QCD Scattering Amplitudes

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UO

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Nonabelian Gauge Theory

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- This necessitates the inclusion of an $SU(N)$ *gauge field* $A_\mu(x)$ (an NxN traceless hermitian matrix field).
- Also, we must promote normal derivatives ∂_μ to *covariant derivatives* $D_\mu = \partial_\mu - igA_\mu$.

Nonabelian Gauge Theory (cont.)

$SU(N)$ Generators

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- The gauge field can be expanded in the basis of the generators:

$$A_\mu(x) = A_\mu^a(x)T^a$$

where

$$A_\mu^a(x) = 2\,\mathrm{Tr}[A_\mu(x)T^a]$$

Nonabelian Gauge Theory (cont.)

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- The field strength is given by

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F_{\mu\nu} &:= \frac{i}{g}[D_\mu, D_\nu] \\
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- Note that this term includes interactions among the gauge fields.
    - A theory of this type is called Yang-Mills theory.

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$$\mathcal{L} = i\bar{\Psi}_{iI} \not{D}_{ij} \Psi_{jI} - m_I \bar{\Psi}_{iI} \Psi_{iI} - \frac{1}{2} \mathrm{Tr}[F^{\mu\nu} F_{\mu\nu}]$$

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- The path integral is given by

$$Z(J) \propto \int \mathcal{D}A \; e^{iS_{\text{YM}}(A,J)}$$

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$$S_{\text{YM}} = \int \mathrm{d}^4 x \left[-\frac{1}{4} F^{a\mu\nu} F^a_{\mu\nu} + J^{a\mu} A^a_\mu \right]$$

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- Overcomming this difficulty will require...*ghosts!!!*

The Path Integral in Nonabelian Gauge Theory (cont.)

Fixing the Gauge

Fixing the Gauge

- The general form of the gauge-fixed path integral is

$$Z(J) \propto \int \mathcal{D}A \ \det\left(\frac{\delta G^a(x)}{\delta\theta^b(y)}\right) \prod_{x,a} \delta(G) e^{iS_{\mathrm{YM}}}$$

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- The functional determinant can be expressed as a path integral over complex Grassmann variables:

$$\det \frac{\delta G^a(x)}{\delta\theta^b(y)} \propto \int \mathcal{D}c\mathcal{D}\bar{c}e^{iS_{\mathrm{gh}}}$$

  where the fields c fields are known as *Faddeev-Popov ghosts*, and $S_{\mathrm{gh}} = \int \mathrm{d}^4x\bar{c}^a\partial^\mu D_\mu^{ab}c^b$ is the *ghost action*.

The Full Path Integral

The Path Integral in Nonabelian Gauge Theory (cont.)

The Full Path Integral

- The full, gauge-fixed, path integral for Yang-Mills theory is thus

$$Z(J) \propto \int \mathcal{D}A\mathcal{D}\bar{c}\mathcal{D}c \ \exp[i(S_{\mathrm{YM}} + S_{\mathrm{gh}} + S_{\mathrm{gf}})]$$

where, in R_ξ gauge,

$$\begin{aligned}
S_{\mathrm{YM}} &= \int \mathrm{d}^4x \left[-\frac{1}{4}F^{a\mu\nu}F^a_{\mu\nu} + J^{a\mu}A^a_\mu\right] \\
S_{\mathrm{gh}} &= \int \mathrm{d}^4x \ \bar{c}^a\partial^\mu D^{ab}_\mu c^b \\
&= \int \mathrm{d}^4x \left[-\partial^\mu\bar{c}^a\partial_\mu c^a + gf^{abc}A^c_\mu\partial^\mu\bar{c}^a c^b\right] \\
S_{\mathrm{gf}} &= \int \mathrm{d}^4x \left[-\frac{1}{2\xi}\partial^\mu A^a_\mu\partial^\nu A^a_\nu\right]
\end{aligned}$$

Expanding the Lagrangian

Expanding the Lagrangian

Consider just the Yang-Mills and gauge-fixing parts of the path integral:

$$
\begin{aligned}
\mathcal{L}_{\mathrm{YM}} + \mathcal{L}_{\mathrm{gf}} &= -\frac{1}{4} F^{e\mu\nu} F^{e}_{\mu\nu} - \frac{1}{2\xi} \partial^{\mu} A^{e}_{\mu} \partial^{\nu} A^{e}_{\nu} \\
&= \frac{1}{2} A^{e\mu} (g_{\mu\nu} \partial^2 - \partial_{\mu}\partial_{\nu}) A^{e\nu} + \frac{1}{2\xi} A^{e\mu} \partial_{\mu}\partial_{\nu} A^{e\nu} \\
&\quad + g f^{abc} A^{a\mu} A^{b\nu} \partial_{\mu} A^{c}_{\nu} \\
&\quad - \frac{1}{4} g^2 f^{abe} f^{cde} A^{a\mu} A^{b\nu} A^{c}_{\mu} A^{d}_{\nu}
\end{aligned}
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Feynman Rules for Nonabelian Gauge Theory (cont.)

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- The gluon propagator is given by

$$\tilde{\Delta}^{ab}_{\mu\nu}(k) = \frac{\delta^{ab}}{k^2 - i\epsilon}\left(g_{\mu\nu} + (\xi - 1)\frac{k_\mu k_\nu}{k^2}\right)$$

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- The three-gluon vertex factor is

$$iV^{abc}_{\mu\nu\rho}(p,q,r) = gf^{abc}\left[(q-r)_\mu g_{\nu\rho} + (r-p)_\nu g_{\rho\mu} + (p-q)_\rho g_{\mu\nu}\right]$$

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- The four-gluon vertex factor is

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- These vertex factors are crazy complicated! Tree-level spin/color summed/averaged $gg \to gg$ cross section has 12,996 terms!!

Gervais-Neveu Gauge

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    - The ghost fields only appear in loops, thus need not be considered.
- As we saw in the previous slide, even tree-level computations in Yang-Mills theory seem to be rather involved.
- A clever choice of gauge, the Gervais-Neveu gauge, can greatly simplify calculations.

Gervais-Neveu Gauge (cont.)

Change Convention

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- We begin with a *slight* redefinition of the generator norm and commutation relation:

$$\mathrm{Tr}\left[T^a T^b\right] = \delta^{ab} \implies \left[T^a, T^b\right] = i\sqrt{2} f^{abc} T^c$$

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- The field strength tensor and the Yang-Mills lagrangian can now be written in terms of $H_{\mu\nu}$:

$$F_{\mu\nu} = H_{\mu\nu} - H_{\nu\mu}; \quad \mathcal{L}_{\mathrm{YM}} = -\frac{1}{2}\mathrm{Tr}[H^{\mu\nu}H_{\mu\nu} - H^{\mu\nu}H_{\nu\mu}]$$

Gervais-Neveu Gauge (cont.)

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- Note that $\mathcal{L}_{\text{gf}}$ is *not* hermitian, but this is acceptable as its role is merely to fix the gauge.
- Adding the gauge-fixing and Yang-Mills lagrangians gives

$$\mathcal{L} = -\frac{1}{2}\,\text{Tr}\big[H^{\mu\nu}H_{\mu\nu} - H^{\mu\nu}H_{\nu\mu} + H^{\mu}_{\mu}H^{\nu}_{\nu}\big]$$

Feynman rules for $N \times N$ Matrix Fields

A Slicker Formalism

Feynman rules for $N \times N$ Matrix Fields

A Slicker Formalism

- The (expanded) lagrangian for $SU(N)$ Yang-Mills theory in Gervais-Neveu gauge is

$$\mathcal{L} = \mathrm{Tr}\left[-\frac{1}{2}\partial^\mu A^\nu \partial_\mu A_\nu - i\sqrt{2}g\partial^\mu A^\nu A_\nu A_\mu + \frac{1}{4}g^2 A^\mu A^\nu A_\mu A_\nu\right]$$

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- Consider the simpler example of a hermitian *non*-traceless $N \times N$ matrix $B(x)$ with a lagrangian of the form

$$\begin{aligned}\mathcal{L} &= \text{Tr}\left[-\frac{1}{2}\partial^\mu B \partial_\mu B + \frac{1}{3}gB^3 - \frac{1}{4}\lambda B^4\right] \\ &= -\frac{1}{2}\partial^\mu B^a \partial_\mu B^a + \frac{1}{3}\,\text{Tr}\left[T^a T^b T^c\right] B^a B^b B^c \\ &\quad - \frac{1}{4}\lambda\,\text{Tr}\left[T^a T^b T^c T^d\right] B^a B^b B^c B^d\end{aligned}$$

where we've expanded in the generators in the second equality.

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- The propagator for $B_i^{\ j}$ then has the form

$$\tilde{\Delta}_{i\ k}^{\ j\ l}(k^2) = \frac{(T^a)_i^{\ j}(T^a)_k^{\ l}}{k^2 - i\epsilon} = \frac{\delta_i^{\ l}\delta_k^{\ j}}{k^2 - i\epsilon}$$

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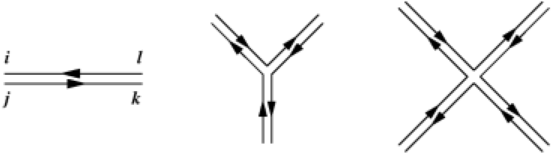
- We also adopt a double-line convetion for Feynman diagrams:



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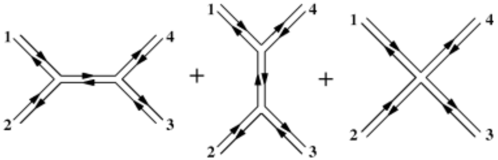
$2 \rightarrow 2$ Scattering

$2 \to 2$ Scattering

- The contributing tree-level diagrams are given by



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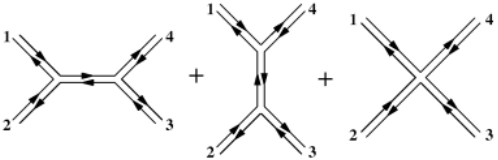

plus all permutations of 2, 3, and 4.

- The resulting ampluted is

$$i\mathcal{T} = \mathrm{Tr}[T^{a_1}T^{a_2}T^{a_3}T^{a_4}]\left(\frac{(ig)^2(-i)}{(k_1+k_2)^2} + \frac{(ig)^2(-i)}{(k_1+k_4)^2} - i\lambda\right) + \text{Perms}$$

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Evaluating $|\mathcal{T}|^2$

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    - products of traces e.g.

$$\mathrm{Tr}[T^{a_{i_1}} \dots T^{a_{i_n}}]\,\mathrm{Tr}\left[T^{a_{i'_1}} \dots T^{a_{i'_n}}\right]^*$$

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Evaluating $|\mathcal{T}|^2$

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  - products of momentum factors e.g.

$$\left(\frac{g^2}{(k_{i_1}+k_{i_2})^2} + \frac{g^2}{(k_{i_3}+k_{i_4})^2} - \lambda\right)\left(\frac{g^2}{(k_{i'_1}+k_{i'_2})^2} + \frac{g^2}{(k_{i'_3}+k_{i'_4})^2} -\right.$$

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Evaluating $|\mathcal{T}|^2$ cont.

- After summing over repeated indices, the trace products yield factors of N^2 and N^4.
- Using momentum conservation, there are only three distinct momentum factors:

$$A_2 := \frac{g^2}{(k_1 + k_4)^2} + \frac{g^2}{(k_1 + k_3)^2} - \lambda$$
$$A_3 := \frac{g^2}{(k_1 + k_2)^2} + \frac{g^2}{(k_1 + k_4)^2} - \lambda$$
$$A_4 := \frac{g^2}{(k_1 + k_3)^2} + \frac{g^2}{(k_1 + k_2)^2} - \lambda$$

Feynman Rules for $N \times N$ Matrix Fields (cont.)

Evaluating $|\mathcal{T}|^2$ cont.

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- The summed amplitude squared is then

$$\sum_{a_1,a_2,a_3,a_4} |\mathcal{T}|^2 = (2N^2 + 2N^2) \sum_j |A_j|^2 + 4N^2 \sum_{j \neq k} A *_j A_k$$
$$= (2N^2 - 2N^2) \sum_j |A_j|^2 + 4N^2 \left(\sum_j A*_j \right) \left(\sum_k A_k \right)$$

where j and k are summed over 2, 3, 4.

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- This would seem to again make computations more involved, but we will see that this winds up not being the case.

The Twister Formalism

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- Recall that calculations in spinor electrodynamics for massless fermions can be greatly simplified by introducing the notation of *twistors*:

$$|p] := u_-(p) = v_+(p); \quad [p| := \bar{u}_+(p) = \bar{v}_-(p)$$

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- We can also express boson polarization vectors (with respect to an arbitrary reference momentum q) in twistor notation:

$$\epsilon_+^\mu(k;q) = -\frac{\langle q|\,\gamma^\mu|k]}{\sqrt{2}\,\langle qk\rangle}; \quad \epsilon_-^\mu(k;q) = -\frac{[q|\gamma^\mu\,\langle k|}{\sqrt{2}[qk]}$$

The Twister Formalism (cont.)

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$$\epsilon_+(k;q)\cdot\epsilon_+(k';q') = \frac{\langle qq'\rangle\,[kk']}{\langle qk\rangle\,\langle q'k'\rangle}$$

$$\epsilon_-(k;q)\cdot\epsilon_-(k';q') = \frac{[qq']\,\langle kk'\rangle}{[qk][q'k']}$$

$$\epsilon_+(k;q)\cdot\epsilon_-(k';q') = \frac{\langle qk'\rangle\,[kq']}{\langle qk\rangle\,[q'k']}$$

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Tree-Level N-Gluon Scattering

Back to Amplitudes

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- We return now to the lagrangian for $SU(N)$ Yang-Mills theory in Gervais-Neveu gauge:

$$\mathcal{L} = \mathrm{Tr}\left[-\frac{1}{2}\partial^\mu A^\nu \partial_\mu A_\nu - i\sqrt{2}g\partial^\mu A^\nu A_\nu A_\mu + \frac{1}{4}g^2 A^\mu A^\nu A_\mu A_\nu\right]$$

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- The tree-level n-gluon scattering amplitude may be written as

$$\mathcal{T} = g^{n-2} \sum_{\pi \in \tilde{S}_n} \mathrm{Tr}[T^{a_{\pi_1}} \dots T^{a_{\pi_n}}] A(\pi_1, \dots, \pi_n)$$

where the sum is over all non-cyclic permutations of n elements, and the $A()$ are *partial amplitudes* that are computed with the color-ordered Feynman rules.

Tree-Level N-Gluon Scattering (cont.)

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  3) "The Third Symmetry":

$$A(1, 2, 3, 4) = -A(1, 2, 4, 3) - A(1, 4, 2, 3)$$

Tree-Level N-Gluon Scattering (cont.)

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$$iV_{123} = -i\sqrt{2}g\left[(\epsilon_1\epsilon_3) + (\epsilon_2\epsilon_3)(k_2\epsilon_1) + (\epsilon_3\epsilon_1)(k_3\epsilon_2)\right]$$

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- Clearly, our calculations will be vastly simplified if we can get the majority of these polarization dot products to vanish, and we will see that is in fact easily achievable.

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- Becuase the polarization vectors are transverse, this restricts the possible combinations of helicities that can have non-zero partial amplitudes.
- Specifically, if all, or all but one, of the external gluons have the same helicity, the amplitude is zero, i.e.

$$A(1^{\pm}, 2^{+}, \ldots, n^{+}) = A(1^{\pm}, 2^{-}, \ldots, n^{-}) = 0$$

Calculating a Partial Amplitude

Tree-Level N-Gluon Scattering (cont.)

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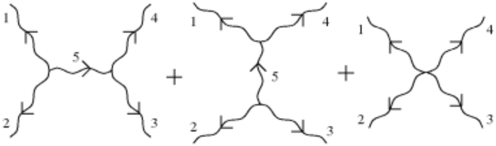
- Consider the partial amplitude $A(1^-, 2^-, 3^+, 4^+)$, given by the diagrams



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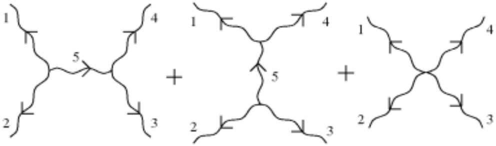

- Choosing our reference momenta to be $q_1 = q_2 = k_3$ and $q_3 = q_4 = k_2$ causes all polarization products to vannish except for

$$\epsilon_1 \cdot \epsilon_4 = \frac{\langle 21 \rangle [43]}{\langle 24 \rangle [31]}$$

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- Note that ϵ_5 is just a placeholder for the internal propagator, and $k_{5'} = -k_5$.

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- Note that ϵ_5 is just a placeholder for the internal propagator, and $k_{5'} = -k_5$.
- After substituting in the propagator and take the product of these vertex factors, only one term winds up being nonvanishing, giving

$$ig^2 A(1^-, 2^-, 3^+, 4^+) = (-i\sqrt{2}g)^2 (i/s_{12})(\epsilon_1\epsilon_4)(k_5\epsilon_2)(k_4\epsilon_3)$$

where $s_{12} = -(k_1 + k_2)^2 = \langle 12\rangle\,[21]$.

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- we can now express our partial amplitude as

$$A(1^-,2^-,3^+,4^+) = \frac{\langle 21\rangle\,[43]^2}{[21][32]\,\langle 23\rangle} = \frac{\langle 12\rangle^4}{\langle 12\rangle\,\langle 23\rangle\,\langle 34\rangle\,\langle 41\rangle}$$

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- Note that this is a specific case of the *Parke-Taylor Maximum Helicity Violoating* amplitudes:

$$A(1^+ \dots i^- \dots j^- \dots n^+) \propto \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle \cdots \langle (n-1)n \rangle \langle n1 \rangle}$$

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- We can now use the cyclic symmetry of the partial amplitudes to get any other amplitude in which the helicities with the same sign sit next to each other (e.g. $A(1^-, 2^+, 3^+, 4^-)$).

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- Amplitudes with alternating helicities can be obtained using "the third symmetry" (e.g. $A(1^+, 2^-, 3^+, 4^-)$).
- The only distinct amplitudes that must be calculated may be denoted

$$A_2 := A(1, 4, 2, 3);\; A_3 := A(1, 2, 3, 4);\; A_4 := A(1, 4, 3, 2)$$

Tree-Level N-Gluon Scattering (cont.)

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$$\sum_{\text{colors}} |\mathcal{T}|^2 = 2N^2(N^2-1)g^4 \left(|A_2|^2 + |A_3|^2 + |A_4|^2\right)$$

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- If we take gluons 1 and 2 to be incoming, and 3 and 4 outgoing, we can express this in terms of the usual Mandelstam variables:

$$\sum_{\text{colors}} |\mathcal{T}|^2 = 2N^2(N^2-1)g^4 s^4\left(\frac{1}{s^2t^2} + \frac{1}{t^2u^2} + \frac{1}{u^2s^2}\right)$$

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- Our final result is then

$$\boxed{\sum_{\substack{\text{colors}\\\text{helicities}}} |\mathcal{T}|^2 = \frac{N^2}{N^2-1} g^4 (s^4+t^4+u^4)\left(\frac{1}{s^2t^2}+\frac{1}{t^2u^2}+\frac{1}{u^2s^2}\right)}$$