Homework 1

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Math 4320

[2]

(a) Let $z \in \mathbb{C}$ such that $z = x + iy$ for some $x, y \in \mathbb{R}$. If we perform $iz = ix + i^2y = -y + ix$ and then $Re(iz) = -y$. Similarly, $Im(z) = y$ and therefore $-Im(z) = -y$. Thus, $Re(iz) = -Im(z) = -y$ as desired.

(b) Similarly to part a, $iz = -y = ix$ which implies that $Im(iz) = x$ and $Re(z) = x$. Thus $Im(iz) = Re(z) = x$ as desired.

[10]

It will be useful to first recall the multiplication identity for reference purposes. $z_1\dot{z}_2 = (x_1, y_1)(x_2, y_2) = (x_1x_2 - y_1y_2, y_1x_2 + x_1y_2) = (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2)$. We will now calculate $-(iy) = -((0, 1)(y, 0)) = -(0 - 0, 1y + 0) = (0, -y)$. Similarly, $(-i)y = (0, -1)(y, 0) = (0 - 0, -y + 0) = (0, -y)$. This would indicate what we desire that the additive inverse of a complex number $z = x + iy$ for $x, y \in \mathbb{R}$ can be written as $-z = -x - iy$. We will verify this.

Proof. Let $z \in \mathbb{C}$, that is to say that $z = x_1 + iy_1$ for some $x_1, y_1 \in \mathbb{R}$ and we will denote a proposed additive inverse of a number as $(-z) = x_2 + iy_2$ for $x_2, y_2 \in \mathbb{R}$. Then, by the definition of the additive inverse of a complex number, that $z + (-z) = (x_1 + iy_1) + (x_2 + iy_2) = 0 + 0i = 0$. We then have that $x_1 + x_2 = 0$ and $iy_1 + iy_2 = (y_1 + y_2)i = 0$ which we can extend to say that $y_1 + y_2 = 0$ since $i \neq 0$. By the additive inverse rule of real numbers we then know that $x_1 + x_2 = 0$ and $y_1 + y_2 = 0$ implies that x_2, y_2 are the additive inverses of x_1, y_1 respectively. Therefore, we can denote them as $-x_1, -y_1$ which means that we can write $(-z) = -x_1 + i(-y_1) = -x_1 - iy_1$. Therefore we can then explicitly write that the additive inverse of a complex number $z = x + iy$ as $-z = -x - iy$. □

[6]

Proof. We can then write,

$$
\begin{aligned}
(\frac{z_1}{z_3})(\frac{z_2}{z_4}) &= z_1z_3^{-1}z_2z_4^{-1} \\
&= z_1z_2z_3^{-1}z_4^{-1} \\
&= (z_1z_2)(z_3^{-1}z_4^{-1}) \\
&= \frac{z_1z_2}{z_3z_4}.
\end{aligned}
$$

Therefore, we know that $(\frac{z_1}{z_3})(\frac{z_2}{z_4}) = \frac{z_1z_2}{z_3z_4}$ as desired. □

[3]

Proof. Let $z_1, z_2, z_3, z_4 \in \mathbb{C}$. We know that $|z_3 + z_4| \geq ||z_3| - |z_4||$ which would imply that $\frac{Re(z_1+z_2)}{|z_3+z_4|} \leq \frac{Re(z_1+z_2)}{||z_3|-|z_4||}$. Then we also know that $Re(z_1 + z_2) \leq |z_1 + z_2| \leq |z_1| + |z_2|$. Then we can extend our statement $\frac{Re(z_1+z_2)}{|z_3+z_4|} \leq \frac{Re(z_1+z_2)}{||z_3|-|z_4||} \leq \frac{|z_1+z_2|}{||z_3|-|z_4||} \leq \frac{|z_1|+|z_2|}{||z_3|-|z_4||}$. Thus we know that $\frac{Re(z_1+z_2)}{|z_3+z_4|} \leq \frac{|z_1|+|z_2|}{||z_3|-|z_4||}$ as desired. □

[4]

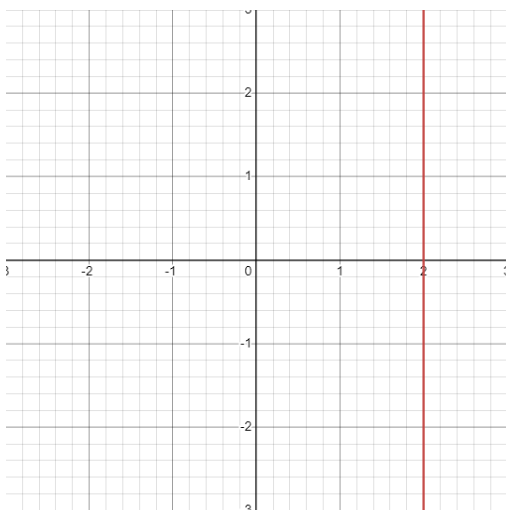
Proof. Let $z \in \mathbb{C}$, then if we consider the original problem $\sqrt{2}|z| \geq |Rez| + |Imz|$ we could then say that $(\sqrt{2}|z|)^2 \geq (|Rez| + |Imz|)^2$. Then we can say $(\sqrt{2}|z|)^2 = 2|z|^2 = 2(\sqrt{x^2+y^2})^2 = 2x^2 + 2y^2$. Similarly, if we consider $(|Rez| + |Imz|)^2 = (|x| + |y|)^2 = x^2 + y^2 + 2|xy|$. Thus we can say that $(\sqrt{2}|z|)^2 - (|Rez| + |Imz|)^2 = 2x^2 + 2y^2 - x^2 - y^2 - 2|xy| = x^2 + y^2 - 2|xy| = (|x| - |y|)^2 \geq 0$. Therefore, this then implies that $\sqrt{2}|z| \geq |Rez| + |Imz|$ as desired. □

[8]

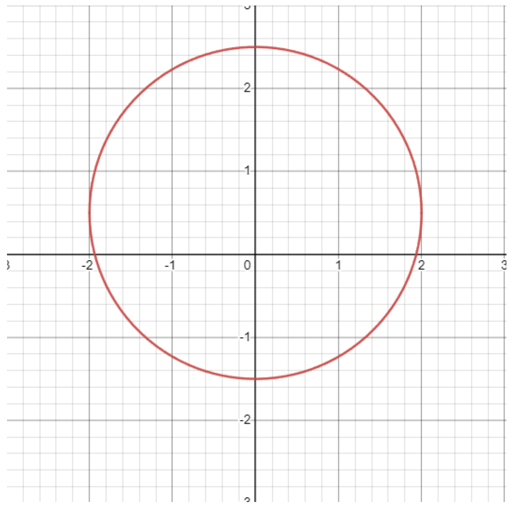
Proof. Let $z_1, z_2 \in \mathbb{C}$ such that $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then we have that $z_1 z_2 = (x_1x_2 - y_1y_2) + i(y_1x_2 + y_2x_1)$. Then it would follow that $|z_1z_2| = \sqrt{(x_1x_2 - y_1y_2)^2 + (y_1x_2 + y_2x_1)^2} = \sqrt{(x_1x_2)^2 - 2(x_1x_2y_1y_2) + (y_1y_2)^2 + (y_1x_2)^2 + (x_1y_2)^2 + 2(x_1x_2y_1y_2)} = \sqrt{(x_1x_2)^2 + (y_1y_2)^2 + (y_1x_2)^2 + (x_1y_2)^2} = \sqrt{x_1^2x_2^2 + y_1^2y_2^2 + y_1^2x_2^2 + x_1^2y_2^2} = \sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)}$ as desired. It would then follow that $\sqrt{(x_1^2 + y_1^2)(x_2^2 + y_2^2)} = \sqrt{(x_1^2 + y_1^2)}\sqrt{(x_2^2 + y_2^2)} = |z_1||z_2|$ proving the identity. □

[2]

(a) $Re(\bar{z} - i) = Re(\bar{z}) = Re(z) = 2$. Therefore if $z = x + iy$, then we know that $x = 2$. Therefore, we know that this is a vertical line at $x = 2$.



(b) We will first consider $|2\bar{z} + i| = 4$ and divide the whole equation by 2 to get $|\bar{z} - (-\frac{i}{2})| = 2$. Literally, this means that the distance from $\bar{z}$ and $(-\frac{i}{2})$ is 2. If we then take the conjugate of this, we then know that the distance between z and $\frac{i}{2}$ is 2 which would imply that this is a circle of radius 2 centered at $\frac{i}{2}$



[6]

(a) *Proof.* We know that $\overline{\frac{z_1}{z_2}} = \frac{\bar{z_1}}{\bar{z_2}}$. Then we know that $\overline{\frac{z_1}{z_2 z_3}} = \frac{\bar{z}}{\overline{z_2 z_3}}$. And since we also know that $\overline{z_1 z_2} = \bar{z_1}\bar{z_2}$ then we can say that $\overline{\frac{z_1}{z_2 z_3}} = \frac{\bar{z_1}}{\bar{z_2}\bar{z_3}}$ as desired. $\square$

(b) *Proof.* In a similar fashion to part a of this problem we have that $|\frac{z_1}{z_2}| = \frac{|z_1|}{|z_2|}$ and we have that $|z_1 z_2| = |z_1||z_2|$. Therefore, we have that $|\frac{z_1}{z_2 z_3}| = \frac{|z_1|}{|z_2 z_3|} = \frac{|z_1|}{|z_2||z_3|}$ as desired. $\square$

[11]

(a) *Proof.* Base Case: Let $n = 2$, then we have that $\overline{x_1 + x_2} = \bar{x_1} + \bar{x_2}$ as we know from the identity and then we know that the statement is true for $n = 2$.
Inductive Step: Let the statement be true for $n = k$, that is to say that $\overline{x_1 + ... + x_k} = \bar{x_1} + ... + \bar{x_k}$. We now need to prove the statement for $n = k + 1$. We can then show $\overline{z_1 + ... + z_k + z_{k+1}}$ we will define $z = z_1 + ... + z_k$ and then show $\overline{z + z_{k+1}} = \bar{z} + \bar{z_{k+1}}$ and since we know that the statement is true for $n = k$ then we know that $\bar{z} = \overline{z_1 + ... + z_k} = \bar{z_1} + ... + \bar{z_k}$ which then means that $\overline{z_1 + ... + z_k + z_{k+1}} = \bar{z_1} + ... + \bar{z_k} + \bar{z_{k+1}}$. Thus, given the statement is true for $n = k$ and we were then able to show that the statement was true for $n = k + 1$ then we have proved that the statement is true for all $n = 2, 3, ...$ as desired. $\square$

(b) *Proof.* Base Case: Let $n = 2$, then we have that $\overline{z_1 z_2} = \bar{z_1}\bar{z_2}$ from the section and the statement is verified for $n = 2$.
Inductive Step: Let the statement be true for $n = k$, that is to say that $\overline{x_1 ... x_k} = \bar{x_1} ... \bar{x_k}$. We then want to show that the statement is true for $n = k + 1$. Consider if we set that $z = z_1 ... z_k$, we can then say that $\overline{z z_k} = \bar{z}\bar{z_k}$, and by the inductive hypothesis since we know that the statement is true for $n = k$, then we know that $\bar{z} = \overline{z_1 ... z_k} = \bar{z_1} ... \bar{z_k}$ which allows us to say that $\overline{z_1 ... z_k z_{k+1}} = \bar{z_1} ... \bar{z_k}\bar{z_{k+1}}$ as desired. Therefore, since we knew that the statement was true for $n = k$, we were able to show that the statement was true for $n = k + 1$ and we know that the statement is further true for $n = 2, 3, ...$ as desired. $\square$

[1]

(a) $z^{-1} = \frac{1+\sqrt{3}i}{-2} = \frac{-1}{2} + \frac{-\sqrt{3}}{2}i$ and that $z^{-1} = \frac{1}{r}e^{-i\theta}$. If we are only consider about finding $Arg(z)$ then we only need to find θ. Therefore, we have that $\theta = \tan^{-1}(\frac{\frac{-\sqrt{3}}{2}}{\frac{-1}{2}}) = \tan^{-1}(\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}) = \tan^{-1}(\sqrt{3})$ which we would then know the say that $\theta = -2\pi/3$ and since we know that the θ is negative, then we know that $\theta = 2\pi/3$

(b) We know that $z^6 = r^6 e^{i6\pi}$ so then we know that $Arg(\sqrt{3} - i) = -\pi/6$ so then $6 * -\pi/6 = -\pi$. However, the principal argument $Arg(z)$ is defined $-\pi < \theta \leq \pi$ so then $Arg(z) = \pi$.

[6]

Proof. We already know from the section that $arg(z_1 z_2) = arg(z_1) + arg(z_2)$; therefore, we need to show that when we take this sum for the z_1, z_2 in the problem statement that the resulting angle lies in the interval $(-\pi, \pi)$. Given that we know that z_1 and z_2 have positive real parts, then we know that their respective principal arguments are in the interval $(-\pi/2, \pi/2)$. Therefore, the sum of their principal arguments is in the interval $(-\pi, \pi)$ meaning that $Arg(z_1 z_2) = Arg(z_1) + Arg(z_2)$ when z_1, z_2 have positive real components. $\square$

[2]

We are going to note $z^3 = r^3 e^{i3\theta}$. So if we set $z^3 = -8i = 8e^{i(-\pi/2)}$. Then we know that we can say that $z = (8)^{1/3} e^{i(-\pi/6)} = 2e^{i(-\pi/6)}$. So if we then convert the complex number to its standard form from the polar form, then we find that the cubed roots of $-8i$ are $\sqrt{3} - i, -\sqrt{3} - i, 2i$.

[5]

Proof. Let $z_0 = -4\sqrt{2} + 4\sqrt{2}i$, if we convert this to the polar form, we get that $z_0 = 8e^{i(-\pi/4)}$ or that $z_0 = 8e^{i(-\pi/4+2n\pi)}$ for $n \in \mathbb{Z}$. Then we know that the cube roots are given by $2e^{i(-\pi/12+2\pi/3)}$. Therefore, there are three solutions in a revolution, those being $2e^{i(-\pi/12)}, 2e^{i(7\pi/12)}, 2e^{i(15\pi/12)}$ which converted back into standard form are the other two solutions as desired. $\square$