MATH 4317 Homework 10 Due Friday, 4/22 at 11PM to Gradescope

Homework guidelines:

- Each problem I assign, unless otherwise stated, is asking you to prove something. Give a full mathematical proof using only results from class or Wade.
- Submit a PDF or JPG to gradescope. The grader has ~ 250 proofs to grade: please make his job easier by submitting each problem on a different page.
- If you submit your homework in Latex, you get 2% extra credit.

Problems (5 total, 10 pts each)

Problem 1. Let $E = [0,1] \cap \mathbb{Q}$. Find $\overline{E}, E^\circ$ and ∂E.

Proof. Let us first understand what the set E really is, it is the rational numbers between 0 and 1. We will start by finding the interior of this set. If we take any $x \in E$ and $r \in \mathbb{R}$ then it is not possible for us to construct an open ball centered at x that will contain only rational numbers. This then implies that $E^\circ = \emptyset$. Further the closure of this set for every $x \in [0,1]$ there are rationals q arbitrarily close to x. For $n \in \mathbb{N}$ choose $a_n \in E$ such that $|x - a_n| < \frac{1}{n}$. This implies that we have $lim_{n \to \infty} a_n = x$ which then implies that $\overline{E} = [0,1]$. Finally, since $\partial E = \overline{E} \backslash E^\circ = [0,1] \backslash \emptyset = [0,1]$. $\square$

A set $K \subset \mathbb{R}^n, n \geq 1$ is called *convex* if for all $x, y \in K$ and $t \in [0,1]$, we have that

$$tx + (1-t)y \in K\,.$$

Problem 2. Show that any nonempty convex subset $K \subset \mathbb{R}^n$ is connected.

Hint: Assume for the sake of contradiction that K is not connected, and let U, V be disjoint open sets separating K. Fix $x \in U, y \in V$ and define $\phi : [0,1] \to \mathbb{R}^n, \phi(t) := tx + (1-t)y$. Show that the contradiction hypothesis implies $\phi([0,1])$ is disconnected and derive a contradiction.

Claim: If F is a family of connected sets such that $\cap F \neq \emptyset$ then $\cup F$ is connected.

Proof. Let $x \in \cap F$. Suppose that $x \in X$ so if $y \in Y$, then there is $f \in F$ such that $y \in f$ so $f = (A \cap f) \cup (B \cap f)$ which is a contradiction since A is open in $\cup F, A \cap f$ is open in f and $B \cap f$ is open in f □

Now for the question.

Proof. Let K be a non-empty convex subset and let $a \in K$. Let $L_v = \{x \in X | x = y + \lambda v : \lambda \in \mathbb{R}\}$. As K is convex, $X = \bigcup_{\|v\|=1} L_v$. We then need to show that L_v is connected, but since L_v is homeomorphic to some real interval, since K is convex, L_v is also connected. Since $\{a\} \subset \cap_{\|v\|=1} L_v$, the previous claim implies that K is in fact connected. □

Problem 3. Let $E \subset \mathbb{R}^n$ be finite. Prove, directly from the open-cover definition of compactness, that E is compact.

Proof. Let U be an open cover of E, there is some $A_0 \in U$ such that $0 \in A_0$. Since A_0 is open and $\lim_{n\to\infty} \frac{1}{n} = 0$, $\frac{1}{n} \in A_0$ as n is large enough. So there is some $N \in \mathbb{N}$ such that $n \geq N$ that implies that $\frac{1}{n} \in A_0$. For each $n \in \{1, 2, ..., N-1\}$, let $A_n \in U$ be such that $\frac{1}{n} \in A_n$. Then $\{A_n | n \in \{0, 1, ..., N-1\}\}$ is a finite subcover of U. Since this open cover has finite subcovers, then E is compact as desired. $\square$

Problem 4. Prove that $\mathbb{R}$ is not compact by finding an explicit open covering of $\mathbb{R}$ admitting no finite subcover. Make sure to prove that for the open cover you specify, no finite subcover exists.

Proof. Let $U_n = (-n, n)$. If we take $\bigcup_{n\in\mathbb{N}} U_n = \mathbb{R}$ and if $F \subset \bigcup_{n\in\mathbb{N}} U_n$, then F contains an element U_k such that $k \geq i$ for each $U_i \in F$. But then $\cup F = U_k = (-k, k) \subsetneq \mathbb{R}$ so F cannot be an open cover for $\mathbb{R}$. This implies that the open cover U_n does not have any subcoverings as desired. $\square$

For a set $S \subset \mathbb{R}^n$ and a point $x \in \mathbb{R}^n$, define the *minimal distance*

$$d(x, S) = \inf\{\|x - y\| : y \in S\}\,.$$

Problem 5. Assume $E \subset \mathbb{R}^n$ is compact. Show that for any $x \in \mathbb{R}^n$ there exists $\hat{x} \in E$ such that

$$\|x - \hat{x}\| = d(x, E)\,.$$

Hint: Show that the function $g : \mathbb{R}^n \to \mathbb{R}$ defined by $g(y) := \|x - y\|$ is continuous.

Proof. Let assumptions be as above. For $x \in \mathbb{R}^n$ and for each $\hat{x} \in E$. Consider the ball $B(\hat{x} : \|\hat{x} - x\|)$. Then $\{B(\hat{x} : \|\hat{x} - x\| : \hat{x} \in E\}$ is an open cover for E. Since E is compact, there are points $\hat{x_1}, \hat{x_2}, ..., \hat{x_n}$ such that $E = \bigcup_{i=1}^n B(\hat{x_i} : \|\hat{x_i} - x\|)$. Then $d(x)$ will be minimum for all $\hat{x} \in E$ as desired. $\square$