Homework 2

Sean Eva

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1 Exercises

2. 10. The image of 1_E is $\{0, 1\}$. Therefore, since the image of 1_E is countable (2) it is also a discrete random variable.

   11. The image of U is $\{1, 2, 3, 4, 5, 6\}$ which is countable; therefore, U is a discrete random variable on the probability space. The image of V is $\{0, 1\}$ which is countable amount of outcomes; therefore, V is a discrete random variable on the probability space. The image of W is $\{1, 4, 9, 16, 25, 36\}$ which, similar to U, is countable; therefore, W is a discrete random variable on the probability space.

   24. In order to show this we assume that there are k failures which means that this would occur with probability $(1-p)^x$. Then we have that $\sum_{n=k}^{\infty}(1-p)^n p$ instead of proceeding from 1 we proceed from k. If we consider k as the starting point, it would be equivalent to say we are starting from 1 just the same. That means that the probability is $\frac{p}{1-(1-p)}$. Therefore, $P(X > k) = \frac{(1-p)^x p}{1-(1-p)} = (1-p)^x$

3. 8. There are several different outcomes in this situation. With a total of $\binom{52}{2}$ outcomes.
   Two kings, zero aces $= \binom{4}{2}\binom{4}{0}\binom{44}{0} = (2, 0) = \frac{1}{221}$
   Zero kings, two aces $= \binom{4}{0}\binom{4}{2}\binom{44}{0} = (0, 2) = \frac{1}{221}$
   One king, one ace $= \binom{4}{1}\binom{4}{1}\binom{44}{0} = (1, 1) = \frac{8}{663}$
   One king, zero aces $= \binom{4}{1}\binom{4}{0}\binom{44}{1} = (1, 0) = \frac{88}{663}$
   Zero kings, one ace $= \binom{4}{0}\binom{4}{1}\binom{44}{1} = (0, 1) = \frac{88}{663}$
   Zero kings, zero aces $= \binom{4}{0}\binom{4}{0}\binom{44}{2} = (2, 0) = \frac{473}{663}$

2 Problems for Chapter 2

4. In order for this to be a probability mass functions we would require that $\sum_{k=1}^{\infty} ck^{\alpha} = 1$, or similarly that $c\sum_{k=1}^{\infty} k^{\alpha} = 1$. Therefore $c = \frac{1}{\sum_{k=1}^{\infty} k^{\alpha}}$. This will apply for any value of $\alpha > 1$

5. Assume that X has the 'lack-of-memory property', that is that $p(x > m+n|x > m) = p(x > n)$ for $m, n = 0, 1, 2, ...$ where X has geometric distribution. Then $p(X \leq n) = \sum_{k=1}^{n} p(1-p)^{k-1} = p(1 + (1-p) + (1-p)^2 + ... + (1-p)^{n-1} = p(\frac{1-(1-p)^n}{1-(1-p)} = 1 - (1-p)^n$. So, $p(X > n) = 1 - p(X \leq n) = (1-p)^n$. Then, $p(X > m+n|X > m) = \frac{p(X?m+n, X>m}{p(X>m)}$. $\frac{p(X>m+n)}{p(X>m)} = \frac{(1-p)^{n+m}}{(1-p)^m} = (1-p)^n = p(X > m)$. Therefore, $p(X > m+n|X > m) = p(x, n)$. Thus, the geometric distribution has the 'lack-of-memory property'.

7. This problem can be solved using an inclusion-exclusion approach. there are $\binom{c}{n}$ ways to choose n sets. The probability to complete n sets is the probability to complete a standard coupon collection. This leads to the expectation of $\sum_{n=1}^{c}(-1)^{n-1}\binom{c}{n}\frac{c}{n}njH_{nj} = cj\sum_{n=1}^{c}(-1)^{n-1}\binom{c}{n}H_{nj} = cj\sum_{n=1}^{c}(-1)^{n-1}\binom{c}{n}(log(j) + log(n) + \gamma + \frac{1}{2nj}) + O(\frac{c}{j}) = cj(log(n) + \gamma) + \frac{1}{2}cH_c + cj\sum_{n=1}^{c}(-1)^{n-1}\binom{c}{n}log(n)O(\frac{c}{j}) = cjH_n + \frac{1}{2}cH_c - c + cj\sum_{n=1}^{c}(-1)^{n-1}\binom{c}{n}log(n) + O(\frac{c}{j})$.