Homework 1

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Problem 1

(a)

$$\frac{\hbar c}{r_J} \approx 2.82 \times 10^{-21} \text{ MeV}$$

(b)

$$\frac{\hbar c}{\lambda_W} \approx 2.45 \times 10^{-16} \text{ cm}$$

(c)

$$\frac{\hbar c}{\sqrt{s}} \approx 9.9 \times 10^{-17} \text{ cm}$$

Problem 2

Conservation of energy implies

$$E + m_e = E' + E'_e, \tag{2.1}$$

while conservation of momentum implies

$$\begin{aligned} E' \sin\theta - |p'_e| \sin\theta' &= 0 \\ E' \cos\theta - |p'_e| \cos\theta' &= E. \end{aligned} \tag{2.2}$$

Now we note that the final electron energy is given by

$$(E'_e)^2 = |p'_e|^2 + m_e^2. \tag{2.3}$$

Plugging (2.3) into (2.1), then solving the result along with (2.3) for $|p'_e|^2$, we find

$$\begin{aligned} |p'_e|^2 &= (E - E' + m_e)^2 - m_e^2 \\ |p'_e|^2 &= E^2 + (E')^2 - 2EE' \cos\theta. \end{aligned} \tag{2.4}$$

Equating the two right hand sides of (2.4) gives

$$\begin{aligned}
& E^2+(E')^2-2EE'\cos\theta = E^2+(E')^2+m_e^2+2(E-E')m_e-2EE'-m_e^2 \\
\Longrightarrow \quad & 2(E-E')m_e-2EE' = -2EE'\cos\theta \\
\Longrightarrow \quad & (\frac{1}{E'}-\frac{1}{E})m_e = 1-\cos\theta
\end{aligned}$$

Finally, using the photon wavelength/energy relation $E = \frac{2\pi}{\lambda}$, we arrive at the desired equation for wavelength shift:

$$(\frac{\lambda'}{2\pi}-\frac{\lambda}{2\pi})m_e = (1-cos\theta) \implies \boxed{\lambda'-\lambda = \frac{2\pi}{m_e}(1-\cos\theta)}$$

Problem 3

Given that

$$(p')^\mu = \Lambda^\mu_{\ \nu} p^\nu \tag{3.1}$$

we have that

$$\begin{aligned}
(p')_\mu (p')^\mu &= \Lambda_{\mu\nu} p^\nu \Lambda^\mu_{\ \rho} p^\rho \\
&= \eta_{\mu\alpha} \Lambda^\alpha_{\ \nu} \Lambda^\mu_{\ \rho} p^\nu p^\rho
\end{aligned} \tag{3.2}$$

Now, if

$$\eta_{\mu\alpha} \Lambda^\alpha_{\ \nu} \Lambda^\mu_{\ \rho} = \eta_{\nu\rho}, \tag{3.3}$$

then the right hand side of (3.2) becomes

$$\eta_{\nu\rho} p^\nu p^\rho = p_\nu p^\nu, \tag{3.4}$$

which is the result we wish to show. Now, (3.4) is equivalent to $\Lambda^\intercal \eta \Lambda$. Calculating $\eta\Lambda$, we find

$$\eta \begin{pmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & 0 \\ -\gamma\beta_x & 1+(\gamma-1)\frac{\beta_x^2}{|\beta|^2} & (\gamma-1)\frac{\beta_x\beta_y}{|\beta|^2} & 0 \\ -\gamma\beta_y & (\gamma-1)\frac{\beta_x\beta_y}{|\beta|^2} & 1+(\gamma-1)\frac{\beta_y^2}{|\beta|^2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta_x & -\gamma\beta_y & 0 \\ \gamma\beta_x & -1-(\gamma-1)\frac{\beta_x^2}{|\beta|^2} & (1-\gamma)\frac{\beta_x\beta_y}{|\beta|^2} & 0 \\ \gamma\beta_y & (1-\gamma)\frac{\beta_x\beta_y}{|\beta|^2} & -1-(\gamma-1)\frac{\beta_y^2}{|\beta|^2} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Finally, multiplying on the left by $\Lambda^\intercal = \Lambda$ and simplifying, we arrive back at the metric:

$$\Lambda^\intercal \eta \Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \eta$$

. Thus, (3.3) is satisfied and therefore $(p')_\mu (p')^\mu = p_\mu p^\mu$.

Problem 4

In the center of mass-energy frame, we have

$$(M,0,0,0) \to \begin{cases} (\frac{M}{2}, \frac{1}{4}M^2 - m^2, 0, 0) \\ (\frac{M}{2}, m^2 - \frac{1}{4}M^2 0, 0), \end{cases} \tag{4.1}$$

while in the lab frame we have

$$(\gamma M, 0, 0, \gamma\beta) \to \begin{cases} (\sqrt{m^2 + \gamma^2 |\beta'_{\mathrm{Lab}}|}, \gamma|\beta'_{\mathrm{Lab}}| \sin\frac{\theta}{2}, 0, \gamma|\beta'_{\mathrm{Lab}}|\cos\frac{\theta}{2}) \\ (\sqrt{m^2 + \gamma^2 |\beta'_{\mathrm{Lab}}|}, -\gamma|\beta'_{\mathrm{Lab}}| \sin\frac{\theta}{2}, 0, \gamma|\beta'_{\mathrm{Lab}}|\cos\frac{\theta}{2}), \end{cases} \tag{4.2}$$

where θ is the angle between the final state particles in the lab frame. Boosting from the CoM/E frame to labe frame (i.e. boosting by $-\beta\hat{z}$), we find

$$\left.\begin{matrix} (\frac{M}{2}, \frac{1}{4}M^2 - m^2, 0, 0) \\ (\frac{M}{2}, m^2 - \frac{1}{4}M^2 0, 0) \end{matrix}\right\} \to \begin{matrix} (\frac{1}{2}\gamma M, \frac{1}{4}M^2 - m^2, 0, -\gamma\beta m) \\ (\frac{1}{2}\gamma M, m^2 - \frac{1}{4}M^2, 0, -\gamma\beta m) \end{matrix}. \tag{4.3}$$

By conservation of energy, we have that

$$\sqrt{m^2 + \gamma^2 |\beta'_{\mathrm{Lab}}|^2} = \frac{1}{2}\gamma M \implies |\beta'_{\mathrm{Lab}}|^2 = \frac{1}{4}M^2 - (\frac{m}{\gamma})^2 \tag{4.4}$$

Now, comparing the z components of (4.2) and (4.3), we see that

$$\gamma|\beta_{\mathrm{Lab}}| \cos\frac{\theta}{2} = -\gamma\beta \implies \boxed{\cos\frac{\theta}{2} = \frac{-\beta}{\frac{1}{4}M^2 - (\frac{m}{\gamma})^2}} \tag{4.5}$$

Oops, this can't be right...