Midterm

Sean Ericson
Phys 610

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Problem 1

The metric and its inverse are given by:

$$g_{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{2M}{\sqrt{r^2+b^2}}\right) & 0 & 0 & 0 \\ 0 & \left(1 - \frac{2M}{\sqrt{r^2+b^2}}\right)^{-1} & 0 & 0 \\ 0 & 0 & r^2 + b^2 & 0 \\ 0 & 0 & 0 & (r^2 + b^2)\sin^2\theta \end{pmatrix}$$

$$g^{\mu\nu} = \begin{pmatrix} -\left(1 - \frac{2M}{\sqrt{r^2+b^2}}\right)^{-1} & 0 & 0 & 0 \\ 0 & \left(1 - \frac{2M}{\sqrt{r^2+b^2}}\right) & 0 & 0 \\ 0 & 0 & \frac{1}{r^2+b^2} & 0 \\ 0 & 0 & 0 & \frac{1}{(r^2+b^2)\sin^2\theta} \end{pmatrix}$$

Problem 2

The Christoffel symbols were calculated using the attached Mathematica code. Their reduction to those of the Schwarzschild metric when $b \to 0$ is clear.

$$
\begin{aligned}
\Gamma^t_{tr} &= \frac{M}{r^2+b^2}\left(1-\frac{2M}{\sqrt{r^2+b^2}}\right)^{-1} \\
\Gamma^r_{tt} &= \frac{Mr}{(r^2+b^2)^{\frac{3}{2}}}\left(1-\frac{2M}{\sqrt{r^2+b^2}}\right) \\
\Gamma^r_{rr} &= -\frac{M}{r^2+b^2}\left(1-\frac{2M}{\sqrt{r^2+b^2}}\right)^{-1} \\
\Gamma^r_{\theta\theta} &= -r\left(1-\frac{2M}{\sqrt{r^2+b^2}}\right) \\
\Gamma^r_{\phi\phi} &= -r\left(1-\frac{2M}{\sqrt{r^2+b^2}}\right)\sin^2\theta \\
\Gamma^\theta_{\theta r} &= \frac{r}{r^2+b^2} \\
\Gamma^\theta_{\phi\phi} &= -\sin\theta\cos\theta \\
\Gamma^\phi_{\phi r} &= \frac{r}{r^2+b^2} \\
\Gamma^\phi_{\phi\theta} &= \cot\theta
\end{aligned}
$$

Problem 3

The geodesic equations were calculated with the attached Mathematica code.

$$
\begin{aligned}
\ddot{t} &= -\frac{2Mr}{(r^2+b^2)^{\frac{3}{2}}}\left(1-\frac{2M}{\sqrt{r^2+b^2}}\right)\dot{t}\dot{r} \\
\ddot{r} &= r\left(1-\frac{2M}{\sqrt{r^2+b^2}}\right)\left(\sin^2\theta\dot{\phi}^2+\dot{\theta}^2\right)-\left(1-\frac{2M}{\sqrt{r^2+b^2}}\right)\frac{Mr\dot{t}^2}{(r^2+b^2)^{\frac{3}{2}}}+\left(1-\frac{2M}{\sqrt{r^2+b^2}}\right)^{-1}\frac{Mr\dot{r}^2}{(r^2+b^2)^{\frac{3}{2}}} \\
\ddot{\theta} &= \sin\theta\cos\theta\dot{\phi}^2-\frac{2r\dot{\theta}\dot{r}}{r^2+b^2} \\
\ddot{\phi} &= -2\dot{\phi}\left(\cot\theta\dot{\theta}+\frac{r\dot{r}}{r^2+b^2}\right)
\end{aligned}
$$

Problem 4

The metric is independent of t and ϕ, so we have two conserved quantities:

$$
\begin{aligned}
e &:= -\xi_t^{\ \mu}u_\mu = \left(1-\frac{2M}{\sqrt{r^2+b^2}}\right)\dot{t} \\
l &:= -\xi_\phi^{\ \mu}u_\mu = -(r^2+b^2)\sin^2\theta\dot{\phi}
\end{aligned}
$$

The ϕ independence also implies that orbits lie in a plane, so we can consider an orbit in the $\theta = \pi/2$ plane, with $\mathrm{d}\theta = 0$.

$$\begin{aligned}
u^2 = -1 &= -\left(1 - \frac{2M}{\sqrt{r^2+b^2}}\right)\dot{t}^2 + \left(1 - \frac{2M}{\sqrt{r^2+b^2}}\right)^{-1}\dot{r}^2 + (r^2+b^2)\dot{\phi}^2 \\
&= -\left(1 - \frac{2M}{\sqrt{r^2+b^2}}\right)^{-1} e^2 + \left(1 - \frac{2M}{\sqrt{r^2+b^2}}\right)^{-1}\dot{r}^2 + \frac{l^2}{r^2+b^2} \\
\implies \qquad e^2 &= \dot{r}^2 + \left(1 - \frac{2M}{\sqrt{r^2+b^2}}\right)\left(1 + \frac{l^2}{r^2+b^2}\right) \\
\implies \qquad \frac{1}{2}(e^2-1) &= \frac{1}{2}\dot{r}^2 + \left(1 - \frac{2M}{\sqrt{r^2+b^2}}\right)\left(1 + \frac{l^2}{r^2+b^2}\right) - 1 \\
\implies \qquad \mathcal{E} &= \frac{1}{2}\dot{r}^2 + V_{\text{eff}}(r) \qquad (1)
\end{aligned}$$

where

$$\begin{aligned}
V_{\text{eff}}(r) &\coloneqq \frac{1}{2}\left[\left(1 - \frac{2M}{\sqrt{r^2+b^2}}\right)\left(1 + \frac{l^2}{r^2+b^2}\right) - 1\right] \\
&= -\frac{M}{\sqrt{r^2+b^2}} + \frac{l^2}{2(r^2+b^2)} - \frac{Ml^2}{(r^2+b^2)^{\frac{3}{2}}} \qquad (2)
\end{aligned}$$

Problem 5

No, for $M = 0$ bound orbital solutions do not exist. In this case, the effective potential is of the form

$$V_{\text{eff}}(r)\big|_{M\to 0} = \frac{l^2}{2(r^2+b^2)},$$

which has a single maximum at $r = 0$ and no local minima, as illustrated in Figure 1.

Figure 1: An example plot of V_{eff} with $M = 0$.

Problem 6

Using the chain rule to turn $\mathrm{d}r/\mathrm{d}\tau$ into $\mathrm{d}r/\mathrm{d}\phi$, along with equation (1) above, we can determine a differential equation for the shape of orbits:

$$
\begin{aligned}
\frac{\mathrm{d}r}{\mathrm{d}\tau} &= \frac{\mathrm{d}r}{\mathrm{d}\phi}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} \\
&= \frac{\mathrm{d}r}{\mathrm{d}\phi}\frac{-l}{r^2+b^2} \\
\implies \qquad \mathcal{E} &= \frac{1}{2}\left(\frac{\mathrm{d}r}{\mathrm{d}\phi}\frac{-l}{r^2+b^2}\right)^2 + V_{\text{eff}}(r) \\
\implies \qquad \frac{\mathrm{d}r}{\mathrm{d}\phi} &= \pm\frac{r^2+b^2}{l}\sqrt{2(\mathcal{E}-V_{\text{eff}})}
\end{aligned}
$$

See the attached python notebook for an attempted application of the Runge-Kutta method for integrating this equation.

Problem 7

$$V_{\text{eff}}(r) = \frac{-1}{\sqrt{r^2+1}} + \frac{25}{2(r^2+1)} - \frac{25}{(r^2+1)^{\frac{3}{2}}}$$

Minimum occurs ar $r \approx 21.5$ Bound orbits exist.

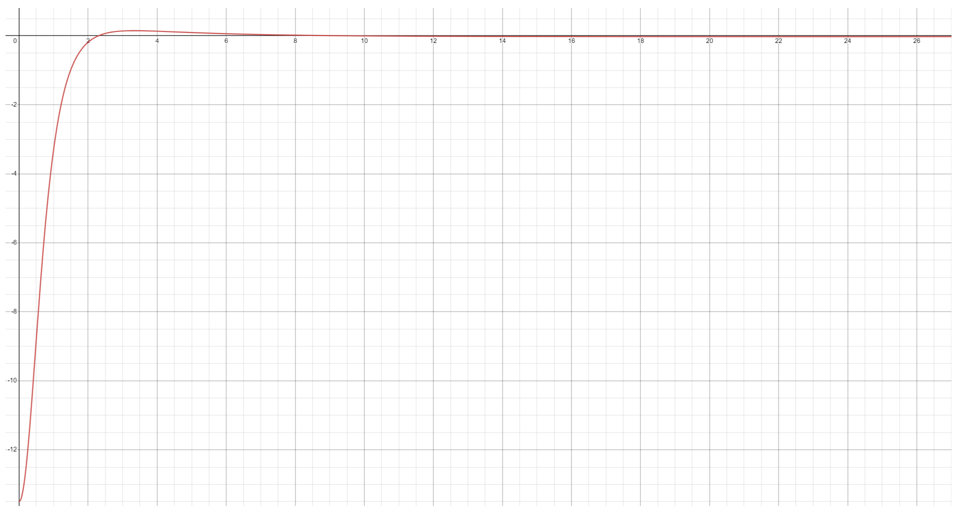


Figure 2: $V_{\text{eff}}(r)$ with $M=1$, $b=1$, $l=5$.

Problem 8

$$V_{\text{eff}}(r) = \frac{-1}{\sqrt{r^2+9}} + \frac{4}{2(r^2+9)} - \frac{4}{(r^2+9)^{\frac{3}{2}}}$$

$V_{\text{eff}} = \mathcal{E}$ when $r \approx 17.8$. With $\mathcal{E} = -0.05$, orbits will not extend beyond this maximum radius.

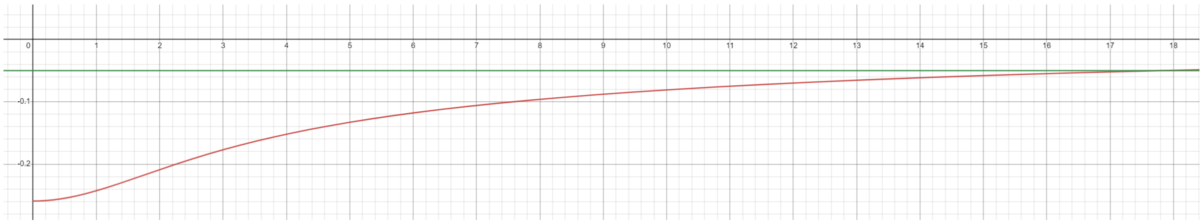


Figure 3: $V_{\text{eff}}(r)$ with $M = 1$, $b = 3$, $l = 2$.

Numeric Integration

Phys 610 Midterm

Problems 6-8

Sean Ericson

In [ ]:
```
# Imports
import matplotlib as mpl
import matplotlib.pyplot as plt
import numpy as np
from scipy.integrate import solve_ivp

mpl.rcParams['figure.dpi'] = 200
```

In [ ]:
```
def effective_potential(r, M, b, l):
    tmp = r*r + b*b
    return 0.5*((1 - 2*M/np.sqrt(tmp))*(1 + l*l/tmp) - 1)

# Equation to integrate (dr/dphi = r'(r; M, b, l, E))
def r_prime(r, M, b, l, E):
    return -1*(r*r + b*b) * np.sqrt(2 * (E - effective_potential(r, M, b, l))) / l
```

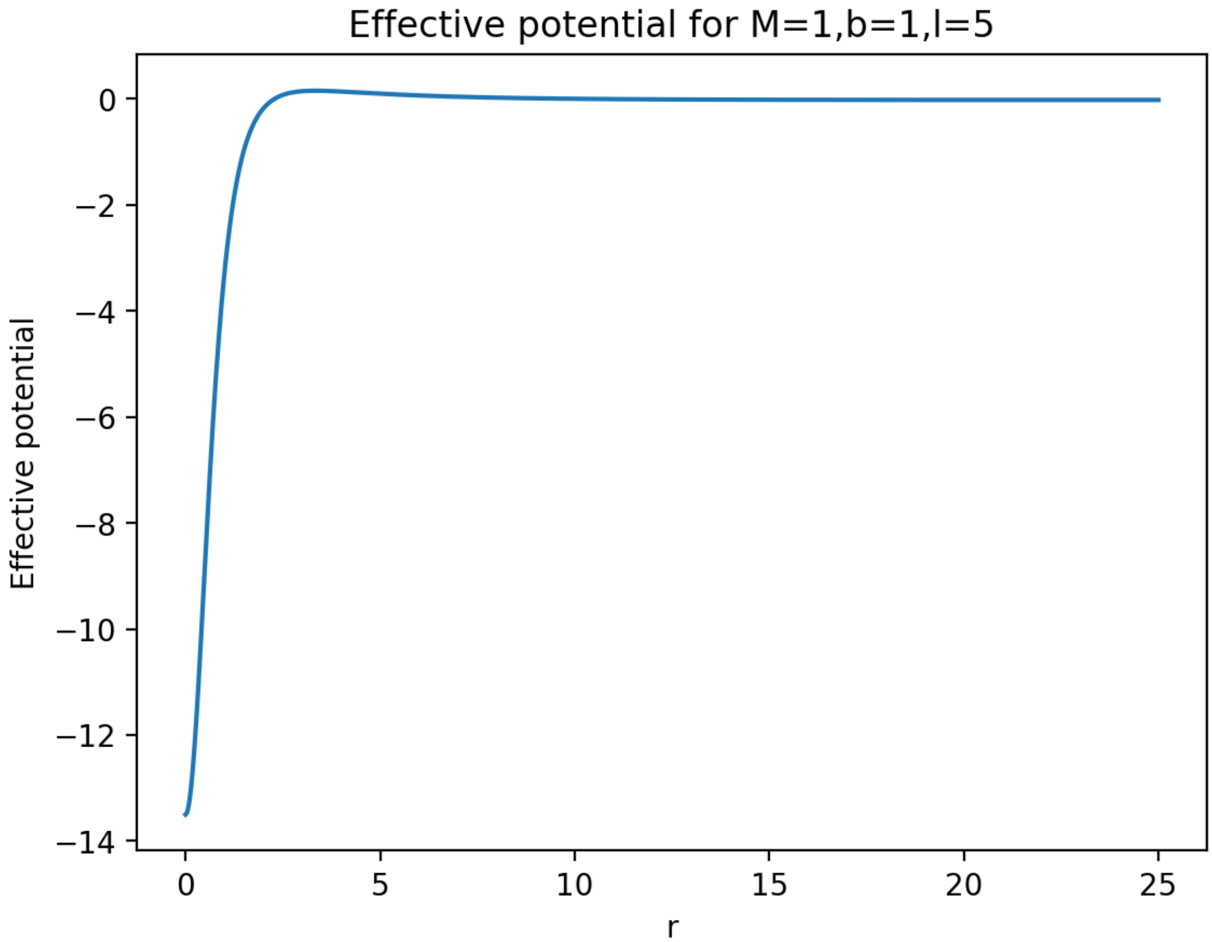
In [ ]:
```
def RungeKutta(f, y0, step_size, num_steps):
    ys = [y0]
    while len(ys) < num_steps:
        k1 = f(ys[-1])
        k2 = f(ys[-1] + step_size * k1 / 2)
        k3 = f(ys[-1] + step_size * k2 / 2)
        k4 = f(ys[-1] + step_size * k3)
        ys.append(ys[-1] + step_size * (k1 + 2*k2 + 2*k3 + k4) / 6)
    return ys
```

Problem 7

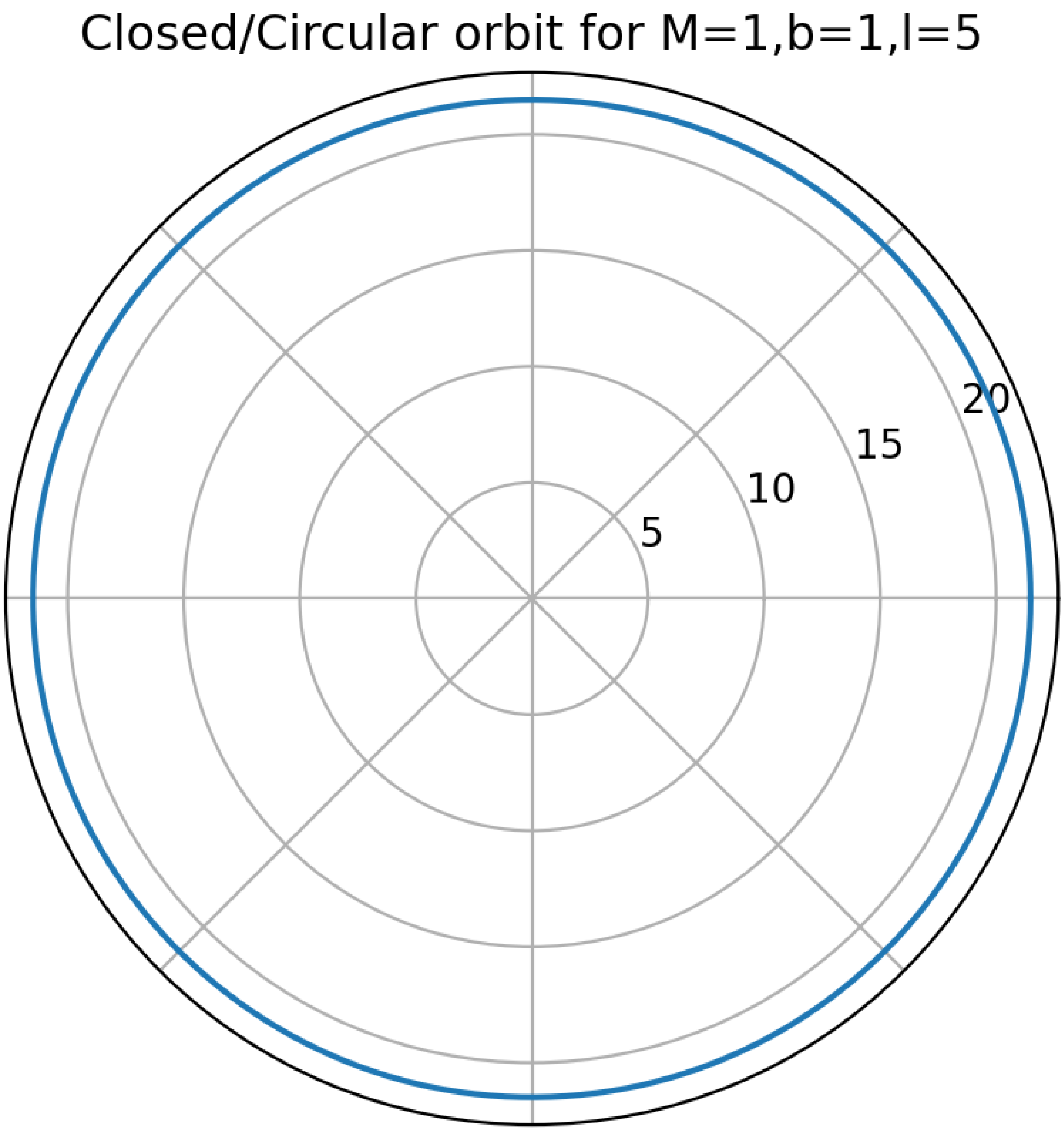
In [ ]:
```
M = 1
b = 1
l = 5
```

In [ ]:
```
rs = np.linspace(0, 25, 1000)
Vs = [effective_potential(r, M, b, l) for r in rs]
plt.plot(rs, Vs)
plt.title("Effective potential for M=1,b=1,l=5")
plt.xlabel("r")
plt.ylabel("Effective potential")
```

Out[ ]:
```
Text(0, 0.5, 'Effective potential')
```



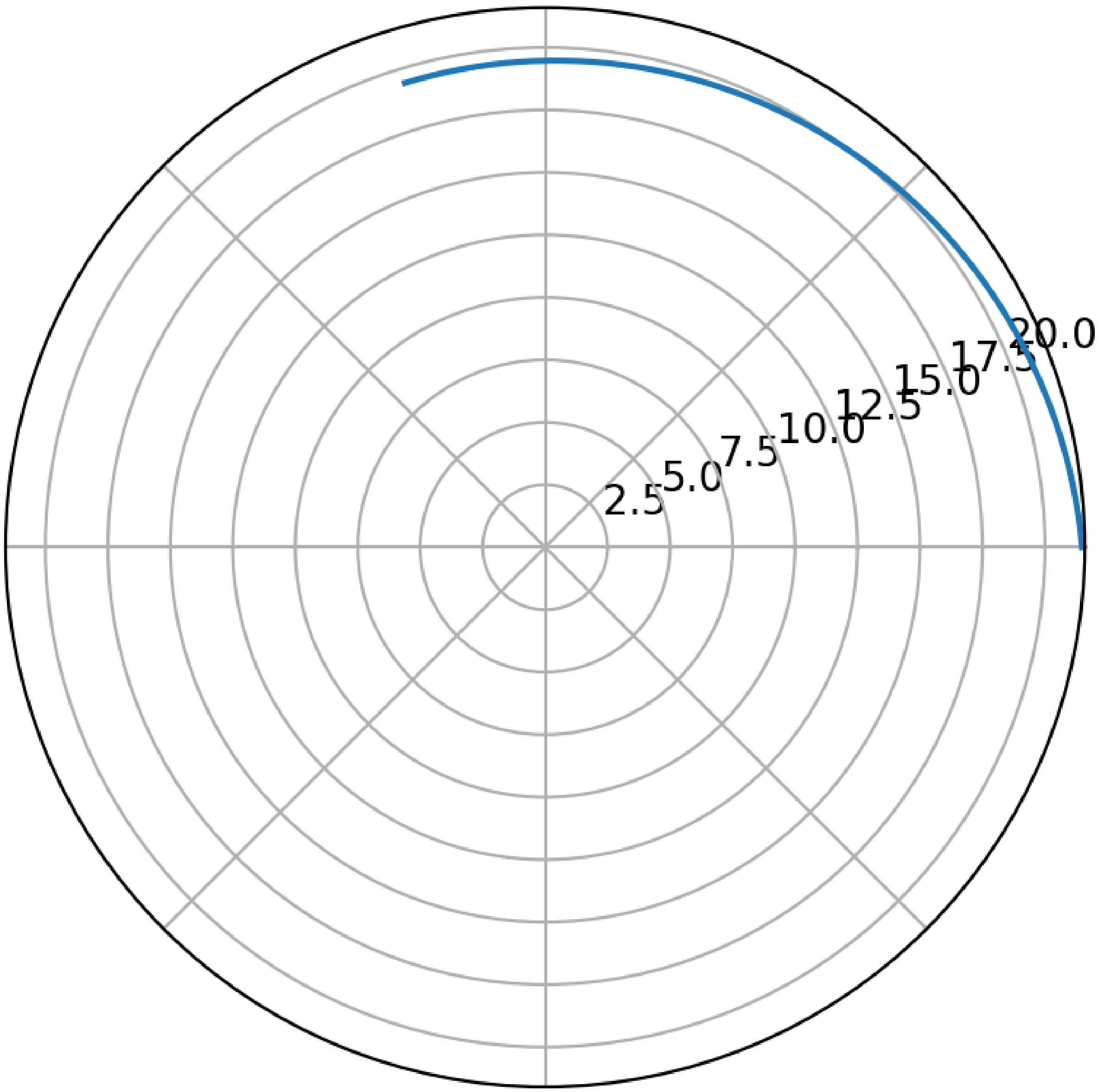
In [ ]:
```
r0 = 21.4906 # Start at local minimum calculated in Mathematica
E = effective_potential(r0, M, b, l) # Give test particle 0 kinetic energy
step_size = 0.01
final_angle = 2*np.pi
num_step = int(final_angle / step_size)
angles = np.linspace(0, final_angle, num_step)

func = lambda y: r_prime(y, M, b, l, E)
rs = RungeKutta(func, r0, step_size=step_size, num_steps=num_step)

#Make plt figure
fig = plt.figure()

#Make sub-plot with attribute "polar"
ax = fig.add_subplot(polar=True)

#Plot function
ax.plot(angles, rs)
ax.set_xticklabels([])

plt.title("Closed/Circular orbit for M=1,b=1,l=5")
plt.show()
```



In [ ]:
```
r0 = 21.4906 # Start at local minimum calculated in Mathematica
E = effective_potential(r0, M, b, l)*0.99  # Give test particle a little kinetic energ
step_size = 0.01
final_angle = 2*np.pi
num_step = int(final_angle / step_size)
angles = np.linspace(0, final_angle, num_step)

func = lambda y: r_prime(y, M, b, l, E)
rs = RungeKutta(func, r0, step_size=step_size, num_steps=num_step)

#Make plt figure
fig = plt.figure()

#Make sub-plot with attribute "polar"
ax = fig.add_subplot(polar=True)

#Plot function
ax.plot(angles, rs)
```

```
ax.set_xticklabels([])

plt.title("")
plt.show()
```

```
C:\Users\Sean\AppData\Local\Temp\ipykernel_15476\4176942516.py:7: RuntimeWarning: inv
alid value encountered in sqrt
  return -1*(r*r + b*b) * np.sqrt(2 * (E - effective_potential(r, M, b, l))) / l
```



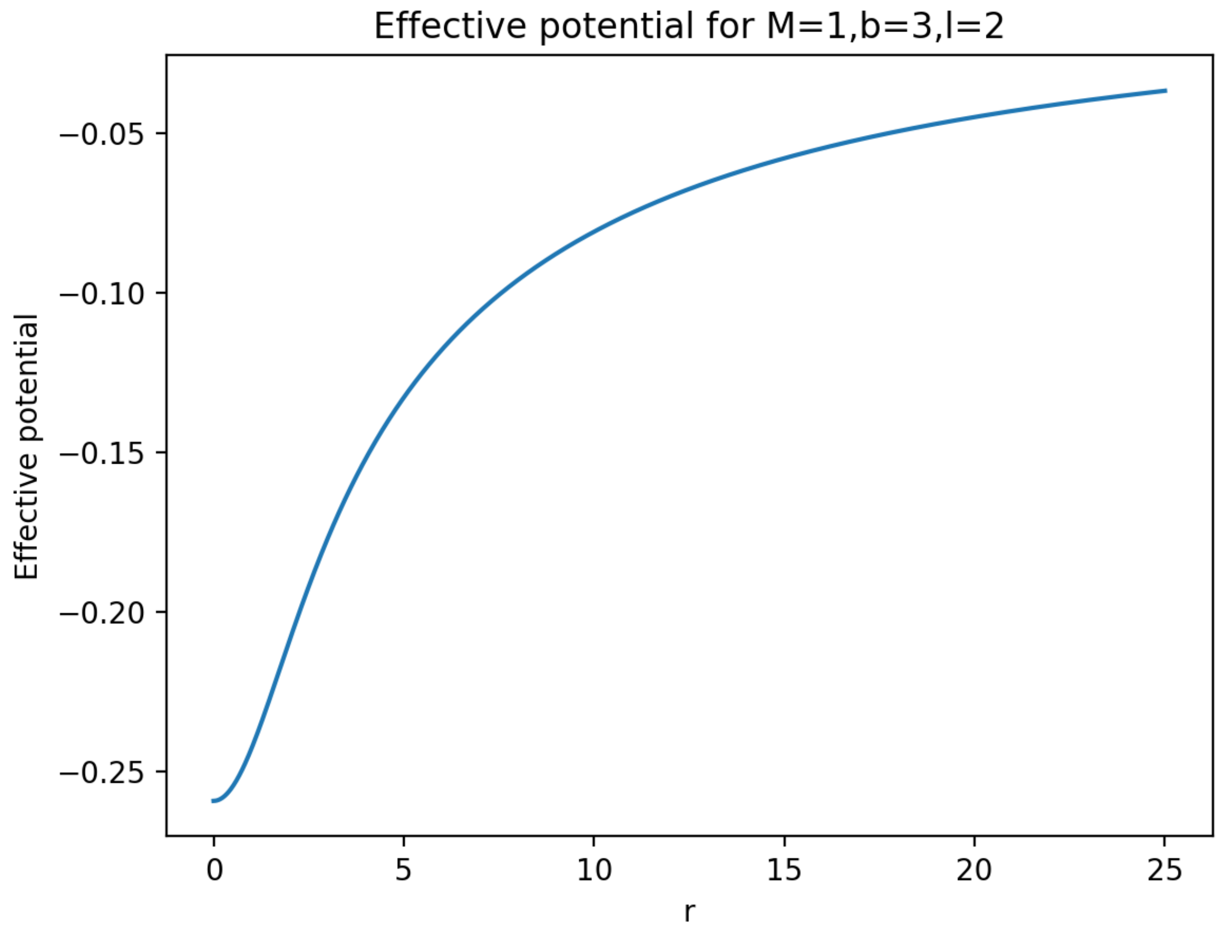
Integrator seems to break when I try a non-circular orbit

Problem 8

In [ ]:
```
M = 1
b = 3
l = 2
E = -0.05
```

In [ ]:
```
rs = np.linspace(0, 25, 1000)
Vs = [effective_potential(r, M, b, l) for r in rs]
```

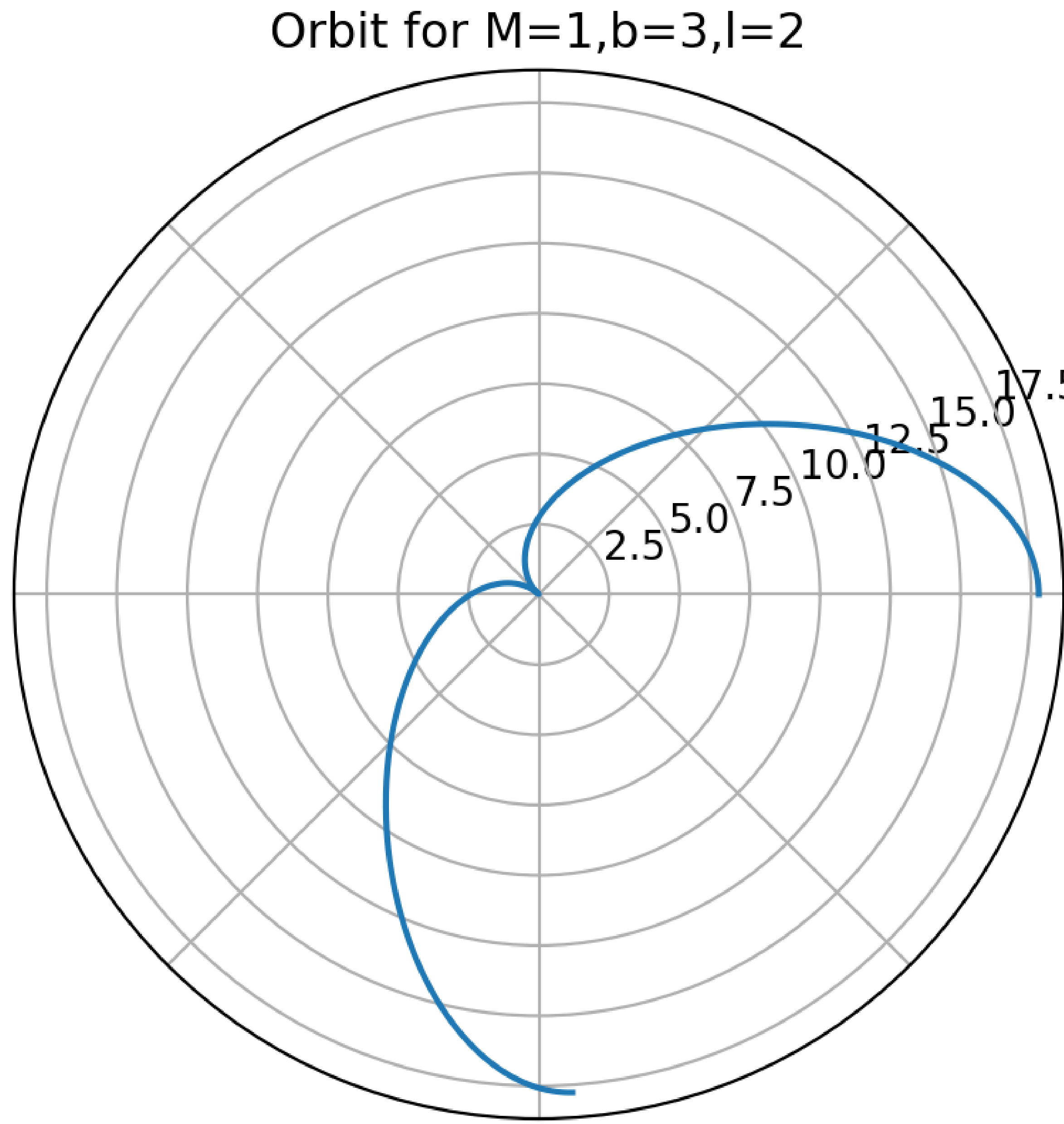
```
plt.plot(rs, Vs)
plt.title("Effective potential for M=1,b=3,l=2")
plt.xlabel("r")
plt.ylabel("Effective potential")
plt.show()
```



In [ ]:

```
r0 = 17.7759
step_size = 0.01
final_angle = 2*np.pi
num_step = int(final_angle / step_size)
angles = np.linspace(0, final_angle, num_step)

func = lambda y: r_prime(y, M, b, l, E)
rs = RungeKutta(func, r0, step_size=step_size, num_steps=num_step)
rs = [abs(r) for r in rs]

#Make plt figure
fig = plt.figure()

#Make sub-plot with attribute "polar"
ax = fig.add_subplot(polar=True)

#Plot function
ax.plot(angles, rs)
ax.set_xticklabels([])

plt.title("Orbit for M=1,b=3,l=2")
plt.show()
```

```
C:\Users\Sean\AppData\Local\Temp\ipykernel_15476\4176942516.py:7: RuntimeWarning: inv
alid value encountered in sqrt
  return -1*(r*r + b*b) * np.sqrt(2 * (E - effective_potential(r, M, b, l))) / l
```




```
In [ ]:
```

Calculations

In[44]:= **Definitions and Assumptions;**

```
In[45]:= $Assumptions = {M > 0, r > 0, θ ∈ ℝ, t ∈ ℝ, ϕ ∈ ℝ, τ ∈ ℝ};
n = 4;
coord = {t, r, θ, ϕ};
vel = {Dt[t, τ], Dt[r, τ], Dt[θ, τ], Dt[ϕ, τ]};

In[49]:= affine :=
  affine = FullSimplify[Table[1/2 * Sum[(inversemetric[[i, s]]) * (D[metric[[s, j]], coord[[k]]] +
         D[metric[[s, k]], coord[[j]]] - D[metric[[j, k]], coord[[s]]]),
      {s, 1, n}], {i, 1, n}, {j , 1, n}, {k, 1, n}]];

In[50]:= geodesic := geodesic = FullSimplify[
    Table[-Sum[affine[[i, j, k]] × vel[[j]] × vel[[k]], {j, 1, n}, {k, 1, n}], {i, 1, n}]];

In[51]:= listaffine :=
  Table[If[UnsameQ[affine[[i, j, k]], 0], {ToString[Γ[coord[[i]], coord[[j]], coord[[k]]]],
     affine[[i, j, k]]}], {i, 1, n}, {j, 1, n}, {k, 1, j}];

In[52]:= listgeodesic := Table[{Dt[vel[[i]], τ], "=", geodesic[[i]]}, {i, 1, n}];

In[53]:=
```

Problem 1;

In[57]:= `metric = DiagonalMatrix[{-(1 - 2 M/Sqrt[r^2 + b^2]), (1 - 2 M/Sqrt[r^2 + b^2])^-1, r^2 + b^2, (r^2 + b^2) Sin[θ]^2}];`

$$\text{metric} = \text{DiagonalMatrix}\left[\left\{-\left(1-\frac{2M}{\sqrt{r^2+b^2}}\right),\ \left(1-\frac{2M}{\sqrt{r^2+b^2}}\right)^{-1},\ r^2+b^2,\ \left(r^2+b^2\right)\text{Sin}[\theta]^2\right\}\right];$$

```
inversemetric = Simplify[Inverse[metric]];

In[59]:= metric // MatrixForm
inversemetric // MatrixForm
```

Out[59]//MatrixForm=

$$\begin{pmatrix} -1+\frac{2M}{\sqrt{b^2+r^2}} & 0 & 0 & 0 \\ 0 & \frac{1}{1-\frac{2M}{\sqrt{b^2+r^2}}} & 0 & 0 \\ 0 & 0 & b^2+r^2 & 0 \\ 0 & 0 & 0 & \left(b^2+r^2\right)\text{Sin}[\theta]^2 \end{pmatrix}$$

Out[60]//MatrixForm=

$$\begin{pmatrix} \frac{\sqrt{b^2+r^2}}{2M-\sqrt{b^2+r^2}} & 0 & 0 & 0 \\ 0 & 1-\frac{2M}{\sqrt{b^2+r^2}} & 0 & 0 \\ 0 & 0 & \frac{1}{b^2+r^2} & 0 \\ 0 & 0 & 0 & \frac{\text{Csc}[\theta]^2}{b^2+r^2} \end{pmatrix}$$

In[61]:=

Problem 2;

In[65]:=
```
TableForm[Partition[DeleteCases[Flatten[listaffine], Null], 2], TableSpacing → {2, 2}]
```

Out[65]//TableForm=

Γ[t, r, t]	$\frac{M\,r}{(b^2+r^2)\left(-2M+\sqrt{b^2+r^2}\right)}$
Γ[r, t, t]	$\frac{M\,r\left(-2M+\sqrt{b^2+r^2}\right)}{(b^2+r^2)^2}$
Γ[r, r, r]	$-\frac{M\,r}{(b^2+r^2)\left(-2M+\sqrt{b^2+r^2}\right)}$
Γ[r, θ, θ]	$r\left(-1+\frac{2M}{\sqrt{b^2+r^2}}\right)$
Γ[r, ϕ, ϕ]	$r\left(-1+\frac{2M}{\sqrt{b^2+r^2}}\right)\mathrm{Sin}[\theta]^2$
Γ[θ, θ, r]	$\frac{r}{b^2+r^2}$
Γ[θ, ϕ, ϕ]	$-\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]$
Γ[ϕ, ϕ, r]	$\frac{r}{b^2+r^2}$
Γ[ϕ, ϕ, θ]	$\mathrm{Cot}[\theta]$

In[66]:=
```
TableForm[Partition[DeleteCases[Flatten[listaffine], Null], 2],
   TableSpacing → {2, 2}] /. {b → 0} // FullSimplify
```

Out[66]//TableForm=

Γ[t, r, t]	$\frac{M}{r\,(-2M+r)}$
Γ[r, t, t]	$\frac{M\,(-2M+r)}{r^3}$
Γ[r, r, r]	$\frac{M}{2Mr-r^2}$
Γ[r, θ, θ]	$2M-r$
Γ[r, ϕ, ϕ]	$(2M-r)\,\mathrm{Sin}[\theta]^2$
Γ[θ, θ, r]	$\frac{1}{r}$
Γ[θ, ϕ, ϕ]	$-\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]$
Γ[ϕ, ϕ, r]	$\frac{1}{r}$
Γ[ϕ, ϕ, θ]	$\mathrm{Cot}[\theta]$

In[67]:=

Problem 3;

In[70]:= `TableForm[listgeodesic, TableSpacing → {2, 2}] // FullSimplify // TraditionalForm`

Out[70]//TraditionalForm=

$$\frac{d^2t}{d\tau^2} = -\frac{2Mr\frac{dr}{d\tau}\frac{dt}{d\tau}}{(b^2+r^2)\left(\sqrt{b^2+r^2}-2M\right)}$$

$$\frac{d^2r}{d\tau^2} = r\left(\frac{\left(\sqrt{b^2+r^2}-2M\right)\left((b^2+r^2)^{3/2}\left(\sin^2(\theta)\left(\frac{d\phi}{d\tau}\right)^2+\left(\frac{d\theta}{d\tau}\right)^2\right)-M\left(\frac{dt}{d\tau}\right)^2\right)}{(b^2+r^2)^2}+\frac{M\left(\frac{dr}{d\tau}\right)^2}{(b^2+r^2)\left(\sqrt{b^2+r^2}-2M\right)}\right)$$

$$\frac{d^2\theta}{d\tau^2} = \sin(\theta)\cos(\theta)\left(\frac{d\phi}{d\tau}\right)^2-\frac{2r\frac{d\theta}{d\tau}\frac{dr}{d\tau}}{b^2+r^2}$$

$$\frac{d^2\phi}{d\tau^2} = 2\frac{d\phi}{d\tau}\left(-\frac{r\frac{dr}{d\tau}}{b^2+r^2}-\cot(\theta)\frac{d\theta}{d\tau}\right)$$

In[71]:=

Problem 4;

In[74]:=
```
e = -(metric.vel)[[1]];
l = -(metric.vel)[[4]];
```

In[76]:=
```
e // FullSimplify
l // FullSimplify
```

Out[76]=

$$\left(1-\frac{2M}{\sqrt{b^2+r^2}}\right)\text{Dt}[t,\tau]$$

Out[77]=

$$-\left(\left(b^2+r^2\right)\text{Dt}[\phi,\tau]\,\text{Sin}[\theta]^2\right)$$

In[78]:= `vel`$^{\mathsf{T}}$`.metric.vel // FullSimplify // TraditionalForm`

Out[78]//TraditionalForm=

$$\left(\frac{2M}{\sqrt{b^2+r^2}}-1\right)\left(\frac{dt}{d\tau}\right)^2+\frac{\left(\frac{dr}{d\tau}\right)^2}{1-\frac{2M}{\sqrt{b^2+r^2}}}+\left(b^2+r^2\right)\left(\sin^2(\theta)\left(\frac{d\phi}{d\tau}\right)^2+\left(\frac{d\theta}{d\tau}\right)^2\right)$$

Problem 7;

In[87]:= effectivePotential $= \frac{1}{2}\left(\left(1-\frac{2M}{\sqrt{r^2+b^2}}\right)\left(1+\frac{L^2}{r^2+b^2}\right)-1\right)$;

In[88]:=
```
conditions = {M → 1, b → 1, L → 5};
```

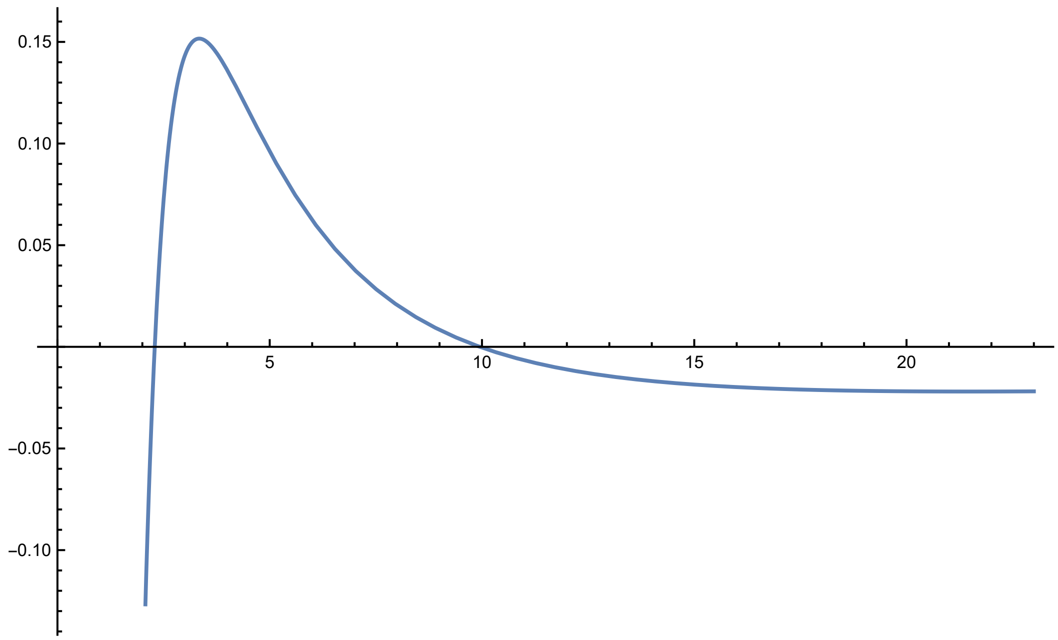
In[89]:=
```
extrema = FullSimplify[
   Solve[D[effectivePotential /. conditions, r] == 0, r], Assumptions → {L > 0}] // N
```

Out[89]=
```
{{r → 3.33962}, {r → 21.4906}}
```

In[90]:=
```
D[D[effectivePotential, r], r] /. extrema /. conditions // N
```

Out[90]=
```
{-0.112016, 0.0000839707}
```

In[91]:=
```
Plot[effectivePotential /. conditions, {r, 0, 23}]
```
Out[91]=



Problem 8;

In[94]:=
```
conditions = {M → 1, b → 3, L → 2};
```
In[95]:=
```
Solve[(effectivePotential /. conditions) == -0.05, r]
```
Solve: Solve was unable to solve the system with inexact coefficients. The answer was obtained by solving a corresponding exact system and numericizing the result.

Out[95]=
```
{{r → 17.7759}}
```
In[96]:=
```
Plot[(effectivePotential /. conditions), {r, 0, 18}]
```
Out[96]=