Homework 5

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1. It is trivial to see that there are two eigenvalues for the matrix A, 3 and 2. In order for this matrix to be diagonalizable, the geometric multiplicity of $\lambda = 2$ would need to be 1 and the geometric multiplicity of $\lambda = 3$ would need to be 2. That means that the eigenspace of $\lambda = 2$ would be of dimension 1 and the eigenspace of $\lambda = 3$ would be of dimension 2. In order to solve for the eigenvectors for each eigenvalue, we need to solve $Nul(A - \lambda I)$ for $\lambda = 2, 3$.
   $\lambda = 2$: $(A - 2I) = \begin{bmatrix} 1 & a & b \\ 0 & 0 & c \\ 0 & 0 & 1 \end{bmatrix}$. When solving for $(A - 2I) = 0$ we have, $\begin{bmatrix} 1 & a & b & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ So for eigenvector $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, the equation gives that $v_3 = 0, v_1 = -a * v_2 - b * v_3 = -av_2$ and v_2 as a free variable. So the eigenspace would be $v_2 \begin{bmatrix} -a \\ 1 \\ 0 \end{bmatrix}$
   $\lambda = 3$: $(A - 3I) = \begin{bmatrix} 0 & a & b \\ 0 & -1 & c \\ 0 & 0 & 0 \end{bmatrix}$. Therefore, when solving $(A - 3I) = 0$ we have, $\begin{bmatrix} 0 & a & b & 0 \\ 0 & -1 & c & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, for eigenvector $v = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$, we get $av_2 + bv_3 = 0, v_2 = cv_3$, and v_1 can be a free variable. Since this eigenspace must have dimension 2 we will require that $c = 0$ which allows $v_2 = 0$ and v_3 becomes a free variable. Then the eigenspace will become $v_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + v_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.
   Therefore, $c = 0$ and a, b can be any arbitrary real numbers.

2. Let A be an $n \times n$ matrix where it has $\lambda_1, \lambda_2, ..., \lambda_n$ distinct eigenvalues where $v_1, v_2, ..., v_n$ are the corresponding eigenvectors. Since there are n distinct eigenvalues, and n corresponding eigenvectors, the geometric multiplicity of each of the n eigenvalues is 1. Since each eigenvalue has geometric multiplicity of 1, then the eigenspace of each of the eigenvalues

is of dimension 1. Therefore, if any vector space generated by one non-zero vector $v_1(1 \leq i \leq n)$, all vectors of w_i are scalar multiple of v_i. This means that $w_i = \{u_i : u = kv_i\}k \in$ field. Then any eigenvector v of A will be an element of any $1-$dim eigenspace w_i generated by v_i corresponding to eigenvalue λ_i. Then $v = k * v_i$ for some $v_i \in \{v_1, v_2, ..., v_n\}$ and $k \in$ Field.

3. So given that these vectors would be existing in $\mathbb{R}^n$, since the set of vectors $\{e_1, e_2, ..., e_n\}$ are the standard basis for $\mathbb{R}^n$, the Gram-Schmidt process would first convert the first vectors $\{u_1, u_2, ..., u_k\}$ to an orthonormal collection of vectors, then once it reaches the set $\{e_1, e_2, ..., e_n\}$ it would simplify some of those vectors to become orthonormal, but since there are, in total, more than n vectors $(k + n > n)$ for nonzero, positive k, some of the vectors would have to result as 0 which would be skipped. Therefore, the Gram-Schmidt process would result in a set of n orthonormal vectors which would also be a basis for $\mathbb{R}^n$.

4.

	-4	-8	-14	-16
GS	$1 * 10^{-7} * 0.3815$	0.5760	0.5000	1.0000
GS twice	$1 * 10^{-11} * 0.1140$	$1 * 10^{-8} * 0.6729$	$1 * 10^{-15} * 0.2220$	1.0000
QR	$1 * 10^{-15} * 0.6661$	$1 * 10^{-15} * 0.2282$	$1 * 10^{-15} * 0.2220$	$1 * 10^{-15} * 0.2220$