Homework 4

Sean Ericson
Phys 632

February 1, 2022

Problem 1

Firstly,

$$[A, J_x] = [A, J_y] = 0 \implies [A, J_+] = 0$$

Therefore,

$$[A, J_+J_-] = 0$$
$$\implies \left[A, J_x^2 + J_y^2 - \hbar J_z\right] = \hbar[A, J_z] = 0$$

Problem 2

(a) Consider $[J^2, J_\alpha^2]$. We know every individual component commutes with J^2, so

$$\left[J^2, J_\alpha^2\right] = 0 \implies \left[J_x^2, J_\alpha^2\right] + \left[J_y^2, J_\alpha^2\right] + \left[J_z^2, J_\alpha^2\right] = 0$$

When we substitute one of $\{x, y, z\}$ for α, one of the terms above will disappear and another will be "out of order". For example, $\alpha = x$ gives

$$\left[J_y^2, J_x^2\right] + \left[J_z^2, J_x^2\right] = 0 \implies \left[J_z^2, J_x^2\right] = \left[J_x^2, J_y^2\right]$$

Substituting the remaining values of α gives other two requisite equations.

(b) First let

$$J_\pm^2 \left|j, m\right\rangle = c_{j,m}^\pm \left|j, m \pm 2\right\rangle.$$

Then, for $j = 1$,

$$J_z^2(J_+^2 + J_-^2) \left|1, 1\right\rangle = J_z^2 J_-^2 \left|1, 1\right\rangle = \hbar^2 c_{1,1}^- \left|1, -1\right\rangle$$
$$(J_+^2 + J_-^2) J_z^2 \left|1, 1\right\rangle = \hbar^2 J_-^2 \left|1, 1\right\rangle = \hbar^2 c_{1,1}^- \left|1, -1\right\rangle$$
$$J_z^2(J_+^2 + J_-^2) \left|1, -1\right\rangle = J_z^2 J_+^2 \left|1, -1\right\rangle = \hbar^2 c_{1,-1}^+ \left|1, 1\right\rangle$$
$$(J_+^2 + J_-^2) J_z^2 \left|1, -1\right\rangle = \hbar^2 J_+^2 \left|1, -1\right\rangle = \hbar^2 c_{1,-1}^+ \left|1, 1\right\rangle$$
$$J_z^2(J_+^2 + J_-^2) \left|1, 0\right\rangle = (J_+^2 + J_-^2) J_z^2 \left|1, 0\right\rangle = 0$$

Obviously for $j = 1/2$ and $j = 0$ the operator vanishishes similarly to the $|1,0\rangle$ case. Therefore it is the case that, for $j \in \{0, \frac{1}{2}, 1\}$,

$$\left[J_z^2, J_+^2 + J_-^2\right] = 0$$

Now, since

$$J_+^2 + J_-^2 = 2J_x^2 - 2J_y^2 = 2J^2 - 2J_z^2 - 4J_y^2$$

we see that

$$\left[J_z^2, J_+^2 + J_-^2\right] = 0 \implies \left[J_z^2, 2J^2 - 2J_z^2 - 4J_y^2\right] = -4\left[J_z^2, J_y^2\right] = 0$$

This combined with the result from part (a) give the desired result.

Problem 3

In the case that $l = 1/2$, we *should* have that

$$\Theta_{1/2}^{-1/2}(\theta) = \Theta_{1/2}^{1/2}(\theta) \propto \sqrt{\sin\theta}$$

It should also be the case that

$$L_+\Theta_{1/2}^{-1/2}(\theta)e^{-i\phi/2} \propto \Theta_{1/2}^{1/2}(\theta)e^{i\phi/2}$$

However, applying L_+ to $\Theta_{1/2}^{-1/2}(\theta)e^{-i\phi/2}$, we see that

$$\begin{aligned} L_+\Theta_{1/2}^{-1/2}(\theta)e^{-i\phi/2} &= \hbar e^{i\phi/2}\left(i\cot\theta\partial_\phi + \partial_\theta\right)\sqrt{\sin\theta}e^{-i\phi/2} \\ &= i\hbar e^{i\phi/2}\cot\theta\partial_\phi\sqrt{\sin\theta}e^{-i\phi/2} + \hbar e^{i\phi/2}\partial_\theta\sqrt{\sin\theta}e^{-i\phi/2} \\ &= \frac{\hbar}{2}e^{i\phi/2}\cot\theta\sqrt{\sin\theta}e^{-i\phi/2} + \frac{\hbar}{2}e^{i\phi/2}\cos\theta\sqrt{\sin\theta}e^{-i\phi/2} \\ &= \frac{\hbar}{2}e^{i\phi/2}\frac{\cos\theta}{\sqrt{\sin\theta}}e^{-i\phi/2} \end{aligned}$$

Which is *not* proportional to $\sqrt{\sin\theta}$.

Problem 4

$$|11\rangle = \sqrt{\frac{3}{7}}\,|2,2;2-1\rangle - \sqrt{\frac{1}{14}}\,|2,2;1,0\rangle - \sqrt{\frac{1}{14}}\,|2,2;1,0\rangle + \sqrt{\frac{3}{7}}\,|2,2;-1,2\rangle$$

$$\implies P(m_1 = 0 \text{ OR } m_2 = 0) = 2 \times \frac{1}{14} = \frac{1}{7}$$

Problem 5

Let $\alpha = \langle 1,0;j,0|j,0\rangle$. The symmetry relation

$$\langle j_1, m_1; j_2, m_2|j_3, m_3\rangle = (-1)^{j_1+j_2-j_3}\,\langle j_1, -m_1; j_2 - m_2|j_3, -m_3\rangle$$

implies that $\alpha = -\alpha$, therefore it must be that $\alpha = 0$.