Homework 1

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January 25, 2021

1. Let $Ax = b$ be a linear system with real coefficients such that A is an $m \times n$ matrix and $Ax = b$ has a unique solution. What can you say about m, n? Explain how to pick a subsystem of n equations in $Ax = b$ such that the two systems have the same solutions. (Just present a correct algorithm; you don't have to prove that the algorithm works).

   In this situation, m is the number of equations that are present in the linear system and n is the number of unknowns the equations have. For example, if A has dimensions 3×4, then the system would contain 3 equations in 4 unknowns.
   In order to come up with a pair of systems that will produce the same solutions, you simply need to put the matrix A into Row Echelon Form, then take the equations whose rows contain the pivots. There should be n equations given that the system as a unique solution. Use exclusively these equations to form an $n \times n$ matrix that would have the same solution.

2. Read the essay on floating point arithmetic in Appendix A (in the course notes to be found in Canvas) and write a paragraph (under 50 words) on why round-off errors are inevitable in the floating point arithmetic.

   In floating point arithmetic, the rewritten number can only have a certain degree of precision. If a number contains more information that extends beyond the degree of precision, that information will be lost. This means that round-off errors are inevitable because it will almost always lose information.

3. Read the essay on row reduction with partial pivoting in Appendix D and fill details in the claim that the equation $-9999x_2 = -9998$ can be written as $-(.1) \times 10^5 x_2 = -(.1) \times 10^5$.

   Through the use of rounded arithmetic, we can estimate the values of $-9999x_2 = -9998$ to be of base 10, precision 3, and exponent range $[-16, 16]$. Rounded arithmetic desires for numbers to be written in the form $\pm 0.d_1d_2d...d_t \times \beta^e$ for $d_1, d_2, ..., d_t$ being non-negative integers between 0 and 9 where $d_1 \neq 0$ and e is an integer between a defined exponent range. This allows for the original equation to be written as $-(0.1) \times 10^5 x_2 = -(0.1) \times 10^5$.

4. The MATLAB command $hilb(n)$ calls the $n \times n$ Hilbert matrix that is famously illconditioned. Let's play with it a bit. Find the smallest n such that $inv(hilb(n)) - invhilb(n)$ has an entry with absolute value at least 1. Here $inv(hilb(n))$ is the numerical inverse of $hilb(n)$ and $invhilb(n)$ is the exact inverse. Discuss what this has to do with errors in solving numerically the linear system $Ax = b$ where $A = hilb(n)$ and b is a column of the identity matrix.

   Given that there was a difference between $inv(hilb(n))$ and $invhilb(n)$, that means that there is an amount of error that occurs during the calculation process, depending on which inverse you find.

5. Let A be a real matrix such that the linear system Ax = 0 has a nonzero complex solution. Show that it has a nonzero real solution.

   Proof. Suppose for the sake of contradiction that the real matrix A does not have real solutions to the linear system $Ax = 0$. However, we know the system has a solution matrix x that has a nonzero complex solution. If this is the only solution, that means that there must be a complex component of matrix A in order for the product of complex numbers to result in 0. That is a contradiction with the definition of matrix A in that the matrix is real. ∎