Summer Examinations 2012

Exam Codes	1BE1, 1BEEE1, 1BEI1, 1BG1, 1BLE1, 1BM1, 1BP1, 1BSE1, 1BV1, 1EG1
Exams	First Year Engineering
Module	Mathematical Methods for Engineers
Module Code	MM120
External Examiner	Dr C.M. Campbell Prof. G. Saccomandi
Internal Examiner(s)	Dr M. Krnjajić Dr M.P Tuite Dr J.J.J. Ward
Instructions:	**Answer any SIX questions from eight.**
Duration	3 Hours
No. of Pages	2+ pages, including this one
School	Mathematics, Statistics and Applied Mathematics
Requirements:	No special requirements
Release to Library:	Yes
Statistical Tables/ Log Tables	Yes

1. (a) Find the inverse of the matrix

$$A = \begin{pmatrix} 3 & 4 & 4 \\ 2 & 3 & 2 \\ 6 & 4 & 3 \end{pmatrix}$$

(b) A brewery produces Brown Ale, Dark Brown Ale and Porter. The following table summarizes the amount of malt, hops and yeast used to produce one keg of beer together with the total amount of these resources available per day.

Resource	Brown Ale	Dark Brown Ale	Porter	Daily available
Yeast	3 kg	4 kg	4 kg	72 kg
Hops	2 kg	3 kg	2 kg	43 kg
Malt	6 kg	4 kg	3 kg	74 kg

Let x, y, z be the number of kegs of Brown Ale, Dark Brown Ale and Porter produced daily. Write down a system of three linear equations which hold precisely when all three resources are fully used.
Find the values of x, y, z which ensure that the daily supply of hops, malt and yeast are fully used.

(c) Give one value of t for which the system of equations

$$\begin{array}{rcrcl} x & + & 3y & = & ty \\ 3x & + & y & = & tx \end{array}$$

has infinitely many solutions.

2. (a) Evaluate the determinant of the matrix

$$A = \begin{pmatrix} 7 & 12+2t & t+27 & 3 \\ 4 & 7+t & t+14 & 2 \\ 2 & 4 & t+4 & 3 \\ 3 & 5+t & 13 & t \end{pmatrix}$$

and find all values of t for which A has no inverse.

(b) Find three vectors P, V, $W \in \mathbb{R}^4$ such that the solutions to the system

$$\begin{array}{rcrcrcrcr} w & + & 3x & + & 3y & + & 2z & = & 1 \\ 6w & + & 18x & + & 27y & + & 15z & = & 15 \\ 2w & + & 6x & + & 24y & + & 10z & = & 20 \end{array}$$

have the form $P + \lambda V + \mu W$ with $\lambda, \mu \in \mathbb{R}$.

3. (a) Express

$$\frac{4-3i}{1+2i} - \frac{3+i}{1-2i}$$

in the form $x + iy$ for real numbers x and y, and $i = \sqrt{-1}$.

(b) Show that the set of complex numbers z which satisfy

$$|z| = |z - 1 + 2i|$$

(where $i = \sqrt{-1}$), is a line in the Argand Plane, and sketch this line.

(c) State de Moivre's Formula and use it to find integers A and B such that

$$\cos 3\theta = A\cos^3\theta + B\cos\theta.$$

(d) Write down the 5th roots of unity in the form $a + ib,\ a, b \in \mathbb{R}$. Hence or otherwise write $z^5 - 1$ as a product of irreducible linear or quadratic factors over $\mathbb{R}$.

4. (a) Show that the characteristic polynomial $\chi_A(\lambda)$ of the matrix

$$\begin{pmatrix} 0 & b & 0 \\ b & a & b \\ 0 & b & a \end{pmatrix}$$

is given by $\chi_A(\lambda) = -\lambda^3 + 2a\lambda^2 + (2b^2 - a^2)\lambda - b^2 a$.

(b) Find the characteristic polynomial, the eigenvalues and corresponding eigenvectors of the matrix

$$\begin{pmatrix} -5 & -6 & 0 \\ 9 & 10 & 0 \\ -9 & -18 & -2 \end{pmatrix}.$$

Hence or otherwise find a diagonal matrix D and a matrix E such that $E^{-1}AE = D$.

(c) Let $\mathbf{v}$ be an eigenvector of the matrix M, with corresponding eigenvalue λ. Show that $\mathbf{v}$ is also an eigenvector of the matrix M^2 with corresponding eigenvalue λ^2.

5. (a) Sketch the direction field of the differential equation:

$$\frac{dx}{dt} = 2t$$

By integrating both sides, find the general solution of the equation. Sketch the particular solution for which $x(0) = 1$, and for which $x(1) = 0$, and demonstrate that these are consistent with your direction field.

(b) Find the solution of each of the following differential equations for a function $x = x(t)$:

(i) $$\frac{dx}{dt} = 3xt^2$$

(ii) $$\frac{dx}{dt} - 2\frac{x}{t} = t^2 - 1$$ (Hint: Find an integrating factor)

6 (a) A chemical reaction is governed by the differential equation:

$$\frac{dx}{dt} = K(1-x)^2$$

where $x(t)$ is the concentration of the chemical at time t seconds (s). The concentration of the chemical is zero at time $t = 0$ s and its concentration is found to be $\frac{1}{2}$ at time $t = 1$ s. Determine the reaction constant K and find $x(t)$ for all t.

(b) Solve the following initial value problem for a function $x = x(t)$:

$$\frac{d^2x}{dt^2} + 7\frac{dx}{dt} + 12x = 0, \quad t > 0,$$

where $x(0) = 0, \quad \frac{dx}{dt}(0) = 1.$

7 Find the general solution of each of the following nonhomogeneous differential equations:

$$\text{(i)} \qquad \frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 3x = e^{-t}$$

$$\text{(ii)} \qquad \frac{d^2x}{dt^2} + x = t^2$$

8 (a) Calculate $P(A|B)$ when (i) $A \cap B = \emptyset$, (ii) $A \subset B$, and (iii) $B \subset A$.

(b) A laboratory blood test of a certain disease is positive 90 percent of time when the disease is, in fact, present. In terms of probability we write: $P($ test + | disease $) = 0.9$. However, the test also gives a "false positive" with probability 0.02 for the healthy persons tested, meaning that $P($test + | no disease $) = 0.02$. Assume that one percent of the population actually has the disease, that is, $P($desease$) = 0.01$. Calculate $P($ disease | test + $)$, the probability that a randomly chosen person has the disease, given that his test has come out positive.

(c) The number of customers arriving at a post office during any 10-minute interval is known to be a Poisson random variable with $\lambda = 4$. What is the probability that more than two customers will arrive during a 10-minute period? Recall the Poisson distribution: $P(X = k) = \mathrm{e}^{-\lambda}\lambda^k/k!$, where $k = 0, 1, 2, 3, ...$