1

1\. a. $A = \begin{pmatrix} 1 & -2 & e^2 \\ 1 & -1 & e^1 \\ 1 & 0 & e^0 \\ 1 & 1 & e^{-1} \\ 1 & 2 & e^{-2} \end{pmatrix}$ $b = \begin{pmatrix} -7.1474 \\ -3.3021 \\ 1.2274 \\ 6.1006 \\ 11.0370 \end{pmatrix}$

$A\beta = b$

$A^TA\beta = A^Tb$

$\beta = (A^TA)^{-1}A^Tb$

$\min\{\|A\beta - b\|_2^2\}$ $\beta = (A^TA)^{-1}A^Tb$

b. $\beta^* = (A^TA)^{-1}A^Tb = \begin{pmatrix} 0.9975 \\ 5.0022 \\ 0.2522 \end{pmatrix}$

$f(x) = 0.9975 + 5.0022x + 0.2522e^{-x}$

c. $\hat{f}(-3) = -8.9433$ $\hat{f}(0.5) = 3.6516$ $\hat{f}(2.5) = 13.5237$

$f(-3) = -8.9786$ $f(0.5) = 3.6516$ $f(2.5) = 13.5205$

$\frac{|\hat{f}(-3) - f(3)|}{|f(3)|} = 0.054$ $\frac{|\hat{f}(0.5) - f(0.5)|}{|f(0.5)|} = 0$ $\frac{|\hat{f}(2.5) - f(2.5)|}{|f(2.5)|} = 0.000236$

2a

2.a $f(x) = \frac{1}{2}(x_1 - \alpha_1)^2 + \frac{1}{2}(x_2 - \alpha_2)^2$

$= \frac{1}{2}(x_1^2 - 2x_1\alpha_1 + \alpha_1^2) + \frac{1}{2}(x_2^2 - 2x_2\alpha_2 + \alpha_2^2)$

$= \frac{1}{2}(x_1^2 + x_2^2) - (\alpha_1 x_1 + \alpha_2 x_2) + \frac{1}{2}(\alpha_1^2 + \alpha_2^2)$

$Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $q = \begin{bmatrix} -\alpha_1 \\ -\alpha_2 \end{bmatrix}$ $A = \begin{bmatrix} 1 & 1 \end{bmatrix}$ $b = 1$

$\begin{bmatrix} Q & A^T \\ A & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -q \\ b \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ y \end{bmatrix} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ 1 \end{bmatrix}$

$x_1 + y = \alpha_1$

$\Rightarrow x_2 + y = \alpha_2$

$x_1 + x_2 = \alpha_1$

$y = \alpha_1 - x_1 = \alpha_2 - x_2$

$\cancel{x_1 + x_2 = x_1} = \alpha_2 - x_2$

$2x_2 = \alpha_2$ $x_2 = \frac{\alpha_2}{2}$

$x_1 = \alpha_1 - \frac{\alpha_2}{2}$

$y = \alpha_2 - \frac{\alpha_2}{2} = \frac{\alpha_2}{2}$

$\boxed{x_1 = \alpha_1 - \frac{\alpha_2}{2} \quad x_2 = \frac{\alpha_2}{2} \quad y = \frac{\alpha_2}{2}}$

2b

b. $Q = \begin{bmatrix} 1 & 1 & -0.5 \\ 1 & 4 & 0 \\ -0.5 & 0 & 1 \end{bmatrix}$ $q = \begin{bmatrix} -2 \\ -1 \\ 2 \end{bmatrix}$ $A = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ $b = 1$

$$\begin{bmatrix} Q & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -q \\ b \end{bmatrix} = \begin{bmatrix} 1 & 1 & -0.5 & 1 \\ 1 & 4 & 0 & 1 \\ -0.5 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -2 \\ 1 \end{bmatrix}$$

$x_1 + x_2 - 0.5x_3 + y = 2$

$x_1 + 4x_2 + y = 1$

$-0.5x_1 + x_3 + y = -2$

$x_1 + x_2 + x_3 = 1$

$$\left[\begin{array}{cccc|c} 1 & 1 & -0.5 & 1 & 2 \\ 1 & 4 & 0 & 1 & 1 \\ -0.5 & 0 & 1 & 1 & -2 \\ 1 & 1 & 1 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & -0.2 \\ 0 & 0 & 1 & 0 & -0.8 \\ 0 & 0 & 0 & 1 & -0.2 \end{array}\right]$$

$\boxed{x_1 = 2,\ x_2 = -0.2,\ x_3 = -0.8,\ y = -0.2}$

2c

c. $Q = \begin{bmatrix} 1 & 1 & -0.5 & 0 \\ 1 & 4 & 0 & 0 \\ -0.5 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ $q = \begin{bmatrix} -2 \\ -1 \\ 2 \\ -2 \end{bmatrix}$ $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$$\begin{bmatrix} Q & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -q \\ b \end{bmatrix} = \begin{bmatrix} 1 & 1 & -0.5 & 0 & 1 & 1 \\ 1 & 4 & 0 & 0 & 1 & 1 \\ -0.5 & 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 & 1 & -1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & -1 & -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -2 \\ 2 \\ 1 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cccccc|c} 1 & 1 & -0.5 & 0 & 1 & 1 & 2 \\ 1 & 4 & 0 & 0 & 1 & 1 & 1 \\ -0.5 & 0 & 1 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 1 & 1 & -1 & 2 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0.5652 \\ 0 & 1 & 0 & 0 & 0 & 0 & -0.0652 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1.6086 \\ 0 & 0 & 0 & 1 & 0 & 0 & 2.1087 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0.2935 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0.4022 \end{array}\right]$$

$\boxed{x_1 = 0.5652,\ x_2 = -0.0652,\ x_3 = -1.6086,\ x_4 = 2.1087,\ y_1 = 0.2935,\ y_2 = 0.4022}$

3a

3a. $x_1 + x_2 + s_1 = 1$ $\quad x_1, s_1 \geq 0$ $\quad Ax = b$, $Gx \leq h$

$A = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ $\quad b = 1$

$$Q = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad p = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \quad h = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad G = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

```
#3a

from cvxopt import matrix, solvers

Q= matrix([[1.0, -1.0, 0.0],
             [-1.0, 2.0, 0.0],
             [0.0, 0.0, 0.0]])

p= matrix([1.0, -1.0, 0.0])

G= matrix([[-1.0, 0.0, 0.0],
           [0.0, 0.0, 0.0],
           [0.0, -1.0, 0.0]])

h=matrix([0.0, 0.0, 0.0])

A=matrix([1.0, 1.0, 1.0], (1, 3))

b=matrix(1.0)

sol = solvers.qp(Q, p, G, h, A, b)
print(sol["primal objective"])
print(sol["x"])
```

```
     pcost       dcost       gap    pres   dres
 0: -2.2222e-01 -5.5556e-01  4e+00  2e+00  1e+00
 1: -1.5511e-01 -4.7625e-01  3e-01  2e-02  1e-02
 2: -2.3105e-01 -2.5436e-01  2e-02  2e-04  1e-04
 3: -2.4975e-01 -2.5007e-01  3e-04  2e-06  1e-06
 4: -2.5000e-01 -2.5000e-01  3e-06  2e-08  1e-08
 5: -2.5000e-01 -2.5000e-01  3e-08  2e-10  1e-10
Optimal solution found.
-0.2499999752186499
[ 4.96e-08]
[ 5.00e-01]
[ 5.00e-01]
```

The optimal solution is x1=4.96e-08, x2=5.0e-01, s1=5.0e-01. The optimal value is -0.250 which is minimized by the optimal solution. This is the minimum of the objective function.

3b

b. $x_1+x_2+x_3 \geq 1$

$\Rightarrow x_1+x_2+x_3-s_1=1$ $\quad x_1, x_2, s_1, s_2 \geq 0$ $\quad -x_1 \leq 0$

$x_1 \leq 1 \Rightarrow x_1+s_2=1$ $\quad x_3 \leq 0$ $\quad -x_2 \leq 0$

$x_3 \leq 0$

$-s_1 \leq 0$

$-s_2 \leq 0$

$$A=\begin{bmatrix} 1 & 1 & 1 & -1 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \quad b=\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$Q=\begin{bmatrix} 1 & -1 & 0 & 0 & 0 \\ -1 & 1 & 2 & 0 & 0 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad p=\begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad h=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad G=\begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}$$

```python
#3b

from cvxopt import matrix, solvers

Q= matrix([[1.0, -1.0, 0.0, 0.0, 0.0],
           [-1.0, 1.0, 2.0, 0.0, 0.0],
           [0.0, 2.0, 2.0, 0.0, 0.0],
           [0.0, 0.0, 0.0, 0.0, 0.0],
```

```
             [0.0, 0.0, 0.0, 0.0, 0.0]])

p= matrix([1.0, 1.0, -1.0, 0.0, 0.0])

G= matrix([[-1.0, 0.0, 0.0, 0.0, 0.0],
           [0.0, -1.0, 0.0, 0.0, 0.0],
           [0.0, 0.0, 1.0, 0.0, 0.0],
           [0.0, 0.0, 0.0, -1.0, 0.0],
           [0.0, 0.0, 0.0, 0.0, -1.0]])

h=matrix([0.0, 0.0, 0.0, 0.0, 0.0])

A=matrix([1.0, 1.0, 1.0, -1.0, 0.0,
          1.0, 0.0, 0.0, 0.0, 1.0], (2, 5))

b=matrix([1.0, 1.0])

sol = solvers.qp(Q, p, G, h, A, b)
print(sol["primal objective"])
print(sol["x"])
```

```
     pcost       dcost       gap    pres   dres
 0:  1.0425e+00 -6.2500e-02  1e+01  3e+00  2e+00
 1:  1.1152e+00  2.1995e-01  9e-01  3e-02  2e-02
 2:  1.0288e+00  9.3043e-01  1e-01  2e-03  2e-03
Terminated (singular KKT matrix).
1.028796956467899
[ 3.66e-01]
[ 6.34e-01]
[-2.38e-01]
[ 1.28e+00]
[ 1.51e+00]
```

The optimal solution was found at x1=3.66e-01, x2=6.34e-01, x3=-2.38e-01, s1=1.28, s2=1.51. The optimal value (minimum of the objective function, minimized by the optimal solution) is 1.0288.

4a

4a $\bar{R}_1 = E[R_1] = 1.078$ $\bar{R}_2 = 1.1524$ $\bar{R}_3 = 1.15945$

$\bar{R}_4 = 1.16545$ $\bar{R}_5 = 1.0997$

$Z = R - \bar{R}$ $S = Z^T Z$

$$\min f(x) = -\sum_{j=1}^{5} \bar{R}_j x_j + \lambda x^T S x$$

$\lambda x^T S x = \frac{1}{2} x^T Q x$ $\lambda S = \frac{1}{2} Q$ $Q = 2\lambda S$

s.t. $\sum_{j=1}^{5} x_j = 1$

$x_j \geq 0,\ j = 1 \ldots 5$

$Q = 2\lambda S$

$$p = \begin{bmatrix} -R_1 \\ -R_2 \\ -R_3 \\ -R_4 \\ -R_5 \end{bmatrix} \quad A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \end{bmatrix} \quad b = 1$$

$$G = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \quad h = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$-x_j \leq 0$

Using R-studio to calculate the centered data matrix and sample covariance matrix:

```{r}
PortSelectData <- read.csv("PortSelectData.csv")

mu_1 <- mean(PortSelectData[, 2], na.rm=T)
mu_2 <- mean(PortSelectData[, 3], na.rm=T)
mu_3 <- mean(PortSelectData[, 4], na.rm=T)
mu_4 <- mean(PortSelectData[, 5], na.rm=T)
mu_5 <- mean(PortSelectData[, 6], na.rm=T)

paste("E[R_1]: ", mu_1)
paste("E[R_2]: ", mu_2)
paste("E[R_3]: ", mu_3)
paste("E[R_4]: ", mu_4)
paste("E[R_5]: ", mu_5)
```

```
[1] "E[R_1]:  1.078"
[1] "E[R_2]:  1.1524"
[1] "E[R_3]:  1.15945"
[1] "E[R_4]:  1.16545"
[1] "E[R_5]:  1.0997"
```

```{r}
clean_port <- PortSelectData[-c(21, 22), -c(1, 7, 8, 9)]
R_j <- c(mu_1, mu_2, mu_3, mu_4, mu_5)


for (i in 1:ncol(clean_port)){
  clean_port[, i]=clean_port[, i]-R_j[i]
}

X <- as.matrix(clean_port)

S <- 1/(nrow(clean_port)-1) * t(X) %*% X

colnames(S) <- NULL
rownames(S) <- NULL

S
```

```
              [,1]        [,2]         [,3]          [,4]         [,5]
[1,]  0.0010230526 0.000195000 0.0001472105 -0.0002766842 0.0003207895
[2,]  0.0001950000 0.018573621 0.0189497579  0.0195837053 0.0052181263
[3,]  0.0001472105 0.018949758 0.0201629974  0.0225924711 0.0045108789
[4,] -0.0002766842 0.019583705 0.0225924711  0.0326845763 0.0043572474
[5,]  0.0003207895 0.005218126 0.0045108789  0.0043572474 0.0064332737
```

4b

```
#4 lambda = 1
from cvxopt import matrix, solvers

lamb = 1

Q=  2* lamb * matrix([[0.0010230526, 0.000195000, 0.0001472105, -0.0002766842,
0.0003207895],
                    [0.0001950000, 0.018573621, 0.0189497579,  0.0195837053,
0.0052181263],
                    [0.0001472105, 0.018949758, 0.0201629974,  0.0225924711,
0.0045108789],
                    [-0.0002766842, 0.019583705, 0.0225924711,  0.0326845763,
0.0043572474],
                    [0.0003207895, 0.005218126, 0.0045108789,  0.0043572474,
0.0064332737]])

p= matrix([-1.07800, -1.15240, -1.15945, -1.16545, -1.09970])

G= matrix([[-1.0, 0.0, 0.0, 0.0, 0.0],
           [0.0, -1.0, 0.0, 0.0, 0.0],
           [0.0, 0.0, -1.0, 0.0, 0.0],
           [0.0, 0.0, 0.0, -1.0, 0.0],
           [0.0, 0.0, 0.0, 0.0, -1.0]])

h=matrix([0.0, 0.0, 0.0, 0.0, 0.0])

A=matrix([1.0, 1.0, 1.0, 1.0, 1.0], (1, 5))

b=matrix(1.0)

sol = solvers.qp(Q, p, G, h, A, b)
print(sol["primal objective"])
print(sol["x"])
```

```
     pcost       dcost       gap    pres   dres
 0: -1.1244e+00 -2.1419e+00  1e+00  7e-16  1e+00
 1: -1.1253e+00 -1.1515e+00  3e-02  1e-16  3e-02
 2: -1.1355e+00 -1.1399e+00  4e-03  1e-16  2e-03
 3: -1.1391e+00 -1.1398e+00  7e-04  3e-16  2e-16
 4: -1.1393e+00 -1.1393e+00  2e-05  2e-16  2e-16
 5: -1.1393e+00 -1.1393e+00  6e-07  1e-16  1e-16
Optimal solution found.
-1.1393294355920018
[ 5.02e-07]
[ 9.52e-07]
[ 9.25e-01]
[ 7.49e-02]
[ 6.95e-07]
```

For lambda=1 as the penalization/tuning parameter on the variance of return, the optimal allocation of the investment in the portfolio is to put 5.02e-07 of the investment in US 3-month T-Bills, 9.52e-07 of the investment in S&P 500, 9.25e-01 of the investment into Wilshire 5000, 7.49e-02 of the investment into NASDAQ Composite, and 6.95e-07 of the investment into EAFE. The optimal value is -1.1393 which means that the previous allocation along with the penalization of lambda=1 minimized the objective function to get to this value.