Homework 7

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Problem 1

We assume the only change in the physical distance between the test particles is due to the change in the scale factor. In a vacuum dominated universe, the scale factor grows as

$$a(t) = a_0 e^{Ht},$$

where H is the Hubble parameter. The physical distance between the test particles is

$$d(t) = a(t)r(t),$$

where $r(t)$ is the coordinate separation, which does not change under evolution due only to the scale factor. The physical separation between the particles is then

$$d(t) = r_0 e^{Ht}.$$

The time for a given relative change in physical separation is therefore

$$t = H^{-1} \ln\left(\frac{d}{r_0}\right).$$

For movement on the order of the initial separation size (i.e. $d \to 2\text{cm}$), this is a time scale on the order of 10^{10} years. However, if we were able to measure the separation of the particles to the accuracy of LIGO ($\sim 10^{-22}\text{m}$), the associated timescale would be on the order of seconds! So, laboratory detection seems like it would be extremely challenging, but not necessarily impossible.

Problem 2

(a) Units of physical quantities form a $\mathbb{Q}$-vector space. Let's work in the Length, Mass, Time or $\{\mathcal{L}, \mathcal{M}, \mathcal{T}\}$. The units of the constants can then be expressed as

$$[c] = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad [\hbar] = \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}, \quad [G] = \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}.$$

Calculating the units of the proposed quantities follows as

$$\begin{aligned}\left[\frac{c^5}{\hbar G^2}\right] &= 5\begin{pmatrix}1\\0\\-1\end{pmatrix} - \begin{pmatrix}2\\1\\-1\end{pmatrix} - 2\begin{pmatrix}3\\-1\\2\end{pmatrix}\\ &= \begin{pmatrix}-3\\1\\0\end{pmatrix}\\ &= \frac{\mathcal{M}}{\mathcal{L}^3},\end{aligned}$$

and we can see the quantity indeed as units of mass density.

(b) Plugging in the values of the constants and the observed vacuum energy density, we get a ration of $\rho_\Lambda/\rho_{\mathrm{Pl}} \approx 10^{-123}$, as illustrated in Figure 1.

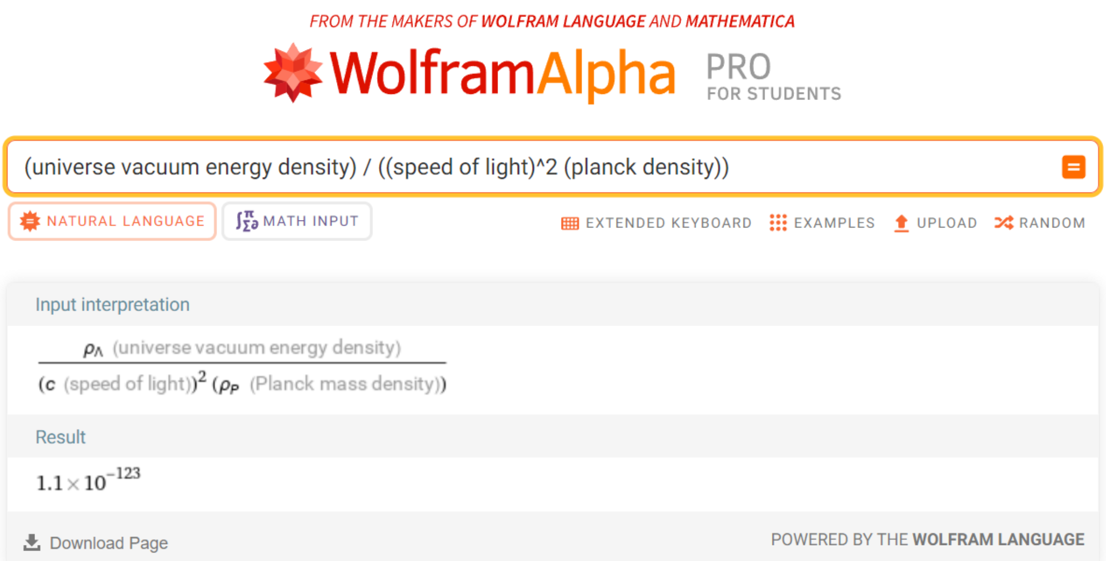

Figure 1: Calculation of $\rho_\Lambda/\rho_{\mathrm{Pl}}$ via WolframAlpha.

Problem 3

(a) Matter density is proportional to a^{-3}, and the temperature is proportional to a^{-1}, so

$$p \propto T\rho \implies \boxed{p \propto a^{-4}}$$

(b) The energy of the neutrinos is proportional to their temperature, and we seek the temperature at which this energy is comparable to their rest mass. Therefore,

$$\frac{T}{T_0} \propto \left(\frac{a}{a_0}\right)^{-1} = \frac{a_0}{a} = z + 1$$

$$\implies z \approx \frac{T}{T_0} - 1 \approx \frac{m_\nu}{T_0} - 1 \approx \boxed{579}$$

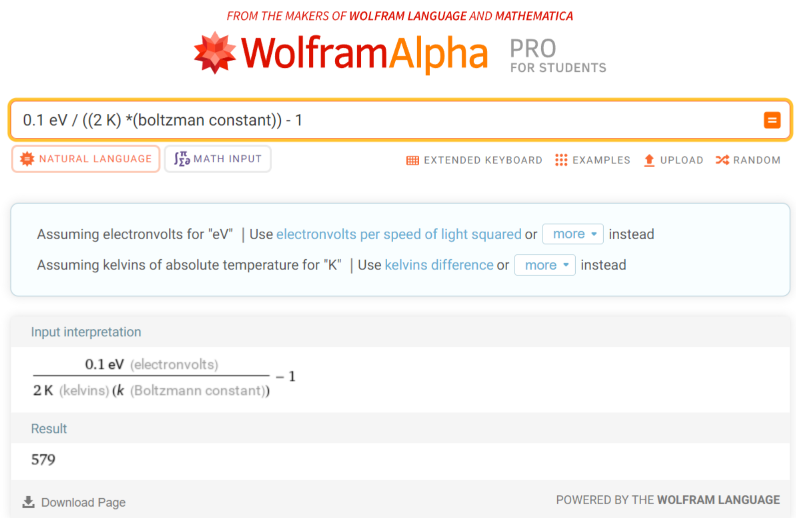

Figure 2: Calculation of z for the neutrino transition via WolframAlpha.