Homework 1

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Problem 1

Let

$$\Lambda^\mu_\nu = \delta^\mu_\nu + \delta\omega^\mu_\nu$$

Then, (keeping only terms linear in $\delta\omega$)

$$\begin{aligned} g_{\rho\sigma} &= g_{\mu\nu}\Lambda^\mu{}_\rho\Lambda^\nu{}_\sigma \\ &= g_{\mu\nu}\left(\delta^\mu_\rho + \delta\omega^\mu_\rho\right)\left(\delta^\mu_\rho + \delta\omega^\nu{}_\sigma\right) \\ &= g_{\mu\nu}\delta^\mu_\rho\delta^\mu_\rho + g_{\mu\nu}\delta^\mu_\rho\delta\omega^\nu{}_\sigma + g_{\mu\nu}\delta\omega^\mu_\rho\delta^\mu_\rho + O(\delta\omega^2) \\ &= g_{\rho\sigma} + \delta\omega_{\rho\sigma} + \delta\omega_{\sigma\rho} \end{aligned}$$

$$\implies 0 = \delta\omega_{\rho\sigma} + \delta\omega_{\sigma\rho}$$

$$\implies \boxed{\delta\omega_{\rho\sigma} = -\delta\omega_{\sigma\rho}}$$

Problem 2

Starting with

$$U(\Lambda)^{-1}U(\Lambda')U(\Lambda) = U(\Lambda^{-1}\Lambda'\Lambda)$$

we can expand Λ' to $\delta^\mu_\nu + \delta\omega'^\mu{}_\nu$. On the left hand side this gives

$$\begin{aligned} U(\Lambda)^{-1}U(\Lambda')U(\Lambda) &= U(\Lambda)^{-1}U(\delta^\mu_\nu + \delta\omega'^\mu{}_\nu)U(\Lambda) \\ &= U(\Lambda)^{-1}\left(\mathbb{I} + \frac{i}{2}\delta\omega'^\mu{}_\nu\right)U(\Lambda) \\ &= \mathbb{I} + \frac{i}{2}\delta\omega'{}_{\mu\nu}U(\Lambda)^{-1}M^{\mu\nu}U(\Lambda) \end{aligned}$$

On the right hand side we get

$$
\begin{aligned}
U(\Lambda^{-1}\Lambda'\Lambda) &= U((\Lambda^{-1})^{\mu}_{\ \alpha}\Lambda'^{\alpha}_{\ \beta}\Lambda^{\beta}_{\ \nu}) \\
&= U(\Lambda_{\alpha}^{\ \mu}(\delta^{\alpha}_{\beta} + \delta\omega'^{\alpha}_{\ \beta})\Lambda^{\beta}_{\ \nu}) \\
&= U(\delta^{\nu}_{\ \mu} + \Lambda_{\alpha}^{\ \mu}\Lambda^{\beta}_{\ \nu}\delta\omega'^{\alpha}_{\ \beta}) \\
&= U(\delta^{\nu}_{\ \mu} + \delta\tilde{\omega}^{\mu}_{\ \nu}) \\
&= \mathbb{I} + \frac{i}{2}\delta\tilde{\omega}_{\mu\nu}M^{\mu\nu}
\end{aligned}
$$

Where

$$\delta\tilde{\omega}^{\mu}_{\ \nu} = \Lambda_{\alpha}^{\ \mu}\Lambda^{\beta}_{\ \nu}\delta\omega'^{\alpha}_{\ \beta})$$

Now,

$$
\begin{aligned}
\delta\tilde{\omega}_{\mu\nu} &= g_{\mu\rho}\delta\tilde{\omega}^{\rho}_{\ \nu} \\
&= g_{\kappa\sigma}\Lambda^{\kappa}_{\ \mu}\Lambda^{\sigma}_{\ \rho}\Lambda_{\alpha}^{\ \rho}\Lambda^{\beta}_{\ \nu}\delta\omega'^{\alpha}_{\ \beta} \\
&= g_{\kappa\sigma}\Lambda^{\kappa}_{\ \mu}\delta^{\sigma}_{\ \alpha}\Lambda^{\beta}_{\ \nu}\delta\omega'^{\alpha}_{\ \beta} \\
&= g_{\kappa\sigma}\Lambda^{\kappa}_{\ \mu}\Lambda^{\beta}_{\ \nu}\delta\omega'^{\alpha}_{\ \beta} \\
&= \Lambda^{\kappa}_{\ \mu}\Lambda^{\beta}_{\ \nu}\delta\omega'_{\ \kappa\beta}
\end{aligned}
$$

So we have that

$$
\begin{aligned}
\mathbb{I} + \frac{i}{2}\delta\omega'_{\ \mu\nu}U(\Lambda)^{-1}M^{\mu\nu}U(\Lambda) &= \mathbb{I} + \frac{i}{2}\Lambda^{\kappa}_{\ \mu}\Lambda^{\beta}_{\ \nu}\delta\omega'_{\ \kappa\beta}M^{\mu\nu} \\
\implies \delta\omega'_{\ \mu\nu}U(\Lambda)^{-1}M^{\mu\nu}U(\Lambda) &= \delta\omega'_{\ \kappa\beta}\Lambda^{\kappa}_{\ \mu}\Lambda^{\beta}_{\ \nu}M^{\mu\nu} \\
\implies \delta\omega'_{\ \mu\nu}U(\Lambda)^{-1}M^{\mu\nu}U(\Lambda) &= \delta\omega'_{\ \mu\nu}\Lambda^{\mu}_{\ \kappa}\Lambda^{\nu}_{\ \beta}M^{\kappa\beta}
\end{aligned}
$$

Now we can equate the antisymmetric parts on both sides. Since $M^{\alpha\beta}$ is already antisymmetric, we can simply cancel $\delta\omega'_{\ \mu\nu}$, giving (after swapping out the contracted indicies to match the desired form):

$$\boxed{U(\Lambda)^{-1}M^{\mu\nu}U(\Lambda) = \Lambda^{\mu}_{\ \rho}\Lambda^{\nu}_{\ \sigma}M^{\rho\sigma}}$$

Problem 3

Starting with the solution from Problem 2, we can again expand Λ. On the left-hand side this gives

$$
\begin{aligned}
U(\Lambda)^{-1}M^{\mu\nu}U(\Lambda) &= U(\delta^{\rho}_{\sigma} + \delta\omega^{\rho}_{\ \sigma})^{-1}M^{\mu\nu}U(\delta^{\alpha}_{\beta} + \delta\omega^{\alpha}_{\ \beta}) \\
&= \left(\mathbb{I} - \frac{i}{2}\delta\omega_{\rho\sigma}M^{\rho\sigma}\right)M^{\mu\nu}\left(\mathbb{I} + \frac{i}{2}\delta\omega_{\alpha\beta}M^{\alpha\beta}\right) \\
&= M^{\mu\nu} - \frac{i}{2}\delta\omega_{\rho\sigma}M^{\rho\sigma}M^{\mu\nu} + \frac{i}{2}\delta\omega_{\alpha\beta}M^{\mu\nu}M^{\alpha\beta} + O(\delta\omega^2)
\end{aligned}
$$

The right-hand side gives

$$\begin{aligned}\Lambda^{\mu}_{\ \rho}\Lambda^{\nu}_{\ \sigma}M^{\rho\sigma} &= (\delta^{\mu}_{\ \rho}+\delta\omega^{\mu}_{\ \rho})(\delta^{\nu}_{\ \sigma}+\delta\omega^{\nu}_{\ \sigma})M^{\rho\sigma} \\ &= M^{\mu\nu}+\delta\omega^{\nu}_{\ \sigma}M^{\mu\sigma}+\delta\omega^{\mu}_{\ \rho}M^{\rho\nu}+O(\delta\omega^2)\end{aligned}$$

Re-equating the two sides, canceling the first term in each, and swapping some contracted indicies gives

$$\frac{i}{2}\left(\delta\omega_{\alpha\beta}M^{\mu\nu}M^{\alpha\beta}-\delta\omega_{\alpha\beta}M^{\alpha\beta}M^{\mu\nu}\right)=\delta\omega^{\nu}_{\ \alpha}M^{\mu\alpha}+\delta\omega^{\mu}_{\ \alpha}M^{\alpha\nu}$$

To get the index locations on the right to agree with the left, we can toss on some metrics:

$$\begin{aligned}\delta\omega^{\nu}_{\ \alpha}M^{\mu\alpha}+\delta\omega^{\mu}_{\ \alpha}M^{\alpha\nu} &= g^{\nu\beta}\delta\omega_{\beta\alpha}M^{\mu\alpha}+g^{\mu\alpha}\delta\omega_{\beta\alpha}M^{\beta\nu} \\ &= -g^{\nu\beta}\delta\omega_{\alpha\beta}M^{\mu\alpha}+g^{\mu\beta}\delta\omega_{\alpha\beta}M^{\alpha\nu}\end{aligned}$$

In the first term of the second line above, the antisymmetry of $\delta\omega$ was used to swap its indicies. In the second term, the contracted indicies were simply relabeled. The full expression is now

$$\begin{aligned}\delta\omega_{\alpha\beta}M^{\mu\nu}M^{\alpha\beta}-\delta\omega_{\alpha\beta}M^{\alpha\beta}M^{\mu\nu} &= -2i\left(-g^{\nu\beta}\delta\omega_{\alpha\beta}M^{\mu\alpha}+g^{\mu\beta}\delta\omega_{\alpha\beta}M^{\alpha\nu}\right) \\ &= -i\left(-g^{\nu\beta}\delta\omega_{\alpha\beta}M^{\mu\alpha}+g^{\mu\beta}\delta\omega_{\alpha\beta}M^{\alpha\nu}+g^{\nu\alpha}\delta\omega_{\alpha\beta}M^{\mu\beta}-g^{\mu\beta}\delta\omega_{\alpha\beta}M^{\alpha\nu}\right)\end{aligned}$$

Just as in Problem 2, the antisymmetry of $M^{\alpha\beta}$ allows us to cancel all the $\delta\omega_{\alpha\beta}$. Swapping contracted indicies again to match the desired result, we have

$$\begin{aligned}M^{\mu\nu}M^{\rho\sigma}-M^{\rho\sigma}M^{\mu\nu} &= i\left(g^{\nu\beta}M^{\mu\alpha}-g^{\mu\beta}M^{\alpha\nu}-g^{\nu\alpha}M^{\mu\beta}+g^{\mu\beta}M^{\alpha\nu}\right) \\ \Longrightarrow &\ \boxed{[M^{\mu\nu},M^{\rho\sigma}]=(g^{\mu\rho}M^{\nu\sigma}-(\mu\leftrightarrow\nu))-(\rho\leftrightarrow\sigma)}\end{aligned}$$

Problem 4

To show that

$$[J_i,J_j]=i\varepsilon_{ijk}J_k$$

we can start by expressing the Js in terms of the Ms:

$$\begin{aligned}J_1 &= -\frac{1}{2}\varepsilon_{1jk}M^{jk} = -\frac{1}{2}\left(M^{23}-M^{32}\right)=M^{32} \\ J_2 &= -\frac{1}{2}\varepsilon_{2jk}M^{jk} = -\frac{1}{2}\left(M^{31}-M^{13}\right)=M^{13} \\ J_3 &= -\frac{1}{2}\varepsilon_{3jk}M^{jk} = -\frac{1}{2}\left(M^{12}-M^{21}\right)=M^{21}\end{aligned}$$

Now, using the result from Problem 3, we have that

$$\begin{aligned}[J_1,J_2] &= \left[M^{32},M^{13}\right] \\ &= i\left(g^{31}M^{23}-g^{21}M^{33}-g^{33}M^{21}+g^{23}M^{31}\right) \\ &= iM^{21} \\ &= iJ_3\end{aligned}$$

Taking cyclic permutations of the above result gives

$$\boxed{[J_i, J_j] = i\varepsilon_{ijk} J_k}$$

For the next identity, consider

$$\begin{aligned}
[J_1, K_2] &= \left[M^{32}, M^{20}\right] \\
&= i\left(g^{32}M^{20} - g^{22}M^{30} - g^{30}M^22 + g^{20}M^{32}\right) \\
&= iM^{30} \\
&= iK_3
\end{aligned}$$

Again, taking cyclic permutations of the above result gives

$$\boxed{[J_i, K_j] = i\varepsilon_{ijk} K_k}$$

Finally,

$$\begin{aligned}
[K_i, K_j] &= \left[M^{10}, M^{20}\right] \\
&= i\left(g^{12}M^{00} - g^{02}M^{10} - g^{10}M^{02} + g^{00}M^{12}\right) \\
&= iM^{12} \\
&= -iJ_3
\end{aligned}$$

taking cyclic permutations of the above result then gives

$$\boxed{[K_i, K_j] = -i\varepsilon_{ijk} J_k}$$

Problem 5

(a) Starting with

$$U(\Lambda)^{-1}\partial^\mu \phi(x) U(\Lambda) = \Lambda^\mu{}_\rho \bar{\partial}^\mu \phi(\Lambda^{-1}x)$$

we can expand both sides using $\Lambda = 1 + \delta\omega$. Starting with the left-hand side this gives

$$\begin{aligned}
U(\Lambda)^{-1}\partial^\mu \phi(x) U(\Lambda) &= \partial^\mu U(\Lambda)^{-1}\phi(x)U(\Lambda) \\
&= \partial^\mu \left(\mathbb{I} - \delta\omega_{\alpha\beta}M^{\alpha\beta}\right)\phi(x)\left(\mathbb{I} + \delta\omega_{\alpha\beta}M^{\alpha\beta}\right) \\
&= \partial^\mu \left[\phi(x) + \frac{i}{2}\delta\omega_{\alpha\beta}\left(\phi(x)M^{\alpha\beta} - M^{\alpha\beta}\phi(x)\right)\right] \\
&= \partial^\mu \left[\phi(x) + \frac{i}{2}\left[\phi(x), M^{\alpha\beta}\right]\right] \\
&= \partial^\mu \phi(x) + \frac{i}{2}\left[\partial^\mu \phi(x), M^{\alpha\beta}\right]
\end{aligned}$$

while the right-hand side gives

$$\begin{aligned}
\Lambda^{\mu}{}_{\rho}\bar{\partial}^{\mu}\phi(\Lambda^{-1}x) &= \partial^{\mu}\phi(\Lambda^{-1}x) \\
&= \partial^{\mu}\phi((1-\delta\omega)x) \\
&= \partial^{\mu}\phi(x - x\delta\omega) \\
&= \partial^{\mu}\left(\phi(x) - \delta\omega_{\mu\nu}x^{\nu}\partial^{\mu}\phi(x)\right)
\end{aligned}$$

Canceling the first terms, we now have that

$$\frac{i}{2}\delta\omega_{\alpha\beta}\left[\partial^{\mu}\phi(x), M^{\alpha\beta}\right] = \partial^{\mu}\left(\delta\omega_{\alpha\beta}x^{\beta}\delta^{\alpha}\phi(x)\right)$$

Rearranging terms, we have that

$$\begin{aligned}
\delta\omega_{\alpha\beta}\left[\partial^{\mu}\phi(x), M^{\alpha\beta}\right] &= -2i\partial^{\mu}\left(\delta\omega_{\alpha\beta}x^{\beta}\delta^{\alpha}\phi(x)\right) \\
&= -i\partial^{\mu}\left(\delta\omega_{\alpha\beta}x^{\beta}\delta^{\alpha}\phi(x) + \delta\omega_{\beta\alpha}x^{\alpha}\partial^{\beta}\phi(x)\right) \\
&= -i\partial^{\mu}\left(\delta\omega_{\alpha\beta}x^{\beta}\partial^{\alpha}\phi(x) - \delta\omega_{\alpha\beta}x^{\alpha}\partial^{\beta}\phi(x)\right) \\
&= -i\partial^{\mu}\delta\omega_{\alpha\beta}\left(x^{\beta}\partial^{\alpha}\phi(x) - x^{\alpha}\partial^{\beta}\phi(x)\right) \\
&= \delta\omega_{\alpha\beta}\partial^{\mu}\mathcal{L}^{\alpha\beta}\phi(x)
\end{aligned}$$

Now we can cancel the $\delta\omega_{\alpha\beta}$, giving

$$\left[\partial^{\mu}\phi(x), M^{\alpha\beta}\right] = \partial^{\mu}\mathcal{L}^{\alpha\beta}\phi(x)$$

We can now use the chain rule to expand the right-hand side further

$$\begin{aligned}
\partial^{\mu}\mathcal{L}^{\alpha}\beta\phi(x) &= \mathcal{L}^{\alpha}\beta\partial^{\mu}\phi(x) + \left(\partial^{\mu}\mathcal{L}^{\alpha\beta}\right)\phi(x) \\
&= \mathcal{L}^{\alpha\beta}\partial^{\mu}\phi(x) - i\partial^{\mu}\left(x^{\alpha}\partial^{\beta} - x^{\beta}\partial^{\alpha}\right)\phi(x) \\
&= \mathcal{L}^{\alpha\beta}\partial^{\mu}\phi(x) - i\left(g^{\mu\alpha}\partial^{\beta} - g^{\mu\beta}\partial^{\alpha}\right)\phi(x) \\
&= \mathcal{L}^{\alpha\beta}\partial^{\mu}\phi(x) - i\left(g^{\alpha\mu}\delta^{\beta}_{\tau} - g^{\beta\mu}\delta^{\alpha}_{\tau}\right)\partial^{\tau}\phi(x) \\
&= \mathcal{L}^{\alpha\beta}\partial^{\mu}\phi(x) + \left(S_V^{\alpha\beta}\right)^{\mu}{}_{\tau}\partial^{\tau}\phi(x)
\end{aligned}$$

which finally gives the desired result

$$\boxed{\left[\partial^{\mu}\phi(x), M^{\alpha\beta}\right] = \mathcal{L}^{\alpha}\beta\partial^{\mu}\phi(x) + \left(S_V^{\alpha\beta}\right)^{\mu}{}_{\tau}\partial^{\tau}\phi(x)}$$

(c) To begin, let's right out S_V^{12} as a matrix:

$$\begin{aligned}
S_V^{12} &= -i\left(g^{1\rho}\delta^2_{\ \tau} - g^{2\rho}\delta^1_{\ \tau}\right) \\
&= -i\left[\begin{pmatrix}0\\-1\\0\\0\end{pmatrix}\begin{pmatrix}0 & 0 & 1 & 0\end{pmatrix} - \begin{pmatrix}0\\0\\-1\\0\end{pmatrix}\begin{pmatrix}0 & 1 & 0 & 0\end{pmatrix}\right] \\
&= -i\left[\begin{pmatrix}0 & 0 & 0 & 0\\0 & 0 & -1 & 0\\0 & 0 & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix} - \begin{pmatrix}0 & 0 & 0 & 0\\0 & 0 & 0 & 0\\0 & -1 & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix}\right] \\
&= \begin{pmatrix}0 & 0 & 0 & 0\\0 & 0 & i & 0\\0 & -i & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix}
\end{aligned}$$

Now let $A = i\,(S_V^{12})$. Powers of A are given by

$$\begin{aligned}
A^1 &= \begin{pmatrix}0 & 0 & 0 & 0\\0 & 0 & -1 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix} \\
A^2 &= \begin{pmatrix}0 & 0 & 0 & 0\\0 & -1 & 0 & 0\\0 & 0 & -1 & 0\\0 & 0 & 0 & 0\end{pmatrix} \\
A^3 &= \begin{pmatrix}0 & 0 & 0 & 0\\0 & 0 & 1 & 0\\0 & -1 & 0 & 0\\0 & 0 & 0 & 0\end{pmatrix} \\
&= -A
\end{aligned}$$

Now we can expand $\exp(-i\theta S_V^{12})$ as follows:

$$\begin{aligned}
\exp\!\left(-i\theta S_V^{12}\right) &= e^{-\theta A} \\
&= \mathbb{I} - \theta A + \frac{1}{2!}\theta^2 A^2 - \frac{1}{3!}\theta^3 A^3 + \frac{1}{4!}\theta^4 A^4 - \frac{1}{5!}\theta^5 A^5 + \frac{1}{6!}\theta^6 A^6 + \cdots \\
&= \mathbb{I} - \theta A + \frac{1}{2!}\theta^2 A^2 + \frac{1}{3!}\theta^3 A - \frac{1}{4!}\theta^4 A^2 - \frac{1}{5!}\theta^5 A + \frac{1}{6!}\theta^6 A^2 + \cdots \\
&= \mathbb{I} + \left(-\theta + \frac{1}{3!}\theta^3 - \frac{1}{5!}\theta^5 + \cdots\right) A + \left(\frac{1}{2!}\theta^2 - \frac{1}{4!}\theta^4 + \frac{1}{6!}\theta^6 + \cdots\right) A^2 \\
&= \mathbb{I} - (\sin\theta)A + (1 - \cos\theta)A^2
\end{aligned}$$

Plugging in the above matrix expressions for A and A^2 gives the desired result:

$$\boxed{\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}}$$

(d) As in part c, we can begin by writing out S_V^{30} as a matrix:

$$\begin{aligned} S_V^{30} &= -i\left(g^{3\rho}\delta^0_{\ \tau} - g^{0\rho}\delta^3_{\ \tau}\right) \\ &= -i\left[\begin{pmatrix} 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}\right] \\ &= -i\left[\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right] \\ &= \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

Now, letting $A = i\,(S_V^{30})$, powers of A are given by

$$\begin{aligned} A &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \\ A^2 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ A^3 &= \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \\ &= A \end{aligned}$$

Now we can expand $\exp(i\eta S_V^{30})$ as follows:

$$\begin{aligned}
\exp\left(i\eta S_V^{30}\right) &= e^{\eta A} \\
&= \mathbb{I} + \eta A + \frac{1}{2!}\eta^2 A^2 + \frac{1}{3!}\eta^3 A^3 + \frac{1}{4!}\eta^4 A^4 + \frac{1}{5!}\eta^5 A^5 + \cdots \\
&= \mathbb{I} + \eta A + \frac{1}{2!}\eta^2 A^2 + \frac{1}{3!}\eta^3 A + \frac{1}{4!}\eta^4 A^2 + \frac{1}{5!}\eta^5 A + \cdots \\
&= \mathbb{I} + \left(\eta + \frac{1}{3!}\eta^3 + \frac{1}{5!}\eta^5 + \cdots\right) A + \left(\frac{1}{2!}\eta^2 + \frac{1}{4!}\eta^4 + \cdots\right) A^2 \\
&= \mathbb{I} + (\sinh\eta) A + (\cosh\eta - 1) A^2
\end{aligned}$$

Plugging in the above matrix expressions for A and A^2 gives the desired result:

$$\boxed{\Lambda = \begin{pmatrix} \cosh\eta & 0 & 0 & \sinh\eta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \sinh\eta & 0 & 0 & \cosh\eta \end{pmatrix}}$$