Homework 10

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1. Consider arbitrary singular value for the matrix A, that is to say that for arbitrary singular value σ and eigenvector v, then $(A^*A)v = \sigma^2 v$. Then,
$$\begin{aligned} A^{-1}Av &= A^{-1}\sigma v \\ Iv &= \sigma^2(A^*A)^{-1}v \\ \frac{1}{\sigma^2}v &= (A^*A)^{-1}v. \end{aligned}$$
Therefore, the singular value for the inverse of A is $\sqrt{\frac{1}{\sigma^2}} = \frac{1}{\sigma}$. Thus, the singular values for A^{-1} are $\{\frac{1}{\sigma_n}, ..., \frac{1}{\sigma}\}$.

2. Consider the matrix A such that $A^2 = 0$. The characteristic polynomial of A^2 would then be $(x^3)^2 = 0$. This then implies that the characteristic polynomial of A would be $x^3 = 0$. This then implies that all three eigenvalues of the matrix A are $\lambda = 0$. Therefore, consider the following set of matrices, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$.

3. (a) Let$B = \{v, Av, ..., A^{n-1}v\} \subseteq \mathbb{R}^n, |B| = n$, so it is sufficient to show that B is linearly independent. Suppose that $a_0v + a_1(Av) + ... + a_{n-1}(A^{n-1}v) = 0$. If we apply A^{n-1} we get $a_i \in \mathbb{F}^n$
$$\begin{aligned} A^{n-1}(a_0v + ... + a_{n-1}A^{n-1}v) &= 0 \\ a_0A^{n-1}v + a_1A^nv + ... + a_{n-1}A^{2n-2}v &= 0 \\ a_0A^{n-1}v &= 0 \\ a_0 &= 0. \end{aligned}$$
By applying subsequent As we will get that $a_i = 0$ for $0 \leq i \leq n-1$. Therefore, $a_0 = a_1 = ... = a_{n-1} = 0$ which means that B is linearly independent and is therefore a basis of $\mathbb{F}^n$.

   (b) Let $x \in \mathbb{F}^n \Rightarrow x = a_0v + a_1Av + ... + a_{n-1}A^{n-1}v, a_i \in \mathbb{F}^n$. Then,
$$\begin{aligned} A^nx &= a_0A^nv + ... + a_{n-1}A^{2n-1}v \\ &= a_0(0) + ...+)a_{n-1}(0) \\ A^nx &= 0, \forall x \in \mathbb{F}^n. \end{aligned}$$

Therefore, $A^n = 0$.

(c) If λ is an eigenvalue of $A \Rightarrow \exists v \neq 0 \in \mathbb{F}^n$ such that $Av = \lambda v$. Then,

$$\begin{aligned} A^n v &= \lambda^n \\ 0 &= \lambda^n v \\ \lambda^n &= 0. \end{aligned}$$

Therefore, $v \neq 0$. Thus, $\lambda = 0$ is the only eigenvalue of A.

(d)

$$\begin{aligned} A^{n-1}v &\neq 0 \\ A(A^{m-1}v) \neq 0, \forall m \leq n-1. \end{aligned}$$

Therefore, $A^{m-1}v$ is not an eigenvector for A $m \leq n-1$. If $m \leq n-1 \Rightarrow A^{n-1}v \neq 0$ and $A(A^{n-1}v) = A^n v = 0$. A^{n-1} is an eigenvector of A.

(e) Let P be a matrix whose columns are $A^{n-1}v, ..., Av, v$. Then,

$$\begin{aligned} Ap &= PI \\ P^{-1}AP &= I. \end{aligned}$$

Then, $J = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$.

4. Task 1: Increasing to $n = -50$ does not affect the Jordan form of B. However, it will make the size of the Jordan form of A become 5 times smaller, decreasing its factor from 10^{-5} to 10^{-25}

Task 2: The Jordan form of each random matrix is always $J = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

The matrix of B^4 does confirm the value of J because of how as each successive product $B, B^2, ...,$ the amount of data being stored decreases. This is indicative of why J comes out this way because B continues to lose information through iterations.