Lecture_02

September 15, 2022

1 Microstates versus Macrostates

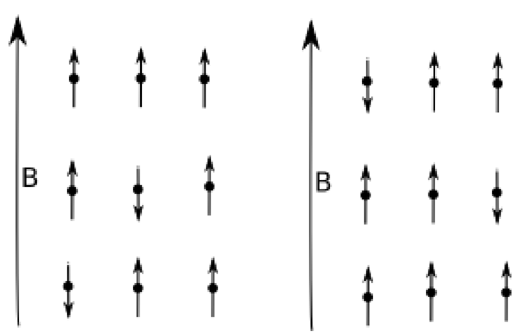
Imagine we have some dipoles in a magnetic field. Each dipole contributes an interaction energy of $-\mu \cdot B$. Quantum mechanics tells us that there are only specific orientations of the dipole in the magnetic field allowed. This is simplest when the dipoles possess angular momentum of $\hbar/2$, which is referred to as "spin 1/2".

The dipoles can thus line up in one of two directions: parallel, or anti-parallel to the magnetic field. This leads to an interaction energy of $\mp\mu B$ per dipole. Imagine now we have N dipoles. If n are oriented parallel to the field, then $(N-n)$ must be anti-parallel. Thus, the total energy of the system is given by:

$$E(n)n(-\mu B) + (N-n)(\mu B) = (N-2n)\mu B$$

This means that if we have $N = 9$, and $n = 7$, we would have an interaction energy of $-6\mu B$. So why is this interesting

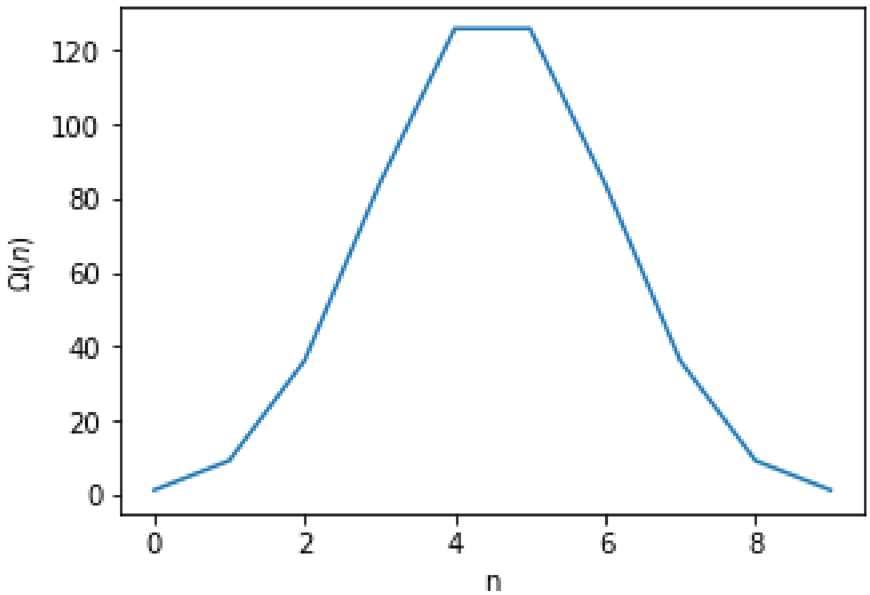
Well, consider examples A and B below. The both have two dipoles which are oriented anti-parallel to the magnetic field, while the other 7 are all in alignment. From a microscopic point of view, the two scenarios are different, as the individual atoms which are anti-parallel are different in both cases. However, from a macroscopic point of view, the interaction energy of both with the magnetic field (which is what we can measure) are the same. These means that a **given macrostate can be described by a large number of equally likely of microstates**.



For the above example, there are 2^N microstates. The number of microstates which can give the same interaction energy $E(n)$ is

$$\Omega(n) = \frac{N!}{n!(N-n)!}$$

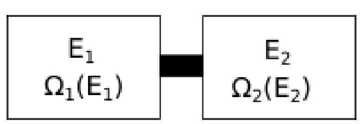
This is referred to as the statistical weight. It's worth considering exactly what it means - see Section 1.4 of Blundell & Blundell for a more robust discussion of where this term comes from. For the moment, the below plot shows the statistical weight for our above toy problem, where we vary n.



What this shows us is that the statistical weight is highest for when n is between 4 and 5. This should make sense as this is when half of the dipoles are oriented parallel to the B field, and half anti-parallel, and so we have the most freedom with placing them in the grid. This also means that the macrostate given by $n = 4/5$ is the most probable state to occur, as these macrostates have the largest number of microstates.

2 Temperature

We can now use the above expresssion for statistical weight to arrive at a defintion of the temperature of an object. First, let's consider 2 bodies which are in thermal contact.



For the following, we are going to assume that this is a closed system (that is, we're not losing any energy to the surroundings) and so, at all times, the total energy of the system is given by $E = E_1 + E_2$. As such, all we need to do is specify E_1 and the macrostate of the system will be determined. The first system can be in any one of $\Omega_1(E_1)$ microstates and the second system can be in any one of $\Omega_2(E_2)$ microstates. Thus, the system as a whole can be in any number of $\Omega_1(E_1)\Omega_2(E_2)$ microstates.

Now, if we let the system reach thermal equilibrium, then the system will appear to be in the macrostate which maximises the number of microstates (consider the above graph to convince yourself of this). As such, the most probable division of energy between the two systems is the one which maximises $\Omega_1(E_1)\Omega_2(E_2)$. So we end up with

$$\frac{\mathrm{d}}{\mathrm{d}E_1}\left(\Omega_1(E_1)\Omega_2(E_2)\right) = 0$$

This gives

$$\Omega_2(E_2)\frac{\mathrm{d}\Omega_1(E_1)}{\mathrm{d}E_1} + \Omega_1(E_1)\frac{\mathrm{d}\Omega_2(E_2)}{\mathrm{d}E_2}\frac{\mathrm{d}E_2}{\mathrm{d}E_1} = 0$$

Now recalling that $E = E_1 + E_2$ and is fixed gives $\mathrm{d}E_1 = -\mathrm{d}E_2$, which in turn gives $\frac{\mathrm{d}E_1}{\mathrm{d}E_2} = -1$. This simplifies our above expression to

$$\Omega_2(E_2)\frac{\mathrm{d}\Omega_1(E_1)}{\mathrm{d}E_1} + \Omega_1(E_1)\frac{\mathrm{d}\Omega_2(E_2)}{\mathrm{d}E_2}\frac{\mathrm{d}E_2}{\mathrm{d}E_1} = 0$$

$$\frac{1}{\Omega_1}\frac{\mathrm{d}\Omega_1}{\mathrm{d}E_1} - \frac{1}{\Omega_2}\frac{\mathrm{d}\Omega_2}{\mathrm{d}E_2} = 0$$

from which we get

$$\frac{\mathrm{d}\ln\Omega_1}{\mathrm{d}E_1} = \frac{\mathrm{d}\ln\Omega_2}{\mathrm{d}E_2}$$

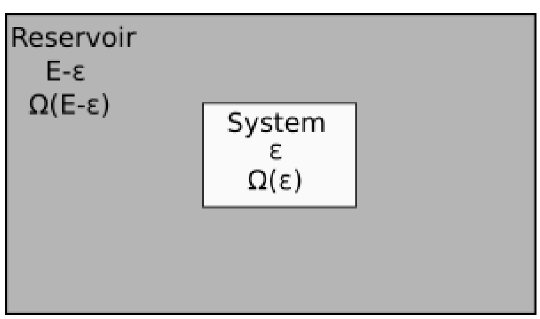
This condition defines the most likely division of energy between the objects which we would call "being at the same temperature". As such, we can now define

$$\frac{1}{k_{\mathrm{B}}T} = \frac{\mathrm{d}\ln\Omega}{\mathrm{d}E}$$

where the choice of $k_{\mathrm{B}}T$ will be motivated later.

3 The Boltzmann Distribution

Ok, now let's consider the above example again. However, we're going to change one of the systems such that it acts as a reservoir (or heat bath). We then place the other system into the reservoir. The reservoir has such an enormour amount of energy compared to the system that any change in its temperature as it heats the smaller system is completely negligible.



Now, we're going to assume that for each energy of the system, there is only a single allow microstate. As such, $\Omega(\epsilon) = 1$.

So, the probability that the system has energy ϵ is proportional to the number of microstates of the reservoir times the number of microstates of the system:

$$P(\epsilon) \propto \Omega(E - \epsilon)\Omega(\epsilon) = \Omega(E - \epsilon)$$