MATH 4032: HOMEWORK #6
DUE APRIL 21 AT 1:59PM

This assignment is on combinatorial design theory. Relevant background material for this assignment is covered in Chapters 5 and 12 of the Jukna book and Chapters 9 and 13 of the Matoušek–Nešetřil book. However, note that some material covered in class is not in either text.

- You are strongly encouraged to typeset your homework solutions using LaTeX.
- Acknowledge collaborations as noted in the syllabus.
- The following problems are optional exercises not to be turned in. Problems to be turned in for a grade begin on the next page.

Exercise 1. Find a set of three MOLS of order 4.

Exercise 2.

(a) Construct a symmetric Latin square of order 5 with $1, 2, 3, 4, 5$ appearing in that order on the diagonal.

(b) Construct Steiner triple system of orders 13 and 15. *Hint: For the order* 15*, use the Latin square from part (a). For the order* 13*, use the Latin square which is the addition table of* $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Exercise 3. Use three MOLS of order 4 to construct an affine plane of order 4.

Exercise 4. Use the operation described in Problem 4 to construct an order-3 projective plane from the order-3 affine plane depicted in Problem 3.

Exercise 5. Show that there is no 5-$(16, 7, 1)$ design.

Recall that a *transversal* in a Latin square $L = (\ell_{i,j})_{i,j\in[n]}$ of order n is a set of n entries $(i_1, j_1), \ldots, (i_n, j_n) \in [n] \times [n]$ such that each row, column, and symbol appears exactly once, i.e. $i_1, \ldots i_n$ are distinct, $j_1, \ldots, j_n$ are distinct, and $\ell_{i_1,j_1}, \ldots, \ell_{i_n,j_n}$ are distinct. A *perfect matching* in a hypergraph $(V, \mathcal{B})$ (sometimes called a *parallel class* in the context of design theory) is a set $M \subseteq \mathcal{B}$ of pairwise disjoint edges such that that every vertex is in exactly one edge in M, i.e. $V = \bigcup_{A\in M} A$. The hypergraph is *resolvable* if there is a decomposition of $\mathcal{B}$ into pairwise disjoint perfect matchings / parallel classes.

Problem 1. Let $L = (\ell_{i,j})_{i,j\in[n]}$ be an order-n Latin square. Prove that the following are equivalent:

(a) L has an orthogonal mate;

(b) L has a decomposition into n pairwise disjoint transversals;

(c) the graph $G = ([n] \times [n], E)$, where

$$E = \left\{ \{(i, j), (i', j')\} \in \binom{[n] \times [n]}{2} : i = i',\ j = j',\ \text{or } \ell_{i,j} = \ell_{i',j'} \right\},$$

has chromatic number n.

(d) the hypergraph $(V, \mathcal{B})$ where $V = \{(\{\mathrm{ROW}\} \times [n]) \cup (\{\mathrm{COL}\} \times [n]) \cup (\{\mathrm{SYM}\} \times [n])\}$ and

$$E = \{\{(\mathrm{ROW}, i), (\mathrm{COL}, j), (\mathrm{SYM}, k)\} : \ell_{i,j} = k\}$$

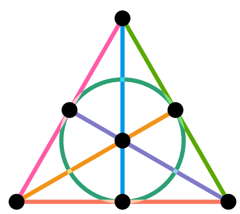
is resolvable.

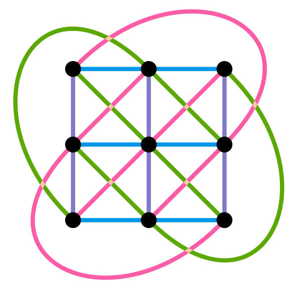
An *isomorphism* from a hypergraph $(V_1, \mathcal{B}_1)$ to a hypergraph $(V_2, \mathcal{B}_2)$ is a bijection $f : V_1 \to V_2$ such that $B_1 \in \mathcal{B}_1$ if and only if $\{f(v) : v \in B_1\} \in \mathcal{B}_2$.

Problem 2.

(a) Prove that every Steiner triple system of order 7 is isomorphic to the Fano plane (below, left). *Hint: Prove that any two triples of an order-*7 *Steiner triple system intersect.*

(b) Prove that every Steiner triple system of order 9 is isomorphic to the order-3 affine plane (below, right). *Hint: Prove that for every triple in an order-*9 *Steiner system, there are only two triples disjoint from it, and moreover, these two are also disjoint.*





Recall the definition of *parallel class* and *resolvable* from Problem 1. An *affine plane of order q* is a hypergraph $(V, \mathcal{B})$ where

- $|V| = q^2$,
- $\mathcal{B} \subseteq \binom{V}{q}$ (the elements of $\mathcal{B}$ are called *lines*), and
- every two distinct $u, v \in V$ are in a unique line $B \in \mathcal{B}$.

(An order-3 affine plane is depicted to the right in Problem 2.)

Problem 3. Prove that every affine plane is resolvable. More generally, prove that

- there are $q + 1$ parallel classes containing q lines each and
- any two lines from different parallel classes intersect.

Hint: Prove that every vertex is in $q + 1$ lines and that there are $q^2 + q$ lines. Each vertex is contained in one line from every parallel class, so the set of $q+1$ lines containing some fixed vertex contains a representative for each parallel class.

A *projective plane* of order q is a hypergraph $(V, \mathcal{B})$ where

- $|V| = q^2 + q + 1$,
- $\mathcal{B} \subseteq \binom{V}{q+1}$ (the elements of $\mathcal{B}$ are called *lines*), and
- every two distinct $u, v \in V$ are in a unique line $B \in \mathcal{B}$.

(An order-2 projective plane is depicted to the left in Problem 2.)

Problem 4. This problem shows that a projective plane of order q exists if and only if an affine plane of order q exists.

(1) Let $(V, \mathcal{B})$ be a projective plane of order q, let $L \in B$, let $V' = V \setminus L$, and let $\mathcal{B}' = \{L' \setminus L : L' \in \mathcal{B}, L' \neq L\}$. Prove that $(V', \mathcal{B}')$ is an affine plane of order q. *Note: The removed line is called the "line at infinity".*

(2) Let $(V, \mathcal{B})$ be an affine plane of order q, and let Y be the set of its parallel classes (which has size $q + 1$ by the previous problem). For each $L \in \mathcal{B}$, let $L^* = L \cup \{C\}$, where C is the parallel class containing L. Let $V' = V \cup Y$, and let

$$\mathcal{B}' = \{Y\} \cup \{L^* : L \in \mathcal{B}\}.$$

Prove that $(V', \mathcal{B}')$ is a projective plane of order q. *Note: This operation is called 'adding a line at infinity'.*

Problem 5. Let $(V, \mathcal{B})$ be a 2-(n, k, λ) design, and let $\mathcal{B}' = \{V \setminus B : B \in \mathcal{B}\}$. Prove that $(V, \mathcal{B}')$ is a 2-$(n, n-k, \lambda')$ design for some λ'. What is λ' (in terms of n, k, and λ)?