Homework 6

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1. (a) Consider we have two unitary matrices U and V, that is two say that $U^{-1} = \overline{U}^T$ and $V^{-1} = \overline{V}^T$. The product of these two is UV. Therefore,

$$\begin{aligned}(UV)^{-1} &= V^{-1}U^{-1}\\ &= \overline{V}^T\overline{U}^T\\ &= (\overline{U}\,\overline{V})^T\\ &= (\overline{UV})^T.\end{aligned}$$

Thus, the product of unitary matrices is also unitary.

(b) Suppose that U is a unitary matrix, that is to say that $U^{-1} = \overline{U}^T$. Then,

$$\begin{aligned}U * U^{-1} &= I\\ U * \overline{U}^T &= I\\ U^{-1} * (\overline{U}^T)^{-1} &= I\\ U * U^{-1} * (\overline{U}^T)^{-1} * U^{-1} &= I\\ (\overline{U}^T)^{-1} * U^{-1} &= I\\ \overline{U^{-1}}^T * U^{-1} &= I.\end{aligned}$$

This implies that $(U^{-1})^{-1} = \overline{U^{-1}}^T$ which means that U^{-1} is also unitary.

(c) Suppose that U is a unitary matrix, that is to say that $U^{-1} = \overline{U}^T$. Then,

$$\begin{aligned}\overline{U}^{-1} &= \overline{U^{-1}}\\ &= \overline{\overline{U}^T}\\ &= \overline{\overline{U}}^T.\end{aligned}$$

Therefore, the complex conjugate of a unitary matrix is also unitary.

(d) Suppose that U is a unitary matrix, that is to say that $U^{-1} = \overline{U}^T$. Then,

$$\begin{aligned}(U^T)^{-1} &= (U^{-1})^T \\ &= (\overline{U}^T)^T \\ &= (\overline{U^T})^T.\end{aligned}$$

Therefore, the transpose of a unitary matrix is also unitary.

2. ($\Rightarrow$) Let $R : \mathbb{C}^n \to \mathbb{R}^{2n}$ be the $\mathbb{R}$-linear isomorphism defined by $\begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix} \to \begin{bmatrix} x_1 \\ y_1 \\ \vdots \\ x_n \\ y_n \end{bmatrix}$, where $z_k = x_k + iy_k$. Let $z = \begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}, w = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} \in \mathbb{C}, z_k = x_k + iy_k, w_k = x_k^{'} + iy_k^{'}$. Suppose that z and w are orthogonal in $\mathbb{C}^n$. That is to say that $z \cdot w = 0 \Rightarrow \sum_{k=1}^n \overline{z_k} w_k = 0$. Next, $R(z) = \begin{bmatrix} x_1 \\ y_1 \\ \vdots \\ x_n \\ y_n \end{bmatrix}, R(w) = \begin{bmatrix} x_1^{'} \\ y_1^{'} \\ \vdots \\ x_n^{'} \\ y_n^{'} \end{bmatrix}$.

Then, $R(iw) = \begin{bmatrix} -y_1^{'} \\ x_1^{'} \\ \vdots \\ -y_n^{'} \\ x_n^{'} \end{bmatrix}$. Therefore, $R(z) \cdot R(iw) = \sum_{k=1}^n (-x_k y_k^{'} + y_k x_k^{'})$ and $R(z) \cdot R(w) = x_1 x_1^{'} + y_1 y_1^{'} + ... + x_n x_n^{'} + y_n y_n^{'} = \sum_{k=1}^n (x_k x_k^{'} + y_k y_k^{'})$. Since we know,

$$\begin{aligned}\sum_{k=1}^n \overline{z_k} w_k &= 0 \\ \sum_{k=1}^n (x_k - iy_k)(x_k^{'} + iy_k^{'}) &= 0 \\ \sum_{k=1}^n (x_k x_k^{'} + y_k y_k^{'}) + i(x_k y_k^{'} - y_k x_k^{'}) &= 0 \\ \sum_{k=1}^n (x_k x_k^{'} + y_k y_k^{'}) + i \sum_{k=1}^n (x_k y_k^{'} - y_k x_k^{'}) &= 0.\end{aligned}$$

This implies that both $\sum_{k=1}^{n}(x_k x_k^{'}+y_k y_k^{'})=0$ and $\sum_{k=1}^{n}(x_k y_k^{'}-y_k x_k^{'})=0$. This therefore means that $R(z)\cdot R(w)=0$ and $-\sum_{k=1}^{n}(-x_k y_k^{'}+y_k x_k^{'}=$, which means that $R(z)\cdot R(iw)=0$. Thus, $R(z)$ is orthogonal to both $R(w)$ and $R(iw)$ in $\mathbb{R}^{2n}$

$(\Leftarrow)$ Let $R(z)$ be orthogonal to both $R(w)$ and $R(iw)$ in $\mathbb{R}^{2n}$ That is to say that $\sum_{k=1}^{n}(x_k x_k^{'}+y_k y_k^{'})=0$ and $\sum_{k=1}^{n}(-x_k y_k^{'}+y_k x_k^{'})=0$ Therefore, $\sum_{k=1}^{n}(x_k x_k^{'}+y_k y_k^{'})-i\sum_{k=1}^{n}(-x_k y_k^{'}+y_k x_k^{'})=\sum_{k=1}^{n}(x_k x_k^{'}+y_k y_k^{'})-i(-x_k y_k^{'}+y_k x_k^{'})=\sum_{k=1}^{n}(x_k-iy_k)(x_k^{'}+iy_k^{'})=0$, Thus, $\sum_{k=1}^{n}\overline{z_k}w_k=0$, which means that $z\cdot w=0$ which means that z and w are orthogonal in $\mathbb{C}^n$.

Let $z=\begin{bmatrix} z_1 \\ \vdots \\ z_n \end{bmatrix}\in\mathbb{C}^n$. Where $z_k=x_k+iy_k$. Then $R(z)=\begin{bmatrix} x_1 \\ y_1 \\ \vdots \\ x_n \\ y_n \end{bmatrix}$, $R(iz)=\begin{bmatrix} -y_1 \\ x_1 \\ \vdots \\ -y_n \\ x_n \end{bmatrix}$. Therefore, $R(z)\cdot R(iz)=-x_1y_1+y_1x_1-x_2y_2+y_2x_2-...-x_ny_n+y_nx_n=0$. Thus, $R(z)$ is orthogonal to $R(iz)$ for any $z\in\mathbb{C}^n$

3. $(\Rightarrow)$ Let $<su,v>=<u,sv>$. Let $A=(a_{ij})_{n\times n}$. Then, $\overline{A}=A^T$ if and only if $\overline{a_{ij}}=a_{ji}$. Let, e_i be an orthonormal basis for $\mathbb{R}^n$. Then we know that $a_{ij}=<Ae_j,e_i>$. Therefore,

$$\begin{aligned}
\overline{a_ij} &= \overline{<Ae_j,e_i>} \\
&=<e_i,Ae_j> \\
&=<e_i,TST^{-1}e_j> \\
&=<T^{-1}e_i,ST^{-1}e_j> \\
&=<u_i,su_j> \text{ for } u_i=T^{-1}e_i, u_j=T^{-1}e_j \\
&=<su_i,u_j> \\
&=<ST^{-1}e_i,T^{-1}e_j> \\
&=<TST^{-1}e_i,e_j> \\
&=<Ae_i,e_j> \\
&=a_{ji} \\
\overline{a_{ij}} &= a_{ji} \\
\overline{A} &= A^T
\end{aligned}$$

($\Leftarrow$) Alternatively, let $\overline{A} = A^T$. Therefore,

$$\begin{aligned}
\overline{a_{ij}} &= a_{ji} \\
\overline{< Ae_j, e_i >} &=< Ae_i, e_j > \\
< e_i, Ae_j > &=< Ae_i, e_j > \\
< e_i, TST^{-1}e_j > &=< TST^{-1}e_i, e_j > \\
< T^{-1}e_i, TST^{-1}e_j > &=< ST^{-1}e_i, T^{-1} > \\
< u_i, su_j > &=< su_i, u_j > \text{ where } u_i = T^{-1}e_i, u_j, T^{-1}e_j.
\end{aligned}$$

Now, let $u, v \in V$, since T is an isomorphism, $\exists x, y \in \mathbb{C}^n$ such that $T(u) = x$ and $T(v) = y$. Therefore, $u = T^{-1}(x)$ and $v = T^{-1}(y)$. Since $\{e_i\}_{i=1:n}$ is an orthonormal basis for $\mathbb{R}^n$ so $< T^{-1}x, ST^{-1}y >=< ST^{-1}x, T^{-1}y > \Rightarrow < u, sv >=< su, v > \forall u, v \in V$

4. Given the initial equation, we can solve for these two sequences by,

$$\begin{aligned}
y_{k+2} &= \frac{5}{2}y_{k+1} - y_k \\
y_{k+2} - \frac{5}{2}y_{k+1} + y_k &= 0 \\
s^2 - \frac{5}{2}s + 1 &= 0 \\
(s-2)(s-\frac{1}{2}) &= 0 \\
s &= 2, 2^{-1}
\end{aligned}$$

Therefore, we know that the two sequences that solve the series of values are 2^k and 2^{-k} to provide the general equation $y_k = C_1 2^k + C_2 2^{-k}$. In order to solve for the specific solution given y_1 and y_2, we would set $k = 1, 2$ equal to y_1, y_2 respectively, and solve the systems of equation for the values of C_1 and C_2. The effect of noise as the initial condition of the recursive equation as n increases, the little bit of noise causes the sequence to grow once it had decayed to what was rounded down to 0.0000.