Homework 3

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1 Exercises from Chapter 3

25. Let A and B be two events such that A and B are independent. Therefore, $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$, $\mathbb{P}(A \cap B) = 0$ which means that $A \cap B = \emptyset$. Consider,

$$\begin{aligned} 1_{A\cap B}(\omega) &= \begin{cases} 1 & x \in A \cap B \\ 0 & x \notin A \cap B \end{cases} \\ &= \begin{cases} 1 & x \in A \text{ and } x \in B \\ 0 & x \notin A \cap B \end{cases} \\ &= \begin{cases} 1 & x \in A \\ 0 & x \notin A \cap B \end{cases} * \begin{cases} 1 & x \in B \\ 0 & x \notin A \cap B \end{cases} \\ &= 1_A(\omega) * 1_B(\omega). \end{aligned}$$

Since A and B are independent events, their random variables 1_A and 1_B are independent.

42. N could be redefined by using the indicator function for $A_1, A_2, ..., A_n$ as $1_{A_1} + 1_{A_2} + ... + 1_{A_n}$. We could then apply the expected value of this sum to receive, $\mathbb{E}(1_{A_1} + 1_{A_2} + ... + 1_{A_n}) = \mathbb{E}(1_{A_1}) + \mathbb{E}(1_{A_2}) + ... + \mathbb{E}(1_{A_n}) = \mathbb{P}(A_1) + \mathbb{P}(A_2) + ...\mathbb{P}(A_n) = \sum_{i=1}^{n}(A_i)$.

2 Problems from Chapter 3

4. It is simple to see that $\mathbb{P}(x_i \leq k) = \frac{k}{N}$ as $\mathbb{P}(x_i = 1) + \mathbb{P}(x_i = 2) + ... + \mathbb{P}(x_i = k) = \frac{1}{N} + \frac{1}{N} + ... + \frac{1}{N} = \frac{k}{N}$. Therefore, $\mathbb{P}(U_n \leq k) = \mathbb{P}(x_1 \leq k) * \mathbb{P}(x_2 \leq x) * ... * \mathbb{P}(x_K \leq k) = (\frac{k}{N}) * (\frac{k}{N}) * ... * (\frac{k}{N}) = \frac{k^n}{N^n}$. Then, $\mathbb{P}(U_n = k) = \mathbb{P}(U_n \leq k) - \mathbb{P}(U_n \leq k-1) = \frac{k^n}{N^n} - \frac{(k-1)^n}{N^n} = \frac{k^n-(k-1)^n}{N^n}$

$$\mathbb{P}(U_n = k) = \tfrac{k^n-(k-1)^n}{N^n}$$

Similarly, $\mathbb{P}(x_1 > k) = (1 - \frac{k}{N})$. Then, $\mathbb{P}(V_n > k) = (1 - \frac{k}{N})(1 - \frac{k}{N})...(1 - \frac{k}{N} = (1 - \frac{k}{N})^n$. Therefore,

$$\mathbb{P}(V_n = k) = (1 - \tfrac{k}{N})^n - (1 - \tfrac{k-1}{N})^n.$$

7.

$$\begin{aligned}
\mathbb{E}(X_1 + X_2 + ... + X_N) &= \mathbb{E}(\mathbb{E}(X_1 + X_2 + ... + X_N \mid N)) \\
&= \mathbb{E}(\mathbb{E}(\sum_{i=1}^{N} X_i \mid N)) \\
&= \mathbb{E}(\sum_{i=1}^{N} \mathbb{E}(X_i) \mid N) \\
&= \mathbb{E}(\sum_{i=1}^{N} \mu \mid N) \\
&= \mathbb{E}(N\mu N) \\
&= \mu\mathbb{E}(N \mid N) \\
&= \mu\mathbb{E}(N).
\end{aligned}$$

3 Exercises from Chapter 4

18.

$$\begin{aligned}
G_Y(s) &= \mathbb{E}(s^Y) \\
&= \mathbb{E}(s^k X) \\
&= \mathbb{E}((s^k)^X) \\
&= G_X(s^k).
\end{aligned}$$

$$\begin{aligned}
G_Z(s) &= \sim^{\mathbb{Z}} \\
&= \mathbb{E}(s^{X+k}) \\
&= \mathbb{E}(s^X s^k) \\
&= s^k \mathbb{E}(s^X) \\
&= s^k G_X(s).
\end{aligned}$$

41. Let $(B_k)_k$ be a sequence of a random variable with Bernoulli distribution such that $\mathbb{P}(B_k = 1) = p$ for $k = 1, 2, ...$ Then, $X = B_1 + ... + B_n$ for parameters n, p and $Y = B_{n+1} + ... + B_{n+m}$ for parameters m, p. Therefore, $X + Y = B_1 + ... + B_n + B_{n+1} + ... + B_{n+m}$. This sum is binomally distributed with parameters $n + m, p$.

4 Problems from Chapter 4

5. The tree will produce $N = n$ flowers following a binomial distribution with parameters n and $p = \frac{1}{2}$.

1. The probability that the tree will have r ripe fruits is,
   $\mathbb{P}(R = r) = \sum_{n=r}^{\infty} \mathbb{P}(N = n)\mathbb{P}(R = r \mid N = n) = \sum_{n=r}^{\infty}(1 - p)p^n \binom{n}{r} \frac{1}{2^n}$
   $\mathbb{P}_R(z) = \sum_{r=0}^{\infty} \mathbb{P}(R = r)z^r = \sum_{r=0}^{\infty} \sum_{n=r}^{\infty} \mathbb{P}(R = r \mid N = n)\mathbb{P}(N = n)z^r = \sum_{n=0}^{\infty} \mathbb{P}(N = n) \sum_{r=r}^{n} \mathbb{P}(R = r \mid N = n)\mathbb{P}(N = n)z^r = \sum_{n=0}^{\infty} \mathbb{P}(N = n)\mathbb{P}_{R|N=n}(z) = \sum_{n=0}^{\infty}(1 - p)p^n(\frac{1+z}{2})^n = \frac{1-p}{1-p\frac{1+z}{2}} = \frac{2-2p}{2-p-pz} = \sum_{r=0}^{\infty} \frac{2-2p}{2-p}(\frac{p}{2-p})^r z^r$.
   Therefore, $\mathbb{P}(R = r) = \frac{2-2p}{2-p}(\frac{p}{2-p})^r$

2. $\mathbb{P}(N = n \mid R = r) = \frac{\mathbb{P}(N=n,R=r)}{\mathbb{P}(R=r)} = \frac{\mathbb{P}(R=r|N=n)\mathbb{P}(N=n)}{\mathbb{P}(R=r)} = \frac{\binom{n}{r}2^{-n}(1-p)p^n}{\frac{2-2p}{2-p}(\frac{p}{2-p})^r}$