Homework 4

Sean Eva

March 2021

1 Exercises

14. There are four scenarios to be considered for $\mathbb{P}(a < X \leq b, c < Y \leq d)$. This is the same as considering the four separate probabilities $\mathbb{P}(b, d)$, $\mathbb{P}(b, c)$, $\mathbb{P}(a, d)$, and $\mathbb{P}(a, c)$. We can consider all cases of $\mathbb{P}(b, d)$. We will then need to remove bad scenarios such as $\mathbb{P}(a, d)$ and $\mathbb{P}(b, c)$ which will include situations like $\mathbb{P}(X \leq a, Y \leq d)$ and $\mathbb{P}(X \leq b, y \leq c)$. However, when we remove both of these situations we double remove the case when $\mathbb{P}(X \leq a, Y \leq c)$; therefore, we will need to add back this case with $\mathbb{P}(a, c)$. This results in $\mathbb{P}(a < X \leq b, c < Y \leq d) = F(b, d) + F(a, c) - F(a, d) - F(b, c)$

26. Solving this using the first requirement of $\mathbb{P}(X + Y \leq 1)$ would be solving $\int_{-\infty}^{\infty} \int_{-\infty}^{1-X} f(x, y)dydx$ which will edit to, $\int_0^1 \int_0^{1-X} f(x, y)dydx$ due to the bounds of $f(x, y)$ being such that $x, y > 0$. Therefore, $\int_0^1 \int_0^{1-X} f(x, y)dydx = \frac{e-2}{e}$.
    For the second requirement $\mathbb{P}(X > Y)$, $f(x, y)$ is bounded to $0 < x < \infty, 0 < y < x$. Therefore, $\int_0^{\infty} \int_0^X f(x, y)dydx = \frac{1}{2}$.

36. Joint density functions integrate to 1 so that $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, z)dxdydz = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (8xyz)dxdydz = x^2y^2z^2$. X, Y, Z are independent since f can be represented as a product of a function of X, a function of Y, and function of Z as $f_X(x) = 2x, f_Y(y) = 2y, f_Z(z) = 2z$.
    $\mathbb{P}(X > Y) = \int_0^1 \int_0^1 \int_Y^1 (8xyz)dxdydz = \frac{1}{2}$
    $\mathbb{P}(Y > Z) = \int_0^1 \int_Z^1 \int_0^1 (8xyz)dxdydz = \frac{1}{2}$

45. We can define random variables $u = X + Y$ and $v = Y$ which would therefore mean that $X = u - v$ and $Y = v$. Therefore, the Jacobian is $J = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = 1$. Therefore, $g(u, v) = f(x, y) * J = \frac{1}{2}(x + y)e^{-x-y} = \frac{1}{2}ue^{-u}$.
    Thus, the density function for $u = X + Y$ is $g_{X+Y}(u) = \int_0^u \frac{1}{2}ue^{-u}dv = \frac{1}{2}u^2e^{-u}$. Therefore, the density function for $X + Y$ is $g_{X+Y}(u) = \frac{1}{2}u^2e^{-u}$.

55. Given the substitutions $U = \frac{1}{2}(X - Y), V = Y$, then $X = 2U + V, Y = V$. Then the Jacobian is $J = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} = 2$. Then $f_{U,V} = \frac{1}{4}e^{-\frac{1}{2}(2u+v+v)} = \frac{1}{2}e^{-u-v}$. Therefore, $f_U(u) = \begin{cases} \int_{-2u}^{\infty} \frac{1}{2}e^{-u-v}dv & u < 0 \\ \int_0^{\infty} \frac{1}{2}e^{-u-v}dv & u \geq 0 \end{cases} = \begin{cases} \frac{1}{2}e^{-(-u)} & u < 0 \\ \frac{1}{2}e^{-u} & u \geq 0 \end{cases} = \frac{1}{2}e^{-}|u|$

61. Given that the solution is x and y are independent random variables each having the exponential distribution with parameter λ. The joint density function of x and y is $f(x, y) = \lambda^2 e^{-\lambda(x+y)}, x, y > 0$. We want to find the joint probability density function of x and $x + y$. Let $u = x + y, v = x$, then $x = v, y = u - v$. The Jacobian is then, $J = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = 1. f(u, v) = f(x = y, y = u - v) * J = \lambda^2 e^{-\lambda(v+u-v)} * 1 = \lambda^2 e^{-\lambda u}$. Therefore, the joint probability density function of x ad $x + y$ is $f(u = x + y, v + x) = \begin{cases} \lambda^2 e^{-\lambda(x+y)} & 0 < x \leq x + y, x + y > 0 \\ 0 & \text{otherwise} \end{cases}$. Now we need to find the general density function of $u = x + y$ as $f(u) = \int_0^u \lambda^2 e^{-\lambda u} dv = \lambda^2 e^{-\lambda u}(v)_0^u = \lambda^2 u e^{-\lambda u}$. Then we need to find the conditional probability density function of x given a point $x + y = a$, $f(x|x + y = a) = f(v|u = a) = \frac{f(v,u-a)}{f(u)} = \frac{\lambda^2 e^{-\lambda a}}{\lambda^2 a e^{-\lambda a}} = \frac{1}{a}$. Therefore, the conditional probability density function of x given a point $x + y = a$ is $f(x|x + y = a) = \begin{cases} \frac{1}{a} & 0 \leq x \leq a, a > 0 \\ 0 & otherwise \end{cases}$

70. It will make this problem much simpler if we apply change of variables to this problem to work in polar coordinates.
$\mathbb{E}\sqrt{X^2 + Y^2} = \int_0^{2\pi} \int_0^1 (r * \pi^{-1} * r) dr d\theta = \frac{2}{3}$
$\mathbb{E}\sqrt{X^2 + Y^2} = \int_0^{2\pi} \int_0^1 (r^2 * \pi^{-1} * r) dr d\theta = \frac{1}{2}$

80. If (X, Y) have distribution $BN(\mu_1, \mu_2, \sigma_1^2, \sigma_2^2)$ then the Mg of the bivariate normal distribution is $M(x, y) = \mathbb{E}(e^{tx+tu}) = e^{\mu_1 t_t + u_2 t_2 + \frac{1}{2}(t_1^2\sigma_1^2 + 2\rho t_1 t_2 \sigma_1 \sigma_2 + t_2^2 \sigma_2^2)}$. Let's assume that (X, Y) has distribution $Bn(\mu_1, \mu_2, \sigma_1^2, \sigma_2)$ to show that ax+by has normal distribution. Therefore, consider the Mg of $(ax + by = z)$, $m_z(t) = \mathbb{E}(e^{tz}) = \mathbb{E}(e^{t(ax+by)}) = \mathbb{E}(e^{atx+bty}) = \mathbb{E}(e^{t_1 x + t_2 y})$ for $t_1 = at, t_2 = bt$. Since (x, y) has bivariate normal distribution, the Mg is given as $m_{X,Y}(t_1, t_2) = \mathbb{E}(e^{t_1 1x + t_2 y}) = e^{u_1 t_1 + u_2 t \frac{1}{2}(t_1^2\sigma_1^2 + 2\rho\sigma_1\sigma_2 t_1 t_2 + t_2^2 \sigma_2^2)}$. Replace $t_1 = at, t_2 = bt$ so that $m_{ax+by}(t) = m_z(t) = e^{u_1 at + u_2 bt + \frac{1}{2}(a^2 t^2 \sigma_1^2 + 2\rho\sigma_1\sigma_2 atbt + b^2 t^2 \sigma_2^2)}$. Therefore, $m_{X,Y}(t) = e^{t(a\mu_1 + b\mu_2) + \frac{1}{2}t^2(a^2\sigma_1^2 + s\rho ab\sigma_1\sigma_2 + b^2\sigma_2^2)}$ which is the Mg of the univeriate normal distribution with mean $au_1 + bu_2$. Thus, $ax + by$ has normal distribution $N(a\mu_1 + b\mu_2, a_1^2\sigma_1^2 + 2\rho ab\sigma_1\sigma_2 + b^2\sigma_2^2)$.

2 Problems

6. We can start by defining the cdf of U as

$$\begin{aligned}
G(u) &= \mathbb{P}(U \leq u) \\
&= \mathbb{P}(min\{X_1, X_2, ..., X_n\} \leq u) \\
&= 1 - \mathbb{P}(min\{X_1, X_2, ..., X_n\} > u) \\
&= 1 - \mathbb{P}(X_1 > u, X_2 > u, ..., X_n > u) \\
&= 1 - \mathbb{P}(X_1 > u)\mathbb{P}(X_2 > u)...\mathbb{P}(X_n > u) \\
&= 1 - (\mathbb{P}(X_1 > u))^n \\
&= 1 - (1 - \mathbb{P}(X_1 \leq u))^n \\
&= 1 - (1 - F(u))^n.
\end{aligned}$$

Similarly for the distribution of $V = max\{X_1, X_2, ..., X_n\}$

$$\begin{aligned}
H(v) &= \mathbb{P}(V \leq v) \\
&= \mathbb{P}(max\{X_1, X_2, ..., X_n\} \leq v) \\
&= \mathbb{P}(X_1 \leq v, X_2 \leq v, ..., X_n \leq v) \\
&= (F(v))^n.
\end{aligned}$$

Then,

$$\begin{aligned}
g(u) &= \frac{dG(u)}{du} \\
&= nf(u)(1 - F(u))^{n-1} \\
h(v) &= \frac{dH(v)}{dv} \\
&= nf(v)(F(v))^{n-1}.
\end{aligned}$$

We can therefore write the CDF of random variable V as $\mathbb{P}(V \leq v) = \mathbb{P}(U \leq u, V \leq v) + \mathbb{P}(U > u, V \leq v)$ where $K(u, v) = \mathbb{P}(U \leq u, V \leq v)$. Then,

$$\begin{aligned}
\mathbb{P}(U \leq u, V \leq v) &= \mathbb{P}(V \leq v) - \mathbb{P}(U > u, V \leq v) \\
&= (F(v))^n - \mathbb{P}(u < X_1 \leq v, u < X_2 \leq v, ..., u < X_n \leq v) \\
&= (F(v))^n - \mathbb{P}(u < X_1 \leq v)^n \\
&= (F(v))^n - (F(v) - F(u))^n.
\end{aligned}$$

The joint density function is then: $h(u, v) = \frac{\delta (F(v))^n - (F(v) - F(u))^n}{\delta u \delta v} = n(n-1)f(u)f(v)(F(v - F(u)))^{n-2}$

20. The joint probability density of X and Y is $F_{X,Y}(x, y) = \begin{cases} 1 & 0 < x < 1, 0 < y < 1. \\ 0 & \text{otherwise} \end{cases}$.

Then,

$$\begin{aligned}
\mathbb{P}(X+y<1) &= \mathbb{P}(X<1-y)\\
&= \int_{x=0}^{1-y}\int_{y=0}^{1} f(x,y)dxdy\\
&= \int_{y=0}^{1}\int_{x=0}^{1-y} 1dxdy = \int_{y=0}^{1}(x)_0^{1-y}dy\\
&= \int_{y=0}^{1}(1-y)dy = (y=\frac{y^2}{2})_0^1\\
&= 1-\frac{1}{2}=\frac{1}{2}.
\end{aligned}$$

The marginal of X is then,

$$\begin{aligned}
f_X(x) &= \int_y f(x,y)dy = \int_0^1 dy = (y)_0^1 = 1\\
f_X(x) &= \begin{cases} 1 & 0<x<1\\ 0 & \text{otherwise}\end{cases}\\
f_Y(y) &= \int_x f(x,y)dx = \int_0^1 1dx = (x)_0^1 = 1\\
f_Y(y) &= \begin{cases} 1 & 0<y<1\\ 0 & \text{otherwise}\end{cases}.
\end{aligned}$$

$\mathbb{E}(x) = \int_0^1 xf(x)dx = \int_0^1 x1dx = (\frac{x^2}{2})_0^1 = \frac{1}{2}$.
$f_{Y|X}(y|x) = \frac{f(x,y)}{f(x)} = \frac{1}{1} = 1;\ 0<y<1$.
$\mathbb{E}(Y|X=x) = \int_0^1 yf(y|x)dy = \int_0^1 y*1dy = (\frac{y^2}{2})_0^1 = \frac{1}{2}$