MATH 4032: HOMEWORK #1
DUE JANUARY 20 AT 1:59PM

You are strongly encouraged to typeset your homework solutions using $\LaTeX$.

The following problems are optional exercises not to be turned in. Problems to be turned in for a grade begin on the next page.

Exercise 1. Let $n \in \mathbb{N}$. Prove that every n-element set contains 2^{n-1} subsets of odd size and 2^{n-1} subsets of even size.

Exercise 2. Determine how many permutations of $\{1, \ldots, n\}$ have a single cycle.

Exercise 3. Provide an 'algebraic proof' of the following formula:

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}.$$

That is, show how the formula follows from the definition $\binom{n}{k} = \left(\prod_{i=0}^{k-1}(n-i)\right)/k!$ by rearranging terms.

Exercise 4. Determine how many ways are there to arrange 7 elves and 5 goblins in a row in such a way that no goblins stand next to each other.

Exercise 5. Determine whether each of the following are true or false:

(a) $n^2 = O(n^2 \ln n)$
(b) $n^2 = o(n^2 \ln n)$
(c) $n^2 + 5n \ln n = n^2(1 + o(1)) \sim n^2$
(d) $n^2 + 5n\ lnn = n^2 + O(n)$
(e) $\sum_{i=1}^{n} i^8 = \Theta(n^9)$
(f) $\sum_{i=1}^{n} \sqrt{i} = \Theta(n^{3/2})$

Exercise 6. Determine whether each of the following are true or false:

(a) $n! \sim ((n+1)/2)^n$
(b) $n! \sim ne(n/e)^n$
(c) $n! = O((n/e)^n)$
(d) $\ln(n!) = \Omega(n \ln n)$
(e) $\ln(n!) \sim n \ln n$

Problem 1. Among the numbers $1, 2, \ldots, 10^{10}$, are there more of those containing the digit 9 in their decimal notation, or of those with no 9? Prove your answer is correct.

Claim: There are more numbers between 1 and 10^{10} that have the digit 9 in them.

Proof. We can simply calculate the numbers that do NOT have the digit 9 in them by considering that there are 10 digits that cannot be the digit 9 which leaves 9 digit options for each of those spots. That is to say that there are 9^10 numbers that do not have the digit 9 in them. So then we have $10^10 - 9^10 = 6513215600$ which is more than half of the numbers in the range so we have shown that there are more numbers that contain the digit 9. $\square$

Problem 2. Let $n \in \mathbb{N}$. Prove that for every $r \in [n]$,

$$\sum_{k=r}^{n} \binom{k}{r} = \binom{n+1}{r+1}.$$

Proof. We will proved by induction.
Base Case: Let n = r, then we have that $\sum_{k=r}^{n} \binom{k}{r} = \binom{r}{r} = 1 = \binom{r+1}{r+1} = \binom{n+1}{r+1}$. Thus the statement holds for $n = r$.
Inductive step: Suppose that for some $j \in \mathbb{N}, j \geq r$ the statement is true, that is to say that $\sum_{k=r}^{j} \binom{k}{r} = \binom{k+1}{r+1}$. Then we want to show that the statement is true for $j + 1$. Then $\sum_{k=r}^{j+1} \binom{k}{r} = \sum_{k=r}^{j} \binom{k}{r} + \binom{j+1}{r} = \binom{j+1}{r+1} + \binom{k+1}{r} = \binom{j+2}{r+1}$ by pascal's identity. Thus, since $j \to j+1$ we know that the statement holds for all $n \in \mathbb{N}$. $\square$

Problem 3. Calculate the following sums. (That is, express by a simple formula not containing a summation.)

(a) $\sum_{i=2}^{n} i(i-1)$
Claim: $= \frac{k(k+1)(k-1)}{3}$

Proof. We can show that $\sum_{i=2}^{n} i(i-1) = \sum_{i=2}^{n} i^2 - i = \sum_{i=2}^{n} i^2 - \sum_{i=2}^{n} i = \frac{n(n+1)(2n+1)}{6} - 1 - (\frac{n}{n+1}2 - 1)$ by part b. Then we know that $\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} = \frac{k(k+1)(k-1)}{3}$ as desired. $\square$

(b) $\sum_{i=2}^{n} i^2$
Claim: $= \frac{n(n+1)(2n+1)}{6} - 1$

Proof. Base Case: Consider $n = 2$, then $2^2 = 4$ and $\frac{2(2+1)(2(2)+1)}{6} - 1 = 4$. Therefore, the statement is true for $n = 2$.
Inductive Step: Consider that the statement is true for $n = k$, then we want to show that it is true for $n = k + 1$. Then we have that $\sum_{i=2} i^2 = 2^2 + ... + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} - 1 + (k+1)^2 = \frac{2k^3+3k^2+k}{6} + \frac{6k^2+12k+6}{6} - 1 = \frac{2k^3+9k^2+13k+6}{6} - 1 = \frac{(k+1)((k+1)+1)(2(k+1)+1)}{6} - 1$ as we desire. Then by the inductive hypothesis, since $k \to k+1$ we know that the statement is for all n. $\square$

(c) $\sum_{i=1}^{n} i^3$
Claim: $= \frac{n^2(n+1)^2}{4} - 1$

Proof. Base Case; Consider when $n = 2$ then $n^3 = 8$ and $\frac{2^2(2+1)^2}{4} - 1 = 8$. Therefore, the statement is verified for $n = 2$.
Inductive step: Assume that the statement is true for $n = k$, that is to say that $\sum_{i=2}^{k} i^3 = \frac{k^2(k+1)^2}{4} - 1$. We want to show that it is true for $n = k + 1$. We have that $\sum_{i=2}^{k+1} i^3 = 2^3 + ... + k^3 + (k+1)^3 = \frac{k^2(k+1)^2}{4} - 1 + (k+1)^3 = \frac{k^4+2k^3+k^2}{4} + \frac{4k^3+12k^2+12k+4}{4} - 1 = \frac{k^4+6k^3+13k^2+12k+4}{4} - 1 = \frac{(k+1)^2((k+1)+1)^2}{4} - 1$ as desired. Thus, by the inductive hypothesis $k \to k+1$ and we know that the statement is true for all n. $\square$

Prove your answers are correct. *Hint: Use the previous problem.*

Problem 4. Let $n \in \mathbb{N}$. Prove that

$$\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}.$$

Proof. We can say that $\sum_{k=1}^{n} k\binom{n}{k} = n\sum_{k=1}^{n}\binom{n-1}{k-1} = n\sum_{k=0}^{n-1}\binom{n-1}{k} = n2^{n-1}$ by pascal's identity. Thus, we have that $\sum_{k=1}^{n} k\binom{n}{k} = n2^{n-1}$ as desired. $\square$

Problem 5. Prove (without using Stirling's formula) that every $n \in \mathbb{N}$ satisfies

$$n! \geq e\left(\frac{n}{e}\right)^n.$$

Hint: It is possible to prove both via induction and via integration.

Proof. We will proceed by induction.
Base Case: Let $n = 1$.Then we have that $1! = 1$ and $e(\frac{1}{e})^1 = 1$. Thus, the statement is proved for $n = 1$.
Inductive Step: Assume the statement is true for $n = k$ that is to say that we know that $k! \geq e(\frac{k}{e})^k$. We then need to show that the statement is true for $n = k + 1$. Then we have that $(k+1)! = (k+1)k! \geq (k+1)e(\frac{k}{e})^k = e(k+1)(\frac{k}{k+1})^k(\frac{(k+1)^k)}{e^k}) = e(\frac{k}{k+1})^k(\frac{(k+1)^{k+1})}{e^k})$. We also know that $\lim_{x\to\infty}(1+\frac{1}{x})^x = e$ which implies then that $(\frac{k}{k+1})^k = \frac{1}{(1+1/k)^k} = \frac{1}{e}$. Then we have that $e(k+1)(\frac{k}{k+1})^k(\frac{(k+1)^k)}{e^k}) = e(\frac{1}{e})(\frac{(k+1)^{k+1})}{e^k}) = e(\frac{(k+1)^{k+1}}{e^{k+1}}) = e(\frac{k+1}{e})^{k+1}$ as desired. Thus, since $k \to k+1$ we know that the statement is true for all $n \in \mathbb{N}$. $\square$