Homework 6 due Wed, Oct 27th by 11am in Gradescope

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INSERT a "pagebreak" command between each problem (integer numbers). Problem subparts (letter numbered) can be on the same page.

REMOVE all comments (within "textit{}" commands) before submitting solutions.

DO NOT include any identifying information (name, GTID) except on the first/cover page.

1. Problem 4.3 # 9

   (a) Let assumptions be as in the problem statement. We know that $A = \{a \in R | \phi(a) \in A'\}$. Consider $a_1, a_2 \in A$, then we know that $\phi(a_1), \phi(a_2) \in A'$. Since A' is a subring of R' we know that $\phi(a_1) + phi(a_2) \in A'$. Similarly, since phi is a ring homomorphism, we know that $\phi(a_1) + \phi(a_2) = \phi(a_1 + a_2)$. Likewise, since A' is a ring, we know that $\phi(a_2) + \phi(a_1) \in A'$ and it is simple to show that $a_2 + a_1 \in A$ and it would follow that $\phi(a_1 + a_2) = \phi(a_1) + \phi(a_2) = \phi(a_2) + \phi(a_1) = \phi(a_2 + a_1)$ and we then know that $a_1 + a_2 = a_2 + a_1$. Since we know that $\phi(a_1) + \phi(a_2) = \phi(a_1 + a_2)$, we know $\phi(a_1 + a_2) \in A'$; therefore, by the definition of the set A we have that $a_1 + a_2 \in A$. Similarly, since ϕ is a ring homomorphism, we know that $\phi(a_1 a_2) = \phi(a_1 a_2)$. Since $\phi(a_1), \phi(a_2) \in A'$, we also know that $\phi(a_1 a_2) \in A'$ and by the definition of A we know that $a_1 a_2 \in A$. Therefore, since for elements $a_1, a_2 \in A$ we know that $a_1 + a_2 = a_2 + a_1 \in A$ and $a_1 a_2 \in A$, then A is a subring of R. Now if we take $k \in K$, since K is the kernel of ring homomorphism ϕ then it implis that $\phi(k) = 0$ but since A' is a subring of R' therefore, $0 \in A'$. Therefore, since $\phi(k) \in A'$ and from the definition of A we have that $k \in A$, we know that $K \subset A$.

   (b) Let assumptions be as in the problem statement. We can define a new function $\phi_A : A \to R'$ which is a ring homomorphism since it is just a restriction of a ring homomorphism to a subring of R. We define that for any $a \in A$ we have that $\phi_A(a) = \phi(a)$. The kernel of ϕ_A is K since $K \subset A$. We know that $\phi_A(a) = \phi(a) \in A'$ which implies that the image of $\phi_A \subset A'$. We also know that for any $a' \in A'$ that there exists an $a \in A$, such that $\phi(a) = \phi_A(a) = a' \in \text{Im}\phi_A$. This then implies that $A' \subset \text{Im}\phi_A$. Therefore, $\text{Im}\phi_A = A'$. Then by the First Homomorphism Theorem, we know that $A/K \simeq A'$.

   (c) Let assumptions be as in the problem statement. We are given that A' is a left ideal of R', which tells us that for $r' \in R', a' \in A'$. We want to show that for $r \in R, a \in A$ that $ra \in A$. Since $a \in A$, then we know that $\phi(a) \in A'$. If we take $\phi(ra) = \phi(r)\phi(a)$ since ϕ is a homomorphism. Since A' is a left ideal we know that $\phi(r)\phi(a) \in A'$. Therefore, by the definition of A we know that $ra \in A$. Therefore, A is a left ideal of R.

2. Problem 4.3 # 18

   Let assumptions be as in the problem statement. In order to show that $R \oplus S$ is a ring we need to show that for elements $a, b \in R \oplus S$ that $ab, a+b, a+(-b) \in R \oplus S$. Since R, S are rings by definition, we know that elements $r+t, r+(-t), rt \in R$ and that $s+u, s+(-u), su \in S$. Then if we define $a = (r,s), b = (t,u) \in R \oplus S$ we know that $a+b, a+(-b), ab \in R \oplus S$. Therefore, we know that $R \oplus S$ is a ring. Next we will show that the subring $\{(r,0)|r \in R\}$ is an ideal of $R \oplus S$. Since we are given that this is a subring of $R \oplus S$ we know that it is an additive subgroup. We then need to show that this subring absorbs multiplication on the left and right. Let us define $(a,0),(b,0) \in R \oplus S$ and $a, b \in R$. We know that $ab, ba \in R$ since R is a ring. This means that $(a,0)(b,0) = (ab,0) \in \{(r,0)|r \in R\}$ and $(b,0)(a,0) = (ba,0) \in \{(r,0)|r \in R\}$. We then want to show that this ideal is isomorphic to R. We can show this if we can develop an isomorphism $\phi : \{(r,0)|r \in R\} \to R$. We will define $\phi : \{(r,0)|r \in R\} \to R$ as for $a \in \{(r,0)|r \in R\}$ where $a = (r,0)$ that $\phi(a) = \phi((r,0)) = r \in R$. Let $a = (r,0), b = (t,0) \in \{(r,0)|r \in R\}$, we know that $rt \in R$ by definition, then we can show $\phi(a) + \phi(b) = r + t = \phi(a+b)$ and $\phi(a)\phi(b) = rt = \phi(ab)$. This showed that ϕ is a homomorphism. Then we have that if $\phi(a) = \phi(b)$ we know that $r = t$ which implies that $a = (r,0) = (t,0) = b$ which shows that ϕ is one to one and an monomorphism. Lastly we want to check to see if ϕ is onto. Then for $r \in R$ we know that $(r,0) \in \{(r,0)|r \in R\}$ would map as $\phi((r,0)) = r \in R$. This applies the exact same to $\{(0,s)|s \in S\}$ without loss of generality.

3. Problem 4.3 # 19

(a) Let assumptions be as in the problem statement. In order prove that R is a ring we want to show that for $a_1, a_2 \in R$ that $ab, a+b, a+(-b) \in R$. Let $a_1 = \begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}, a_2 = \begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix} \in R$. Then it is simple to see that $a_1a_2 = \begin{pmatrix} a_1a_2 & a_1b_2 + b_1c_2 \\ 0 & c_1c_2 \end{pmatrix}$ then $a_1a_2 \in R$. Similarly, $a_1 + a_2 = \begin{pmatrix} a_1 + a_2 & b_1b_2 \\ 0 & c_1 + c_2 \end{pmatrix} \in R$, and $a_1 + (-a_2) = \begin{pmatrix} a_1 - a_2 & b_1 - b_2 \\ 0 & c_1 - c_2 \end{pmatrix} \in R$. Since $a_1a_2, a_1 + a_2, a_1 + (-a_2) \in R$ we know that R is a ring.

(b) Let assumptions be as in the problem statement. In order to show that I is an ideal of R we need to show that I is an additive subgroup of R and absorbs multiplication from the right and left. Since R and subsequently I follows normal matrix addition properties we know that for any if for any $i_1 = \begin{pmatrix} 0 & a_1 \\ 0 & 0 \end{pmatrix}, i_2 = \begin{pmatrix} 0 & a_2 \\ 0 & 0 \end{pmatrix} \in I$ we know that $i_1 + i_2 = \begin{pmatrix} 0 & a_1 + a_2 \\ 0 & 0 \end{pmatrix} \in I$. Therefore, we know that I is an additive subgroup of R. Then we want to show that I absorbs multiplication on the left and right. Consider the same $i_1, i_2 \in I$ as before. Then it is simple to see that $i_1i_2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ and $i_2i_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. And since $0 \in \mathbb{R}$, then $i_1i_2, i_2i_1 \in I$. This implies that I absorbs multiplication on the left and the right and is therefore an ideal of R.

(c) Let assumptions be as in the problem statement. In order to show these are isomorphic we will define a mapping $\phi : R/I \to F \oplus F$ as $\phi(\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}) = (a, c)$. Let us define $A = \begin{pmatrix} a_1 & b_1 \\ 0 & c_1 \end{pmatrix}, B = \begin{pmatrix} a_2 & b_2 \\ 0 & c_2 \end{pmatrix} \in R$. We are able to define A and B simply like this because the definition of $R/I = \{a + I | a \in R\}$ does not affect elements a_1, c_1. Then we have that $A + B = \begin{pmatrix} a_1 + a_2 & b_1 + b_2 \\ 0 & c_1 + c_2 \end{pmatrix}$ and $\phi(A + B) = (a_1 + a_2, c_1 + c_2) = (a_1, c_1) + (a_2, c_2) = \phi(A) + \phi(B)$ therefore, we know that ϕ is a homomorphism. We then want to show that ϕ is one to one, Let us say that $\phi(A) = \phi(B)$, that is to say that $(a_1, c_1) = (a_2, c_2)$. This implies then that $a_1 = a_2, c_1 = c_2$. Then it would follow that $A = B$. This means that ϕ is one to one and is therefore a monomorphism. Lastly, we want to show that ϕ is onto. Let us define an element of $F \oplus F$ as (a, c), then the element of R/I the corresponds to (a, c) is $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$. This implies that ϕ is onto and is therefore an isomorphism. Therefore, since we have an isomorphism $\phi : R/I \to F \oplus F$ we know that $R/I \simeq F \oplus F$.

4. Problem 4.3 # 20

   Let assumptions be as in the problem statement. Then for all $r_1, r_2 \in R$, we know that $\phi(r_1 + r_2) = (r_1 + r_2 + I, r_1 + r_2 + J) = ((r_1 + I) + (r_2 + I), (r_1 + J) + (r_2 + J)) = (r_1 + I, r_1 + J) + (r_2 + I, r_2 + J) = \phi(r_1) + \phi(r_2)$. This is true since I, J are ideals. Then we also know that $\phi(r_1 r_2) = (r_1 r_2 + I, r_1 r_2 + J) = ((r_1 + I)(r_2 + I), (r_1 + J)(r_2 + J)) = (r_1 + I, r_1 + J)(r_2 + I, r_2 + J) = \phi(r_1)\phi(r_2)$. Therefore, we know that ϕ is a homomorphism. Then $\operatorname{Ker}\phi = \{r \in R | \phi(r) = (I, J)\} = \{r \in R | (r + I, r + J) = (I, J)\} = \{r \in R | r + I = I, r + J = J\} = \{r \in R | r \in I, r \in J\} = \{r \in R | r \in I \cap J\} = I \cap J$.

5. Problem 4.3 # 21.

   Let assumptions be as in the problem statement. We have to show that $\mathbb{Z}_{15} \simeq \mathbb{Z}_3 \oplus \mathbb{Z}_5$. We know that $\mathbb{Z}_1 5$ and $\mathbb{Z}_3 \oplus \mathbb{Z}_5$ are cyclic groups and since $\gcd(3, 5) = 1$ we know that they are isomorphic since there is a unique cyclic group of given order up to isomorphism.