## Homework 2 due Wed, Sept 8th by 11am in Gradescope

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**Collaborators:**
**Outside resources:**

INSERT a "pagebreak" command between each problem (integer numbers). Problem subparts (letter numbered) can be on the same page.

REMOVE all comments (within "textit{}" commands) before submitting solutions.

DO NOT include any identifying information (name, GTID) except on the first/cover page.

1. Let $G$ be a group with subgroups $A$ and $B$. Let $AB = \{ab \mid a \in A,\ b \in B\}$.

   (a) If $G$ is abelian, prove that $AB$ is a subgroup of $G$.

   *Proof.* Let $G$ be a group such that $G$ is abelian and $A, B$ are subgroups of $G$. Let $AB = \{ab \mid a \in A, b \in B\}$. Since $A, B$ are subgroups, $e \in A, e \in B$, so $e = e \circ e \in AB$. Let $a_1b_1, a_2b_2 \in AB$, then we know that $(a_1b_1)(a_2b_2) = (a_1a_2)(b_1b_2) \in AB$ since $a_1a_2 \in A, b_1b_2 \in B$ and $G$ is abelian, which implies that $AB$ is closed under the inherited operation from $G$. If $ab \in AB$, then $(ab)^{-1} = a^{-1}b^{-1} \in AB$ which implies that $AB$ contains inverses. Therefore, $AB$ is a subgroup of G. ■

   (b) Suppose $b^{-1}Ab \subseteq A$ for all $b \in B$. Show $AB$ is a subgroup of $G$.

   *Proof.* Let $G$ be a group such that $A, B$ are subgroups of $G$. Let $AB = \{ab \mid a \in A, b \in B\}$. Since $A, B$ are subgroups, $e \in A, e \in B$, so $e = e \circ e \in AB$. Consider $a \in A, b \in B$ then $(ab)^{-1} = b^{-1}a^{-1} = (b^{-1}a^{-1}b)b^{-1}$ and since $b^{-1}Ab \subseteq A$ when we know that $(b^{-1}a^{-1}b) \in A$ and that $AB$ contains inverses. Additionally, consider $a, a' \in A$ and $b, b' \in B$. Consider $aba'b' = a(ba'b^{-1})bb'$. Since $b^{-1}Ab \subseteq A$ we know that $ba'b^{-1} \in A$ and we know that $a(ba'b^{-1}) \in A$ and $bb' \in B$ then we know that $AB$ is closed under the operation and that $AB$ is a subgroup of $G$. ■

   (c) Disprove that $AB$ is always a subgroup of $G$.

   Consider if $G = S_3$, $A =< (12) >$ and $B =< (23) >$ . A and B are subgroups of $G$. However, $AB = \{1, (12), (23), (132)\}$. By Lagrange's Theorem, since $|AB| = 4$ and $|G| = 6$ and 4 does not divide 6 then $AB$ is not a subgroup of $G$.

2. Let $G$ be a group and $H$ a subgroup of $G$.

   (a) Suppose $a^{-1}Ha \subseteq H$ for all $a \in G$. Prove $a^{-1}Ha = H$.

   *Proof.* Consider $h \in H$ and $a \in G$. There exists some $h' \in H$ such that $ha = ah'$. Then $a^{-1}ha = h' \in H$. Since this applies to any $h \in H$, $a^{-1}Ha = H$. ■

   (b) Suppose every right coset of $H$ in $G$ is also a left one. Prove $aHa^{-1} = H$.

   *Proof.* Consider $aha^{-1}$ for $h \in H$. We know that $ah \in aH = Ha$, so therefore $ah = h'a$ for some $h' \in H$. Therefore, $aha^{-1} = h'$. Since this applies to any $h \in H$, then $aHa^{-1} = H$. ■

3. Let $(G, *)$ and $(G', \circ)$ be two groups with identity $e$ and $e'$ respectively. Let $\phi : G \to G'$ be a homomorphism.

   (a) Prove that $\phi(G)$, the image of $G$, is a subgroup of $G'$.

   *Proof.* In order to prove that the image of $G$ is a subgroup of $G'$ we need to prove that the image is closed, contains the identity, and contains inverses. Consider $a, b \in G$. Then $\phi(a), \phi(b) \in \phi(G)$. Therefore, since $\phi$ is a homomorphism, $\phi(ab) = \phi(a)\phi(b) \in \phi(G)$ which implies that the image of $G$ is closed. Let $a \in G$. Then $a = ae$, but then $\phi(a) = \phi(ae) = \phi(a)\phi(e)$ by the definition of a homomorphism so by left cancellation in $G'$ we have that $e' = \phi(e)$. Also, since $e' = \phi(e) = \phi(aa^{-1}) = \phi(a)\phi(a^{-1})$. Therefore, the inverse of $\phi(a)$ in $G'$ is $\phi(a^{-1})$; $\phi(a)^{-1} = \phi(a^{-1})$. Therefore, the image of $G$ is a subgroup of $G'$. ■

   (b) Prove that $\phi$ is a monomorphism if and only if $\operatorname{Ker}\phi = (e)$.

   *Proof.* ($\Rightarrow$) Let $\phi$ be a monomorphismm that is to say that for $a, b \in G$ such that $\phi(a) = \phi(b)$ that $a = b$ and since $\phi$ is a homomorphism $\phi(e) = e'$. If $g \in \operatorname{Ker}(f)$, then we have that $\phi(g) = e'$. Therefore, $\phi(g) = \phi(e)$. Since $\phi$ is a monomorphism then $g = e$ and then $\operatorname{Ker}(\phi) = \{e\}$.
   ($\Leftarrow$) Let $\operatorname{Ker}(\phi) = \{e\}$. Let $a, b \in G$ such that $\phi(a) = \phi(b)$. Then we have that

$$\begin{aligned} \phi(ab^{-1}) &= \phi(a)\phi(b^{-1}) \\ &= \phi(a)\phi(b)^{-1} \\ &= \phi(a)\phi(a)^{-1} \\ &= e'. \end{aligned}$$

   Thus, the element $ab^{-1}$ is in the $\operatorname{Ker}(\phi) = \{e\}$ and hence $ab^{-1} = e$. This implies that $a = b$ and that $\phi$ is a monomorphism. ■

4. Let $G$ be a group and $H$ a subgroup of $G$. Let $S$ be the set of all distinct left cosets of $H$ in $G$ and $T$ the set of all distinct right cosets of $H$ in $G$.

   (a) Prove or disprove: If $aH, bH \in S$ and $aH \neq bH$, then $Ha \neq Hb$.

   This statement is false. Consider $H = \{e, (12)\}$ from $S_3$. Then, $(13)H = (123)H = \{(13), (123)\}$. While, $H(13) = \{(13), (132)\}$ and $H(123) = \{(23), (123)\}$. Therefore, $H(123) \neq H(13)$.

   (b) Prove there is a 1-1 mapping of $S$ onto $T$.

   Consider the mapping $x \to x^{-1}$. Any element of the form $xh$ for $h \in H$ gets mapped to $h^{-1}x^{-1}$ which lies in $Hx^{-1}$. Therefore, the image of $xH$ under the mapping is inside $Hx^{-1}$. Then, any element in $Hx^{-1}$ comes as the image of the unique element under the inverse mapping and that element is in $xH$ if $hx^{-1}$ is an element, it comes as the image of the element $xh^{-1} \in xH$