STOR415: INTRODUCTION TO OPTIMIZATION
DEPARTMENT OF STATISTICS AND OPERATIONS RESEARCH
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HOMEWORK 9: QUADRATIC PROGRAMMING

Question 1. (**25 points**): **(Least-Squares Problem)** The following table provides some data observed from the function $f(x) = \beta_0 + \beta_1 x + \beta_2 e^{-x}$ with a little noise, where β_0, β_1, and β_2 are given parameters.

x	-2	-1	0	1	2
$\hat{f}(x)$	-7.1474	-3.3021	1.2274	6.1006	11.0370

Here, $\hat{f}(x)$ is an approximation of $f(x)$ at x with a little noise (i.e., $\hat{f}(x) = f(x) + \text{noise}$).
Your tasks are as follows:

(a) Formulate the problem of estimating $\beta = (\beta_0, \beta_1, \beta_2)^T$ as a linear least-squares problem.

(b) Mathematically solve this least-squares problem to find the optimal solution β^* and reform the function $f(x)$ from this solution.

(c) Using this function $f(x)$ to predict its value at $x = -3$, $x = 0.5$, and $x = 2.5$. Compare the predicted values and the actual values: $f(-3) = -8.9786$, $f(0.5) = 3.6516$, and $f(2.5) = 13.5205$, respectively by computing the relative error $\frac{|\hat{f}(x) - f(x)|}{|f(x)|}$, where $\hat{f}(x)$ is the predicted value.

Question 2. (**25 points**): **(QPs with equality constraints)** Solve the following QP with equality constraints:

$$(a)\ \begin{cases} \min\limits_{x\in\mathbb{R}^2} & \frac{1}{2}[(x_1-\alpha_1)^2 + (x_2-\alpha_2)^2] \\ \text{subject to} & x_1 + x_2 = 1. \end{cases}, \text{where } \alpha_1, \alpha_2 \in \mathbb{R} \text{ are given constants.}$$

$$(b)\ \begin{cases} \min\limits_{x\in\mathbb{R}^3} & \frac{1}{2}(x_1^2 + 2x_1x_2 + 4x_2^2 - x_1x_3 + x_3^2) - 2x_1 - x_2 + 2x_3 \\ \text{subject to} & x_1 + x_2 + x_3 = 1. \end{cases}$$

$$(c)\ \begin{cases} \min\limits_{x\in\mathbb{R}^4} & \frac{1}{2}(x_1^2 + 2x_1x_2 + 4x_2^2 - x_1x_3 + x_3^2 + x_4^2) - 2x_1 - x_2 + 2x_3 - 2x_4 \\ \text{subject to} & x_1 + x_2 + x_3 + x_4 = 1 \\ & x_1 + x_2 - x_3 - x_4 = 0. \end{cases}$$

To get full credits, you need to show your work in detail. Do not use software to solve them.

Question 3. (**25 points**): **(General QPs)** Transform the following QP problems into standard form (show your work in detail):

$$(a)\begin{cases} \min\limits_{x\in\mathbb{R}^2} & \frac{1}{2}x^TQx + x_1 - x_2 \quad \text{with} \quad Q = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \\ \text{subject to} & x_1 + x_2 \le 1, \quad x_1 \ge 0. \end{cases}$$

$$(b)\begin{cases} \min\limits_{x\in\mathbb{R}^3} & \frac{1}{2}x^TQx + x_1 + x_2 - x_3 \quad \text{with} \quad Q = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 2 \\ 0 & 2 & 2 \end{bmatrix} \\ \text{subject to} & x_1 + x_2 + x_3 \ge 1, \quad 0 \le x_1 \le 1, \ x_2 \ge 0, \ x_3 \le 0. \end{cases}$$

Solve these QPs by Matlab and Python, and report the results. Please also print out and attach the code and screenshots of the results you ran with your solution.

Question 4. (25 points): (Portfolio optimization problem) We want to model a *Markowitz portfolio optimization problem* by defining the total return $r = \sum_{j=1}^n x_jR_j$ at the end of the given time period, where x_j is the proportion of the investment for the j-th asset (or the j-th stock), and R_j is the relative price change of the j-th asset at the current time period. Here, R_j ($j = 1, \cdots, n$) are random variables, which imply that R is also a random variable.

The goal is to maximize the expected return $\mathbb{E}(r)$ while minimizing the risk. It leads to the following optimization problem:

$$\begin{cases} \min\limits_{x\in\mathbb{R}^n} & f(x) = -\sum_{j=1}^n x_j\mathbb{E}(R_j) + \lambda\text{Var}(r) \\ \text{s.t.} & \sum_{j=1}^n x_j = 1, \\ & x_j \ge 0, \quad j = 1, \cdots, n. \end{cases}$$

Here, $\lambda > 0$ is a tuning parameter (given) to trade-off between two objective terms: the expected return $\mathbb{E}(r)$, and the variance $\text{Var}(r)$ of r.

In practice, we cannot compute $\mathbb{E}(R_j)$ and $\text{Var}(r)$. Therefore, we replace them by the sample mean $\bar{R}_j$, and the sample covariance S, respectively for a given dataset. The data set `PortSelectData.xlsx` contains the data (relative return) of 5 US stocks over year from 1975 to 1994.

You are asked to perform the following tasks:

(a) Compute $\bar{R}_j$ and S from `PortSelectData.xlsx` and form the corresponding QP problem for the above portfolio optimization problem.

(b) Solve this problem by using `quadprog` in Matlab, `cvxopt` in Python, or any QP solver that you know, and interpret your results. You can try different values of λ, e.g., $\lambda \in \{2, 1, 0.5, 0.25\}$.

Note that you need to print your code, screenshots of the results you ran, and staple them with the solution.

Hint. How compute $\bar{R}_j$ and S?

- The formula to compute sample mean $\bar{R}_j$ is $\bar{R}_j = \frac{1}{N}\sum_{i=1}^N R_{ij}$ for every column $j = 1, 2, \cdots, n$. where R_{ij} is the relative price change of asset j w.r.t. time period i (given in the dataset).
- The matrix $S = (S_{jk})_{n\times n}$ is computed by $S_{jk} := \frac{1}{N-1}\sum_{i=1}^N (R_{ij} - \bar{R}_j)(R_{ik} - \bar{R}_k)$ for $j, k = 1, \cdots, n$.
- Replace the objective function $f(x) = -\sum_{j=1}^n x_j\mathbb{E}(R_j) + \lambda\text{Var}(r)$ by $f(x) = -\sum_{j=1}^n \bar{R}_jx_j + \lambda x^TSx$.

The end