AAE 590ACA PS1.A

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AI Disclaimer: ChatGPT was used to plot the moon path when plotting trajectories in the ECI frame. This edit was adopted to show the spacecraft trajectory more clearly in relation to the Earth and Moon in ECI.

I. PS1.A: Three-Body Problem

A. PS1.B Part a

Given: Table 7 provides the dynamical parameters used in the Earth-Moon system. Table 8 provides the non-dimensional initial conditions used in the circular restricted three-body problem (CR3BP) simulation.

Table 1 Assumed dynamical parameter values

Parameter	Symbol	Value	Unit
Earth-Moon distance	$d_{\text{Earth-Moon}}$	3.8475×10^5	km
Earth-Moon barycenter GM	GM_*	4.0350×10^5	km^3/s^2
Mass ratio	μ	1.2151×10^{-2}	-

Table 2 Assumed dynamical parameter values

IC #	x_0	y_0	z_0	$\dot{x}_0$	$\dot{y}_0$	$\dot{z}_0$	Propagation Time
IC-1	1.2	0	0	0	-1.06110124	0	6.20628
IC-2	0.85	0	0.17546505	0	0.2628980369	0	2.5543991
IC-3	0.05	-0.05	0	4.0	2.6	0	15.0

Assumptions: System is simulated using CR3BP. Assumes the Moon and Earth have circular trajectories around the common barycenter.

Find: Simulate the trajectory using CR3BP with the initial conditions given in Table 8. Plot the dimensional trajectories in the synodic frame and include Earth and Moon location markers.

The equations of motion describing CR3BP are given by the following:

$$
\begin{aligned}
\ddot{x} &= 2\dot{y} + x - \frac{(1-\mu)(x+\mu)}{r_{13}^3} - \frac{\mu(x-1+\mu)}{r_{23}^3} \\
\ddot{y} &= -2\dot{x} + y - \frac{(1-\mu)\,y}{r_{13}^3} - \frac{\mu y}{r_{23}^3} \\
\ddot{z} &= -\frac{(1-\mu)\,z}{r_{13}^3} - \frac{\mu z}{r_{23}^3}
\end{aligned} \tag{1}
$$

where $r_{13} = \|\mathbf{r} - \mathbf{r}_1\| = \sqrt{(x+\mu)^2 + y^2 + z^2}$ and $r_{23} = \|\mathbf{r} - \mathbf{r}_2\| = \sqrt{(x-1+\mu)^2 + y^2 + z^2}$. $\mathbf{r}$ is the spacecraft position vector, $\mathbf{r}_1$ is the first body position vector, $\mathbf{r}_2$ is the second body position vector all measured from the barycenter, and $\mu = \frac{m_2}{m_1+m_2}$ is the mass ratio. All quantities in Equation 1 are non-dimensional and expressed in the synodic frame. The non-dimensional quantities can be dimensionalized by the following:

$$ {}^{R}\mathbf{r}_{\text{dim}} = l_*^{R}\mathbf{r} \tag{2} $$

$$ t = t_*\tau \tag{3} $$

$$ {}^{R}\mathbf{v}_{\text{dim}} = \frac{l_*}{t_*}{}^{R}\mathbf{v} \tag{4} $$

where ${}^{R}\mathbf{r}_{\text{dim}}$, t, and ${}^{R}\mathbf{v}_{\text{dim}}$ are the spacecraft's dimensional position vector, time, and velocity vector in the rotating frame respectively, $l_* = \|\mathbf{r}_{1,\text{dim}}\| + \|\mathbf{r}_{2,\text{dim}}\|$ is the characteristic length, $t_* = \sqrt{\frac{l_*^3}{GM_*}}$, is the characteristic time, and τ is the non-dimensional propagation time. Converting the state to ECI is performed by a single translation and rotation as described by the following:

$$ \theta = \frac{t}{t_*} \tag{5} $$

$$ R = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{6} $$

$$ {}^{I}\mathbf{r}_{\text{dim}} = R\left({}^{R}\mathbf{r}_{\text{dim}} + \begin{bmatrix} -\mu & 0 & 0 \end{bmatrix}^T\right) \tag{7} $$

$$ {}^{I}\mathbf{v}_{\text{dim}} = R\left({}^{R}\mathbf{v}_{\text{dim}} + \begin{bmatrix} 0 & 0 & \frac{1}{t_*} \end{bmatrix}^T \times {}^{R}\mathbf{r}_{\text{dim}}\right) \tag{8} $$

where θ is the angular position of the Moon, ${}^{I}\mathbf{r}_{\text{dim}}$ and ${}^{I}\mathbf{v}_{\text{dim}}$ are the dimensional position and velocity vectors in ECI respectively. The position vector is shifted by $-\mu$ to account for the barycenter location.

Figure 1 - 3 below show the spacecraft trajectory using CR3BP in the synodic and Earth-Centered Inertial (ECI) frames using the initial conditions given in Table 8.

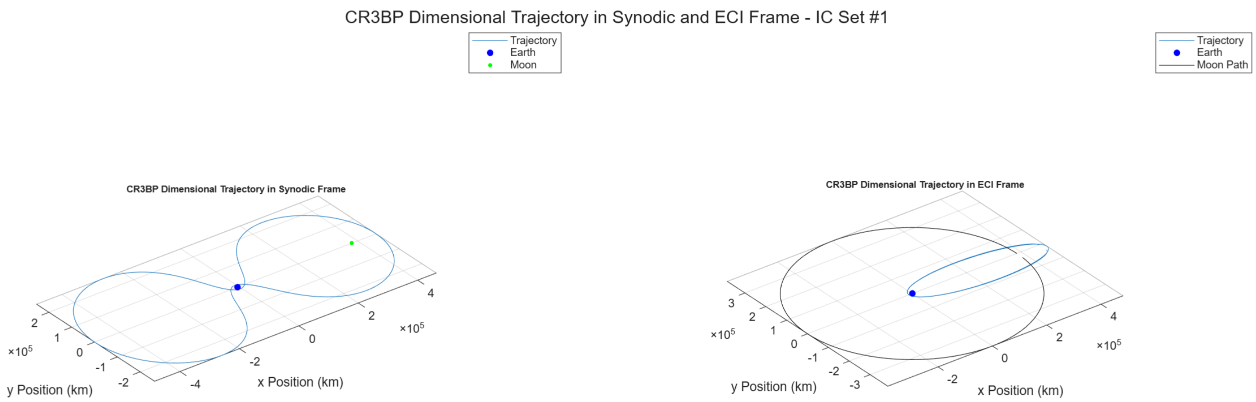

Fig. 1 Initial Conditions Set 1 simulated using CR3BP and represented in the synodic and ECI frames.

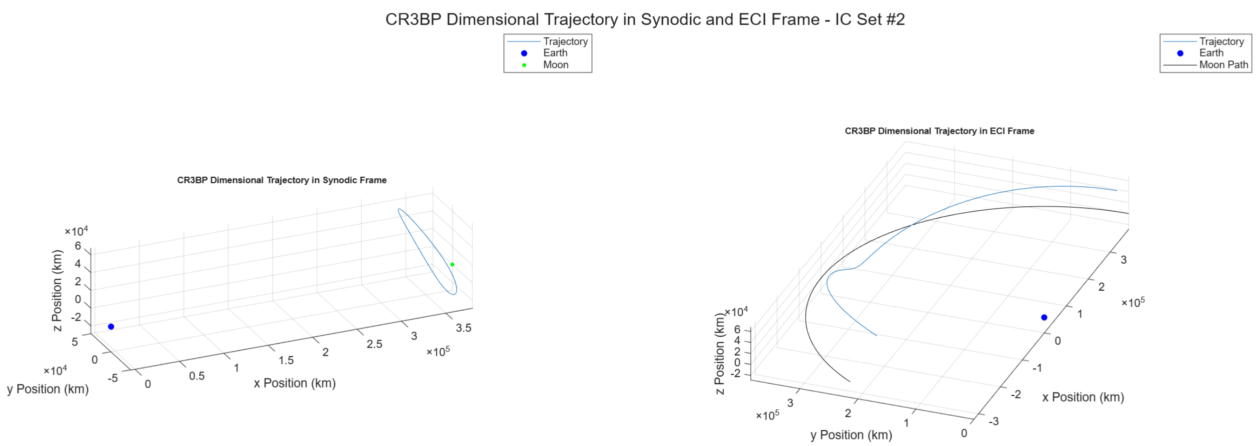

Fig. 2 Initial Conditions Set 2 simulated using CR3BP and represented in the synodic and ECI frames.

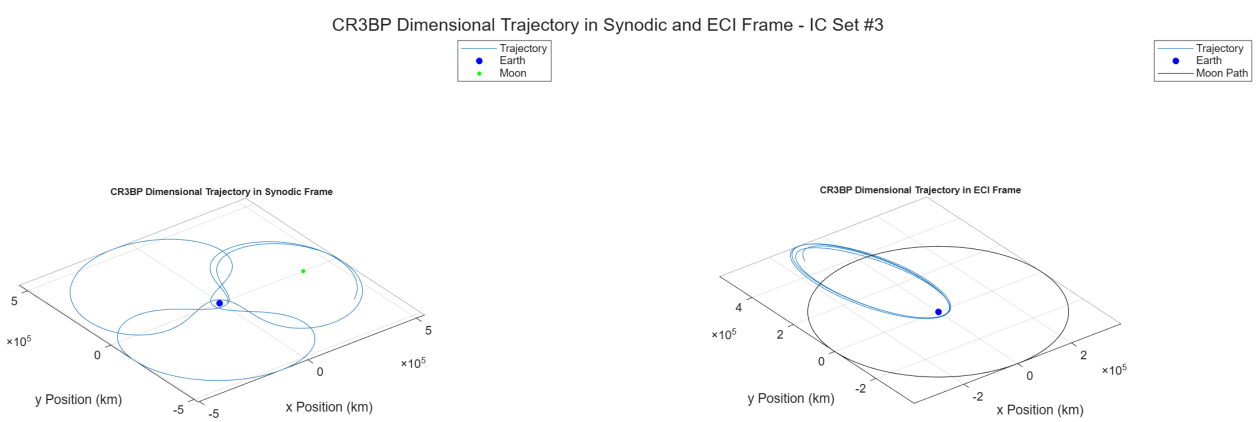

Fig. 3 Initial Conditions Set 3 simulated using CR3BP and represented in the synodic and ECI frames.

B. PS1.B Part b

Given: Table 7 provides the dynamical parameters used in the Earth-Moon system. Table 8 provides the non-dimensional initial conditions used in the circular restricted three-body problem (CR3BP) simulation. Re-define the

ECI frame such that $\hat{\mathbf{n}}_1$ is aligned with the Earth-Moon line at the eopch and $\hat{\mathbf{n}}_3$ is aligned with the normal vector of the Earth-Moon orbital plane. Use $\mu_{\text{moon}} = 4.9028 \times 10^3$ km^3/s^2 and $\mu_{\text{earth}} = 3.986 \times 10^5$ km^3/s^2 for the Moon and Earth's gravitational parameters respectively.

Table 3 Assumed dynamical parameter values

Parameter	Symbol	Value	Unit
Earth-Moon distance	$d_{\text{Earth-Moon}}$	3.8475×10^5	km
Earth-Moon barycenter GM	GM_*	4.0350×10^5	km^3/s^2
Mass ratio	μ	1.2151×10^{-2}	-

Table 4 Assumed dynamical parameter values

IC #	x_0	y_0	z_0	$\dot{x}_0$	$\dot{y}_0$	$\dot{z}_0$	Propagation Time
IC-1	1.2	0	0	0	-1.06110124	0	6.20628
IC-2	0.85	0	0.17546505	0	0.2628980369	0	2.5543991
IC-3	0.05	-0.05	0	4.0	2.6	0	15.0

Assumptions: System is simulated in ECI using third-body perturbations. Assumes the Moon has a circular orbit around Earth, with Earth at the origin of the coordinate system.

Find: Simulate the trajectory using third-body perturbations with the initial conditions given in Table 8. Plot the dimensional trajectories in the synodic and ECI frames and include Earth and Moon location markers.

Third-body perturbations with respect to the ECI frame is given by the following:

$$\mathbf{a}_{\text{moon}} = -\mu_{\text{moon}} \left(\frac{\mathbf{r} - \mathbf{r}_{\text{moon}}}{\|\mathbf{r} - \mathbf{r}_{\text{moon}}\|_2^3} + \frac{\mathbf{r}_{\text{moon}}}{\|\mathbf{r}_{\text{moon}}\|_2^3} \right) \tag{9}$$

where $\mathbf{a}_{\text{moon}}$ are the third-body perturbations in ECI, $\mathbf{r}$ is the spacecraft position vector in ECI, and $\mathbf{r}_{\text{moon}}$ is the moon's position vector in ECI. The simulated trajectory using the perturbations in Equation 9 can be converted to the dimensional synodic frame by the following:

$$\theta = \frac{t}{t_*} \tag{10}$$

$$R = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{11}$$

$${}^{R}\mathbf{r}_{\text{dim}} = R\,{}^{I}\mathbf{r}_{\text{dim}} + \begin{bmatrix} -\mu l_* & 0 & 0 \end{bmatrix}^T \tag{12}$$

$$
{}^{R}\mathbf{v}_{\text{dim}} = R^{I}\mathbf{v}_{\text{dim}} - \begin{bmatrix} 0 & 0 & \frac{1}{t_*} \end{bmatrix}^{T} \times {}^{R}\mathbf{r}_{\text{dim}} \tag{13}
$$

Figure 4 - 6 below show the spacecraft trajectory using third-body perturbations in the synodic and Earth-Centered Inertial (ECI) frames using the initial conditions given in Table 8.

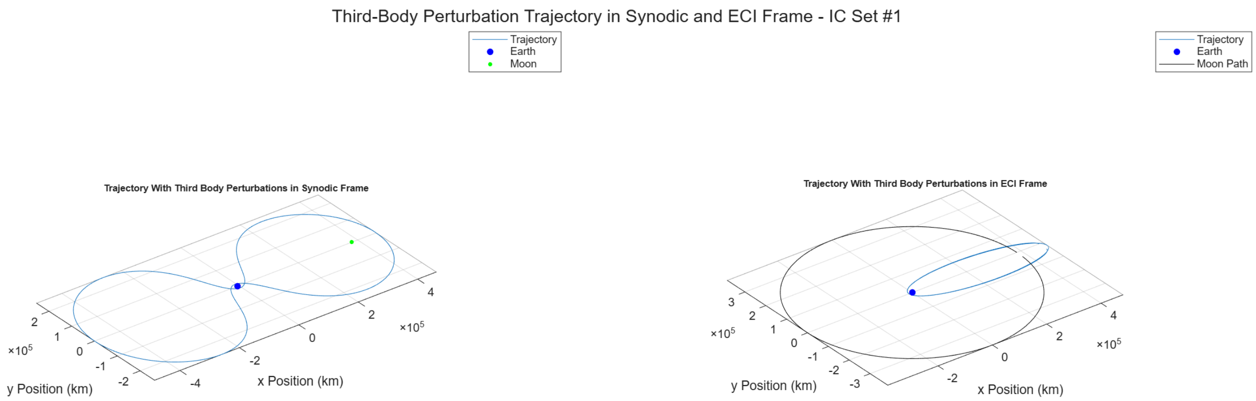

Fig. 4 Initial Conditions Set 1 simulated using third-body perturbations and represented in the synodic and ECI frames.

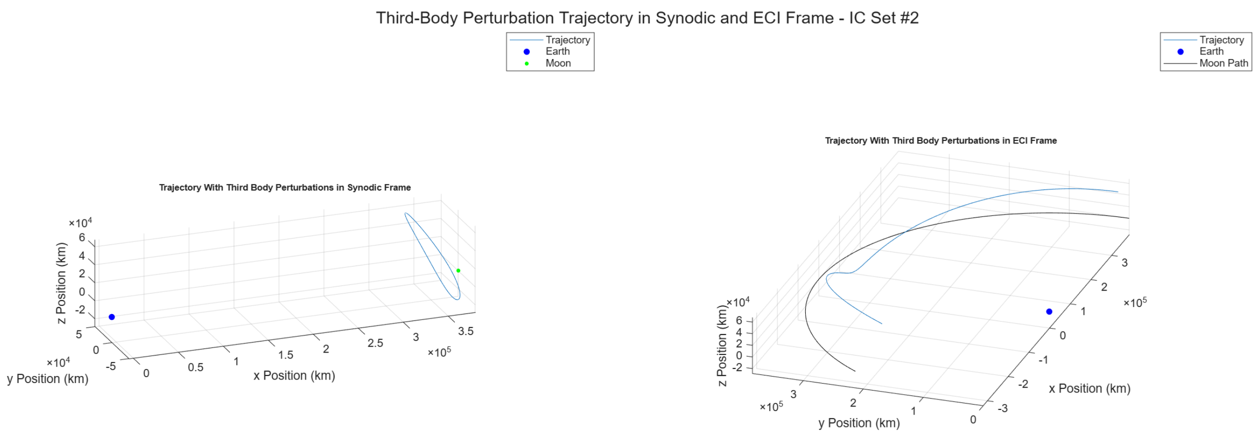

Fig. 5 Initial Conditions Set 2 simulated using third-body perturbations and represented in the synodic and ECI frames.

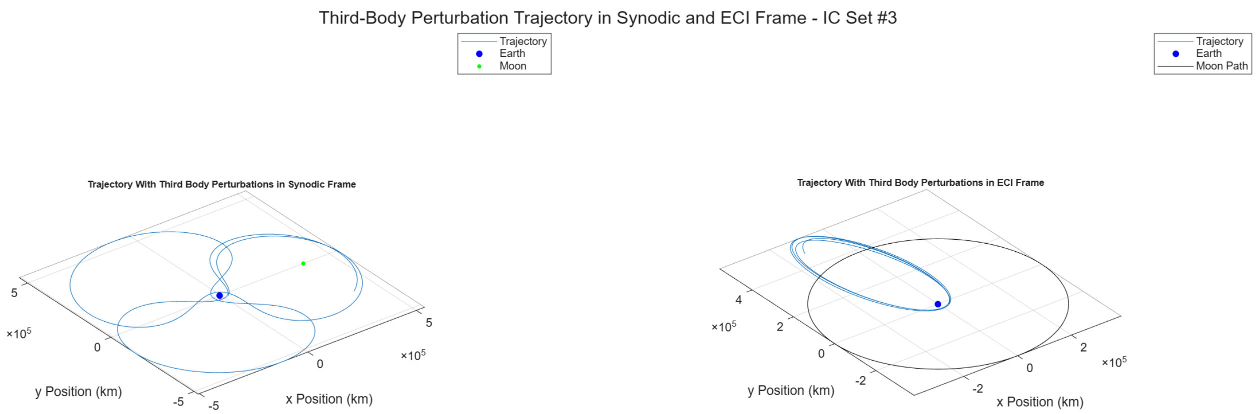

Fig. 6 Initial Conditions Set 3 simulated using third-body perturbations and represented in the synodic and ECI frames.

C. PS1.B Part C

Given: Table 7 provides the dynamical parameters used in the Earth-Moon system. Table 8 provides the non-dimensional initial conditions used in the circular restricted three-body problem (CR3BP) simulation. Re-define the ECI frame such that $\hat{\mathbf{n}}_1$ is aligned with the Earth-Moon line at the eopch and $\hat{\mathbf{n}}_3$ is aligned with the normal vector of the Earth-Moon orbital plane. Use $\mu_{\text{moon}} = 4.9028 \times 10^3$ km^3/s^2 and $\mu_{\text{earth}} = 3.986 \times 10^5$ km^3/s^2 for the Moon and Earth's gravitational parameters respectively.

Table 5 Assumed dynamical parameter values

Parameter	Symbol	Value	Unit
Earth-Moon distance	$d_{\text{Earth-Moon}}$	3.8475×10^5	km
Earth-Moon barycenter GM	GM_*	4.0350×10^5	km^3/s^2
Mass ratio	μ	1.2151×10^{-2}	-

Table 6 Assumed dynamical parameter values

IC #	x_0	y_0	z_0	$\dot{x}_0$	$\dot{y}_0$	$\dot{z}_0$	Propagation Time
IC-1	1.2	0	0	0	-1.06110124	0	6.20628
IC-2	0.85	0	0.17546505	0	0.2628980369	0	2.5543991
IC-3	0.05	-0.05	0	4.0	2.6	0	15.0

Assumptions: Trajectories are simulated using both CR3BP and third-body perturbations. Assumes Earth and Moon have circular trajectories around the barycenter for CR3BP. Assumes the Moon has a circular orbit around Earth, with Earth at the origin of the coordinate system for third-body perturbations.

Find: Demonstrate the discrepancies between the CR3BP and third-body perturbations and discuss possible causes for those discrepancies.

Figure 7 - 12 show the differences between the simulated position and velocity for each set of initial conditions in

the ECI frame.

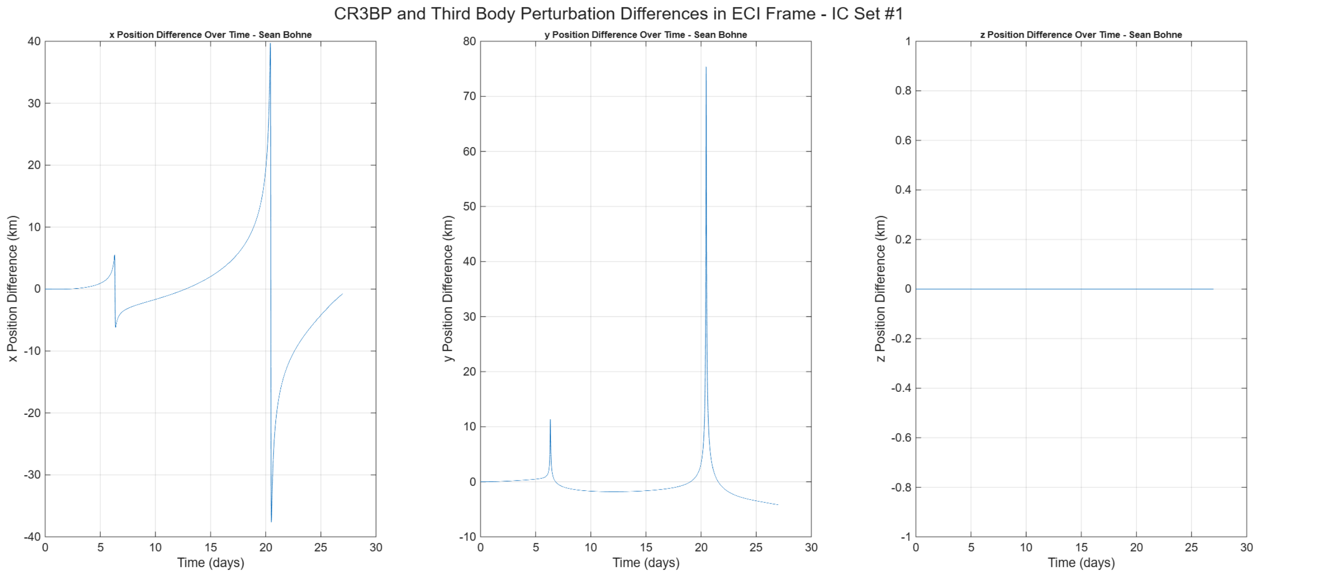

Fig. 7 Initial Conditions Set 1 ECI position differences between CR3BP and third-body perturbation simulations.

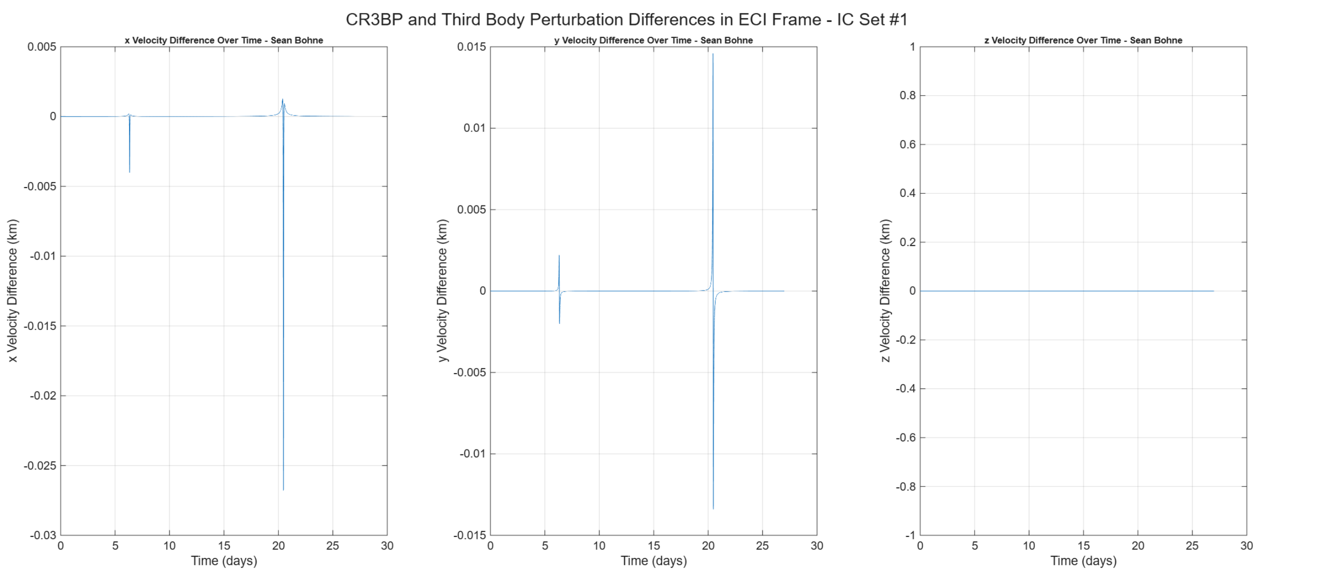

Fig. 8 Initial Conditions Set 1 ECI velocity differences between CR3BP and third-body perturbation simulations.

Figure 7 and Figure 8 show the differences between the CR3BP and third-body perturbation simulations in the ECI frame using initial conditions set 1. Since the motion is constricted to the x-y plane, the differences in the z positions will always be zero. However, the differences spike twice in both the x and y positions, likely due to the mismatch in the Moon's position in the CR3BP and third-body simulations. As a result, any large increases in spacecraft velocity amplify the position mismatch, leading to increased deviations in the simulated position and velocities. However, the differences are relatively small compared to the orbit scale because the first set of initial conditions do not have extreme velocities.

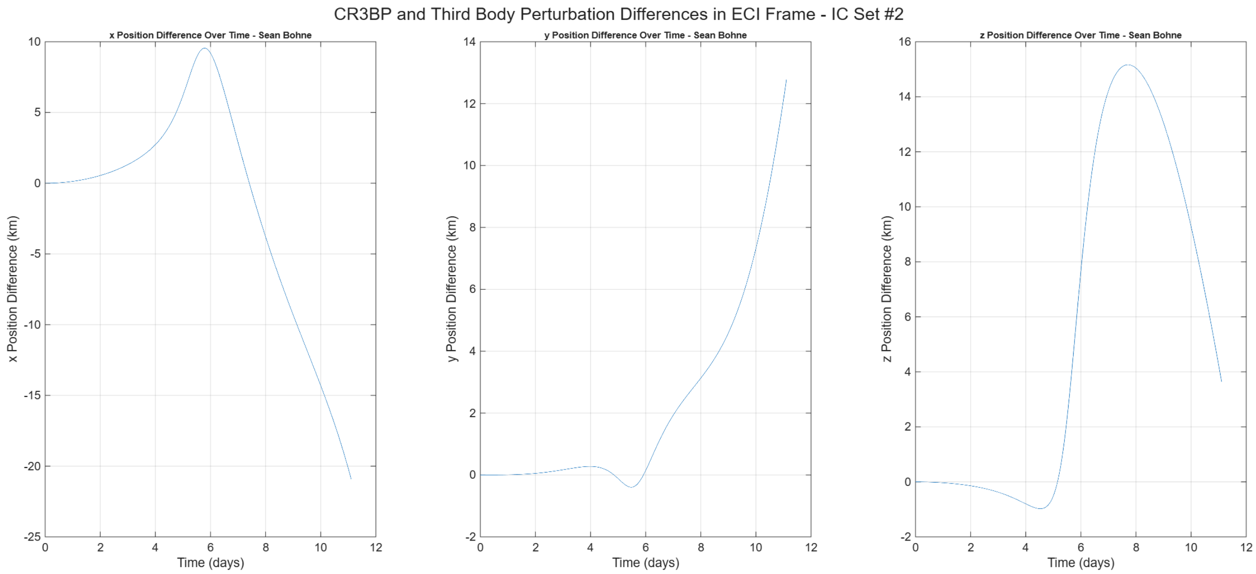

Fig. 9 Initial Conditions Set 2 ECI position differences between CR3BP and third-body perturbation simulations.

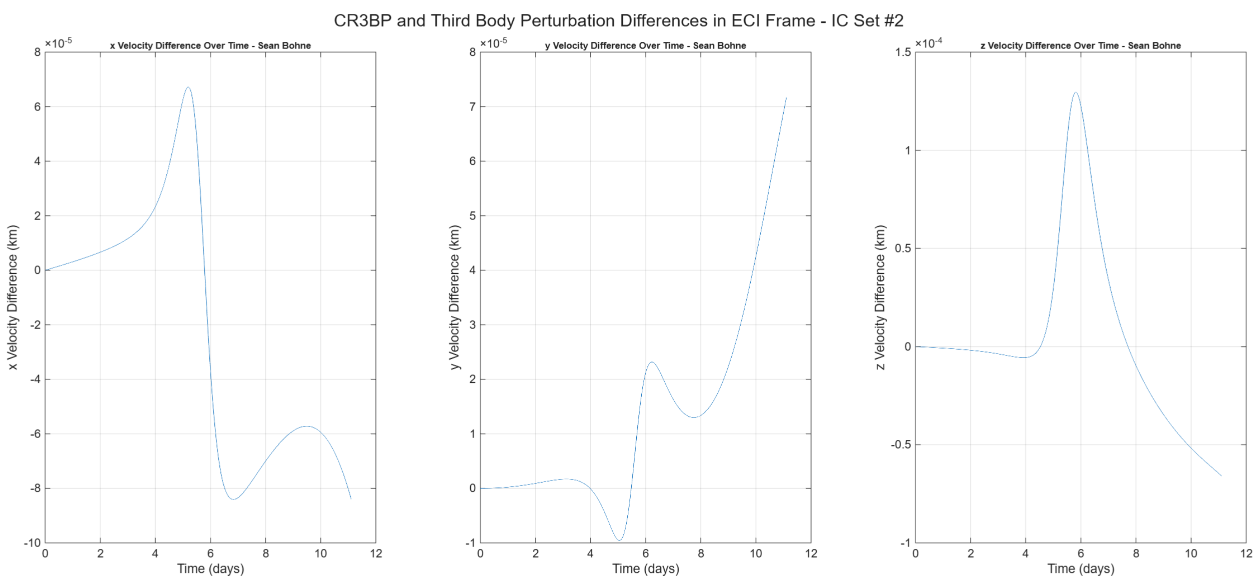

Fig. 10 Initial Conditions Set 2 ECI velocity differences between CR3BP and third-body perturbation simulations.

Figure 9 and Figure 10 show the differences between the CR3BP and third-body perturbation simulations in the ECI frame using initial conditions set 2. The differences are much smaller compared to initial conditions set 1 because the initial velocity is much smaller. As a result, the Moon's position mismatch between the CR3BP and third-body simulations do not lead to large differences in the simulated position and velocity.

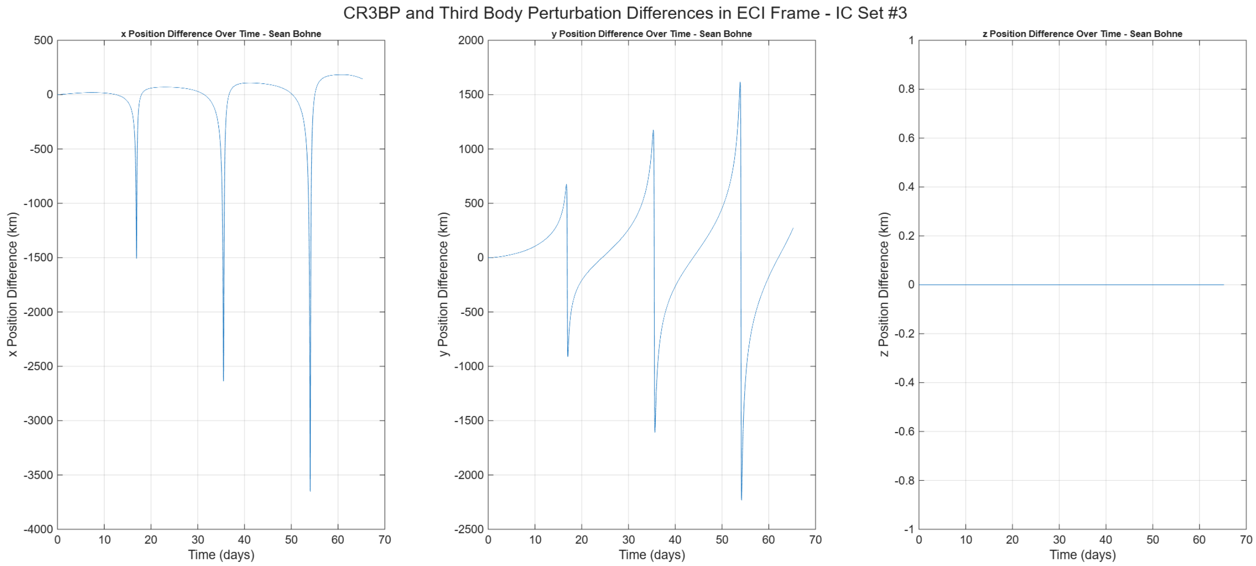

Fig. 11 Initial Conditions Set 3 ECI position differences between CR3BP and third-body perturbation simulations.

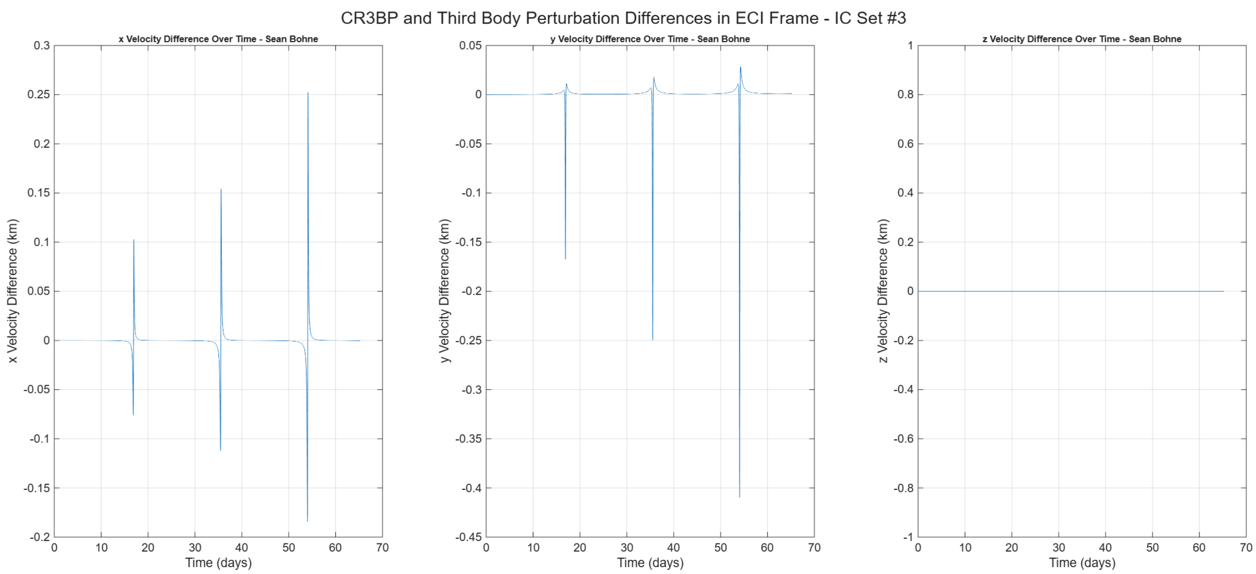

Fig. 12 Initial Conditions Set 3 ECI velocity differences between CR3BP and third-body perturbation simulations.

Figure 11 and Figure 12 show the differences between the CR3BP and third-body perturbation simulations in the ECI frame using initial conditions set 3. The differences are significant due to the large initial velocity. The spacecraft approaches the Earth three times, each time traveling at high speeds. These approaches lead to large spikes in the position and velocity differences and is clearly shown by Figure 11 and Figure 12.

D. PS1.B Part D

Given: Table 7 provides the dynamical parameters used in the Earth-Moon system. Table 8 provides the non-dimensional initial conditions used in the circular restricted three-body problem (CR3BP) simulation. Re-define the ECI frame such that $\hat{\mathbf{n}}_1$ is aligned with the Earth-Moon line at the eopch and $\hat{\mathbf{n}}_3$ is aligned with the normal vector of the Earth-Moon orbital plane. Use $\mu_{\text{moon}} = 4.9028 \times 10^3$ km^3/s^2 and $\mu_{\text{earth}} = 3.986 \times 10^5$ km^3/s^2 for the Moon and Earth's gravitational parameters respectively. Use $r_{\text{earth}} = 6378.1$ km for Earth radius and $J_2 = 1.0826 \times 10^{-3}$ for J2 perturbation parameter.

Table 7 Assumed dynamical parameter values

Parameter	Symbol	Value	Unit
Earth-Moon distance	$d_{\text{Earth-Moon}}$	3.8475×10^5	km
Earth-Moon barycenter GM	GM_*	4.0350×10^5	km^3/s^2
Mass ratio	μ	1.2151×10^{-2}	-

Table 8 Assumed dynamical parameter values

IC #	x_0	y_0	z_0	$\dot{x}_0$	$\dot{y}_0$	$\dot{z}_0$	Propagation Time
IC-1	1.2	0	0	0	-1.06110124	0	6.20628
IC-2	0.85	0	0.17546505	0	0.2628980369	0	2.5543991
IC-3	0.05	-0.05	0	4.0	2.6	0	15.0

Assumptions: Trajectories are simulated using third-body and J2 perturbations. Assumes Assumes the Moon has a circular orbit around Earth, with Earth at the origin of the coordinate system for third-body perturbations.

Find: Demonstrate the discrepancies between third-body perturbations with and without J2 perturbations and discuss possible causes for those discrepancies.

The effects of J2 perturbations can be characterized by including the following acceleration term in ECI:

$$\mathbf{a}_{J2} = -\frac{3\mu_{\text{earth}} J_2 r_o^2}{2r^5}\left[\left(1 - 5\frac{z^2}{r^2}\right) x\hat{\mathbf{n}}_1 + \left(1 - 5\frac{z^2}{r^2}\right) y\hat{\mathbf{n}}_2 + \left(3 - 5\frac{z^2}{r^2}\right) z\hat{\mathbf{n}}_3\right] \tag{14}$$

where $\mathbf{a}_{J2}$ is the J2 perturbation acceleration vector, $r_0 = 6378.1$ km is the radius of the Earth, and $r = \|\mathbf{r}\|_2$. $\mathbf{r}$ is the spacecraft position vector in ECI. Figure 13 - 18 show the differences between the simulated position and velocity under third-body perturbations with and without J2 effects for each set of initial conditions in the ECI frame.

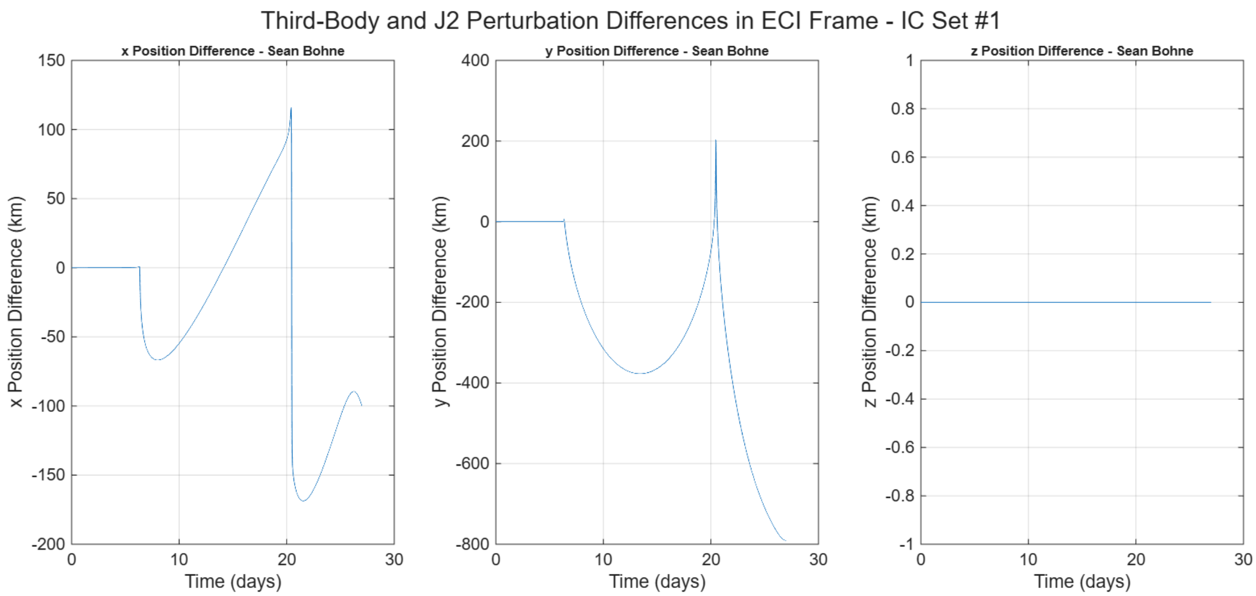

Fig. 13 Initial Conditions Set 1 ECI position differences between third-body perturbation simulations with and without J2 perturbations.

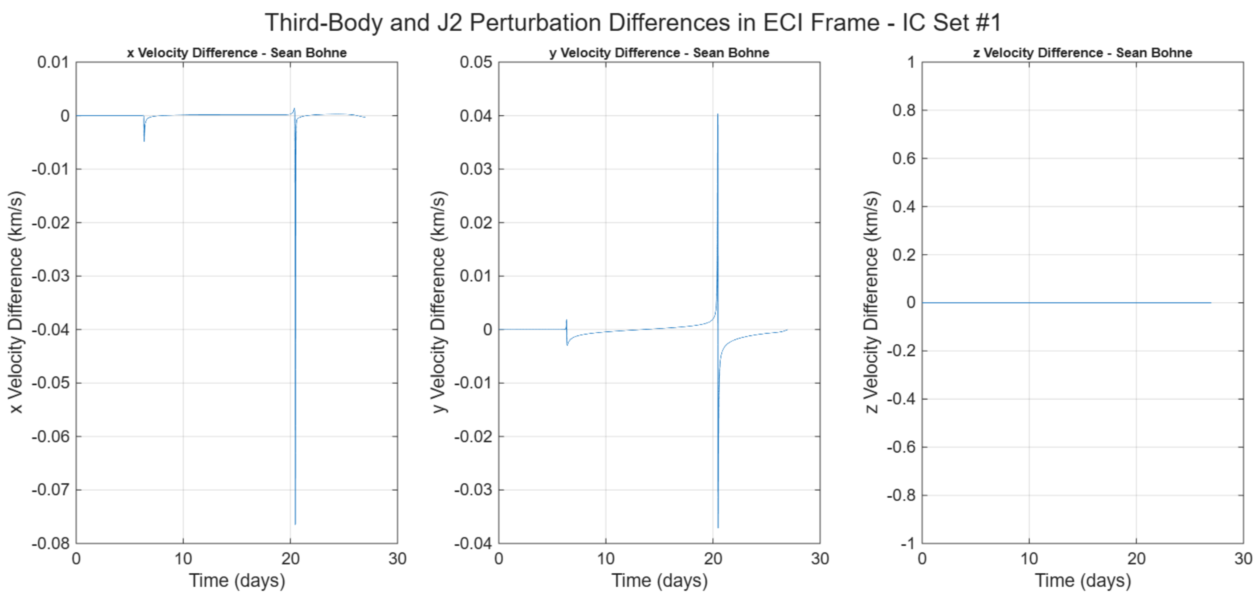

Fig. 14 Initial Conditions Set 1 ECI velocity differences between third-body perturbation simulations with and without J2 perturbations.

Figure 13 and Figure 14 show the differences between the third-body perturbation simulations with and without J2 effects in the ECI frame using initial conditions set 1. Since the motion is constricted to the x-y plane, the differences in the z positions will always be zero. However, the differences spike as the spacecraft speeds up on its approach around Earth, leading to increased mismatch between the two simulations. Relative to the orbit scale, the differences are small because the initial conditions do not have extreme velocities and the initial position of the spacecraft is far from Earth.

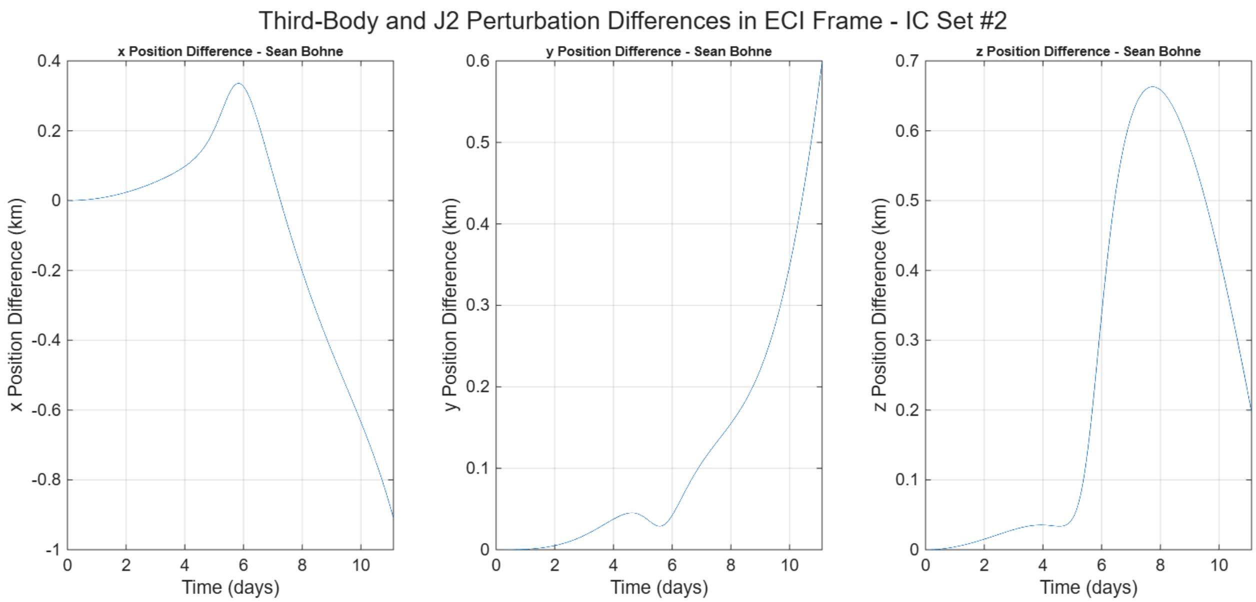

Fig. 15 Initial Conditions Set 2 ECI position differences between third-body perturbation simulations with and without J2 perturbations.

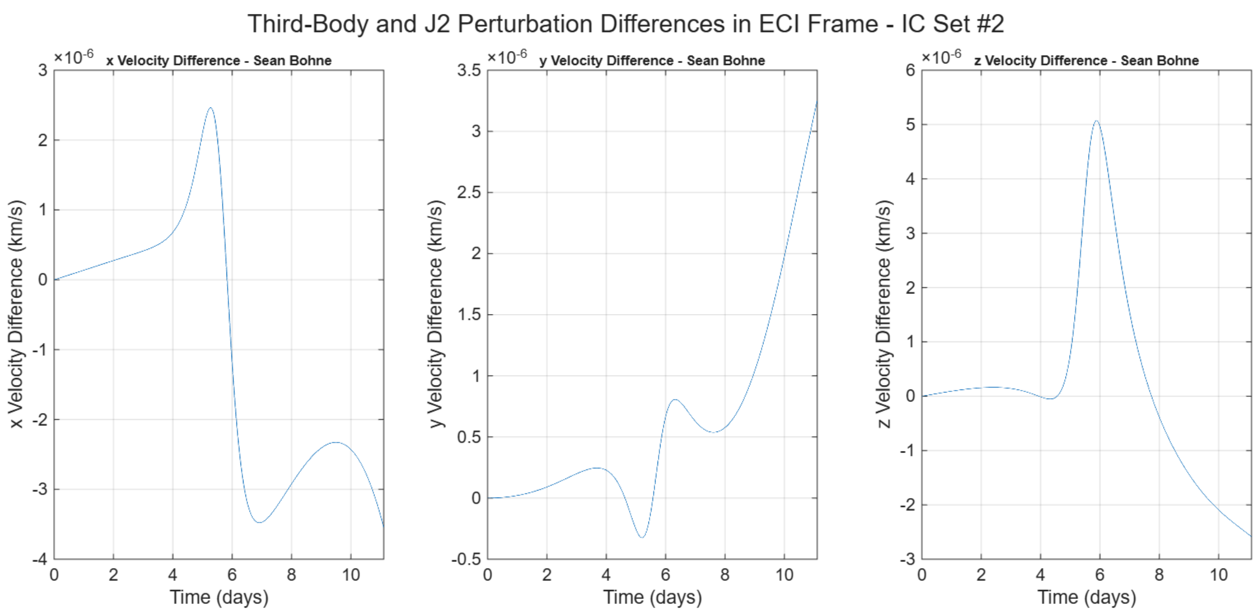

Fig. 16 Initial Conditions Set 2 ECI velocity differences between third-body perturbation simulations with and without J2 perturbations.

Figure 15 and Figure 16 show the differences between the third-body perturbation simulations with and without J2 effects in the ECI frame using initial conditions set 2. The differences for this set of initial conditions is almost negligible because the initial velocity is relatively small and the initial position of the spacecraft is far from the Earth. As a result, the J2 perturbations do not play a significant role in the spacecraft's dynamics.

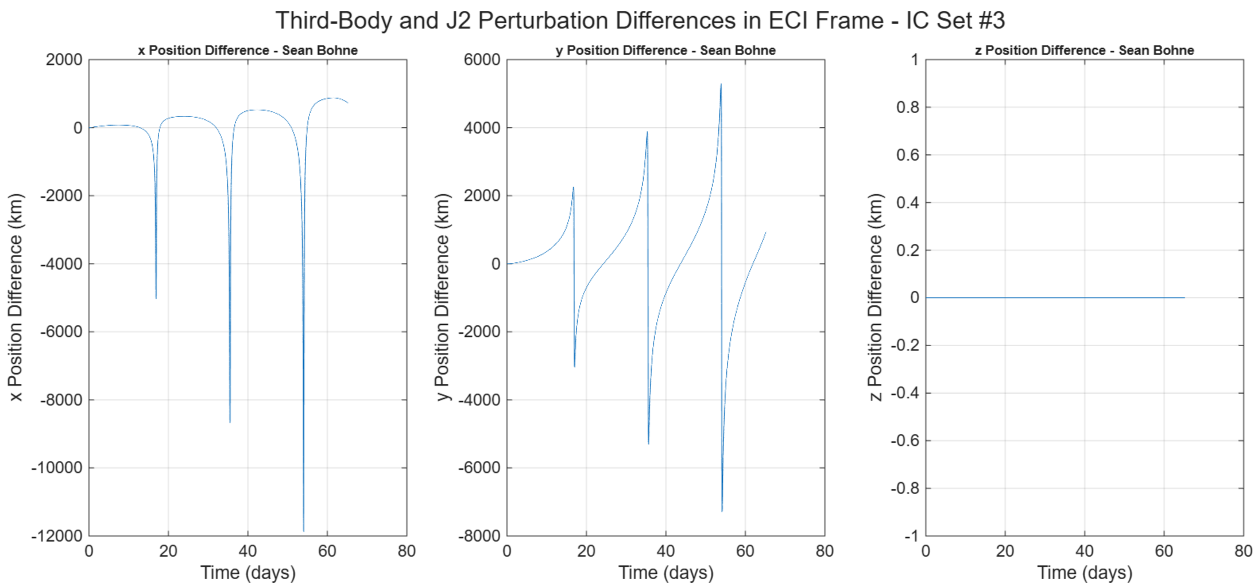

Fig. 17 Initial Conditions Set 3 ECI position differences between third-body perturbation simulations with and without J2 perturbations.

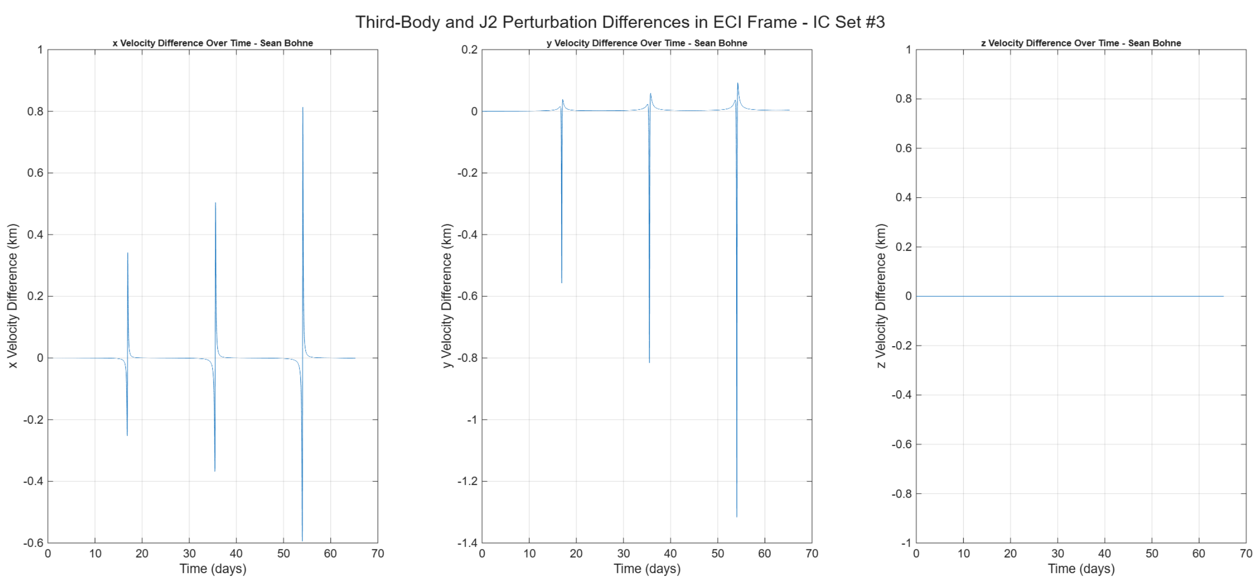

Fig. 18 Initial Conditions Set 3 ECI velocity differences between third-body perturbation simulations with and without J2 perturbations.

Figure 11 and Figure 12 show the differences between the third-body perturbation simulations with and without J2 perturbations in the ECI frame using initial conditions set 3. The differences are very significant due to the large initial velocity and the spacecraft's initial position is close to Earth. During each approach close to Earth, the spacecraft experiences significant effects due to Earth's non-spherical gravity, which can be seen at the spikes in the position and velocity differences shown in Figure 11 and Figure 12.