The LSZ Reduction Formula

Sean Ericson

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- It *reduces* the problem of calculating scattering amplitudes to calculating correlation functions of fields.
- Originally published by German physicists Harry Lehmann, Kurt Symanzik and Wolfhart Zimmermann in 1955.
- Not to be confused with Lysergic acid 2,4-dimethylazetidide, an analog of LSD

What is the LSZ Reduction Formula?

The LSZ Reduction Formula

$$\langle \mathbf{k_{n+1}} \cdots \mathbf{k_{n+m}} | \mathbf{k_1} \cdots \mathbf{k_n} \rangle =$$
$$\left(\prod_{i=1}^{n} \int \mathrm{d}^4 x_i e^{-ik_i x_i} (\Box_i + m^2) \right) \left(\prod_{i=n+1}^{n+m} \int \mathrm{d}^4 x_i e^{ik_i x_i} (\Box_i + m^2) \right)$$
$$\times \ \langle \Omega | T\{\phi(x_1) \cdots \phi(x_{n+m})\} | \Omega \rangle$$

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- Solution: The *Key Identity*:

The Key Identity

$$a_{\mathbf{k}}(+\infty) - a_{\mathbf{k}}(-\infty) = i \int \mathrm{d}^4x \; e^{ikx}(\Box + m^2)\phi(x)$$

Where does it Come From?

Proof of the Key Identity

$$
\begin{aligned}
a_{\mathbf{k}}(+\infty) - a_{\mathbf{k}}(-\infty) &= \int_{-\infty}^{\infty} \mathrm{d}t\ \partial_0 a_{\mathbf{k}}(t) \\
&= \int_{-\infty}^{\infty} \mathrm{d}t\ \partial_0 \int \mathrm{d}^3x\ e^{-ikx}\,(i\partial_0 + \omega)\,\phi(x) \\
&= i \int \mathrm{d}^4x\ e^{ikx}\,\left(\partial_0^2 + \omega^2\right)\phi(x) \\
&= i \int \mathrm{d}^4x\ e^{ikx}\,\left(\partial_0^2 + \mathbf{k}^2 + m^2\right)\phi(x) \\
&= i \int \mathrm{d}^4x\ e^{ikx}\,\left(\partial_0^2 - \overleftarrow{\nabla}^2 + m^2\right)\phi(x) \\
&= i \int \mathrm{d}^4x\ e^{ikx}\,\left(\partial_0^2 - \vec{\nabla}^2 + m^2\right)\phi(x) \\
&= i \int \mathrm{d}^4x\ e^{ikx}\,\left(\Box + m^2\right)\phi(x)
\end{aligned}
$$

How does that Help?

- Goal: $\langle\Omega|a_{n+1}(\infty)\cdots a_{n+m}(\infty)a_1^\dagger(-\infty)\cdots a_n^\dagger(-\infty)|\Omega\rangle$

How does that Help?

- Key Identity Lemma:

$$
\begin{aligned}
a_i(+\infty) &= a_i(-\infty) + \Phi_i^+ \\
a_i^\dagger(-\infty) &= a_i^\dagger(+\infty) + \Phi_i^- \\
\Phi_i^\pm &= \int \mathrm{d}^4 x_i \; e^{\pm i k_i x_i} (\Box_i + m^2)
\end{aligned}
$$

How does that Help?

- Insert into scattering amplitude:

$$
\begin{aligned}
&\langle\Omega| a_{n+1}(\infty)\cdots a_{n+m}(\infty) a_1^\dagger(-\infty)\cdots a_n^\dagger(-\infty)|\Omega\rangle \\
&= \langle\Omega| T\{a_{n+1}(\infty)\cdots a_{n+m}(\infty) a_1^\dagger(-\infty)\cdots a_n^\dagger(-\infty)\}|\Omega\rangle \\
&= \langle\Omega| T\{\cdots \left[a_{n+m}(-\infty) + \Phi_{n+m}^+\right]\cdots \left[a_n^\dagger(\infty) + \Phi_n^-\right]\}|\Omega\rangle \\
&= \langle\Omega| T\{\Phi_{n+1}^+ \cdots \Phi_{n+m}^+ \Phi_1^- \cdots \Phi_n^-\}|\Omega\rangle \\
&=^* \left(\prod_{i=1}^{n}\int \mathrm{d}^4 x_i e^{-ik_i x_i}(\Box_i + m^2)\right)\left(\prod_{i=n+1}^{n+m}\int \mathrm{d}^4 x_i e^{ik_i x_i}(\Box_i + m^2)\right) \\
&\times \langle\Omega| T\{\phi(x_1)\cdots\phi(x_{n+m})\}|\Omega\rangle
\end{aligned}
$$

*Aside: $\Box$ and $T\{\}$

- Technically

$$\Box_x \langle\Omega| T\{\phi_x\phi_1\cdots\phi_n\}|\Omega\rangle \neq \langle\omega| T\{\Box_x\phi_x\phi_1\cdots\phi_n\}|\Omega\rangle$$

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- In fact,

$$\begin{aligned}\Box_x \langle\Omega|T\{\phi_x\phi_1\cdots\phi_n\}|\Omega\rangle = \\ &\langle\Omega|T\{\Box_x\phi_x\phi_1\cdots\phi_n\}|\Omega\rangle \\ &- i\sum_j \delta^4(x-x_j)\langle\Omega|T\{\phi_1\cdots\phi_{j-1}\phi_{j+1}\phi_n\}|\Omega\rangle\end{aligned}$$

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- However,
    - These so-called *contact terms* don't contribute to the connected part of the scattering amplitude

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- Correlation functions are extremely versitile:
  - Calculated using Feynmann rules
  - Contain much more information than just scattering
  - LSZ projects out the single-particle asymptotic states

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  - Assuming sufficiently low energy scattering