MATH 4317 Homework 8 Due Friday, 4/8 at 11PM to Gradescope

Homework guidelines:

- Each problem I assign, unless otherwise stated, is asking you to prove something. Give a full mathematical proof using only results from class or Wade.
- Submit a PDF or JPG to gradescope. The grader has ~ 250 proofs to grade: please make his job easier by submitting each problem on a different page.
- If you submit your homework in Latex, you get 2% extra credit.

Problems (5 total, 10 pts each)

Problem 1. For each $p \in \mathbb{N}$ and $\mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$, define

$$\|\mathbf{x}\|_p := \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} .$$

Prove that for each such $\mathbf{x}$, we have that

$$\lim_{p \to \infty} \|\mathbf{x}\|_p = \|\mathbf{x}\|_\infty ,$$

where

$$\|\mathbf{x}\|_\infty := \max\{|x_i| : 1 \le i \le n\} .$$

You may freely use the following facts: (i) if $a \in (0, \infty)$, then $\lim_{m\to\infty} a^{1/m} = 1$; and (ii) if $a \in (0,1)$ then $\lim_{m\to\infty} a^m = 0$.

Proof. Let assumptions be as the above problem statement. We can write $\lim_{p\to\infty} \|x\|_p = \lim_{p\to\infty}(\sum_{i=1}^n |x_i|^p)^{1/p}$. Consider if $x_1 = x_2 = ... = x_n$ then we get $(nx_i^p)^{1/p} = (n)^{1/p}x_i$ and since $n \ge 1$ then $n^{1/p} \to 1$ as $n \to \infty$. Therefore, $\lim_{p\to\infty} \|x\|_p = x_i$ if $x_1 = x_2 = ... = x_n = \max\{|x_i| : 1 \le i \le n\}$. Now consider if $x_1 > x_2, x_3, ..., x_n$ without loss of generality. We have that $(x_1^p + x_2^p + ... + x_n^p)^{1/p} = (1 + \frac{x_2^p}{x_1^p} + ... + \frac{x_n^p}{x_1^p})^{1/p} x_1$ and similarly since $(1 + \frac{x_2^p}{x_1^p} + ... + \frac{x_n^p}{x_1^p}) \ge 1$ we have that $(1 + \frac{x_2^p}{x_1^p} + ... + \frac{x_n^p}{x_1^p}) \to 1$ as $p \to \infty$ which further implies that $(1 + \frac{x_2^p}{x_1^p} + ... + \frac{x_n^p}{x_1^p})^{1/p} x_1 \to x_1$ as $p \to \infty$ which means that $\lim_{p\to\infty} \|x\|_p = \max\{|x_i| : 1 \le i \le n\}$. Since this is true for both cases then we know that the original statement that $\lim_{p\to\infty} \|x\|_p = \|x\|_\infty$ as desired. $\square$

Problem 2. Prove that the set

$$U = \{\mathbf{x} \in \mathbb{R}^n : \|\mathbf{x}\|_1 < 1\}$$

is open. You may freely use the triangle inequality for $\|\cdot\|_1$, namely that

$$\|\mathbf{x} + \mathbf{y}\|_1 \leq \|\mathbf{x}\|_1 + \|\mathbf{y}\|_1 \qquad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n\,,$$

as well as the fact (Remark 8.7 in Wade) that $\|\mathbf{x}\| \leq \|\mathbf{x}\|_1 \leq \sqrt{n}\|\mathbf{x}\|$ for all $\mathbf{x} \in \mathbb{R}^n$.

Proof. Let $U = \{x \in \mathbb{R}^n : \|x\|_1 < 1\}$. Then we can see that $\|x - y + y\|_1 \leq \|x - y\|_1 + \|y\|_1$. It follows then that $\|x\| - \|y\| \leq \|x - y\|$ by symmetry, one gets $|\|x\| - \|y\|| \leq \|x - y\|$. Therefore, this set U is a continuous preimage of an open set, $[0, 1)$, which means that it is open. $\square$

Problem 3. Prove that $\mathbb{Z} \subset \mathbb{R}$ is closed.

Proof. Consider the compliment to $\mathbb{Z}$ which we will denote $\mathbb{Z}^c = \cup_{k \in \mathbb{Z}}(k, k+1)$ which is an open set because it is a union of open sets. Therefore, since the compliment $\mathbb{Z}^c$ is an open subset, we know that $\mathbb{Z} \subset \mathbb{R}$ is closed. $\square$

Problem 4. Let $a, b, c, d \in \mathbb{R}$ and assume $a < b$ and $c < d$. Prove that the set $U = (a, b) \times (c, d)$ is open.

Proof. Let assumptions be as in the problem statement. Let $x = (x_1, x_2)$ where $a < x_1 < b$ and $c < x_2 < d$. If we let $\epsilon > \min(x_1 - a, b - x_1, x_2 - c, d - x_2)$ which is the minimum distance from (x_1, x_2) to the edge of the rectangle $(a, b) \times (c, d)$. Then the open ball $B_\epsilon(x_1, x_2) \subset (a, b) \times (c, d)$ which implies that $(a, b) \times (c, d)$ is an open rectangle. $\square$

Problem 5. Prove that the set

$$E = \{(x, y) \in \mathbb{R}^2 : |y| > |x| + 2\}$$

is disconnected.

Proof. Let $E = \{(x, y) \in \mathbb{R}^2 : |y| > |x| + 2\}$. Consider the set $U = \{(x, y) \in \mathbb{R}^2 : |x| + 3 > |y| > |x| + 2\}$ and $V = \{(x, y) \in \mathbb{R}^2 : |y| \geq |x| + 3\}$. For these sets, because of the way they are defined we know that $E \cap U \cap V = \emptyset, E \cap U = \emptyset$ primarily that we know that $U \cap V = \emptyset$. Similarly if we take $U \cup V = E$ then we know that $E \subseteq U \cup V$. Lastly, we know that $E \cup U = U \neq \emptyset$ and that $E \cup V = V \neq \emptyset$. Given these conditions we know that the set $E = \{(x, y) \in \mathbb{R}^2 : |y| > |x| + 2\}$ is disconnected. $\square$