Homework 8

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1. First it would be helpful to show that if $Bv = \lambda v$ then $BAv = \lambda Av$ for arbitrary eigenvalue λ and corresponding eigenvector v.

$$\begin{aligned} BAv &= ABv \\ &= A(Bv) \\ &= A(\lambda v) \\ &= \lambda Av. \end{aligned}$$

Then to show that B and A share an eigenvector. Since $AB = BA$, we know that $B^{-1}AB = A$. Since we know that $BAv = \lambda Av$,

$$\begin{aligned} BAv &= \lambda Av \\ [B - \lambda I]Av &= 0. \end{aligned}$$

Therefore, Av is also an eigenvector of B. We know that it will be a multiple of v which means that it will also be an eigenvector of A.

2. Since P is an orthogonal projection of $\mathbb{R}^n$ onto W, we know that $Px = x, \forall x \in W$ and $py = 0, \forall y \in W^{\perp}$. If we let $dim(W) = k$ for $0 < k < n$, we can define $\{u_1, u_2, ..., u_k\}$ to be an orthonormal basis of W. We could then extend this basis to be a basis of $\mathbb{R}^n$ as $\{u_1, u_2, ..., u_k, u_{k+1}, ..., u_n\}$ where $\{u_{k+1}, ..., u_n\}$ will form a basis for $W^{\perp}$. Therefore $Pu_i = u_i$ for $1 \leq i \leq k$ and $Pu_j = 0$ for $k+1 \leq j \leq n$. Therefore, we could say that $\{u_1, u_2, ..., u_k\}$ are eigenvectors of P corresponding to the eigenvalue 1. Similarly, $\{u_{k+1}, ..., u_n\}$ are eigenvectors of P corresponding to eigenvalue 0. Since Q is orthogonal, we know that the columns of Q are orthonormal and we are given that $Q^T = Q^{-1}$. Let us define diagonal matrix D to have the eigenvalues of P as the elements of the diagonal where $a_{i,i} = \begin{cases} 1 & 1 \leq i \leq k \\ 0 & k+1 \leq i \leq n \end{cases}$. Therefore, $PQ = QD, P = QDQ^{-1} = QDQ^T$. Thus, $Px = QDQ^Tx$.

3. The maximum value of n such that norm $< 10^{-2}$ is when $n = 69$. The Schur decomposition is quite robust as we can see in this problem, even when we calculate when the norm is $< 10^{-2}$, the complexity of A is well

beyond a reasonable scope that can be achieved by hand. Therefore, it will help in creating simplified representations of a matrix like A as a product of three matrices that can provide information on the matrix A.

4. The largest n for $B^n D B^{-n}$ is $n = 9$. The largest n for $Q^n D Q^{-1}$ is around $1.25 * 10^{17}$. This means that the second process of orthogonalizing the columns of B is much more reliable and accurate to the production of the eigenvalues.