Homework 7 due Wed, Nov 17th by 11am in Gradescope

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Outside resources:

INSERT a "pagebreak" command between each problem (integer numbers). Problem subparts (letter numbered) can be on the same page.

REMOVE all comments (within "textit{}" commands) before submitting solutions.

DO NOT include any identifying information (name, GTID) except on the first/cover page.

1. Problem 4.5 # 11.

   Let assumptions be as in problem statement. If we assume that $p(x)$ is irreducible over F then $p(x)$ cannot be factored as a product of two polynomials of positive degree. Consider then that there is an element $r \in F$ such that $p(r) = 0$. Then we would know that $p(r)$ has a root of $(x - r)$ which we could factor out and $p(r)$ would therefore be reducible which contradicts our assumption that $p(r)$ is irreducible. Therefore, if $p(x)$ is irreducible over F then there is no element $r \in F$ such that $p(r) = 0$.

2. Problem 4.5 # 13.

   Let assumptions be as in the problems statement. Let us define a function $\phi : \mathbb{R}[x] \to \mathbb{C}$ as $\phi(f(x)) = f(i)$ for $f(x) \in \mathbb{R}[x]$ essentially swapping x with i. First we want to prove that ϕ is a ring homomorphism. Let $f(x), g(x) \in \mathbb{R}[x]$, then $\phi((f+g)(x)) = (f+g)(i) = f(i) + g(i) = \phi(f(x)) + \phi(g(x))$, and $\phi((fg)(x)) = (fg)(i) = f(i)g(i) = \phi(f(x))\phi(g(x))$. We then want to show that ϕ is onto. $\forall a + bi \in \mathbb{C}$, we then choose $a + bx \in \mathbb{R}[x]$. Then, $\phi(a + bx) = \phi(a) + \phi(bx) = a + bi$. Now we are going to find $\operatorname{Ker}\phi = \{f(x) | \phi(f(x)) = 0\}$, it is important to note that $\phi(x^2 + 1) = i^2 + 1 = -1 + 1 = 0$. Therefore, $x^2 + 1 \in \operatorname{Ker}\phi$, and $< x^2 + 1 > \subseteq \operatorname{Ker}\phi$. Let $f(x) \in \operatorname{Ker}\phi$ and then by the division algorithm we would have that $f(x) = q(x)(x^2 + 1) + r(x)$ where $r = 0$ or $\deg r(x) \leq 1$. Therefore,

$$
\begin{aligned}
\phi(f(x)) &= \phi(q(x)(x^2+1)) + \phi(r(x)) \\
0 &= phi(q(x))(0) + r(i) \\
0 &= 0 + r(i) \\
0 &= r(i).
\end{aligned}
$$

   Since $\deg r(x) \leq 1$ then $r(x) = ax + b$ and $0 = ai + b$ and this implies that $a = 0, b = 0$. Therefore, $r(x) = 0$. Therefore, $f(x) = q(x)(x^2 + 1)$ which implies that $f(x) \in < x^2 + 1 >$, then $\operatorname{Ker}\phi \subseteq < x^2 + 1 >$, and $\operatorname{Ker}\phi = < x^2 + 1 >$. Thus, by the first homomorphism theorem, $\mathbb{R}[x] / < x^2 + 1 > \simeq \mathbb{C}$.

3. Problem 4.5 # 16.

   Let assumptions be as in the problem statement. Since $q(x)$ is irreducible, then $F[x]/q(x)$ is a field, and can be expressed as $a_0 + a_1x + ... + a_{n-1}x^{n-1}, a_i \in F$. From the division theorem we are able to say that for some $g(x) \in F[x]$, then $g(x) = f(x)q(x) + r(x)$ where $q(x), r(x) \in F[x]$. Then by counting, since there are p options for each coefficient a_i and there are n coefficients, we have p^n total combinations, and therefore that are p^n elements in $F[x]/q(x)$.

4. Problem 4.5 # 20.

   Let assumptions be as in the problem statement. Given that R is a Euclidean ring, then we know that it is an integral domain and that there is a function d from the nonzero elements of R to the nonnegative integers that satisfies for $a \neq 0, b \neq 0 \in R, d(a) \leq d(ab)$ and there exists $q, r \in R$ such that $b = qa + r$, where $r = 0$ or $d(r) < d(a)$. Let I be an ideal of R. If I is the ideal of only the additive identity then it is obviously a principal ideal. Suppose I is an ideal with more than the additive identity in R, then I contains at least one nonzero element. Let $b \in I, b \neq 0$ such that $d(b)$ is minimum. We have to show that $I =< b >$. Let $a \in I$ be any element, then by the division algorithm, we have that $a = qb + r$ since R is a euclidean ring. If $r \neq 0$ then since $b \in I$ then $qb \in I$ and since $a \in I$ we know that $a - qb \in I$ and thus, $r \in I$. Since $r \neq 0$, it would imply that $d(r) < d(b)$ by the definition of a euclidean ring which is not possible since we defined $d(b)$ to be a minimum. Therefore, $r = 0$, and $a - qb = 0$ and $a = qb \forall a \in I$. Thus, I is a principal ideal generated by b.

5. Problem 4.5 # 21.

   Let assumptions be as in the problem statement. Since R is a Euclidean Ring, it is an ideal of itself generated by an element a and we have that $R = (a)$ for some $a \in R$ since every ideal of R is a principal ideal from problem 20. Then we can define any element in R as a multiple of a, so we will write $a = a * b$ for some $b \in R$, similarly we can define some other element $c \in R$ as $c = da$ for some $d \in R$. Then $cb = dab = da = c$ which implies that $cb = c$ and b is then a unit element. Therefore, if R is a Euclidean Ring then R has a unit element.

6. Problem 4.5 # 22.

   Let assumptions be as in the problem statement. Consider $2, 6 \in R$, that means that there should be $q, r \in R$ such that $6 = 2q + r$. However, it is trivial to see that $6 = 2 * 3 + 0$ where $q = 3, r = 0$ and it is obvious that $3 \notin R$. Therefore, Euclid's algorithm is false for the ring of even integers.