Homework 9

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1. Part 1: Consider the following theorem,

   Theorem 0.1. *Let T be a normal operator on a finite dimensional inner product space V. Let x be an eigenvector of T corresponding to eigenvalue λ of T. Then x is an eigenvector of T^* corresponding to eigenvalue λ'.*

   Proof. Let $u = T - \lambda I$. Then, $uu^* = u^*u$, then u is normal. Also,

   $$\begin{aligned} ||u^*x||^2 &=< u^*x, u^*x > \\ &=< x, uu^*x > \\ &=< x, u^*ux > \\ &=< ux, ux > \\ &= ||ux||^2. \end{aligned}$$

   This implies that $||u^*x|| = 0 \iff ||ux|| = 0$. So, $0 = ||(T^* - \lambda' I)x|| \iff ||(T - \lambda I)x|| = 0$. This implies that x is an eigenvector of T corresponding to eigenvalue λ iff x is an eigenvector of T^* corresponding to eigenvalue λ'. ■

   In other words, we can say that eigenvectors of A and A^* are equal.
   Part 2: Let $Tx = \lambda x$ and $ty = \mu y$ for $\lambda \neq \mu$. Consider,

   $$\begin{aligned} \lambda < x, y > &=< \lambda x, y > \\ &=< Tx, y > \\ &=< x, T^*y > \\ &=< x, \bar{\mu} y > \\ &= \mu < x, y > . \end{aligned}$$

   Since $\lambda \neq \mu \Rightarrow < x, y >= 0 \Rightarrow x \perp y$. Therefore, we can say that there is an orthonormal basis in $\mathbb{C}^n$ such tat each vector in the basis is an eigenvector of both A and A^*.

2. (a) We know that A is normal iff it is diagonalizable by some unitary matrix U. Given that U is unitary, $UU^* = U^*U = I$ which therefore

means that $U^{-1} = U^*$. Therefore, $A = U^{-1}DU$ where D is a diagonal matrix containing the eigenvalues of A along the diagonal. Given that the the eigenvalues of A are real. Then,

$$\begin{aligned} A^* &= (U^*DU)^* \\ &= U^*D^*U \\ &= U^*DU \\ &= A. \end{aligned}$$

Therefore, $A^* = A$ which means that A is self-adjoint.

(b) ($\Rightarrow$): Let A be a normal matrix that is to say that $AA^* = A^*A = I$. Then, $Au = \lambda u$, and by taking the conjugate transpose, $u^*A^* = \lambda^* u^*$. If we multiply both those statements together.

$$\begin{aligned} u^*A^*Au &= \lambda^* u^* \lambda u \\ u^*Iu &= (\lambda^*\lambda)(u^*u) \\ ||u||^2 &= |\lambda|^2 ||u||^2 \\ |\lambda|^2 &= 1. \end{aligned}$$

Therefore, the eigenvalues of A have absolute value of 1. Thus, if A is a normal, unitary matrix, then it has eigenvalues with absolute value 1.
($\Leftarrow$): Let A be a normal matrix with eigenvalues that have absolute value equal to 1. Then, $Au = du$ and $u^*A^* = d^*u^*$. Then,

$$\begin{aligned} u^*A^*Au &= \lambda^*\lambda u^*u \\ u^*(A^*A)u &= |\lambda|^2 ||u||^2 \\ u^*(A^*A)u &= ||u||^2. \end{aligned}$$

This then implies that $A^*A = AA^* = I$ which means that A is unitary. Thus, if A is a normal matrix whose eigenvalues have absolute value 1, then it is also unitary.
Therefore, a normal matrix A is unitary iff its eigenvalues have absolute value 1.

(c) If the normal matrix is Hermitian, then its eigenvalues must be real, but if the normal matrix is not Hermitian, then this restriction does not apply.

3. (a) Given that A is normal, that is to say that $AA^* = A^*A$. Let $P(x) =$

Ax. Then,

$$\begin{aligned}
A^2 &= A \\
A &= A^* \\
A^*A &= A^* \\
(A^*A)^* &= A^* \\
A^*A &= A \\
(I - A^*)A &= 0 \\
(I - A)^*A &= 0 \\
y^*(I - A)^*Ax &= 0 \\
(Ax, (I - A)y) &= 0 \\
Ax &\perp (I - A)y \\
Ax &\perp Col(I - A) \\
(I - A)x &\in Col(I - A)
\end{aligned}$$

for all $x, y \in \mathbb{C}^n$. Therefore, if we say that $P = (I - A)$ then P is the orthogonal projection of $\mathbb{C}^n$ onto the column space of A.

(b) Consider the matrix, $B = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$. Since, B has two distinct eigenvalues, B is diagonalizable. Additionally, $B^2 = B$. However, $BB^T = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $B^TB = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} \neq BB^T$. Therefore, B is not normal.

4. $cond(B) = 4.1804e + 16$,
$rank(B) = 1$,
$norm(B - U(1:m, 1) * S(1, 1) * V(1:5, 1)') = 6.6385e - 14$
As the value of k increases, the noise size approaches the value of the compression error. Particularly the values for when $k = 19, k = 20$ the rounded values are the same. That means that when the noise is a very small then the SVD compression process is largely unaffected; however, when the noise is larger, then the SVD compression is affected by very noticeable amounts (the value of N is 11.3416 when $k = 1$).