Homework 1

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Problem 1

We can get the electric field amplitude from the intensity, as

$$I = \frac{P}{A} = \frac{1}{2} c\epsilon_0 E_0^2 \implies E_0 = \sqrt{\frac{2P}{c\epsilon_0 A}} \approx 868 \text{ V/m}.$$

A rough but simple estimate for the dipole moment is just $ea_0 \approx 2.5$ Debye. The Rabi frequency is then

$$\Omega_0 = \frac{\mu E_0}{\hbar} \approx 70 \text{ MHz}$$

See the end of the document for a printout of the Mathematica notebook used for these calculations.

Problem 2

Under the rotating wave approximation, we neglect the counter-rotating term and get as our differential equation (neglecting bars on the 'c's)

$$\dot{c}_1 = -\frac{1}{2} i\Omega_0 e^{i\delta t} c_2$$
$$\dot{c}_2 = -\frac{1}{2} i\Omega_0 e^{-i\delta t} c_1.$$

The Rabi frequency is directly proportional to the applied electric field. In the weak-field limit, we can perturbatively expand the amplitudes as

$$c_i = c_i^{(0)} + \Omega_0 c_i^{(1)} + \Omega_0^2 c_i^{(2)} + \cdots .$$

To zero-th order, the amplitudes are given by the initial conditions $c_1^{(0)} = c_1(0) = 1$, and $c_2^{(0)} = c_2(0) = 0$. Now we go to first order and plug into the differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(c_1^{(0)} + \Omega_0 c_1^{(1)}\right) = -\frac{i}{2}\Omega_0 e^{i\delta t}\left(c_2^0 + \Omega_0 c_2^{(1)}\right)$$
$$\implies \qquad \Omega_0 \dot{c}_1^{(1)} = -\frac{i}{2}\Omega_0^2 e^{i\delta t} c_2^{(1)}$$

Matching terms proportional to Ω_0 gives

$$\begin{aligned} & \dot{c}_1^{(1)} = 0 \\ \implies \quad & c_1^{(1)} = c_1^{(1)}(0) = 0. \end{aligned}$$

For c_2, we find

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(c_2^{(0)} + \Omega_0 c_2^{(1)}\right) &= -\frac{i}{2}\Omega_0 e^{-i\delta t}\left(c_1^{(0)} + \Omega_0 c_1^{(1)}\right) \\ \implies \qquad \Omega_0 \dot{c}_2^{(1)} &= -\frac{i}{2}\Omega_0 e^{-i\delta t} + O(\Omega^2) \\ \implies \qquad c_2^{(1)} &= -\frac{i}{2}\int_0^t \mathrm{d}t' e^{-i\delta t'} \\ &= \frac{1}{2\delta}\left(e^{-i\delta t} - 1\right). \end{aligned}$$

So, to first order we have that

$$\left.\begin{aligned} c_1 &\approx 1 \\ c_2 &\approx \tfrac{\Omega_0}{2\delta}\left(e^{-i\delta t} - 1\right) \end{aligned}\right\} \implies \begin{aligned} \left|c_1\right|^2 &\approx 1 \\ \left|c_2\right|^2 &\approx \frac{\Omega_0^2}{2\delta^2}\left(1 - \cos\delta t\right) \end{aligned}$$

Repeating the process for second order (matching terms proportional to Ω_0^2),

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(1 + \Omega_0^2 c_1^{(2)}\right) &= -\frac{i}{2}\Omega_0 e^{i\delta t}\frac{\Omega_0}{2\delta}\left(e^{-i\delta} - 1\right) \\ \implies \qquad \dot{c}_1^{(2)} &= -\frac{i}{4\delta}\left(1 - e^{i\delta t}\right) \\ \implies \qquad c_1^{(2)} &= -\frac{i}{4\delta}\int_0^t \mathrm{d}t'\left(1 - e^{i\delta t}\right) \\ &= -\frac{i}{4\delta}\left[t - \frac{1}{i\delta}\left(e^{i\delta t} - 1\right)\right] \end{aligned}$$

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\Omega_0}{2\delta}\left(e^{-i\delta t} - 1\right) + \Omega_0^2 c_2^{(2)}\right) &= -\frac{i}{2}\Omega_0 e^{-i\delta t}\left(1 + \Omega_0^2 c_1^{(2)}\right) \\ \implies \qquad \dot{c}_2^{(2)} &= 0 \\ \implies \qquad c_2^{(2)} &= 0 \end{aligned}$$

So, to second order we have

$$\begin{aligned} c_1 &\approx 1 - \frac{i\Omega_0^2}{4\delta}\left[t - \frac{1}{i\delta}\left(e^{i\delta t} - 1\right)\right] \\ c_2 &\approx \frac{\Omega_0}{2\delta}\left(e^{-i\delta t} - 1\right) \end{aligned}$$

$$\implies \begin{aligned} \left|c_1\right|^2 &\approx 1 - \frac{\Omega_0^2}{2\delta}\left(1 - \cos\delta t\right) + \frac{\Omega_0^4}{8\delta^4}\left(1 + \frac{t^2}{2\delta^2} - \cos\delta t - 2t\sin\delta t\right) \\ \left|c_2\right|^2 &\approx \frac{\Omega_0^2}{2\delta^2}\left(1 - \cos\delta t\right) \end{aligned}$$

Problem 3

(a) We consider a two-level atom with states denoted $|1\rangle$, $|2\rangle$ and corresponding energies $E_1 = \hbar\omega_1 = -\omega_0/2$ and $E_2 = \hbar\omega_2 = \omega_0/2$. The atom interacts with a linearly polarized optical field described by

$$\vec{E} = \mathrm{Re}[E_0 e^{-i\omega t}]\ \hat{z}.$$

The interaction between the atom and the field is given to lowest order by the electric dipole interaction

$$V = -\vec{\mu}\cdot\vec{E} = ez|E_0|\cos(\omega t - \phi).$$

Assuming the two atomic states have opposite parity, the diagonal interaction matrix elements vanish:

$$V_{11} \propto \langle 1|z|1\rangle = 0 = V_{22}.$$

By a choice of phase for the wavefunction, we can take the off-diagonal elements to be real (and hence equal):

$$V_{12} = e|E_0|z_{12}\cos(\omega t - \phi)$$

The hamiltonian for the combined system is then

$$\begin{aligned} H &= H_0 + V \\ &= \frac{\hbar\omega_0}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \hbar\Omega_0\cos(\omega t - \phi)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= -\frac{\hbar\omega_0}{2}\sigma_z + \hbar\Omega_0\cos(\omega t - \phi)\sigma_x, \end{aligned}$$

where we define the Rabi frequency $\Omega_0 = \frac{ez_{12}E_0}{\hbar}$.

In the Schrödinger representation, the state of the atom is described by

$$|\psi(t)\rangle_{\mathrm{S}} = c_1(t)\,|1\rangle + c_2(t)\,|2\rangle \qquad \left(\left|c_1\right|^2 + \left|c_2\right|^2 = 1\right).$$

In the absence of the external field, the state would evolve as

$$|\psi(t)\rangle_{\mathrm{S}} = c_1(0)e^{-i\omega_1 t}\,|1\rangle + c_2(0)e^{-i\omega_2 t}\,|2\rangle\,.$$

In the interaction representation, we factor out this free phase evolution by writing

$$|\psi(t)\rangle_{\mathrm{I}} = \bar{c}_1(t)e^{-i\omega_1 t}\,|1\rangle + \bar{c}_2(t)e^{-i\omega_2 t}\,|2\rangle\,,$$

that is, we make the (time-dependent) unitary transformation

$$|\psi(t)\rangle_{\mathrm{S}} \to |\psi(t)\rangle_{\mathrm{I}} = U(t)\,|\psi(t)\rangle_{\mathrm{S}}\,,$$

where

$$U(t) = \begin{pmatrix} e^{-i\omega_1 t} & 0 \\ 0 & e^{-i\omega_2 t} \end{pmatrix}.$$

We get the effective interaction hamiltonian by making the inverse transformation on V:

$$V_{\mathrm{I}} = U^\dagger V U = \hbar\Omega_0\cos(\omega t)\begin{pmatrix} 0 & e^{-i\omega_0 t} \\ e^{i\omega_0 t} & 0 \end{pmatrix},$$

where the phase ϕ has been absorbed into the (complex) Rabi frequency.

(b) To make the rotating wave approximation, we re-write the effective hamiltonian as

$$\begin{aligned} V_{\mathrm{I}} &= \frac{1}{2}\hbar\Omega_0 \left(e^{i\omega t} + e^{-i\omega t}\right) e^{i\omega_0 t}\sigma_+ + \text{h.c.} \\ &= \frac{1}{2}\hbar\Omega_0 \left(e^{i(\omega+\omega_0)t} + e^{-i(\omega-\omega_0)t}\right)\sigma_+ + \text{h.c.} \end{aligned}$$

where σ_+ is the raising operator $|2\rangle\langle 1|$. Making the RWA amounts to neglecting the counter-rotating term (i.e. the term with $\omega_0 + \omega$), leaving

$$V_{\mathrm{I}} \approx \frac{1}{2}\hbar\Omega_0 e^{i\delta t}\sigma_+ + \text{h.c.},$$

where we have switched to using the detuning $\delta = \omega_0 - \omega$.

(c) The Bloch Siegert shift effectively decreases $\omega_2 - \omega_1 = \omega_0$, and therefore becomes relevant when the level spacing is already small, such as in magnetic field interactions.

Problem 4

Making the simplifying assumptions of a constant Rabi frequency and zero detuning, we have that

$$\left.\begin{aligned} \dot{c}_1 &= -\tfrac{i}{2}\Omega_0 c_2 - \tfrac{\gamma_1}{2}c_1 \\ \dot{c}_2 &= -\tfrac{i}{2}\Omega_0 c_1 - \tfrac{\gamma_2}{2} \end{aligned}\right\} \implies \begin{pmatrix}\dot{c}_1\\ \dot{c}_2\end{pmatrix} = -\frac{1}{2}\begin{pmatrix}\gamma_1 & i\Omega_0\\ i\Omega_0 & \gamma_2\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix},$$

or, more simply,

$$\dot{\psi} = M\psi; \qquad M = -\frac{1}{2}\begin{pmatrix}\gamma_1 & i\Omega_0\\ i\Omega_0 & \gamma_2\end{pmatrix}.$$

For $c_1(0) = 1,\ c_2(0) = 0$, the solution to this first order differential equation is

$$\begin{aligned} \psi(t) &= e^{Mt}\psi(0) \\ &= \frac{1}{2\chi}\begin{pmatrix} e^{-(\gamma_1+\gamma_2+\chi)t}\left[\chi\left(e^{\chi t/2}+1\right) + (\gamma_1+\gamma_2)\left(e^{\chi t/2}-1\right)\right] \\ -2ie^{-(\gamma_1+\gamma_2+\chi)/4}\left(e^{\chi t/2}-1\right)\end{pmatrix} \end{aligned}$$

Problem 1

In[ ]:=
```
P = 1 mW; A = 1 mm^2;

μ = e a0; UnitConvert[μ, "Debyes"]
```

Out[ ]=

2.541746473 D

In[ ]:=
$$E_0 = \sqrt{\frac{2P}{c\ \varepsilon_0\ A}};\ \text{UnitConvert}[E_0, \text{"V/m"}]$$

Out[ ]=

868.021098 V/m

In[ ]:=
$$\Omega_0 = \frac{\mu E_0}{\hbar};\ \text{UnitConvert}[\Omega_0, \text{"MHz"}]$$

Out[ ]=

69.7855727 MHz

Problem 2

In[9]:=

$$c_1 = 1 - \frac{i\,\Omega_0{}^2}{4\,\delta}\left(t - \frac{1}{i\,\delta}\left(e^{i\,\delta\,t} - 1\right)\right);$$

$c_1\,c_1{}^*$ // ComplexExpand // FullSimplify

Out[10]=

$$1 + \frac{8\,\delta^2\,(-1 + \text{Cos}[t\,\delta])\,\Omega_0^2 + \left(2 + t^2\,\delta^2 - 2\,\text{Cos}[t\,\delta] - 2\,t\,\delta\,\text{Sin}[t\,\delta]\right)\,\Omega_0^4}{16\,\delta^4}$$

Problem 3

In[23]:=
```
σx = PauliMatrix[1]; σz = PauliMatrix[3];
σ- = (PauliMatrix[1] + i PauliMatrix[2]) / 2;
σ+ = (PauliMatrix[1] - i PauliMatrix[2]) / 2;
```

In[80]:=
$$H_0 = -\frac{\hbar\,\omega_0}{2}\,\sigma_z;$$

$V = \hbar\Omega_0\,\text{Cos}[\omega\,t - \phi]\,\sigma_x;$

$U = \text{MatrixExp}[-i\,H_0\,t\,/\,\hbar];$

In[78]:=
```
V_I = U†.V.U // ComplexExpand // TrigToExp // FullSimplify;
V_I // MatrixForm
```

Out[79]//MatrixForm=

$$\begin{pmatrix} 0 & e^{-i\,t\,\omega_0}\,\mathrm{Cos}[\phi - t\,\omega]\,\hbar\Omega_0 \\ e^{i\,t\,\omega_0}\,\mathrm{Cos}[\phi - t\,\omega]\,\hbar\Omega_0 & 0 \end{pmatrix}$$