MATH 4317 Homework 3 Due Friday, 2/4/2022 at 11PM to Gradescope

Homework guidelines:

- Each problem I assign, unless otherwise stated, is asking you to prove something. Give a full mathematical proof using only results from class or Wade.
- Submit a PDF or JPG to gradescope. The grader has ~ 250 proofs to grade: please make his job easier by submitting each problem on a different page.
- If you submit your homework in Latex, you get 2% extra credit.

Problems (5 total, 10 pts each)

Problem 1. Prove, directly from the definition of convergence of a sequence, that $\frac{1}{n^2} \to 0$ as $n \to \infty$. You may use the fact that if $E \subset \mathbb{Z}$ is bounded from above, then $\sup E \in E$. **You may not use the results of Wade Section 2.2, nor may you use the fact that $\frac{1}{n} \to 0$ as $n \to \infty$.**

Hint: Try to emulate the proof that $\frac{1}{n} \to 0$ as $n \to \infty$.

Proof. Let $\epsilon > 0$ and by using the Archimedean Principle we will choose $N \in \mathbb{N}$ such that $N > \sqrt{\frac{1}{\epsilon}}$. Therefore, we have that if $n > N > \sqrt{\frac{1}{\epsilon}}$ implies that $n^2 > \frac{1}{\epsilon}$ implies that $\frac{1}{n^2} < \epsilon$.. Since $\frac{1}{n^2}$ is positive for all n it follows that $|\frac{1}{n^2}| < \epsilon$ for all $n \geq N$. $\square$

Problem 2. Let $a, b \in \mathbb{R}, a < b$. Prove, directly from the definition of convergence of a sequence, that if (x_n) is such that $a \leq x_n \leq b$ for all n and $x_n \to x$ for some $x \in \mathbb{R}$, then $x \in [a, b]$. **You may not use the results of Wade Section 2.2.**

Proof. Let $\epsilon > 0$. We know from above that $a \leq x_n \leq b$ for all n. By the definition of convergence we know that for some $N \in \mathbb{N}$ and then for $n > N$ we have that $|x_n - x| < \epsilon$. From this we can then write that $x < x_n + \epsilon \leq b + \epsilon$ and similarly $a - \epsilon \leq x_n - \epsilon < x$. Therefore, this implies that $a - \epsilon < x < b + \epsilon$. Since this holds for any $\epsilon > 0$ we have that $a \leq x \leq b$ as desired. $\square$

For the remaining problems, you are allowed to use the results of Wade Section 2.2:

Problem 3. Assume $x > 0$ and (x_n) is a sequence of nonnegative numbers such that $x_n \to x$. Prove that $\sqrt{x_n} \to \sqrt{x}$.

Hint: You may find it useful to use the following identity:

$$\sqrt{a} - \sqrt{b} = \frac{a-b}{\sqrt{a}+\sqrt{b}}$$

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Proof. Let $\epsilon > 0$, then by Archimedean Principle we will choose $N \in \mathbb{N}$ such that $N > \epsilon\sqrt{x}$. Since we are given that $|x_n - x| < \epsilon$ we also know that $|x_n - x| < \epsilon\sqrt{x}$. Then we have that $|\sqrt{x_n} - \sqrt{x}| = |\frac{x_n - x}{\sqrt{x_n}+\sqrt{x}}|$ and since the sequence is nonnegative numbers we know that $|\sqrt{x_n} + \sqrt{x}| = \sqrt{x_n} + \sqrt{x}$. Then it follows that $|\frac{x_n - x}{\sqrt{x_n}+\sqrt{x}}| = \frac{|x_n - x|}{\sqrt{x_n}+\sqrt{x}} < \frac{|x_n - x|}{\sqrt{x}} < \frac{\epsilon\sqrt{x}}{\sqrt{x}} = \epsilon$. Therefore for $n > N$ we have that $|\sqrt{x_n} - \sqrt{x}| < \epsilon$ then we know that $\sqrt{x_n} \to \sqrt{x}$. $\square$

Problem 4. Let (x_n) be defined recursively by

$$x_{n+1} = 2 + \sqrt{x_n - 2}\,,$$

where x_1 is assumed to be > 3. Note that the sequence (x_n) is defined. Prove that if $x_n \to x$ for some $x \in \mathbb{R}$, then $x = 3$.

By theorem 2.8 we know that every convergent sequence is bounded, and since x_n is convergent it is also bounded. I am first going to show that x_n is bounded from below by 3.

Proof. We are going to prove this statement by mathematical induction.
Base case: We are given that $x_1 > 3$ by definition so it is true for $n = 1$.
Inductive step: Let's assume that $x_n > 3$ we then want to show that $x_{n+1} > 3$. Then $x_{n+1} = 2 + \sqrt{x_n - 2} > 2 + \sqrt{3-2} = 2 + \sqrt{1} = 3$. Therefore, $x_{n+1} > 3$ and further that x_n is bounded from below by 3. $\square$

Given this we are going to prove the original problem statement.

Proof. Given that $x_n \to x$ for $x \in \mathbb{R}$, we can say that as $n \to \infty$ then we can say that $x_n \to x$ and $x_{n+1} \to x$. Then we can continue to say that,

$$\begin{aligned}
x &= 2 + \sqrt{x-2} \\
x - 2 &= \sqrt{x-2} \\
(x-2)^2 &= x - 2 \\
x^2 - 4x + 4 &= x - 2 \\
x^2 - 5x + 6 &= 0 \\
(x-2)(x-3) &= 0 \\
x &= 2, 3.
\end{aligned}$$

But since we know that x_n is bounded below by 3, we can eliminate the fact that $x = 2$ as it is not the biggest lower bound. So we can then determine that if $x_n \to x$ for some $x \in \mathbb{R}$, then $x = 3$. $\square$

Problem 5. Does the sequence
$$x_n = \frac{(-1)^n n^3 + 2n^2 + 1}{3n^3 + (-1)^n n + 13}$$
converge? Prove your answer.

Hint: Consider the sequences $y_n = x_{2n}$ *and* $w_n = x_{2n+1}$ *for* $n \in \mathbb{N}$. *What can you say about* $(y_n), (w_n)$*?*

Proof. In order to attempt to solve for this convergence we are going to multiply the top and bottom of this fraction by $\frac{1}{n^3}$. Then we have that $\lim_{n\to\infty} \frac{(-1)^n n^3+2n^2+1}{3n^3+(-1)^n n+13} = \lim_{n\to\infty} \frac{(-1)^n+\frac{2}{n}+\frac{1}{n^3}}{3+\frac{(-1)^n}{n^2}+\frac{13}{n^3}}$. By Theorem 2.12 (iv) we have that $\lim_{n\to\infty} \frac{(-1)^n+\frac{2}{n}+\frac{1}{n^3}}{3+\frac{(-1)^n}{n^2}+\frac{13}{n^3}} = \frac{\lim_{n\to\infty}(-1)^n+\frac{2}{n}+\frac{1}{n^3}}{\lim_{n\to\infty} 3+\frac{(-1)^n}{n^2}+\frac{13}{n^3}}$. By simplifying this we know that $(-1)^n$ does not converge in the numerator and in the denominator we can conclude that $\lim_{n\to\infty} 3 + \frac{(-1)^n}{n^2} + \frac{13}{n^3} = 3 + \lim(\frac{(-1)^n}{n^2}) + 0 = 3 + \lim((-1)^n \frac{1}{n^2}) = 3 + 0 = 3$. Therefore we are left with $\lim_{n\to\infty} \frac{(-1)^n}{3}$ and we know from example 2.3 that $(-1)^n$ does not converge so we know that the whole sequence does not converge. $\square$