Homework 1

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Problem 1

(a) The norm of a four-velocity is given by

$$\begin{aligned}
u^\mu u_\mu &= \frac{\partial x^\mu}{\partial \tau}\frac{\partial x_\mu}{\partial \tau} \\
&= \eta_{\mu\nu}\frac{\partial x^\mu}{\partial \tau}\frac{\partial x^\nu}{\partial \tau} \\
&= -\left(\frac{\partial t}{\partial \tau}\right)^2 + \left(\frac{\partial x}{\partial \tau}\right)^2 + \left(\frac{\partial y}{\partial \tau}\right)^2 + \left(\frac{\partial z}{\partial \tau}\right)^2 \\
&= -\gamma^2 + \gamma^2(v_x^2 + v_y^2 + v_z^2) \\
&= \frac{-1}{1-v^2} + \frac{v^2}{1-v^2} \\
&= \frac{v^2-1}{1-v^2} \\
&= -1,
\end{aligned}$$

where

$$\frac{\partial t}{\partial \tau} = \gamma; \quad \frac{\partial x^i}{\partial \tau} = \gamma v_i$$

was used between the third and fourth lines. Clearly, the zero component of the four-velocity is $\gamma > 0$.

(b) The fact that for any timelike vector there exists a frame in which that vector has zero spatial components is most easily illustrated with a spacetime diagram as in Figure 1. Timelike vectors reside in the future and past light cones. The effect of a boost on the spacetime diagram is to 'rotate' the x and t axes into each other. For any line through the origin with slope between -1 and 1, a boost can put the t axis on that line, and in the boosted frame all events on the t axis have zero spatial component. All points in the future/past light cones lie on lines through the origin with slope between -1 and 1, so there exists a boost that takes their spatial components to zero. Mathematically,

$$x' = \gamma(x - vt) = 0 \implies v = \frac{x}{t},$$

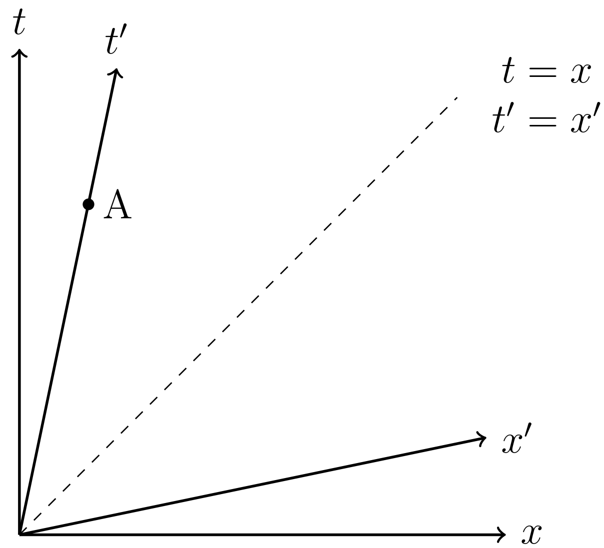

Figure 1: A spacetime diagram illustrating a timelike event which occurs at $x' = 0$.

where $|x/t| < 1$ by the fact that the vector is timelike.

(c) Let $V^\mu V_\mu > 0$, $W^\mu W_\mu > 0$, and $V^\mu W_\mu = 0$. Then,

$$\begin{aligned}(V+W)^\mu (V+W)_\mu &= V^\mu V_\mu + W^\mu W_\mu + 2V^\mu W_\mu \\ &= V^\mu V_\mu + W^\mu W_\mu \\ &> 0\end{aligned}$$

(d) Let A be a nonzero timelike vector, and let B be a nonzero null vector. Then, by the result of part (b), there exists a frame in which we can write these vectors as

$$A^\mu = (t', 0, 0, 0), \quad B^\mu = (t, x, y, z) \quad (x^2 + y^2 + z^2 = t^2),$$

with both t and t' nonzero. Assume for the sake of contradiction that their inner product vanishes. Then,

$$\begin{aligned} A^\mu B_\mu &= 0 \\ \implies \quad tt' &= 0, \end{aligned}$$

thus either $t = 0$ or $t' = 0$, a contradiction.

Problem 2

The magnitude of the centrifugal force is given by

$$|F_c| = m_I R \omega^2 \sin\theta,$$

where R is the radius of the Earth, ω is the Earth's rotational velocity, and θ is the polar angle. For the purpose of the Eötvös experiment, we want to maximize the component of the centrifugal force parallel to the ground (i.e. perpendicular to $\hat{r}$). The maximum value of this component is just the magnitude times $\cos\theta$, so we need to maximize $\sin\theta\cos\theta$, which occurs at $\boxed{\theta = \pi/4}$. The 45^{th} parallel(s) is(are) the best place(s) to do the experiment, then.

Problem 3

(a)

$$\begin{aligned}
S &= -m\int \mathrm{d}\tau + q\int \mathrm{d}x^\mu A_\mu(x) \\
&= -m\int \sqrt{-\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu} + q\int \mathrm{d}x^\mu A_\mu(x) \\
&= \int \mathrm{d}\lambda \left[-m\sqrt{-\eta_{\mu\nu}\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\frac{\mathrm{d}x^\nu}{\mathrm{d}\lambda}} + qA_\mu(x)\frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}\right]
\end{aligned}$$

(b) Let $\dot{x}^\mu := \frac{\mathrm{d}x^\mu}{\mathrm{d}\lambda}$. We can write the action above as the integral of a Lagrangian,

$$S = \int \mathrm{d}\lambda\; L(x,\dot{x}),$$

where

$$L(x,\dot{x}) = -m\sqrt{-\eta_{\mu\nu}\dot{x}^\mu\dot{x}^\nu} + qA_\mu(x)\dot{x}^\mu.$$

The canonical momentum is given by

$$\begin{aligned}
p_\mu &= \frac{\partial L}{\partial \dot{x}^\mu} \\
&= m\frac{\dot{x}_\mu}{\sqrt{-\eta_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta}} + qA_\mu(x) \\
&= m\frac{\frac{\partial x_\mu}{\partial\tau}\frac{\partial\tau}{\partial\lambda}}{\sqrt{-\eta_{\alpha\beta}\frac{\partial x^\alpha}{\partial\tau}\frac{\partial x^\beta}{\partial\tau}\frac{\partial\tau}{\partial\lambda}}} \\
&= m\frac{u_\mu}{-u\cdot u} + qA_\mu(x) \\
&= mu_\mu + qA_\mu(x)
\end{aligned}$$

(c) The Euler-Lagrange equation is

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}\frac{\partial L}{\partial\dot{x}^\mu} = \frac{\partial L}{\partial x^\mu}.$$

Note that the left hand side is already $\frac{\mathrm{d}}{\mathrm{d}\lambda}p_\mu$. The only x^μ dependence is in the gauge filed term,

$$\frac{\partial L}{\partial x^\mu} = q\frac{\partial A_\nu}{\partial x^\mu}\dot{x}^\nu.$$

Then, after relabeling indices,

$$\frac{\mathrm{d}p_\mu}{\mathrm{d}\lambda} = q\frac{\partial A_\mu}{\partial x^\gamma}\frac{\partial x^\mu}{\partial\lambda}.$$

(d)

$$
\begin{aligned}
&& \frac{\mathrm{d}p_\mu}{\mathrm{d}\lambda} &= q\frac{\partial A_\mu}{\partial x^\gamma}\frac{\partial x^\mu}{\partial \lambda} \\
\Longrightarrow && \frac{\mathrm{d}}{\mathrm{d}\lambda}\left(m u_\gamma + q A_\gamma\right) &= q\frac{\partial A_\mu}{\partial x^\gamma}\frac{\partial x^\mu}{\partial \lambda} \\
\Longrightarrow && m\frac{\mathrm{d}u_\gamma}{\mathrm{d}\lambda} + q\frac{\partial A_\gamma}{\partial x^\mu}\frac{\partial x^\mu}{\partial \lambda} &= q\frac{\partial A_\mu}{\partial x^\gamma}\frac{\partial x^\mu}{\partial \lambda} \\
\Longrightarrow && m\frac{\mathrm{d}u_\gamma}{\mathrm{d}\lambda} &= q\left(\frac{\partial A_\mu}{\partial x^\gamma}\frac{\partial x^\mu}{\partial \lambda} - \frac{\partial A_\gamma}{\partial x^\mu}\frac{\partial x^\mu}{\partial \lambda}\right) \\
\Longrightarrow && m\frac{\mathrm{d}u_\gamma}{\mathrm{d}\tau}\frac{\partial \tau}{\partial \lambda} &= qF_{\gamma\mu}\frac{\partial x^\mu}{\partial \tau}\frac{\partial \tau}{\partial \lambda} \\
\Longrightarrow && m\frac{\mathrm{d}u_\gamma}{\mathrm{d}\tau} &= qF_{\gamma\mu}u^\mu
\end{aligned}
$$

(e) The antisymmetry of $F_{\gamma\mu}$ is clear from its definition:

$$F_{\mu\gamma} = \partial_\mu A_\gamma - \partial_\gamma A_\mu = -(\partial_\gamma A_\mu - \partial_\mu A_\gamma) = -F_{\gamma\mu}.$$

To check its transformation properties, we first note that under the transformation

$$x \to x' = \Lambda x,$$

the field $A(x)$ transforms as

$$A \to A' = \Lambda A(\Lambda^{-1}x).$$

Then,

$$
\begin{aligned}
F'_{\mu\nu} &= \frac{\partial A'_\nu}{\partial x'^\mu} - \frac{\partial A'_\mu}{\partial x'^\nu} \\
&= \Lambda^\alpha_{\ \nu}\frac{\partial A_\alpha}{\partial x'^\mu} - \Lambda^\beta_{\ \mu}\frac{\partial A_\beta}{\partial x'^\nu} \\
&= \Lambda^\alpha_{\ \nu}\frac{\partial A_\alpha}{\partial x^\gamma}\frac{\partial x^\gamma}{\partial x'^\mu} - \Lambda^\beta_{\ \mu}\frac{\partial A_\beta}{\partial x^\gamma}\frac{\partial x'^\gamma}{\partial x^\nu} \\
&= \Lambda^\alpha_{\ \nu}\Lambda^\gamma_{\ \mu}\frac{\partial A_\alpha}{\partial x^\gamma} - \Lambda^\beta_{\ \mu}\Lambda^\gamma_{\ \nu}\frac{\partial A_\beta}{\partial x^\gamma} \\
&= \Lambda^\alpha_{\ \nu}\Lambda^\gamma_{\ \mu}F_{\alpha\gamma},
\end{aligned}
$$

where in the last step the dummy index β was relabeled to α. The field strength tensor therefore has the correct transformation properties.

(f) The Lorentz force law gives the force on a charge particle from electric and magnetic fields as

$$\vec{F} = q\left(\vec{E} + \vec{v}\times\vec{B}\right).$$

In components, this is

$$\begin{aligned} F_x &= q\left(E_x + v_y B_z - v_z B_y\right) \\ F_y &= q\left(E_y + v_z B_x - v_x B_z\right) \\ F_z &= q\left(E_z + v_x B_y - v_y B_x\right) \end{aligned}$$

The result of part (d) gives mass times proper acceleration as equal to the field strength tensor times the four-velocity,

$$ma_\mu = qF_{\mu\nu}u^\nu = q\begin{pmatrix} F_{00} & F_{01} & F_{02} & F_{03} \\ F_{10} & F_{11} & F_{12} & F_{13} \\ F_{20} & F_{21} & F_{22} & F_{23} \\ F_{30} & F_{31} & F_{32} & F_{33} \end{pmatrix}\begin{pmatrix} 1 \\ v_x \\ v_y \\ v_z \end{pmatrix},$$

where the four-velocity has been expressed as that of a slow-moving particle. Comparing to the Lorentz force law, we can see that the rows F_{1i}, F_{2i}, and F_{3i} must be

$$\begin{pmatrix} E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}.$$

The antisymmetry of F (and the lack of any constant terms in the force law) then fixes the 0^{th} row:

$$F_{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

(g)

$$\begin{aligned} F_{10} = E_x &= \frac{\partial A_0}{\partial x} - \frac{\partial A_1}{\partial t} = \left(-\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}\right)_x \\ F_{23} = B_x &= \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} = (\vec{\nabla} \times \vec{A})_x \end{aligned}$$

These are consistent with the usual definitions of the fields in terms of the electrostatic and vector potentials,

$$\vec{B} = \vec{\nabla} \times \vec{A}; \quad \vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t},$$

given that A^0 is the electrostatic potential ϕ (and hence $A_0 = -\phi$).