Homework 5

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Problem 1

(a) The rotation matrix for a $\pi/2$ rotation about the y-axis (in the standard basis) is

$$\mathbf{d}^{(1)}(\hat{y}\pi/2) = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & \sqrt{2} & 1 \end{pmatrix}$$

Applying this rotation to $|1,0\rangle$ gives

$$\sum_m |1,m\rangle\, d^{(1)}_{m,0}(\hat{y}\pi/2) = \frac{1}{\sqrt{2}}\left(|1,1\rangle - |1,-1\rangle\right)$$

(b) The rotation matrix for a $-\pi/2$ rotation about the x-axis is

$$\mathbf{d}^{(1)}(-\hat{x}\pi/2) = \frac{1}{2}\begin{pmatrix} 1 & -\sqrt{2}i & -1 \\ -\sqrt{2}i & 0 & -\sqrt{2}i \\ -1 & -\sqrt{2}i & 1 \end{pmatrix}$$

Applying this rotation to $|1,1\rangle$ gives

$$\sum_m |1,m\rangle\, d^{(1)}_{m,0}(-\hat{x}\pi/2) = \frac{-i}{\sqrt{2}}\left(|1,1\rangle + |1,-1\rangle\right)$$

Problem 2

(a) The transformation from cartesian to spherical coordinates is given by

$$\begin{aligned} A_1 &= \frac{-1}{\sqrt{2}}A_x - \frac{i}{\sqrt{2}}A_y \\ A_0 &= A_z \\ A_{-1} &= \frac{1}{\sqrt{2}}A_x - \frac{i}{\sqrt{2}}A_y \end{aligned}$$

or, in matrix form,

$$\begin{pmatrix} A_1 \\ A_0 \\ A_{-1} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & -i & 0 \\ 0 & 0 & 1 \\ 1 & -i & 0 \end{pmatrix} \begin{pmatrix} A_x \\ A_y \\ A_z \end{pmatrix}$$

(b) The cartesian rotation matricies about the y and z axes are given by

$$R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \quad R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The composite passive-rotation operator, $P = R_z(-\gamma)R_y(-\beta)R_z(-\alpha)$, in cartesian coordinates is

$$\begin{pmatrix} \cos\alpha\cos\beta\cos\gamma - \sin\alpha\sin\gamma & \cos\beta\cos\gamma\sin\alpha + \cos\alpha\sin\alpha & -\cos\gamma\sin\beta \\ -\cos\gamma\sin\alpha - \cos\alpha\cos\beta\sin\gamma & \cos\alpha\cos\gamma - \cos\beta\sin\alpha\sin\gamma & \sin\beta\sin\gamma \\ \cos\alpha\cos\beta & \sin\alpha\sin\beta & \cos\beta \end{pmatrix}$$

Transforming to the spherical basis gives

$$UPU^\dagger = \begin{pmatrix} \frac{1}{2}(\cos\beta + 1)e^{-i\alpha}e^{-i\gamma} & \frac{1}{\sqrt{2}}\sin\beta e^{-i\gamma} & \frac{1}{2}(1-\cos\beta)e^{i\alpha}e^{-i\gamma} \\ -\frac{1}{\sqrt{2}}\sin\beta e^{-i\alpha} & \cos\beta & \frac{1}{\sqrt{2}}\sin\beta e^{i\alpha} \\ \frac{1}{2}(1-\cos\beta)e^{-i\alpha}e^{i\gamma} & -\frac{1}{\sqrt{2}}\sin\beta e^{i\gamma} & \frac{1}{2}(1+\cos\beta)e^{i\alpha}e^{i\gamma} \end{pmatrix}$$

(c) Finally,

$$\left(UPU^\dagger\right)^\intercal = \begin{pmatrix} \frac{1}{2}(\cos\beta + 1)e^{-i\alpha}e^{-i\gamma} & -\frac{1}{\sqrt{2}}\sin\beta e^{-i\alpha} & \frac{1}{2}(1-\cos\beta)e^{-i\alpha}e^{i\gamma} \\ \frac{1}{\sqrt{2}}\sin\beta e^{-i\gamma} & \cos\beta & -\frac{1}{\sqrt{2}}\sin\beta e^{i\gamma} \\ \frac{1}{2}(1-\cos\beta)e^{i\alpha}e^{-i\gamma} & \frac{1}{\sqrt{2}}\sin\beta e^{i\alpha} & \frac{1}{2}(1+\cos\beta)e^{i\alpha}e^{i\gamma} \end{pmatrix})$$

Problem 3

$$\begin{aligned}
T_1^{(1)} &= \frac{1}{\sqrt{2}}\left(A_1B_0 - A_0B_1\right) \\
&= \frac{1}{2}\left[\left(-A_x - iA_y\right)B_z - A_z\left(-B_x - iB_y\right)\right] \\
&= \frac{1}{2}\left[A_zB_x - A_xB_z + i\left(A_zB_y - A_yB_z\right)\right] \\
&= \frac{-i}{\sqrt{2}}\left[\left(\vec{A}\times\vec{B}\right)_x + i\left(\vec{A}\times\vec{B}\right)_y\right] \\
&= \frac{i}{\sqrt{2}}\left(\vec{A}\times\vec{B}\right)_1 \\
T_0^{(1)} &= \frac{1}{\sqrt{2}}\left(A_{-1}B_1 - A_1B_{-1}\right) \\
&= \frac{1}{2^{3/2}}\left[\left(A_x - iA_y\right)\left(-B_x - iB_y\right) - \left(-A_x - iA_y\right)\left(B_x - iB_y\right)\right] \\
&= \frac{i}{\sqrt{2}}\left(A_xB_y - A_yB_x\right) \\
&= \frac{i}{\sqrt{2}}\left(\vec{A}\times\vec{B}\right)_0 \\
T_{-1}^{(1)} &= \frac{1}{\sqrt{2}}\left(A_{-1}B_0 - A_0B_{-1}\right) \\
&= \frac{1}{2}\left[\left(A_x - iA_y\right)B_z - A_z\left(B_x - iB_y\right)\right] \\
&= \frac{1}{2}\left[A_xB_z - A_zB_x + i\left(A_zB_y - A_yB_z\right)\right] \\
&= \frac{i}{\sqrt{2}}\left[\left(\vec{A}\times\vec{B}\right)_x - i\left(\vec{A}\times\vec{B}\right)_y\right] \\
&= \frac{i}{\sqrt{2}}\left(\vec{A}\times\vec{B}\right)_{-1}
\end{aligned}$$

Problem 4

Consider $|\theta_1,\phi_1\rangle$, $|\theta_2,\phi_2\rangle$, $|\theta_1',\phi_1'\rangle$, $|\theta_2',\phi_2'\rangle$ such that

$$|\theta_\alpha',\phi_\alpha'\rangle = R\,|\theta_\alpha,\phi_\alpha\rangle$$

for some rotation R.
Consider now the quantity

$$C_l = \sum_m \langle\theta_2,\phi_2|l,m\rangle\,\langle l,m|\theta_1,\phi_1\rangle = \sum_m \left[Y_l^m(\theta_1,\phi_1)\right]^* Y_l^m(\theta_2,\phi_2)$$

This quantity my be expressed in terms of the primed coordinates as so:

$$C_l' = \sum_{m,m',m''} d_{m',m}^{(l)}\left(d_{m,m''}^{(l)}\right)^*\langle\theta_2,\phi_2|l,m'\rangle\,\langle l,m''|\theta_1,\phi_1\rangle = \sum_m \left[Y_l^m(\theta_1',\phi_1')\right]^* Y_l^m(\theta_2',\phi_2'),$$

where $\mathbf{d}^{(l)}$ is the matrix representation for R in the standard basis. Note that, due to the unitarity of R,

$$\sum_m d^{(l)}_{m',m} d^{(l)}_{m,m''} = \delta_{m',m''}.$$

The tripple sum above therefore reduces to

$$C'_l = \sum_{m'} \langle \theta_2, \phi_2 | l, m' \rangle \langle l, m' | \theta_1, \phi_1 \rangle = C_l$$

and we see that the primed and un-primed quantities are equal.
Next consider the case where (θ'_1, ϕ'_1) points along the z-axis (i.e. $\theta'_1 = 0$), and $\phi'_2 = 0$. The value of the spherical harmonic along the z-axis is

$$Y_l^m(0, \phi) = \sqrt{\frac{2l+1}{4\pi}} \delta_{m,0},$$

from the definition of the spherical harmonics. This eliminates all but one of the terms in the quantity above:

$$C'_l = \sqrt{\frac{2l+1}{4\pi}} Y_l^0(\theta'_2, 0).$$

However, since this is equal to the rotated quantity, we have

$$Y_l^0(\theta, 0) = \sqrt{\frac{4\pi}{2l+1}} \sum_m \left[Y_l^m(\theta_1, \phi_1)\right]^* Y_l^m(\theta_2, \phi_2)$$

where θ is the angle between the directions given by (θ_1, ϕ_1) and (θ_2, ϕ_2). Letting $\theta_1 = \theta_2$ and $\phi_1 = \phi_2$, we get

$$\sqrt{\frac{2l+1}{4\pi}} = \sqrt{\frac{4\pi}{2l+1}} \sum_m \left|Y_l^m(\theta, \phi)\right|^2$$

$$\implies \sum_m \left|Y_l^m(\theta, \phi)\right|^2 = \frac{2l+1}{4\pi}$$