Krabby Patty Hunting

Expected Value Casino Assignment

Ms. Martin

MDM4U-01

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Game Information

Sales Pitch

On April 5, 2020, Spongebob Squarepants was fired from the Krabby Patty. He misses making and eating the restaurant's delicious hamburgers, which are fittingly called "Krabby Patties". While walking home, he senses the presence of a Krabby Patty burger across Bikini Bottom, and he becomes desperate to eat it. Unfortunately, it's dark outside, and Plankton has filled the city with traps! Can you help Spongebob safely find the Krabby Patty? If you manage to get the job done, you'll get paid a lot of money!

Rules

- The object of the game is to make it to the top right corner of a 4 x 4 grid, starting from the bottom left corner. There is another option in which the player has 2 lives to make it across a 5 x 5 grid.
- At the beginning of the game, a path is randomly generated through the grid. There are $\frac{6!}{3!3!}$ = 20 possible paths (the player must make 6 moves, 3 up and 3 to the right, so we use permutations with identical items), with each path having an equal chance of being generated.
- On each move, you must select one of two possible directions, up or right. Press the corresponding arrow key on the keyboard to do so.
- If you move onto a square containing a trap, the game will be over and you will receive a payout depending on the number of successful moves you were able to make.
- If you move onto a square without a trap, you will be able to make another move.
- You cannot move past the boundaries of the grid. For instance, if you're on the top row of the grid, you cannot move up; you can only move right.
- The game ends if you reach the top right square of the grid, in which case you'll receive the top payout.

Payouts

Players will receive payouts depending on the number of successful moves they have made prior to the game-ending.

4 x 4 Grid, 1 Life

Successful Moves (moves)	Payout ($)
0	0
1	0
2	10
3	20
4	30
6	100

Please note that 5 moves are not possible as the player would be forced to get the grand prize.

5 x 5 Grid, 2 Lives

Lives Left and Successful Moves (lives, moves)	Payout ($)
2 lives left, 8 moves	150
1 life left, 8 moves	50
0 lives left, 6 moves	30
0 lives left, 5 moves	10
2 lives left, 8 moves	150
1 life left, 8 moves	50

Please note that 5 moves are not possible as the player would be forced to get the grand prize.

Option 1: 4 x 4 grid, 1 life

First, before determining what the payouts should be, we must calculate the probability of the player moving a certain number of spaces. We can do this by splitting all 20 possible paths into 3 groups. Paths in the first group take the player to a square along the top or right edge of the grid in 3 moves. Paths in the second group do this in 4 moves, and paths in the third group do this in 5 moves. Note that if the player has reached the top or right edge of the grid, he or she will only have one possible direction to move and thus is certain to reach the top right square. Probabilities will be calculated separately for each group, and then added together to produce totals.

X1	X2	X3	
			X3
			X2
			X1

Figure 1: Squares marked 'X' are the squares along the top or right edge of the 4 x 4 grid. Note that these squares do not include the square at the top right. If the player reaches this square, the game is over.

Group 1: X1

This group consists of paths for which the player can get to squares along the top or right edge in three moves. Note that, in three moves, the only squares along these edges that the player can reach are two corner squares; the one at the top left and the one at the bottom right of the grid. As such, there are only 2 paths in this group. Since there are 20 possible paths, a path in group 1 has a 2/20 = 1/10 chance of being generated.

Immediately after the player starts the game, he or she is forced to choose between going right or going up. Since only one of these directions is correct, and the player does not know which path to take, he or she has a ½ chance of making 0 successful moves.

Assuming the player guesses correctly on the first move, he or she is faced with a similar decision for his or her second move, with equal chances of success and failure. Since the player has a ½ chance of guessing incorrectly, he or she has a (½) (½) = ¼ chance of making 1 successful move. Note that the second factor of ½ comes from having to guess the first move correctly in order to guess the second move incorrectly.

Using similar logic, we observe that the player has a (½)(½)(½) = ⅛ chance of making 2 successful moves.

If the player has made 3 successful moves, he or she will reach the top or right edge of the grid, and will have a 100% chance of reaching the top right square, given that there is only one possible direction for the player to move. Thus, the player will have made 6 successful moves, and this has a (½)(½)(½) = ⅛ chance of happening. Each factor of ½ comes from making a successful move.

Finally, all the above probabilities of successful moves must be multiplied by 1/10, the probability of a Group 1 path being generated.

This allows us to come up with the grid shown in Figure 2, below:

Successful Moves, x	Probability of generating a Group 1 path and making this number of successful moves, $P(x)$
0	0.05
1	0.025
2	0.0125
6	0.0125

Figure 2: The probabilities of generating a Group 1 path and making *x* successful moves. Probabilities have been converted into decimals.

Group 2: X2

Group 2 paths require the player to make 4 moves before reaching the top or right edge of the grid. There are $2(\frac{4!}{3!}) - 2$, or 6 such paths. To explain, each path in this group will involve the player either making 3 moves right and 1 move up, or 3 moves up and 1 move right. This explains the factor of 2 at the beginning, as well as the factor of $\frac{4!}{3!}$, because the player is making 4 total moves: 3 in one direction and one in another, to reach the top or right edge of the grid. The - 2 comes from the fact that Group 1 paths also satisfy the criteria of a Group 2 path, and thus must be subtracted out.

As such, a Group 2 path has a 6/20 = 3/10 chance of being generated.

As before, immediately after the game starts, the player is faced with a decision as to whether to move up or right. Again, the player has a ½ chance of guessing correctly and thus has a ½ chance of making 0 successful moves.

As well, using similar logic, the player has a ¼ chance of making 1 successful move and a ⅛ chance of making 2 successful moves. However, the player is not in the clear yet after having made 3 successful moves; he or she must make yet another decision. Only then will the player have a certain path to the top right square. Hence, the player has a 1/16 chance of making 3 successful moves, and a 1/16 chance of making 6 successful moves and winning the grand prize.

Again, all probabilities of making a specific number of moves are multiplied by 3/10 (the probability of generating a Group 2 path) to produce the following table (Figure 3):

Successful Moves, x	Probability of generating a Group 2 path and making this number of successful moves, $P(x)$
0	0.15
1	0.075
2	0.0375
3	0.01875
6	0.01875

Figure 3: The probabilities of generating a Group 2 path and making *x* successful moves. Probabilities have been converted into decimals.

Group 3: X3

Group 3 paths require 5 moves for the player to reach the top or right edge of the grid. The number of paths in Group 3 can simply be determined by subtracting the number of paths in Group 1 or Group 2 from the total number of paths. This means that the number of paths in Group 3 is 20 - 2 - 6, or 12.

As such, a Group 3 path has a 12/20 or 3/5 chance of being generated.

Knowing that the player will have to make 5 successful moves to be assured of winning, we use similar logic as before to calculate that the player has a ½ chance of making 0 successful moves, a ¼ chance of making 1 successful move, a ⅛ chance of making 2 successful moves, a 1/16 chance of making 3 successful moves, a 1/32 chance of making 4 successful moves and a 1/32 chance of making 6 successful moves. Multiplying these probabilities by the 3/5 chance of generating a Group 3 path, we can create the table shown below.

Successful Moves, x	Probability of generating a Group 3 path and making this number of successful moves, $P(x)$
0	0.3
1	0.15
2	0.075
3	0.0375
4	0.01875
6	0.01875

Figure 4: The probabilities of generating a Group 3 path and making *x* successful moves. Probabilities have been converted into decimals.

Total Probabilities

Adding together all the probabilities of making a certain number of moves (for each group), we arrive at the following probabilities (Figure 5):

Successful Moves, x	Probability of making this number of successful moves, $P(x)$
0	0.5
1	0.25
2	0.125
3	0.05625
4	0.01875
6	0.05

Figure 5: The probability of making *x* successful moves for the game as a whole. Probabilities have been converted into decimals.

Probability Distribution and Expected Profit

Given the probabilities of making any number of successful moves, we can determine appropriate payouts for the player. Note that, for this assignment, we were instructed to modify the payouts such that we would expect to make an average profit between $1 and $4 each time the player bets $10 and plays the game, over many games. Calculations are made under the assumption that the player always bets $10. Playing around with the numbers, we can come up with the payouts below:

Number of successful moves	Payout to the player ($)	Casino's Profit or Loss, x ($)	Probability of this payout, $P(x)$
0	0	10	0.5
1	0	10	0.25
2	10	0	0.125
3	20	-10	0.05625
4	30	-20	0.01875
6	100	-90	0.05

Figure 6: The payouts given to the player.

The casino's expected average profit per play is then calculated below:

$$E(X) = 10(0.5) + 10(0.25) + 0(0.125) - 10(0.05625) - 20(0.01875) - 90(0.05)$$

$$= 5 + 2.5 + 0 - 0.5625 - 0.375 - 4.5$$

$$= \$2.0625$$

Thus, for each time the player bets $10 and plays the game, we'd expect to make an average profit of about $2.06 per game over many trials.

Option 2: 5 x 5 grid, 2 lives

For this option, the player will have two lives to make it across a 5 x 5 grid. If the player runs into a trap with 2 lives remaining, he or she will lose a life and will have to restart at the bottom left square. If he or she does so with 1 life remaining, the game would be over, and the player would receive a payout based on how many successful moves he or she has made. Payouts for this option will be different from the other option.

The probabilities of making a certain number of successful moves will be calculated in a similar manner to what was done with the other option. The difference is that, for this option, the player will have an extra life.

For this section, there will be 4 groups of paths. Each group will require 4, 5, 6 or 7 moves, respectively, to reach the top or right edge of the grid. Calculations are done under the assumption that the player knows what he or she is doing and will not repeat the same mistakes.

There are C(8, 4), or 70 possible paths, for this option.

Group 1

This time, a path in group 1 will require 4 moves to reach the top or right edge of the grid. Like before, a group 1 path will pass through either the top left square or the bottom right square of the grid, and there are 2 paths in group 1. As such, a group 1 path has a 2/70, or 1/35 chance of being generated.

First, consider the possibility that the player will make it to the top right square without losing a life. The player has to make 4 successful decisions, each with a 50% chance of success or failure, of reaching a corner square and being assured of making it to the end. This has a (½)(½)(½)(½) = 1/16 chance of happening.

Next, consider the possibility that the player has made 3 successful moves on the first life, and then loses a life. The probability of making the 3 successful moves on the first life is (½)(½)(½)(½) = 1/16. (3 successes, 1 failure). The player, knowing not to make the same wrong move on the second life, will then be assured of making it to the top right square on the second life.

Third, consider the possibility that the player has made 2 successful moves on the first life, and then loses a life. The probability of making the 2 successful moves on the first life is (½)(½)(½) = 1/8 .The player is then assured of making at least 3 successful moves on the second life. To win the grand prize, the player has to guess the fourth move correctly. Since the player had to make 3 uninformed decisions, this case has a (½)(½)(½)(½) = 1/16 chance of happening. Failing to make the fourth move also has a 1/16 chance of happening, since the probability of failing to make a correct move is ½.

Fourth, consider the possibility that the player has made 1 successful move on the first life, and then loses a life. The probability of making the 1 successful move on the first life is (½)(½) = ¼. The player is then assured of making at least 2 successful moves on the second life. Using the same logic as before, the player has a (½)(½)(½) = ⅛ chance of failing to make a third successful move, a (½)(½)(½)(½) = 1/16 chance of making 3 successful moves, and a (½)(½)(½)(½) = 1/16 chance of winning on the second life.

Last, consider the possibility that the player has made 0 successful moves on the first life, and then loses a life. The probability of making the 0 successful moves on the first life is ½. The player is then assured of making at least 1 successful move on the second life. The probability of failing to make a second successful move is (½)(½) = ¼. The probability of making 2 successful moves is (½)(½)(½) = ⅛. The probability of making 3 successful moves is (½)(½)(½)(½) = 1/16, and the probability of winning in the second life is also 1/16.

Equipped with this knowledge, we can make the table below (by summing the probabilities of a specific outcome across all cases), similar to the previous section and with the same caveats (multiplying each outcome by the probability of generating a group 1 path):

Lives Left	Successful Moves	Probability of generating a Group 1 path and making this number of successful moves
2	8	0.00178571429
1	8	0.00714285714
0	3	0.00535714286
0	2	0.00714285714
0	1	0.00714285714

Figure 7: The probabilities of generating a Group 1 path and making any possible number successful moves, and making it to the end with 1 or 2 lives remaining. Probabilities have been converted into decimals.

Group 2

Using the same logic as before, we construct the table below, in Figure 8, describing the probability of obtaining each individual possible outcome for group 2 paths, in which 5 moves are required to reach the top or right edge of the grid.

	Outcome (lives left and successful moves made)						
Successful Moves During First Life (Probability of Occurring)		2 lives left, 8 moves	1 life left, 8 moves	0 lives left, 4 moves	0 lives left, 3 moves	0 lives left, 2 moves	0 lives left, 1 move
	8 (1/32)	1/32					
	4 (1/32)		1/32				
	3 (1/16)		1/32	1/32			
	2 (⅛)		1/32	1/32	1/16		
	1 (¼)		1/32	1/32	1/16	1/8	
	0 (½)		1/32	1/32	1/16	1/8	1/4
	TOTAL	1/32	5/32	⅛ (4/32)	3/16 (6/32)	¼ (8/32)	¼ (8/32)

Figure 8: The probabilities of the player getting a certain outcome on a Group 2 path, given that such a path has been generated. Probabilities are represented by fractions. Note that, after having made 5 successful moves, the player would be forced to get the grand prize.

Probabilities of making ‘n’ moves within the first life can be generated using the formula $P(n) = 1/(2^{n+1})$. The ‘+1’ comes from having to make an incorrect move, which has a ½ chance of happening.

Now, to figure out the probability of generating a group 2 path *and* attaining a certain possible outcome, we must first calculate the number of group 2 paths. The number of group 2 paths is equal to 2 (C(5, 4)) - 2, or 8. This is because, when the 5 moves are made to reach the top or right edge, 4 of them are identical, and there are 2 possible edges to choose from, explaining the factor of 2. One could use permutations with identical items to perform this calculation, but combinations are also an option. Disregarding the factorials, the sum of the factors in the denominator (using permutations) would be equal to that of the numerator. The ‘-2’ comes from the overlap with group 1 paths. Thus, a group 2 path has an 8/70, or 4/35 chance of being generated. This allows us to construct Figure 9, below:

Lives Left	Successful Moves	Probability of generating a Group 2 path and making this number of successful moves
2	8	0.00357142857
1	8	0.01785714286
0	4	0.01428571429
0	3	0.02142857143
0	2	0.02857142857
0	1	0.02857142857

Figure 9: The probabilities of generating a Group 2 path and making any possible number successful moves, and making it to the end with 1 or 2 lives remaining. Probabilities have been converted into decimals.

Group 3

Group 3 paths require 6 moves to make it to the top or right edge of the grid. There are 2 (C(6, 4)) - 8 - 2, or 20 such paths. Thus, a group 3 path has a 20/70, or 2/7 chance of being generated. We use similar logic as before to construct the following tables (similar to before):

	Outcome (lives left and successful moves made)							
Successful Moves During First Life (Probability of Occurring)		2 lives left, 8 moves	1 life left. 8 moves	0 lives left, 5 moves	0 lives left, 4 moves	0 lives left, 3 moves	0 lives left, 2 moves	0 lives left, 1 move
	8 (1/64)	1/64						
	5 (1/64)		1/64					
	4 (1/32)		1/64	1/64				
	3 (1/16)		1/64	1/64	1/32			
	2 (⅛)		1/64	1/64	1/32	1/16		
	1 (¼)		1/64	1/64	1/32	1/16	1/8	
	0 (½)		1/64	1/64	1/32	1/16	1/8	1/4

Figure 10: The probabilities of the player getting a certain outcome on a Group 3 path, given that such a path has been generated. Probabilities are represented by fractions. Note that, after having made 6 successful moves, the player would be forced to get the grand prize.

Lives Left	Successful Moves	Probability of generating a Group 3 path and making this number of successful moves
2	8	0.00446428571
1	8	0.02678571429
0	5	0.02232142857
0	4	0.03571428571
0	3	0.05357142857
0	2	0.07142857143
0	1	0.07142857143

Figure 11: The probabilities of generating a Group 3 path and making any possible number of successful moves, and making it to the end with 1 or 2 lives remaining. Probabilities have been converted into decimals.

Group 4

Group 4 paths require 7 moves to reach the top or right edge of the screen. There are 70 - 20 - 8 - 2, or 40 such paths. Hence, a group 4 path has a 40/70 or 4/7 chance of being generated. As before, we construct two tables:

	Outcome (Lives/Successful Moves)								
Successful Moves During First Life (Probability of Occurring)		2 lives left, 8 moves	1 life left. 8 moves	0 lives left, 6 moves	0 lives left, 5 moves	0 lives left, 4 moves	0 lives left, 3 moves	0 lives left, 2 moves	0 lives left, 1 move
	8 (1/128)	1/128							
	6 (1/128)		1/128						
	5 (1/64)		1/128	1/128					
	4 (1/32)		1/128	1/128	1/64				
	3 (1/16)		1/128	1/128	1/64	1/32			
	2 (⅛)		1/128	1/128	1/64	1/32	1/16		
	1 (¼)		1/128	1/128	1/64	1/32	1/16	1/8	
	0 (½)		1/128	1/128	1/64	1/32	1/16	1/8	1/4
	TOTALS FOR EACH OUTCOME	1/128	7/128	6/128	10/128	16/128	24/128	32/128	32/128

Figure 12: The probabilities of the player getting a certain outcome on a Group 4 path, given that such a path has been generated. Probabilities are represented by fractions. Note that, after having made 7 successful moves, the player would be forced to get the grand prize.

Lives Left	Successful Moves	Probability of generating a Group 4 path and making this number of successful moves
2	8	0.00446428571
1	8	0.03125
0	6	0.02678571429
0	5	0.04464285714
0	4	0.07142857143
0	3	0.10714285714
0	2	0.14285714286
0	1	0.14285714286

Figure 13: The probabilities of generating a Group 4 path and making any possible number successful moves, and making it to the end with 1 or 2 lives remaining. Probabilities have been converted into decimals.

Total Probabilities

Adding the probabilities of each possible outcome, we arrive at the following:

Lives Left	Successful Moves	Probability of making this number of successful moves
2	8	1/70
1	8	93/1120
0	6	3/112
0	5	15/224
0	4	17/140
0	3	3/16
0	2	1/4
0	1	1/4

Figure 13: The probabilities of making any possible number successful moves for the game as a whole, and making it to the end with 1 or 2 lives remaining. Probabilities are represented by fractions.

Probability Distribution and Expected Profit

As before, we were instructed to keep the casino's expected profit per game between \$1 and \$4. Playing around with the numbers, we arrive at the following:

Outcome (Lives Left and Successful Moves)	Payout to the player (\$)	Casino's Profit or Loss, x (\$)	Probability of this payout, $P(x)$
2 lives left, 8 moves	150	-140	1/70
1 life left, 8 moves	50	-40	93/1120
0 lives left, 6 moves	30	-20	3/112
0 lives left, 5 moves	10	0	15/224
0 lives left, 4 moves	0	10	17/140
0 lives left, 3 moves	0	10	3/16
0 lives left, 2 moves	0	10	1/4
0 lives left, 0 or 1 move(s)	0	10	1/4

Figure 14: The payouts given to the player.

The expected profit is then calculated as follows:

$$E(X) = 10(1/4) + 10(1/4) + 10(3/16) + 10(17/140) + 0(15/224) - 20(3/112) - 40(93/1120) - 140(1/70)$$

$$E(X) \simeq \$2.23$$

Thus, we'd expect to make an average profit of about \$2.23 every time the player bets \$10 and plays the game. We settled on these payouts because the player might be attracted by the higher top payout for this variation, as opposed to the 4 x 4 grid with 1 life. However, unbeknownst to the player (hopefully), he or she will actually lose more money in the long run by playing this game mode. However, the average profit for this mode isn't that much above the other mode; the difference in expected profit is about 17 cents.

Analysis

A simulation was run to prove the finding shown above. The simulation approximated the results with 50,000 iterations. For both of the sections below, the following statistics are shown:

Stats

1. **Games Played**: The total rounds played (how accurate the results are)
2. **Total Revenue**: The total revenue earned. This value equals to the #of_rounds * cost_per_game. So in our situation, $10 * $50000 = $500000
3. **Total Expenses**: How much our casino machine was losing for all of the rounds. For example, if a player paid $10 to play and won $30, the expense for the round would be $-30. However, the P&L (Profit and Loss) of the round would be $(10-30) = $-20.
4. **Total P&L**: This is equivalent to Revenue - Expenses. This is the total profit or loss of the casino.
5. **Average Return per Round**: This is equivalent to Total_P&L / Total_Games_Played
6. **σ(SD)**: The measure of the amount of variation of the histogram (the lower graph)
7. **%Deviations From Theoretical Expected Return**: How much does the theoretical expectations different from the experimental one.

Occurrences

This is another way to measure accuracy. For each profit and loss, the total number of occurrences and the probability of the outcome is given. For example, the theoretical probability of gaining $10 for a 4 by 4 grid is 0.5 + 0.25 = 0.75 (0 and 1 successful moves). From the simulation below, it can be seen that the probability for that event is 0.75026. Indeed close to the expected value.

Graphs

The graphs below show three important visualizations: the revenue, accumulated P&L and current P&L over time (playing rounds). As seen in Fig. 16 and Fig. 17, running the simulation for many rounds flattens out the curve, but with magnifying the graph, such as in Fig. 15, interesting visualizations could be seen.

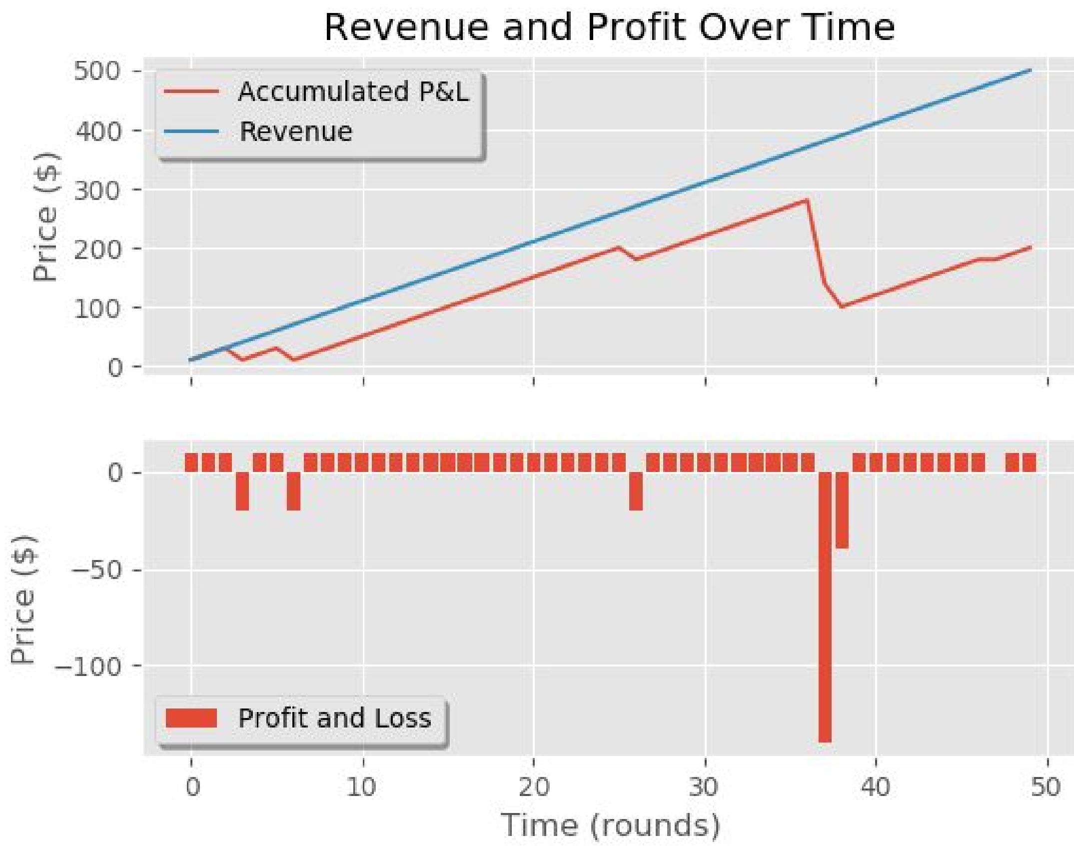

Figure 15: A simulation visualization ran for 50 rounds.

4 x 4 grid, 1 life

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Time to run 50000 games: 28 minutes

Stats
Games Played: 50000
Total Revenue: $500000
Total Expenses: $-398550
Total P&L: $101450
Average Return per Round: $2.03
σ(SD): 22.16
%Deviations From Theoretical Expected Return($2.06): 1.5%

Occurrences
$-90: 2512 Occurrences, Prob(0.05024)
$-20: 947 Occurrences, Prob(0.01894)
$-10: 2866 Occurrences, Prob(0.05732)
$0: 6162 Occurrences, Prob(0.12324)
$10: 37513 Occurrences, Prob(0.75026)
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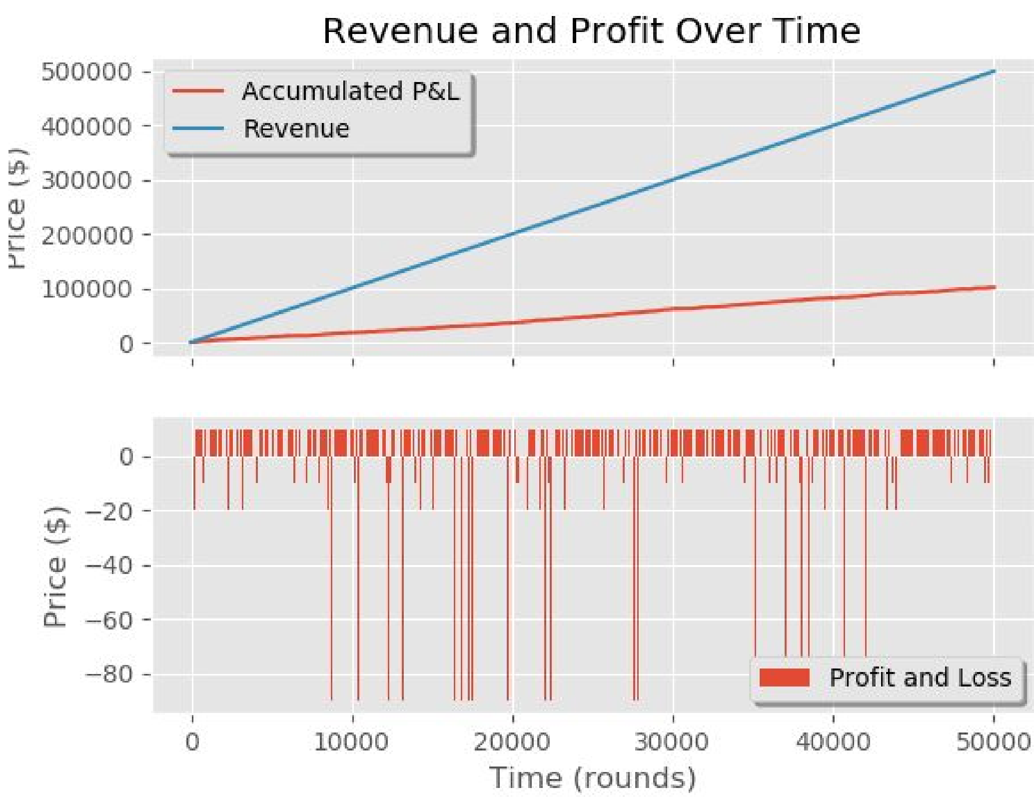

Figure 16: A simulation ran for 50000 iterations on a 4x4 grid, with one life.

5 x 5 grid, 2 lives

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Time to run 50000 games: 55 minutes

Stats
Games Played: 50000
Total Revenue: $500000
Total Expenses: $-387550
Total P&L: #112450
Average Return per Round: $2.25
σ(SD): 22.33
% Deviations From Theoretical Expected Return($2.23): 0.89%

Occurrences
$-140: 714 Occurrences, Prob(0.01428)
$-40: 4147 Occurrences, Prob(0.08294)
$-20: 1286 Occurrences, Prob(0.02572)
$0: 3452 Occurrences, Prob(0.06904)
$10: 40401 Occurrences, Prob(0.80802)
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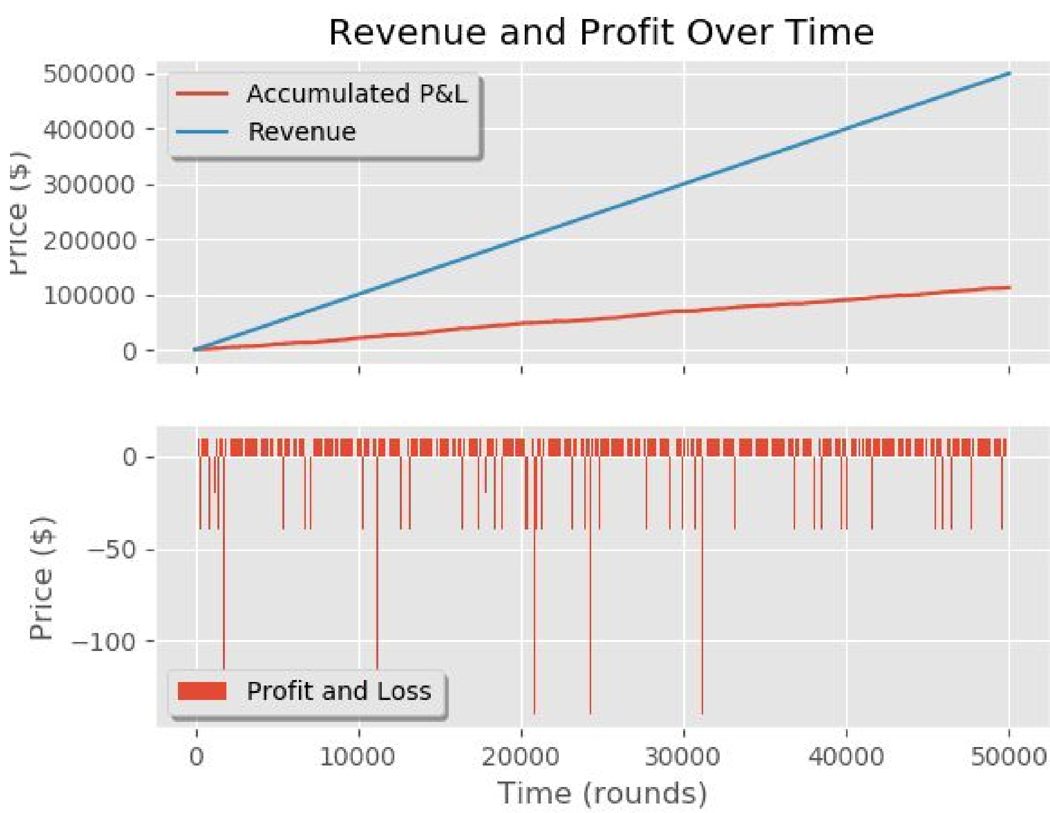

Figure 17: A simulation ran for 50000 iterations on a 4x4 grid, with one life.