MATH 4317 Homework 6 Due Friday, 3/11/2022 at 11PM to Gradescope

Homework guidelines:

- Each problem I assign, unless otherwise stated, is asking you to prove something. Give a full mathematical proof using only results from class or Wade.
- Submit a PDF or JPG to gradescope. The grader has $\sim$ 250 proofs to grade: please make his job easier by submitting each problem on a different page.
- If you submit your homework in Latex, you get 2% extra credit.

Problems (5 total, 10 pts each)

Problem 1. Consider, for each $p \in \mathbb{R}$, the series

$$S_p := \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(1+k)^{p/2}}.$$

For which values of $p \in \mathbb{R}$ is the series S_p convergent? For each p for which S_p is convergent, determine whether S_p is absolutely or conditionally convergent. Provide rigorous arguments for all statements.

Proof. We will first rewrite $S_p := \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(1+k)^{p/2}}$ as $S_p := \sum_{k=1}^{\infty}(-1)^{k+1} * \frac{1}{k(1+k)^{p/2}}$. Then by the alternating series test we know a series of the form $\sum_{k=1}^{\infty}(-1)^{k} a_k$ if $a_k \downarrow$ as $k \to \infty$ converges; therefore, for S_p it will converge when $\frac{1}{k(1+k)^{p/2}} \downarrow 0$ as $k \to \infty$. For $p > 0$ we have $\frac{1}{k(1+k)^{p/2}}$ will allows us to use the comparison test with $\frac{1}{k^n}$ where $n > 1$ since $p > 0$ which means that by the p-test this series will converge and that $\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k(1+k)^{p/2}}$ will converge absolutely since $\sum_{k=1}^{\infty} |\frac{(-1)^{k+1}}{k(1+k)^{p/2}}| = \sum_{k=1}^{\infty} \frac{1}{k(1+k)^{p/2}}$ as we were using. For $p \in (-2, 0]$ we have that by the alternating series test, that we need to evaluate $\frac{1}{k(1+k)^{p/2}}$ and we can see that $\frac{1}{k(1+k)^{p/2}} \downarrow 0$ as $k \to \infty$ therefore, we know that the series would converge. However, this series does not converge absolutely on this range, if we consider the series $\sum_{k=1}^{\infty} \frac{1}{k(1+k)^{p/2}}$ with negative p as it is in this range we get that $\sum_{k=1}^{\infty} \frac{(k+1)^{p/2}}{k}$ for $p \in (-2, 0]$. If we consider $p = 0$ we get $\sum_{k=1}^{\infty} \frac{1}{k}$ which is the harmonic series which we already know is divergent. For other values $p \in (-2, 0)$ we get we get similarly divergent series. Lastly, for $p \in (-\infty, -2)$ this series diverges because $\frac{1}{k(1+k)^{p/2}}$ does not approach 0. Therefore, this series converges absolutely for $p > 0$, converges conditionally for $p \in (-2, 0]$, and the diverges for $p \leq -2$. $\square$

Given a (possibly divergent) series $\sum_{k=1}^{\infty} a_k$ with partial sums $s_n, n \geq 1$, the *Cesaro averages* are the sequence of values σ_n given by

$$\sigma_n = \frac{s_1 + \cdots + s_n}{n} .$$

We say that S is *Cesaro summable* to some value L if $\lim_n \sigma_n = L$.

Problem 2. Prove the following version of Tauber's theorem: if $a_k \geq 0$ for all $k \geq 1$ and $\sum_1^{\infty} a_k$ is Cesaro-summable to some L, then $\sum_1^{\infty} a_k$ converges to L.

Hint: It will be useful to first prove that

$$\sigma_n = \sum_{k=1}^{n} \left(1 - \frac{k-1}{n}\right) a_k$$

for all n.

Claim $\sigma_n = \sum_{k=1}^{n} \left(1 - \frac{k-1}{n}\right) a_k$

Proof. Let us consider the value of $\sigma_n = \frac{s_1+s_2+...+s_n}{n} = \frac{1}{n}(s_1+s_2+...+s_n) = \frac{1}{n}(\sum_{k=1}^{1} a_k + \sum_{k=1}^{2} a_k + ... + \sum_{k=1}^{n} a_k) = \frac{1}{n}((a_1)+(a_1+a_2)+...(a_1+a_2+...+a_n)) = \frac{1}{n}(n(a_1)+(n-1)(a_2)+...2(a_{n-1})+a_n) = a_1 + \frac{(n-1)a_2}{n} + ... + \frac{2a_{n-1}}{n} + \frac{a_n}{n} = (a_1 - 0) + (a_2 - \frac{(2-1)a_2}{n}) + (a_3 - \frac{(3-1)a_3}{n}) + ...(a_n - \frac{(n-1)a_n}{n}) = \sum_{k=1}^{n}(1 - \frac{k-1}{n})a_k$. Therefore, we have that $\sigma_n = \sum_{k=1}^{n} \left(1 - \frac{k-1}{n}\right) a_k$ as desired. □

Now we will prove the original statement.

Proof. Let assumptions be as above, that is to say that $\lim_{\infty} \sigma_n = L$ for our series. This implies that $\lim_n \sigma_n = \lim_n \sum_{k=1}^{\infty}(1 - \frac{k-1}{n})a_k = L$. Therefore if we take $\sum_{k=1}^{\infty} a_k - L = \sum_{k=1}^{\infty} a_k - \sum_{k=1}^{\infty}(1 - \frac{k-1}{n})a_k = \sum_{k=1}^{\infty} a_k + \sum_{k=1}^{\infty}(\frac{k-1}{n} - 1)a_k = \sum_{k=1}^{\infty} a_k + (\frac{k-1}{n} - 1)a_k = \sum_{k=1}^{\infty} a_k(1 + \frac{k-1}{n} - 1) = \sum_{k=1}^{\infty} a_k(\frac{k-1}{n})$. Therefore, we know by Dirichlet's test, since $\frac{k-1}{n} \downarrow 0$ as $k \to \infty$ implies that the whole series converges. Therefore, this all implies that $\sum_{k=1}^{\infty}$ converges to L. □

Problem 3. Given the formula $\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$, find the exact value of

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$$

and prove your answer.

Proof. We know that $\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + ...$ and that $\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = 1 + \frac{1}{3^3} + \frac{1}{5^2} + ...$, so we will recognize that $\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2}$ are the odd terms of $\sum_{k=1}^{\infty} \frac{1}{k^2}$. Then,

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{1}{k^2} &= \sum_{k=1}^{\infty} \frac{1}{2k^2} + \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \\
\sum_{k=1}^{\infty} \frac{1}{k^2} &= \frac{1}{4} \sum_{k=1}^{\infty} \frac{1}{k^2} + \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \\
\frac{3}{4} \sum_{k=1}^{\infty} \frac{1}{k^2} &= \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} \\
\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} &= \frac{3}{4} * \frac{\pi^2}{6} \\
\sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} &= \frac{\pi^2}{8}.
\end{aligned}$$

$\square$

Problem 4. Let $f_n, f : E \to \mathbb{R}, n \geq 1$ be continuous functions defined on some set $E \subset \mathbb{R}$. Show that f_n converges to f uniformly on E if and only if

$$\lim_{n\to\infty} \sup_{x\in E} |f_n(x) - f(x)| = 0\,.$$

Proof. Let assumptions be as above. Let us assume that f_n converges to f uniformly on E, that is to say that for every $\epsilon > 0$ there is an $N \in \mathbb{N}$ such that $n \geq N$ implies $|f_n(x) - f(x)| < \frac{\epsilon}{2}$ for all $x \in E$. This implies that $0 \leq \sup |f_n(x) - f(x)| \leq \frac{\epsilon}{2} < \epsilon$ for all $n > N$. It then follows that $\lim_{n\to\infty} \sup |f_n(x) - f(x)| = 0$.
Conversely, suppose that $\lim_{n\to\infty} \sup_{x\in E} |f_n(x) - f(x)| = 0$, and let $\epsilon > 0$. We want to show that there exists $N \in \mathbb{N}$ such that $n > N$ implies that $|f_n(x) - f(x)| < \epsilon$ for all $x \in E$. Since $\lim_{n\to\infty} \sup_{x\in E} |f_n(x) - f(x)| = 0$, there exists N such that $n > N$ implies that $\sup |f_n(x) - f(x)| - 0 = \sup |f_n(x) - f(x)| < \frac{\epsilon}{2}$. Then, by the definition of a supremum, we have that $|f_n(x) - f(x)| \leq \frac{\epsilon}{2} < \epsilon$ for $x \in E$ and $n > N$ as desired.
Therefore, for $f_n, f : E \to \mathbb{R}, n \geq 1$ where f_n is a continuous function on E, then f_n converges to f uniformly on E if and only if $\lim_{n\to\infty} \sup_{x\in E} |f_n(x) - f(x)| = 0$. $\square$

Problem 5. Let $f_n, f, g : [a, b] \to \mathbb{R}, n \geq 1$ be functions. Assume the following:

(a) there exists $M > 0$ such that $|f_n(x)| \leq M$ for all $n \geq 1, x \in [a, b]$ (i.e., (f_n) is *uniformly bounded*);

(b) $f_n \to f$ uniformly on any closed subinterval $[c, d] \subset (a, b)$;

(c) g is continuous on $[a, b]$ and satisfies $g(a) = g(b) = 0$.

Prove that under these conditions, $f_n g \to fg$ uniformly.

Proof. Let assumptions be as above. Let us define $h = f_n g$; therefore, we want to show that h converges uniformly to fg. Let $\epsilon > 0$, since g is continuous on $[a, b]$, it has a maximum value C on $[a, b]$. There is $N \in \mathbb{R}$ such that if $n > N$ and $x \in [a, b]$ then $|f_n - f| < \frac{\epsilon}{C}$. So if $n > N$ and $x \in [a, b]$ then $|h - fg| = |f_n g - fg| = |g||f_n - f| < C(\frac{\epsilon}{C} = \epsilon$. Therefore, $f_n g$ converges uniformly to fg as desired. $\square$