Homework 3, 4, & 5

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Math 4320

[10]

(a) To show that $\lim_{z\to\infty} \frac{4z^2}{(z-1)^2} = 4$ we can show that $\lim_{z\to 0} \frac{4(\frac{1}{z})^2}{(\frac{1}{z}-1)^2} = \lim_{z\to 0} \frac{4\frac{1}{z^2}}{\frac{1}{z^2}-\frac{2}{z}+1} = \lim_{z\to 0} \frac{4}{1-2z+z^2} = \lim_{z\to 0} \frac{4}{(z-1)^2} = 4$ as desired.

(b) We know that $\lim_{z\to 1} \frac{1}{(z-1)^3} = \infty$ since we can easily show that $\lim_{z\to 1} \frac{(z-1)^3}{1} = \frac{0}{1} = 0$.

(c) We are able to show that

$$\begin{aligned}\lim_{z\to\infty} \frac{z^2+1}{z-1} &= \lim_{z\to\infty} \frac{z^2(1+\frac{1}{z^2})}{z(1-\frac{1}{z})} \\ &= \lim_{z\to\infty} \frac{z(1+\frac{1}{z^2})}{(1-\frac{1}{z})} \\ &= \lim_{z\to\infty} z\frac{\lim_{z\to\infty}(1+\frac{1}{z^2})}{\lim_{z\to\infty}(1-\frac{1}{z})} \\ &= \infty(\frac{1+0}{1-0} \\ &= \infty * 1 \\ &= \infty\end{aligned}$$

[3]

(a) We can consider $P(z) = a_0 + f_1(z) + f_2(z) + ...f_n(z)$ where each $f_j(z) = a_j z^j$. Therefore we have that $P'(z) = \frac{d}{dz}(a_0 + f_1(z) + f_2(z) + ...f_n(z)) = 0 + f_1'(z) + f_2'(z) + ...f_n'(z) = 0 + a_1 + 2a_2 z + ... + na_n z^{n-1}$ as desired.

(b) *Proof.* We will proceed by mathematical induction:
Base Case: If we consider the case when $m = 0$ then we have that $P(0) = a_0 + a_1(0) + ... + a_n(0)^n = a_0$ as desired. Therefore, the statement is verified for $m = 0$.
Inductive Step: Assume that the statement is true for $m = k$, we then want to show that the statement is true for $m = k+1$. Then we know that $\frac{d}{dz}P^{k+1}(z) = \frac{d}{dz}(P^{(k)}(z)) = \frac{d}{dz}(\frac{k!}{0!}a_{k+1} + \frac{(k+1)!}{1!}a_{k+1}z + ... + \frac{n!}{(n-k)!}z^{n-k}) = 0 + \frac{(k+1)!}{0!}a_{k+1} + ... + \frac{n!}{(n-k+1)!}z^{n-k+1}$. Therefore, we see that $P^{(k)}(0) = k!a_k$ implies then that $a_k = \frac{P^{(k)}(0)}{k!}$ as desired. $\square$

[8]

(a) *Proof.* Consider $f'(z) = \lim_{\Delta z\to 0} \frac{Re(z+\Delta z) - Re z}{\Delta z} = \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{x+\Delta x - x}{\Delta x + i\Delta y} = \lim_{(\Delta x,\Delta y)\to(0,0)} \frac{\Delta x}{\Delta x + i\Delta y}$. Then, if we find the limit of this along the line $(\Delta x, 0)$, the limit is 1. However, if we take the limit along the line $(0, \Delta y)$, the limit is 0. Therefore, since the limit of the function is different from two different directions, $1 \neq 0$, then we know that the limit does not exist and further that f is not differentiable at any point. $\square$

(b) *Proof.* Consider $f'(z) = \lim_{\Delta z \to 0} \frac{Im(z+\Delta z) - Im z}{\Delta z} = \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{y + \Delta y - y}{\Delta x + i\Delta y} = \lim_{(\Delta x, \Delta y) \to (0,0)} \frac{\Delta y}{\Delta x + i\Delta y}$. Then, if we find the limit of this along the line $(\Delta x, 0)$, the limit is 0. However, if we take the limit along the line $(0, \Delta y)$, the limit is 1. Therefore, since the limit of the function is different from two different directions, $1 \neq 0$, then we know that the limit does not exist and further that f is not differentiable at any point. □

[4]

Note: for $z \neq 0$ we can write $z = re^{i\theta}$ with $r > 0$ and $-\pi < \theta \leq \pi$ and $f(z) = u(r, \theta) + iv(r, \theta)$.

(a) We know then that $f(z) = \frac{1}{z^4}$ is the same as $f(r, \theta) = r^{-4}e^{i(-4\theta)}$ which we can then split up into $z = r^{-4}\cos(4\theta) - ir^{-4}\sin(4\theta)$ to get that $u(r, \theta) = r^{-4}\cos(4\theta)$ and $v(r, \theta) = -r^{-4}\sin(4\theta)$. We will then calculate the first order partial derivatives with respect to r and θ for both of these u and v to get $u_r = -4r^{-5}\cos(4\theta), u_\theta = -4r^{-4}\sin(4\theta), v_r = 4r^{-5}\sin(4\theta), v_\theta = -4r^{-4}\cos(4\theta)$. The polar form of the Cauchy-Riemann equations state that $ru_r = v_\theta$ and $-rv_r = u_\theta$. Then, we can apply this to the problem to get that, $ru_r = -4r^{-4}\cos(4\theta) = v_\theta, -rv_r = -4r^{-4}\sin(4\theta) = u_\theta$. Therefore, the Cauchy-Riemann condition is satisfied indicating that $f'(z)$ does exist and $f'(z) = e^{-i\theta}(-4r^{-5}\cos(4\theta) + i4r^{-5}\sin(4\theta)) = -4r^{-5}e^{-i5\theta} = \frac{-4}{z^5}$.

(b) We need not to worry about converting this function to polar form. We can note that $u(r, \theta) = e^{-\theta}\cos(\ln(r)), v(r, \theta) = e^{-\theta}\sin(\ln(r))$. We can then take the partial derivatives, $u_r = \frac{-e^{-\theta}\sin(\ln(r))}{r}, u_\theta = -e^{-\theta}\cos(\ln(r)), v_r = \frac{e^{-\theta}\cos(\ln(r))}{r}, v_\theta = -e^{-\theta}\sin(\ln(r))$. We note that the Cauchy-Riemann condition is satisfied for this scenario and we have verified that $f'(z)$ does exist. Therefore, $f'(z) = e^{-i\theta}(\frac{-e^{-\theta}\sin(\ln(r))}{r} + i(\frac{e^{-\theta}\cos(\ln(r))}{r})) = i(\frac{e^{-\theta}\cos(\ln(r))}{re^{i\theta}} + i(\frac{e^{-\theta}\sin(\ln(r))}{re^{i\theta}})) = i\frac{f(z)}{z}$ as desired.

[4]

(a) The singular points are when $z(z^2 + 1) = 0$, which is true when $z = 0, +i, -i$. Therefore, the function is analytic for $z \in \mathbb{C}$ where $z \neq 0, +i, -i$ since the function is not continuous at these values of z.

(b) The singular points are when $z^2 - 3z + 2 = 0$ which is when $z = 1, 2$. Therefore, the function is analytic when $z \in \mathbb{C}$ for $z \neq 1, 2$ since the function is not continuous at these values of z.

(c) The singular points are when $(z + 2)(z^2 + 2z + 2) = 0$ which is when $z = -2, -1 + i, -1 - i$. Therefore, the function is analytic when $z \in \mathbb{C}$ for $z \neq -2, -1 + i, -1 - i$ since the function is not continuous at these values of z.

[7]

Proof. Let us assume we have a function $f(z)$ as described in the problem statement. Then we can write $f(z) = f(x + iy) = u(x, y) + iv(x, y)$. Since we know that $f(z)$ is real-valued for all $z \in D$, then we know that $v(x, y) = 0$ for all $x + iy \in D$. Given that the function f is analytic in D, then we know that the Cauchy-Riemann criteria is met which implies that $u_x = v_y$ and since we know that for $x + iy \in D$ that $v(x, y) = 0$ in the region, then we know that $u_x = v_y = 0$ and similarly $u_y = -v_u = 0$. Therefore, we know that both partial derivatives of $u(x, y)$ equal 0, which then means that $u(x, y) = c$ where c is a constant and $c \in \mathbb{R}$ as it's partial derivatives with respect to x and y and similarly for v(x, y). This then implies that $f(z) = c_1 + ic_2$ which means that $f(z)$ is constant for all $z \in D$ as desired. □

[2]

Proof. Let assumptions be as in the problem statement. Let $z_0 \in \mathbb{C}$ such that $z_0 = x_0 + iy_0$ is a point in the domain D and $c_1 = u(x_0, y_0)$ and $c_2 = v(x_0, y_0)$ specifically. Since we know that the function $f(z)$ is analytic in D, then we know that the Cauchy-Riemann conditions are satisfied which means that $f'(z)$ exists in D and that $u_x = v_y$ and $u_y = -v_x$ and we will allow $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0) = u_x(x_0, y_0) - iu_y(x_0, y_0) = v_u(x_0, y_0) + v_x(x_0, y_0)$ by definition. Let's define the matrices of partial derivatives $\mathbf{n}_1 = \begin{bmatrix} u_x(x_0,y_0) \\ u_y(x_0,y_0) \end{bmatrix}, \mathbf{n}_2 = \begin{bmatrix} v_x(x_0,y_0) \\ v_y(x_0,y_0) \end{bmatrix}$. We know that $\mathbf{n}_1$ is orthogonal to the tangent line of the level curve $u(x,y) = c_1$ at the point (x_0,y_0) and $\mathbf{n}_2$ is similarly orthogonal to the tangent line of the level curve $v(x,y) = c_2$ at the point (x_0,y_0). Then we know that the two tangent lines will only be orthogonal to each other if $\mathbf{n}_1$ and $\mathbf{n}_2$ are orthogonal to each other. This implies then that we need $\mathbf{n}_1 \cdot \mathbf{n}_2 = 0$. Then we have $\mathbf{n}_1 \cdot \mathbf{n}_2 = u_x(x_0,y_0)v_x(x_0,y_0) + u_y(x_0,y_0)v_y(x_0,y_0) = -u_x(x_0,y_0)u_y(x_0,y_0) + u_y(x_0,y_0)u_x(x_0,y_0) = 0$ since the Cauchy-Riemann equations we know that $v_x = -u_y$ and $v_y = u_x$. Therefore we know that the tangent lines to the level curves are orthogonal at the point (x_0,y_0) as desired. $\square$

[4]

We will first show that this function is exact, which is to say that it is analytic for all $z \in \mathbb{C}$. We will let $g(z) = e^z$ and $h(z) = z^2$ be two functions such that we know that $(g \circ h)(z) = e^{z^2} = f(z)$. We know that $g(z)$ and z^2 are exact themselves. Therefore, we know that $(g \circ h) = f(z)$ is also exact. Another way to find if $f(z)$ is to check the Cauchy-Riemann equations. We can say that $f(z) = e^{z^2} = e^{(x+iy)^2} = e^{x^2-y^2}\cos(2xy) + ie^{x^2-y^2}\sin(2xy)$ which means that $u(x,y) = e^{x^2-y^2}\cos(2xy), v(x,y) = e^{x^2-y^2}\sin(2xy)$. Therefore, we can find that $u_x = 2xe^{x^2-y^2}\cos(2xy) - 2ye^{x^2-y^2}\sin(2xy), u_y = -2ye^{x^2-y^2}\cos(2xy) - 2xe^{x^2-y^2}\sin(2xy), v_x = 2xe^{x^2-y^2}\sin(2xy) + 2ye^{x^2-y^2}\cos(2xy), v_y = -2ye^{x^2-y^2}\sin(2xy) + 2xe^{x^2-y^2}\cos(2xy)$. This then shows that $u_x = v_y$ and $u_y = -v_x$. So since the Cauchy-Riemann equations are satisfied for all $z \in \mathbb{C}$, and we again know that $f(z)$ is exact. Both of these methods have shown that $f(z)$ is differentiable everywhere and therefore exact. We then find that $f'(z) = 2ze^{z^2}$

[7]

($\Rightarrow$) Since we know that $|e^z| = e^x$, we then know that $|e^{-2z}| = e^{-2x}$ and to find when $e^{-2x} < 1$ we need to find when $-2x < 0$ which means $e^{-2x} < 1$ when $x > 0$.
($\Leftarrow$) If we know that $Re(z) > 0$ then we know that $x > 0$. Then we know that $e^{-2x} < 1$. Since we know that $|e^z| = e^x$ which then means that we know that $|e^{-2z}| = e^{-2x}$. Therefore, we know that $|e^{-2z}| < 1$ as desired.
Since the statement is true in both directions, we then know that $|e^{-2z}| < 1$ if and only if $Re(z) > 0$ as desired.

[3]

First we will note that $i^3 = -i = e^{i(\frac{-\pi}{2})}$. Therefore, we know that $Log(i^3) = Log(-i) = ln|1| + i(\frac{-\pi}{2}) = -i\frac{\pi}{2}$. However, we know that $i = e^{i(\frac{\pi}{2})}$, and then we know that $3Log(i) = 3(ln|1| + i\frac{\pi}{2}) = 3(\frac{\pi}{2}i) = \frac{3\pi}{2}i$ which is in another branch of the logarithmic function. Therefore, we know that $-i\frac{\pi}{2} \neq \frac{3\pi}{2}i$ as they are located within different branches.

[8]

If we take $e^{log(z)} = z = e^{i\frac{\pi}{2}} = \cos(\frac{\pi}{2}) + i\sin(\frac{\pi}{2}) = 0 + i * 1 = i$.

[3]

Equation 4 states that $\log(\frac{z_1}{z_2}) = \log(z_1) - \log(z_2)$. Let us assume that equation 4 holds when log is changed for Log that would be to say that $Log(\frac{z_1}{z_2}) = Log(z_1) - Log(z_2)$. Let us take that $z_1 = 1, z_2 = -1$. That then means that $Log(z_1) - Log(z_2) = (0 + i0) - (0 + i\pi) = -i\pi$. However, $Log(\frac{z_1}{z_2}) = Log(-1) = i\pi$. Which implies that $i\pi = -i\pi$ which is not true since they would be on different branches of the logarithm and leads to our assumption of equation 4 being true for Log to be a contradiction. Therefore we know that $Log(\frac{z_1}{z_2}) \neq Log(z_1) - Log(z_2)$ as desired.

[8]

(a) Let us say that $z^{c_1} z^{c_2} = a$ for some $a \in \mathbb{C}$. Let us apply log to both sides, we get

$$\begin{aligned} \log(z^{c_1} z^{c_2}) &= \log(a) \\ \log(z^{c_1}) + \log(z^{c_2}) &= \\ c_1 \log(z) + c_2 \log(z) &= \\ (c_1 + c_2) \log(z) &= \\ (c_1 + c_2) \log(z) &= \\ \log(z^{c_1 + c_2}) &= . \end{aligned}$$

Then if we exponentiate this result we get that $z^{c_1+c_2} = a$ which implies then that $z^{c_1} z^{c_2} = z^{c_1+c_2}$ as desired.

(b) Let us say that $\frac{z^{c_1}}{z^{c_2}} = b$. Then,

$$\begin{aligned} \log(\frac{z^{c_1}}{z^{c_2}}) &= \log(b) \\ \log(z^{c_1}) - \log(z^{c_2}) &= \\ c_1 \log(z) - c_2 \log(z) &= \\ (c_1 - c_2) \log(z) &= \\ \log(z^{c_1 - c_2}) &= . \end{aligned}$$

Which then if we exponentiate, we find that $\frac{z^{c_1}}{z^{c_2}} = z^{c_1 - c_2}$ as desired.

(c) Let us say that $(z^c)^n = d$. Then,

$$\begin{aligned} \log(z^c)^n &= d \\ \log(z^{cn} &= . \end{aligned}$$

Then if we expenonentiate this, we get that $(z^c)^n = z^{cn}$ as desired.

[2]

(a)

$$\begin{aligned} e^{iz_1} e^{iz_2} &= (\cos(z_1) + i\sin(z_1))(\cos(z_2) + i\sin(z_2)) \\ &= \cos(z_1)\cos(z_2) + \cos(z_1)(i\sin(z_2)) + (i\sin(z_1))\cos(z_2) + i^2 \sin(z_1)\sin(z_2) \\ &= \cos(z_1)\cos(z_2) + i\cos(z_1)\sin(z_2) + i\sin(z_1)\cos(z_2) - \sin(z_1)\sin(z_2) \\ &= cos(z_1)\cos(z_2) - \sin(z_1)\sin(z_2) + i(\sin(z_1)\cos(z_2) + \cos(z_1)\sin(z_2)). \end{aligned}$$

Then we have that $e^{-i\theta} = \cos(\theta) - i\sin(\theta)$ then,

$$\begin{aligned} e^{-iz_1} e^{iz_2} &= (\cos(z_1) - i\sin(z_1))(\cos(z_2) - i\sin(z_2)) \\ &= \cos(z_1)\cos(z_2) - i\cos(z_1)\sin(z_2) - i\sin(z_1)\cos(z_2) + i^2 \sin(z_1)\sin(z_2) \\ &= \cos(z_1)\cos(z_2) - i(\cos(z_1)\sin(z_2) + \sin(z_1)\cos(z_2)) - \sin(z_1)\sin(z_2) \\ &= \cos(z_1)\cos(z_2) - \sin(z_1)\sin(z_2) - i(\cos(z_1)\sin(z_2) + \sin(z_1)\cos(z_2)). \end{aligned}$$

And this is as desired.

(b) If we know that $\sin(z_1 + z_2) = \frac{1}{2i}(e^{iz_1} e^{iz_2} - e^{-iz_1} e^{-iz_2})$. If we substitute $e^{iz_1} e^{iz_2}, e^{-iz_1} e^{-iz_2}$ in the equation, which are obtained in part a. Then we know that

$$\begin{aligned}
\sin(z_1+z_2) &= \frac{1}{2i}((\cos(z_1)\cos(z_2)-\sin(z_1)\sin(z_2)+i(\sin(z_1)\cos(z_2)+\cos(z_1)\sin(z_2))) \\
&\quad -((\cos(z_1)\cos(z_2))-(\sin(z_1)\sin(z_2))-i(\sin(z_1)\cos(z_2)+\cos(z_1)\sin(z_2))) \\
&= \frac{1}{2i}(\cos(z_1)\cos(z_2)-\sin(z_1)\sin(z_2)+i(\sin(z_1)\sin(z_2)+\cos(z_1)\sin(z_2)) \\
&\quad -\cos(z_1)\cos(z_2)+\sin(z_1)\sin(z_2)+i(\sin(z_1)\cos(z_2)+\cos(z_1)\sin(z_2))) \\
&= \frac{1}{2i}(i(\sin(z_1)\cos(z_2)+\cos(z_1)\sin(z_2))+i(\sin(z_1)\cos(z_2)+\cos(z_1)\sin(z_2)) \\
&= \frac{1}{2i}(2i(\sin(z_1)\cos(z_2)+\cos(z_1)\sin(z_2))) \\
&= \sin(z_1)\cos(z_2)+\cos(z_1)\sin(z_2)
\end{aligned}$$

[5]

(a) We know the identity $\sin^2(z)+\cos^2(z)=1$, then if we divide each side by $\cos^2(z)$ we get that $\frac{\sin^2(z)}{\cos^2(z)}+\frac{\cos^2(z)}{\cos^2(z)}=\frac{1}{\cos^2(z)} \Rightarrow \tan^2(z)+1=\sec^2(z)$ as desired.

(b) We know the identity $\sin^2(z)+\cos^2(z)=1$, then if we divide each side by $\sin^2(z)$, we get that $\frac{\sin^2(z)}{\sin^2(z)}+\frac{\cos^2(z)}{\sin^2(z)}=\frac{1}{\sin^2(z)} \Rightarrow 1+\cot^2(z)=\csc^2(z)$ as desired.

[9]

(a) We know that $\cos(z)=\frac{e^{iz}+e^{-iz}}{2}$. Therefore we know that

$$\begin{aligned}
|\cos(z)| &= |\frac{e^{iz}+e^{-iz}}{2} \\
&\le |\frac{e^{iz}}{2}|+|\frac{e^{-iz}}{2}| \\
&\le |\frac{e^{i(x+iy)}}{2}|+|\frac{e^{-i(x+iy)}}{2}| \\
&= |\frac{e^{ix-y}}{2}|+|\frac{e^{-ix+y}}{2}| \\
&= |\frac{e^{ix}e^{-y}}{2}|+|\frac{e^{-ix}e^{y}}{2}| \\
&\le \frac{e^{-y}}{2}+\frac{e^{y}}{2} \\
&= \frac{e^{-y}+e^{y}}{2} \\
&= \cosh(y) \\
|\cos(z)| &\le \cosh(y).
\end{aligned}$$

Similarly, since we know that $\sin(z) = \frac{e^{iz}-e^{-iz}}{2i}$

$$\begin{aligned}
|\sin(z)| &= |\frac{e^{iz}-e^{-iz}}{2i}| \\
&= |\frac{e^{iz}-e^{-iz}}{2}| \\
&\leq |\frac{e^{iz}}{2}| + |\frac{e^{-iz}}{2}| \\
&= |\frac{e^{i(x+iy)}}{2}| + |\frac{e^{-i(x+iy)}}{2}| \\
&= |\frac{e^{ix-y}}{2}| + |\frac{e^{-ix+y}}{2}| \\
&= |\frac{e^{ix}e^{-y}}{2}| + |\frac{e^{-ix}e^{y}}{2}| \\
&\leq \frac{e^{-y}}{2} + \frac{e^{y}}{2} \\
&= \cosh(y) \\
|\sin(z)| &\leq \cosh(y).
\end{aligned}$$

Then given that $|\sin(z)|^2 = \sin^2(x) + \sinh^2(y) \Rightarrow |\sinh(y)| \leq |\sin(z)|$ which implies that $|\sinh(y)| \leq |\sin(2)| \leq \cosh(y)$ as desired.

(b) Since we already know that $|\cos(z)| \leq |\cosh(y)|$. Given that $|\cos(z)|^2 = \cos^2(x) + \sinh^2(y) \Rightarrow \sinh^2(y) \leq |\cos(z)|^2 \Rightarrow \sqrt{\sinh^2(y)} \leq \sqrt{|cos(z)|^2} \Rightarrow |\sinh(y)| \leq |\cos(z)|$. Therefore, we know that $|\sinh(y)| \leq |\cos(z)| \leq |\cosh(y)|$ as desired.

[5]

By the properties of sinh we have that $\sinh(z) = -i\sin(iz)$. Therefore, we have that $|\sinh(z)|^2 = |-i\sin(i(x+iy))|^2 = |-i|^2|\sin(-y+xi)|^2 + |\sin(-y+xi)|^2$. (15) states that $|\sin(x+iy)|^2 = \sin^2(x) + \sinh^2(y)$. So we then have that $|\sin(-y+xi)|^2 = \sin^2(y) + \sinh^2(x)$ as desired.

[16]

(a) Given that

$$\begin{aligned}
\sinh(z) &= i \\
\sinh(x+iy) &= \\
\sinh(x)\cos(y) + i\cosh(x)\sin(y) &= \\
\sinh(x)\cos(y) &= 0 \\
\cosh(x)\sin(y) &= 1.
\end{aligned}$$

We have that either $x = 0, y = (2n-1)\frac{\pi}{2}$ or both. When $x = 0, y = \frac{\pi}{2} + 2n\pi$, then there is no solution if $y = (2n-1)\frac{\pi}{2}$. Solutions are $z = (2n + \frac{1}{2})\pi i$.

(b) Given $\cosh(z) = \frac{1}{2} \Rightarrow \cosh(z) = \frac{e^z+e^{-z}}{2} = \frac{1}{2} \Rightarrow e^z + e^{-z} = 1$. Thus, $e^z + e^{-z} = 2\cos(y)\cosh(x) + 2i\sin(y)\sinh(x) = 1 + 0i$. Therefore, $\sin(y)\sinh(x) = 0$ and $\cos(y)\cosh(x) = \frac{1}{2}$. Therefore, $\sinh(z) = 0, \cos(y) = \frac{1}{2} \Rightarrow x = 0, y = 2n\pi \pm \frac{\pi}{y} \Rightarrow z = (2n\pi \pm \frac{\pi}{y})i \Rightarrow z = (2n \pm \frac{1}{y})\pi i$.

[2]

(a) We will first write that $\sin(z) = \sin(x+iy) = \sin(x)\cosh(y) + \cos(x)\sinh(iy) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$. Therefore, $\sin(x)\cosh(y) + i\cos(x)\sinh(y) = 2$. If we compare the real and imaginary parts, we get that $\sin(x)\cosh(y) = 2$ and $\cos(x)\sinh(y) = 0$. If $y = 0$ then the real part becomes $\sin(x) = 2$ which is not possible because the range of $\sin(x)$ is $[-1, 1]$ so $y \neq 0$,

$\sinh(y) \neq 0$, and $\cos(x) = 0$ from the imaginary part. Therefore, $x = 2n\pi - \frac{\pi}{2}$. In this case, the real part becomes $(-1)^n \cosh(y) = 2$. Since $\cosh(y)$ is always positive, then x has to be $2n\pi + \frac{\pi}{2}$. Then, $\cosh(y) = 2$ or $y = \cosh^{-1}(2)$. Therefore, the only roots of $\sin(z) = 2$ are $z = 2n\pi + \frac{\pi}{2} + i\cosh^{-1}(2)$. Now we want to show that $\cosh^{-1}(2) = \pm\ln(2+\sqrt{3}$. We have that $y = \cosh^{-1}(2) \Rightarrow \cosh(y) = 2 \Rightarrow \frac{e^y+e^{-y}}{2} = 2 \Rightarrow e^y + \frac{1}{e^y} = y \Rightarrow (e^y)^2 - ye^y + 1 = 0$. This is the in the form of a quadratic equation in terms of e^y. Then by using the quadratic equation we find that the roots are $2 \pm \sqrt{3}$. Then if we take the logarithm of this, we find that $\ln(2-\sqrt{3} = -\ln(2+\sqrt{3}$ which leads to $y = \pm\ln(2+\sqrt{3}$, and $\cosh^{-1}(2 = \pm\ln(2+\sqrt{3})$ since $y = \cosh^{-1}(2)$. Therefore, the only roots of $\sin(z) = 2$ are $z = (2n\pi + \frac{\pi}{2}) \pm i\ln(2+\sqrt{3})$.

(b) We have that $\sin^{-1}(w) = -i\log(iw + \sqrt{1-w^2})$ from the data we knwo that $\sin(z) = 2$ as $z = \sin^{-1}(2)$. But putting $w = 2$ in $\sin^{-1}(w) = -i\log(iw+\sqrt{1-w^2})$ we get that $z = -i\log(2i \pm i\sqrt{3})$. Now we find $\log((2+\sqrt{3})i$ and $\log((2-\sqrt{3})i)$ as if $z = re^{i\theta}$ is a nonzero complex number, the argument θ has only one of the values of $\theta = \Theta + 2n\pi$ where $\Theta = Arg(z)$ and $\log(z) = \ln r + i(\Theta + 2n\pi)$. Let $(2+\sqrt{3})i = re^{i\theta}$. Then, $(2+\sqrt{3})i = r\cos(\theta) + ir\sin(\theta) \Rightarrow r\cos(\theta) = 0, r\sin(\theta) = (2+\sqrt{3})$. Then we get that $r = (2+\sqrt{3})$ and therefore, $\cos(\theta) = 0, \sin(\theta) = 1$, then $\theta = \frac{\pi}{2}, Arg(i) = \frac{\pi}{2}, \Theta = \frac{\pi}{2}$. Therefore, $\log((2+\sqrt{3})i) = \ln(2+\sqrt{3}) + i(\frac{\pi}{2} + 2n\pi)$. Now, $z = -i\ln(2+\sqrt{3}) + \frac{\pi}{2} + 2n\pi$. Let $(2-\sqrt{3})i = re^{i\theta} = r\cos(\theta) = ir\sin(\theta)$. Then we know that $r\cos(\theta) = 0$, $r\sin(\theta) = 1$ which implies that $\theta = \frac{\pi}{2}, Argh(i) = \frac{\pi}{2}, \Theta = \frac{\pi}{2}$. Thus $\log((2-\sqrt{3})i) = \ln(2-\sqrt{3}) + i(\frac{\pi}{2} + 2n\pi)$. Now, $z = -i\ln(2-\sqrt{3}) + \frac{\pi}{2} + 2n\pi$. Then we have that $\ln(2-\sqrt{3} = -\ln(2+\sqrt{3})$. Then we know that $z = -i\ln(2+\sqrt{3}) + \frac{\pi}{2} + 2n\pi$. Then we arrive to $z = (\pm i\ln(2+\sqrt{3}) + (\frac{\pi}{2} + 2n\pi))$.

[2]

(a) $\int_0^1 (1+it)^2 dt = \int_0^1 1 - t^2 + 2itdt = t|_0^1 + \frac{1}{3}|_0^1 + it^2|_0^1 = 1 - \frac{1}{3} + i = \frac{2}{3} + i$ as desired.

(b) $\int_1^2 (\frac{1}{t} - i)^2 dt = \int_1^2 (\frac{1}{t^2} - \frac{2i}{t} + i^2)dt = \int_1^2 (\frac{1}{t^2} - 1)dt - 2i\int_1^2 \frac{dt}{t} = -\frac{1}{t}|_1^2 - t|_1^2 - 2i\ln(t)|_1^2 = -(\frac{1}{2} - 1) - (2-1) - 2i(\ln(2) - \ln(1))$. So $\in_1^2 (\frac{1}{t} - i)^2 dt = -\frac{1}{2} - 2i\ln(2) = -\frac{1}{2} - i\ln(4)$ as desired.

(c) $\int_0^{\frac{\pi}{6}} e^{i2t}dt = \int_0^{\frac{\pi}{6}} (\cos(2t) + i\sin(2t)dt = \int_0^{\frac{\pi}{6}} \cos(2t)dt + i\int_0^{\frac{\pi}{6}} \sin(2t)dt = \frac{1}{2}\sin(2t)|_0^{\frac{\pi}{6}} + i(-\frac{1}{2}\cos(2t)|_0^{\frac{\pi}{6}} = \frac{1}{2}(\frac{\sqrt{3}}{2} - 0) - \frac{i}{2}(\frac{1}{2} - 1)$. Therefore, we have that $\int_0^{\frac{\pi}{6}} e^{i2t}dt = \frac{\sqrt{3}}{4} + \frac{i}{4}$.

(d) If $M > 0$, we have that

$$\int_0^M e^{-zt}\,dt = \int_0^M e^{-(x+iy)t}\,dt = \int_0^M e^{-xt}e^{-iyt}\,dt = \int_0^M e^{-xt}\cos(yt)\,dt - i\int_0^M e^{-xt}\sin(yt)\,dt.$$

Then if we say $M \to \infty$,

$$\int_0^\infty e^{-zt}\,dt = \int_0^\infty e^{-xt}\cos(yt)\,dt - i\int_0^\infty e^{-xt}\sin(yt)\,dt,$$

where both of these integrals converge since $x = \Re(z) > 0$. Since $\frac{d}{dt}(e^{-zt}) = -ze^{-zt}$, then for $M > 0$, we know that

$$\int_0^M e^{-zt}\,dt = -\frac{1}{z}e^{-zt}\Big|_0^M = \frac{1}{z}(1 - e^{-Mz}),$$

and since $|e^{-Mz}| = e^{-Mx}|e^{-iMy}| = e^{-Mx} \to 0$ as $M \to \infty$ provided that $x > 0$, then we have that

$$\int_0^\infty e^{-zt}\,dt = \lim_{M\to\infty}\int_0^M e^{-zt}\,dt = \frac{1}{z}$$

provided that $x = \Re(z) > 0$. If we then equate the real and imaginary parts, we get that

$$\int_0^\infty e^{-xt}\cos(yt)\,dt = \frac{x}{x^2+y^2}$$

and

$$\int_0^\infty e^{-xt}\sin(yt)\,dt = -\frac{y}{x^2+y^2}.$$

[3]

Let $m, n \in \mathbb{Z}$ such that $m \neq n$, then we have that $\int_0^{2\pi} e^{im\theta} e^{-in\theta} d\theta = \int_o^{2\pi} e^{i(m-n)\theta} = \frac{1}{i(m-n)} e^{i(m-n)\theta}|_0^{2\pi} = \frac{1}{i(m-n)}(e^{i(m-n)2\pi} - e^{i(m-n)0}) = \frac{1}{i(m-n)}(1-1)$ since $e^{i(m-n)2\pi} = e^{i0} = 1$. Therefore, we have that $\int_0^{2\pi} e^{im\theta} e^{-in\theta} d\theta = 0$ if $m \neq n$. Also, if $m = n$, then we have that $e^{im\theta} e^{-in\theta} = 1$, so then $\int_0^{2\pi} e^{im\theta} e^{-in\theta} d\theta = \int_0^{2\pi} 1 dt = 2\pi$ if $m = n$.