Homework 9 & 10

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[2]

(a) Let $f(z) = e^z$, then we know that $f^{(n)}(z) = e^z$ for all $n \geq 0$, so then we know that $f^{(n)}(1) = e$ for all $n \geq 0$. Thus, $f(z) = e^z = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n^1}(z-1)^n = e\sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}$ for $|z-1| < \infty$.

(b) Let us first replace z with $z-1$ in the Maclaurin series for e^z, then we have that $e^{z-1} = \sum_{n=1}^{\infty} \frac{(z-1)^n}{n!}$. Thus, we have that $e^z = e\sum_{n=0}^{\infty} \frac{(z-1)^n}{n!}$.

[10]

(a) We will first recall that $\sinh(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$ when $|z| < \infty$. If $z \neq 0$ then we have that $\frac{\sinh(z)}{z^2} = \frac{1}{z^2}\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = \frac{1}{z^2}(z + \frac{z^3}{3!} + \frac{z^5}{5!} + ...) = \frac{1}{z} + \frac{z^2}{z*3!} + \frac{z^4}{z*5!} + ... = \frac{1}{z}(1 + \frac{z^1}{3!} + \frac{z^3}{5!} + ...) = \frac{1}{z}\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+3)!}$ as desired.

(b) We will first recall that $\sin(z) = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$ when $|z| < \infty$. If $z \neq 0$, then we have that $\frac{\sin(z^2)}{z^4} = \frac{1}{z^4}\sum_{n=0}^{\infty} \frac{(-1)^n z^{4n+2}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n z^{4n-2}}{(2n+1)!} = \frac{1}{z^2} - \frac{z^2}{3!} + ...$ as desired.

[4]

For $0 < |z| < 1$ we have that $\frac{1}{z^2(1-z)} = \frac{1}{z^2}\sum_{n=0}^{\infty} z^n = \sum_{n=0}^{\infty} z^{n-2} = \frac{1}{z^2} + \frac{1}{z} + \sum_{n=2}^{\infty} z^{n-2} = \sum_{n=0}^{\infty} z^n + \frac{1}{z} + \frac{1}{z^2}$. Then we also have for $1 < |z| < \infty$, $\frac{1}{z^2(1-z)} = -\frac{1}{z^3}\frac{1}{1-\frac{1}{z}} = -\frac{1}{z^3}\sum_{n=0}^{\infty} \frac{1}{z^n} = -\sum_{n=0}^{\infty} \frac{1}{z^{n+3}} = -\sum_{n=3}^{\infty} \frac{1}{z^n}$ as desired.

[6]

First we have that $\frac{z}{(z-1)(z-3)} = \frac{1}{z-3}\frac{z}{z-1} = \frac{1}{z-3}\frac{z-1+1}{z-1} = \frac{1}{z-3}(1 + \frac{1}{z-1})$. Then for $|z-1| < 2$, $\frac{1}{z-3} = -\frac{1}{2-(z-1)} = -\frac{1}{2}11 - \frac{z-1}{2} = -\frac{1}{2}\sum_{n=0}^{\infty} \frac{(z-1)^n}{2^n} = -\sum_{n=0}^{\infty} \frac{(z-1)^n}{2^{n+1}}$. Hence, $\frac{z}{(z-1)(z-3)} = -\sum_{n=0}^{\infty} \frac{(z-1)^n}{2^{n+1}} - \sum_{n=0}^{\infty} \frac{(z-1)^{n-1}}{2^{n+1}} = -\frac{1}{2(z-1)} - \sum_{n=0}^{\infty} \frac{(z-1)^n}{2^{n+1}} - \sum_{n=1}^{\infty} \frac{(z-1)^{n-1}}{2^{n+1}} = -\frac{1}{2(z-1)} - \sum_{n=0}^{\infty} \frac{(z-1)^n}{2^{n+1}} - \sum_{n=0}^{\infty} \frac{(z-1)^n}{2^{n+2}} = -\frac{1}{2(z-1)} - 3\sum_{n=0}^{\infty} \frac{(z-1)^n}{2^{n+2}}$ for $0 < |z-1| < 2$.

[2]

First we have that $\frac{1}{(1-z)^2} = \sum_{n=0}^{\infty}(n+1)z^n$ for $|z| < 1$. If we replace z with $\frac{1}{1-z}$ we get, $\frac{1}{(1-\frac{1}{1-z})^2} = \sum_{n=0}^{\infty}(n+1)(\frac{1}{1-z})^n$. This is the same as $\frac{(1-z)^2}{z^2} = \sum_{n=0}^{\infty} \frac{(n=1)}{(1-z)^n}$ for $|\frac{1}{z-1}| < 1$ or that is for $|z-1| > 1$. Thus, we have that $\frac{1}{z^2} = \sum_{n=0}^{\infty} \frac{(n+1)}{(1-z)^{n+2}} = \sum_{n=2}^{\infty} \frac{(n-1)}{(1-z)^n} = \sum_{n=2}^{\infty} \frac{(-1)^n(n-1)}{(z-1)^n}$ for $1 < |z-1| < \infty$ as desired.

[1]

We have $e^z = 1 + z + \frac{z^2}{2!} + ...$ and we also have that $\frac{1}{(z^2+1)} = 1 - z^2 + z^4 - z^6 + ...$ Thus we have that $\frac{e^z}{z(z^2+1)} = \frac{1}{z}(e^z)(\frac{1}{z^2+1}) = \frac{1}{z}(1 + z + \frac{z^2}{2!} + ...)(1 - z^2 + z^4 - z^6 + ...) = \frac{1}{z} + 1 + \frac{z}{2} - z... = \frac{1}{z} + 1 - \frac{z}{2} - \frac{5}{6}z^2 + ...$ as desired.

[3]

We have that $\int_C \frac{4z-5}{z(z-1)}dz = 2\pi i Res_{z=0}(\frac{1}{z^2}f(\frac{1}{z})) = 2\pi i Res(\frac{-5z+4}{z(1-z)})$. Then we have that $\frac{-5z+4}{z(1-z)} = \frac{1}{z}\frac{-5z+4}{(-z+1)}$ for which the residue is 4. Therefore, we have $\int_C \frac{4z-5}{z(z-1)}dz = 2\pi i * 4 = 8\pi i$.

[6]

Let the assumptions be as in the problem statement. That is that we have f which is analytic over a closed contour C except for a finite number of singular points within the contour. By Cauchy's Residue Theorem $\frac{1}{2\pi i}\int_C f(z)dz = \sum_{z_1}^{z_n} R = Res_{z=z_1}f(z) + Res_{z=z_2}f(z) + ...Res_{z=z_n}f(z)$ and by the residue theorem at infinity, we have that since $f(z)$ is analytic on $|z| \geq R$. $-Res_{z=\infty}f(z) = \frac{1}{2\pi i}\int_C f(z)dz = -\frac{1}{2\pi i}\int_C f(z)dz$. Therefore we have that, $-Res_{z=\infty}f(z) = Res_{z=z_1}f(z)+Res_{z=z_2}f(z)+...+Res_{z=z_n}f(z)$ which is the same as $Res_{z=z_1} + Res_{z=z_2} + ... + Res_{z=z_n} + Res_{z=\infty} = 0$ as desired.