -
        Sum[η[[s, s]] × D[h[[i, j]], {coord[[s]], 2}], {s, 1, n}]) / 2, {i, 1, n}, {j, 1, n}]];
```

In[15]:=
```
linearRicci // MatrixForm // TraditionalForm
```

Out[15]//TraditionalForm=

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$