MATH 4317 Homework 1 Due Friday, 1/21/2022 at 11PM to Gradescope

Homework guidelines:

- Each problem I assign, unless otherwise stated, is asking you to prove something. Give a full mathematical proof using only results from class or Wade.
- Submit a PDF or JPG to gradescope. The grader has ~ 250 proofs to grade: please make his job easier by submitting each problem on a different page.
- If you submit your homework in Latex, you get 2% extra credit.

Problem 1. Prove the following directly from the field axioms (Postulate 1 in Section 1.2): For all $x \in \mathbb{R}$ and $n \in \mathbb{N}$, we have that

$$(x-1)\cdot\left(\sum_{k=0}^{n-1} x^k\right) = (x^n - 1) \tag{1}$$

Explain each step, including uses of the distributive properties for addition and multiplication. You do not have to explain uses of the associative properties for addition and multiplication.

Hint: Use mathematical induction, see Section 1.4 in Wade or the notes from MATH 2106 on Canvas. Remember that you have to do two things: first, prove (1) *at* $n = 1$*, then show that* (1) *for some* n *implies* (1) *holds for* $n+1$.

Let us first consider when $n = 1$. Then $(x-1) * (\sum_{k=0}^{1-1} x^k) = (x-1)(1)$ by the definition of the problem statement. Then by the additive identity we know then that $(x-1)(1) = (x-1)$ which then implies that $(x-1) = (x^1 - 1)$ which means the statement is true for $n = 1$. So we will then assume that this statement is true for $n = j$. That is to say that we know that $(x-1)(\sum_{k=0}^{j-1} x^k) = (x^k - 1)$. We now want to show it works for $n = j+1$. That is to say that we want to show $(x-1)(\sum_{k=0}^{j+1-1} x^k) = (x^{j+1} - 1)$. We could then say by the distributive law $(x-1)(\sum_{k=0}^{j-1} x^k + x^j) = (x-1)(\sum_{k=0}^{j-1} x^k) + (x-1)(x^j) = (x^j - 1) + (x-1)(x^j)$. Then we could again invoke the distributive law to say $(x^j - 1) + (x^{j+1} - x^j) = x^{j+1} - 1$ which is what we wanted to show. Then we know it is then true that $(x-1)(\sum_{k=0}^{n-1} x^k) = (x^n - 1)$ by induction.

Problem 2. Write the set

$$E = \{\cdots, -3, 1, 5, 9, 13, \cdots\}$$

of integers equal to 1 modulo 4 in set-builder notation. You do not have to prove your answer.

$E = \{x \in \mathbb{Z} : x \equiv 1 (\mod 4)\}$ or alternatively, $E = \{x \in \mathbb{Z} : 4|x + 3\}$.

Problem 3. Let X, Y be sets.

(a) Show that if $f : X \to Y$ is one-to-one, then $f(A \cap B) = f(A) \cap f(B)$ for all sets $A, B \subset X$.

(b) Is the assertion in (a) true if we drop the assumption that $f : X \to Y$ is one-to-one? If yes, give a proof, and if no, give a counterexample.

(a) Since $A \cap B \subset A$ we then know that $f(A \cap B) \subset f(A)$; similarly, $A \cap B \subset B$ we know that $f(A \cap B) \subset f(B)$. From this we then get $f(A \cap B) \subset f(A) \cap f(B)$. We then want to show that $f(A) \cap f(B) \subset f(A \cap B)$. Let $x \in f(A) \cap f(B)$, then we know that $x \in f(A)$ and $x \in f(B)$. Since $y \in f(A)$ and $y \in f(B)$, then there exists $a \in A$ such that $y = f(a)$ and $b \in B$ such that $y = f(b)$. Therefore, $y = f(a) = f(b)$ implies that $f(a) = f(b)$ and since f is one to one we know that $a = b$. Let us then define $x = a = b$ and then $x \in A$ and $x \in B$ such that $y = f(x)$. Then $x \in A \cap B$ and $f(x) \in f(A \cap B)$ then $y \in f(A \cap B)$. Therefore, $f(A) \cap f(B) \subset f(A \cap B)$. Since $f(A \cap B) \subset f(A) \cap f(B)$ and $f(A) \cap f(B) \subset f(A \cap B)$ we know then that $f(A \cap B) = f(A) \cap f(B)$.

(b) Let us define $A = \{1, 2\}$ and $B = \{2, 3\}$ such that $f(1) = 4, f(2) = 5, f(3) = 4$. Then $f(A \cap B) = f(\{2\}) = \{5\}$, but $f(A) = \{4, 5\}$ and $f(B) = \{4, 5\}$ and therefore, $f(A) \cap f(B) = \{4, 5\}$. Under this example $f(A \cap B) \neq f(A) \cap f(B)$.

For the next problem: If E is a finite set, then the cardinality $|E|$ is defined to be the number of elements in E.

Problem 4. Let $n \in \mathbb{N}$ and let $f : \{1, 2, \cdots, n\} \to \{1, 2, \cdots, n\}$ be a function. Prove that f is one-to-one if and only if f is onto.

For this problem, you may use the following freely:

(a) If X, Y are finite sets and $f : X \to Y$, then f is one-to-one if and only if $|f(X)| = |X|$.

(b) If A, B are finite sets and $A \subset B$, then $|A| = |B|$ if and only if $A = B$.

First let us assume that the function f is onto. That is to say that for every element x' in the codomain of f there is an element x in the domain of f such that $f(x) = x'$. Given that the codomain and the domain of f are defined the same we can conclude that they are equal and then by condition (b) we are able to conclude that they have equal cardinality. This then implies that the range of f is also equal to the codomain of f and then by using condition (a) we know then that f is one-to-one. Therefore, if f is onto then it is also one-to-one. Alternatively, if f is one-to-one, that is to say that for a, b is the domain of f if $f(a) = f(b)$ then we know that $a = b$. From condition (a) we know that the cardinality of the domain and the range are equal. However, since we know that f is one-to-one we know that each a in the domain of f has a mapping in the codomain of f or else there would be two elements in the domain of f that are equal and as defined this is not the case. Therefore, we know that the cardinality of the range of f is n which the cardinality of the codomain of f which implies that the codomain is equal to the range. Therefore, this then shows that each element of the codomain of f has an element in the domain of f that maps to it meaning then that if f is one-to-one then it is also onto. Therefore f is one-to-one if and only if f is onto.

For the last problem: If A, B are (possibly infinite) sets, we say that *the cardinality of* A *equals the cardinality of* B, written $|A| = |B|$ for short, if there is a one-to-one and onto mapping $f : A \to B$. We say that a set A is countable if $|A| = |\mathbb{Z}|$.

Problem 5. Let $A = \{2k + 1 : k \in \mathbb{Z}\}$ be the set of odd integers. Prove that A is countable.

In order to show that A is countable, we want to show that $|A| = |\mathbb{Z}|$ and order to do this we need to construct a function $f : A \to \mathbb{Z}$ that is both one-to-one and onto. We will define this function as $f(x) = \frac{x-1}{2}$. Let us say that there exists $a, b \in A$ such that $f(a) = f(b)$ this implies that $\frac{a-1}{2} = \frac{b-1}{2}$, which we can rearrange to show then that $a = b$ and f is therefore, one-to-one. Now we will assume that there is a $z \in \mathbb{Z}$, we will consider the element $y = 2z + 1 \in A$, we can then take $f(y) = \frac{(2z+1)-1}{2} = z$ which implies that f is onto. Since f is one-to-one and onto we can then conclude that $|A| = |\mathbb{Z}|$ and then that A is countable.