Homework 11, 12 & 13

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November 2022

[1]

(a) If we isolate this function around $z = 0$ then we can write it as the series $ze^{\frac{1}{z}} = \frac{1}{2z} + \frac{1}{6z^2} + \frac{1}{24z^3} + ...$ which would indicate that $z = 0$ is an essential singular point.

(b) The principal part of this function is isolated at $z = -1$ which is $\frac{1}{1+z}$, then we know that this is a simple pole.

(c) The principal part of the function is at $z = 0$. We can write out $\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - ....$ This allows us to see that $z = 0$ is a removable singular point.

(d) The principal part of this function is isolated at $z = 0$, but unlike the previous question, this one is not removable because the taylor expansion of $\cos(z)$ has a 1 so $\frac{1}{z}$ is a simple pole.

(e) The principal part of this function is isolated at $z = 2$ and is the function in and of itself, and is non-removable. Therefore, it is a pole of order 3.

[5]

(a) In the contour $C : |z| = 2$, which is the circle centered at $z = 0$ with radius 2, contains the isolated singular point of $z = 0$. This singular point is a pole of order 3 and we can rewrite $f(z) = \frac{1}{z^3(z+4)} = \frac{\phi(z)}{z^3}$ where $\phi(z) = \frac{1}{z+4}$. Then we know by the residue theorem that $Res_{z=0} = \frac{\phi^{(3-1)}(0)}{(3-1)!} = \frac{\phi^{(2)}(0)}{2} = \frac{2}{(z+4)^3 * 2} = \frac{1}{(0+4)^3} = \frac{1}{64}$. Then we have that $\int_C \frac{dz}{z^3(z+4)} = 2\pi i(\frac{1}{64}) = \frac{\pi i}{32}$.

(b) If we now consider the contour $C : |z + 2| = 3$ is a circle centered at $z = -2$ with radius 3. This contour then includes both of the singularity points of $z = 0$ and $z = -4$. We then know that $\int_C f(z)dz = 2\pi i * \sum Res(f(z))$. We already know the $Res_{z=0} = \frac{1}{64}$, and the $Res_{z=-4} f(z) = \frac{1}{(-4^3)} = \frac{-1}{64}$. Thus, we have that $\int_C f(z)dz = 2\pi i(\frac{1}{64} + \frac{-1}{64}) = 0$.

[7]

We can manipulate the denominator such that $(z^2 - 1)^2 + 3 = 0$ to be such that

$$\begin{aligned} (z^2 - 1)^2 + 3 &= 0 \\ (z^2 - 1)^2 &= -3 \\ z^2 - 1 &= \pm\sqrt{3}i \\ z^2 &= 1 \pm \sqrt{3}i \\ z &= \pm\sqrt{1 \pm \sqrt{3}i}, \end{aligned}$$

so we need to find the roots of $1 + \sqrt{3}i$. When we find the roots of $1 + \sqrt{3}i$ we get $\pm\sqrt{\frac{3}{2}} \mp \frac{1}{\sqrt{2}}i$ and we find that only $z_0 = \sqrt{\frac{3}{2}} + \frac{1}{\sqrt{2}}$ lies in the contour C. Therefore the function $f(z)$ has a simple pole of order 1 and then by the residue theorem, we know that $\int_C \frac{dz}{(z^2-1)^2+3} = 2\pi i * Res_{z=\sqrt{\frac{3}{2}}+\frac{1}{\sqrt{2}}}(f(z))$. Then $Res_{z=\sqrt{\frac{3}{2}}+\frac{1}{\sqrt{2}}}(f(z)) = \lim_{z\to\frac{\sqrt{3}+1}{\sqrt{2}}} \frac{1}{((\frac{\sqrt{3}+i}{\sqrt{2}})^2-1)^2+3} = \lim_{z\to\frac{\sqrt{3}+1}{\sqrt{2}}} \frac{1}{2(z^2-1)*2z} = \frac{1}{4\sqrt{3}i(\frac{\sqrt{3}+1}{\sqrt{2}})} = \frac{\sqrt{2}}{4\sqrt{3}i}$. Then $Res_{z=\frac{-\sqrt{3}+1}{\sqrt{2}}} f(z) = -\frac{\sqrt{2}}{4i\sqrt{3}(i-\sqrt{3})}$. Then the contour integral is equal to $2\pi i(\frac{\sqrt{2}}{4\sqrt{3}i}\frac{1}{\sqrt{3}+1} - \frac{\sqrt{2}}{4i\sqrt{3}(i-\sqrt{3}} = \frac{\pi}{2\sqrt{2}}$

[3]

If we consider $\int_0^\infty \frac{dx}{x^4+1}$ we can consider the associate complex function $f(z) = \frac{1}{z^4+1}$. If we then want to find $\int_C f(z)dz$ where C is the semicircle contour from $R \to -R$ where $0 \leq \theta \leq \pi$, then we know $\int_C f(z) = 2\pi i \sum_{i=0}^n Res_{z=z_i} f(z)$ and in this contour we have the singularity points $z = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i, -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$. Therefore, we need to evaluate $Res_{z=\frac{1}{\sqrt{2}}(1+i)} f(z)$ and $Res_{z=\frac{1}{\sqrt{2}}(-1+i)} f(z)$. We get $Res_{z=\frac{1}{\sqrt{2}}(-1+i)} f(z) = \frac{1}{4(\frac{1}{\sqrt{2}}(-1+i))^3} = \frac{1}{2\sqrt{2}(1+i)}$ and $Res_{z=\frac{1}{\sqrt{2}}(1+i)} f(z) = \frac{1}{4(\frac{1}{\sqrt{2}}(1+i))^3} = \frac{1}{2\sqrt{2}(-1+i)}$. Then we know that $\int_C = 2\pi i(\frac{1}{2\sqrt{2}(1+i)} + \frac{1}{2\sqrt{2}(-1+i)} = 2\pi i(\frac{\pi}{2\sqrt{2}}) = \frac{\pi}{\sqrt{2}}$. Then since this is an even function we need to divide this solution by 2 to arrive that $\int_0^\infty \frac{dx}{x^4+1} = \frac{\pi}{2\sqrt{2}}$ as desired.

[4]

In order to solve $\int_{-\infty}^\infty \frac{x\sin(ax)}{x^4+4}dx$ we are going to solve the associate $f(z) = \frac{xe^{az}}{x^4+4}$. If we map the semi-circle from $-R \to R$ for $0 \leq \theta \leq \pi$. Then we want to solve the contour integral $\int_C \frac{xe^a x}{(z-(1+i))(z-(-1-i))(z-(1-i))(z-(-1+i))}$ and in this contour we have the two singularity points $z = 1+i, -1+i$. Therefore, to calculate this contour we need to find the associated residues. Then we have that $Res_{z=1+i} f(z) = \frac{1}{8i}e^{-a}e^{ia}$ and $Res_{z=-1+i} f(z) = -\frac{1}{8i}e^{-a}e^{-ia}$. Therefore we know that $\int_C f(z)dz = 2\pi i(\frac{1}{8i}e^{-a}e^{ia} + -\frac{1}{8i}e^{-a}e^{-ia}) = \frac{\pi i}{2}e^{-a}\sin(a)$. Since we use the form e^{az} as an event representation of $\sin(z)$ we need to take the imaginary portion of the solution to the contour integral which would lead to the solution of $\frac{\pi}{2}e^{-a}\sin(a)$. as desired.

[2]

If we let $f(z) = \frac{1}{\sqrt{z}(z^2+1)} = \frac{z^{\frac{-1}{2}}}{z^2+1} = \frac{e^{\frac{-1}{2}\log(z)}}{z^2+1}$. This function has $z = \pm i$ and if we define the contour C to be the upper semicircle centered at $z = 0$ with radius R, then only $z = i$ lies inside C. Then we want to find $Res_{z=i} f(z) = \frac{1}{\sqrt{i}2i} = \frac{1}{\frac{1+i}{\sqrt{2}}2i} = \frac{1}{\sqrt{2}i(1+i)} = \frac{1-i}{2\sqrt{2}i}$. Therefore, we know that $\int_0^\infty = Re(2\pi i(\frac{1-i}{2\sqrt{2}i})) = Re(\frac{\pi}{\sqrt{2}}(1-i)) = \frac{\pi}{\sqrt{2}}$. Therefore, $\int_0^\infty \frac{dx}{\sqrt{x}(x^2+1)} = \frac{\pi}{2}$ as desired.

[2]

In order to solve this integral, we first need to rewrite the function, we are going to recognize that $\sin(\theta) = \frac{e^{i\theta}-e^{-i\theta}}{2i}$ and if we say that $z(\theta) = e^{i\theta}$ then we have that $z(\theta) = \frac{z-z^{-1}}{2i}$. Then we find that $dz = ie^{i\theta}d\theta = izd\theta \Rightarrow d\theta = \frac{dz}{iz}$. We can now consider the contour integral $\int_C \frac{1}{1+(\frac{z-z^{-1}}{2i})^2}\frac{dz}{iz} = \int_C \frac{-4zdz}{i(z^4-6z^2+1)}$. Then we need to solve this integral, but first we are going to make a substitution to make it simpler to solve. If we let $u = z^2 \Rightarrow du = 2zdz$, then we get $2\int_C \frac{-2du}{i(u^2-6u+1)} = 4i\int_C \frac{du}{u^2-6u+1}$. The contour we are going to use is the unit circle and with the singularities of $u = 2 \pm 2\sqrt{2}$ we only need to worry about $u = 3-2\sqrt{2}$. Then the residue about $z = 3-2\sqrt{2}$ is $\frac{-1}{4\sqrt{2}}$. Then the value of $4i\int_C = 4i(2\pi i(\frac{-1}{4\sqrt{2}}) = \sqrt{2}\pi$ as desired.