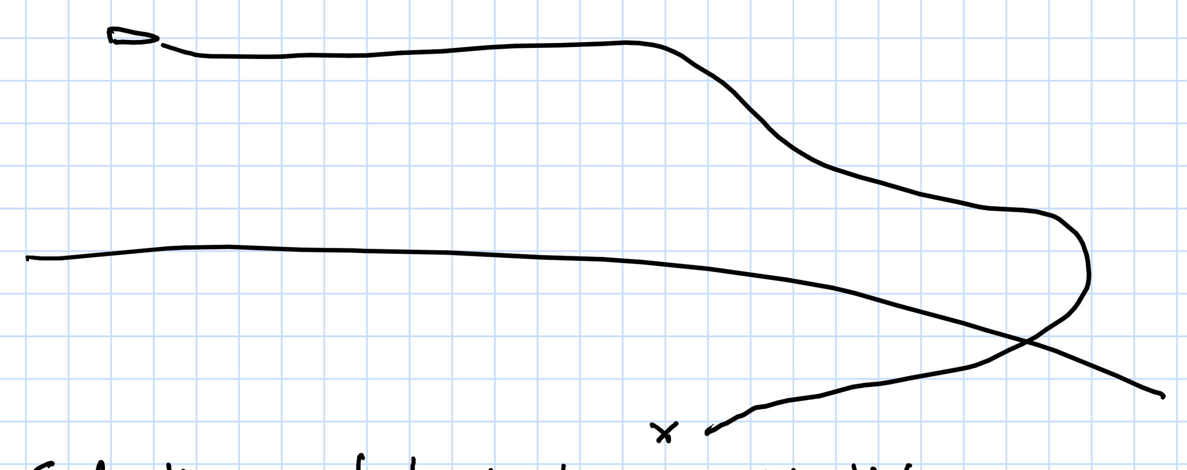
JWJ

Reminder motivation problem



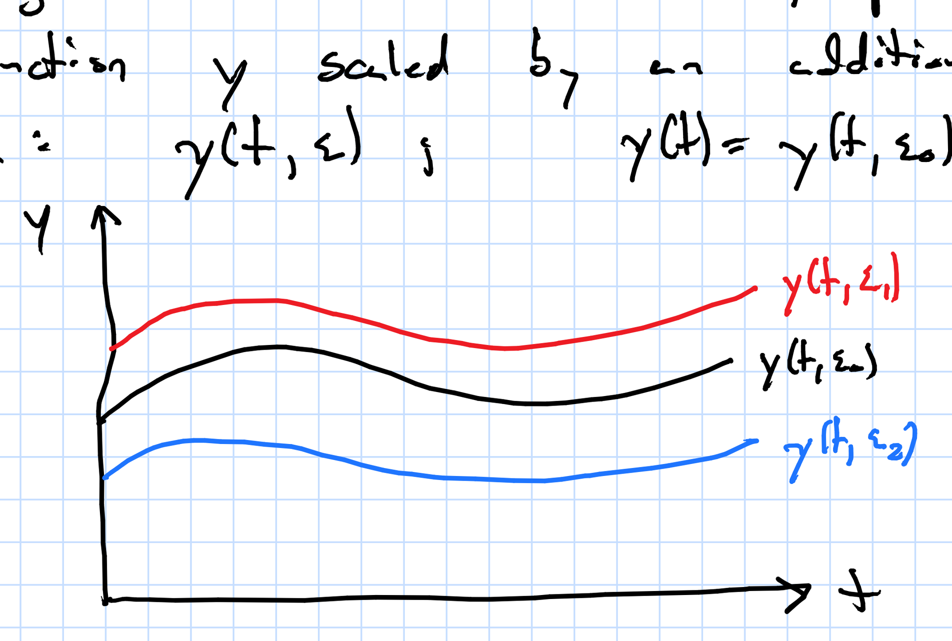
Find the control history $u(t)$ that reaches some terminal constraints while minimizing some scalar cost (-energy, time, effort...)

- Controls are continuous therefore we must
  - Discretize the problem before optimizing
    $\rightarrow$ Direct Methods
  - Optimize the continuous problem
    $\rightarrow$ Indirect Method
- As calculus can minimize functions of discrete parameters $J(\bar{u})$
  Calculus of variations minimizes functionals ("functions" of functions) dependent on continuous variables
  $$J(t, \bar{u}(t))$$

Derivative vs. Variations

- In calculus, a derivative measures the change of a function when a variables changes
$$\Delta J(u) = \frac{\partial J}{\partial u}\Big|_{u^*} du + \frac{1}{2}\frac{\partial^2 J}{\partial u^2}\Big|_{u^*} du^2 + \dots$$
- Likewise in calculus of variations, a variation is the measure of a change in a functional
$$J(y(t) + \delta y(t)) - J(y(t)) = \underset{1^{st}\text{ order}}{\delta J} + \underset{2^{nd}\text{ order}}{\tfrac{1}{2}\delta^2 J} + \dots$$

To get a sense of variations, picture a function y scaled by an additional parameter ε: $y(t, \varepsilon)$; $y(t) = y(t, \varepsilon_0)$



$$y(t,\varepsilon) = y(t,\varepsilon_0) + \underbrace{\frac{\partial y}{\partial \varepsilon}\Big|_{\varepsilon_0}\overset{d\varepsilon}{(\varepsilon - \varepsilon_0)}}_{\delta y} + O[(\varepsilon-\varepsilon_0)^2]$$

If $d\varepsilon \to 0$
$$\delta y = y(t,\varepsilon) - y(t,\varepsilon_0) = \frac{\partial y}{\partial \varepsilon}\Big|_{\varepsilon_0} d\varepsilon$$

Optimization of Continuous System

- We want to minimize some cost functional $J(t, u(t))$ wrt $u(t)$
$$\min J = \int_{t_0}^{t_f} \underset{\text{path cost}}{L(t,u)}\, dt$$
where t_0, t_f fixed
- Like parameter optimization, necessary condition is
$$\delta J = 0$$
- To analyze δJ, we need Leibniz rule:
$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(t,x)\,dt = f(b(x),x)\frac{db(x)}{dx} - f(a(x),x)\frac{da(x)}{dx} + \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(t,x)\,dt$$

- Apply to J:
$$\frac{dJ}{d\varepsilon} = \frac{d}{d\varepsilon}\int_{t_0}^{t_f} L(t,u)\,dt$$
$$= \int_{t_0}^{t_f} \frac{\partial L}{\partial \varepsilon}\,dt \qquad t_0, t_f \text{ fixed}$$
$$= \int_{t_0}^{t_f} \frac{\partial L}{\partial u}\frac{\partial u}{\partial \varepsilon}\,dt$$
$$\Downarrow$$
$$\frac{\partial J}{\partial \varepsilon}\Big|_{\varepsilon_0}(\varepsilon - \varepsilon_0) = \int_{t_0}^{t_f} \frac{\partial L}{\partial u}\frac{\partial u}{\partial \varepsilon}\Big|_{\varepsilon_0}(\varepsilon-\varepsilon_0)\,dt$$
$$\downarrow d\varepsilon \to 0 \qquad \Downarrow \varepsilon - \varepsilon_0 \to 0$$
$$\delta J = \int_{t_0}^{t_f} L_u\, \delta u\, dt = 0$$
$$\Downarrow$$
$$L_u = 0$$

Continuous System s.t. Algebraic Equality Constraints
$$\min J = \int_{t_0}^{t_f} L(t,x,u)\,dt$$
s.t.
$$f(t,x,u) = 0$$
$$t_0, t_f \text{ fixed}$$

- We adjoin constraints to path cost L with $\lambda = \lambda(t)$ varies continuously
$$\min J = \int_{t_0}^{t_f}\left[L(t,x,u) + \underset{\lambda(t)}{\lambda^T} f(t,x,u)\right]dt$$
$$0 = \delta J' = \int_{t_0}^{t_f}\Big[\underbrace{\Big(\frac{\partial L}{\partial x} + \lambda^T\frac{\partial f}{\partial x}\Big)}_{=0}\delta x + \underbrace{\Big(\frac{\partial L}{\partial u} + \lambda^T\frac{\partial f}{\partial u}\Big)}_{=0}\delta u + \overset{0}{\cancel{f^T}}\delta\lambda\Big]dt$$

Therefore,
$$\frac{\partial L}{\partial x} + \lambda^T\frac{\partial f}{\partial x} = 0 \qquad n \text{ eqns}$$
$$\frac{\partial L}{\partial u} + \lambda^T\frac{\partial f}{\partial u} = 0 \qquad m \text{ eqns}$$
$$f = 0 \qquad n \text{ eqns}$$
$2n+m$ eqns for $2n+m$ unknowns $(u(t), x(t), \lambda(t))$

Optimal control of dynamical systems
$$\min J = \underset{\text{terminal cost}}{\phi(t_f, x(t_f))} + \int_{t_0}^{t_f} L(t,x,u)\,dt$$
subject to: t_0 & t_f fixed
$$\dot{x} = f(t,x,u) \qquad \text{(dynamics/eom)}$$
Adjoin dynamics to path cost
$$J' = \phi + \int_{t_0}^{t_f}[L + \lambda^T(\overset{=0}{f - \dot{x}})]\,dt$$
$$H = L + \lambda^T f \qquad \underline{\text{Hamiltonian}}$$
$$J' = \phi + \int_{t_0}^{t_f}[H - \lambda^T\dot{x}]\,dt$$
Take variation and set to 0
$$0 = \delta J' = \frac{\partial\phi}{\partial x(t_f)}\delta x(t_f) + \int_{t_0}^{t_f}\Big[\frac{\partial H}{\partial x}\delta x + \frac{\partial H}{\partial u}\delta u + \Big(\frac{\partial H}{\partial\lambda}\delta\lambda - \overset{0}{\cancel{\dot{x}^T\delta\lambda}}\Big) - \lambda^T\underline{\delta\dot{x}}\Big]dt$$
$$=(f-\dot{x})^T\delta\lambda = 0$$

We eliminate $\delta\dot{x}$ via integration by parts

Note: $\int_a^b u\frac{dv}{dt}\,dt = [uv]_a^b - \int_a^b \frac{du}{dt}v\,dt$
$$\Rightarrow \int_{t_0}^{t_f}\lambda^T\delta\dot{x}\,dt = \int_{t_0}^{t_f}\frac{d}{dt}(\lambda^T\delta x)\,dt - \int_{t_0}^{t_f}\dot{\lambda}^T\delta x\,dt$$
$$= [\lambda^T\delta x]_{t_0}^{t_f} - \int_{t_0}^{t_f}\dot{\lambda}^T\delta x\,dt$$
$$\Rightarrow \delta J' = \frac{\partial\phi}{\partial x(t_f)}\delta x(t_f) - \lambda^T(t_f)\delta x(t_f) + \lambda^T(t_0)\delta x(t_0) + \int_{t_0}^{t_f}[(H_x + \dot{\lambda}^T)\delta x + H_u\delta u]\,dt = 0$$
$$\delta J' = \overbrace{\Big[\frac{\partial\phi}{\partial x(t_f)} - \lambda^T(t_f)\Big]\delta x(t_f)}^{①} + \overbrace{\lambda^T(t_0)\delta x(t_0)}^{②} + \int_{t_0}^{t_f}[\underbrace{(H_x + \dot{\lambda}^T)}_{③}\underline{\delta x} + H_u\delta u]\,dt = 0$$

Term ②:

If $x_i(t_f)$ is fixed then $\delta x_i(t_f) = 0$

Therefore,

if $x_i(t_0)$ is fixed $\leftarrow$ constraint
$\lambda_i(t_0)$ is free
else if $x_i(t_0)$ is free
$\lambda_i(t_0) = 0$ $\leftarrow$ constraint

Term ①:

Likewise for ①

if $x_i(t_f)$ is fixed
$\lambda_i(t_f)$ is free
else if $x_i(t_f)$ is free
$\lambda_i(t_f) = \frac{\partial\phi}{\partial x(t_f)}$

Term ③

To eliminate δx in the integral:
$$H_x + \dot{\lambda}^T = 0$$
$$\Rightarrow \dot{\lambda}^T = -H_x$$

λ is referred as the <u>costates</u>

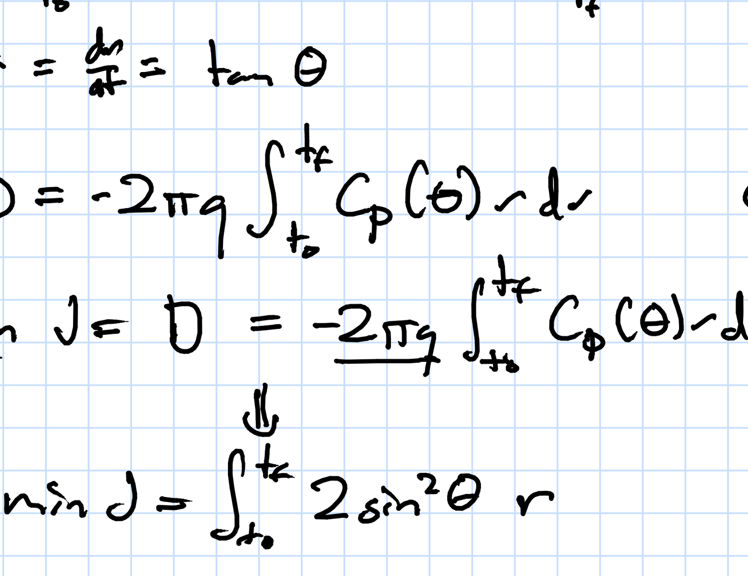
As we eliminated dependence on δx
$$\Rightarrow J' = \int_{t_0}^{t_f} H_u\,\delta u\,dt = 0$$
$$\Rightarrow \underline{\underline{H_u}} = 0 \qquad \text{provides optimal control law}$$

In summary,
$$\left.\begin{aligned}\dot{\lambda}^T &= -\frac{\partial H}{\partial x} && n \text{ diff eq}\\ \dot{x} &= f && n \text{ diff eq}\end{aligned}\right\}\text{need } 2n \text{ BC}$$
$$H_u = 0 \qquad m \text{ algebraic eq.}$$

BC

x_0: If $x_i(t_0)$ fixed, $\lambda_i(t_0)$ free; else $\lambda_i(t_0) = 0$ $\Big\}$ n BC

x_f: If $x_i(t_f)$ fixed, $\lambda_i(t_f)$ free; else $\lambda_i(t_f) = \frac{\partial\phi}{\partial x(t_f)}$; $\Big\}$ n BC



$u = \theta$

$$\dot{r} = \frac{dr}{dt} = \tan\theta$$
$$D = -2\pi q\int_{t_0}^{t_f} C_p(\theta)\, r\, dr \qquad C_p = 2\sin^2\theta, \ \theta \geq 0$$
$$\min_\theta J = D = \underline{-2\pi q}\int_{t_0}^{t_f} C_p(\theta)\, r\, dr$$
$$\Downarrow$$
$$\min J = \int_{t_0}^{t_f} 2\sin^2\theta\, r$$

$$\underset{\text{time fixed}}{\underline{H_f\,dt_f}} - \underline{\lambda_f^T dx_f} + d\phi + \lambda_0^T dx_0 = 0$$