Homework 1

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Problem 1

(a)

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma_x & -\gamma_x V_x & 0 & 0 \\ -\gamma_x V_x & \gamma_x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

(b)

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \mu & \nu & 0 & 0 \\ \sigma & \gamma & 0 & 0 \\ 0 & 0 & \rho & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

Problem 2

$$\begin{aligned}
\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} &= \begin{pmatrix} \gamma_y & 0 & \gamma_y V_y & 0 \\ 0 & 1 & 0 & 0 \\ \gamma_y V_y & 0 & \gamma_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_x & \gamma_x V_x & 0 & 0 \\ \gamma_x V_x & \gamma_x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_y & 0 & -\gamma_y V_y & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma_y V_y & 0 & \gamma_y & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_x & -\gamma_x V_x & 0 & 0 \\ -\gamma_x V_x & \gamma_x & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix} \\
&= \begin{pmatrix} -\gamma_x^2\gamma_y^2\left(\frac{V_x^2}{\gamma_y^2}+\frac{V_y^2}{\gamma_x^2}-1\right) & \gamma_x^2\gamma_y^2 V_x\left(\frac{V_y^2}{\gamma_x}+\frac{1}{\gamma_y}-1\right) & -(\gamma_x-1)\gamma_y^2 V_y & 0 \\ (\gamma_y-1)\gamma_x^2 V_x & \gamma_x^2\left(1-\gamma_y V_x^2\right) & -\gamma_x\gamma_y V_x V_y & 0 \\ -\gamma_x^2\gamma_y^2 V_y\left(\frac{V_x^2}{\gamma_y}+\frac{1}{\gamma_x}-1\right) & \gamma_x^2\gamma_y^2 V_x V_y\left(\frac{1}{\gamma_x}+\frac{1}{\gamma_y}-1\right) & -\gamma_y^2\left(\gamma_x V_y^2-1\right) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}
\end{aligned}$$

Composing the four given boosts results in the more complicated lorentz transformation given above. This transformation is not symmetric, so it's clearly not a pure boost. It leaves the z-component untouched, so it must be a combination of a boost in the xy-plane and a rotation about the z-axis (or vice-versa).
Expanding this to lowest order in V_x and V_y (using Mathematica), I found that this transformation is just the identity, with corrections that are second-order in the Vs.

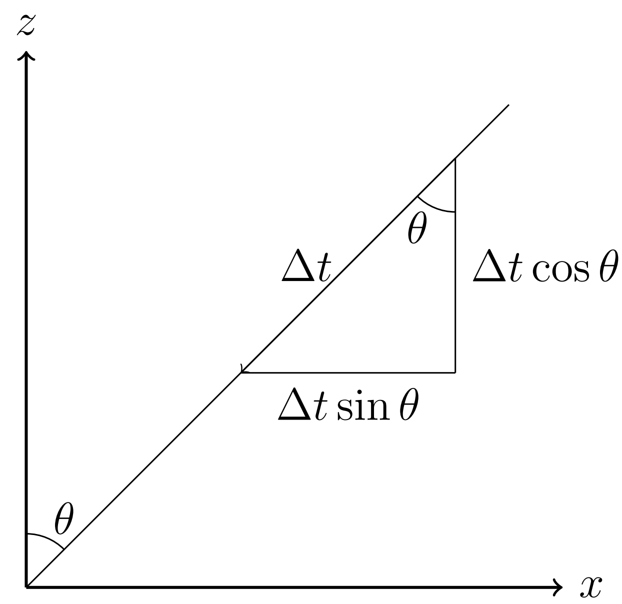

Figure 1: The photon's path in the xz-plane.

Problem 3

(a) From the frame of the Earth, the photon is traveling towards the origin, making an angle θ with the z-axis, as shown in Figure 1. In a time Δt, the photon travels a distance Δt ($c = 1$). Its x-displacement is therefore $\Delta x = -\Delta t \sin\theta$, while the z-displacement is $\Delta z = -\Delta t \cos\theta$.

(b) Using the transformation for a boost along the z-direction,

$$\begin{aligned} t' &= \gamma t - \gamma V z \\ x' &= x \\ y' &= y \\ z' &= \gamma z - \gamma V t, \end{aligned}$$

we can transform the displacement 4-vector to the ship's frame:

$$\begin{aligned} \Delta t' &= \gamma \Delta t - \gamma V \Delta t \sin\theta \\ &= -\gamma \Delta t (V + \cos\theta) \\ \Delta x' &= \Delta x \\ \Delta y' &= \Delta y \\ \Delta z' &= -\gamma \Delta t \cos\theta - \gamma V \Delta t \\ &= -\gamma \Delta t (1 + V \cos\theta). \end{aligned}$$

An observer from the ship could draw the exact same diagram as Figure 1 (with $\theta \to \overline{\theta}$), so the ship-based observer can calculate $\cos\overline{\theta}$ as $\Delta z'/\Delta t'$, thus

$$\cos\overline{\theta} = \frac{\Delta z'}{\Delta t'} = \frac{V + \cos\theta}{1 + V\cos\theta}$$

(c)

$$\begin{aligned} \theta &= \frac{\pi}{2} \\ \Longrightarrow \cos\theta &= 0 \\ \Longrightarrow \cos\overline{\theta} &= \frac{V+0}{1+0} = V \\ \Longrightarrow \overline{\theta} &= \cos^{-1} V \end{aligned}$$

If $V = 0.9c$, this gives a $\overline{\theta}$ of about 0.45 rad (25.8°), while for $V = 0.99c$ this gives about 0.14 rad (8°).

(d)

$$\begin{aligned} \cos\overline{\theta} &= \frac{V+\cos\theta}{1+V\cos\theta} \\ \Longrightarrow \quad 1-\frac{1}{2}\overline{\theta}^2 &\approx \frac{V+1-\frac{1}{2}\theta^2}{1+V-\frac{1}{2}V\theta^2} \\ \Longrightarrow \quad &\approx (V+1-\frac{1}{2}\theta^2)\left(\frac{1}{1+V}+\frac{\frac{1}{2}V\theta^2}{(1+V)^2}\right) \\ \Longrightarrow \quad &\approx 1+\frac{\frac{1}{2}V\theta^2}{V+1}-\frac{\frac{1}{2}\theta^2}{V+1} \\ \Longrightarrow \quad -\overline{\theta}^2 &\approx \frac{V-1}{V+1}\theta^2 \\ \Longrightarrow \quad \overline{\theta} &\approx \sqrt{\frac{1-V}{1+V}}\theta \end{aligned}$$