MATH 4317 Homework 2 Due Friday, 1/28/2022 at 11PM to Gradescope

Homework guidelines:

- Each problem I assign, unless otherwise stated, is asking you to prove something. Give a full mathematical proof using only results from class or Wade.
- Submit a PDF or JPG to gradescope. The grader has ~ 250 proofs to grade: please make his job easier by submitting each problem on a different page.
- If you submit your homework in Latex, you get 2% extra credit.

Problems (5 total, 10 pts each)

Definition 1. Given two nonempty sets $A, B \subset \mathbb{R}$, define their *Minkowski sum* $A + B$ to be

$$A + B = \{a + b : a \in A, b \in B\}.$$

Problem 1. Prove or disprove (provide a counterexample) for the following statement: For all nonempty, bounded sets $A, B \subset \mathbb{R}$, we have that

$$\sup(A + B) = \sup(A) + \sup(B).$$

Proof. Assume that we have two bounded sets A, B and assume that we have a third set C that is the Minkowski sum of $A + B$. That is to say that $C = A + B = \{a + b : a \in A, b \in B\}$. Let $\sup(A) = a_1, \sup(B) = b_1$. Let $a \in A, b \in B$ then we have that $a \leq a_1, b \leq b_1$ by the definition of a supremum. Therefore, we know that $a + b \leq a_1 + b_1$. We then know that C is bounded above and that $a_1 + b_1$ is an upper bound of the set. We then need to prove that $a_1 + b_1$ is not just the upper bound of the set C but that it is also $\sup(C) = a_1 + b_1$. Let $\epsilon > 0$ be an arbitrary number. Then there exists an element $a \in A$ such that $a_1 - \frac{\epsilon}{2} < a \leq a_1$ and similarly there is an element $b \in B$ such that $b_1 - \frac{\epsilon}{2} < b \leq b_1$. Therefore, if we add these two statements together we get $(a_1 + b_1) - \epsilon < a + b \leq (a_1 + b_1)$. This then shows that $\sup C = a_1 + b_1$. $\square$

Problem 2. Find the sup and inf of the set

$$E = \left\{ 2 + \frac{1}{n} : n \in \mathbb{N} \right\} .$$

Claim 1. sup(E) = 3

Proof. We first want to show that 3 is an upper bound of E. We want to show that $2 + \frac{1}{n} \leq 3 : \forall n \in \mathbb{N}$. We can say that $2 + \frac{1}{n} \leq 3 \Rightarrow \frac{1}{n} \leq 1$. Then for all $n \in \mathbb{N}$ we have that $n \geq 1$, and further that $1 \geq \frac{1}{n}$. Therefore we have that $2 + \frac{1}{n} \leq 3$. Now we want to show that 3 is the least upper bound of E. Let $\epsilon > 0$ if we then take that $3 - \epsilon < 3$ we know that that $3 - \epsilon$ is not an upper bound of the set E. Therefore, we know that $\sup(E) = 3$. □

Claim 2. inf(E) = 2

Proof. We first want to show that 2 is a lower bound of E. We want to show that $2+\frac{1}{n} \geq 2 : \forall n \in \mathbb{N}$. We know that $2 + \frac{1}{n} \geq 2 \Rightarrow \frac{1}{n} \geq 0$, and if we take $1 \geq 0$, then for all $n \in \mathbb{N}$ we know then that $\frac{1}{n} \geq 0$. Therefore, 2 is a lower bound on E. We now want to show that 2 is a greatest lower bound on E. Let $\epsilon > 0$ then we have that $2 + \epsilon > 2$ we know that $2 + epsilon$ is not a lower bound of the set E. Therefore, we then know that $\inf(E) = 2$. □

Problem 3. Prove that
$$\bigcap_{k\in\mathbb{N}} \left[\frac{k-1}{k}, \frac{k+1}{k}\right] = \{1\}\,.$$

Proof. Let $E = \bigcap_{k\in\mathbb{N}} \left[\frac{k-1}{k}, \frac{k+1}{k}\right]$. We will first show that $\{1\} \subset E$. We will first manipulate the definition of E, we then have that $\left[\frac{k-1}{k}, \frac{k+1}{k}\right] = \left[\frac{k}{k} - \frac{1}{k}, \frac{k}{k} + \frac{1}{k}\right] = \left[1 - \frac{1}{k}, 1 + \frac{1}{k}\right]$. From examples in class we know that $\sup(\frac{1}{n}) = 1, \inf(\frac{1}{n}) = 0$ for $n \in \mathbb{N}$. Therefore we have that $1 - \frac{1}{n} \leq 1 \leq 1 + \frac{1}{n}$. Thus, we know that $\{1\} \in \left[\frac{k-1}{k}, \frac{k+1}{k}\right]$ and further that $\{1\} \subset E$. Next let $x \in E$, that is to say that $\frac{k-1}{k} \leq x \leq \frac{k+1}{k}$ for all $k \in \mathbb{N}$. We can then manipulate this as earlier to arrive at $1-\frac{1}{k} \leq x \leq 1+\frac{1}{k}$. If we again recall that $\sup(\frac{1}{n}) = 1, \inf(\frac{1}{n}) = 0$, then we need x such that $0 \leq x \leq 2 \cap 1 \leq x \leq 1 = \{1\}$. This implies that $E \subset \{1\}$. Since $\{1\} \subset E, E \subset \{1\}$ we then know that $E = \{1\}$ $\square$

Definition 2. Let $f : \mathbb{R} \to \mathbb{R}$. We call f *monotone increasing* if $a \leq b$ implies $f(a) \leq f(b)$.

Problem 4. Let $E \subset \mathbb{R}$ be a bounded set (from above and below). Prove or disprove (provide a counterexample) each of the following statements:

(i) If f is monotone increasing, then $\sup f(E) \leq f(\sup E)$.

(ii) If f is monotone increasing, then $\sup f(E) = f(\sup E)$.

Hint: For (ii), consider a function f with a jump discontinuity.

(i) Let $s = \sup(E)$. By definition of supremum we know that for $n \in \mathbb{N}$ that we can pick an $x_n \in E$ such that $s - \frac{1}{n} < x_n \leq s$. Then by the squeeze theorem we know that $x_n \to s$ as $n \to \infty$ and therefore, $f(x_n) \to f(s)$. Let $\epsilon > 0$ and pick N such that $n \geq N$ implies that $|f(x_n) - f(s)| < \epsilon$. Then, $n \geq N$ implies that $f(s) = f(s) - f(x_n) + f(x_n) \leq |f(s) - f(x_n)| + f(x_n) < \epsilon + f(x_n) \leq \epsilon + \sup f(E)$. Since $\epsilon > 0$ was arbitrary we know that $f(\sup A) = f(y) \leq \sup f(A)$.

(ii) Let $f(x) = \begin{cases} x & x < 2 \\ x+1 & 2 \leq x \end{cases}$, and let $E = (1, 2)$. It is easy to see that $\sup(E) = 2$. Then $f(\sup(E)) = f(2) = 3$. Alternatively if we take $\sup(f(E))$ we get that $\sup(f(E)) = \sup((1, 2)) = 2$. Since $3 \neq 2$ we have disproven this statement for monotone increasing function f that $\sup f(E) = f(\sup E)$.

Let $\sim$ denote the relation on $\mathbb{Z} \times \mathbb{Z} \setminus \{0\}$ defined by

$$(p, q) \sim (m, n) \quad \iff \quad pn = mq\,.$$

Given $(p, q) \in \mathbb{Z} \times \mathbb{Z} \setminus \{0\}$, let $[(p, q)]$ denote the *equivalence class* of (p, q), i.e.,

$$[(p, q)] = \{(m, n) \in \mathbb{Z} \times \mathbb{Z} \setminus \{0\} : (m, n) \sim (p, q)\}\,.$$

Recall from class that the set of equivalence classes $\{[(p, q)]\}$ can be identified with $\mathbb{Q}$, the set of rational numbers.

Problem 5. Prove that $\mathbb{Q}$ is countable.

Hint: You may freely use the following facts about cardinality in your proof:

(a) An infinite subset of a countable set is countable.

(b) If A, B are two countable sets, then the Cartesian product $A \times B$ is also countable.

(c) If A is countable, B is a set, and $f : A \to B$ is an onto mapping from A to B, then B is either finite or countable.

Proof. Given that we know that $\mathbb{Z}$ is countable, if we take the Cartesian product of $\mathbb{Z} \times \mathbb{Z}$ we arrive at another countable set by (b). Let us consider the set of all equivalence classes of $\mathbb{Z} \times \mathbb{Z}$ as $E = \{[(p_1, q_1)], [(p_2, q_2)], ...\}$ with the definition of each equivalence class be as above. Since $|E| = \infty$ and $E \subset \mathbb{Z} \times \mathbb{Z}$ we know that E is countable by (a). Let $f : E \to \mathbb{Q}$ such that for $z = [(a_1, b_1)] \in E$ we have $f(z) = \frac{a_1}{b_1}$. We want to show that f is onto. Let $q = \frac{q_1}{q_2} \in \mathbb{Q}$ be reduced as much as possible; consider the element $x = [(q_1, q_2)] \in E$, then $f(x) = \frac{q_1}{q_2}$. Therefore since we can find an element of E that can map to an arbitrary element of $\mathbb{Q}$ we know that f is onto. Therefore by (c) we know that $\mathbb{Q}$ is either finite or countable. Since $\mathbb{Z} \subset \mathbb{Q}$ and $\mathbb{Z}$ is infinite, we know then that $\mathbb{Q}$ is also infinite which further implies that $\mathbb{Q}$ is countable. $\square$