JMJ

- Beluga is a general trajectory optimization package written in Python
  - Unlike other available trajectory, beluga uses indirect methods
  - Indirect methods produce very high quality solutions free from numerical artefact of other methods

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States

altitude	velocity
latitude	FPA
longitude	heading

Controls AoA bank

Objective (cost)

max terminal → min $-v_f$

Parameter Optimization

The most basic class of optimization problem concerns finding $\bar{u}$ of m parameters such that a cost $J(\bar{u})$ is minimized.

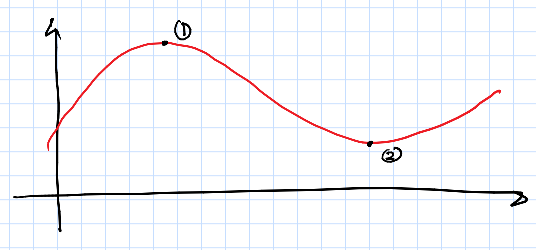
$\bar{u}$ is the control vector

$$\min_{\bar{u}} J(\bar{u})$$

If $J(\bar{u})$ has 1st & 2nd derivatives everywhere, the necessary conditions for a local minimum are

$$\frac{\partial J}{\partial \bar{u}} = 0 \qquad (1.1)$$

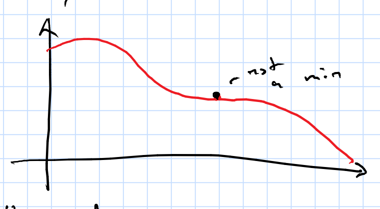
$$\frac{\partial^2 J}{\partial \bar{u}^2} \geq 0 \qquad (1.2) \qquad \text{positive semidefinite}$$



① Maximum $\frac{\partial J}{\partial u} = 0$ $\frac{\partial^2 J}{\partial u^2} < 0$

② Minimum $\frac{\partial J}{\partial u} = 0$ $\frac{\partial^2 J}{\partial u^2} > 0$

- (1.1) we get m algebraic equation for our m unknowns, $\bar{u}$
- Points that satisfy (1.1) are called stationary points
- If $\frac{\partial^2 J}{\partial u^2} = 0$, that is a singular point and more information is need to determine optimality



- If (1.1) and

$$\frac{\partial^2 J}{\partial \bar{u}^2} > 0 \quad \text{(positive definite)} \quad (1.4)$$

then sufficient conditions of optimality exist.

Problems with Equality Constraints

- A more general class of optimization involves minimizing a scalar $J(\bar{x}, \bar{u})$ where the state vector

$$\bar{x} = [x_1, x_2, \dots x_n]^T$$

contains n parameters determined by n constraint functions

$$\bar{f}(\bar{x}, \bar{u}) = \begin{bmatrix} f_1(\bar{x}, \bar{u}) \\ \vdots \\ f_n(\bar{x}, \bar{u}) \end{bmatrix} = 0 \qquad (1.5)$$

Therefore the resulting problem is

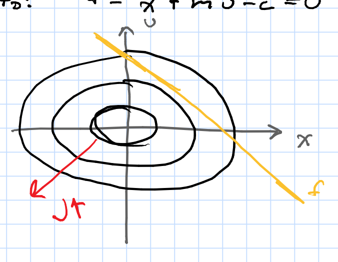
$$\min_{\bar{u}} J(\bar{x}, \bar{u})$$
$$\text{subject to} \qquad (1.6)$$
$$\bar{f}(\bar{x}, \bar{u}) = 0$$

Example

$$\min \; J = \frac{1}{2}\left(\frac{x^2}{a^2} + \frac{u^2}{b^2}\right)$$

subject to: $f = x + mu - c = 0$

Contour Plot



Unadjoined Method I

For a stationary, $dJ = 0$ for an arbitrary $d\bar{u}$

$$\Rightarrow dJ = \frac{\partial J}{\partial \bar{x}} d\bar{x} + \frac{\partial J}{\partial \bar{u}} d\bar{u} = 0 \quad (1.7) \qquad \frac{\partial J}{\partial \bar{x}} = J_{\bar{x}}$$

Also, for $d\bar{x}$ to change without changing $\bar{f}$

$$\Rightarrow d\bar{f} = \frac{\partial \bar{f}}{\partial \bar{x}} d\bar{x} + \frac{\partial \bar{f}}{\partial \bar{u}} d\bar{u} = 0 \quad (1.8) \qquad = \bar{f}_{\bar{x}} d\bar{x} + \bar{f}_{\bar{u}} d\bar{u}$$

$$\bar{f}_{\bar{x}} dx = -\bar{f}_{\bar{u}} d\bar{u}$$

Rearranging (1.8) yields

$$d\bar{x} = -\bar{f}_{\bar{x}}^{-1} \bar{f}_{\bar{u}} d\bar{u} \qquad (1.9)$$

Substitute (1.9) into (1.7)

$$dJ = (J_{\bar{u}} - J_{\bar{x}} \bar{f}_{\bar{x}}^{-1} \bar{f}_{\bar{u}}) d\bar{u} = 0$$

$$\Rightarrow J_u - J_x f_x^{-1} f_u \, du = 0$$

m equations
+
n equations from $f = 0$

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m+n equations for m+n unknown (x, u)

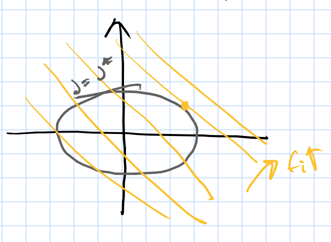
Note:

Symmetry of (1.7) & (1.8) indicates that we interchange the object with a constraint

$$\min f_i(x, u)$$

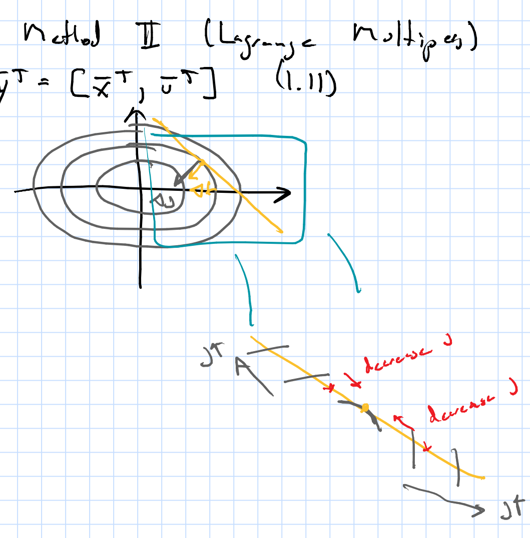
$$\text{s.t.} \begin{bmatrix} J(x,u) - J^* \\ f_{j \neq i} \\ \text{for } j = 1 \dots n \end{bmatrix} = 0$$

is equivalent to problem in 1.6



Unadjoined Method II (Lagrange Multipliers)

$$\bar{y}^T = [\bar{x}^T, \bar{u}^T] \qquad (1.11)$$



Tangency results in gradient of J and f_i being scalar multiples of each other

$$\nabla J = \frac{dJ}{dy} = \sum -\lambda_i \frac{\partial f_i}{\partial y}$$
$$= -\lambda^T \frac{\partial f}{\partial y}$$

$$\Rightarrow J_y + \lambda^T f_{\bar{y}} = 0$$

λ are Lagrange Multipliers

Separate y

$$J_{\bar{x}} + \lambda^T f_{\bar{x}} = 0 \qquad \text{n equations} \quad (1.12)$$
$$J_{\bar{u}} + \lambda^T f_{\bar{u}} = 0 \qquad \text{m equations} \quad (1.13)$$

Solving (1.12) for $\bar{\lambda}$ yields

$$\lambda^T = -J_x f_x^{-1} \qquad (1.14)$$

Note

From 1.14, we can infer an interpretation of λ by setting $du = 0$

$$\lambda^T = -J_x f_x^{-1} = -\frac{\partial J}{\partial x} \frac{\partial x}{\partial f}\bigg|_u = -\frac{\partial J}{\partial f}\bigg|_u \qquad 1.15$$

Lagrange indicate how much the constraint effects the cost

Adjoined Method

Because $\bar{f} = 0$ at the solution we can adjoin $\bar{f}$ to the cost without changing its optimal value

Here $\lambda_1 \dots \lambda_n$ are not yet specified

$$J'(x, u) = J(x, u) + \sum_{i=1}^{n} \lambda_i f_i(\bar{x}, \bar{u})$$

augmented cost

$$= J(x, u) + \lambda^T f \qquad (1.16)$$

Differential change in J' or

$$dJ' = \frac{\partial J'}{\partial x} dx + \frac{\partial J'}{\partial u} du \qquad (1.17)$$

We are about how du affect dJ', so we pick λ to eliminate dx

$$\frac{\partial J'}{\partial x} = \frac{\partial J}{\partial x} + \lambda^T \frac{\partial f}{\partial x} = 0$$

$$\Rightarrow \lambda^T = -\frac{\partial J}{\partial x}\left(\frac{\partial f}{\partial x}\right)^{-1} \qquad (1.18)$$

Same as (1.15)

Therefore

$$dJ' = \frac{\partial J'}{\partial u} du = \frac{\partial J'}{\partial u} = 0$$

the resulting nec conditions are

$$f(x, u) = 0 \qquad \text{(n eqns)}$$
$$\frac{\partial J'}{\partial x} = 0 \qquad \text{(n eqns)} \qquad \text{2n+m eqns for}$$
$$\frac{\partial J'}{\partial u} = 0 \qquad \text{(m eqns)} \qquad \text{2n+m unknowns } (x, \lambda, u)$$

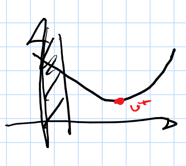
Problems with Inequality Constraints

Problems may also contain constraints of the form

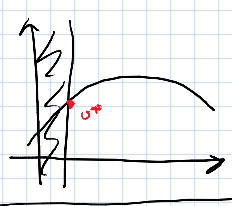
$$\bar{g}(\bar{x}, \bar{u}) \leq 0$$

There are 2 cases that can exist

① Inactive:
- solution on the constraint boundary
- choice of x does not depend on u
- constraint can be neglected



② Active:
- Solution is on constraint boundary
- Constraint can be treated as equality constraint



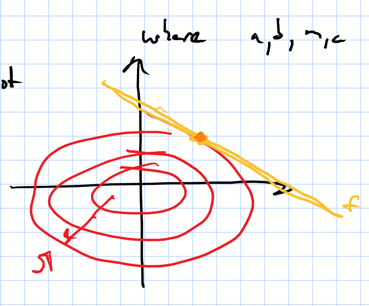
Eg Example

$$\min J = \frac{1}{2}\left(\frac{x^2}{a^2} + \frac{u^2}{b^2}\right)$$

s.t. $f(x, u) = x + mu - c = 0$

where a, b, m, c are constant

Contour plot



Augment cost

$$J' = J + \lambda^T f = \frac{1}{2}\left(\frac{x^2}{a^2} + \frac{u^2}{b^2}\right) + \lambda(x + mu - c)$$

$$\left.\begin{aligned} \frac{\partial J'}{\partial u} &= \frac{u}{b^2} + \lambda m = 0 \\ \frac{\partial J'}{\partial x} &= \frac{x}{a^2} + \lambda = 0 \\ f &= x + mu - c = 0 \end{aligned}\right\} \text{3 eqns for 3 unknowns}$$

Solve linear equations

$$\Rightarrow x = \frac{a^2 c}{a^2 + m^2 b^2}, \quad u = \frac{b^2 m c}{a^2 + m^2 b^2}, \quad \lambda = -\frac{c}{a^2 + m^2 b^2}$$

$$J = \frac{c^2}{2(a^2 + m^2 b^2)} \quad \text{(is min)}$$