On The Relationship Between Rotations of Angular Momentum Eigenstates in $\mathcal{H}$ and Rotations of Elements of $\mathbb{R}^3$

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The Problem

Wigner, being the champ that he was, kindly calculated for our utilization and enjoyment the representation of $R(\mathbf{v}) = e^{-i|\mathbf{v}|J_{\hat{v}}/\hbar}$ in the basis of eigenstates of J_z for $l = 1$. In particular, if the general rotation is decomposed into rotations about the standard Euler angles, $R(\mathbf{v}) = R_z(\alpha)R_y(\beta)R_z(\gamma)$, then

$$D^{(1)}_{m'm} = \begin{pmatrix} e^{-i\alpha}\cos^2(\beta/2)e^{-i\gamma} & -\frac{1}{\sqrt{2}}e^{-i\alpha}\sin\beta & e^{-i\alpha}\sin^2(\beta/2)e^{i\gamma} \\ \frac{1}{\sqrt{2}}\sin\beta e^{-i\gamma} & \cos\beta & -\frac{1}{\sqrt{2}}\sin\beta e^{i\gamma} \\ e^{i\alpha}\sin^2(\beta/2)e^{-i\gamma} & \frac{1}{\sqrt{2}}e^{i\alpha}\sin\beta & e^{i\alpha}\cos^2(\beta/2)e^{i\gamma} \end{pmatrix}$$

Now, this rotation operator acts on elements of an abstract Hilbert space. It is defined as the exponential of the Anuglar Momentum component operator along the the direction of $\hat{v}$.

We intuitively expect this abstract operator to relate to the more familiar rotation operators for $\mathbb{R}^3$. In particular, we might expect that the representations of the Hilbert space operator in the J_z eigenbasis to be identical to that of the operator on $\mathbb{R}^3$ wich affects the same rotation expressed in the spherical basis. That is, if

$$R \in \mathcal{H} = R_z(\alpha)R_y(\beta)R_z(\gamma) = e^{-i\alpha J_z/\hbar}e^{-i\beta J_y/\hbar}e^{-i\gamma J_z/\hbar}$$

$$D^{(1)}_{m'm} = \langle lm'|R|lm\rangle$$

and

$$\tilde{R} \in \mathbb{R}^3 = \tilde{R}_z(\alpha)\tilde{R}_y(\beta)\tilde{R}_z(\gamma)$$

then we expect

$$P_{q'q} = \langle e_{q'}|\tilde{R}|e_q\rangle = D^{(1)}_{q'q}$$

However, when we actually calculate $P_{q'q}$, we find that

$$P_{q'q} = \left[D^{(1)}_{q'q}\right]^*$$

This tells us that our naive assumption that $(R) \leftrightarrow (\tilde{R})$ was simply incorrect.

So let's try to figure out what rotation operator on $\mathbb{R}^3$ really does correspond to the operator on the Hilbert space. To do so we must establish some sort of connection between the spaces $\mathcal{H}$ and $\mathbb{R}^3$ in order to understand what the "correspondence" really means. Consider the following definitions:

$$\begin{aligned} Y_1^m &= \langle \theta, \phi | 1, m \rangle \\ \tilde{Y}_1^m &= \langle e_m | \mathbf{r} \rangle \, , \end{aligned}$$

where $|e_m\rangle$ are the spherical basis vectors. The requirement that $\tilde{Y}_1^m = Y_1^m$ gives us a link between the two spaces. Further, let

$$\begin{aligned} [Y_1^m]' &= \langle \theta, \phi | R | 1, m \rangle = \sum_{m'} \langle \theta, \phi | 1m' \rangle \, \langle 1m' | R | 1m \rangle = D_{m'm} Y_1^m \\ \left[\tilde{Y}_1^m\right]' &= \langle e_m | \tilde{R} | \mathbf{r} \rangle = \sum_{m'} \langle e_m | R | e_{m'} \rangle \, \langle e_{q'} | \mathbf{r} \rangle = P_{mm'} Y_1^m \end{aligned}$$

Thus we see that, whatever $\tilde{R}$ on $\mathbb{R}^3$ really corresponds to R, it's matrix representation in the spherical basis is the *transpose* of the Wigner D matrix.

Now we can put both of these observations together. If $R(\mathbf{v}) = R_z(\alpha) R_y(\beta) R_z(\gamma)$ is the operator on the Hilbert space corresponding to the rotation $\mathbf{v}$, and we go through the perscription of taking the transpose of $\tilde{R}$'s matrix in the spherical basis, we will wind up with

$$P_{q'q} = \left[D^{(1)}_{mm'}\right]^* = \left[D^{(1)}_{m'm}\right]^\dagger$$

This tells us that the rotation on $\mathbb{R}^3$ which corresponds (via the equivalency of the Y_1^m) to the rotation on the Hilber space is infact

$$(R) \overset{\mathrm{T}}{\leftrightarrow} (\tilde{R}(-\mathbf{v})) = (\tilde{R}_z(-\gamma) \tilde{R}_y(-\beta) \tilde{R}_z(-\alpha))$$

i.e. that it is the *inverse* rotation which corresponds (again, via the equivalency of the Y_1^m) to rotation on the Hilbert space.