# Homework 5

Sean Eva

April 2021

## 1 Chapter 7 Exercises

36. Since $X_1, X_2, \ldots$ are uncorrelated, $Cov(X_i, X_j) = 0$ if $i \neq j$

$$\begin{aligned}
Cov(S_m, S_n) &= Cov(X_1 + X_2 + \ldots + X_m, X_1 + X_2 + \ldots + X_m + X_{m+1} + \ldots + X_{n-1} + X_n) \\
&= Cov(X_1, X_1) + \ldots + Cov(X_m, X_m) \\
&= V(X_1) + \ldots + V(X_m) \\
&= V(X_1 + \ldots + X_m) \\
&= V(S_m) \\
&= V(X_1) + \ldots + V(X_m) \\
&= m\sigma^2.
\end{aligned}$$

Since all $X$ have the same variance.

60. Since the odd central moment of the normal distribution is 0. Then,

$$\begin{aligned}
0 &= \mathbb{E}[(X - \mu)^3] \\
&= \mathbb{E}(X^3 - 3\mu X^2 + 3\mu^2 X - \mu^3) \\
&= \mathbb{E}(X^3) - 3\mu\mathbb{E}(X^2) + 3\mu^2\mathbb{E}(C) - \mu^3.
\end{aligned}$$

Where $\mathbb{E}(X) = \mu$ and $\mathbb{E}(X^2) = Var(X) + [\mathbb{E}(X)]^2 = \sigma^2 + \mu^2$.

$$\begin{aligned}
0 &= \mathbb{E}(X^3) - 3\mu(\sigma^2 + \mu^2) + 3\mu^3 - \mu^3 \\
&= \mathbb{E}(X^3) - 3\mu\sigma^2 - \mu^3 \\
\Rightarrow \mathbb{E}(x^3) &= 3\mu\sigma^2 + \mu^3
\end{aligned}$$

75. Given that we know, $Am \geq \omega m$. We have that $\frac{x_1+x_2+\ldots+x_n}{\eta} \geq \sqrt[n]{\eta x_2 \ldots x_n}$.
If we apply this, $\frac{\sqrt[n]{\prod_{i=1}^n x_i}}{x_1}, \frac{\sqrt[n]{\prod_{i=1}^n x_i}}{x_2}, \ldots, \frac{\sqrt[n]{\prod_{i=1}^n x_i}}{x_n}$. Then we have, $\frac{\sum_{i=1}^n \frac{\sqrt[n]{x_1 x_2 \ldots x_n}}{x_i}}{n} \geq (\frac{\sqrt[n]{x_1 x_2 \ldots x_n}}{x_1} * \frac{\sqrt[n]{x_1 x_2 \ldots x_n}}{x_2} * \ldots * \frac{\sqrt[n]{x_1 x_2 \ldots x_n}}{x_n})^{\frac{1}{n}}$
$\Rightarrow \frac{1}{2}\sqrt[n]{x_1 x_2 \ldots x_n} \sum_{i=1}^n \frac{1}{x_i} \geq 1$
$\Rightarrow \sqrt[n]{x_1 x_2 \ldots x_n} > \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \ldots + \frac{1}{x_n}}$

## 2 Chapter 7 Problems

8. Let $M_X(t)$ be the m.g.f. of $X_i$ and $P_N(*)$ be the p.g.f. of $N$. Then the m.g.f. of $X_1 + X_2 + ... + X_N$ is

$$\begin{aligned} M_{X_1+X_2+...+X_N}(t) &= \mathbb{E}(e^{t(X_1+X_2+...+X_n)}) \\ &= \mathbb{E}_N \mathbb{E}_X((e^{t(X_1+X_2+...+X_N)})|N=n) \\ &= \mathbb{E}_N[(M_X(t))^N] \\ &= P_N(M_X(t)) \end{aligned}$$

## 3 Chapter 8 Exercises

10. Let $X_i$ be a random variable such that $X_i = \begin{cases} 1 & \text{if the throw is a 5 or 6} \\ 0 & \text{if the throw is a 1, 2, 3, or 4} \end{cases}$.

Therefore,

$$\begin{aligned} \mathbb{P}(X_i = 1) &= \frac{1}{6} + \frac{1}{6} = \frac{1}{3} \\ \mathbb{P}(X_i = 0) &= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3}. \end{aligned}$$

Also, $X_i$s are independent and identically distributed random variables. Let's define $N_n = \sum_{i=1}^{n} X_i$. Then,

$$\begin{aligned} \mathbb{E}(N_n) &= \mathbb{E}(X_1) + \mathbb{E}(X_2) + ... + \mathbb{E}(X_n) \\ &= \frac{1}{3} + \frac{1}{3} + ... + \frac{1}{3} \\ &= \frac{n}{3} \\ var(N_n) &= var(X_1 + X_2 + ... + X_n) \\ &= var(X_1) + var(X_2) + ... + var(X_n) \\ &= p(1-p) + p(1-p) + ... + p(1-p) \\ &= np(1-p) \\ &= n\frac{1}{3}(1 - \frac{1}{3}) \\ &= \frac{2n}{9} \\ \mathbb{E}(\frac{N_n}{n}) &= \frac{n}{3n} = \frac{1}{3} \\ var(\frac{N_n}{n}) &= \frac{1}{n^2} var(N_n) \\ &= \frac{1}{n^2}\frac{2n}{9} \\ &= \frac{2}{9n} \to 0 \text{ as } n \to \infty. \end{aligned}$$

Therefore, $\frac{N_n}{n} \to \mathbb{E}(\frac{N_n}{n}) = \frac{1}{3}$ as $n \to \infty$

21. Let $X$ be a random variable that equals 1 with probability $p = \frac{1}{6}$ and 0 with probability $q = 1-p = \frac{5}{6}$. We have that $\mathbb{E}(X) = \frac{1}{6}$ and $var(X) = \frac{5}{36}$. Let us define $Y = X + ... + X$ ($n$ times) as a random variable accounting for the number of sixes. $Y$ has a mean value of $\mu = \mathbb{E}(Y) = \frac{n}{6}$ and $\sigma^2 = var(Y) = \frac{5n}{36}$. By using Chebyshev's inequality,

$$\mathbb{P}[|Y - \mu| > k\sigma] \leq \frac{1}{k^2}.$$

Therefore, by choosing $k = \sqrt{\frac{36}{5}}$ we get,

$$\mathbb{P}[|Y - \frac{n}{6} > \sqrt{n}] \leq \frac{5}{36}.$$

So,

$$\mathbb{P}[Y \in (\frac{n}{6} - \sqrt{n}, \frac{n}{6} + \sqrt{n}] \geq \frac{31}{36}.$$

32. Let $S$ be a random variable with binomial distribution on $n$ trials where $p = \frac{1}{6}$. Then,

$$\begin{aligned}
\mathbb{E}(S) &= np = 2000 \\
var(S) &= np(1-p) = \frac{5000}{3} \\
Z &= \frac{S - np}{\sqrt{np(1-p)}} \\
\mathbb{P}(1900 < S < 2200) &= \mathbb{P}(\frac{1900 - 2000}{\sqrt{\frac{5000}{3}}} < \frac{S - 2000}{\sqrt{\frac{5000}{3}}} < \frac{2200 - 2000}{\sqrt{\frac{5000}{3}}} \\
&= \mathbb{P}(-\frac{100\sqrt{6}}{100} < Z < \frac{200\sqrt{6}}{100}) \\
&= \int_{-\sqrt{6}}^{2\sqrt{6}} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du.
\end{aligned}$$

Therefore, $a = -\sqrt{6}, b = 2\sqrt{6}$.

# 4 Chapter 8 Problems

14. Sorry, I got stumped on this one.