**Fundamentals of Artificial Intelligence [H02C1a]**

**Xinhai Zou (r0727971)**

# Contents

# 1 Intro to AI

## 1.1 Overview

AI starts from **1955** in Bell Telephone Laboratories. What is AI today?

- Planning & Scheduling
- Declarative Programming
- Robotics
- Computer vision
- Machine learning
- language processing
- Human computer interaction
- theorem proving

Sometime, we must make a decision when faced with insufficient information. Being rational means maximizing your expected utility.

## 1.2 Summary

1. One summer was not enough to shape AI -> but many significant advances have been made since!
2. Movie AI, News AI and Research AI differ
3. Modern AI systems combine 'reasoning' with 'learning' -> this course: 'resoning' focus
4. Much to say about ethics and fairness of AI systems

# 2 Rational agent

## 2.1 Overview

- Rational agent
  - Agent: an entity that perceives (from env.) and acts (to env.)
  - Rational: maximizing expected utility
  - Rational agent: selects actions that maximize its (expected) utility, i.e., they do the right thing.
- Different environments
  - Accessible vs. inaccessible (fully observable vs. partially observable)
    - ∗ environment is fully observed by agent?
  - dterministic vs. stochasitic
    - ∗ is the next state of the envioronment completely determined by the current state?
  - episodic vs. sequential
    - ∗ can the quality of an action be evaluated within **an episode** (perception + action)?
  - static vs. dynamic
    - ∗ can the environment change while the agent is deliberating?
    - ∗ dynamic - not turn-based game, static - mostly turn-based game
  - discrete vs. continuous
    - ∗ is the value discrete (nominal) or continuous (numerical)
  - single vs. multi-agent
    - ∗ which entities have to regarded as agents? (competitive or cooperative scenarios?)

Example

- Reflex agents
  - choose (respond) action only based on current state
  - do not consider the future consequences of their actions
- Planning/ Goal based agent
  - choose action based on consequences of actions (future)
  - consider how the world will be

Table 1: Examples using different environments

| Env. | Observable | Agents | Deterministic | Episodic | Static | Discrete |
|---|---|---|---|---|---|---|
| **Crossword puzzle** | Fully | Single | Deterministic | sequential | Static | Discrete |
| **Chess with a clock** | | | | | | |
| **Poker** | | | | | | |
| **Backgammon** | | | | | | |
| **Taxi driving** | | | | | | |
| **Medical diagnosis** | | | | | | |
| **Image analysis** | | | | | | |
| **Part-picking robot** | | | | | | |
| **Refinery controller** | | | | | | |
| **English tutor** | | | | | | |
| **Pac-Mac** | Observable | Multi | Stochasitic | Sequential | Dynamic | Discrete |

### 2.2 Summary

1. an agent is something that perceives and acts, being rational is to obtain maximal expected value. An rational agent will take the action that maximizes its expected performance given the percept sequence and its knowledge of the environment.

2. an agent program maps from a percept to an action

   (a) reflex agents respond based on current state

   (b) goal-based agents respond based on current and future state

   (c) learning agents improve their behaviour over time

3. some environments are more demanding than others

# 3 State space

## 3.1 Overview

How to represent a problem in machine? For a basic search problem, there are five ingredients.

1. States: represent the states in a clear way
2. Actions: define the actions you can execute and change the world state, normally $\delta : X \to Y$
3. Initial state: what is the initial state, the **starting** point
4. Goal formulation: define which states you want to end
5. Specification of the search cost: the execution costs

Take Pac-man - path-finding as an example, which is shown below:

- States: $(x, y)$, where x is x-axis and y is y-axis
- Initial state: starting point $(x_0, y_0)$
- Actions: different actions
  - east: $f_{east} : (x, y) \to (x + 1, y)$
  - west: $f_{west} : (x, y) \to (x - 1, y)$
  - north: $f_{north} : (x, y) \to (x, y + 1)$
  - south: $f_{south} : (x, y) \to (x, y - 1)$
- Goal test: goal point/state $(x^*, y^*)$
- Path cost: each step cost 1 step unit, only have <specific number> state unit

Sometimes, we need a check-loop detection. We have two different structure for state-space representation: graph and search tree.

- graph
  - For graph, during searching, we often keep track of the already visited nodes to avoid loops and redundancies.
- search tree
  - nodes show states, each node in the search tree is an entire PATH in the state space graph.

Some key issues:

- definition of state space and transition model

- choose implicit graph or implicit tree
- search forward or backward
- use optimal solution or any solution
- decompose difficult problem to simpler problem (problem reduction/ decomposability)

For forward and backward search, the definition is shown as belwo. The efficiency can be improved based on different branching factor. However, backward is not always possible, e.g., playing chess.

1. Forward: start from initial state and search towards goal
2. Backward: start from the goal state and search toward initial state

## 3.2 Summary

1. Before an agent can start to perform on searching problem, 5 ingredients are important:
   (a) State
   (b) Initial state
   (c) Goal state
   (d) Action
   (e) Cost

# 4 Uniformed search

## 4.1 Overview

Some terminologies for searching problem in graph:

1. node expansion: generating all successor node considering the availble actions (the whole process)
2. explored nodes: nodes that have already been expanded
3. frontier/ fringe: set of all nodes available for expansion (candidate for expansion)
4. search strategy: defines which node is expanded next (the strategies)

## 4.2 Graph search

Graph search, frontier = <s,a>, <s,b>, its basic algorithm:

```
Input:
    a graph,
    a set of start nodes,
    Boolean procedure goal(n) that tests if $n$ is a goal node.
frontier := $< s >$, $s$ is a start node
while frontier is not empty:
    select and remove path $< n_0, ..., n_k >$ from frontier for every neighbor $n$ of $n_k$
        if goal(n)
            return $< n_0, ..., n_k, n >$
        else add $< n_0, ..., n_k, n >$ to frontier
end while
```

For loop detection, the loop-detection algorithm will be:

```
Input:
    a graph,
    a set of start nodes,
    Boolean procedure goal(n) that tests if $n$ is a goal node.
frontier := $< s >$, $s$ is a start node
while frontier is not empty:
    select and remove path $< n_0, ..., n_k >$ from frontier for every neighbor $n$ of $n_k$
        if goal(n)
            return $< n_0, ..., n_k, n >$
        else if $n \notin n_0, ..., n_k$
            then add $< n_0, ..., n_k, n >$ to frontier
end while
```

Two basic searching strategies for selecting nodes to explore: depth-first search and breadth-first search.

## 4.3 DFS and BFS

### 4.3.1 Depth-First Search (DFS)

Its selecting strategy is **LIFO = "Last In First Out"**, we work with **stack**.

- Time complexity: $O(b^m)$
  - with b: branching factor, m: maximum depth
- Space complexity: $O(b \times m)$
  - with b: branching factor, m: maximum depth

The property of DFS:

- Time complexity: $O(b^m)$
- Space complexity: $O(b \times m)$
- Completeness: is complete if finite m and prevent loop
- Optimal: no guarantee, it finds the leftmost solution

### 4.3.2 Breadth-First Search (BFS)

Its selecting strategy is **FIFO = "First In First Out"**, we work with **queue**.

- Time complexity: $O(b^s)$
  - with b: branching factor, s: shallowest-solution depth
- Space complexity: $O(b^s)$
  - with b: branching factor, s: shallowest-solution depth
- Completeness: no matter loop or not, it is complete
- Optimal: optimal only if **cost between each node = 1**

### 4.3.3 Comparison of DFS & BFS

DFS uses less memory, but needs loop detection; while BFS use more memory, but does not need loop detection. More details are shown as below:

- Depth-First Search
  - Space is restricted
  - Many solutions exist, solutions are long

  - no infinite paths

- Breadth-First Search
  - Space is not a problem
  - Solution containing fewest/shallowest depth

## 4.4 Iterative Deepening

Iterative Deepening is a combinatino of DFS and BFS, the idea is to get advantages from both DFS (space complexity) and BFS (time complexity, optimal, completeness). Run a DFS with depth limit 0/1/2/3/.... (P.S.:LIFO-stack, FIFO-queue).
Furthermore, the **outter** algorithm:

```
procedure iterative-deepening (
    Input:
    a graph,
    a set of start nodes,
    Boolean function goal(n) that tests if n is a goal mode
)
depthlimit = 1
while goal not found do
    call depth-limited-search(graph, starting nodes, goal(n), depthlimit)
    depthlimit = depthlimit + 1
```

The **inner** algorithm:

```
procedure depth-limited-search(
    Input:
    a graph,
    a set of start nodes,
    Boolean procedure goal(n) that tests if n is a goal node,
    depthlimit: natural number
)
frontier := < s > : s is a start node
while frontier is not empty:
    select and remove first path $< n_0, ..., n_k >$ from frontier
    if $k <$ depthlimit
        for every neighbor $n$ for $n_k$
            if goal(n)
                then return $n_0, ..., n_k, n$
            else add $n_0, ..., n_k, n$ to start of frontier
end while
```

The property of iterative deepening (ID) is similar to BFS, the only difference is the memory usage which is better than BFS ($O(b \times s) \geq O(b^s)$). *Time complexity almost is the same as BFS, the only difference is ID needs iterative* *reconstruction each limit of 0,1,2,3 needs to start from start node (root) and to rebuild the search tree*

- Time complexity: $O(b^s)$
  - with b: branching factor, s: shallowest-solution depth
- Space complexity: $O(b \times s)$
  - with b: branching factor, s: shallowest-solution depth
- Completeness: no matter loop or not, it is complete
- Optimal: optimal only if **cost between each node = 1**

## 4.5 Bi-directional search

Forward and backward search together. But it is not always possible *because backward is not always known.* The algorithm is shown below:

```
Input:
    a graph,
    a set of start nodes,
    Boolean procedure goal(n) that tests if n is a goal node.
$frontier_0 := < s >$: $s$ is a start node
$frontier_1 := < g >$: $g$ is a goal node
$i := 0$;
while $frontier_1$ and $frontier_0$ not empty:
    $i := i + 1$ mode 2; $j := i + 1$ mod 2; (swap $i$ and $j$)
    select and remove first path $< n_0, ..., n_k >$ from $frontier_i$
    for every neighbor $n$ of $n_k$ in direction $i$
        if $n$ occurs in path $< n'_0, ..., n'_l >$ of $frontier_j$ with l the length of n in $frontier_j$
            return solution based on $< n'_0, ..., n'_l >$ and $< n_0, ..., n_k >$
        else add $< n_0, ..., n_k, n >$ to end of $frontier_i$
end while
```

The property of Bi-directional search:

- Time complexity: $O(2 \times b^{\frac{d}{2}})$
  - with b: branching factor, d:maximum depth
- Space complexity: $O(2 \times b^{\frac{d}{2}})$
  - with b: branching factor, d:maximum depth
- Completeness: yes
- Optimal: yes <- **why?**

## 4.6 Uniform Cost Search (UCS)

BFS finds the shortest path in terms of number of actions. However, it does not find the least-cost path. We will now consider the cost between 2 nodes.
**The formula:** $cost(< n_0, n_1, ..., n_k >) = \sum_{i=1}^{k} cost(< n_{i-1}, n_i >)$
**SELECT** becomes **"priority queue, select element with lowest cost first"**
Two possibilities in graph searching: first version and last version.

1. First version: after selecting first lowest-cost node, done.
2. Last version: after checking every lowest-cost node, then done (more optimal).

Then we can have a further optimization: **Branch-and-bound principle:** when a goal is found along a path with cost C, prune all paths on the frontier with cost > C, where these cost can never be optimal since their cost has already > C.
The properties of UCS are:
The property of Bi-directional search:

- Time complexity: $O(b^{1+\frac{C^*}{\epsilon}})$
  - with b: branching factor, $C^*$: solution cost, arc cost (cost between 2 neighbor nodes) at least $\epsilon$
- Space complexity: $O(b^{1+\frac{C^*}{\epsilon}})$
  - with b: branching factor, $C^*$: solution cost, arc cost (cost between 2 neighbor nodes) at least $\epsilon$
- Completeness: yes (minimum arc cost is positive, $\epsilon > 0$)
- Optimal: yes

Remember: UCS explains increasing **cost contours**.

- Pros:
  - Complete and Optimal!
- Cons:
  - Explores option in every "direction", just like BFS
  - No information about goal location, only accumulation of cost (advanced algorithm should have advanced function/info providing)

## 4.7 Summary

1. The difference between each algorithm: **which node on the frontier will be explored next**.

2. Search algorithm is judged by its properties: **time complexity, space complexity, completeness, optimality**.

# 5 Informal search

## 5.1 Overview

Different from uninformed search and informed search.

- Uninformed search
  - use only information on **current path**, that is, from goal to fringe
- Informed search
  - use also **heuristics**, that is, information about how close nodes on the fringe are to a goal

A heuristics is a function that estimates **how close a state is to a goal**, which needs **domain knowledge**

There is different heuristic function: manhattan distance, euclidean distance, ....

## 5.2 Greedy search

The difference between greedy search and uniformed search is **SELECT = "priority queue, order elements by** $h(n)$, which is called "Best-first search". The difference between greedy search and uniformed cost-sensitive search is:

- Greedy Search uses heuristic to order.
- UCS uses accumulated cost to order.

Is it optimal? No, it just takes **the most obvious result** as next step, but may not be the best one. (Short-term vs. Long-term) Thus, the properties of greedy search is: (Time and space complexity is just badly-guide DFS)

- Time complexity: $O(b^m)$
  - with b: branching factor, m: maximum depth
- Space complexity: $O(b^m)$
  - with b: branching factor, m: maximum depth
- Completeness: no, like badly DFS
- Optimal: no, it just choose the most obvious result instead of the optimal one

## 5.3 Beam search

A variant of ~~greedy search~~ and breadth-first search. It is similar to breadth-first, but only expend $k$ times **each tier**. First build the whole tier, then keep $k$ elements in frontier. The properties of beam search is:

- Time complexity: $O(k \times b \times s)$
  - with k: hyperparameter, b: branching factor, s: shallowest-solution depth
- Space complexity: $O(k \times b)$
  - with k: hyperparameter, s: shallowest-solution depth
- Completeness: no, may drop the goal
- Optimal: no, not completeness not even optimal

When k = 1, it is called **"Hill-Climbing search"**.

## 5.4 Combination of UCS and Greedy Search

As we known, UCS uses **path cost = "Backward cost"** = $g(n)$ (Figure 1), and Greedy search uses **heuristic = "Forward cost** = $h(n)$ (Figure 2). And if we combine those two cost to estimate the quality of our current state: $f(n) = g(n) + h(n)$. The **SELECT = "priority queue, order elements by $f(n)$"**

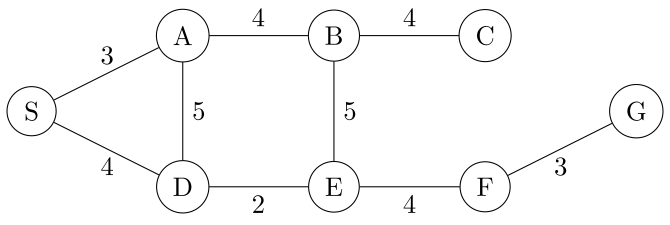


Figure 1: Path cost $g(n)$

However, this is not enough for optimal. We need to do something on the heuristic cost. Otherwises, the actual bad goal cost < estimated good goal cost. **Admissible heuritics** is needed, they **underestimate the actual cost**. Saying that $h$ is admissible iff for all nodes $n \leq h(n) \leq h^*(n)$ where $h^*(n)$ is the actual cost to the closest goal.
How to come up with admissible heuristics? The cost of an optimal solution to a **relaxed problem** is admissible heuristics, an understimate, for the original problem.

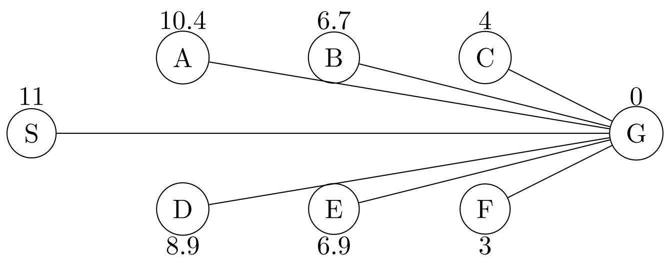


Figure 2: Heuristic $h(n)$

Larger admissible heuristics, better estimation, closer to actual value, but cannot larger than actual value (upper bound). For example, $h2$ dominates $h1$ means for all nodes $n : h2(n) \geq h1(n)$. In this case, $h2$ is always better than $h1$, but for all nodes $h1(n) \leq h2(n) \leq h^*(n)$. For two heuristics $h1$ and $h2$, $h3(n) = max(h1(n), h2(n))$ always dominates both $h1$ and $h2$, because for all nodes n, $h3(n) \geq h1(n)$ and $h3(n) \geq h1(n)$.

## 5.5 UCS + Greedy Search

**Generally, UCS + Greedy = $A^*$ without redundant path elimination.** For redundant paths: $A^*$, for memory usage $IDA^*$. The properties of combination between UCS and Greedy search is shown:

- Time complexity: $O(b^{1+\frac{C'}{\epsilon}})$worst case $h = 0$
  - with b: branching factor, $C^*$: solution cost, arc cost (cost between 2 neighbor nodes) at least $\epsilon$ shallowest-solution depth
- Space complexity: $O(b^{1+\frac{C'}{\epsilon}})$worst case $h = 0$
  - with b: branching factor, $C^*$: solution cost, arc cost (cost between 2 neighbor nodes) at least $\epsilon$
- Completeness: yes due to optimal
- Optimal: yes, if admissible heuristics

**Proof of optimality of UCS + G is shown:**

1. Imagine B (suboptimal) is on the fringe
2. Imagine some ancestor n of A (optimal) is on the fringe too
3. n will expaned before B
   (a) $f(n)$ is less or equal to $f(A)$ due to admissible heuristics
   (b) $f(A)$ is less or equal to $f(B)$ due to admissble heuristics

    (c) n expands before B

4. All ancestors of A expand before B
5. A (optimal) expands before B (suboptimal)
6. UCS + G search is optimal

### 5.6 Redundant Paths

Key: failure to detect **repeated states** can cause exponentially more work. **Idea: never expand a state twice.** One way to implement is shown:

1. **Keep track** of set of expanded states ”closed set”
2. Expand the search tree node-by-node
3. Before expanding a node, **check** to make sure its state has never been expanded before
4. **If not new, skip it, if new add to closed set.**

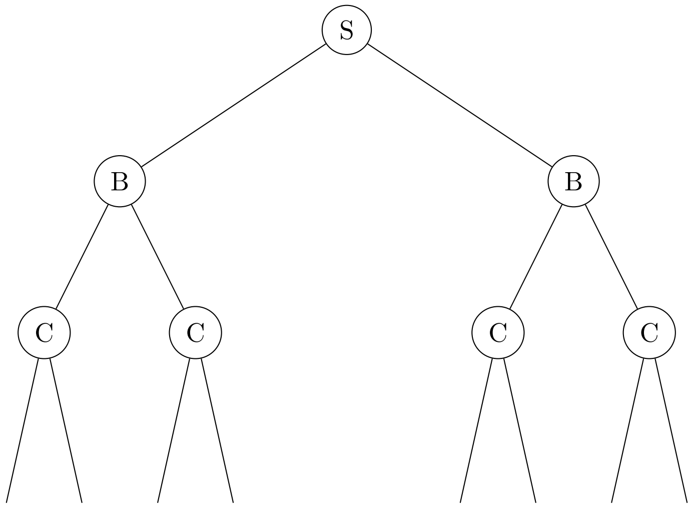


Figure 3: Repeated state without redundant path elimination

Generally,

- UCS + Greedy search = $A^*$ without redundant path elimination
- $A^*$ = UCS + Greedy search + redundant path elimination

The algorithm of redundant path elimination *(First version: the closed set, not the path deletion version)* is shown as below:

```
Input:
    a graph,
    a set of start nodes,
    Boolean procedure $goal(n)$ that tests if $n$ is a goal node.
frontier := $< s >$: $s$ is a start node
closed := {}
while frontier is not empty:
    select and remove path $< n_0, ..., n_k >$ from frontier
    if $n_k \notin closed$ then
        add $n_k$ to closed
        if goal($n_k$)
            return $n_0, ..., n_k$
        for every neighbor $n$ of $n_k$
            add $n_0, ..., n_k$ to frontier
end while
```

## 5.7 $A^*$ went wrong

Problem: When $A^*$ gone wrong?. If the heuristics is **not consistent**. There are two options: consistent heuristics or repair the frontier

1. Consistent heuristics: ensure that the first "selected" path to a node always has lowest cost, thus, we can directly prune node by closed set without considering. *(This can only work for consistent admissble heuristics)*
2. Repair the frontier: if there is a path $p =< s, ..., n, ..., m >$ on the frontier and a path $p'$ to $n$ such that $g(p' \geq g(p))$ then remove $p'$ from the frontier. *(This can work for any admissible heuristics)*

### 5.7.1 Consistent heuristics

Difference between admissibility and consistency. But the main idea for both is "estimated heuristic cost $\leq$ actual consts". Generally, **consistency $\rightarrow$ admissibility**. Consistency makes $A^*$ having a **f-contour**, and increase this **"f-contour"** one by one.

- Admissibility: from A to G, heuristic cost $\leq$ actual cost to goal (for whole)
  - $h(A) \leq$ actual cost from A to G
- Consistency: for each path from power set of from A to G, heuristic "arc" cost $\leq$ actual cost for each arc
  - $h(A) - h(C) \leq$ cost (A to C)

### 5.7.2 repair frontier

For repairing the frontier, we need to compare new repeated path, and the state in the current path, the algorithm is shown as below: **(no closed set exist)**

```
Input:
    a graph,
    a set of start nodes,
    Boolean procedure $goal(n)$ that tests if $n$ is a goal node.
$frontier := < s >$: $s$ is a start node
~~$closed := \{\}$~~
while $frontier$ is not empty:
    select and remove path $< n_0, ..., n_k >$ from $frontier$
    ~~if $n_k \notin closed$ then~~
        ~~add $n_k$ to $closed$~~
    if $goal(n_k)$
        return $n_0, ..., n_k$
    for every neighbor $n$ of $n_k$
        add $n_0, ..., n_k$ to $frontier$
    if there are $p =< s, t_1, t_2, ..., t_k, i, ..., m >$ and $p' =< s, s_1, s_2, ..., s_l, i >$ on the frontier and $g(p') \geq g(p)$
        then remove all such $p'$
end while
```

## 5.8 Iterative Deepening $A^*$

Idea from $A^*$ with consistent heuristics, with increasing "f-contour". Can we do this **iteratively**? Get DFSś space advantages with $A^*$ time. The difference between ID and ID$A^*$ is **how to choose "contour limit"**. The ID$A^*$ usually combined with loop breaking instead of redundant path elimination (this is not $A^*$), as this requires to store all generated nodes (closed set).

- For ID, just do a DFS, if no solution, just limit = limit + 1
- For ID$A^*$, if no solution, **use the smallest f value as the next contour.**

The algorithm of ID$A^*$ is shown as below:

```
procedure f-limited-search (
    input: a graph,
    a set of start nodes,
    Boolean procedure $goal(n)$ that tests if $n$ is a goal node,
    $f_{bound}$: positive number
)
$frontier := < s >$: $s$ is a start node
$f_{next} := \infty$
```

```
while frontier is not empty:
    select and remove first $p =< n_0, ..., n_k >$ from frontier
        if $goal(n_k)$ then return $p$
        for every neighbor $n$ of $n_k$
            compute $f_n = f(< n_0, ..., n_k, n >)$
            if $f_n \leq f_{bound}$
                then add $< n_0, ..., n_k, n >$ to front of frontier
            if $f_n > f_{found}$ and $f < f_{next}$
                then $f_{next} := f_n$
end while
return $f_{next}$
```

The properties of ID$A^*$ is shown as below:

- Time complexity: worst case like $b^0 + (b^0 + b^1) + (b^0 + b^1 + b^2) + ... + (b^0 + b^1 + ... + b^N) = ...$, which is $O(b^{\frac{C^*}{\epsilon}})$, as ID is $O(b^d)$
  - with b: branching factor, $C^*$: solution cost, arc cost (cost between 2 neighbor nodes) at least $\epsilon$ shallowest-solution depth, s: shallowest-solution depth
- Space complexity: $O(b \times \frac{C^*}{\epsilon})$ worst case $h = 0$, as ID is $O(b \times s)$
  - with b: branching factor, $C^*$: solution cost, arc cost (cost between 2 neighbor nodes) at least $\epsilon$, s: shallowest-solution depth
- Completeness: yes due to optimal
- Optimal: yes, under same situation as UCS + Greedy Search *(Consistent admissible heuristics or redundant path elimination)*

## 5.9 Summary

1. The best heuristics $f(n) = g(n) + h(n)$ is an **admissble heuristics** with the combination of Uniformed Cost Search and Greedy Search.
2. Coming up with a good admissible heuristics is important (larger -> better -> closer to the $h^*(n)$ (actual value), but also should be (for all nodes n) smaller than $h^*(n)$)
3. **Heuristics design** is key for searching problem!
4. Uniformed search vs. Informed Search = with or without **"heuristics"**
5. $f(n) = g(n)+h(n)$, **difference choices → difference algorithms**, where without $g(n)$ is Greedy search, and without $h(n)$ is UCS, with both $g(n)$ and $h(n)$ is $A^*$ without redundant path elimination.
6. Important **FOUR** properties for an algorithm: Time complexity, Space complexity, Completeness, Optimality.

# 6 Constraint Sastification Problems

## 6.1 Overview

Two different tasking: planning and identification

1. Planning: sequences of actions: path is important
2. Identification: assignments to variables: goal is important (CSP)

What is CSP?

- A special subset of search problems
- State is defined by varaible $X_i$ with values from a domain $D$
- Goal is a set of **constraints** specifying allowable combination of values for subsets of variables

For example, the N-Queens problem:

- Variables: $X_{i,j}$
- Domains: $D = 0, 1$
- Constraints: **(not allow touch together (1,1))**
  - $\forall i, j, k(X_{i,j}, X_{i,k} \in (0,0), (0,1), (1,0))$
  - $\forall i, j, k(X_{i,j}, X_{k,j} \in (0,0), (0,1), (1,0))$
  - $\forall i, j, k(X_{i,j}, X_{i+k,j+k} \in (0,0), (0,1), (1,0))$
  - $\forall i, j, k(X_{i,j}, X_{i+k,j-k} \in (0,0), (0,1), (1,0))$
  - $\sum_{i,j} X_{i,j} = N$

## 6.2 Variaties of CSP

Variables can be classified as: **discrete variables** and **continuous variables**. And the discrete variable can be further classified as **finite domain** and **infinite domains**.
Constrains can be classified as **hard constraints** and **soft constraints**. Hard constraint includes **unary constraint** involing a single variable: $SA \neq green$, or **binary constraint** involing pairs of variable: $SA \neq WA$, or even **higher-order constraints** involing 3 or more variables: $O+O = R+10\times X_1$. **Preferences** (soft constraint) is also important, which will be used in Constrained Optimization Problem.

## 6.3 Solution to CSP

The standard search formulation is defined as below (initial → assigning values to unassigned varaibles → check goal test):

1. Initial state: the empty assignment, {}
2. Successor funtion: assign a value to an unassigned variables
3. Goal test: the current assignment is complete and satisfies all constraints

## 6.4 Backtracking

Backtracking search is the basic uniformed algorithm for solving CSPs. DSP with two improvements below is called ”backtracking search”. In CSP, DFS is better than BFS since BFS will check layer by layer which is unnecessary.

1. One variable at time - only consider assignments to a single variable at each step
2. Check constraints as you go - consider only values which do not conflict previous assignments

The algorithm of backtracking search is shown as below:

```
function BACKTRACKING-SEARCH(csp) returns solution/ failure
    return RECURSIVE-BACKTRACKING({},csp)
function RECURSIVE-BACKTRACKING(assignment, csp) return solution/
failure
    if assignment is complete then return assignment
    var ← SELECT-UNASSIGNED-VARIABLE(VARIABLE[csp], assignment,
csp)
    for each value in ORDER-DOMAIN-VALUES(var, assignment, csp) do
        if value is consistent with assignment given CONSTRAINTS[csp] then
            add {var = value} to assignment
            var ← RECURSIVE-BACKTRACKING(assignment, csp)
            if result ≠ failure
                then return result % Find answer, so return result
                else remove {var = value} from assignment
    return failure % No completed solution found, so return failure
```

As we can see above, Backtracking = DFS + variable-ordering *(select one variable)* + fail-on-violation *(check constraint)*

## 6.5 Improvement on Backtracking

The improvement can be applied on three aspects: Ordering, Filtering and Structure:

1. Ordering - which should be next, similar to heuristics

2. Filtering - can we detect inevitable failure in advance?
3. Structure - can we exploit the problem structure?

### 6.5.1 Filtering

Filtering: keep track of domains for unassigned variables and cross off bad options
**Forward checking (FC)**: Corss off values that violate a constraint when added to the existing assignment; but not provide early detection for all failures. **Assign values to unassigned variables to check, and backtrack. Constraint propagation**: reason from constraint to constraint.
**Consistency of a single arc (AC-3)**: An arc X $\rightarrow$ Y is consistent iff for *every* x in the tail there is *some* y in the head which could be assigned without violating the constraint. Important: if X loses a value, neighbors of X need to be rechecked! Arc consistency detects failure earlier than forward checking. It checks **all** arcs are consistency once time.
Both constraint propagation and consistency of a single arc can be run as a **preprocessor** or **after each assignment**.
The algorithm is shown as below:

```
**function** AC-3(csp) **returns** false if an inconsistency is found and true otherwises
    queue ← a queue of arcs, initially all the arcs in csp % arcs: double direction ($X_i, X_j$) and ($X_j, X_i$) are different
    **while** queue is not empty **do**
        ($X_i, X_j$) ← POP(queue)
        **if** REVISE(csp, $X_i$, $X_j$) **then**
            **if** size of $D_i = 0$ **then return** false
            **for each** $X_k$ **in** $X_i$**.NEIGHBORS - {**$X_j$**} do**
                add ($X_k$, $X_i$) to queue
    **return** true

**function** REVISE(csp, $X_i$, $X_j$) **returns** true iff we revise the domain of $X_i$
    revised ← false
    **for each** $x$ **in** $D_i$ **do**
        **if** no value $y$ in $D_j$ allows ($x$,$y$) to satisfy the constraint between $X_i$ and $X_j$ **then**
            delete $x$ from $D_i$
            revised ← true
    **return** revised
```

Difference between forward checking and arc consistency: forward check is global and arc consistency is local (may not detect failure as more as forward check (the global)). More further comparison is shown as below:

- Filtering with Forward Checking (FC)

  - after assinging one variable, check the constraints that involve this variables

- Filtering with Arc Consistency (AC3)
  - like FC, but if a variable changed, also check all constraints involing this variable (except itself)

### 6.5.2 Ordering

There are two types of ordering, as shown below. But there is a common principle: **fail-first**.

1. Variable ordering - which one to assign first?
   (a) Minimum Remaining Values
   (b) Most Constratining Variables
2. Value ordering - which value to try first?
   (a) Least Constraining Value

Minimum Remaining Values: choose the variable with the fewest legal left values in its domain $\rightarrow$ less values means less steps to iterate its remaining values $\rightarrow$ easy to check whether fail or success, fail-first. It is also called ”Most Constrained Variable”

Most Constraining Variables: choose the variable with the most constraints (only counting constraints involving other unassigned variables) $\rightarrow$ most constraints $\rightarrow$ easier to fail $\rightarrow$ fail-first. It is also called ”max degree heuristics”

Least Constraining Value: given a choice of variable, choose the least constraining value.Least value $\rightarrow$ closer to threshold $\rightarrow$ easier to delete. $\leftarrow$ My guessing. Why least rather than most? You can do this with most, actually.

### 6.5.3 Structure

**Independent substructure:** Some structure can be broken into subproblems, which can significantly improve the efficiency. Similar to Bi-direction search, from $O(b^m)$ to $O(2 \times b^{\frac{m}{2}})$.
**Value symmetry:** in many CSP problems, there are many symmetries, we can reduce the search space by using **symmetry breaking constraints.**

## 6.6 Constraint Optimization

COP = Constraint Optimization Problem. It uses ”branch-and-bound”, similar to uniformed search.
**Key idea:**

1. maintain the ’best’ objective value so far
2. constrain solutions to be better than the ’best’

## 6.7 Summary

1. CSPs are a special kind of search problem
   (a) States are partial assignments
   (b) Goal test defined by constraints
2. Basic solution: backtracking search
3. Speed-up
   (a) Filtering - FC, AC3
   (b) Ordering - Minimum Remaining Values, Most Constraining Variables, Least Constraining Value
   (c) Structure - substructure, symmetry breaking constraints
   (d) Optimization - branch-and-bound

# 7 Pattern mining

## 7.1 Overview

Is to find interesting and actionable patterns and models in the data.

Example: Recommender System
Terminology of datasets:

- Itemsets
  - the special case where all attributes are boolean
  - a set of possible items $\mathscr{I}$
  - each example $e \subseteq \mathscr{I}$
  - each hypothesis $h \subseteq \mathscr{I}$
  - hypothesis $h$ covers example $e$ iff $h \subseteq e$
  - $h_1$ is more general than $h_2$ iff $h_1 \subseteq h_2$, $h_1$ has less specific items, which means it is less specific
  - **hypothesis = pattern**

## 7.2 Frequent itemset mining

**Given:**
a set of possible item $\mathscr{I} = i_1, ..., i_n$,
a dataset $D$
a threshold $c$
**Find** all itemsets $I \subseteq \mathscr{I}$ such that $I$ is frequent in $D$, i.e., $freq(I, D) \geq c$.

**"covers":**

- $h_1$ covers $h_2$ means $h_1$ is more general than $h_2$, i.e., {s,b,c} covers {s,b,c}, {s,b,c,m}
- more examples, {s,b} covers {s,b}, {s,b,c}, {s,b,c,m}

## 7.3 Search space

Adding more definition on *Itemsets*

- most general element: {}
- most specific element: $\mathscr{I}$

As we see from Figure 4 and Table 2, if the minimal freq threshold $c = 1$, $D = \{< s, b >\}$

- $freq(\{\}, D) = 1$

- $freq(\{s\}, D) = 1$
- $freq(\{b\}, D) = 0$
- $freq(\{s, b\}, D) = 0$

For minimal freq threshold $c = 1$, the answer would be **{}** and **{s}**.

Table 2: Table for generality ordering lattice for {s,b}

| **s** | **b** |
|---|---|
| True | False |

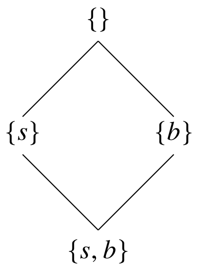


Figure 4: Figure for generality ordering lattice for {s,b}

”Anti-monotonicity” principle: if $I_1 \subseteq I_2$ (that is, $I_1$ is more general than $I_2$ then $freq(I_1, D) \geq freq(I_2, D)$, since it has more supersets)

There are two borders: specific borders and general borders.

1. S-set = $\{I \subseteq \mathscr{I} | I$ satisfies min and max frequency thresholds, and there is no $J \subseteq \mathscr{I}$ that is strictly more specific than $I$ and also satisfies the thresholds$\}$. $\rightarrow$ meaning that **S is the most specific solution**, it contains all maximall solutions, solution that cannot be extended with extra items.

2. G-set = $\{I \subseteq \mathscr{I} | I$ satisfies min and max frequency thresholds, and there is no $J \subseteq \mathscr{I}$ that is strictly more geeneral than $I\}$. $\rightarrow$ meaning that **G is the most general solution**, it contains all minimal solutions, solution that cannot be deleted.

Concept-learning = 100% frequency on the positives and 0% frequency on the negatives.

## 7.4 Enumeration Algorithms

The naive algorithm for frequent itemset mining is shown as below:

```
Input:
    a set of transactions $D$,
    a set of items $\mathcal{I}$,
    a frequency threshold $c$.
Queue = $\{\{\}\}$ % Empty set is also a itemset
for all $I \in Queue$ do
    if $freq(I, D) \geq c$
        then output I
        add all $I \cup \{i\}$ to Queue (with $i \in \mathcal{I} - I$) % Add, s,c,b,m to {} to make {s}, {c}, {b}, {m}
end for
```

Improvement: the naive algorithm has too many redundant examples, "lexicographic order" can be used to improve the efficiency. Symmetry breaking: impose the order $s > m > c > b$, and never add an element to a set $I$ if it smaller than the smallest element in the set. *Every set is only generated once and we get a tree instead of a graph.* The algorithm is shown as below **(min + max + lexicographic)**:

```
Input:
    a set of pos transactions $Pos$,
    a set of neg transactions $Neg$,
    a set of items $\mathcal{I}$,
    frequency thresholds $t_1, t_2$
Queue = $\{\{\}\}$
for all $I \in Queue$ do
    if $freq(I, Pos) > t_1$
        then if $freq(I, Neg) \leq t_2$ % This Neg can be Pos, too, but the truth meaning is $t' \geq freq(I, Pos) \leq t''$
            then output $I$
                add all $I \cup \{i\}$ to Queue (with $i \in \mathcal{I} - I$ and $i$ lexicographically smaller than elements of $I$)
end for
```

## 7.5 Summary

1. The basic concept-learning - frequent itemsets mining
2. min + max is just ont type of constraints
3. 100% frequency on positives and 0% on negatives = concept-learning with version spaces
4. Frequent patterns can be turned in association rules to make predictions - similar to Naive Bayes (Probabilistic Machine Learning)

# 8 Game tree

## 8.1 Games

There are different kind of games:

- Deterministic or stochastic game
- Multi-player or single player game
- Zero-sum or general game
- Board game ← we will focus on this!

A good game program/ strategy should have the properties that

1. delete **irrelevant branches** of the search tree
2. use **goof evaluation functions** for in-between states
3. **look ahead** as many moves as possible

## 8.2 Minimax

### 8.2.1 Terminology

Soem terminology of two-person board games:

- Players are **min** and **max**, where max begins
- **Initial** position
- **Operators**, actions
- **Game tree** is the search tree generated from the possible moves
- **Termination** test, determines when the game is over and what the **value of final state** is
- **Strategy**

### 8.2.2 Algorithm

For minimax algorithm, we use DFS to generate the game tree. We will apply **utility function**/ heuristics function to each terminal state. During the minimax, max should have the highest value, and min will always choose the worst-value step for max (they are opponents!). The algorithm is shown as below:

```
function MINIMAX-SEARCH(game, state) returns an action
    player ← game.TO-MOVE(state)
    value,move ← MAX-VALUE(game,state)
    return move
```

```
function MAX-VALUE(game,state) returns a (utility,move) pair
    if game.IS-TERMINAL(state) then return game.UTILITY(state,player),null
    v ← $-\infty$
    for each $a$ in game.ACTIONS(state) do
        v2,a2 ← MIN-VALUE(game,game.RESULT(state,a))
        if v2 > v then
            v,move ← v2,a
    return v,move

function MIN-VALUE(game,state) returns a (utility,move) pair
    if game.IS-TERMINAL(state) then return game.UTILITY(state,player),null
    v ← $+\infty$
    for each $a$ in game.ACTION(state) do
        v2,a2 ← MAX-VALUE(game,game.RESULT(state,a))
        if v2 < v then
            v,move ← v2,a
    return v,move
```

### 8.2.3 Evaluation function

The evaluation function is important! Some examples are shown below:

- Material value: pawn/1, knight/3, queen/9
- Other: king safety, good pawn structure
- Rule of thumb: three-point advantage

The preffered evaluation functions are **weighted, linear functions**: $w_1 f_1 + w_2 f_2 + ... + w_n fn$, where $w_i$ are the weights and the $f_i$ are the features of the board. However, it has a strong assumption that all of features are **independent**, which is unrealistic.

### 8.2.4 Stopping criteria

Stopping critiera can be fixed-depth or iterative deepening (when time is over.) However, only stop and evaluate in **quiescent positions** that will not cause large fluctuations. Because we do not know what happen if so large fluctuations happen may cause → **horizon effect:** try to delay lose but fail and with more cost.

## 8.3 Alpha-Beta Search

It is an optimization of minimax algorithm, which helps improve its efficiency. Instead of generating tree → propogating the values upwards in the tree. We try to **interleave these two steps**, meaning that exploit → propogate → prune → exploit → propogate → prune. Some terminology is shown as below:

- The temporary values at MAX-nodes are alpha($\alpha$) values
- The temporary values at MIN-nodes are beta($\beta$) values

Minimax algorithm with DFS, it provides the **same result** as the complex minimax search to the same depth because only irrelevant nodes are eliminated.

- ALPHA = the value of the best choice we have found so far at any choice point along the path for MAX
  - ALPHA value can **never devrease**
- BETA = the value of the best choice we have found so far at any choice point along the par for MIN
  - BETA value can **never increase**

The Alpha-Beta (pruning) principle is shown as below:

- If an **ALPHA-value is larger or equal than the BETA-value** of a descendant node: stop generation of the children of the descentdant
- If a **BETA-value is smaller or equal than the ALPHA-value** of a descendant node: stop generation of the children of the descendant

The algorithm for ALPHA-BETA is shown as below:

```
function MINIMAX-SEARCH(game,state) returns an action
    player ← game.TO-MOVE(state)
    value,move ← MAX-VALUE(game,state,$-\infty$,$+\infty$)
    return move

function MAX-VALUE(game,state,$\alpha$,$\beta$) returns a (utility,move) pair
    if game.IS-TERMINAL(state) then return game.UTILITY(state,player),null
    v ← $-\infty$
    for each $a$ in game.ACTIONS(state) do
        v2,a2 ← MIN-VALUE(game,game.RESULT(state,a),$\alpha$,$\beta$)
        if v2 > v then
            v,move ← v2,a
            $\alpha$ ← MAX($\alpha$,v) % prepare $\alpha$ for MIN
        if v > $\beta$ then reutrn v,move then return result % ALPHA-BETA pruning, just return
    return v,move

function MIN-VALUE(game,state,$\alpha$,$\beta$) returns a (utility,move) pair
    if game.IS-TERMINAL(state) then return game.UTILITY(state,player),null
    v ← +∞
    for each $a$ in game.ACTION(state) do
        v2,a2 ← MAX-VALUE(game,game.RESULT(state,a),$\alpha$,$\beta$)
        if v2 < v then
            v,move ← v2,a
            $\beta$ ← MIN($\beta$,v) % prepare $\beta$ for MAX
        if v ≤ $\alpha$ then return v,move % ALPHA-BETA pruning, just return
    return v,move
```

The Gain: best case. If at every layer: the best node is the left-most one **(perfectly ordered tree)**. In the best case, the search expenditure is reduced to $O(b^{\frac{d}{2}})$

## 8.4 Expectiminimax

If there is posibility, stochastic instead of deterministic.
$d_i$: possible dice roll
$P(d_i)$: probability of obtaining that roll
$EXPECTIMINIMAX(s) =$
$UTILITY(s)$ if $IsTerminal(s)$
$max_a EXPECTIMINIMAX(Result(s,a))$ if $ToMove(s) = MAX$
$min_a EXPECTIMINIMAX(Result(s,a))$ if $ToMove(s) = MIN$
$\sum_d P(d) \times EXPECTIMINIMAX(Result(s,d))$ if $ToMove(s) = CHANCE$
$Result(s,d)$: attainable positions from $C$ with $d$

## 8.5 Summary

1. A **game** can be defined by the initial state, the operator/actions, a terminal test, a utility function/ outcome of the game
2. In two-player board game, **minimax algorithm** can be used.
3. The **alpha-beta algorithm** produces the **same result** but it is more efficient because it just prunes away irrelevant branches.
4. **Utility/heuristics** of some states will be determined by an **evaluation function**.
5. Games of chance can be handled by some extension, EXPECTIMINIMAX.
6. The success for different games is based on quite different methodologies.

# 9 Planning

## 9.1 Overview

Some lectures overview:

- Algorithm fundamentals of search
  - Intro AI and Rational agents
  - Uniformed Search
  - Informed Search
- Applications of search in deterministic environments
  - CSP
  - Pattern Mining
  - Game Tree
  - Planning
- Application of search in stochastic environments
  - Markov Decision Processes
- BDA
  - Intro to logic
  - SAT solving
  - Local Search

The planning we learn is **Goal Oriented Action Planning: STRIPS Planning.** Search for a plan that achieves that goal in the current state.

## 9.2 STRIPS planning

### 9.2.1 Overview

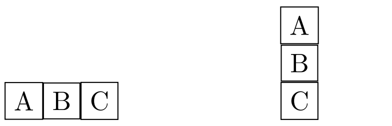
Typical description of a planning problem:

- Initial state
- Goal
- Available actions

Typical solution of a planning problem: a sequence of actions when excuted in the initial state results in a state that satisfies the goal condition. (**A sequence of actions** that achieves the goal for **every application domain**)
Planning: the automatic discovery of such solutions. (**Find** such sequence of actions that achieves the goal for every application domain)

### 9.2.2 Example: Blocks world domain

- Initial state: $s_0$
- Goal: $g$
- Available actions:
  - $move(A, table, B)$: move A from table to top of B
  - $move(A, B, table)$: move A from top of B to table
  - $move(A, C, B)$: move A from top of C to top of B (not from table)



(a) Initial state $s_0$ (b) Goal state $g$

Figure 5: Initial state and Goal state for Blocks world domain

In the beginning, we need to find a representation for the state and actions. For the state in blocks world is shown as below. For each state, the state should be **completely** specified based on the **Closed-World Assumption: any literal not mentioned in the description of the state are assumed to be false!**.

- On(b,x):
  - block b is on top of x, where x is another block
- Clear(x):
  - a block can be placed on the top of x

Thus, the initial state in Figure 5a are: $on(a, table)$, $on(b, table)$, $on(c, table)$, $clear(a)$, $clear(b)$, $clear(c)$. (Closed-World Assumption) Additionally, the goal state in Figure 5b are: $clear(a)$, $on(a, b)$, $on(b, c)$, $on(c, table)$. (No Closed-World Assumption), we only need to satify the goal. (A state $s$ satisfies goal $g$ if it contains all literals in $g$, containing is okay, not exactly identified.)
Next is the availble actions. It consists of **preconditions** and **effects**.

- Preconditions: literals denoting what needs to be in the state for the action to be availble
- Effects: literals denoting how the state is changing when action is applied

One of the avialble action is $move(b, x, y)$, meaning that move b from top of x to top of y. Anonther action can be $moveToTable(b, x)$ and $moveFromTable(b, x)$

$move(b, x, y)$

  Preconditions:

    $on(b, x)$, $clear(b)$, $clear(y)$

  Effects:

    $\neg on(b, x)$, $\neg clear(y)$, $on(b, y)$, $clear(x)$

$moveToTable(b, x)$

  Preconditions:

    $clear(b)$, $on(b, x)$

  Effects:

    $\neg on(b, x)$, $on(b, table)$, $clear(x)$

$moveFromTable(b, x)$

  Preconditions:

    $clear(b)$, $clear(x)$

  Effects:

    $\neg on(b, table)$, $\neg clear(x)$, $on(b, x)$

Why do we like formalism? Because it is easy for representation. STRIPS planning: finding a solution to the planning problem following a **state-based search:**

- Init(where to start from)
- Goal(when to stop searching)
- Action(how to generate the "graph")

When planning, there are two planning:

- Progression planning: forward state-based search ← mainly focus on
- Regression planning: backward state-based search

The **pseudocode** for progressing planning: - similar to backtracking

```
1. Start from the initial state
2. Check if the current state satisfies the goal
3. Compute availble actions to the current state
4. Compute the successor states
5. Pick one of the (not-visited) successor states as the current state
6. Repeat until a solution is found or the state space is exhausted ← similar to backtracking
```

Is it guaranteed that progressive planning will find a solution if one exists? ← yes, if the state-space is finite

But it will be very very slow, if we search the whole space-state (the worst case). ← heuristics needed! We need heuristics that help progression planning pick the most promising states to investigate first

## 9.3 Heuristics for STRIPS planning

Analogy: $A^*$ search, evalution function = $f(s) = g(s) + h(s)$, where $g(s)$ from UCS and $h(s)$ from greedy search. Similar to the $A^*$ search, here we use $f(s)$ for STRIPS planning to pick the most promising one. Some of possible $h(s)$ and $g(s)$ can be as below. *(Similarly, we need to check whether it is admissble heuristics to obtain the optimal solution)*

- $g(s)$: number of literals that exists in the goal
- $h(s)$: number of literals in the goal that are missing from s

The **pseudocode** for **progressing planning** applied with heuristics can be updated as below:

1. Start from the initial state
2. Check if the current state satisfies the goal
3. Compute availble actions to the current state
4. Compute the successor states
5. Pick **(the most promising** successor states as the current state
6. Repeat until a solution is found or the state space is exhausted $\leftarrow$ similar to backtracking

Alternatives: **pseudocode** for **Regression planning** is similar but invert the progressing planning:

1. Start from the **goal** state as current state
2. Check if the **initial state** satisfies the current state
3. Compute the **relevant** and **consistent** actions for current state
4. Compute the **predecessor** states (which is actually next state in this case)
5. Pick **(the most promising** successor states as the current state
6. Repeat until a solution is found from **goal state** to **initial state** or the state space is exhausted (no solution)

## 9.4 Summary

1. A generic formalism (easy to represent difficult space search, similar to ILP) for **representing** planning problem (a specific aspect in searching problem)
2. Generic **search** procedures for finding a plan

   (a) Progressing planning (forward)
   (b) Regression planning (backward, not discussed, only pseudocode)

3. Application in many aspects, e.g., space, flight, robotics, etc.

# 10 Markov Decision Process

## 10.1 Overview

This section is about non-deterministic search, which is called stochastic search. First we can take Grid World as an example, which is shown below:

- A maze-like problem
  - Agent move in the grid
  - Walls block the agent's path
- Noisy movement - stochastic movement
  - 80% $\rightarrow$ correct direction, 20% $\rightarrow$ other (incorrect) directions.
- The agent recives rewards each time step
  - Small living rewards (can be negative)
  - Big rewards at the end (negative or positive)
- Goal: maximize sum of rewards

An MDP is defined as below. The MDP, the "markov" means actions outcomes depend only the current state, and the current, future and past are independent. It is similar to search, where the successor function could only depend on the current state (not the history).

- a set of states $s \in S$
- a set of actions $a \in A$
- a transition function $T(s, a, s')$
- a reward function $R(s, a, s')$
- an initial state
- some goal states (probably not)

Different between deterministic and stochastic search problem. In deterministic search, we want **an optimal path**, while in stochastic search, we want **an optimal policy** $\pi : S \rightarrow A$**,** where makes the agent becomes relfex agent. Each MDP state projects an expectimax-like search tree, similar to minimax but only with max layer and an addition chance state layer.
Some preference exists in MDP: **maximize** the sum of rewards $\rightarrow$ only max layer, prefer rewards **now** instead of later $\rightarrow$ discount factor $\gamma$ $\rightarrow$ indicating that values of rewards decay exponentially.
Stopping criteria for MDP:

1. Finite horizon (similar to depth-limited search), terminate episodes after a fixed T steps (e.g., life)
2. Discounting: use $0 < \gamma < 1$: finally it will converge
3. Absorbing state $\leftarrow$ Question: do not understand!

## 10.2 Solution for MDP

- Basic features
  - Set of states $S$
  - Initial state $s_0$
  - Set of actions $A$
  - Transition $P(s'|s,a)$ (or $T(s,a,s')$) $\leftarrow$ probability
  - Rewards $R(s,a,s')$ (and discount $\gamma$)
- MDP quantities so far
  - Policy = Choice of action for each state (Policy layer)
  - Utility = sum of (discount) rewards (Value layer)

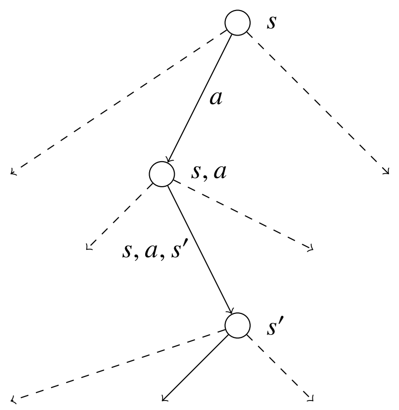


Figure 6: Search tree of MDP

- The value (utility) of *a* state $s$:
  - $V'(s)$ = expected utility starting in $s$ and acting optimally
- The value (utility) of a q-state $(s,a)$:
  - $Q'(s,a)$ = expected utility starting out having taken action $a$ from state $s$ and acting optimally (transition q-state).

- The optimal policy:
    - $\pi'(s)$ = optimal action from every state $s$

Recursive definition of value, **Bellman equations**:

- $V^*(s) = \max_a Q^*(s,a)$
- $Q^*(s,a) = \sum_{s'} T(s,a,s'[R(s,a,s') + \gamma V^*(s')])$
- $V^*(s) = \max_a \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V^*(s')]$

The problem is still there, when to stop? Key idea: **time-limited values** by defining "life". Define $V_k(s)$ to be the optimal value of $s$ if the game ends in $k$ more time steps. Thus, there will be **value iteration** and **policy iteration**.

## 10.3 Value iteration

The complexity of each value iteration is $O(S^2 \times A)$

1. Start with $V_0(s) = 0$: no time steps left means an expected reward sum of zero.
2. Given vector of $V_k(s)$ values, do one ply of expectimax from each state: $V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V_k(s')]$
3. Repeat until convergence

Some problems with value iteration:

1. It is slow - with complexity $O(S^2 \times A)$ per iteration
2. The "max" at each state rarely changes
3. The policy (a set of actions) often **converges long** before the values

## 10.4 Policy iteration

### 10.4.1 Fixed policy

If we try a fixed policy $\pi(s)$, then the tree will be simpler - only one action per state, compared to that the expectimax tree max over all actions to compute the optimal values. Define the utility (V function) of a state $s$, under a fixed policy $\pi$, the value function will be: $V^\pi(s) = \sum_{s'} T(s,\pi(s),s')[R(s,\pi(s),s') + \gamma V^\pi(s')]$. How do we calcualte the V's for a fixed policy *pi*?

- $V_0^\pi(s) = 0$
- $V_{k+1}^\pi(s) \leftarrow \sum_{s'} T(s,\pi(s),s')[R(s,\pi(s),s') + \gamma V_k^\pi(s')]$, with $O(S^2)$

### 10.4.2 Policy extraction

After obtaining values or q-values from the current map, we can extract the policy from a value-map or q-value-map, thus we have two basic algorithm for extracting policy (a set of actions): computing actions from values or computing actions from Q-values. It is obvious that **actions are easier to select from q-values than values**, because $V(s) = arg \max_a Q(a, s)$.

- Computing actions from values:
  - $\pi^*(s) = arg \max_a \sum_{s'} T(s, a, s')[R(s, a, s') + \gamma V^*(s')]$
- Computing actions from Q-values:
  - $\pi^*(s) = arg \max_a Q^*(s, a)$

### 10.4.3 Policy iteration

Alternative approach for optimal values: (policy fixed instead of expectimax search). Optimal and can converge faster under some conditions compared to expectimax search.

1. Step 1: Policy evaluation: calculate utilities for some **fixed policy** (not optimal utilities!) until **convergence**
2. Step 2: Policy improvment: update policy using one-step look-ahead with resulting converged (not optimal but will be finally) utilities as **future values**
3. Repeat steps until policy converges

For more details in policy iteration: evaluation is for finding values by fixed policy $\pi_i : V^{\pi_i}(s) \rightarrow V^{\pi_i}(s')$, and improvment is for extracting policy by fixed values $V^{\pi_i}(s) : \pi_i \rightarrow \pi_{i+1}$. And repeat these two processes until convergence of policy (policy (all actions) is not changed anymore.)

- **Evaluation**: for fixed current policy $\pi$, find values with policy evaluation:
  - $V^{\pi_i}_{k+1}(s) \rightarrow \sum_{s'} T(s, \pi_i(s), s')[R(s, \pi_i(s), s') + \gamma V^{\pi_i}_k(s')]$
- **Improvement**: for fixed values, get a better policy using policy extraction
  - $\pi_{i+1}(s) = arg \max_a \sum_{s'} T(s, a, s')[R(s, a, s') + \gamma V^{\pi_i}(s')]$

## 10.5 Value vs. Policy iteration

Both value iteration and policy iteration compute the same thing (all optimal values), but with different efficiency under various situation. Both are dynamic programs for solving MDPs.

1. In value iteration

   (a) Every iteration updates both the values and **(implicitly)** the policy $\leftarrow$ only value function explicitly
   (b) We do not track the policy, but taking the max over actions implicity recomputes it, just not show a policy map, only show a value map.

2. In policy iteration
   (a) In each turn, we intersectly **update the utility with fixed policy**, and **extract the policy with fixed updated utility** (each turn is fast since we only consider one action not all of them instead of expectimax search/ value interation) $\leftarrow$ both value and policy function explicitly
   (b) After the policy is evaluated, the new policy is chosen
   (c) Using new policy to continue till the stopping criteria (life-limit or threshold)
   (d) Here we not only check the V-value/Q-value but also check the policy, show both V-value/Q-value map and policy map.

## 10.6 Summary

1. Different methods are introduced
   (a) Compute optimal values (whole): **value iteration** or **policy iteration**
   (b) Compute values from a particular policy: use **policy evaluation**
   (c) Extract policy from a set of values: use **policy extraction** (one-step lookahead?)
2. However, both value iteration and policy iteration are similar
   (a) They are basically are variations of **Bellman updates**
   (b) They **all** use **one-step lookahead expectimax fragments** (no matter value iteration, but also policy iteration)
   (c) Difference in **max over actions (value iteration)** or **fixed policy (policy iteration)**, just different efficiency under different situation, sometime value iteration better, sometime policy iteration better, it depends on your own domain knowledge!