Jmy

Optimal Control Problem w/ Path Constraints

Path constraint apply over entire trajectory

Integral Path Constraint

Consider an OCP w/ constraint

$$\int_{t_0}^{t_f} N(t,x,u)\,dt = k \quad = \text{constant}$$

- Enforce by adding another "state"

Let $\dot{y} = N$, $\quad y(t_0) = 0$, $y(t_f) = k$

- Adjoin new "state"

$$H = L + \lambda^T f + \mu N$$

$$\dot{\mu} = -\frac{\partial H}{\partial y} = 0 \Rightarrow \mu \text{ is constant}$$

and

$$\dot{\lambda} = -\frac{\partial H}{\partial x}$$

$$\frac{\partial H}{\partial u} = 0$$

$$\dot{x} = f(t,x,u)$$

$$\dot{y} = N(t,x,u)$$

Equality Path Constraints of Control Variables

$$C(t,u) = 0$$

where u is a higher dimension than C

2 methods

- unadjoined
  - eliminate some control variables through substitution of C equations
- adjoined
  - adjoin constraint equations to path cost

$$\Rightarrow H = L + \lambda^T f + \mu^T C$$

Since $C(t,u)$, the only change the the optimality conditions is

$$0 = \frac{\partial H}{\partial u} = \frac{\partial L}{\partial u} + \lambda^T \frac{\partial f}{\partial u} + \mu^T \frac{\partial C}{\partial u}$$

Equality Path Constraints of Control & State Variables

$$C(t,x,u) = 0$$

Adjoin constraint

$$H = L + \lambda^T f + \mu^T C$$

New optimality conditions

$$\frac{\partial H}{\partial u} = 0 = \frac{\partial L}{\partial u} + \lambda^T \frac{\partial f}{\partial u} + \mu^T \frac{\partial C}{\partial u}$$

$$\dot{\lambda}^T = -\frac{\partial H}{\partial x} = -\frac{\partial L}{\partial x} - \lambda^T \frac{\partial f}{\partial x} - \mu^T \frac{\partial C}{\partial x}$$

Equality constraints of functions of state variables

$$S(t,x) = 0$$

- We could adjoin constraint as before
  - Results in variation to the augmented cost functional being

$$\ldots + \int_{t_0}^{t_f} \left(\dot{\lambda}^T + \mu^T \frac{\partial S}{\partial x} + \frac{\partial H}{\partial x}\right)\delta x + \ldots$$

Difficult to choose λ's and μ's simultaneously

- Easier to convert constraint of state variables into constraint of control and state variables

If $S(t,x) = 0$:

$$\dot{S}(t,x) = \frac{dS}{dt} = \frac{\partial S}{\partial t} + \frac{\partial S}{\partial x}\dot{x} = 0$$

$$= \frac{\partial S}{\partial t} + \frac{\partial S}{\partial x} f(t,x,u) = 0$$

$\vdots$ Continue taking derivatives until control appears explicitly (assume that q times)

$$S^{(q)}(t,x,u) = 0$$

Use this as path constraint which is adjoined

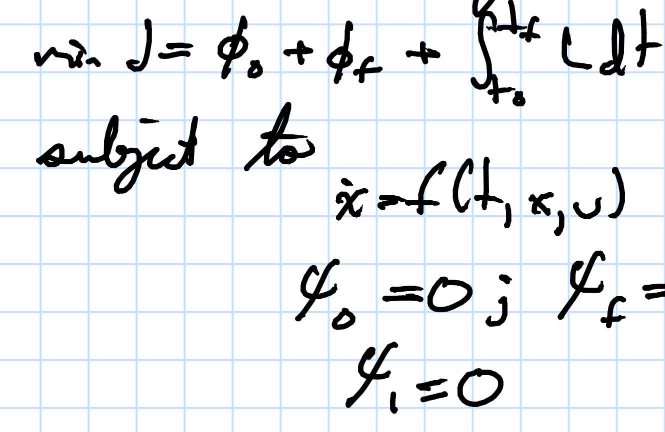
- Enforce original constraint and other derivatives as BC's at $t = t_0$ (or $t = t_f$)

$$\begin{bmatrix} S(t,x) \\ S^{(1)}(t,x) \\ \vdots \\ S^{(q-1)}(t,x) \end{bmatrix} = 0 \qquad (q \text{ BC's}) \qquad q \leq n$$

Interior Point Constraints

- Consider case where a set of condition are specified at an interior point t_1:

$$\psi_1[t_1, x(t_1)] = 0 \quad \text{where } t_0 < t_1 < t_f$$



- Results in three point BVP

$$\min J = \phi_0 + \phi_f + \int_{t_0}^{t_f} L\,dt$$

subject to

$$\dot{x} = f(t,x,u)$$

$$\psi_0 = 0; \quad \psi_f = 0$$

$$\psi_1 = 0$$

Adjoin constraints to cost functional

$$J' = \Phi_0 + \Phi_f + \pi^T \psi_1(t_1, x(t_1)) + \int_{t_0}^{t_f} [H - \lambda^T \dot{x}]\,dt$$

Lagrange mult.

$\Downarrow$ split over two intervals

$$J' = \Phi_0 + \Phi_f + \pi^T \psi_1(t_1, x(t_1)) + \int_{t_0}^{t_1^-} [H - \lambda^T \dot{x}]\,dt + \int_{t_1^+}^{t_f} [H - \lambda^T \dot{x}]\,dt$$

Can derive necessary conditions of optimality by taking differential of J'

$$dJ' = \left[H(t_1^-) - H(t_1^+) + \pi^T \frac{\partial \psi_1}{\partial t_1}\right] dt_1 + \left[\lambda^T(t_1^+) - \lambda^T(t_1^-) + \pi^T \frac{\partial \psi_1}{\partial x(t_1)}\right] dx(t_1) + \int_{t_0}^{t_1^-} \left[\frac{\partial H}{\partial u}\delta u\right] dt + \int_{t_1^+}^{t_f} \left[\frac{\partial H}{\partial u}\delta u\right] dt$$

After setting $\dot{\lambda}$, $\lambda(t_0)$, $\lambda(t_f)$, t_0, t_f

Can make remaining terms vanish

$$\lambda^T(t_1^+) - \lambda^T(t_1^-) + \pi^T \frac{\partial \psi_1}{\partial x(t_1)} = 0$$

&

$$H(t_1^-) - H(t_1^+) + \pi^T \frac{\partial \psi_1}{\partial t_1} = 0$$
