# Homework 8

Sean Ericson
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## Problem 1

(a) The Hamiltonian for the Lambda system in the Schrödinger representation is given by

$$\begin{aligned}
H &= H_0 + V \\
&= -\hbar \begin{pmatrix} \omega_0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \omega_0' \end{pmatrix} + \frac{\hbar}{2} \begin{pmatrix} 0 & \Omega_0^* e^{i\omega t} & 0 \\ \Omega_0 e^{-i\omega t} & 0 & \Omega_0' e^{-i\omega' t} \\ 0 & \Omega_0'^* e^{i\omega' t} & 0 \end{pmatrix} \\
&= \hbar \begin{pmatrix} -\omega_0 & \Omega_0^* e^{i\omega t} & 0 \\ \Omega_0 e^{-i\omega t} & 0 & \Omega_0' e^{-i\omega' t} \\ 0 & \Omega_0'^* e^{i\omega' t} & -\omega_0' \end{pmatrix}.
\end{aligned}$$

Time evolution follows from the Schrödinger equation

$$\begin{aligned}
&\partial_t \left|\psi\right\rangle = -\frac{i}{\hbar} H \left|\psi\right\rangle \\
&\implies \begin{cases} \dot{c}_1 = & i\omega_0 c_1 - \frac{i}{2}\Omega_0^* e^{i\omega t} \\ \dot{c}_2 = & -\frac{i}{2}\Omega_0 e^{-i\omega t} c_1 - \frac{i}{2}\Omega_0' e^{i\omega' t} c_3 \\ \dot{c}_3 = & -\frac{i}{2}\Omega_0'^* e^{-i\omega' t} c_2 i\omega_0' c_3 \end{cases}
\end{aligned}$$

To go to the field interaction representation, we factor out the applied field via

$$c_1 \to \tilde{c}_1 = c_1 e^{-i\omega t}; \quad c_2 \to \tilde{c}_2 = c_2; \quad c_3 \to \tilde{c}_3 = c_3 e^{-i\omega' t}.$$

Then,

$$
\begin{aligned}
\dot{\tilde{c}}_1 &= \left(-i\omega c_1 + \dot{c}_1\right) e^{-i\omega t} \\
&= \left(-i\omega c_1 + i\omega_0 c_1 - \frac{i}{2}\Omega_0^* e^{i\omega t}\right) e^{-i\omega t} \\
&= i\delta \tilde{c}_1 - \frac{i}{2}\Omega_0^* \tilde{c}_2 \\
\dot{\tilde{c}}_2 &= -\frac{i}{2}\left(\Omega_0 \tilde{c}_1 + \Omega_0' \tilde{c}_3\right) \\
\dot{\tilde{c}}_3 &= \left(i\omega' c_3 + \dot{c}_3\right) e^{i\omega' t} \\
&= \left(-i\omega' c_3 + -\frac{i}{2}\Omega_0'^* e^{-i\omega' t} c_2 + i\omega_0' c_3\right) e^{i\omega' t} \\
&= -\frac{i}{2}\Omega_0'^* \tilde{c}_2 + i\delta' \tilde{c}_3.
\end{aligned}
$$

Putting this together, we have

$$
\begin{aligned}
\partial_t |\tilde{\psi}\rangle &= i \begin{pmatrix} \delta \tilde{c}_1 - \frac{1}{2}\Omega_0^* \tilde{c}_2 \\ -\frac{1}{2}\Omega_0 \tilde{c}_1 - \frac{1}{2}\Omega_0' \tilde{c}_3 \\ -\frac{1}{2}\Omega_0'^* \tilde{c}_2 + \delta' \tilde{c}_3 \end{pmatrix} \\
&= -\frac{i}{\hbar}\tilde{H}|\tilde{\psi}\rangle \\
\implies \qquad \tilde{H} &= \hbar \begin{pmatrix} -2\delta & \frac{\Omega_0^*}{2} & 0 \\ \frac{\Omega_0}{2} & 0 & \frac{\Omega_0'}{2} \\ 0 & \frac{\Omega_0'^*}{2} & -2\delta' \end{pmatrix}.
\end{aligned}
$$

(b) Given that

$$
\begin{aligned}
|D\rangle &= \frac{1}{\Omega}\left(\Omega_0' |1\rangle + \Omega_0 |3\rangle\right), \\
|B\rangle &= \frac{1}{\Omega}\left(\Omega_0^* |1\rangle - \Omega_0'^* |3\rangle\right),
\end{aligned}
$$

where $\Omega = \sqrt{|\Omega_0|^2 + |\Omega_0'|^2}$. The transformation between these bases is then (dropping tildes)

$$
\begin{pmatrix} c_d \\ c_2 \\ c_B \end{pmatrix} = \begin{pmatrix} \frac{\Omega_0'}{\Omega} & 0 & \frac{\Omega_0}{\Omega} \\ 0 & 1 & 0 \\ \frac{\Omega_0^*}{\Omega} & 0 & -\Omega_0'^* \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} =: U \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}.
$$

Now we can transform the Hamiltonian into this basis (assuming real Rabi frequencies):

$$
\begin{aligned}
H_{D2B} &= UHU^\dagger \\
&= \hbar \begin{pmatrix} -2\delta & \frac{\Omega_0 \Omega_0'}{\Omega^2} \end{pmatrix}
\end{aligned}
$$

(c)

## Problem 2

## Problem 3

## Problem 4

## Problem 5 (Berman 9.1-2)

In[1]:=

```
Symbolize[ω0]; Symbolize[ω0']; Symbolize[ω']; Symbolize[δ']; Symbolize[Ω0]; Sy
Symbolize[c1]; Symbolize[c2]; Symbolize[c3];
```

In[3]:=

```
$Assumptions = {t > 0, ω ∈ ℝ, ω' ∈ ℝ, ω0 ∈ ℝ, ω0' ∈ ℝ, Ω0 ∈ ℝ, Ω0' ∈ ℝ, γ2,1 > 0, γ3,1 > 0, δ ∈
```

In[4]:=

```
comm[a_, b_] := a.b - b.a;
```

---

# Problem 1

In[6]:=

```
c = Sqrt[Abs[Ω0]^2 + Abs[Ω0']^2];
```

In[9]:=

```
U = 1/c {{Ω0', 0, Ω0}, {0, 1, 0}, {Ω0*, 0, -Ω0'*}}; U // MatrixForm // TraditionalForm
H = ℏ {{-2 δ, Ω0*/2, 0}, {Ω0/2, 0, Ω0'/2}, {0, Ω0'*/2, -2 δ'}}; H // MatrixForm // TraditionalFo
```

Out[9]//TraditionalForm=

$$\begin{pmatrix} \frac{\Omega_0'}{\sqrt{|\Omega_0'|^2+|\Omega_0|^2}} & 0 & \frac{\Omega_0}{\sqrt{|\Omega_0'|^2+|\Omega_0|^2}} \\ 0 & \frac{1}{\sqrt{|\Omega_0'|^2+|\Omega_0|^2}} & 0 \\ \frac{(\Omega_0)^*}{\sqrt{|\Omega_0'|^2+|\Omega_0|^2}} & 0 & -\frac{(\Omega_0')^*}{\sqrt{|\Omega_0'|^2+|\Omega_0|^2}} \end{pmatrix}$$

Out[10]//TraditionalForm=

$$\begin{pmatrix} -2\,\delta\,\hbar & \frac{1}{2}\,\hbar\,(\Omega_0)^* & 0 \\ \frac{\Omega_0\,\hbar}{2} & 0 & \frac{\hbar\,\Omega_0'}{2} \\ 0 & \frac{1}{2}\,\hbar\,(\Omega_0')^* & -2\,\hbar\,\delta' \end{pmatrix}$$

In[11]:=

```
U.H.U† /. {δ' → δ} // FullSimplify // MatrixForm // TraditionalForm
```

Out[11]//TraditionalForm=

$$\begin{pmatrix} -2\,\delta\,\hbar & \frac{\Omega_0\,\hbar\,\Omega_0'}{(\Omega_0')^2+\Omega_0^2} & 0 \\ \frac{\Omega_0\,\hbar\,\Omega_0'}{(\Omega_0')^2+\Omega_0^2} & 0 & \frac{\Omega_0^2\,\hbar}{(\Omega_0')^2+\Omega_0^2}-\frac{\hbar}{2} \\ 0 & \frac{\Omega_0^2\,\hbar}{(\Omega_0')^2+\Omega_0^2}-\frac{\hbar}{2} & -2\,\delta\,\hbar \end{pmatrix}$$

# Problem 2

In[13]:=
```
rho = Table[Subscript[ρ, i, j], {i, 3}, {j, 3}]; rho // MatrixForm
```

Out[13]//MatrixForm=

$$\begin{pmatrix} \rho_{1,1} & \rho_{1,2} & \rho_{1,3} \\ \rho_{2,1} & \rho_{2,2} & \rho_{2,3} \\ \rho_{3,1} & \rho_{3,2} & \rho_{3,3} \end{pmatrix}$$

In[14]:=
$$\text{rhoDot} = \frac{-i}{\hbar}\ \text{comm[H, rho] // FullSimplify; rhoDot // MatrixForm}$$

Out[14]//MatrixForm=

$$\begin{pmatrix} \frac{1}{2} i\, \Omega_0 (\rho_{1,2} - \rho_{2,1}) & \frac{1}{2} i\, (4\,\delta\,\rho_{1,2} + \Omega_0 (\rho_{1,1} - \rho_{2,2}) + \rho_{1,3}\,\Omega_0') & \frac{1}{2} i\, (-\Omega_0\,\rho_{2,3} + \\ -\frac{1}{2} i\, (4\,\delta\,\rho_{2,1} + \Omega_0 (\rho_{1,1} - \rho_{2,2}) + \rho_{3,1}\,\Omega_0') & \frac{1}{2} i\, (\Omega_0 (-\rho_{1,2} + \rho_{2,1}) + (\rho_{2,3} - \rho_{3,2})\,\Omega_0') & -\frac{1}{2} i\, (\Omega_0\,\rho_{1,3} + 4 \\ \frac{1}{2} i\, (\Omega_0\,\rho_{3,2} + 4\,\rho_{3,1} (-\delta + \delta') - \rho_{2,1}\,\Omega_0') & \frac{1}{2} i\, (\Omega_0\,\rho_{3,1} + 4\,\rho_{3,2}\,\delta' + (-\rho_{2,2} + \rho_{3,3})\,\Omega_0') & -\frac{1}{2} i \end{pmatrix}$$

In[18]:=
```
rhoDot[[1, 2]] // TraditionalForm
rhoDot[[1, 3]] // TraditionalForm
rhoDot[[2, 3]] // TraditionalForm
```

Out[18]//TraditionalForm=

$$\frac{1}{2} i\, (4\,\delta\,\rho_{1,2} + \rho_{1,3}\,\Omega_0' + \Omega_0 (\rho_{1,1} - \rho_{2,2}))$$

Out[19]//TraditionalForm=

$$\frac{1}{2} i\, (4\,\rho_{1,3} (\delta - \delta') + \rho_{1,2}\,\Omega_0' - \Omega_0\,\rho_{2,3})$$

Out[20]//TraditionalForm=

$$-\frac{1}{2} i\, (4\,\rho_{2,3}\,\delta' + (\rho_{3,3} - \rho_{2,2})\,\Omega_0' + \Omega_0\,\rho_{1,3})$$

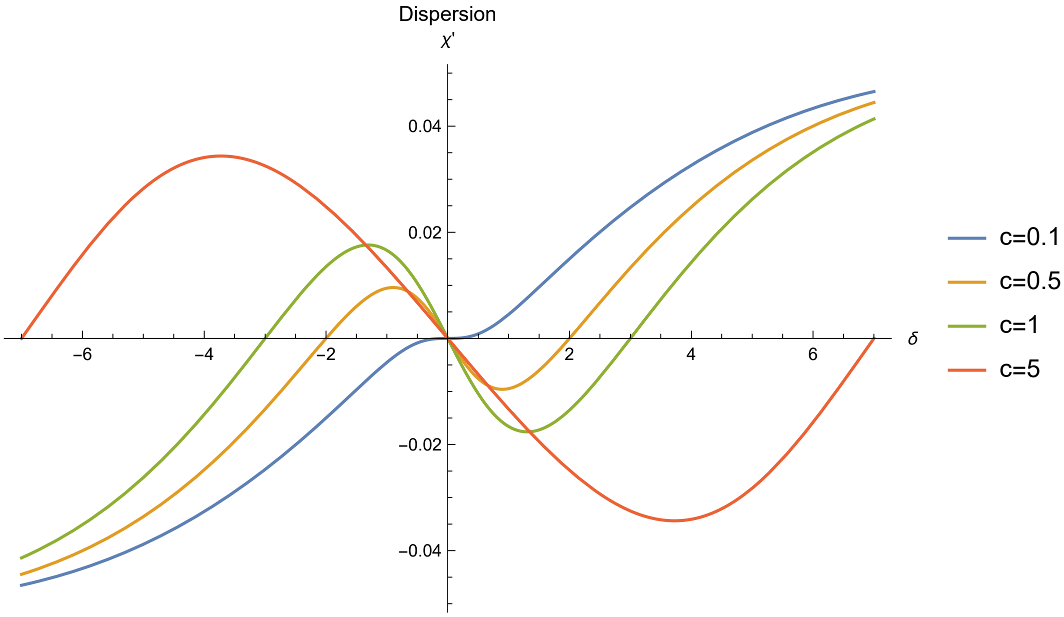
# Problem 3

In[21]:=
```
Clear[c];
```

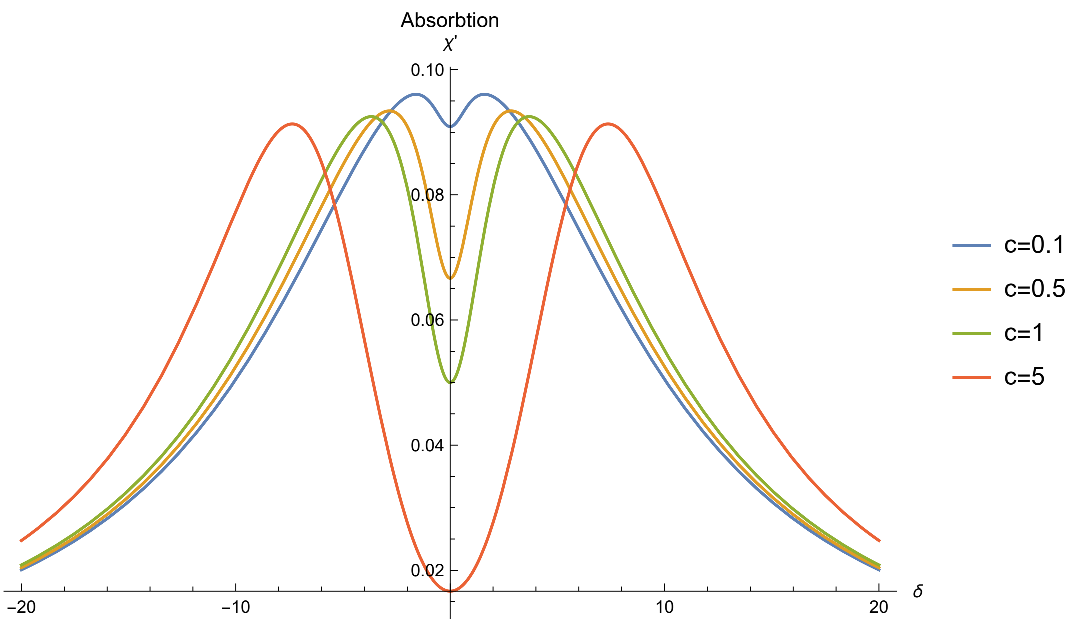
In[22]:=
$$\gamma_{3,1} = 1;$$
$$\gamma_{2,1} = 10\,\gamma_{3,1};$$
$$\Delta = \delta;$$
$$\Omega_0{}' = \sqrt{4\,c\,\gamma_{2,1}\,\gamma_{3,1}};$$
$$\chi = \frac{i}{\gamma_{2,1} + i\,\delta + \frac{(\Omega_0{}')^2}{4\,(\gamma_{3,1} + i\,\Delta)}};$$
```
Cs = {0.1, 0.5, 1, 5};
legends = Table[StringForm["c=``", x], {x, Cs}];
```

```
In[29]:= Plot[Evaluate@Table[Re[χ] /. c → x, {x, Cs}], {δ, -7, 7}, PlotLegends → legends, PlotLabel
         Plot[Evaluate@Table[Im[χ] /. c → x, {x, Cs}], {δ, -20, 20}, PlotLegends → legends, PlotLab
```

Out[29]=




Out[30]=



# Problem 4

# Problem 5 (Berman 9.12)

In[ ]:=
```
inf = 90;
coefficients = {9.5, 10, 11};
delays = {0.75, 0, -0.75};
equations = {c1'[t] == -i a e^(-t^2) c2[t], c2'[t] == -i a (c1[t] e^(-t^2) - c3[t] e^(-(t - τ)^2)), c3'[t] == -i
initialConditions = {c1[-inf] == 1, c2[-inf] == 0, c3[-inf] == 0};
```

In[ ]:=
```
soln = ParametricNDSolveValue[Join[equations, initialConditions], c3, {t, -inf, inf}, {a,
```

In[ ]:=
```
m = Table[Evaluate[Abs[soln[coeff, delay][inf]]], {coeff, coefficients}, {delay, delays}];
TableOfValues1 = Prepend[m, Table[StringForm["a0 = ``", x], {x, coefficients}]];
TableOfValues2 = MapThread[Prepend, {TableOfValues1, Join[{""}, Table[StringForm["τ0 = ``",
Grid[TableOfValues2, Frame → All]
```

Out[ ]=

| | $a_0$ = 9.5` | $a_0$ = 10 | $a_0$ = 11 |
|---|---|---|---|
| $\tau_0$ = 0.75` | 16.8383 | 16.8383 | 16.8383 |
| $\tau_0$ = 0 | 17.7245 | 17.7245 | 17.7245 |
| $\tau_0$ = -0.75 | 19.497 | 19.497 | 19.497 |