Homework 1 due Wed, Sept 1st by 11am in Gradescope

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1. Let G be the dihedral group of order 6. That is to say that the plane S is rotated $\frac{2\pi}{3}$ each rotation. The formula $(f^i h^j)(f^s h^t) = f^{i+s(mod2)} h^{j+t(modn)} = f^{i+s(mod2)} h^{j+t(mod3)}$.

   Claim 0.1. *The dihedral group of order 6 is nonabelian.*

   Proof. Let G be a group such that it is defined as $G =< f, h | f^2 = h^3 = e, fgf^{-1} = g^{-1} >$. In order to show that G is nonabelian we can show that it does not commute. That is to say that $gf \neq fg$. However if we consider the inverse g^{-1}

$$
\begin{aligned}
g^{-1} &= fgf^{-1} \\
fg^{-1} &= ffgf^{-1} \\
fg^{-1} &= gf^{-1} \\
fg^{-1} &= gf,
\end{aligned}
$$

   By left multiplication and the fact that $f = f^{-1}$. This shows that $gf = fg^{-1}$ instead of $gf = fg$. Alternatively,

$$
\begin{aligned}
g^{-1} &= fgf^{-1} \\
g^{-1}f &= fgf^{-1}f \\
fg^{-1}f &= fg.
\end{aligned}
$$

   This shows that $fg = fg^{-1}$. Since $gf \neq fg$, G is nonabelian. ∎

2. (a) f^2 is always the identity when defined for the dihedral group of order $2n$ because when we reflect a plane over the y-axis two times, it is the same as doing nothing to the plane. Similarly, the rotation h is defined as a rotation $\frac{2\pi}{n}$ counterclockwise, which would mean that if we rotated n times, the total rotation would be 2π which is a full rotation, or the same as doing nothing to the plane.

(b) The matrix transformation f is $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, and the matrix transformation for h, a rotation of $\frac{2\pi}{n}$ radians counterclockwise, is $\begin{bmatrix} cos(\frac{2\pi}{n}) & -sin(\frac{2\pi}{n}) \\ sin(\frac{2\pi}{n}) & cos(\frac{2\pi}{n}) \end{bmatrix}$.

Then, $fh = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} cos(\frac{2\pi}{n}) & -sin(\frac{2\pi}{n}) \\ sin(\frac{2\pi}{n}) & cos(\frac{2\pi}{n}) \end{bmatrix} = \begin{bmatrix} -cos(\frac{2\pi}{n}) & sin(\frac{2\pi}{n}) \\ sin(\frac{2\pi}{n}) & cos(\frac{2\pi}{n}) \end{bmatrix}$.

While, $hf = \begin{bmatrix} cos(\frac{2\pi}{n}) & -sin(\frac{2\pi}{n}) \\ sin(\frac{2\pi}{n}) & cos(\frac{2\pi}{n}) \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -cos(\frac{2\pi}{n}) & -sin(\frac{2\pi}{n}) \\ -sin(\frac{2\pi}{n}) & cos(\frac{2\pi}{n}) \end{bmatrix}$.

Simply put $\begin{bmatrix} -cos(\frac{2\pi}{n}) & -sin(\frac{2\pi}{n}) \\ -sin(\frac{2\pi}{n}) & cos(\frac{2\pi}{n}) \end{bmatrix} \neq \begin{bmatrix} -cos(\frac{2\pi}{n}) & sin(\frac{2\pi}{n}) \\ sin(\frac{2\pi}{n}) & cos(\frac{2\pi}{n}) \end{bmatrix}$, which means that $hf \neq fh$.

(c) By using matrix inverses $h^{-1} = \begin{bmatrix} cos(\frac{2\pi}{n}) & sin(\frac{2\pi}{n}) \\ -sin(\frac{2\pi}{n}) & cos(\frac{2\pi}{n}) \end{bmatrix}$. Then, $h^{-1}f = \begin{bmatrix} cos(\frac{2\pi}{n}) & sin(\frac{2\pi}{n}) \\ -sin(\frac{2\pi}{n}) & cos(\frac{2\pi}{n}) \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -cos(\frac{2\pi}{n}) & sin(\frac{2\pi}{n}) \\ sin(\frac{2\pi}{n}) & cos(\frac{2\pi}{n}) \end{bmatrix}$. Similarly, from the previous part, we saw that $fh = \begin{bmatrix} -cos(\frac{2\pi}{n}) & sin(\frac{2\pi}{n}) \\ sin(\frac{2\pi}{n}) & cos(\frac{2\pi}{n}) \end{bmatrix}$. This means that $fh = h^{-1}f$.

(d) Simply, D_n has the identity element e, f, and h. It also has $h^2, h^3, ..., h^{n-1}$ for each rotation until it reaches $h^n = e$. Similarly, once the plane is reflected using f, we can then apply the same rotations to get elements $fh, fh^2, ..., fh^{n-1}$ where $fh^n = fe = f$. Therefore, D_n has 2n distinct elements.

(e) We can express $(f^i h^j)(f^k h^l) = f^a h^b$ where $a = (i+j)(mod 2)$ and $b = (j+l)(mod n)$. Therefore, $(f^i h^j)(f^k h^l) = f^{(i+j)(mod2)} h^{b=(j+l)(modn)}$. This shows that a depends on i and k in the way that it is the sum of them modulo 2 and b depends on h and l in that it is the sum of them modulo n.

3. (a) There is no element $a \in D_3$ where $a \neq e$ such that $ab = be \forall b \in D_3$.

   (b) This can be generalized to all odd n because

   (c) There is only one element in D_4, h^2, where $h^2 b = bh^2$

   (d) This can be generalized to all dihedral groups of even n. Let $n = 2k$ it is easy to see that $(h^k)(h^i) = h^{k+i} = h^{i+k} = (h^i)(h^k)$. Otherwise, consider the element fh^i, then,

$$\begin{aligned}(h^k)(fh^i) &= (h^k f)(h^i) \\ &= (fh^{-k})(h^i) \\ &= (fh^k)(h^i) \\ &= (fh^i)(h^k).\end{aligned}$$

   This is true because k is half of n and the rotations are the same clockwise and counterclockwise and the inverse of h^k is h^k.

4.

Claim 0.2. *Let G be a group. For $a, b \in G$, define the relation $a \sim b$ if there exists an $x \in G$ such that $b = x^{-1}ax$. This relation is an equivalence relation on G.*

Proof. Let G be a group and relation $a \sim b$ if there exists $x \in G$ such that $b = x^{-1}ax$. In order for us to show that this is a equivalence relation on G we need to show that it is reflexive, symmetric, and transitive.
Reflexive: Consider $a \in G$. Since G is a group $a^{-1} \in G$. Therefore if we front multiply a by $e = a^{-1}a$. This allows $a^{-1}aa$ which implies that $a \sim a$. Therefore, the relation is reflexive.
Symmetric: Let $a, b \in G$ such that $a \sim b$, that is to say that there exists $x \in G$ such that $b = x^{-1}ax$. Then if we premultiply by x and postmultiply by x^{-1}. Then $xbx^{-1} = xx^{-1}axx^{-1} = a$. This then shows that $b \sim a$ which means that the relation is symmetric.
Transitive: Let $a, b, c \in G$ such that $a \sim b$ and $b \sim c$. Since the relation is symmetric, we know that there exists some $x, y \in G$ such that $b \sim a$ and $c \sim b$, that is to say that $a = x^{-1}bx$ and $b = y^{-1}y$. Then we can say that $a = x^{-1}y^{-1}cyx$. Let $z = xy$, we get that $a = z^{-1}cz$ which shows that $a \sim c$. Therefore the relation is transitive.
Since the relation is reflexive, symmetric, and transitive, we know that it is an equivalence relation. ■

5. (a) *Proof.* Reflexive: Let G be a group with subgroup H. Consider $a \in H$. Since H is a group, we know that $a^{-1} \in H$ because of the existence of an inverse property. This means that $a \sim a$ and the relation is reflexive.

Symmetric: Let G be a group with subgroup H. Consider $a, b \in H$ where $a \sim b$ which is to say that $a^{-1}b \in H$. Since $a, b \in H$, we know that $a^{-1}, b^{-1} \in H$ by the existence of inverses property of groups. Then we can premultiply $a^{-1}b$ by $(b^{-1}a)^2$ that gives us, $(b^{-1}a)(b^{-1}a)(a^{-1}b) = (b^{-1}a)(b^{-1})(aa^{-1})(b) = (b^{-1}a)(b^{-1})(e)(b) = (b^{-1}a)(b^{-1}b) = (b^{-1}a)(e) = (b^{-1}a) \in H$ since H is closed. Therefore, since $b^{-1}a \in H$ we know that since $a \sim b$ that $b \sim a$ and that the relation is symmetric.

Transitive: Let G be a group with subgroup H. Consider $a, b, c \in H$ where $a \sim b$ and $b \sim c$. That is to say that $a^{-1}b \in H$ and $b^{-1}c \in H$. We could premultiply $b^{-1}c$ by $a^{-1}b$ to get $a^{-1}bb^{-1}c = a^{-1}c \in H$ since H is closed, which means that $a \sim c$. Since $a \sim c$ we know that the relation is transitive.

Since the relation is reflexive, symmetric, and transitive, then the relation is an equivalence relation. ■

(b) The equivalence class of a is all elements for which $a \sim h$ for $h \in H$. This means that for all elements in H that $a \sim h \in H$. Since H is a group, it is closed, and this implies that $a \sim h$ for all $h \in H$.

(c) *Proof.* Consider $a, b \in H$ such that a and b lie in the left coset of H and $a \sim b$ that is to say that $aH = bH$. Since $a \sim b$, we know that $a^{-1}b \in H$ which then means that $b \in aH$ which means that $bH \subset aH$. But since the relation is symmetric, we know that $aH \subset bH$. Therefore, $a \sim b$ implies that $aH = bH$ which means that the left coset does partition G ■

(d) The relation on $a, b \in G$ that yields the right cosets of H is that $a \sim b$ if $b^{-1}a \in H$.

6.

Claim 0.3. *Let G be a group and fix $a \in G$. Define $\phi_a : G \to G$ by $\phi_a(x) = a^{-1}xa$ for all $x \in G$. ϕ_a is an isomorphism of G onto itself.*

Proof. Let G be a group and fix $a \in G$ and define $\phi_a : G \to G$ by $\phi_a(x) = a^{-1}xa$ for all $x \in G$. First we will show that this is a monomorphism, or is one to one. Consider $x, y \in G$ where $\phi_a(x) = \phi_a(y)$. That is to say that

$$\begin{aligned} a^{-1}xa &= a^{-1}ya \\ aa^{-1}xaa^{-1} &= aa^{-1}yaa^{-1} \\ exe &= eye \\ x &= y. \end{aligned}$$

Therefore, the mapping is one to one and is a monomorphism. Now we will show the mapping is an isomorphism by showing that the mapping is onto. Consider the element $x \in G$ then if we use axa^{-1}, then $\phi_a(axa^{-1}) = a^{-1}axa^{-1}a = exe = x$. Therefore, this mapping is onto and the mapping is an isomorphism of G onto G. ■