**MATH 4317 Homework 4** Due Friday, 2/25/2022 at 11PM to Gradescope

Homework guidelines:

- Each problem I assign, unless otherwise stated, is asking you to prove something. Give a full mathematical proof using only results from class or Wade.
- Submit a PDF or JPG to gradescope. The grader has $\sim 250$ proofs to grade: please make his job easier by submitting each problem on a different page.
- If you submit your homework in Latex, you get 2% extra credit.

## Problems (5 total, 10 pts each)

**Problem 1.** Let $f : E \to \mathbb{R}$ be continuous on a nonempty set $E \subset \mathbb{R}$ and let $F \subset E$ be a nonempty subset. Define $g = f|_F$ to be the **restriction** of $f$ to $F$, defined by

$$g(x) = f(x) \quad \text{for } x \in F\,.$$

Note that, as a function, $g : F \to \mathbb{R}$. Prove that $g$ is continuous on $F$.

*Proof.* Let assumptions be as in the problem statement. Since we are given that $f$ is continuous on $E$, then we know for $\epsilon > 0, \delta > 0$, then $|x - a| < \delta$ and $x \in E$ implies that $|f(x) - f(a)| < \epsilon$ for all points $a \in E$. Given the restriction on $g$, we know that $g(x) = f(x)$; therefore, for all $a \in F, a \in E$ for $\epsilon > 0, \delta > 0$, then we have that $|x-a| < \delta$ for $x \in F$ implies that $|g(x)-g(a)| = |f(x)-f(a)| < \epsilon$ for all $x \in F$. Therefore, we know that $g$ is continuous on $F$. $\square$

**Problem 2.** Prove that the function

$$f(x) = \begin{cases} \frac{x^2+x-2}{x-1} & x \neq 1 \\ 3 & x = 1 \end{cases}$$

is continuous on $\mathbb{R}$.

We will prove this in two parts, $\frac{x^2+x-2}{x-1}$ is continuous on $\mathbb{R}$ then that $f(x)$ is continuous at $x = 1$.

**Claim:** $\frac{x^2+x-2}{x-1}$ is continuous on $\mathbb{R}$.

*Proof.* Let us simplify $\frac{x^2+x-2}{x-1} = \frac{(x+2)(x-1)}{x-1} = x + 2$. So we need to show that $x + 2$ is continuous on $\mathbb{R}$. If we fix $a \in \mathbb{R}$, for $x \in \mathbb{R}$ we have that $|f(x) - f(a)| = |x + 2 - (a + 2)| = |x - a|$. Given $\epsilon > 0$ if we choose $\delta = \epsilon$. So we have that $|f(x) - f(a)| = |x - a| < \delta = \epsilon$ $\square$

**Claim:** $f(x)$ is continuous at $x = 1$.

*Proof.* Using the same simplification from the first part of this proof that we can write $\frac{x^2+x-2}{x-1} = \frac{(x+2)(x-1)}{x-1} = x + 2$ for $\epsilon > 0$ if we choose $\delta = \epsilon$. Then if $|x - 1| < \delta$ we get that $|f(x) - f(3)| = |x + 2 - 3| = |x - 1| < \delta = \epsilon$. Therefore, we know that $f(x)$ is continuous at $x = 3$. $\square$

Since we have that $\frac{x^2+x-2}{x-1}$ is continuous on $\mathbb{R}$ and that $f(x)$ is continuous at $x = 1$ then we know that $f(x)$ is continuous on $\mathbb{R}$.

**Problem 3.** Prove or disprove (find a counterexample) to the following statement: Let $E \subset \mathbb{R}$ and let $f, g : E \to \mathbb{R}$. If $f + g : E \to \mathbb{R}$ is continuous on $E$, then both $f$ and $g$ are continuous on $E$.

Let's take $f(x) = \begin{cases} 1 & x \leq 0 \\ 0 & x > 0 \end{cases}$ and $g(x) = \begin{cases} 1 & x \geq 0 \\ 0 & x < 0 \end{cases}$. If we take $f(x) + g(x) = 1$ at all $x$, then in this case $E = \mathbb{R}$, we know it is continuous on $E$. However, $f(x), g(x)$ themselves are not continuous on $E$.

**Problem 4.** Let $f : \mathbb{R} \to \mathbb{R}$, and for $\epsilon > 0$ define $g_\epsilon(x) := \sup_{y \in [x-\epsilon, x+\epsilon]} f(y)$. Prove that if $f$ is continuous at $x \in \mathbb{R}$, then

$$f(x) = \lim_{\epsilon \to 0} g_\epsilon(x) \,.$$

You may freely use the following subsequential characterization of one-sided limits: if $h : (0, \infty) \to \mathbb{R}$, then $\lim_{\epsilon \to 0+} h(\epsilon)$ exists and equals some $L \in \mathbb{R}$ iff for all sequences $(\epsilon_n) \subset (0, \infty)$ with $\epsilon_n \to 0$, we have that $\lim_{n \to \infty} h(\epsilon_n) \to L$.

*Proof.* Let assumptions be as above. Let us consider the interval of $I = [x - \epsilon, x + \epsilon]$ where we have defined $g_\epsilon(x) := \sup_{y \in I} f(y)$. If we take the $\lim_{\epsilon \to 0} g_\epsilon(x) = \lim_{\epsilon \to 0} \sup_{y \in I} f(y)$ then we get that $y \in [x - \epsilon, x + \epsilon] \to [x - 0, x + 0]$ as $\epsilon \to 0$. Therefore, since this interval $I$ is closed and bounded and a subset of $\mathbb{R}$ and given that $f(y)$ is continuous at $x \in \mathbb{R}$ by the Bolzano-Weierstrass theorem we are able to write a subsequence $y_\epsilon$ over the interval $I$ such that $y_\epsilon$ converges to $x$. This then allows us to say that $\lim_{\epsilon \to 0} \sup_{y \in I} f(y) = f(y_\epsilon) = f(x)$ as desired. $\square$

**Problem 5.** Let $f : \mathbb{R} \to \mathbb{R}$ be an everywhere-continuous function such that

$$f(x+y) = f(x) + f(y) \tag{1}$$

for all $x, y \in \mathbb{R}$. Prove that there exists a fixed value $m \in \mathbb{R}$ such that

$$f(x) = m \cdot x \quad \text{for all } x \in \mathbb{R}\,.$$

*Hint: To start, you should expect that $m = f(1)$ (why?). You might also find it useful to show that starting from the identity (1), we have that $f(0) = 0$ and $f(-x) = -f(x)$ for all $x \in \mathbb{R}$. Now, with $m := f(1)$, show first that $f(n) = m \cdot n$ for all $n \in \mathbb{Z}$, and then that $f(q) = m \cdot q$ for $q \in \mathbb{Q}$. To conclude the desired identity for all $x \in \mathbb{R}$, approximate a given $x \in \mathbb{R}$ by a sequence $q_n$ of rational numbers, using the facts that (i) $f$ was assumed continuous; and (ii) $\mathbb{Q}$ is* **dense** *in $\mathbb{R}$: for any $\alpha \in \mathbb{R}$ there exists a sequence $(q_n)$ of rational numbers such that $q_n \to \alpha$.*

*Proof.* Let assumptions be as above. Given that $f(x+y) = f(x)+f(y)$ we can say that $f(x+0) = f(x) + f(0) = f(x)$, this shows us that we can then say that $f(0) = 0$. Similarly we could further say that $f(2x) = f(x+x) = f(x) + f(x) = 2f(x)$ and further we can say that for $f(ax)$ for some $a \in \mathbb{Z}^+$ we get that $f(x+x+...+x) = f(x) + f(x) + ... + f(x) = af(x)$. Alternatively, for given that $0 = f(0)$ we know that $f(0) = f(x+-x) = f(x) + f(-x)$ implying that $-f(x) = f(-x)$. Then we can say $f(-ax) = -f(ax) = -af(x)$. This allows us to generally say that $f(ax) = af(x)$ for $a \in \mathbb{Z}$ (not necessarily positive integers now). Next consider if we have $f(x) = f(\frac{x}{2} + \frac{x}{2}) = f(\frac{x}{2}) + f(\frac{x}{2}) = 2f(\frac{x}{2}) \Rightarrow \frac{1}{2}f(x) = f(\frac{x}{2})$. More generally we are able to then say $f(\frac{x}{n}) = \frac{1}{n}f(x)$ for $n \in \mathbb{Z}$. Combining these two statements we get that $f(\frac{a}{n}x) = af(\frac{1}{n}x) = \frac{a}{n}f(x)$. We can simplify this to say that for $q \in \mathbb{Q}$ that $f(qx) = qf(x)$. This then allows us to take that $f(q) = f(q*1) = qf(1)$; let us then say that $f(1) = m$ for some constant $m \in \mathbb{R}$, then we can say that $f(q) = mq$ for any rational $q$. Lastly, given that $f(x)$ is continuous on $\mathbb{R}$ we have that it can be expressed over a sequence of rational numbers $q_n$. Since $\mathbb{Q}$ is dense in $\mathbb{R}$ we know that we for any $\alpha \in R$ there exists a sequence $q_n$ of rational numbers such that $q_n \to \alpha$. This then implies that for appropriate $q_n$ that we can say that $f(\alpha) = f(q_n) = q_n f(1) = q_n m = \alpha m$. Therefore, we get that we can write $f(x) = m * x$ for $x \in R$ and fixed $m \in \mathbb{R}$ as desired. $\square$