Homework 14, 15 & 16

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[3]

Let us consider the point $z = x + iy$, then we essentially want to exchange the x and y values as well as translate the region over to the left by 1 unit. Therefore the transformation is $w = i(x+iy)+1 = iz+1$.

[2]

Specifically when $c_1 = 0$ we get $\frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{x}{x^2+y^2} - \frac{iy}{x^2+y^2}$ which then means that we have $u = \frac{x}{x^2+y^2}, v = -\frac{y}{x^2+y^2}$ or inversely, we have that $x = \frac{u}{u^2+v^2}, y = -\frac{v}{u^2+v^2}$. Thus, for the given region, we have that $x < c_1 \Rightarrow \frac{u}{u^2+v^2} < c_1 < 0 \Rightarrow \frac{u}{c_1} > u^2 + v^2$. Then, $u^2 + v^2 - \frac{u}{c_1} < 0 \Rightarrow v^2 + u^2 - 2\frac{1}{2c_1}u + \frac{1}{4c_1^2} - \frac{1}{4c_1^2} < 0 \Rightarrow (u - \frac{1}{2c_1^2})^2 + v^2 < (\frac{1}{2c_1})^2$. So we have that this is the interior of a circle centered at $(\frac{1}{2c_1}, 0)$ with radius $\frac{1}{2c_1}$. If $c_1 = 0$ then the radius of the circle becomes infinite, so the image is then the entire plane.

[9]

Given the mapping $w = \frac{i}{z} \Rightarrow u+iv = \frac{i}{x+iy} = \frac{i(x-iy)}{(x+iy)(x-iy)} = \frac{ix+y}{x^2+y^2} = \frac{y}{x^2+y^2} + i\frac{x}{x^2+y^2}$. This then implies that we have $u = \frac{y}{x^2+y^2}, v = \frac{x}{x^2+y^2}$. Then we have that $\frac{u}{v} = \frac{y}{x}$ which shows that $x = \frac{vy}{u}$ which then gives us that $v = \frac{\frac{vy}{u}}{(\frac{vy}{u})^2+y^2} \Rightarrow vy^2(v^2+u^2) = uvy \Rightarrow vy^2(v^2+u^2) - uvy = 0$. This then gives us that $vy(y(v^2+u^2)-u) = 0$ which implies that either $v = 0$ or $y = 0$ or $y = \frac{u}{v^2+u^2}$. Since we know that $y < 1$ we have that $\frac{u}{v^2+u^2} < 1 \Rightarrow v^2+u^2 > u$ or $u^2+v^2-u > 0 \Rightarrow U62 - u + \frac{1}{4} + v^2 > \frac{1}{4} \Rightarrow (u-\frac{1}{2})^2 + v^2 > (\frac{1}{2})^2$ which shows that that outer part of the circle with center at $(\frac{1}{2}, 0)$ and radius $\frac{1}{2}$. Then for $y > 0$ we have that $\frac{u}{v^2+u^2} > 0 \Rightarrow u > 0$ which still falls within the same circle.

[1]

We can use the equation $\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)} = \frac{(w-w_1)(w_2-w_3)}{(w-w_3)(w_2-w_1)} \Rightarrow \frac{(z+1)(0-1)}{(z-1)(0+1)} = \frac{(w+i)(1-i)}{(w-i)(1+i)} \Rightarrow w = \frac{i-z}{i+z}$

[3]

(a) If we want to find the inverse of $w = \frac{i-z}{i+z}$ we get

$$\begin{aligned} w &= \frac{i-z}{i+z} \\ w(i+z) &= i - z \\ wi + wz &= i - z \\ wz + z &= i - wi \\ z(w+1) &= i(1-w) \\ z &= \frac{i(1-w)}{w+1} \\ z &= i\frac{1-w}{w+1}. \end{aligned}$$

Therefore, we find that the function for the problem $w = i\frac{1-z}{1+z}$ is actually the inverse function. Then we inherently know that the transformation $w = i\frac{1-z}{1+z}$ maps the disk $|z| \leq 1$ onto the half plane $Im(w) = \geq 0$.

(b) To verify this transformation $w = iW = i(i\frac{1-Z}{1+Z} = -1\frac{1-Z}{1+Z} = \frac{Z-1}{1+Z}$ then given that $Z = z-1$ we have that $w = \frac{(z-1)-1}{1+z-1} = \frac{z-2}{z}$ which verifies that this modification is indeed valid. Therefore, if we refer to the previous example, we translate z to the right by 1 with $Z = z-1$ and then we apply the inverse mapping from part a to turn the disk to the upper half plane and then when we multiply by i, we rotate this plane 90 degrees to get the half plane $Re(w) \leq 0$.

[4]

We have that $w = e^z = e^{x+iy} = e^x + e^{iy} \Rightarrow |w| = e^x \geq 1, 0 \leq arg(w) \leq \pi$ so that w lies in the portion of the closed upper half plane external to the open unit disk. Therefore, the map is onto as $w = re^{i\theta}$ with $r \geq 1$ and $0 \leq \theta \leq \pi$ then $e^{x+iy} = w$, where $x = \log(r) \geq 0$ and $y = \theta$.

[4]

Here we have that $w = z^2$ which implies that $u + iv = (x+iy)^2 = (x^2 - y^2) + 2xyi$ which then shows us that $u = x^2 - y^2, v = 2xy$. When $x = 1$ we have that $u = 1 - y^2, v = 2y$ which then implies that $y = \frac{v}{2}$ and $y = 1 - \frac{v^2}{4} = \frac{4-v^2}{4} \Rightarrow 4u = 4 - v^2 \Rightarrow v^2 = 4(1-u) = -4(u-1)$. Then when $y = x, u = 0$ and $y = -x, u = 0$. Thus we have that the image of the line $x = 1$ is the parabola $v^2 = -4(u-1)$ and the images of lines $y = \pm x$ is the v-axis, i,e, $u = 0$ as shown in the given figure. Now, for C, $z = 1 \Rightarrow w = 1$. Thus, $C = (1,0)$ corresponds to $C' = (1,0)$. Which then implies that $A = 0 \Rightarrow w = 0$. Thus, $A = (0,0)$ corresponds to $A' = (0,0)$. Then, $D = (1,1) \Rightarrow w = (1+i)^2 = 2i = (0,2)$; thus, $D = (1,1)$ corresponds to $D' = (0,2)$. And finally, $B = (1,-1) \Rightarrow w = (1-i)^2 = -2i = (0,-2)$; thus, $B = (1,-1)$ corresponds to $B' = (0,-2)$.

[6]

(a) We know that $g(w) = \phi^{\frac{1}{2}} e^{i\frac{\theta}{2}}$ and so we need to limit the bounds of θ and for $z = 2$ we set the bounds $-\pi < \theta < \pi$ and $\phi > 0$

(b) We know that $g(w) = \phi^{\frac{1}{2}} e^{i\frac{\theta}{2}}$ and so we need to limit the bounds of θ and for $z = -2$ we set the bounds $\pi < \theta < 3\pi$ and $\phi > 0$

(c) We know that $g(w) = \phi^{\frac{1}{2}} e^{i\frac{\theta}{2}}$ and so we need to limit the bounds of θ and for $z = -i$ we set the bounds $2\pi < \theta < 4\pi$ and $\phi > 0$

[2]

(a) Given that $u(x,y) = xy$, this is harmonic because $u_{xx} + u_{yy} = 0 + 0 = 0$. Therefore, to find the harmonic conjugate of u we have that $v(x,y) = \int_{(x_0,y_0)}^{(x,y)} -u_t(s,t)ds + u_s(s,t)dt = \int_{(x_0,y_0)}^{(x,y)} -u_t(s,t)ds + \int_{(x_0,y_0)}^{(x,y)} u_s(s,t)dt = \int_{(x_0,y_0)}^{(x,y)} -sds + \int_{(x_0,y_0)}^{(x,y)} tdt = \frac{1}{2}(-x^2) + \frac{1}{2}y^2 = -\frac{1}{2}(x^2 + y^2)$ which means that the harmonic conjugate $v(x,y) = -\frac{1}{2}(x^2 - y^2)$. This therefore means that $f(z) = xy + i(-\frac{1}{2}(x^2 - y^2)) = -\frac{i}{2}z^2$.

(b) Given that $u(x,y) = y^3 - 3x^2y$ we can see that this is harmonic because $u_{xx} + u_{yy} = -6y + 6y = 0$. Then to find the harmonic conjugate v, we can do $v(x,y) = \int_{(x_0,y_0)}^{(x,y)} -u_t(s,t)ds + u_s(s,t)dt = \int_{(x_0,y_0)}^{(x,y)} -u_t(s,t)ds + \int_{(x_0,y_0)}^{(x,y)} u_s(s,t)dt = \int_{(x_0,y_0)}^{(x,y)} -(3t^2 - 3s^2)ds + \int_{(x_0,y_0)}^{(x,y)} (-6st)dt = -3xy^2 + x^3$. Therefore, this gives that $f(z) = y^3 - 3x^2y + i(-2xy^2 + x^3) = iz^3$