Homework 7

Sean Ericson
Phys 684

November 18, 2024

Problem 1

For an inhomogeneously broadened system subject to an electric field with magnitude

$$E(z,t) = \frac{1}{2}E_0 e^{-i(\omega t - kz - \phi)} + \text{c.c.},$$

with $E_0 \in \mathbb{R}$, and polarization (taking μ to be real as well)

$$\begin{aligned} P(z,t) &= \frac{1}{2}P_0 e^{-i(\omega t - kz - \phi)} + \text{c.c.} \\ &= \frac{N\mu}{V}\int_0^\infty \mathrm{d}\omega_0 g(\omega_0)\left(\rho_{12} + \text{c.c.}\right), \end{aligned}$$

we know the two are related under the slowly varying amplitude and phase approximation via the Maxwell wave equation

$$\partial_z E_0 + \frac{1}{c}\partial_t E_0 = -\frac{k}{2\epsilon_0}\,\mathrm{Im}[P_0].$$

By integrating with respect to time, we can get this in terms of the pulse area $A(z) = \int_{-\infty}^{\infty}\mathrm{d}t\ \Omega(z,t)$:

$$\begin{aligned} \partial_z E_0 + \frac{1}{c}\partial_t E_0 &= -\frac{k}{2\epsilon_0}\,\mathrm{Im}[P_0] \\ \implies \quad -\frac{\mu}{\hbar}\int_{-\infty}^{\infty}\mathrm{d}t\ \left[\partial_z E_0 + \frac{1}{c}\partial_t E_0\right] &= \int_{-\infty}^{\infty}\mathrm{d}t\ \left[-\frac{k}{2\epsilon_0}\,\mathrm{Im}[P_0]\right] \\ \implies \quad \partial_z A &= \frac{k\mu}{2\epsilon_0\hbar}\int_{-\infty}^{\infty}\mathrm{d}t\ \mathrm{Im}[P_0], \end{aligned}$$

where we used $\Omega_0 = -\frac{\mu}{\hbar}E_0$, and $E_0(-\infty) = 0$. Now, using

$$\rho_{12} = \frac{1}{2}(u + iv)e^{i(\omega t - kz - \phi)},$$

and the expression for the polarization above, we write out the imaginary part of P_0 to get

$$\partial_z A = \frac{N}{V}\frac{\mu^2 k}{2\epsilon_0 \hbar}\int_{-\infty}^{\infty} \mathrm{d}t \int_0^{\infty} \mathrm{d}\omega_0 \; g(\omega_0) v(z,t,\omega_0).$$

After the pulse (or, after a time t_0 such that $\Omega_0 \approx 0$), the Optical Bloch equations give

$$\dot{\vec{R}} = \vec{\Omega} \times \vec{R}, \quad \vec{\Omega} = \begin{pmatrix} \Omega_0 \\ 0 \\ \delta \end{pmatrix} \quad \Longrightarrow \quad \dot{\vec{R}} = \begin{pmatrix} -\delta v \\ \delta u - \Omega_0 w \\ \Omega_0 v \end{pmatrix} \approx \begin{pmatrix} -\delta v \\ \delta u \\ 0 \end{pmatrix},$$

from which we can see that

$$\dot{u} = -\delta v \quad \Longrightarrow \quad v = -\frac{\dot{u}}{\omega_0 - \omega}.$$

Plugging this into the expression for $\partial_z A$ above,

$$\begin{aligned} \partial_z A &= -\frac{N}{V}\frac{\mu^2 k}{2\epsilon_0 \hbar}\int_{-\infty}^{\infty} \mathrm{d}t \int_0^{\infty} \mathrm{d}\omega_0 \; g(\omega_0) \frac{\dot{u}(z,t,\omega_0)}{\omega_0 - \omega} \\ &= -\frac{N}{V}\frac{\mu^2 k}{2\epsilon_0 \hbar}\int_0^{\infty} \mathrm{d}\omega_0 \; g(\omega_0) \frac{u(z,t,\omega_0)}{\omega_0 - \omega}, \end{aligned}$$

since $u(-\infty) = 0$. Now, from the OBE above, we have the simple coupled differential equation

$$\begin{aligned} \dot{u} &= -\delta v \\ \dot{v} &= \delta u. \end{aligned}$$

This can be solved by simply taking another derivative:

$$\ddot{u} = -\delta \dot{v} = -\delta^2 u \implies u(t) = u(t_0)\cos(\delta(t - t_0)) - v(t_0)\sin(\delta(t - t_0))$$

OK, so the expression for $\partial_z A$ has two terms, one proportional to

$$\int_0^{\infty} \mathrm{d}\omega_0 \; g(\omega_0) \frac{\sin[(\omega_0 - \omega)(t - t_0)]}{\omega_0 - \omega},$$

and another proportional to

$$\int_0^{\infty} \mathrm{d}\omega_0 \; g(\omega_0) \frac{\cos[(\omega_0 - \omega)(t - t_0)]}{\omega_0 - \omega}.$$

Now, through some complex analysis tricks it can be shown that for reasonable $g(\omega_0)$ ("reasonable" meaning a test-function, which a gaussian is the prototypical example of), the cos integral vanishes. However, in the $t \to \infty$ limit that we're ultimately interested in, it is well known that

$$\lim_{t\to\infty} \frac{\sin(xt)}{t} = \pi \delta_{\mathrm{D}}(x),$$

so *that* integral turns out to be trivial (δ_{D} being the Dirac delta). Taking the trivial integral, we get

$$\partial_z A = -\frac{N}{V}\frac{\mu^2 k}{2\epsilon_0 \hbar}\pi g(\omega) v(z, t_0, \omega).$$

Ah, but when $\omega_0 = \omega$ (i.e. zero detuning), we know from the OBE that $v(t) = \sin A(t)$, so we finally have

$$\partial_z A = -\alpha \sin(A(z)); \quad \alpha = \frac{N}{V}\frac{\mu^2 k}{\epsilon_0 \hbar}\pi g(\omega).$$

Problem 2

Given that

$$\frac{\partial A}{\partial z} = -\alpha \sin(A),$$

for $n \in \mathbb{Z}$ we have that

$$\begin{aligned} -\alpha \sin((2n+1)\pi + \epsilon) &\approx \alpha \sin(2\pi n) + \alpha\epsilon \cos(2\pi n) \\ &\approx \alpha\epsilon, \end{aligned}$$

$$\begin{aligned} -\alpha \sin(2n\pi + \epsilon) &\approx -\alpha \sin(2\pi n) - \alpha\epsilon \cos(2\pi n) \\ &\approx -\alpha\epsilon. \end{aligned}$$

So, perturbations away from odd multiples of π have positive slope, meaning these values of A_0 are unstable, while the opposite is true for even multiples of π.

Problem 3

Given that the first pulse has area A_1 occurs over the interval $0 \leq t \leq t_1$ and that time between pulses is $\tau = t_2 - t_1$, the Bloch vector starts at $-\hat{z}$, rotates about the x-axis by an angle A_1, then rotates about the z-axis by an angle $\delta\tau$. The Bloch vector at the start of pulse 2 is then

$$\vec{R}(t_2) = \begin{pmatrix} -s_\tau s_1 \\ c_\tau s_1 \\ c_1 \end{pmatrix},$$

where we're using the short-hand notation $\sin(\delta\tau) \to s_\tau$, $\cos A_1 \to c_1$, etc. Given that the second pulse occurs over the interval $t_2 \leq t \leq t_3$, and has area A_2, it's affect is to rotate the Bloch vector about the x-axis by an angle A_2. At the end of the second pulse, the Bloch vector is

$$\vec{R}(t_3) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_2 & -s_2 \\ 0 & s_2 & c_2 \end{pmatrix} \begin{pmatrix} -s_\tau s_1 \\ c_\tau s_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} -s_\tau s_1 \\ c_2 c_\tau s_1 - s_2 c_1 \\ s_2 c_\tau s_1 - c_2 c_1 \end{pmatrix}.$$

After the last pulse, we then have free precession about $\hat{z}$ again:

$$\vec{R}(t > t_3) = \begin{pmatrix} c_{\tau'} & -s_{\tau'} & 0 \\ s_{\tau'} & c_{\tau'} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -s_\tau s_1 \\ c_2 c_\tau s_1 - s_2 c_1 \\ s_2 c_\tau s_1 - c_2 c_1 \end{pmatrix} = \begin{pmatrix} -c_{\tau'} s_\tau s_1 - s_{\tau'}(c_2 c_\tau s_1 - s_2 c_1) \\ -s_{\tau'} s_\tau s_1 + c_{\tau'}(c_2 c_\tau s_1 - s_2 c_1) \\ s_2 c_\tau s_1 - c_2 c_1 \end{pmatrix},$$

where $\tau' = t - t_3$. Looking at the x and y components, we have

$$\begin{aligned} u &= -\cos(\delta\tau')\sin(\delta\tau)\sin A_1 - \sin(\delta\tau')\left(\cos A_2 \cos(\delta\tau)\sin A_1 - \sin A_2 \cos A_1\right), \\ v &= -\sin(\delta\tau')\sin(\delta\tau)\sin A_1 + \cos(\delta\tau')\left(\cos A_2 \cos(\delta\tau)\sin A_1 - \sin A_2 \cos A_1\right). \end{aligned}$$

Now, for $A_2 = \pi$, we have that $u \to 0,\ v \to \sin A_1$ at $\tau = \tau'$. But what about if $A_2 \neq \pi$?

Problem 4

(a) The Bloch vector starts out pointing in the $-\hat{z}$ direction. The first $\pi/2$ pulse rotates the Bloch vector by $\pi/2$ about the x-axis, leaving it pointing in the $+\hat{y}$ direction.

(b) With decays, the OBE is

$$\dot{\vec{R}} = \begin{pmatrix} -\delta v - \gamma u \\ \delta u - \gamma v \\ -\gamma_2(w+1) \end{pmatrix}.$$

With initial condition $R_0 = (0, 1, 0)^\intercal$, the solution to the above differential equation is given by

$$\vec{R}(t) = \begin{pmatrix} -\sin(\delta t)e^{-\gamma t} \\ \cos(\delta t)e^{-\gamma t} \\ e^{\gamma_2 t} - 1 \end{pmatrix},$$

so the position of the Bloch vector at the start of the second pulse is

$$\vec{R} = \begin{pmatrix} -\sin(\delta\tau)e^{-\gamma\tau} \\ \cos(\delta\tau)e^{-\gamma\tau} \\ e^{\gamma_2\tau} - 1 \end{pmatrix}.$$

(c) The second $\pi/2$ pulse again rotates by $\pi/2$ about the x-axis, giving

$$\vec{R} = \begin{pmatrix} -\sin(\delta\tau)e^{-\gamma\tau} \\ 1 - e^{\gamma_2\tau} \\ \cos(\delta\tau)e^{-\gamma\tau} \end{pmatrix},$$

and an upper-state population of

$$\rho_2 2 = \frac{1}{2}(w+1) = \frac{1}{2}(1 + \cos(\delta\tau)e^{-\gamma\tau}).$$

(d) Using $\pi/2$ pulses is not strictly necessary. Any pulse that results in a non-zero $\left|\vec{R}_\perp\right|$ will suffice, but $\pi/2$ pulses simply maximize this transverse component of $\vec{R}$.

Problem 4

```
In[1]:= M = {{-γ, -δ, 0}, {δ, -γ, 0}, {0, 0, -γ2}};
        R[t_] = {{u[t], v[t], w[t]}}ᵀ;

In[4]:= r = (R[t] /. DSolve[
              {D[R[t], t] == M.R[t] - {{0, 0, γ2}}ᵀ, R[0] == {{0, 1, 0}}ᵀ}, {u, v, w}, t])[[1]];

In[8]:= RotationMatrix[π/2, {1, 0, 0}].r // MatrixForm
```

Out[8]//MatrixForm=

$$\begin{pmatrix} -e^{-t\gamma}\,\mathrm{Sin}[t\,\delta] \\ e^{-t\gamma_2}\left(-1+e^{t\gamma_2}\right) \\ e^{-t\gamma}\,\mathrm{Cos}[t\,\delta] \end{pmatrix}$$