MATH 4032: HOMEWORK #2
DUE FEBRUARY 3 AT 1:59PM

You are strongly encouraged to typeset your homework solutions using LaTeX.

The relevant background material for this assignment is covered in Chapters 3.4–3.8 and 7 of the Matoušek–Nešetřil book.

The following problems are optional exercises not to be turned in. Problems to be turned in for a grade begin on the next page.

Exercise 1. In class we proved that $\binom{n}{k} \le (en/k)^k$ for every $n \in \mathbb{N}$ and $k \in [n]$. Provide alternative proofs

(1) using induction on k and
(2) algebraically, using the bounds $e(n/e)^n \le n! \le en(n/e)^n$.

Exercise 2. Using Stirling's formula, prove that $\binom{2m}{m} \sim 2^{2m}/\sqrt{\pi m}$.

Exercise 3. Use the Inclusion-Exclusion Principle to count the number of numbers less than 100 that are not divisible by a square of any integer greater than 1.

Exercise 4. Recall that $D(n)$ is the number of derangements of an n-element set. Prove that the number of permutations of an n-element set with exactly k fixed points is $\binom{n}{k}D(n-k)$.

Exercise 5. Prove that

$$D(n) = n! - \sum_{k=1}^{n} \binom{n}{k} D(n-k).$$

Recall that the nth harmonic number is $H_n = \sum_{i=1}^{n} 1/i$.

Problem 1. Prove that

$$\ln n < H_n \leq 1 + \ln n$$

for every $n \in \mathbb{N}$. *Hint: Use induction on n. Use the fact that $e^x > 1 + x$ to prove that* $\ln n + 1/n < \ln(n+1)$.

Proof. Let $H_n = \sum_{i=1}^{n} \frac{1}{i}$ as is defined above. Consider then,

$$\begin{aligned} \ln(n) &= \int_1^n \frac{1}{x} dx \\ &= \sum_{i=1}^{n-1} \int_i^{i+1} \frac{1}{x} dx \\ &\leq \sum_{i=1}^{n-1} \int_i^{i+1} \frac{1}{i} dx \\ &= H_{n-1} \\ &< H_n. \end{aligned}$$

Similarly, we then have that

$$\begin{aligned} \ln(n) &= \int_1^n \frac{1}{x} dx \\ &= \sum_{i=1}^{n-1} \int_i^{i+1} \frac{1}{x} dx \\ &\geq \sum_{i=1}^{n-1} \int_i^{i+1} \frac{1}{i+1} dx \\ &= H_n - 1. \end{aligned}$$

This then implies that

$$\ln(n) < H_n \leq 1 + \ln(n)$$

as desired. ∎

Recall that $\pi(n)$ is the number of primes in the set $[n]$. In class, we proved $\pi(n) = O(n/\ln n)$. In the following problem, you will prove $\pi(n) = \Omega(n/\ln n)$, which completes the proof of the "Weak Prime Number Theorem".

For $n \in \mathbb{N}$ and a prime p, let $\nu_p(n)$ denote the largest integer k such that $p^k \mid n$. Equivalently, $\nu_p(n)$ is the exponent to which p appears in the prime factorization of n.

Problem 2. Prove the following, and conclude that $\pi(n) = \Omega(n/\ln n)$.

(a) Prove that every prime p and every $n \in \mathbb{N}$ satisfies

$$\nu_p(n!) = \sum_{k=1}^{\infty} \left\lfloor \frac{n}{p^k} \right\rfloor .$$

Proof. Let assumptions be as in the problem statement, given that $\nu_p(n)$ denotes the largest k such that p^k divides n. By the definition $n!$ is the product of $\{1, 2, ..., n\}$ so we know we will have at least one factor of p in $n!$ for each multiple of p in $\{1, 2, ..., n\}$ and there will be $\lfloor \frac{n}{p} \rfloor$ is a multiple of p. Similarly, we will have a factor of p^2 which will contribute an additional factor of p and then $\lfloor \frac{n}{p^2} \rfloor$ will be a multiple of p^2 in the set $\{1, 2, ..., n\}$ and so on. Now, let's say that k is the largest factor of p such that $p^k \mid n!$. Then we know for $j > k$ that p^j will not divide any $\{1, 2, ..., n\}$ implying that $p^j > n \forall j > k$ further implying that $\lfloor \frac{n}{p^j} \rfloor = 0 \forall j > k$. Thus, we have that $\nu_p(n!) = \lfloor \frac{n}{p} \rfloor + \lfloor \frac{n}{p^2} \rfloor + ... + \lfloor \frac{n}{p^k} \rfloor + 0 + 0... \Rightarrow \nu_p(n!) = \sum_{w=1}^{k} \lfloor \frac{n}{p^w} \rfloor + \sum_{\zeta > k} \lfloor \frac{n}{p^\zeta} \rfloor \Rightarrow \nu_p(n!) = \sum_{k=1}^{\infty} \lfloor \frac{n}{p^k} \rfloor$ as desired. □

(b) Using (a), prove that every prime p and every $n \in \mathbb{N}$ satisfies

$$\nu_p\left(\binom{2n}{n}\right) = \sum_{k=1}^{\infty} \left(\left\lfloor \frac{2n}{p^k} \right\rfloor - 2 \left\lfloor \frac{n}{p^k} \right\rfloor \right)$$

Hint: Use that $\binom{2n}{n} = (2n)!/(n!)^2$.

Proof. We know that $\binom{2n}{n} = \frac{(2n)!}{(n!)^2}$. By definition of $\nu_p(n)$ we know that $p^k \mid n$. Then we have that $\nu_p(\binom{2n}{n}) = \nu_p(\frac{(2n)!}{n!^2}) = \nu_p((2n)!) - \nu_p((n!))^2 = \nu_p((2n)!) - (\nu_p(n!) + \nu_p(n!))$. Also note, $\nu_p((n!)^2) = \nu_p(n!) + \nu_p(n!)$ as the power of p will be twice in $(n!)^2$. Then, $\nu_p((2n)!) - 2(\nu_p(n!)) = \sum_{k=1}^{\infty}(\lfloor \frac{2n}{p^k} \rfloor - 2\lfloor \frac{n}{p^k} \rfloor) = \nu_p(\binom{2n}{n})$ as desired. □

(c) Prove that $\lfloor 2n/m \rfloor - 2\lfloor n/m \rfloor \leq 1$ for every $n, m \in \mathbb{N}$.

Proof. Let assumptions be as in the problem statement. We have three cases to consider. If $2n \leq m$ then we have that both floors will be equal to 0 satisfying the inequality. If we have that $2n = m$ then we know that $\lfloor \frac{n}{m} \rfloor = 0$ and $\lfloor \frac{2n}{m} \rfloor = 1$ which would also satisfy the inequality. Then if we have that $n > m$ we will let $\lfloor \frac{n}{m} = j$ which we know will be $\frac{n}{m} \leq j < \frac{n}{m} + 1$ and similarly for $\lfloor \frac{2n}{m} \rfloor \leq 2j + 1$ implying that $2j + 1 - 2(j) \leq 1$ satisfying the inequality. Then for the last case we have that if $2n > m$ but $n < m$ we have that the second floor is equal to 0 and we need to show that $\lfloor \frac{2n}{m} \rfloor \leq 1$ which means that we need to then show that $\frac{2n}{m} < 2 \Rightarrow \frac{n}{m} < 1$ which we know is true since we stated that $n < m$ meaning that the inequality is satisfied. Thus, for all cases the inequality is satisfied as desired. □

(d) Using (b) and (c), prove that

$$\nu_p\left(\binom{2n}{n}\right) \leq \log_p(2n).$$

Proof. We know from part b that $\nu_p(\binom{2n}{n}) = \sum_{k=1}^{\infty}(\lfloor \frac{2n}{p^k} \rfloor - 2\lfloor \frac{n}{p^k} \rfloor)$ and from part c we have that $\lfloor \frac{2n}{m} \rfloor - 2\lfloor \frac{n}{m} \rfloor \leq 1$. If we combine these two statements we know that $\nu_p(\binom{2n}{n}) \leq \sum_{k=1}^{\infty} 1$. We are going to analyze the elements of this sum to find when the elements are equal to 0 and when they are equal to 1. In this sum we have that for all $k \leq \nu_p(2n)$. Thus the value for this summation is the power for which p divides $2n$ which is $\log_p(2n)$ as desired. $\square$

(e) Using (d), prove that $\binom{2n}{n} \leq (2n)^{\pi(n)}$ for every $n \in \mathbb{N}$.

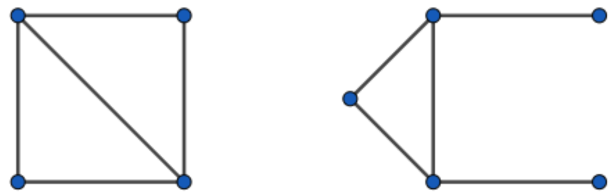
Proof. By using part d, we are able to rewrite the prime factorization of $\binom{2n}{n}$ as the product of $2n$ times itself for every prime factor from $1 \to \binom{2n}{n} \Rightarrow (2n)^{\pi(\binom{2n}{n})}$. Then if we consider $\pi(\binom{2n}{n})$, the prime factors of $\binom{2n}{n}$ will always be less than $2n$ which means that we could then just simply use $\pi(n)$ say that $\binom{2n}{n} \leq (2n)^{\pi(n)}$ as desired. $\square$

(f) Using (e), prove that $\pi(n) = \Omega(n/\log n)$. *Hint: Use the fact that* $\binom{2n}{n} \geq 2^n/(n+1)$.

Proof. If we start with $\binom{2n}{n} \leq (2n)^{\pi(n)}$ we can take the log of both sides to arrive at $\pi(n) = \log_{2n}(\binom{2n}{n})$. Then if we use a change of base we get that $\pi(n) = \frac{\log(\binom{2n}{n})}{\log(2n)}$ and since we know that $\binom{2n}{n} = 2^n$ we can simplify this to be $\frac{\log(2^n)}{\log(2n)} = \frac{n\log(2)}{\log(2)+\log(n)} = \Omega\frac{n}{\log(n)}$ as desired. $\square$

A *proper coloring* of a graph is an assignment of colors to its vertices such that adjacent vertices receive different colors. A *k-coloring* of a graph is a proper coloring that uses at most k colors.

Problem 3. Let $k \geq 5$. How many k-colorings are there of each of the graphs below? Prove your answer is correct. *Hint: Use the Inclusion-Exclusion Principle to count colorings which are NOT proper. Associate with each edge e the set A_e of colorings which assign the same color to its ends.*



I am honestly kind of stumped on this one sorry.

We say a permutation σ of $[2n]$ *has property* P if

$$|\sigma(i) - \sigma(i+1)| = n$$

for some $i \in [2n]$ (in this problem, addition is modulo $2n$, so $\sigma(2n+1)$ is defined to be $\sigma(1)$). For example, the following permutation of [6] has property P

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ 2 & 5 & 4 & 1 & 2 & 3 \end{array}$$

because $\sigma(2) - \sigma(1) = 3$.

Problem 4. Prove that, for each $n \in \mathbb{N}$, there are more permutations with property P than without it. *Hint: Use the Inclusion-Exclusion Principle. For each $i \in [2n]$, consider the set A_i of permutations σ satisfying $|\sigma(i) - \sigma(i+1)| = n$, and show that $|A_i| = 2n(2n-2)!$ and $A_i \cap A_{i+1} = \varnothing$ for all $i \in [2n]$.*

Proof. Let A_k be the set of permutations with k and $k+n$ in neighboring positions as $|\sigma(i) - \sigma(i+1)| = |k - k + n| = |k + n - k| = n$ as per the problem definition. Let A be the set of permutations with property P, so that A is the union of the A_k. Then we have that $|A| = \sum_k |A_k| - \sum_{k<l} |A_k||A_l| + \sum_{k<l<m} |A_k||A_l||A_m| - \ldots$, this is an alternating sequence with decreasing terms, implying that $|A| \geq \sum_k |A_k| - \sum_{k<l} |A_k||A_l|$. Then we have that $|A_k| = 2(2n-1)!$ since we have two orders for k and $k+n$ and then $(2n-1)!$ ways of arranging the $2n-1$ items if we treat k and $k+n$ as a single item. Similarly $|A_k||A_l| = 4(2n-2)!$ so we have that $|A| \geq 2n^2(2n-2)! > (2n)!/2$. This shows then that there are more permuations with the property P than without the property P as desired. $\square$

Problem 5. Let $\mathcal{F} = \{A_1, \dots A_m\}$ be a family of subsets of a finite set X. For $x \in X$, let $d_{\mathcal{F}}(x)$ be the number of members of $\mathcal{F}$ containing x. Prove that

$$\sum_{i=1}^{m} \sum_{j=1}^{m} |A_i \cap A_j| = \sum_{x \in X} d_{\mathcal{F}}(x)^2.$$

Proof. Let assumptions be as in the problem statement. Define a function $d_{\mathcal{F}}(x)$ be the number of elements of $\mathcal{F}$ containing x for any $x \in X$. Then we have that $\sum_{i,j=1}^{m} |A_i \cap A_j| = \sum_{i=j}^{m} |A_i \cap A_j| + \sum_{i \neq j}^{m} |A_i \cap A_j|$ when $x \in A_i$ but $x \notin A_j$ then $A_i \cap A_j = \emptyset$ but when $A_i, A_j \subseteq X \Rightarrow x \in X, x \in A_i, A_j$ except for when $A_i, A_j = \emptyset$. then we know that $\sum_{i,j=1}^{m} |A_i \cap A_j| = \sum_{i,j=1}^{m} |A_i| + \sum_{i,j=1}^{m} |A_i \cap A_j| = \sum_{j=1}^{m} |A_i \cap A_j|$ (for fixed i in the last sum) will have m^2 possibilities as A_j will have m possibilities and A_i will also have m possibilities. Therefore, $\sum_{i=1}^{m} \sum_{j=1}^{m} |A_i \cap A_j| = \sum_{x \in X} d_{\mathcal{F}}(x)^2$ as desired. $\square$