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MAE 207 HW#1 Ex: 1.1 to 1.5; 2.1 to 2.2

Ex: 1.1

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$, $B = \begin{pmatrix} 6 & 5 & 4 \\ 3 & 2 & 1 \end{pmatrix}$ $AB = ?$ $BA = ?$

$$\begin{array}{cc} 3\times 2 & 2\times 3 = 3\times 3 \end{array}$$

$$AB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} \begin{pmatrix} 6 & 5 & 4 \\ 3 & 2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 6\times1+2\times3 & 5\times1+2\times2 & 4\times1+2\times1 \\ 3\times6+4\times3 & 3\times5+4\times2 & 3\times4+4\times1 \\ 5\times6+6\times3 & 5\times5+6\times2 & 5\times4+6\times1 \end{pmatrix}$$

$$\boxed{AB = \begin{pmatrix} 12 & 9 & 6 \\ 30 & 23 & 16 \\ 48 & 37 & 26 \end{pmatrix}}$$

$$\begin{array}{cc} 2\times 3 & 3\times 2 = 2\times 2 \end{array}$$

$$BA = \begin{pmatrix} 6 & 5 & 4 \\ 3 & 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix} = \begin{pmatrix} 6+5\times3+4\times5 & 6\times2+5\times4+4\times6 \\ 3\times1+2\times3+1\times5 & 3\times2+2\times4+1\times6 \end{pmatrix}$$

$$= \begin{pmatrix} 6+15+20 & 12+20+24 \\ 3+6+5 & 6+8+6 \end{pmatrix} = \begin{pmatrix} 41 & 56 \\ 14 & 20 \end{pmatrix}$$

$$\boxed{BA = \begin{pmatrix} 41 & 56 \\ 14 & 20 \end{pmatrix}}$$

Ex. 1.2 Code to build Toeplitz from top row & left column
Hankel & circulant Toeplitz from top rows only

- See Matlab Code

Ex 1.3 $A^2 = A$

a) $B = I - A$ is Idempotent ($B^2 = B$)

$$B^2 = (I-A)(I-A)$$

$$\Downarrow$$

$$II - IA - AI + AA$$

$$II = I \ \& \ IA = AI = A \ \& \ AA = A$$

$$\Downarrow$$

$$I - A - \cancel{A + A}$$

$$\shortparallel$$

$$I - A = B$$

so... $B^2 = I - A = B$
therefore B is also Idempotent!

b) A is invertible & Idempotent show $A = I$

$A^{-1}A = I$ since A is invertible

$A^{-1}AA = IA$ since A is Idempotent $AA = A$
$A^{-1}A = A$

$\Rightarrow \boxed{I = A}$ ✓

Ex 1.4

(a.) Householder reflector matrix $H(2,w)$ reflects vector $x \in \mathbb{R}^3$ through plane normal to $w = (0\ 1\ 0)^T$

$H^T x$ for $x = (3\ 6\ 2)^T$?

$$H = I - \sigma \overset{3\times1}{w}\,\overset{1\times3}{w^H}$$

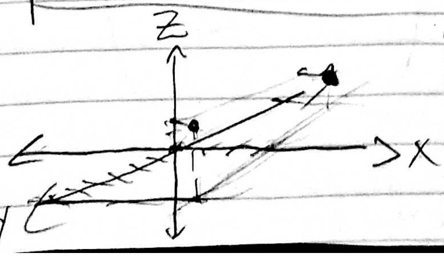
$$= I - 2\begin{bmatrix}0\\1\\0\end{bmatrix}\begin{bmatrix}0 & 1 & 0\end{bmatrix} = I - 2\begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix}$$

$$= \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} - \begin{bmatrix}0&0&0\\0&2&0\\0&0&0\end{bmatrix} = \begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix}$$

$$\boxed{H(2,w) = \begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix}}$$

$$H^T = H = \begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix}$$

$$\boxed{H^T x = \overset{3\times3}{\begin{bmatrix}1&0&0\\0&-1&0\\0&0&1\end{bmatrix}}\overset{3\times1}{\begin{bmatrix}3\\6\\2\end{bmatrix}} = \begin{bmatrix}3\\-6\\2\end{bmatrix}}$$



Since we have $\sigma = 2$ & $\|w\| = 1$ we note that H is symmetric & involutory. We also note that multiplying by $H = H^T$ changes the sign of the 2nd component of x. In the picture drawn previously we see then that multiplying X by H simply reflects X about the x-axis. Similarly multiplying by H again brings us back to our original location

b.) vector $\vec{w}$ = ? H(2, w) = ? in order to reflect $x = (3\ 6\ 2)^T$ to direction parallel to e^1

Find $H^T x$

$$W = \frac{x + ve^1}{\|x + ve^1\|} \qquad V = \text{Sign}(x_1)\|x\| \qquad \text{sign}(a) = \begin{cases} 1 & a \geq 0 \\ -1 & a < 0 \end{cases}$$

$$\|x\| = \sqrt{9+36+4} = \sqrt{49} = 7$$

$$w = \frac{\begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} + \begin{bmatrix} 7 \\ 0 \\ 0 \end{bmatrix}}{\left\| \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix} + \begin{bmatrix} 7 \\ 0 \\ 0 \end{bmatrix} \right\|} \qquad e^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad v = 7$$

$$w = \frac{\begin{bmatrix} 10 \\ 6 \\ 2 \end{bmatrix}}{\left\| \begin{bmatrix} 10 \\ 6 \\ 2 \end{bmatrix} \right\|} \qquad \left\| \begin{bmatrix} 10 \\ 6 \\ 2 \end{bmatrix} \right\| = \sqrt{100+36+4} = 11.8322 = \sqrt{140}$$

$$\boxed{W = \frac{\sqrt{140}}{140} \begin{bmatrix} 10 \\ 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 0.8452 \\ 0.5071 \\ 0.1690 \end{bmatrix}}$$

$$H(2,w) = I - \sigma w w^H$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - 2 \times \frac{\cancel{140}}{140^{\cancel{2}}} \begin{bmatrix} 10 \\ 6 \\ 2 \end{bmatrix} \begin{bmatrix} 10 & 6 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{7\cancel{0}} \begin{bmatrix} 10\cancel{0} & 6\cancel{0} & 2\cancel{0} \\ 6\cancel{0} & 3.6 & 1.2 \\ 2\cancel{0} & 1.2 & 0.4 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{7} \begin{bmatrix} 10 & 6 & 2 \\ 6 & 3.6 & 1.2 \\ 2 & 1.2 & 0.4 \end{bmatrix}$$

$$\boxed{H(2,w) = \begin{bmatrix} -0.4286 & -0.8571 & -0.2857 \\ -0.8571 & 0.4857 & -0.1714 \\ -0.2857 & -0.1714 & 0.9429 \end{bmatrix}}$$

$$H^T x = \begin{bmatrix} -0.4286 & -0.8571 & -0.2857 \\ -0.8571 & 0.4857 & -0.1714 \\ -0.2857 & -0.1714 & 0.9429 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ 2 \end{bmatrix}$$

$$\boxed{H^T x = \begin{bmatrix} -7 \\ 0 \\ 0 \end{bmatrix}}$$

This is the unit vector $e1$ multiplied by -7. This matches with what was given in the text: $H^T x = -\nu e1$

We see then by carefully picking the appropriate w we can construct H so that we reflect x parallel to the desired Cartesian unit vector.

Ex. 1.5

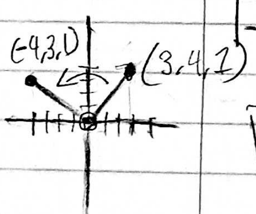
a) $G(1,2,\pi/2)$ rotate arbitrary vector $x \in \mathbb{R}^3$ by $\pi/2$ CCW in x_1-x_2 plane

$$G^T x = ? \quad x = (3\ 4\ 1)^T$$

$$g_{ii} = \cos(\tfrac{\pi}{2}) \quad g_{kk} = \cos(\tfrac{\pi}{2}) \quad g_{ik} = \sin(\tfrac{\pi}{2}) \quad g_{ki} = -\sin(\tfrac{\pi}{2})$$

$$g_{11} = 0 \quad g_{22} = 0 \quad g_{12} = 1 \quad g_{21} = -1$$

$$\boxed{G = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$



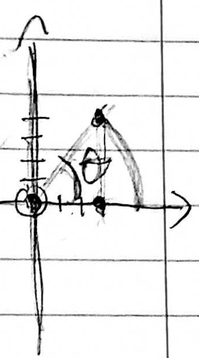
$$\boxed{G^T x = \underbrace{\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{3\times 3} \underbrace{\begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix}}_{3\times 1} = \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}}$$

As shown graphically we see that multiplying x by G^T does indeed rotate our arbitrary x vector by 90° CCW as desired. We also should see that multiplying $\begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}$ by G gives us our original x vector back.

b) $\theta = ?$ $G(1,2,\theta) = ?$

rotates $x = (3\ 4\ 1)^T$ so x_2 component $= 0$

$$G^T x = \begin{bmatrix} ? \\ 0 \\ ? \end{bmatrix}$$



$g_{11} = \cos\theta$ $g_{22} = \cos\theta$ $g_{12} = \sin\theta$ $g_{21} = -\sin\theta$

$$G = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$G^T x = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 3\cos\theta - 4\sin\theta \\ 3\sin\theta + 4\cos\theta \\ 1 \end{bmatrix}$$

$+3\sin\theta + 4\cos\theta = 0$

$3\tan\theta + 4 = 0$

$\tan\theta = -4/3$

$$\boxed{\theta = -0.9273\,\text{rad} = -53.13^\circ}$$

$$\boxed{G(1,2,\theta) = \begin{bmatrix} 0.6 & -0.8 & 0 \\ 0.8 & 0.6 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$\boxed{G^T x = \begin{bmatrix} 0.6 & -0.8 & 0 \\ 0.8 & 0.6 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}}$$

By using a negative θ we are able to rotate clockwise so that the x_2 component after the rotation is 0. The x_1 component however changes as well due to the rotation

Ex 2.1

a) MC where M & C are full matrices all $n \times n$

$$\underset{2\times2}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}} \underset{2\times2}{\begin{bmatrix} e & f \\ g & h \end{bmatrix}} \quad \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

1 flop, 1, 1, 1, 1, 1

1, 1, 4, 1, 1, 1

So... $2\times2\times(2\times2-1)$

Generalizing to $n\times n$

# of Additions $nn(n-1)$
# of multiplications $n\times n\times n$

Total Flops $n\times n\times n + n\times n(n-1) = \boxed{n^2(2n-1)}$

b) MC where M full & C diagonal

# of multiplications $= \boxed{n^2}$

c) Ax A is full

$$\begin{bmatrix} n\times n \end{bmatrix}\begin{bmatrix} n \\ \times \\ 1 \end{bmatrix}$$

# of Mults: $n\times n = n^2$
# of Adds: $n(n-1)$

$n^2 + n(n-1)$

$n(n+n-1) = \boxed{n(2n-1)}$

d) Ax A is diagonal

$$\begin{bmatrix} \circ & & 0 \\ & \circ & \\ 0 & & \circ \end{bmatrix}\begin{bmatrix} n \\ \times \\ 1 \end{bmatrix}$$

# of Adds: 0
# of Mults: n

$\boxed{n}$

e) $Ax = b \Rightarrow$ solve for x where A tridiagonal

$$\left(\begin{array}{ccccc|c} b_1 & c_1 & & & 0 & g_1 \\ a_2 & b_2 & c_2 & & & g_2 \\ & a_3 & b_3 & c_3 & & \vdots \\ 0 & & \ddots & \ddots & \ddots & \vdots \end{array}\right)$$

For Thomas algorithm for tridiagonal A

Forward sweep:	division	mult	addition
Eliminate a_2:	1	2	2

$$-b_1 * \frac{a_2}{b_1} \quad -c_1 * \frac{a_2}{b_1}$$
$$+ a_2 \qquad + b_2$$

To eliminate the entire subdiagonal:

divisions $= n-1$
mults $= 2(n-1)$
add $= 2(n-1)$

Back Substitution
divisions $= n$
mults $= n-1$
adds $= n-1$

$\boxed{\text{Total Flops} = 8n-7}$

f) $Ax=b$ for x but A is full

$$\left[\begin{array}{cccccc} a_1 & d_1 & \cdot & - & - & - \\ b_1 & a_2 & d_2 & - & - & - \\ c_1 & b_2 & a_3 & d_3 & - & - \\ \vdots & c_2 & \ddots & \ddots & \ddots & \ddots \\ \vdots & & \ddots & \ddots & \ddots & \\ 0 & & & \ddots & & \end{array}\right] \left[\begin{array}{c} b_1 \\ \vdots \\ \vdots \\ b_n \end{array}\right]$$

Since diagonally dominant we can use Gaussian elimination w/out pivoting

For the forward sweep we have:

$\#$ of Divisions $= \sum_{k=1}^{n-1}(n-k) = n(n-1)/2$

$\#$ of mults $= \sum_{k=1}^{n-1}(n-k+1)(n-k) = (n^3-n)/3$

$\#$ of adds $= \sum_{k=1}^{n-1}(n-k+1)(n-k) = (n^3-n)/3$

Back substitution:

$\#$ of Divisions $= n$

$\#$ of mults $= \sum_{k=1}^{n-1}(n-k) = n(n-1)/2$

$\#$ of adds $= \sum_{k=1}^{n-1}(n-k) = n(n-1)/2$

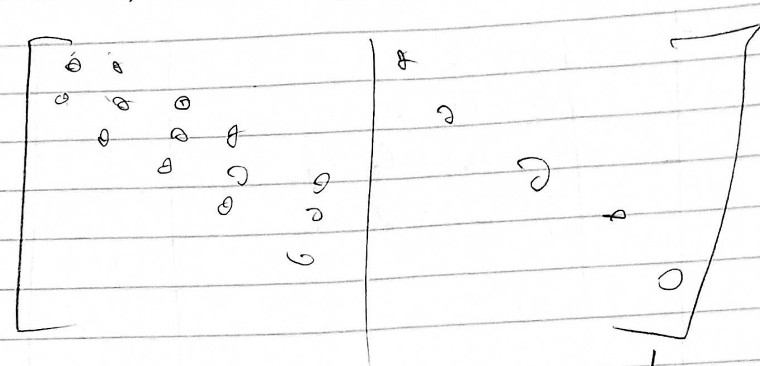
Total $\#$ of Flops $= \dfrac{n(n-1)}{2} + \dfrac{(n^3-n)}{3} + \dfrac{(n^3-n)}{3}$

$\frac{a}{4}$ $\frac{4}{6}$ $-\frac{2}{3}n - \frac{3}{2}n + n$

$+ n + \dfrac{n(n-1)}{2} + \dfrac{n(n-1)}{2}$

$$\frac{2}{3}(n^3-n) + \frac{3}{2}n(n-1) + n = \boxed{\frac{2}{3}n^3 + \frac{3}{2}n^2 - \frac{13}{6}n}$$

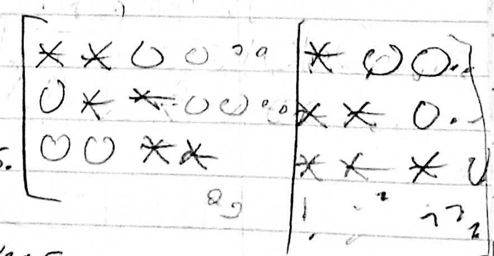
g) $BM = A$ solve for M, B tridiagonal & A diagonal

$[B|A] =$ 

Since B is tridiagonal & diagonally dominant we can use a similar procedure as given for the Thomas algorithm in 2.2.5

Forward sweep! We have 1 division 2 multiplications & 2 additions for a single sweep. In order to get rid of the subdiagonal we need to do this $(n-1)$ times so...

\# of divisions $= (n-1)$
\# of mults $= 2(n-1)$
\# of adds $= 2(n-1)$

Back Substitution:
After the forward sweep we have 

We see we have n $Ax=b$ problems.

Solving each of these n problems results in:
n divisions
$n-1$ mults
$n-1$ additions
for each problem,

So for back substitution we have
n^2 divisions
n^2-n mults
& n^2-n adds

In total we have:

$n-1+2n-2+2n-2+n^2+n^2-n+n^2-n=$

$\boxed{3n^2+3n-5 \text{ Flops}}$ $\sim 3n^2$

h) $BM=A$ where B is full & A is diagonal

Similar to as before we can use Gaussian elimination without pivoting since B is diagonally dominant.
From before the cost of the forward sweep is:
Divisions: $\sum_{k=1}^{n-1}(n-k)=n(n-1)/2$
Mults: $\sum_{k=1}^{n-1}(n-k+1)(n-k)=(n^3-n)/3$
add: $\sum_{k=1}^{n-1}(n-k+1)(n-k)=(n^3-n)/3$
Total: $\frac{2}{3}n^3+\frac{n^2}{2}-\frac{n}{2}-\frac{2}{3}n$

$-\frac{3}{6}-\frac{4}{6}$

$\frac{7}{6}$

For the back substitution process we again have n $Ax=b$ problems to solve, so that:
Divisions: n^2
of Mult: $n\sum_{k=1}^{n-1}(n-k)=n^2(n-1)/2$

adds: $n\sum_{k=1}^{n-1}(n-k)=n^2(n-1)/2$

So the total flops: $(1+\frac{2}{3})n^3+(\cancel{\frac{1}{2}}+1-\frac{1}{2}-\frac{1}{2})n^2-\frac{7}{6}n$

$$\boxed{\frac{5}{3}n^3+\frac{1}{2}n^2-\frac{7}{6}n}$$

Ex. 2.2 See Matlab code