This is a take home midterm. You can use your notes, my online notes on canvas and the textbooks book. You are supposed to work on your own text without external help. I'll be available to answer question in person or via email. Please, write clearly and legibly and take a readable scan before uploading.

Name (print): Sean Eva

Question:	1	2	3	4	5	Total
Points:	20	35	15	15	15	100
Score:						

Question 1 .... *20 point*

Let X be a normal random variable with $\mu = \mathbb{E}(X) = 2$ and $\sigma^2 = \mathrm{var}(X) = 4$.

(a) (10 points) Compute $\mathbb{P}(X < 0)$. Express the result in term of the cumulative distribution function Φ of a Normal Standard r.v.. Φ is usually called the *probability integral.*

Solution: Given that $\mu = \mathbb{E}(x) = 2$ and $\sigma^2 = var(x) = 4, \sigma = 2$. Therefore,

$$\begin{aligned}\mathbb{P}(X < 0) &= \mathbb{P}(Z < \frac{0-2}{2}) \\ &= \mathbb{P}(Z < -1) \\ &= \Phi(-1)\end{aligned}$$

(b) (10 points) Find δ such that

$$\mathbb{P}(2 - \delta < X < 2 + \delta) = 0.95\,.$$

Express the result in term of the α critical value z_α defined as $\Phi(-z_\alpha) = \alpha$.

Solution:

$$\begin{aligned}\mathbb{P}(2 - \delta < X < 2 + \delta) &= 0.95 \\ \mathbb{P}(\frac{2-\delta-2}{2} < Z < \frac{2+\delta-2}{2} &= 0.95 \\ \mathbb{P}(\frac{-\delta}{2} < Z < \frac{\delta}{2}) &= 0.95 \\ \Phi(\frac{\delta}{2}) - \Phi(\frac{-\delta}{2}) &= 0.95 \\ \frac{\delta}{2} &= 1.96 \\ \delta &= 3.92\end{aligned}$$

Question 2 . . . . *35 point*

Let X and Y be two r.v. such that the marginal p.d.f. of X is

$$f_X(x) = \begin{cases} 4xe^{-2x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

and the conditional p.d.f. of Y given X is

$$f_{Y|X}(y|x) = \begin{cases} \frac{1}{x} & 0 < y < x \\ 0 & \text{otherwise}\,. \end{cases}$$

This means that, given $X = x$, Y in uniform in $[0, x]$.

(a) (10 points) Write the joint p.d.f. $f(x, y)$ of X and Y.

Solution:

$$\begin{aligned} f_{X,Y}(x,y) &= f_{Y|X}(y|x)f_X(x) \\ &= \frac{1}{x}4xe^{-2x} \\ &= \begin{cases} 4e^{-2x} & 0 < y < x < \infty \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

(b) (10 points) Compute the marginal p.d.f. of $f_Y(y)$ of Y and the conditional p.d.f. $f_{X|Y}(x|y)$ of X given Y.

Solution:

$$\begin{aligned} f_Y(y) &= \int_y^\infty f_{X,Y}(x,y)dx \\ &= \int_y^\infty 4e^{-2x} \\ &= 4[\frac{e^{-2x}}{-2}]_y^\infty \\ &= \begin{cases} 2e^{-2y} & y > 0 \\ 0 & \text{otherwise} \end{cases} \\ f_{X|Y}(x|y) &= \frac{f_{X,Y}(x,y)}{f_Y(y)} \\ &= \frac{4e^{-2x}}{2e^{-2y}} \\ &= \begin{cases} 2e^{-2(x-y)} & x > y \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

(c) (15 points) Compute $\mathbb{P}(Y > X/2)$.(**Hint**: consider first $\mathbb{P}(Y > X/2|X = x)$.)

Solution:

$$
\begin{aligned}
\mathbb{P}(Y > \frac{x}{2}|X = x) &= \int_{\frac{x}{2}}^{x} f_{Y|X}(y|x)dy \\
&= \int_{\frac{x}{2}}^{x} \frac{1}{x}dy \\
&= \frac{1}{x}(x - \frac{x}{2}) \\
&= \frac{1}{2}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\mathbb{P}(Y > \frac{X}{2}) &= \int_{0}^{\infty} \mathbb{P}(Y > \frac{x}{2}|X = x)f_X(x)dx \\
&= \int_{0}^{\infty} \frac{1}{2}4xe^{-2x}dx \\
&= 2[x\int e^{-2x}dx - \int \frac{d}{dx}x\int e^{-2x}dxdx]_0^\infty \\
&= 2[x\frac{e^{-2x}}{-2} - \frac{e^{-2x}}{4}]_0^\infty \\
&= \frac{2}{4} \\
&= \frac{1}{2}
\end{aligned}
$$

Question 3 .... *15 point*

Let X and Y be two independent Normal Standard r.v., that is the joint p.d.f. of X and Y is

$$f_{X,Y}(x,y) = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}} .$$

Call

$$U = X + Y$$
$$V = X - Y .$$

Compute the joint p.d.f. of U and V. Are they independent?

Solution: We could rewrite our terms of U and V as $X = \frac{U+V}{2}, Y = \frac{V-U}{2}$. Therefore,

$$\begin{aligned}
f_{X,Y}(x,y) &= \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}} \\
f_{U,V}(u,v) &= \frac{1}{2\pi} e^{-\frac{(\frac{u+v}{2})^2+(\frac{v-u}{2})^2}{2}} \\
&= \frac{1}{2\pi} e^{-\frac{(\frac{u^2+2uv+v^2}{4})+(\frac{v^2-2uv+u^2}{4})}{2}} \\
&= \frac{1}{2\pi} e^{-\frac{(\frac{u^2+v^2}{2})}{2}} \\
&= \frac{1}{2\pi} e^{-\frac{u^2+v^2}{4}} .
\end{aligned}$$

U and V are independent if this joint p.d.f. can be represented as a product of a p.d.f. in terms of u and a p.d.f. in terms of v. This is possible as we could write $f_U(u) = \frac{1}{2\pi} e^{\frac{u^2}{-4}}$ and $f_V(v) = e^{\frac{v^2}{-4}}$. Since $f_U(u) * f_V(v) = f_{U,V}(u,v)$, we can say that U and V are independent.

Question 4 .... *15 point*

If X is a continuous r.v., the upper quintile $q(0.8)$ of the p.d.f. of X is defined by

$$\mathbb{P}(X \leq q(0.8)) = 0.8.$$

A Pareto r.v. X with shape α is defined by the p.d.f.

$$f(x) = \begin{cases} \alpha x^{-(\alpha+1)} & x \geq 1 \\ 0 & x < 1 \end{cases}$$

where $\alpha > 1$.

Compute $q(0.8)$ when X is a Pareto r.v. with shape α.

Solution: pdf is given by $f(x) = \begin{cases} \alpha x^{-(\alpha+1)} & x \geq 1 \\ 0 & x < 1 \end{cases}$. Now,

$$\begin{aligned} \int_{-\infty}^{\infty} f(x)dx &= 1 \\ &\Rightarrow \int_{-\infty}^{1} 0 * dx + \int_{1}^{\infty} \alpha x^{-(\alpha+1} dx \\ &= 1 \\ &\Rightarrow [\alpha \frac{x^{-\alpha-1+1}}{-\alpha-1+1}]_1^\infty \\ &= -[\frac{1}{x^\alpha}]_1^\infty \\ &= -[0-1] \\ &= 1. \end{aligned}$$

Now, $\mathbb{P}(X \leq q(0.8))$. Let $q(0.8) = x$. So,

$$\begin{aligned} \mathbb{P}(X \leq x) &= 0.8 \\ &\Rightarrow \int_1^x f(x) = 0.8 \\ &\Rightarrow -[\frac{1}{x^\alpha}]_1^x = 0.8 \\ &\Rightarrow \frac{-1}{x^\alpha} + 1 = 0.8 \\ &\Rightarrow \frac{1}{x^\alpha} = 0.2 \\ &\Rightarrow x^\alpha = 5 \\ &\Rightarrow x = 5^{\frac{1}{\alpha}} \\ &\Rightarrow q(0.8) = 5^{\frac{1}{\alpha}} \qquad (\alpha > 1) \end{aligned}$$

Question 5 .... *15 point*

Let X_1 and X_2 be two independent r.v. uniformly distributed in $[0,1]$. Find the p.d.f. of $Y = X_1 + X_2$.

Solution: $f_Y(y) = \int_{-\infty}^{\infty} f_{X_1}(x) f_{X_2}(y-x)dx$. We could denote both f_{X_1} and f_{X_2} as f_X since they both have the exact same distribution. Therefore, we could rewrite $f_Y(y)$ as $f_Y(y) = \int_{-\infty}^{\infty} f_X(x) f_X(y-x)dx$. The integrand of this function can have values 1 when $0 < x < 1$ and $0 < y - x < 1$ and 0 otherwise. The limits of integration will depend on the value of y.

1. When $0 < y < 1$, $x = 0$ to $x = y$. So $f_Y(y) = \int_0^y 1dx = y$.
2. When $1 < y < 2, x = y - 1$ to $x = 1$, so $f_Y(y) = \int_{y-1}^{1} 1dx = 2 - y$.
3. When $y < 0$ or $y > 2$, the integrand is zero, so $f_Y(y) = 0$

Therefore, the p.d.f. of Y is $f_Y(y) = \begin{cases} y & 0 < y < 1 \\ 2 - y & 1 \leq y < 2 \\ 0 & \text{otherwise} \end{cases}$