Homework 2

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Problem 1. Say whether or not the following the following subsets of $\mathbb{R}^2$ are connected:

$$A = \{(x, y) \in \mathbb{R}^2, \|x\| \leq 1, \|y\| \leq 1\}, \quad B = \{(x, y) \in \mathbb{R}^2, xy = 1\} \cup \{(0, y) \in \mathbb{R}^2, y \in \mathbb{R}\},$$

$$C = \mathbb{R}^2 - \mathbb{Q}^2, \quad D = \{(x, y) \in \mathbb{R}^2; \|x\| > 1, \|y\| > 1\}.$$

Claim. A is connected.

Proof. Let $(a, b), (c, d) \in A$. We will construct a path between them that lies entirely in A. Without loss of generality, assume that $a \leq c$ and $b \leq d$. Let us then define the continuous function $f : [0, 1] \to \mathbb{R}^2$ by $f(x) = ((1 - x)a + xc, (1 - x)b + xd)$ for $0 \leq x \leq 1$. Then for $f(0) = (a, b)$ and $f(1) = (c, d)$, and since $f(x)$ is in A for all x and we have that A is path-connected implying that A is connected as desired. $\square$

Claim. B is not connected.

Proof. By definition of B, we know that $(1, 1)$ and $(-1, -1)$ are in B but there is no path that lies entirely in B. Suppose for the sake of contradiction that there does exists a path $f : [0, 1] \to B$ that connects $(1, 1)$ and $(-1, -1)$. Then $f(t) = (x(t), y(t))$ for some continuous functions $x(t), y(t)$. Since $f(t)$ lies on the $xy = 1$ for all t, we have that $x(t)y(t) = 1$ for all t. This then implies that $y(t) = \frac{1}{x}(t)$. Now consider this as t approaches 0. Since $x(t)$ approaches 1 and $y(t)$ approaches infinity, we have that the path f cannot be continuous at $t = 0$. Therefore, we find that B is not connected. $\square$

Claim. C is connected.

Proof. Suppose for the sake of contradiction that C is not connected. That is to say then that there exists open sets $U, V \in \mathbb{R}^2$ such that $C \in U \cup V, C \cap U \neq \emptyset, C \cap V \neq \emptyset, U \cap V = \emptyset$. Let $(a, b) \in U, (c, d) \in V$. Since the rationals are dense in $\mathbb{R}$, we can choose a rational point (q, r) that is close to (a, b), and an irrational point (p, s) close to (c, d). Then we have that $(q, r) \in U, (p, s) \in V$ and $(q, r), (p, s) \in C$. This contradicts the fact that U, V are disjoint. Thus we have that C is connected. $\square$

Claim. D is connected.

Proof. Let $(a, b), (c, d) \in D$. We will construct a path between these two points that lies entirely in D. Without loss of generality, assume that $|a| < |c|$ and $|b| < |d|$. Let us define a continuous function $f : [0, 1] \to \mathbb{R}^2$ by $f(t) = ((1 - t)a + tc, (1 - t)b + td)$ for $0 \leq t \leq 1$. Then we have that $f(0) = (a, b), f(1) = (c, d)$, and since we have that $|f(t)| > 1$ for all t, we know that $f(t)$ is in D for all t. Thus, we know that D is path-connected and similarly, is connected. $\square$

Problem 2. Show that $S^1, [0,1], [0,1), \mathbb{R}$, and $\mathbb{R}^2$ are not homeomorphic to each other.

Proof. $\left(S^1, [0,1]\right)$: If you remove a point from S^1 we will still have a connected space. However, if we remove a point from $[0,1]$ say $\{1/2\}$ the set is no longer connected. Thus, we know they are not homeomorphic.
$\left(S^1, [0,1)\right)$: If you remove a point from S^1 we will still have a connected space. However, if we remove a point from $[0,1)$ say $\{1/2\}$ the set is no longer connected. Thus, we know they are not homeomorphic.
$\left(S^1, \mathbb{R}\right)$: If you remove a point from S^1 we will still have a connected space. However, if we remove a point from $\mathbb{R}$ say $\{0\}$ the set is no longer connected. Thus, we know they are not homeomorphic.
$\left(S^1, \mathbb{R}^2\right)$: If we remove two distinct points from S^1, it will not necessarily be connected anoymore. However, if you remove two distinct points from $\mathbb{R}^2$ then it will still be connected. Thus, we know they are not homeomorphic.
$([0,1], [0,1))$: Suppose these two are homeomorphic, which implies that we can construct a homeomorphism $g : [0,1) \to [0,1]$. Then if we let $B = [0,1) - \{g^{-1}(0), g^{-1}(1), g_B : B \to (0,1)$ is a homeomorphism. However, $g^{-1}(0) \neq g^{-1}(1)$ so at most one of these can be 1 meaning one must lie in the interval $(0,1)$. Suppose without the loss of generality that $g^{-1}(0) \in (0,1)$. Then it follows that $B = (0, g^{-1}(0)) \cup (g^{-1}(0), 1] - \{g^{-1}(1)\}$ is not connected, whereas $(0,1)$ is connected, so the two cannot be homeomorphic. From this contradiction, then, we conclude that $(0,1]$ and $[0,1]$ are not homeomorphic.
$([0,1], \mathbb{R})$: Suppose we have a continuous bijection $f : \mathbb{R} \to [0,1]$. Let $a \in \mathbb{R}$ be such that $f(a) = 0$. Then if we consider $x = a - 1$ and $y = a + 1$. Since f is injective, it follows that $f(x), f(y) \neq 0$. Let $0 < c < \min\{f(x), f(y)\}$. By the Intermediate Value Theorem, it follows that $f(x') = c$ for some $x < x' < a$ and $f(y') = c$ for some $a < y' < y$; but then $x' \neq y'$ but $f(x') = f(y')$, so f is not injective which is a contradiction. Thus we have that they are not homeomorphic.
$\left([0,1], \mathbb{R}^2\right)$: To show this, it is simple to see that if we remove a point from the range $[0,1]$, for example $\{1/2\}$, this set then becomes unconnected. However, if we remove a point, let's say the origin, from $\mathbb{R}^2$, the set is still connected. Therefore, we find that these are not homeomorphic.
$([0,1), \mathbb{R})$: Let us specifically choose to remove the point $\{0\}$ from $[0,1)$. However, if we remove any point from $\mathbb{R}$ the result will always be disconnected. Thus, we know that they are not homeomorphic.
$\left([0,1), \mathbb{R}^2\right)$: To show this, it is simple to see that if we remove a point from the range $[0,1)$, for example $\{1/2\}$, this set then becomes unconnected. However, if we remove a point, let's say the origin, from $\mathbb{R}^2$, the set is still connected. Therefore, we find that these are not homeomorphic.
$\left(\mathbb{R}, \mathbb{R}^2\right)$: To show this, it is simple to see that if we remove a point from $\mathbb{R}$ take for example $\{0\}$, then the space $\mathbb{R}$ would no longer be connected. However, if we remove the origin from $\mathbb{R}^2$, then the space would still be connected because we could simply go around the hole. Therefore, these two are not homeomorphic.

$\square$

Problem 3. Classify the letters of the alphabet (in capital) by homeomorphic.

There are five homeomorphism classes for the letters of the alphabet $\alpha = \{A, R\}$, $\beta = \{C, I, K, L, M, N, S, U, V, W, Z\}$, $\c{=}\{D, O\}$, $\delta = \{E, F, G, T, Y\}$, $\epsilon = \{H, K\}$. These are decided by simple transformations from one letter to the next to form the homeomorphism.

Problem 4. Show a space X is compact if and only if every collection of closed sets $\{C_\alpha\}_{\alpha\in I}$ having the the finite intersection property has $\cap_{\alpha\in I} C_\alpha \neq \emptyset$.
Hint: Think about the complements of the C_α's

Proof. ($\Rightarrow$) Assume that X is compact. Let C be a collection of closed subsets of X having the finite intersection property. Let $U = \{c^c : c \in C$. Then we know that U is a collection of open sets. Suppose for the sake of contradiction that $\cup U = X$, and then since X is compact, we know that there exists some finite subcover U^* of U. Let us label the sets in $U^* = \{c_1^c, ..., c_n^c\}$ for $c_i \in C$ for all i. Since C has the finite intersection property, we have that $c_1 \cap ... \cap c_n \neq X$ which contradicts the fact that U^* is a cover for X. Then it must be that $\cup U \neq X$ and if we take the complements we get $\cup C \neq \emptyset$.
($\Leftarrow$) Now we will assume that C is a collection of closed subsets of X having the finite intersection property, that is to say that $\cap C \neq \emptyset$. Let U be an open cover of X and let $C = \{u^c : u \in U\}$, so C is a collection of closed subsets. Since U is an open cover, we have that $\cup U = X$ which implies that $\cap C = \emptyset$. By this assumption, we then have that $u_1^c \cap ... \cap u_n^c = \emptyset$ for some finite subset of C. If we then take the compliments, we get that $U_1 \cup ... \cup U_n = X$ for some finite subset of U. Thus, X is compact as desired.
Therefore, we have show that X is compact if and only if every collection of closed sets having the finite intersection property has $\cap C \neq \emptyset$. $\square$

Problem 5. Show that $\{\frac{1}{n}, n \in \mathbb{N}^*\} \cup \{0\}$ is compact.

Proof. Let $S = \{\frac{1}{n} : n \in \mathbb{N}\} \cup \{0\}$ as in the problem statement. We are going to first use that for any $\epsilon > 0$ we have that $\exists n \in \mathbb{N}, m\epsilon \Rightarrow -\epsilon < \frac{1}{m} < \epsilon$. Then, we get that $\frac{1}{m} \in (-\epsilon, \epsilon)$. Thus we have that $(-\epsilon, \epsilon)$ is a neighborhood of 0 that contains a point of S other than 0. This then implies that 0 is the only accumulation point of S. Let us then define $S' = \{x_n : n \in \mathbb{N}, x_n \in S(0, \frac{1}{n}, x_{n+1} \neq x_n\}$. In a similar fashion, we know that 0 is the only accumulation point of S' and for S. Additionally, we know that S is bounded by $[0, 1]$. Therefore, we know that S is compact as desired. $\square$

Problem 6. Show that $\{\frac{1}{n}, n \in \mathbb{N}^*\} \cup \{0\}$ is not homeomorphic to $\mathbb{N}$

Proof. If these two sets were homeomorphic to each other, that would imply that the Hausdorff property would be preserved between the two topologies. For the topology $\{\frac{1}{n}, n \in \mathbb{N}\} \cup \{0\}$ it is easy to show that it is Hausdorff with the limit point of 0. However, the set $\mathbb{N}$ is not Hausdorff. Thus, since the Hausdorff property is preserved under homeomorphism, then we know that these two topologies are not homeomorphic. $\square$