

AAE 590ACA PS1.A

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I. PS1.A: Perturbed two-body problem

A. PS1.A Part a

Given: Earth-centered Inertial (ECI) frame \mathcal{N} : $O, \hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2, \hat{\mathbf{n}}_3$. The component $\hat{\mathbf{n}}_3$ is aligned with Earth's rotation and the origin of the coordinated frame is located at Earth's center. $\mathbf{r} = x\hat{\mathbf{n}}_1 + y\hat{\mathbf{n}}_2 + z\hat{\mathbf{n}}_3$ represents the spacecraft position vector and $\mathbf{v} = v_x\hat{\mathbf{n}}_1 + v_y\hat{\mathbf{n}}_2 + v_z\hat{\mathbf{n}}_3$ represents the spacecraft velocity vector in the ECI frame.

Assumptions: None

Find: Describe and implement an algorithm to solve and numerically verify the following and discuss their physical significance: (a.1): Kepler's equation, (a.2): conversion between Keplerian orbital elements and Cartesian states, (a.3): conversion between modified equinoctial orbital elements and Cartesian states, (a.4): conversion between Milankovitch orbital elements and Cartesian states.

1. PS1.A Part a.1 - Kepler's Equation

Kepler's equation describes the motion of an orbiting body by relating mean anomaly to eccentric anomaly:

$$M = E - e \sin E \quad (1)$$

where M is mean anomaly, E is eccentric anomaly, and e is eccentricity. True anomaly, ν , can be derived from eccentric anomaly using the following equation:

$$\tan \frac{\nu}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2} \quad (2)$$

Solving Kepler's equation for true anomaly requires an iterative method because no analytic expression exists to convert mean anomaly to true anomaly. Therefore, the Newton-Raphson method will be used to convert mean anomaly to true anomaly. First, an initial guess for eccentric anomaly and an error tolerance is provided. In the program, the following initial guess, $E = M$, and tolerance, $tol = 1 \times 10^{-8}$, were used. The first step in the Newton-Raphson method includes moving all terms to one side of the equation and computing the result using the current guess as shown in Equation 3. The derivative is also calculated with respect to eccentric anomaly as shown in Equation 4.

$$f(M, E_k) = E_k - e \sin E_k - M \quad (3)$$

$$f'(M, E_k) = 1 - e \cos E_k \quad (4)$$

Using the results from Equation 3 and Equation 4, the new guess is determined as follows:

$$E_{k+1} = E_k - \frac{f(M, E_k)}{f'(M, E_k)} \quad (5)$$

The algorithm continues iterating until a tolerance is reached where $f'(M, E_k) < \text{tolerance}$. Once the tolerance is reached, a final estimate of eccentric anomaly is determined and can be used to calculate true anomaly as shown in Equation 2.

Converting from true anomaly to mean anomaly is much simpler because an analytical solution exists. First, eccentric anomaly is determined from true anomaly by the following:

$$E = 2 \tan^{-1} \left(\sqrt{\frac{1-e}{1+e}} \tan \frac{\nu}{2} \right) \quad (6)$$

Using the resulting eccentric anomaly, mean anomaly can be simply calculated using Kepler's equation in Equation 1.

To verify the algorithm, the value $\nu = 20^\circ$ was used. The resulting mean anomaly was $M = 13.1526^\circ$ and true anomaly was 20° . This test verified the algorithm's accuracy to convert between mean anomaly and true anomaly.

2. PS1.A Part a.2 - Kepler Elements to Cartesian States

Kepler orbital elements, represented by the state $\mathbf{x} = \begin{bmatrix} a & e & i & \Omega & \omega & M \end{bmatrix}^T$, can be converted to cartesian states in the ECI frame, defined in Section I.A through a series of transformations. In Kepler orbital elements, a is the semi-major axis, e is eccentricity, i is inclination, Ω is the right ascension of the ascending node, ω is the argument of periaxis, and M is mean anomaly. However, true anomaly can also be used in Kepler orbital elements and can be converted between one another using the Newton-Raphson method detailed in Section I.A.1. Firstly the spacecraft's radial position must be calculated and vectorized in the perifocal frame by the following:

$$r = \frac{a(1-e^2)}{1+e \cos \nu} \quad (7)$$

$$\mathbf{r}_{pf} = r \begin{bmatrix} \cos \nu & \sin \nu & 0 \end{bmatrix}^T \quad (8)$$

where r is the radial position from the center of Earth, and \mathbf{r}_{pf} is the spacecraft's position vector in the perifocal frame, which is centered at the center of the attracting body with axes pointing toward periaxis, perpendicular to

the orbital plane, and the final axis completing the right handed system. The velocity vector in the perifocal frame is calculated as follows:

$$h = \sqrt{\mu a (1 - e^2)} \quad (9)$$

$$\mathbf{v}_{pf} = \frac{\mu}{h} \begin{bmatrix} -\sin \nu & e + \cos \nu & 0 \end{bmatrix}^T \quad (10)$$

where h is specific angular momentum and \mathbf{v}_{pf} is the velocity vector in the perifocal frame. Applying a 3-1-3 rotation sequence, shown in Equation 11 converts the position and velocity vectors from the perifocal frame to ECI in Equation 12.

$$R(\phi) R(\theta) R(\psi) = \begin{bmatrix} \cos \psi & -\sin \psi & 0 \\ \sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \phi & -\sin \phi & 0 \\ \sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (11)$$

$$\begin{aligned} \mathbf{r} &= R(\phi) R(\theta) R(\psi) \mathbf{r}_{pf} \\ \mathbf{v} &= R(\phi) R(\theta) R(\psi) \mathbf{v}_{pf} \end{aligned} \quad (12)$$

where \mathbf{r} and \mathbf{v} are the position and velocity vectors in ECI described in Section I.A. Using the Keplerian orbital elements probided in Table 1 of the Problem Set document, where $a = 9000$ km, $e = 0.2$, $i = 50^\circ$, $\Omega = 10^\circ$, $\omega = 20^\circ$, and $\nu = 0^\circ$, the resulting cartesian state in ECI was determined to be $\mathbf{x} = \begin{bmatrix} 6388.1 & 2733.7 & 1886.4 & -3600.2 & 4364.3 & 5867.2 \end{bmatrix}^T$.

3. PS1.A Part a.3 - Equinoctial Elements to Cartesian States

This section describes the conversion between equinoctial orbital elements to cartesian states in ECI. Equinoctial states are $\mathbf{x} = \begin{bmatrix} p & f & g & h & k & L \end{bmatrix}^T$, where p is the semi-parameter and L is the true longitude. The following parameters must first be calculated before conversion:

$$\alpha^2 = h^2 + k^2 \quad (13)$$

$$s^2 = 1 + h^2 + k^2 \quad (14)$$

$$w = 1 + f \cos L + g \sin L \quad (15)$$

$$r = \frac{p}{w} \quad (16)$$

Using the parameters above, the cartesian position and velocity vectors in ECI can be calculated via the following:

$$\mathbf{r} = \frac{r}{s^2} \begin{bmatrix} \cos L + \alpha^2 \cos L + 2hk \sin L \\ \sin L - \alpha^2 \sin L + 2hk \cos L \\ 2h \sin L - 2k \cos L \end{bmatrix} \quad (17)$$

$$\mathbf{v} = \frac{1}{s^2} \sqrt{\frac{\mu}{p}} \begin{bmatrix} -(\sin L + \alpha^2 \sin L - 2hk \cos L + g - 2fhk + \alpha^2 g) \\ -(-\cos L + \alpha^2 \cos L + 2hk \sin L - f + 2ghk + \alpha^2 f) \\ h \cos L + k \sin L + fh + gk \end{bmatrix} \quad (18)$$

In addition, a function was developed to convert equinoctial orbital elements to keplerian elements via the following:

$$a = \frac{p}{1 - f^2 - g^2} \quad (19)$$

$$e = \sqrt{f^2 + g^2} \quad (20)$$

$$i = 2 \tan^{-1} \left(\sqrt{h^2 + k^2} \right) \quad (21)$$

$$\Omega = \tan^{-1} \left(\frac{k}{h} \right) \quad (22)$$

$$\omega = \tan^{-1} \left(\frac{g}{f} \right) - \tan^{-1} \left(\frac{k}{h} \right) \quad (23)$$

$$\nu = L - (\Omega + \omega) \quad (24)$$

To verify the implementation, the same keplerian orbital elements as in Section I.A.2 were used. The resulting cartesian state vector in ECI was the following: $\mathbf{x} = \begin{bmatrix} 6388.1 & 2733.7 & 1886.4 & -3600.2 & 4364.3 & 5867.2 \end{bmatrix}^T$, which

perfectly matches the state found in Section I.A.2, therefore verifying both methods are consistent.

4. PS1.A Part a.4 - Milankovitch Elements to Cartesian States

This section describes the conversion between Milankovitch orbital elements to cartesian states in ECI. Milankovitch states are $\mathbf{x} = \begin{bmatrix} \mathbf{h} & \mathbf{e} & L \end{bmatrix} \in \mathbb{R}^7$, where \mathbf{h} is the angular momentum vector, \mathbf{e} is the eccentricity vector, and L is the true longitude. The unit vectors in the angular momentum and eccentricity directions must first be calculated:

$$\hat{\mathbf{h}} = \frac{\mathbf{h}}{\|\mathbf{h}\|_2} \quad (25)$$

$$\hat{\mathbf{e}} = \frac{\mathbf{e}}{\|\mathbf{e}\|} \quad (26)$$

The right-handed system can be completed by finding the third vector, $\hat{\mathbf{e}}_\perp$:

$$\hat{\mathbf{e}}_\perp = \hat{\mathbf{h}} \times \hat{\mathbf{e}} \quad (27)$$

The true anomaly, ν , semi-parameter, p , and radial position, r can be calculated via the following:

$$\nu = L - \tan^{-1} \left(\frac{\hat{\mathbf{e}}_y}{\hat{\mathbf{e}}_x} \right) \quad (28)$$

$$p = \frac{h^2}{\mu} \quad (29)$$

$$r = \frac{p}{1 + e \cos \nu} \quad (30)$$

Using these unit vectors and quantities, the cartesian position and velocity vectors in ECI are formed by the following:

$$\mathbf{r} = r \left[\cos \nu \hat{\mathbf{e}} + \sin \nu \hat{\mathbf{e}}_\perp \right] \quad (31)$$

$$\mathbf{v} = \sqrt{\frac{\mu}{a(1-e^2)}} \left[-\sin \nu \hat{\mathbf{e}} + (e + \cos \nu) \hat{\mathbf{e}}_\perp \right] \quad (32)$$

Using the same orbital elements as in the previous sections, the resulting cartesian state vector in ECI was the following: $\mathbf{x} = \begin{bmatrix} 6388.1 & 2733.7 & 1886.4 & -3600.2 & 4364.3 & 5867.2 \end{bmatrix}$, which exactly matches the two previous results, and therefore at least proves consistency between all element conversions to cartesian states in ECI.

B. PS1.A Part b

Given: The following parameters will be used to simulate an orbit under no perturbations: $a = 9000$ km, $e = 0.2$, $i = 50^\circ$, $\Omega = 10^\circ$, $\omega = 20^\circ$, and $\nu = 0^\circ$. $\mu = 3.9860 \times 10^5 \text{ km}^2/\text{s}^2$ is used as Earth's gravitation parameter.

Assumptions: No perturbations.

Find: Simulate orbits under no perturbation using four different state representations: (i) Cartesian, (ii), Keplerian orbital elements, (iii) modified equinoctial orbital elements, and (iv) Milankovitch orbital elements for 15 orbits. Plot the constraint in the Milankovitch elements over time and compare the results between integration tolerances of 1.0×10^{-12} , 1.0×10^{-9} , 1.0×10^{-6} , and 1.0×10^{-3} .

This section will discuss methods to simulate the orbit under different state representations and the results. The unperturbed equations of motion in cartesian states, $\mathbf{x} = \begin{bmatrix} x & y & z & \dot{x} & \dot{y} & \dot{z} \end{bmatrix}^T$, are given by the following:

$$\dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x}), \quad \mathbf{f}_0(\mathbf{x}) = \begin{bmatrix} \dot{x} & \dot{y} & \dot{z} & -\frac{\mu}{r^3}x & -\frac{\mu}{r^3}y & -\frac{\mu}{r^3}z \end{bmatrix}^T \quad (33)$$

where $r = \|\mathbf{r}\|_2$. The unperturbed equations of motion in Keplerian orbital elements, $\mathbf{x} = \begin{bmatrix} a & e & i & \Omega & \omega & M \end{bmatrix}^T$, are much simpler compared to the cartesian equations of motion and are given via the following:

$$\dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x}), \quad \mathbf{f}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & n \end{bmatrix}^T \quad (34)$$

where $n = \sqrt{\frac{\mu}{a^3}}$ is the mean motion. The unperturbed equations of motion in equinoctial states, $\mathbf{x} = \begin{bmatrix} p & f & g & h & k & L \end{bmatrix}^T$, are given by the following:

$$\dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x}), \quad \mathbf{f}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \sqrt{\mu p} \left(\frac{q}{p} \right)^2 \end{bmatrix}^T \quad (35)$$

where $q = 1 + f \cos L + g \sin L$. The unperturbed equations of motion in Milankovitch states, $\mathbf{x} = \begin{bmatrix} \mathbf{h}^T & \mathbf{e}^T & L \end{bmatrix}^T$, are given by the following:

$$\dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x}), \quad \mathbf{f}_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \frac{h}{r^2} \end{bmatrix}^T \quad (36)$$

where $h = \|\mathbf{h}\|_2$ and $r = \|\mathbf{r}\|_2$ are angular momentum and radial distance respectively. Using ode45 in MATLAB with varying tolerances given in Section I.B. The results are shown in Figure 1 - 5.

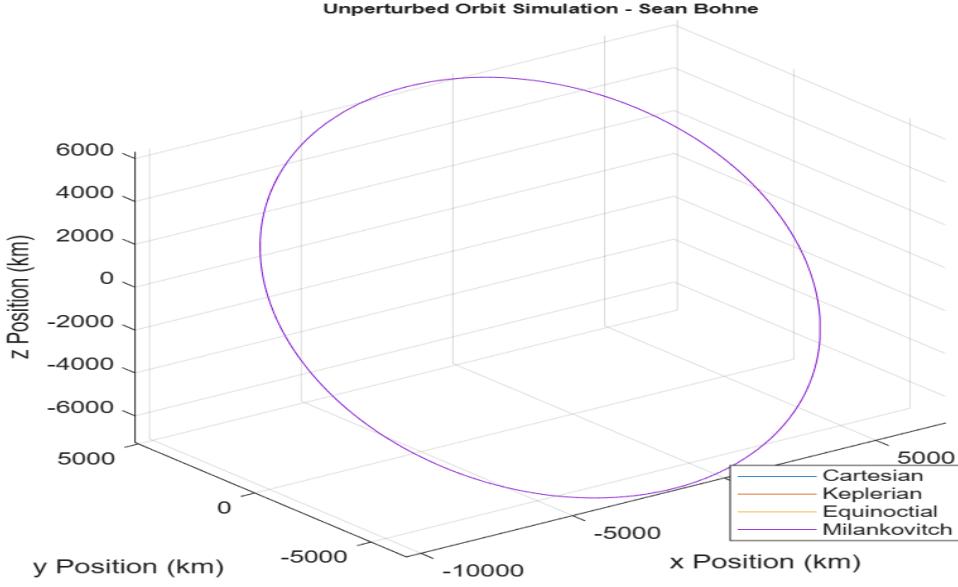


Fig. 1 All state representations simulated under no perturbations with 1.0×10^{-12} integration tolerance.

Figure 1 shows that all state representations agree with each other very well when the integration tolerance is high. This behavior is expected because, although some elements exhibit more nonlinear propagation than others, the tolerance is high enough such that the simulated trajectories agree with each other very closely.

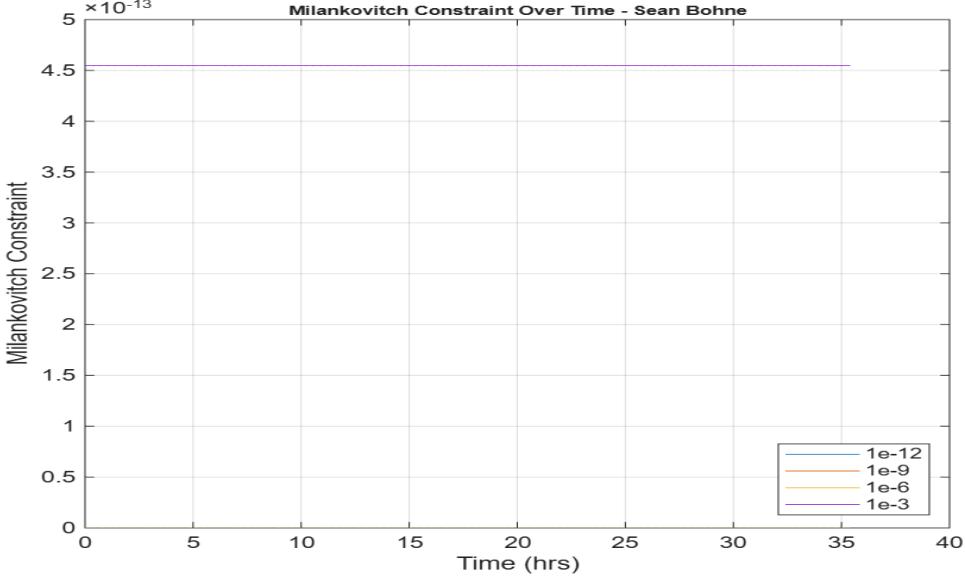


Fig. 2 Milankovitch element constraint plotted over time.

Figure 2 shows the Milankovitch constraint, $\mathbf{h} \cdot \mathbf{e} = 0$, is approximately zero. Numerical round-off in converting between state representations leads to a small error, although the constraint is still close to zero.

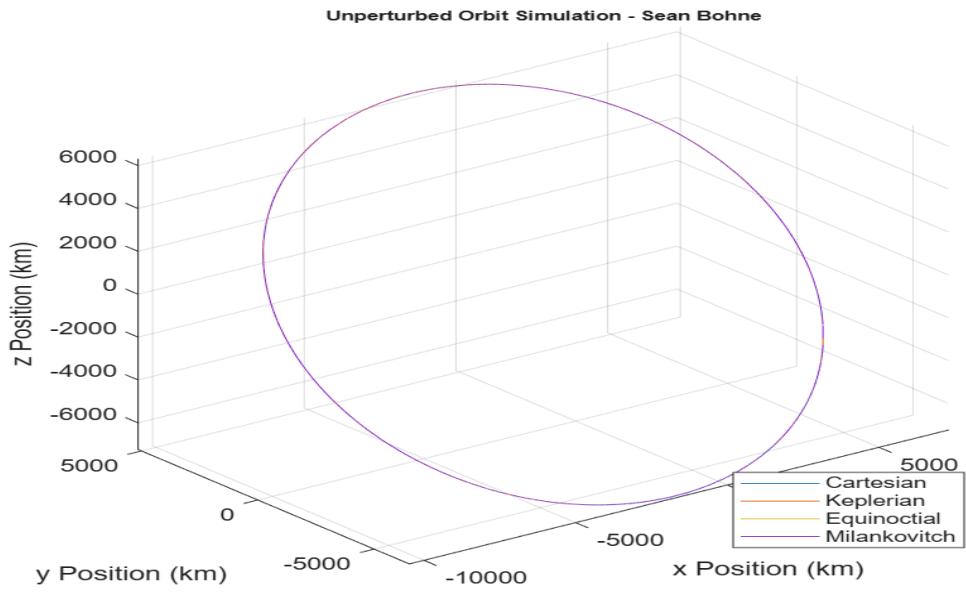


Fig. 3 All state representations simulated under no perturbations with 1.0×10^{-9} integration tolerance.

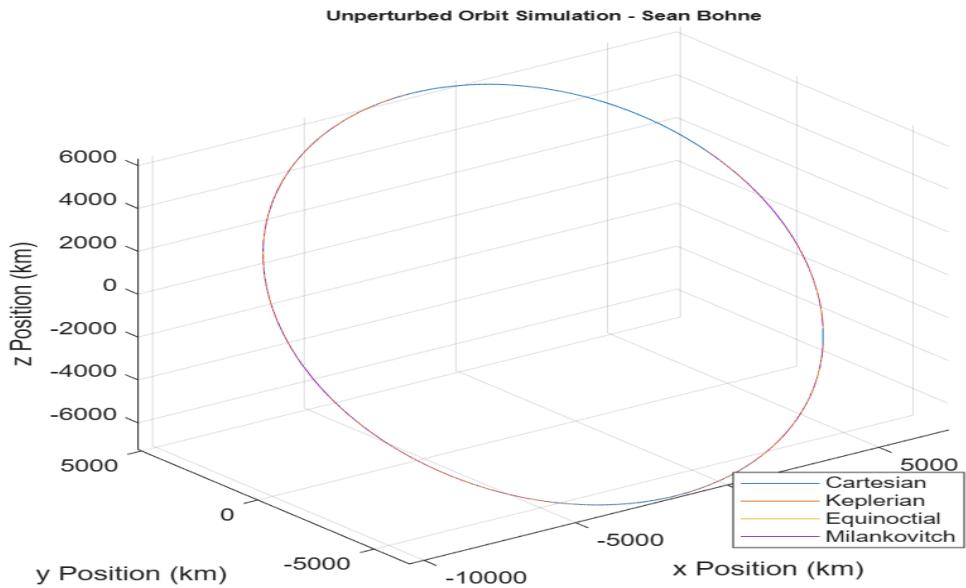


Fig. 4 All state representations simulated under no perturbations with 1.0×10^{-6} integration tolerance.

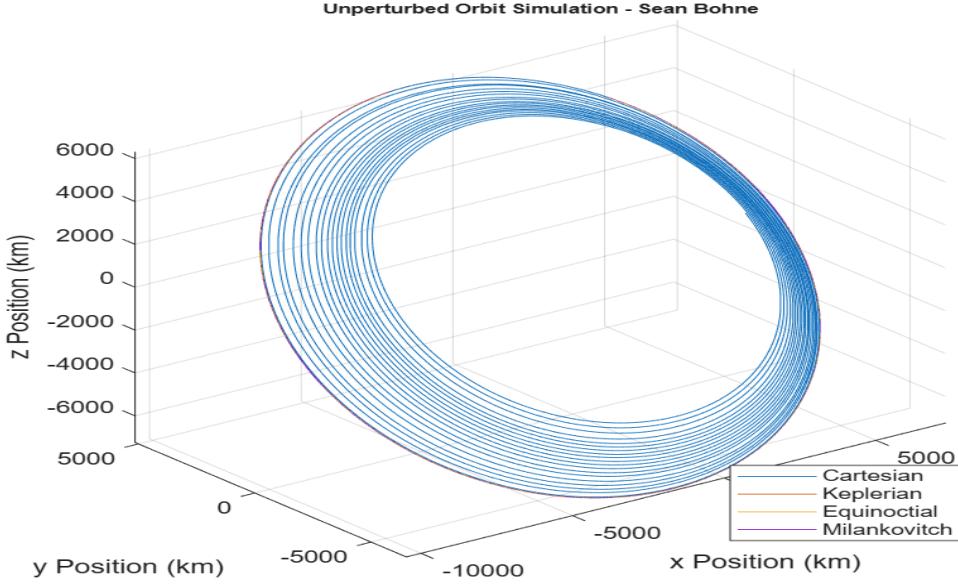


Fig. 5 All state representations simulated under no perturbations with 1.0×10^{-3} integration tolerance.

Figure 3 - 3 demonstrate the effects of decreasing the integration tolerance. In particular, Figure 5 shows that the cartesian state simulation yields very poor performance due to the high nonlinearities in the equations of motion. Figure 3 and Figure 4 show that all state representations perform well at integration tolerances of 1.0×10^{-9} and 1.0×10^{-6} respectively. However, the cartesian states do drift using a tolerance of 1.0×10^{-6} shown in Figure 4 as the nonlinearities begin to increase numerical error.

C. PS1.A Part c

Given: The following parameters will be used to simulate an orbit under Earth J2 perturbations: $a = 9000$ km, $e = 0.2$, $i = 50^\circ$, $\Omega = 10^\circ$, $\omega = 20^\circ$, and $\nu = 0^\circ$. $\mu = 3.9860 \times 10^5 \text{ km}^2/\text{s}^2$ is used as Earth's gravitation parameter.

Assumptions: Only J2 perturbations

Find: Simulate orbits under J2 perturbations using four different state representations: (i) Cartesian, (ii), Keplerian orbital elements, (iii) modified equinoctial orbital elements, and (iv) Milankovitch orbital elements for 15 orbits. Plot the constraint in the Milankovitch elements over time and compare the results between integration tolerances of 1.0×10^{-12} , 1.0×10^{-9} , 1.0×10^{-6} , and 1.0×10^{-3} . Include Earth J2 perturbations in the orbit simulation using $r_o = 6378.1$ km as Earth's radius and $J_2 = 1.0826 \times 10^{-3}$.

This section will discuss methods to simulate the orbit under different state representations and the results. The equations of motion in cartesian states, $\mathbf{x} = \begin{bmatrix} x & y & z & \dot{x} & \dot{y} & \dot{z} \end{bmatrix}^T$, under Earth J2 perturbations are given by the following:

$$\dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x}) + B(\mathbf{x}) \mathbf{a}_{J2}, \quad \mathbf{f}_0(\mathbf{x}) = \begin{bmatrix} \dot{x} & \dot{y} & \dot{z} & -\frac{\mu}{r^3}x & -\frac{\mu}{r^3}y & -\frac{\mu}{r^3}z \end{bmatrix}^T \quad (37)$$

where \mathbf{a}_d is the Earth J2 perturbation acceleration vector in ECI shown in Equation 38 and the perturbation mapping matrix, $B(\mathbf{x})$, is show in Equation 39.

$$\mathbf{a}_{J2} = -\frac{3\mu J_2 r_o^2}{2r^5} \left[\left(1 - 5\frac{z^2}{r^2}\right) x\hat{\mathbf{n}}_1 + \left(1 - 5\frac{z^2}{r^2}\right) y\hat{\mathbf{n}}_2 + \left(3 - 5\frac{z^2}{r^2}\right) z\hat{\mathbf{n}}_3 \right] \quad (38)$$

$$B(\mathbf{x}) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (39)$$

To include Earth J2 perturbations in any of the orbital element sets, the acceleration vector must be expressed in the radial-transverse-normal (RTN) frame for Gauss variational equations. To form the RTN frame, the unit vectors in the radial and normal directions to the plane are formed. The transverse axis is formed by completing the right-handed system and is defined in Equation 40 - 43.

$$\hat{\mathbf{r}} = \frac{\mathbf{r}}{\|\mathbf{r}\|} \quad (40)$$

$$\hat{\mathbf{n}} = \frac{\mathbf{r} \times \mathbf{v}}{\|\mathbf{r} \times \mathbf{v}\|} \quad (41)$$

$$\hat{\mathbf{t}} = \hat{\mathbf{n}} \times \hat{\mathbf{r}} \quad (42)$$

$$R = \begin{bmatrix} \hat{\mathbf{r}} & \hat{\mathbf{t}} & \hat{\mathbf{n}} \end{bmatrix}^T \quad (43)$$

Converting the J2 perturbations from the ECI frame shown in Equation 38 to the RTN frame using Equation 43 provides a way to add the J2 perturbation through Gauss variational equations via the following:

$$\dot{\mathbf{x}} = \mathbf{f}_0(\mathbf{x}) + B(\mathbf{x})^O \mathbf{a}_{J2} \quad (44)$$

where ${}^O \mathbf{a}_{J2}$ is the J2 perturbation in the RTN frame and $B(\mathbf{x})$ varies for each orbital element set. The Keplerian

element perturbation mapping matrix is defined below.

$$B(\mathbf{x}) = \frac{1}{h} \begin{bmatrix} 2a^2 e \sin \nu & \frac{2a^2 p}{r} & 0 \\ p \sin \nu & (p+r) \cos \nu + re & 0 \\ 0 & 0 & r \cos(\nu + \omega) \\ 0 & 0 & \frac{r \sin(\nu + \omega)}{\sin i} \\ -\frac{p \cos \nu}{e} & \frac{(p+r) \sin \nu}{e} & -\frac{r \sin(\nu + \omega)}{\tan i} \\ \frac{bp \cos \nu}{ae} - \frac{2br}{a} & -\frac{b(p+r) \sin \nu}{ae} & 0 \end{bmatrix} \quad (45)$$

where $b = a\sqrt{1-e^2}$, $p = a(1-e^2)$, and $r = \frac{p}{1+e \cos \nu}$. The corresponding perturbation mapping matrix for equinoctial elements is given as the following:

$$B(\mathbf{x}) = \sqrt{\frac{p}{\mu}} \begin{bmatrix} 0 & \frac{2p}{q} & 0 \\ \sin L & \frac{(q+1) \cos L + f}{q} & -\frac{g(h \sin L - k \cos L)}{q} \\ -\cos L & \frac{(q+1) \sin L + g}{q} & \frac{f(h \sin L - k \cos L)}{q} \\ 0 & 0 & \frac{s^2}{2q} \cos L \\ 0 & 0 & \frac{s^2}{2q} \sin L \\ 0 & 0 & \frac{h \sin L - k \cos L}{q} \end{bmatrix} \quad (46)$$

where $q = 1 + f \cos L + g \sin L$ and $s^2 = 1 + h^2 + k^2$. The corresponding perturbation mapping matrix for Milankovitch elements is given as the following:

$$B(\mathbf{x}) = \begin{bmatrix} \tilde{\mathbf{r}} \\ \frac{1}{\mu} (\tilde{\mathbf{v}} \tilde{\mathbf{r}} - \tilde{\mathbf{h}}) \\ \frac{\hat{\mathbf{z}} \cdot \mathbf{r}}{h(h + \hat{\mathbf{z}} \cdot \mathbf{h})} \mathbf{h}^T \end{bmatrix} \quad (47)$$

where $h = \|\mathbf{h}\|_2$, $r = \|\mathbf{r}\|_2$, $\hat{\mathbf{z}} = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T$, and $\tilde{\mathbf{r}} = \begin{bmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{bmatrix}$. Similarly, the matrix, $\tilde{\mathbf{v}}$ can be formed in the same way as $\tilde{\mathbf{r}}$. Using the equations of motions described above, the simulated orbits with varying tolerances are shown in Figure 6 - Figure 10

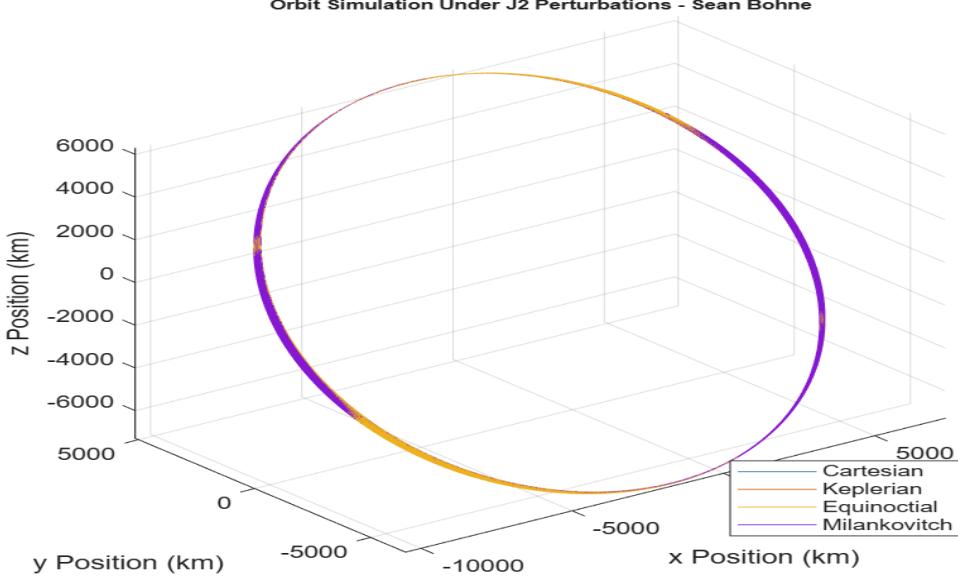


Fig. 6 All state representations simulated under Earth J2 perturbations with 1.0×10^{-12} integration tolerance.

As shown in figure 6, all state representations perform well when a high integration tolerance of 1.0×10^{-12} is used even under J2 perturbations. The tolerance is high enough such that the nonlinearities in the state representations do not significantly impact the numerical accuracy.

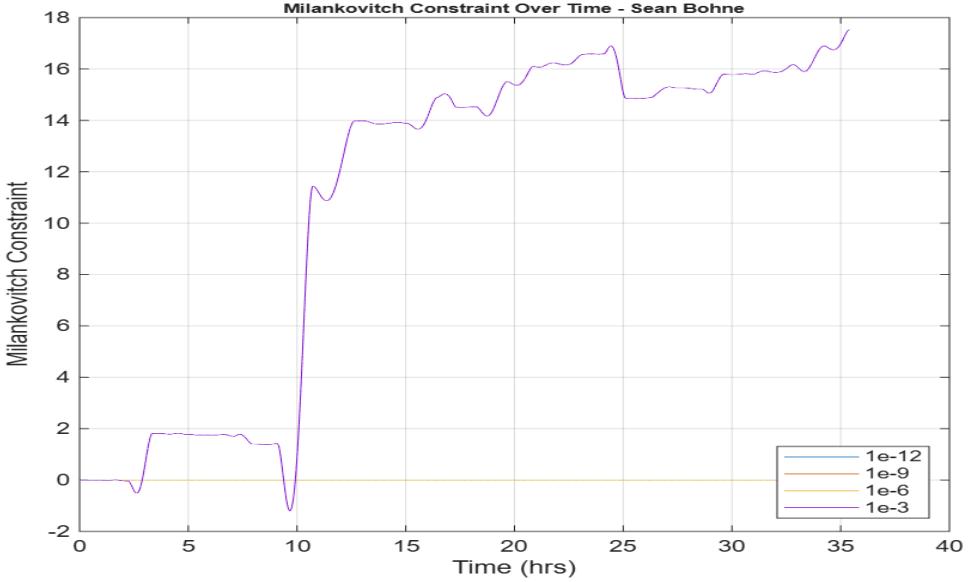


Fig. 7 Milankovitch element constraint plotted over time.

Figure 7 shows the Milankovitch constraint over time for each integration tolerance. All simulations provided constraints equal to zero except for the lowest tolerance of 1×10^{-3} , where numerical error significantly affects the simulation performance.

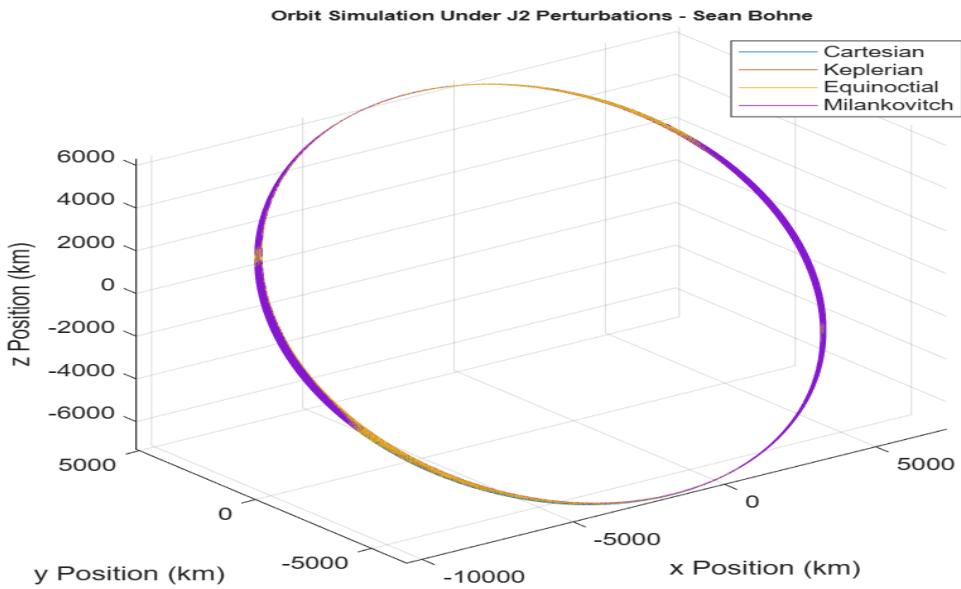


Fig. 8 All state representations simulated under Earth J2 perturbations with 1.0×10^{-9} integration tolerance.

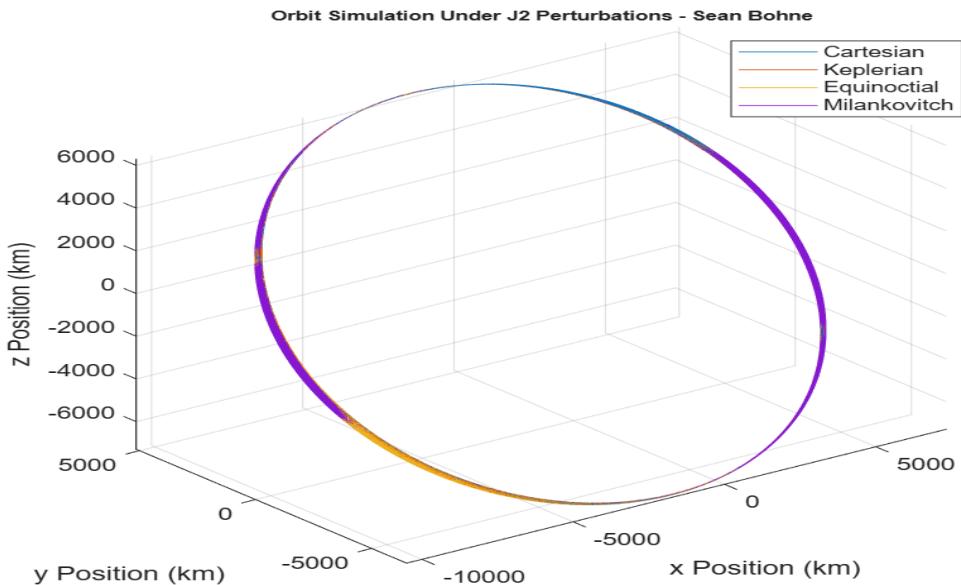


Fig. 9 All state representations simulated under Earth J2 perturbations with 1.0×10^{-6} integration tolerance.

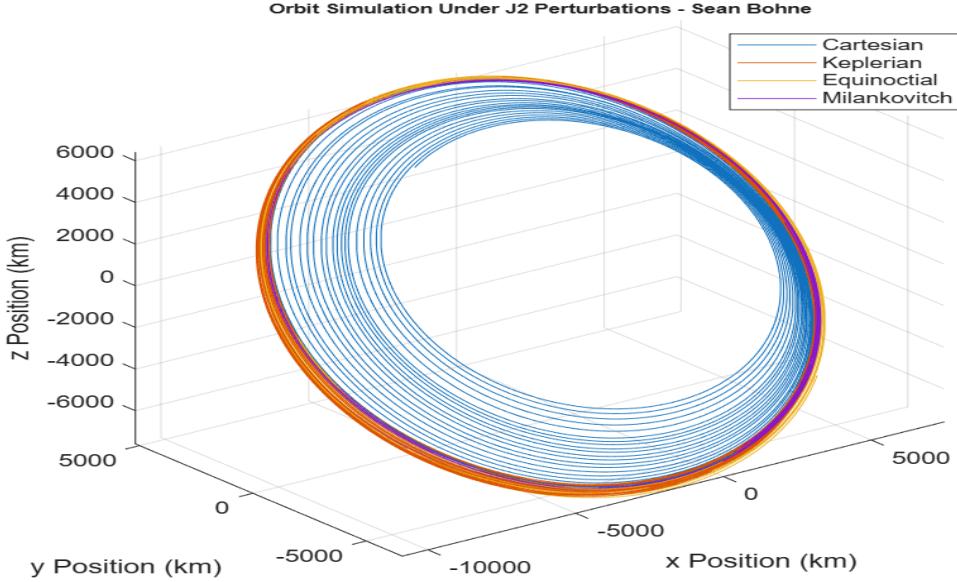


Fig. 10 All state representations simulated under Earth J2 perturbations with 1.0×10^{-3} integration tolerance.

Figure 8 and Figure 9 show that all state representations perform well in the numerical simulation even at moderate integration tolerances. However, Figure 10 shows that the cartesian states do not perform well due to the high nonlinearities in the state representation. As a result, the numerical errors quickly build up and lead the simulation to diverge from the true trajectory.

D. PS1.A Part d

Given: The following parameters will be used to simulate an orbit under Earth J2 perturbations: $a = 9000$ km, $e = 0.2$, $i = 50^\circ$, $\Omega = 10^\circ$, $\omega = 20^\circ$, and $\nu = 0^\circ$. $\mu = 3.9860 \times 10^5 \text{ km}^2/\text{s}^2$ is used as Earth's gravitation parameter.

Assumptions: Only J2 and solar radiation pressure (SRP) perturbations. Use the canon-ball SRP acceleration perturbation representation where the spacecraft is modeled as a sphere. Assume the Earth is far enough away from the sun where the spacecraft's position in ECI is negligible, ignore Earth's rotation pole tilt, and assume that the Earth heliocentric orbit is circular.

Find: Simulate orbits under J2 and solar radiation pressure perturbations using four different state representations: (i) Cartesian, (ii), Keplerian orbital elements, (iii) modified equinoctial orbital elements, and (iv) Milankovitch orbital elements for 15 orbits. Plot the constraint in the Milankovitch elements over time and compare the results between integration tolerances of 1.0×10^{-12} , 1.0×10^{-9} , 1.0×10^{-6} , and 1.0×10^{-3} . Include Earth J2 perturbations in the orbit simulation using $r_o = 6378.1$ km as Earth's radius and $J_2 = 1.0826 \times 10^{-3}$. Include SRP perturbations using $A/m = 5.4 \times 10^{-6} \text{ km}^2/\text{kg}$ as the spacecraft area-to-mass ratio and $G_0 = 1.02 \times 10^{14} \text{ kg} \cdot \text{km}/\text{s}^2$ as the solar flux constant.

The canon-ball SRP acceleration can be calculated using the following:

$$\mathbf{a}_{SRP} = \frac{A}{m} \frac{G_0}{d^2} \hat{\mathbf{s}} \quad (48)$$

where $d = 1\text{AU} = 1.4959 \times 10^8$ is the distance between the Earth and sun and $\hat{\mathbf{s}}$ is the unit vector pointing from the sun toward the Earth. To calculate these quantities, Earth's angular position relative to the sun must first be calculated as follows:

$$\theta = \omega_{sun} \Delta t \quad (49)$$

where θ is the angular position relative to the sun, $\omega_{sun} = \frac{2\pi}{365.25 \times 24 \times 3600} = 1.99102 \times 10^{-7} \text{ rad/s}$ is Earth's angular velocity relative to the sun, and Δt is the elapsed time. Using the angle θ , the unit vector, $\hat{\mathbf{s}}$ is calculated as follows:

$$\hat{\mathbf{s}} = \begin{bmatrix} \cos \theta & \sin \theta & 0 \end{bmatrix}^T \quad (50)$$

These quantities can be used to calculate the solar radiation pressure perturbations using Equation 48 and can be applied directly to the cartesian and Milankovitch state representations or converted to the RTN frame for the Keplerian and equinoctial state representations as described in Section I.C.

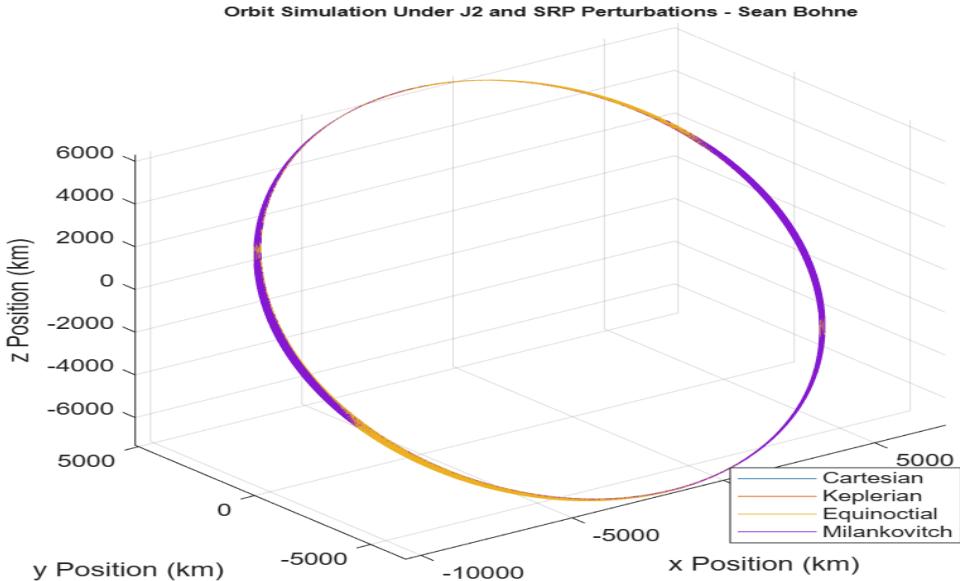


Fig. 11 All state representations simulated under Earth J2 and SRP perturbations with 1.0×10^{-12} integration tolerance.

Figure 11 shows that all state representations perform well at a high integration tolerance even with the addition of SRP perturbations. All state representations seem to be consistent between each other.

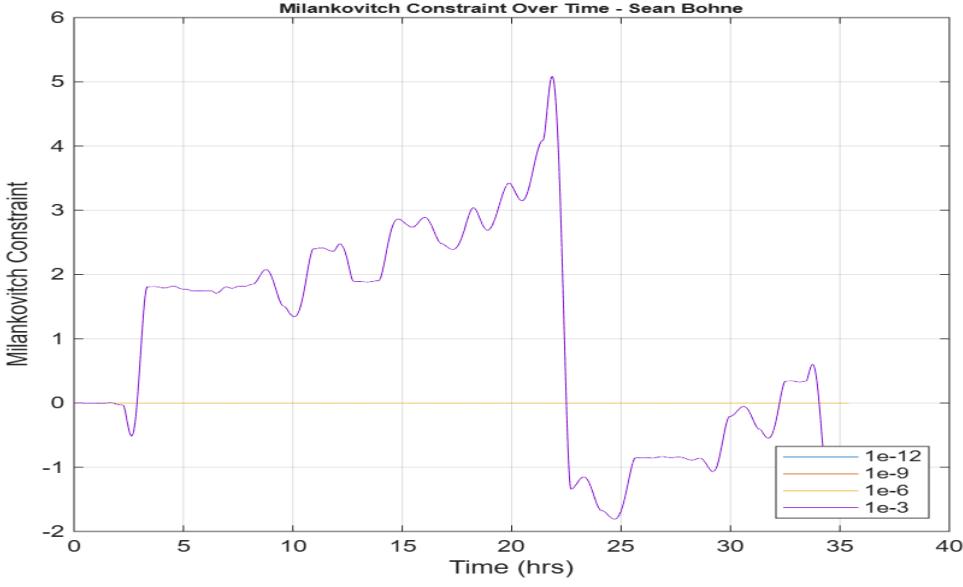


Fig. 12 Milankovitch element constraint plotted over time.

As shown in Figure 12, all simulations yield Milankovitch constraints of zero except when the integration tolerance is 1×10^{-3} . This behavior is expected because numerical errors begin to compound due to the highly nonlinear additions of J2 and SRP perturbations.

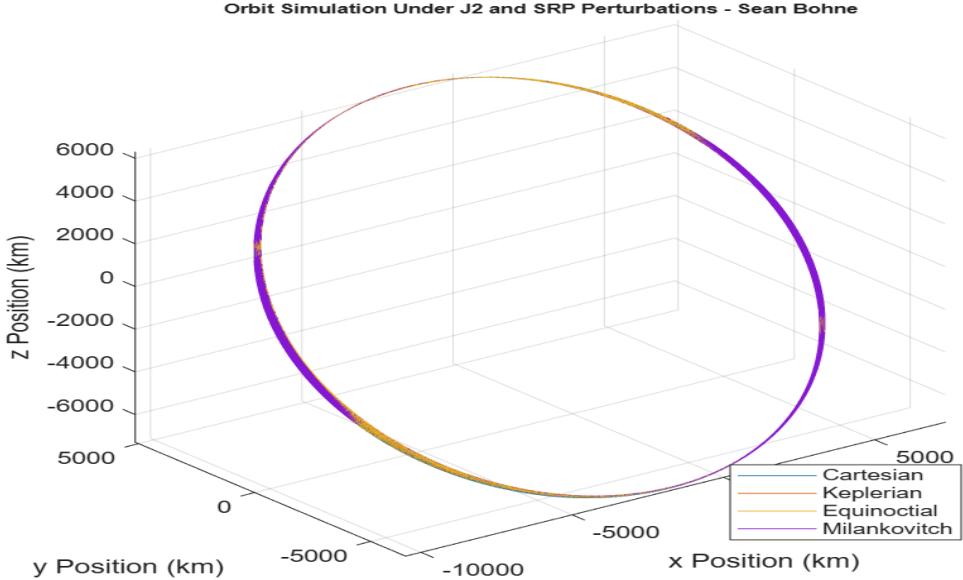


Fig. 13 All state representations simulated under Earth J2 and SRP perturbations with 1.0×10^{-9} integration tolerance.

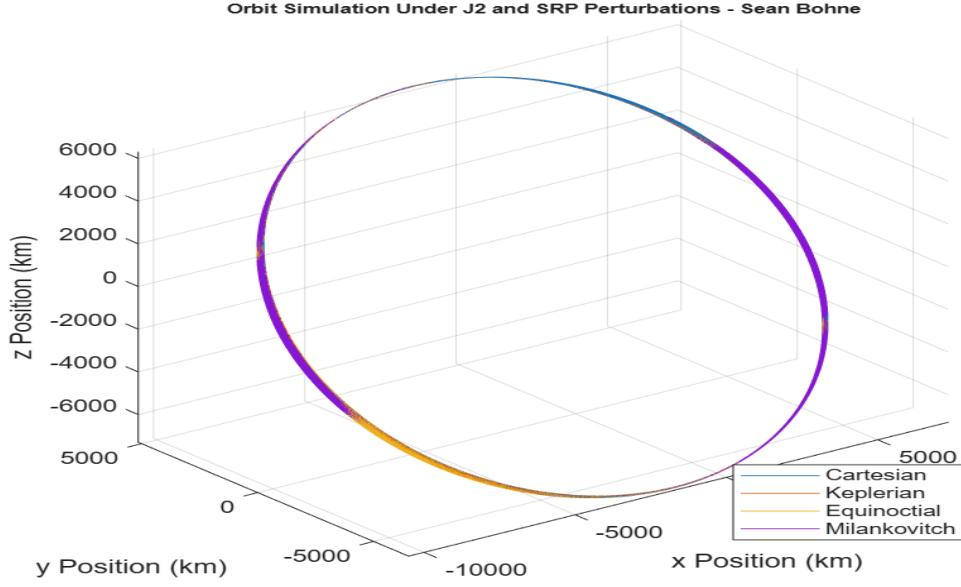


Fig. 14 All state representations simulated under Earth J2 and SRP perturbations with 1.0×10^{-6} integration tolerance.

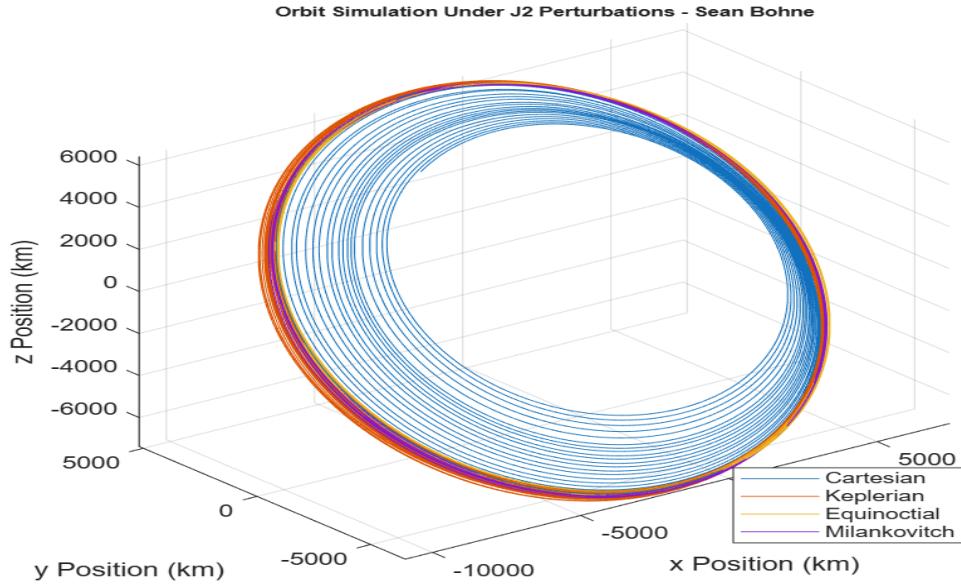


Fig. 15 All state representations simulated under Earth J2 and SRP perturbations with 1.0×10^{-3} integration tolerance.

Figure 18 and Figure 19 show that all state representations perform well and consistently at moderate integration tolerances. However, Figure 20 demonstrates that cartesian states perform exceptionally poorly at low integration tolerances. In addition, the Milankovitch orbital elements slightly deviate from the Keplerian and equinoctial elements.

E. PS1.A Part e

Given: The following parameters will be used to simulate an orbit under Earth J2 perturbations: $a = 9000$ km, $e = 0.2$, $i = 50^\circ$, $\Omega = 10^\circ$, $\omega = 20^\circ$, and $\nu = 0^\circ$. $\mu = 3.9860 \times 10^5 \text{ km}^2/\text{s}^2$ is used as Earth's gravitation parameter.

Assumptions: Only J2, solar radiation pressure (SRP), and drag perturbations. Use the canon-ball SRP acceleration perturbation representation where the spacecraft is modeled as a sphere. Assume the Earth is far enough away from the sun where the spacecraft's position in ECI is negligible, ignore Earth's rotation pole tilt, and assume that the Earth heliocentric orbit is circular. Assume the spacecraft is circular for drag perturbations and atmospheric density follows the exponential density model.

Find: Simulate orbits under J2 and solar radiation pressure perturbations using four different state representations: (i) Cartesian, (ii), Keplerian orbital elements, (iii) modified equinoctial orbital elements, and (iv) Milankovitch orbital elements for 15 orbits. Plot the constraint in the Milankovitch elements over time and compare the results between integration tolerances of 1.0×10^{-12} , 1.0×10^{-9} , 1.0×10^{-6} , and 1.0×10^{-3} . Include Earth J2 perturbations in the orbit simulation using $r_o = 6378.1$ km as Earth's radius and $J_2 = 1.0826 \times 10^{-3}$. Include SRP perturbations using $A/m = 5.4 \times 10^{-6} \text{ km}^2/\text{kg}$ as the spacecraft area-to-mass ratio and $G_0 = 1.02 \times 10^{14} \text{ kg} \cdot \text{km}/\text{s}^2$ as the solar flux constant. Include drag perturbations using $\rho_0 = 1.225 \times 10^9 \text{ kg/km}^3$ as the reference density, $h_0 = 6378.1$ km, $h = \frac{1}{7.8} 1/\text{km}$ as the density scaling parameter [1], and $C_D = 2.1$ as the drag coefficient [2].

This section will describe the method used to model atmospheric drag perturbations. Using the spacecraft position, \mathbf{r} , and velocity, \mathbf{v} , vectors in ECI are used to determine the spacecraft's relative velocity to the atmosphere as follows:

$$\mathbf{v}_{rel} = \mathbf{v} - \begin{bmatrix} 0 & 0 & \omega_{earth} \end{bmatrix}^T \times \mathbf{r} \quad (51)$$

where \mathbf{v}_{rel} is the spacecraft's relative velocity to the atmosphere and $\omega_{earth} = 7.292115 \times 10^{-5} \text{ rad/s}$ is Earth's angular rotation rate in ECI. The atmospheric density is determined using the exponential density model as follows:

$$\rho = \rho_0 \exp(-h(r - h_0)) \quad (52)$$

where ρ is the atmospheric density and the parameters used in the equation are given in the **Find** section at the beginning of the section. The radial distance is defined as $r = \|\mathbf{r}\|$. Using the magnitude of velocity, $v = \|\mathbf{v}\|$ and the parameters described in the **Find** section above, the drag perturbations in ECI can be calculated as follows:

$$\mathbf{a}_{drag} = -\frac{1}{2} \rho C_D A_m v \mathbf{v} \quad (53)$$

The drag perturbation can be directly applied to the cartesian and Milankovitch state representations or transformed to the RTN frame for simulation with Keplerian and equinoctial elements.

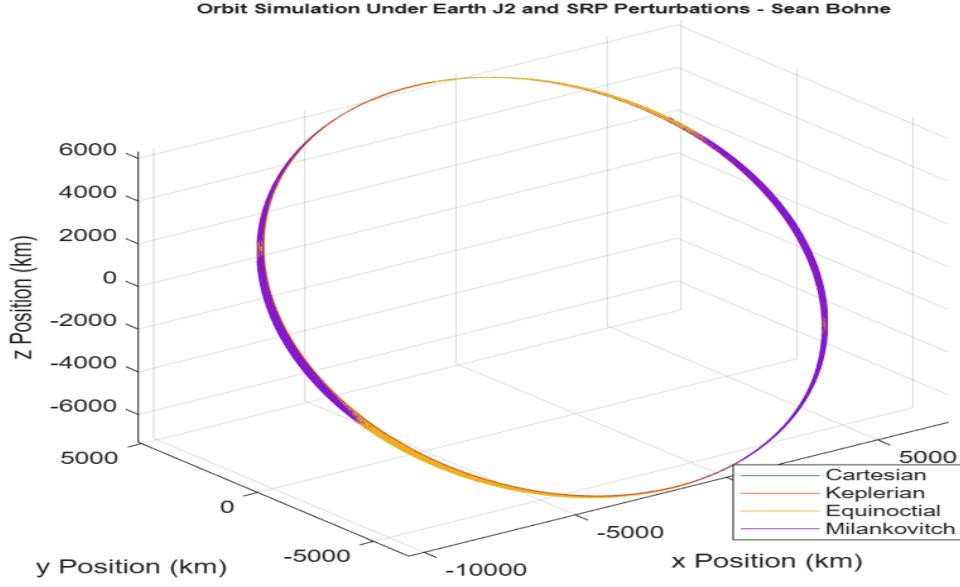


Fig. 16 All state representations simulated under Earth J2, SRP, and drag perturbations with 1.0×10^{-12} integration tolerance.

As shown in Figure 16, all state representations accurately and consistently simulate the trajectory with a high integration tolerance even under J2, SRP, and drag perturbations. When using high integration tolerances, all state representations provide accurate simulations even under many perturbations.

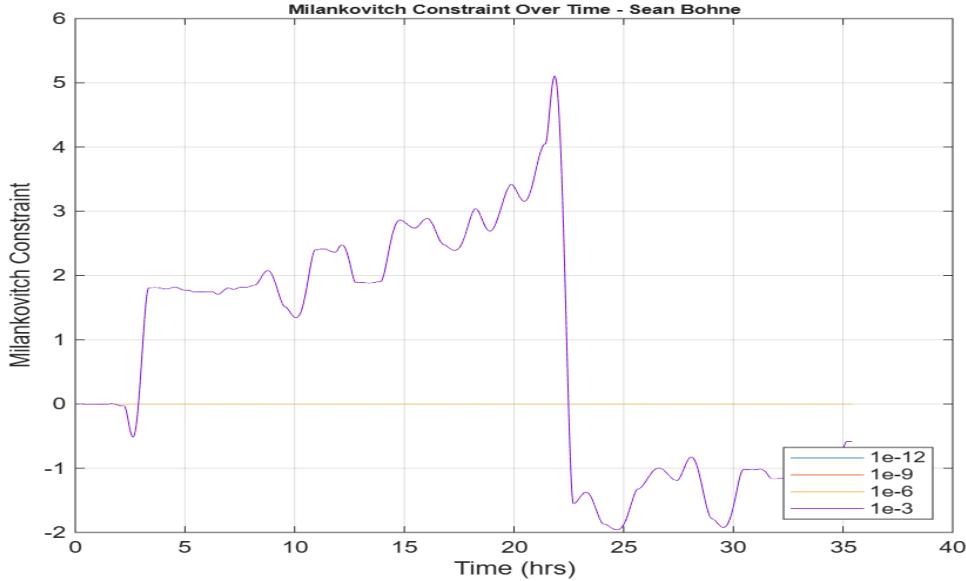


Fig. 17 Milankovitch element constraint plotted over time.

Figure 17 shows that all integration tolerances yield the expected result where the Milankovitch constraint is zero except for the lowest tolerance. This behavior is expected as the nonlinearities impact the simulation accuracy at low tolerances.

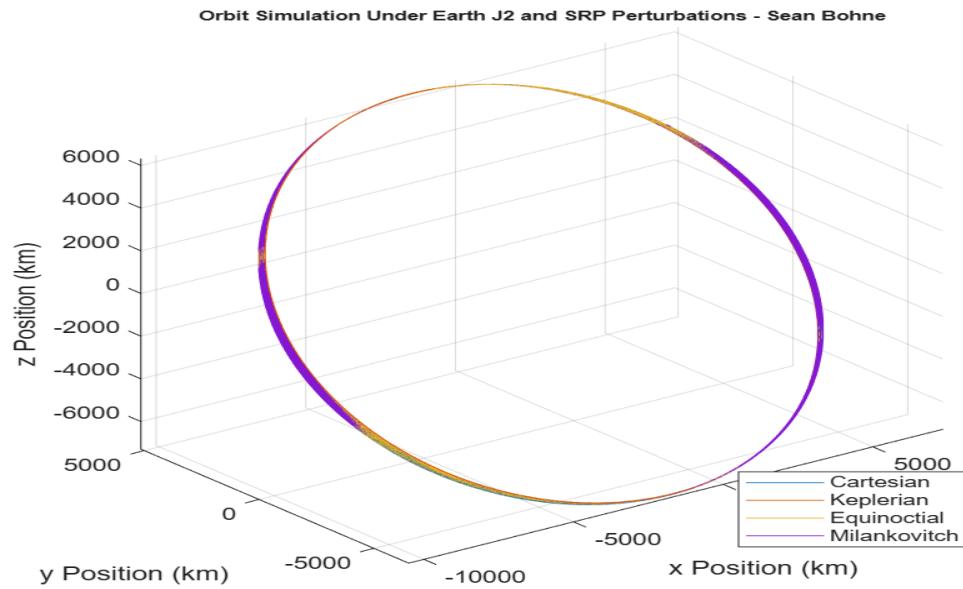


Fig. 18 All state representations simulated under Earth J2, SRP, and drag perturbations with 1.0×10^{-9} integration tolerance.

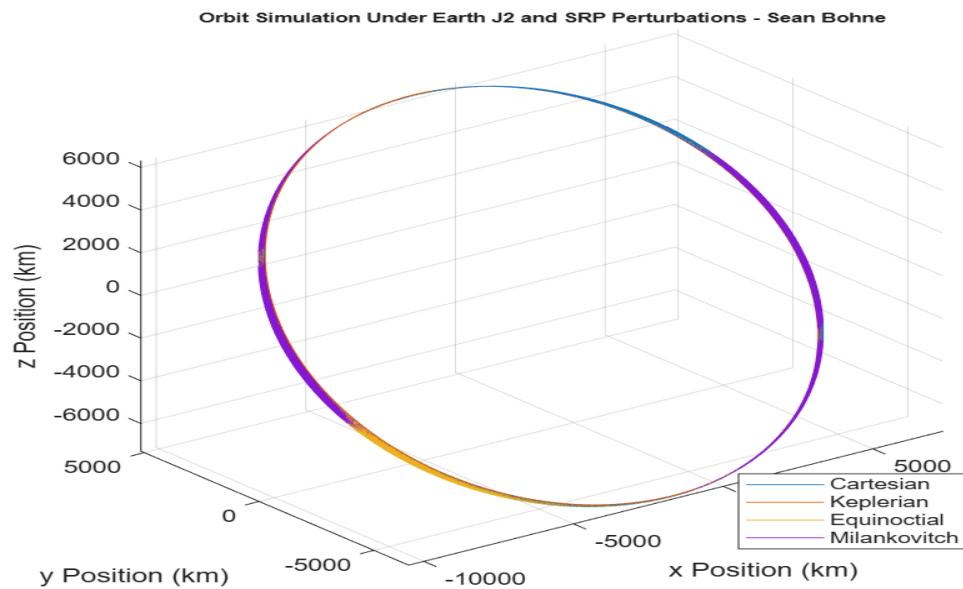


Fig. 19 All state representations simulated under Earth J2, SRP, and drag perturbations with 1.0×10^{-6} integration tolerance.

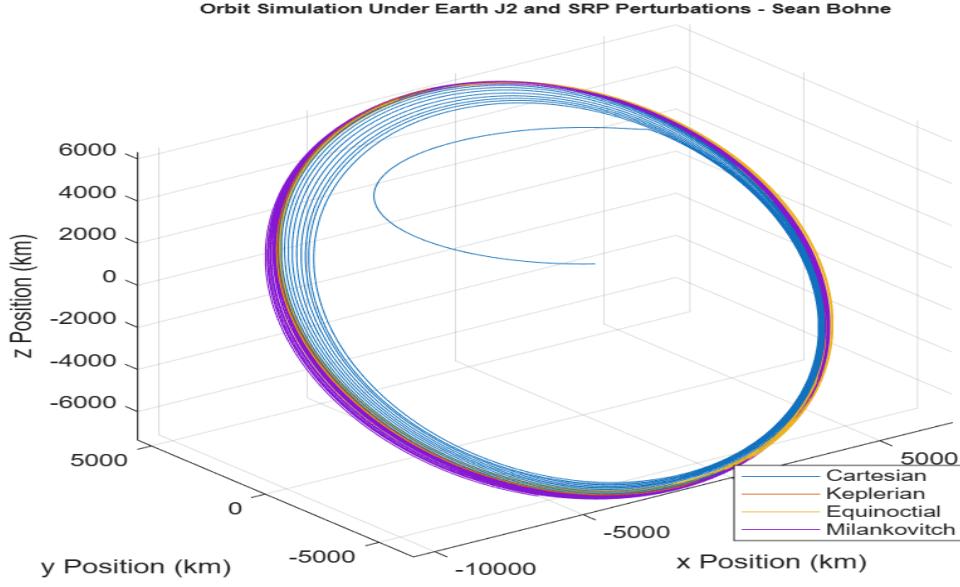


Fig. 20 All state representations simulated under Earth J2, SRP, and drag perturbations with 1.0×10^{-3} integration tolerance.

Figure 18 and Figure 19 demonstrate that all state representations are still accurate and consistent even under J2, SRP, and drag perturbations at moderate integration tolerances. However, as in the simulations discussed in the previous sections, the cartesian states deviate quickly at low tolerances due to the high nonlinearities in the state propagation as shown in Figure 20.

References

- [1] Montenbruck, O., and Gill, E., *Satellite Tracking and Observation Models*, Springer Berlin Heidelberg, Berlin, Heidelberg, 2000, pp. 193–232. https://doi.org/10.1007/978-3-642-58351-3_6, URL https://doi.org/10.1007/978-3-642-58351-3_6.
- [2] Tapley, B. D., Schutz, B. E., and Born, G. H., *Statistical Orbit Determination*, Elsevier: Academic Press, 2004.