JWJ

Optimization of Dynamic Systems II

General initial and terminal constraints with free initial and fixed time

$$\min_u J = \bar{\phi}_0(t_0, \bar{x}_0) + \bar{\phi}_f(t_f, \bar{x}_f) + \int_{t_0}^{t_f} L(t, \bar{x}, \bar{u})\,dt$$

Subject to

$\dot{\bar{x}} = \bar{f}(t, \bar{x}, \bar{u})$ dynamics

$\psi_0(t_0, \bar{x}_0) = 0$ initial BC

$\psi_f(t_f, \bar{x}_f) = 0$ terminal BC

Adjoin constraints to cost functional

$$J' = [\phi_0(t_0, x_0) + \nu_0^T \psi_0(t_0, x_0)] + [\phi_f(t_f, x_f) + \nu_f^T \psi_f(t_f, x_f)] + \int_{t_0}^{t_f} \{L(t, x, u) + \lambda^T[f(t, x, u) - \dot{x}]\}\,dt$$

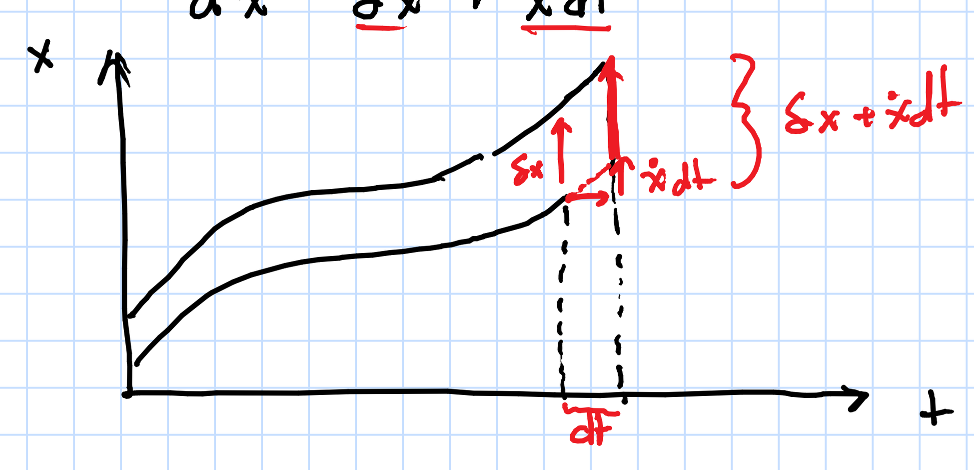
Combine terms for simplicity

$$\Phi_0(t_0, x_0) = \phi_0 + \nu_0^T \psi_0$$
$$\Phi_f(t_f, x_f) = \phi_f + \nu_f^T \psi_f$$
$$H(t, x, u) = L(t, x, u) + \lambda^T f$$
$$\Rightarrow \quad J' = \Phi_0 + \Phi_f + \int_{t_0}^{t_f} [H - \lambda^T \dot{x}]\,dt$$

Note: Now that t_0, t_f are free, we need to account for changes in time as well

- Variations change the variable while holding time fixed
- The differential change consists of the variation and the time varying component

$$dx = \delta x + \dot{x}\,dt$$



Leibniz Rule

$$d\int_{t_0}^{t_f} F(t, x)\,dt = \delta\int_{t_0}^{t_f} F(t, x)\,dt + \int_{t_0}^{t_f} [\dot{F}\,dt]\,dt$$
$$= \delta\int_{t_0}^{t_f} F(t, x)\,dt + [F\,dt]_{t_0}^{t_f}$$

Take differential of J' and set to equal 0

$$J' = \Phi_0 + \Phi_f + \int_{t_0}^{t_f} [H - \lambda^T \dot{x}]\,dt$$

$$0 = dJ' = \left[\frac{\partial \Phi_0}{\partial t}dt + \frac{\partial \Phi_0}{\partial x}dx\right]_{t=t_0} + \left[\frac{\partial \Phi_f}{\partial t}dt + \frac{\partial \Phi_f}{\partial x}dx\right]_{t=t_f} + [(H - \lambda^T \dot{x})\,dt]_{t_0}^{t_f}$$
$$+ \int_{t_0}^{t_f}\left[\frac{\partial H}{\partial x}\delta x + \frac{\partial H}{\partial u}\delta u + \cancel{\left(\frac{\partial H}{\partial \lambda}\delta\lambda - \dot{x}^T\delta\lambda\right)} - \lambda^T\delta\dot{x}\right]dt$$

L

$(f - \dot{x})^T \delta\lambda = 0$

Integration by parts

$$\int_{t_0}^{t_f} -\lambda^T \delta\dot{x}\,dt = [-\lambda^T \delta x]_{t_0}^{t_f} + \int_{t_0}^{t_f} \dot{\lambda}^T \delta x\,dt$$

$$dJ' = 0 = \left[\frac{\partial \phi_0}{\partial t}dt + \frac{\partial \phi_0}{\partial x}dx\right]_{t=t_0} + \left[\frac{\partial \phi_f}{\partial t}dt + \frac{\partial \phi_f}{\partial x}dx\right]_{t=t_f} + [L\,dt]_{t_0}^{t_f} + [-\lambda^T\delta x]_{t_0}^{t_f}$$
$$+ \int_{t_0}^{t_f}\left[\frac{\partial H}{\partial x}\delta x + \frac{\partial H}{\partial u}\delta u + \dot{\lambda}^T\delta x\right]dt$$

Collect terms:

Note $dx = \delta x + \dot{x}\,dt$
$\Rightarrow \delta x = dx - \dot{x}\,dt$

$$dJ' = 0 = \left[\left(\frac{\partial \phi_0}{\partial t} - L\right)dt + \frac{\partial \phi_0}{\partial x}dx\right]_{t=t_0}$$
$$\left[\left(\frac{\partial \phi_f}{\partial t} + L\right)dt + \frac{\partial \phi_f}{\partial x}dx\right]_{t=t_f}$$
$$+ [-\lambda^T(dx - \dot{x}\,dt)]_{t_0}^{t_f}$$
$$+ \int_{t_0}^{t_f}\left[\left(\frac{\partial H}{\partial x} + \dot{\lambda}^T\right)\delta x + \frac{\partial H}{\partial u}\delta u\right]dt$$
$$= \left[\left(\frac{\partial \phi_0}{\partial t} \underbrace{- L - \lambda^T\dot{x}}_{-H}\right)dt + \left(\frac{\partial \phi_0}{\partial x} + \lambda^T\right)dx\right]_{t=t_0}$$
$$+ \left[\left(\frac{\partial \phi_f}{\partial t} \underbrace{+ L + \lambda^T\dot{x}}_{H}\right)dt + \left(\frac{\partial \phi_f}{\partial x} - \lambda^T\right)dx\right]_{t=t_f}$$
$$+ \int_{t_0}^{t_f}\left[\left(\frac{\partial H}{\partial x} + \dot{\lambda}^T\right)\delta x + \frac{\partial H}{\partial u}\delta u\right]dt$$
$$= \left[\left(\frac{\partial \phi_0}{\partial t} - H\right)dt + \left(\frac{\partial \phi_0}{\partial x} + \lambda^T\right)dx\right]_{t=t_0}$$
$$+ \left[\left(\frac{\partial \phi_f}{\partial t} + H\right)dt + \left(\frac{\partial \phi_f}{\partial x} - \lambda^T\right)dx\right]_{t=t_f}$$
$$+ \int_{t_0}^{t_f}\left[\left(\frac{\partial H}{\partial x} + \dot{\lambda}^T\right)\delta x + \frac{\partial H}{\partial u}\delta u\right]dt$$

Setting terms to 0

$\dot{\lambda}^T = \frac{\partial H}{\partial x}$ n diff

$\frac{\partial H}{\partial u} = 0$ m alg eqn

$0 = \frac{\partial \Phi_0}{\partial t} - H_0 = \frac{\partial \phi_0}{\partial t} + \nu_0^T\frac{\partial \psi_0}{\partial t} - H_0$ 1 eqn

$0 = \frac{\partial \phi_f}{\partial t} + H_f = \frac{\partial \phi_f}{\partial t} + \nu_f^T\frac{\partial \psi_f}{\partial t} + H_f$ 1 eqn

$0 = \frac{\partial \Phi_0}{\partial x} + \lambda_0^T = \frac{\partial \phi_0}{\partial x} + \nu_0^T\frac{\partial \psi_0}{\partial x} + \lambda_0^T$ n alg eqn

$0 = \frac{\partial \phi_f}{\partial x} - \lambda_f^T = \frac{\partial \phi_f}{\partial x} + \nu_f^T\frac{\partial \psi_f}{\partial x} - \lambda_f^T$ n alg eqn

$\dot{x} = f$ n diff

$\psi_0 = 0$ p eqns

$\psi_f = 0$ q eqns

$t_0, t_f, x, u, \lambda, \nu_0, \nu_f$

$1 + 1 + n + m + n + p + q = 2 + 2n + m + p + q$ unknowns

$H = L + \lambda^T f$

- Condition for constant Hamiltonian

$$\frac{d}{dt}H = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial x}\dot{x} + \frac{\partial H}{\partial \lambda}\dot{\lambda} + \frac{\partial H}{\partial u}\dot{u}$$

At optimal solution

$\frac{\partial H}{\partial u} = 0 \qquad \frac{\partial H}{\partial \lambda} = f \qquad \dot{\lambda}^T = -\frac{\partial H}{\partial x}$

$$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \underbrace{\cancel{\left(\frac{\partial H}{\partial x}f - \frac{\partial H}{\partial x}^T f\right)}}_{0} + \underbrace{\cancel{\frac{\partial H}{\partial u}}}_{0}\dot{u}$$

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}$$

The Hamiltonian is constant if it is not an explicit function of time

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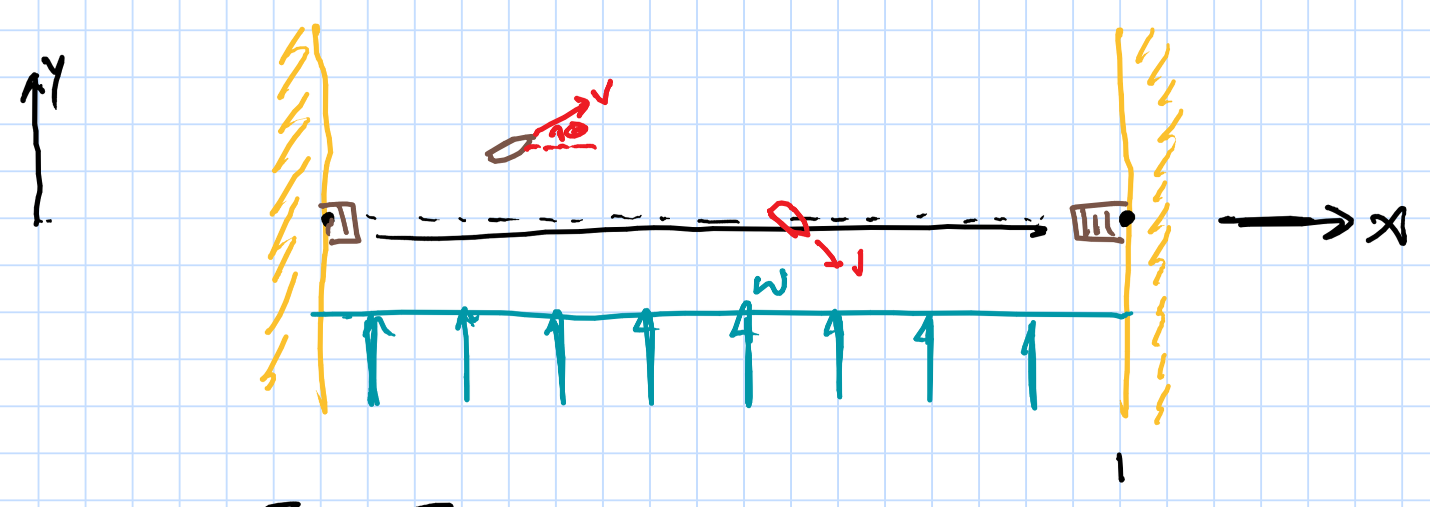
Zermelo's Boat Problem

- Minimize time for a boat to cross a river with constant current w



$\bar{x} = \begin{bmatrix} x \\ y \end{bmatrix} \qquad u = \theta$

$f = \begin{bmatrix} v\cos\theta \\ v\sin\theta + w \end{bmatrix}$

$\psi_0 = \begin{bmatrix} x \\ y \end{bmatrix}$

$\psi_f = \begin{bmatrix} x - x_f \\ y \end{bmatrix}$

$$\min J = t_f = \int_0^{t_f} dt$$
$$H = 1 + \lambda_x v\cos\theta + \lambda_y(v\sin\theta + w)$$
$$\dot{\lambda} = \left(-\frac{\partial H}{\partial x}\right)^T = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} \lambda_x = c_1 \\ \lambda_y = c_2 \end{matrix}$$
$$\frac{\partial H}{\partial u} = 0 = -\lambda_x v\sin\theta + \lambda_y v\cos\theta = 0$$
$$\Rightarrow \tan\theta = \frac{\lambda_y v}{\lambda_x v} = \frac{\lambda_y}{\lambda_x} = \text{constant}$$
$$\theta = \tan^{-1}\frac{\lambda_y}{\lambda_x} = c_3$$

$$\dot{x} = v\cos\theta \Rightarrow x = (v\cos\theta)t + c_4$$
$$x(t=0) = 0 \Rightarrow c_4 = 0$$
$$x = vt\cos\theta$$
$$x(t = t_f) = x_f = (v\cos\theta)t_f$$
$$\dot{y} = v\sin\theta + w$$
$$\Rightarrow y = (v\sin\theta + w)t + c_5$$
$$y(0) = 0 \Rightarrow c_5 = 0$$
$$y(t_f) = 0 \Rightarrow (v\sin\theta + w)t_f = 0$$
$$\Rightarrow \sin\theta = -\frac{w}{v}$$
$$\theta = \sin^{-1}\left(-\frac{w}{v}\right)$$

optimal control

$$x_f = v\cos\left(\sin^{-1}\left(-\frac{w}{v}\right)\right)t_f$$

$$\cos\theta = \sqrt{1 - \sin^2\theta} = \sqrt{1 - \frac{w^2}{v^2}} = \frac{\sqrt{v^2 - w^2}}{v}$$
$$t_f = \frac{x_f}{\sqrt{v^2 - w^2}}$$

Terminal term:

$$0 = \frac{\partial \phi_f}{\partial x} - \lambda_f^T = \frac{\partial \phi_f}{\partial x} + \nu_f^T\frac{\partial \psi_f}{\partial x} - \lambda_f^T$$

If x_i is constrained

$\frac{\partial \psi_f}{\partial x_i} \neq 0$

so $\nu_{f,i}$ can change to make equation 0

so $\lambda_{f,i}$ is free

If x_i is not constrained

$\frac{\partial \psi_f}{\partial x_i} = 0$

so $\nu_{f,i}$ is free and $\lambda_{f,i}$ is constrained