Homework 2

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1. *Proof.* Base: Consider that matrix A is a 1×1 matrix. Then $A = \begin{bmatrix} -a_o \end{bmatrix}$. Then $\lambda I - A = \begin{bmatrix} \lambda + a_0 \end{bmatrix}$. Therefore, $det(\lambda I - A) = \lambda + a_0$. This shows that the general form $\lambda^n + a_{n-1}\lambda^{n-1} + ... + a_1\lambda + a_0$ holds for $n = 1$. Additionally for $n = 2$, $A = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}$ and $\lambda I - A = \begin{bmatrix} \lambda & -1 \\ a_0 & \lambda + a_1 \end{bmatrix}$. Then $det(\lambda I - A) = \lambda^2 + \lambda a_1 + a_0$. Therefore the statement is also true for $n = 2$. We can notice that the matrix for $n = 1$ is present within the matrix for $n = 2$ as the entry in $A_{2,2}$ as $-a_1$. To find the new $det(\lambda I - A)$ Inductive Step: Let the statement be true for matrix A of dimension $m \times m$. That is to say that $det(\lambda I - A) = \lambda^m + a_{n-1}\lambda^{n-1} + ... + a_1\lambda + a_0$. Then for a matrix B of dimension $m + 1 \times m + 1$. We can notice, such as was done during the base case, that the matrix A is present from $A_{2,2}$ to $A_{m+1,m+1}$ simply increase all of the indexes for a_k for all k in the submatrix of A within B to a_{k+1}, we will denote this as A_{k+1} whose determinan will be $det(A_{k+1}) = \lambda^m + a_n\lambda^{n-1} + ... + a_2\lambda + a_1$. This allows us to use the $det(A)$ to help solve $det(B)$. Therefore $det(B) = \lambda * det(A_{k+1}) - (-1) * a_0 = \lambda * (\lambda^m + a_n\lambda^{n-1} + ... + a_2\lambda + a_1) + a_0 = \lambda^{m+1} + a_m\lambda^m + ... + a_1\lambda + a_0$. Therefore, the statement is true for $n = m + 1$ given that it is true for $n = m$. So the statement is true for $n \in \mathbb{N}$ ■

2. $AB = \begin{bmatrix} a_1 & a_2 & ... & a_n \end{bmatrix} \begin{bmatrix} b_1 & b_2 & ... & b_n \end{bmatrix} = \begin{bmatrix} Ab_1 & Ab_2 & ... & Ab_n \end{bmatrix}$. Suppose B is singular, then when AB is calculated, it too will be singular. Similarly, if B is non-singular AB will be non-singular. Therefore, $rank(AB) \leq rank(B)$. If A is singular then $rank(AB) \leq rank(A)$. This means that $rank(AB)$ is capped by $rank(A)$ unless $rank(B)$ is lower. combining those two conditions, then $rank(AB)$ is capped by the smaller of $rank(A)$ and $rank(B)$. Therefore, $rank(AB) \leq min(rank(A), rank(B))$.

3. *Proof.*

$$A^2 + I = 0$$
$$A^2 = -I$$
$$det(A^2) = det(-I).$$

Since the $det(A^2) = det(A) * det(A) = det(A)^2$. This means that for any $det(A)$ it will be positive, therefore, $det(-I)$ must also be positive. The $det(-I)$ is equal to the product along the main diagonal, therefore the dimension of n must be even. ■

Consider $A = \begin{bmatrix} i & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & i \end{bmatrix}$ then $A^2 + I = 0$ and $n = 3$ which is not even.

4. It is better to store the values of L, U, P than it is to store the entries of A^{-1}. While it takes up less individual values to store the entries of A^{-1} by the nature of it only being a single $n \times n$ matrix. However, the data that would be stored for the entries L, U, P are almost undeniably much more trivial. $P = I_n$. U is the matrix where one row over the main diagonal entries, there is a diagonal of -1, then along the main diagonal, the values follow the pattern $\frac{row+1}{col}$ with all other entries equaling 0. Similarly for the matrix L, the entries along the main diagonal are all 1, then one row below the main diagonal the entries follow the pattern $-\frac{col}{row}$ with all other entries equaling 0. So while there are 3 times the number of entries to record, there is a pattern that could be stores to make reproduction much more trivial for larger values of n.