Homework 7

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1.

$$
\begin{aligned}
||A^{-1} - B^{-1}|| &= ||A^{-1} + (-B^{-1})|| \\
&\leq ||A^{-1}|| + || - B^{-1}|| \\
&= ||A^{-1}|| + ||B^{-1}|| \\
&= ||A^{-1}|| * ||B|| * ||B^{-1}|| + ||B^{-1}|| * ||A|| * ||A^{-1}|| \\
&= ||A^{-1}|| * (||B|| * ||B^{-1}|| + ||B^{-1}|| * ||A||) \\
&= ||A^{-1}|| * (||B|| + ||A||) * ||B^{-1}|| \\
&= ||A^{-1}|| * (||B|| + || - A||) * ||B^{-1}|| \\
&= ||A^{-1}|| * ||B - A|| * ||B^{-1}||.
\end{aligned}
$$

2. Hint: It is trivial to see that $\underset{j}{max}|z_j| = ||z||$ if all elements of z are zero or if there is only one non-zero element. It is also easy to see that $\underset{j}{max}|z_j| < ||z||$ if there are multiple nonzero elements in z. Additionally, the maximum value $||z||$ can take on is if all values of z are equal to each other which would result in $\sqrt{\sum_{j=1}^{n} z_j \overline{z_j}} = \sqrt{\sum_{j=1}^{n} z_i \overline{z_i}} = \sqrt{n z_i \overline{z_i}} = \sqrt{n}|z_i|$ for $z_i = \underset{j}{max}|z_j|$. Therefore, $\underset{j}{max}|z_j| \leq ||z|| \leq \sqrt{n} * \underset{j}{max}|z_j|$.
   Actual Question:

$$
\begin{aligned}
\underset{i,j}{max}|a_{ij}| &\leq \sum_{j}{}_{i} = 1^n |a_{ij}| \\
&= max|Ax| \text{ for } x \in \mathbb{C}^n \text{ and } ||x|| = 1 \\
&= ||A||.
\end{aligned}
$$

Therefore,

$$\begin{aligned}\max_{i,j}|a_{ij}| &\leq ||A|| \\ &= max||Ax|| : ||x|| = 1 \\ &\leq n * max||A|| \\ &\leq n * \sqrt{n} * max|a_{ij}| \\ &= n * \sqrt{n} * \max_{i,j}|a_{ij}|.\end{aligned}$$

Thus, $\max_{i,j}|a_{ij}| \leq ||A|| \leq n * \sqrt{n} * \max_{i,j}|a_{i,j}|$.

3. Given that $\kappa(A) = \frac{\max_k|\lambda_k|}{\min_k|\lambda_k|} = \frac{\Lambda}{\lambda}$. Therefore,

$$\begin{aligned}\frac{\Lambda}{\lambda}(\kappa(P))^{-2} &= \kappa(A)(\kappa(P))^{-2} \\ &= \frac{\kappa(A)}{(\kappa(P))^2}.\end{aligned}$$

Since $\kappa \geq 1$ for any matrix,

$$\frac{\kappa(A)}{(\kappa(P))^2} \leq \kappa(A).$$

Since $\kappa(P) \geq 1$,

$$\begin{aligned}\kappa(A) &\leq \kappa(A)(\kappa(P))^2 \\ &= \frac{\Lambda}{\lambda}(\kappa(P))^2.\end{aligned}$$

4. $\kappa(A) = 3.4184, \kappa(A^5) = 149.7032, \kappa(A^{10}) = 1.0116 * 10^4$.
   The growth rate is caused by the growth in $\max_j|\lambda_j|$ and the growth of $\min_j|\lambda_j|$. The largest eigenvalue will grow much faster than the smallest eigenvalue. That means that $\kappa(A) = \frac{\max_j|\lambda_j|}{\min_j|\lambda_j|} = \frac{\Lambda}{\lambda}$ will grow as Λ increases much faster than λ.