Homework 3

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1. (a) $33776925 = 5^2 * 3 * 7^3 * 13 * 101$
   (b) $210733237 = 11^3 * 13 * 19 * 641$
   (c) $1359170111 = 13 * 17 * 19 * 47 * 71 * 97$

20. When $n = 0$ we get that $2^{2^0} + 5 = 7$ which is prime. For all other n we get that $2^{2^n} \equiv_3 1$. Therefore, $2^{2^n} + 5 \equiv_3 0$ which means that all other numbers of this form are divisible by 3 an are greater than 3 would would not be prime. Therefore, the only prime of the form $2^{2^n} + 5$ is 7.

3. $99(63) + 86(41) = 9763$. Therefore, he had 63 US dollars and 41 Canadian dollars.

13. 3 quarters, 2 dimes, and 4 pennies
    And many other combinations of pennies and dimes that are fairly trivial to add together.

20. If we set $x = -1$, then we have that $-a + by = ab - a - b$, so that $y = a - 1$, and a particular solution is $x_0 = -1$, $y_0 = a - 1$ so the general solution is $x = -1 + bt$ and $y = a - 1 - at$ where $t \in \mathbb{Z}$. Now if $x = -1 + bt \geq 0$, then we have $bt \geq 1$, and since $b \geq 1$ and t is an integer, this implies that $t \geq 1$. Also, if $y = a - 1 - at \geq 0$, then we have $t \leq \frac{a-1}{a} = 1 - \frac{1}{a} < 1$, which contradicts the fact that $t \geq 1$. Therefore, if $n = ab - a - b$, there are no nonnegative solutions to the linear diophantine equation $ax = by = n$.

16. Let $m = 5, a = 3, b = 4$, $a + b = 3 + 4 \equiv_5 2$. However, $4 \equiv_5 4$ and $3 \equiv_5 3$ so $4 + 3 = 7 \neq 2$.

28. Given that $\sum_{l=1}^{n-1} l^3 = (\frac{n(n+1)}{2})^2$. If n is *odd* then we can rewrite it as $n = 2k + 1$ for $k \in \mathbb{Z}$ which means that we can do $(\frac{(n-1)(n-1+1)}{2})^2 = (\frac{n(n-1)}{2})^2 = \frac{n^2(n-1)^2}{4} = \frac{(2k+1)^2(2k+1-1)^2}{4} = \frac{(4k^2+4k+1)(4k^2)}{4} = \frac{16k^4+16k^3+4k^2}{4} = 4k^4 + 4k^3 + k^2 = k^2(2k+1)^2$ which implies that $\sum_{l=1}^{n-1} l^3 \equiv 0 \mod (n)$. Similarly, if n is divisible by 4 we can write it as $n = 4k$ which is then $\frac{(4k)^2(4k-1)^2}{4} = \frac{16^2k^4+16*8k^3-16k^2}{4} = 4 * 16 * k^4 + 32k^3 - 4k^2 = 4k^2(4k-1)^2$ which is divisible by $n = 4k$ as desired. However, if the n is even but not divisible by 4 we can write it as $n = 4k + 2$ which means that $\frac{(4k+2)^2(4k+1)^2}{4} = 64k^4 + 96k^3 + 52k^2 + 12k + 1 = (2k+1)^2(4k+1)^2$ which are both odd which means that the number is not divisible by n since n is even.

43. Given the definition of fibonacci numbers we have that $f_n = f_{n-1} + f_{n-2}$. Taking this mod m we have that $f_n \equiv f_{n-1} + f_{n-2} \mod m$ and since there are finitely many possible pairs (f_{n-1}, f_{n-2}) it will repeat eventually. Now we have that $f_{n-2} \equiv f_n - f_{n-1} \mod m$ we can go backwards. Since $f_0 = 0$, there exists at least on $f_i \equiv 0 \mod m$ for each period of repetition, Hence there are infinitely many f_n such that $m|f_n$.

5. From the problem we are given that $x \equiv_{23} 0$ and that $11x \equiv_{24} 17$. The second equation gives that $x \equiv_{24} 19$ since the inverse of $11 \mod 24$ is 11. Therefore, we get that $x \equiv_{23} 0$ and $x \equiv_{24} 19$ which implies that the orbit period of the satellite is $23 * 19 = 437$ hours.

16. Given that $x^2 \equiv 1 \mod 2^k$ we get that $x^2 - 1 \equiv 0 \mod 2^k$. This implies that $(x-1)(x+1) \equiv 0 \mod 2^k$ which implies that x is odd. Let us then write that $x = 2k+1$ for some $k \in \mathbb{Z}$ then we have that $(2k+1-1)(2k+1+1) = 2k(2k+2) = 4k(k+1) \equiv 0 \mod 2^k$. This means that 2^{k-2} divides $m(m+1)$

for $k > 2$.If $k \leq 2$ then there is no condition on m. So all residue classes of odd integer satisfy the above equation. SO now assume that $k > 2$. If m is even then m is divisible by 2^{k-2} and $x = 2^{k-1}t+1$ for $t \in \mathbb{Z}$. But if m is odd, then $m+1$ is divisible by 2^{k-2} and in this case $x = 2(m+1)-1 = 2^{k-1}t-1$ for $t \in \mathbb{Z}$. In the first we shall have only two noncongruent solutions namely $1, 2^{k-1}+1$ whereas in the second case the noncongruent solutions are -1 and $2^{k-1}-1$ as desired.

6. We can use CRT to get that $x = 1014060069938914k + 326741466757708$ for $k \in \mathbb{Z}$. I am sorry but this was just a giant busy work problem and I'm not typing all the numbers for it.

15. We can rewrite these as $x = a_1 + km_1, x = a_2 + lm_2$. If we let $d = \gcd(m_1, m_2)$ we get that $m_1 = dp, m_2 = dq$. Then we get that $a_1 + km_1 = a_2 + lm_2$. This implies that $(a_1 - a_2) = -km_1 + lm_2 \iff (a_1 - a_2) = ldq - kdp = d(lq - kp) \iff d|(a_1 - a_2) \iff (m_1, m_2)|(a_1 - a_2)$ as desired.