FINAL EXAM SOLUTIONS

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Exercise 1.

(1) We are going to note that a subset of a metric space is compact if and only if it is complete and totally bounded. Since $\psi(\mathbb{S}^1)$ is a closed and bounded subset of $\mathbb{R}^2$, it is complete and totally bounded. Therefore, $\psi(\mathbb{S}^1)$ is compact as desired.
(2) A compact subset of a metric space is closed by definition. Thus, $\psi(\mathbb{S}^1)$ is closed as desired.
(3) (a) Since ϕ is a homeomorphism, it is continuous and has a continuous inverse. Thus, $\phi(I)$ is an open subset of $\mathbb{R}$, and since $y \in I \Rightarrow y \in \phi(I)$ as desired.
(b) Since I is an open subset of $\mathbb{S}^1$ and y is arbitrary, it follows that $\mathbb{S}^1$ is an open subset of $\mathbb{R}$ as desired.
(4) Since $\mathbb{S}^1$ is non-empty and open in $\mathbb{R}$, it must have a non-empty interior. Therefore, there exists points in $\mathbb{S}^1$ arbitrarily close to another point in $\mathbb{S}^1$. That is to say that $\mathbb{S}^1$ is dense in $\mathbb{R}$.
(5) The contradiction follows from the fact that a connected, dense subset of $\mathbb{R}$ must be uncountable. However, since $\mathbb{S}^1$ is a 1-dimensional manifold and thus has a countable basis of open sets. Therefore, $\mathbb{S}^1$ cannot be dense in $\mathbb{R}$. Thus, the assumption that $\mathbb{S}^1$ can be embedded in $\mathbb{R}$ leads to a contradiction, and so such an embedding does not exist.

Exercise 2.

(1) Let us define a homotopy between D and $\{0\}$ as $H(D,t) = (1-t)D + t\{0\}$ for $t \in [0,1]$ where $H(D,0) = D, H(D,1) = \{0\}$. It is simple to see that H is well-defined and is an equivalence relation. Therefore we find that $D \simeq \{0\}$ and are homotopically equivalent as desired.
(2) The fundamental group of D based at any point $\pi_1(D,a)$ is just the identity element, i.e. the trivial group. It is easy to see that $D \simeq \{a\}$ for arbitrary constant a. Any two points can shrink to each other to the identity element in any simply connected space that is loop based to a point. For D, every loop can be shrunk to a single loop. Therefore, we find that $\pi_1(D,a) = \{e\}$ where e is the identity element, representing the trivial group.
(3) (a) First, we are going to note that $f(z) = \frac{z}{|z|} = \{(r\cos(2t\pi), r\sin(2t\pi) : r \in (0,1], t \in [0,1]\}$.To show that there is a homotopy between X and $\mathbb{S}^1$ we are going to define a homotopy $H : X \times I \to \mathbb{S}^1$ as $H(X,T) = (1-T)f_0 + Tf_1(z)$ where $H(X,0) = f_0 = (\cos(0), \sin(0) = (1,0)$ and $H(X,1) = f_1 = (\cos(2\pi), \sin(2\pi)) = (0,1)$. This implies then that H sends all the points of X to $\mathbb{S}^1$ and H is an equivalence relation. Therefore, we know that $f : X \to \mathbb{S}^1$ is an equivalence of homotopy.
(b) Since we know that $X \simeq \mathbb{S}^1$ we also know that $\pi_1(\mathbb{S}^1, x_0) = \mathbb{Z}$. Therefore, we know that if $x_0 = 1$, then $\pi_1(X,1) = \mathbb{Z}$.
(c) X is like a disc with a hole in the center (Figure 1). A non-trivial element of $\pi_1(X,1) = \{(\cos(2\pi ns), \sin(2\pi ns) : s \in \mathbb{R}, n \in \mathbb{Z}\}$ is $(\cos(2\pi n), \sin(2\pi n))$ which would just be the outer boundary.

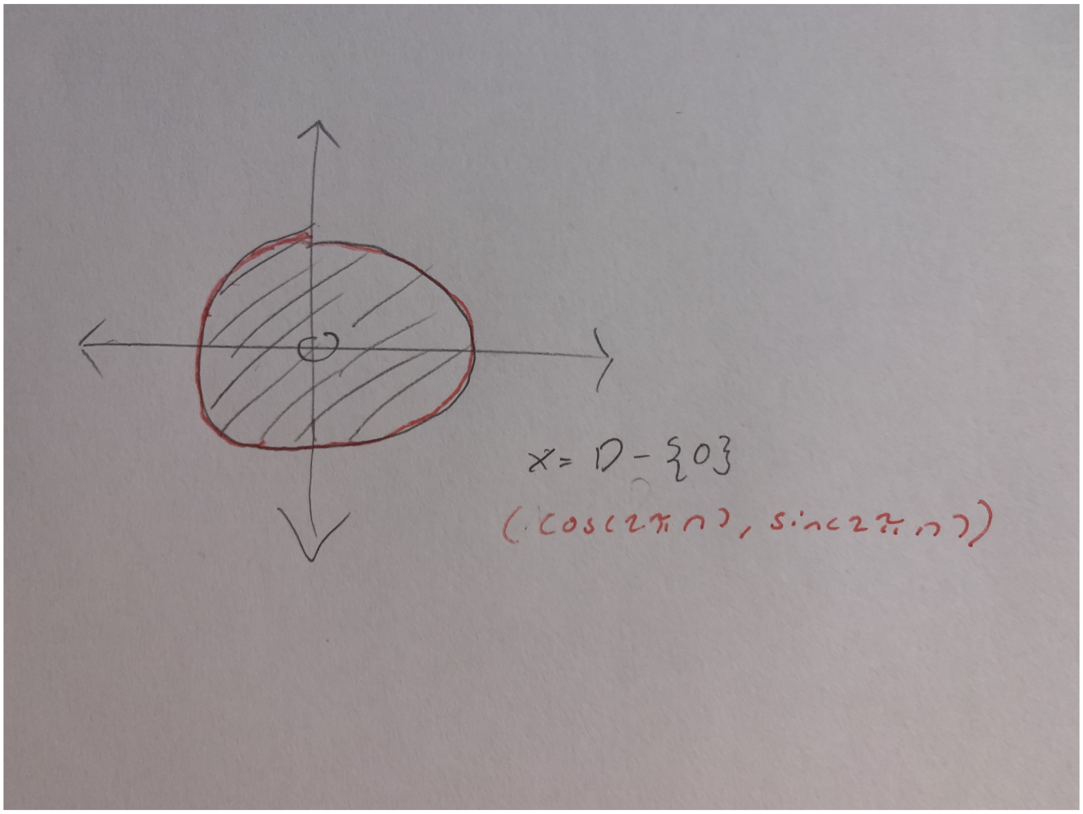

FIGURE 1. Drawing of X with a non-trivial element of $\pi_1(X, 1)$