JMS

Inequality Path Constraints

Pontryagin's Minimum Principle

"H^* must be minimized over the set of all admissible $\bar{u}$"

$$H[t, x^*(t), u^*(t), \lambda(t)] \le H[t, x^*(t), u(t), \lambda(t)]$$

$u \in U$ ← set of admissible control

Note: Does not imply

$$H[t, x(t), u^*(t), \lambda(t)] \le H[t, x(t), u(t), \lambda(t)]$$

Inequality Constraints on Controls

$$C(t, u) \le 0$$

⇓ augment objective functional

$$H = L + \lambda^T f + \underline{\mu^T C}$$

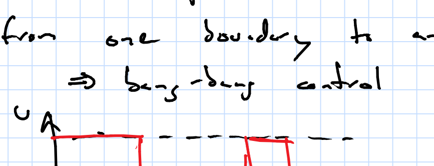
if on constraint ($C=0$) then $\mu \ge 0$
else ($C<0$) then $\mu = 0$

Necessary conditions

$$\frac{\partial H}{\partial u} = \frac{\partial L}{\partial u} + \lambda^T \frac{\partial f}{\partial u} + \mu^T \frac{\partial C}{\partial u} = 0$$

Problem is solved by piecing together constrained and unconstrained arcs

- Bang-Bang control
  - For certain problems, especially with linear control variables, the optimal control will reside on constraint boundary
  - At certain points the control may switch from one boundary to another ⇒ bang-bang control



- Consider:

$$\dot{x} = A(t)x + b(t)u$$
$$-1 \le u \le 1$$

Minimize time from $x(0) = x_0$ to origin $x(t_f) = 0$

$$\Rightarrow L = 1$$

⇓ Consider A and b constant

$$H = 1 + \lambda^T(Ax + bu)$$
$$\frac{\partial H}{\partial u} = \lambda^T b$$

⇒ Cannot determine u^* w/ $\frac{\partial H}{\partial u} = 0$ because u disappears
- Must used Minimum Principle

To minimize H w.r.t. u

$$u(t) = \begin{cases} 1 & \lambda^T b < 0 \\ \text{undefined} & \lambda^T b = 0 \\ -1 & \lambda^T b > 0 \end{cases}$$

- $\lambda^T b$ is a "switching" function
- Also as usual

$$\dot{\lambda} = -\lambda^T f$$

and

$$H_f\,[\lambda^T(Ax+bu)+1]_{t_f} = 0$$

Inequality Constraints on Control and State Variables

$$C(t, x, u) \le 0$$

Handle in same way as previous problem

$$H = L + \lambda^T f + \mu^T C, \quad \begin{matrix} \mu \ge 0 \text{ if } C = 0 \\ \mu = 0 \text{ if } C < 0 \end{matrix}$$

⇓

$$\frac{\partial H}{\partial u} = \frac{\partial L}{\partial u} + \lambda^T \frac{\partial f}{\partial u} + \mu^T \frac{\partial C}{\partial u} = 0$$

and

$$\dot{\lambda}^T = -\frac{\partial H}{\partial x} = -\frac{\partial L}{\partial x} - \lambda^T \frac{\partial f}{\partial x} - \mu^T \frac{\partial C}{\partial x}, \quad C = 0$$
$$= -\frac{\partial L}{\partial x} - \lambda^T \frac{\partial f}{\partial x}, \quad C < 0$$

- Constrained and unconstrained arcs must be pieced together (multi-point BVP)

Singular Arcs

- Where $H_u = 0$ and H_{uu} is singular are refered to as singular arcs
- Neither the normal necessary conditions ($\frac{\partial H}{\partial u} = 0$) nor the minimum principle are adequate to determine the optimal control
- Can use the Generalized Legendre-Clebsch condition

$$(-1)^q \frac{\partial}{\partial u}\left(\frac{d^{2q}}{dt^{2q}} H_u^*\right) \ge 0$$

where $2q$-th derivative is the first even derivative that contains u explicitly

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Inequality constraints on state Variables

$$S(t, x) \le 0$$

- Convert to inequality constraint of control and state variables by taking derivatives of S until u appears explicitly

$$S^{(q)}(t, x, u) \le 0 \quad (\text{acts as } C(t, x, u) \le 0)$$
$$H = L + \lambda^T f + \mu S^{(q)}$$

- To ensure the the system is on constraint boundary, we must also enforce the interior point constraint

$$\psi(t, x) = \begin{bmatrix} S(t, x) \\ S^{(1)}(t, x) \\ \vdots \\ S^{(q-1)}(t, x) \end{bmatrix} = 0$$

at either the beginning or end of constrained arc (Choose beginning)

- λ & H may be discontinuous at application of interior point constraint and continuous at other end of arc

Ex: min $J = \frac{1}{2}\int_0^1 u^2\,dt$

$$\dot{x}_1 = x_2 \qquad x_1(0) = 0\,; \quad x_1(1) = 0$$
$$\dot{x}_2 = u \qquad x_2(0) = 1\,; \quad x_2(1) = -1$$

$$S(t, x) = x_1 - l \le 0$$

- First assume solution does not violate constraint

$$H = \frac{1}{2}u^2 + \lambda_1 x_2 + \lambda_2 u$$
$$\dot{\lambda}_1 = -\frac{\partial H}{\partial x_1} = 0 = \lambda_1 = C$$
$$\dot{\lambda}_2 = -\frac{\partial H}{\partial x_2} = -\lambda_1 = \lambda_2 = -Ct + D$$
$$\frac{\partial H}{\partial u} = 0 \Rightarrow u + \lambda_2 = 0 \Rightarrow u = Ct - D$$

⇓

$$x_2 = \frac{1}{2}Ct^2 - Dt + E$$
$$x_1 = \frac{1}{6}Ct^3 - \frac{1}{2}Dt^2 + Et + F$$

$x_1(0) = 0 \Rightarrow F = 0$

$x_1(1) = 0 \Rightarrow \frac{1}{6}C - \frac{1}{2}D + E = 0$

$x_2(0) = 1 \Rightarrow E = 1$

$x_2(1) = -1 \Rightarrow \frac{1}{2}C - D + E = -1$

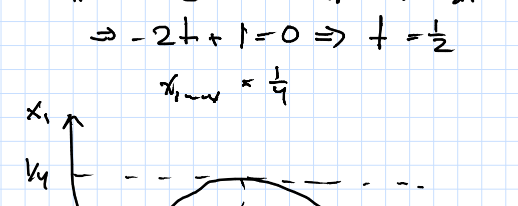
$\Rightarrow C = 0,\ D = 2$

⇓

$$x_1(t) = -t^2 + t$$
$$x_2(t) = -2t + 1$$
$$u(t) = -2$$

- Max value of $x_1(t)$ at $\frac{dx_1}{dt} = 0$

$$\Rightarrow -2t + 1 = 0 \Rightarrow t = \frac{1}{2}$$
$$x_{1max} = \frac{1}{4}$$



Set $l = 1/8$ so constraint is active

For $0 \le t \le t_1$

$$x_1 = \frac{1}{6}C_1 t^3 - \frac{1}{2}D_1 t^2 + E_1 t + F_1$$
$$x_2 = \frac{1}{2}C_1 t^2 - D_1 t + E_1$$
$$u = C_1 t - D_1$$
$$x_1(0) = 0 \Rightarrow F_1 = 0$$
$$x_2(0) = 1 \Rightarrow E_1 = 1$$
$$x_1 = \frac{1}{6}C_1 t^3 - \frac{1}{2}D_1 t^2 + t$$
$$x_2 = \frac{1}{2}C_1 t^2 - D_1 t + 1$$

At $t = t_1$

$$S(t, x) = x_1(t_1) - \frac{1}{8} = 0$$
$$\dot{S}(t, x) = \dot{x}_1 = x_2 \Rightarrow x_2(t_1) = 0$$
$$\ddot{S}(t, x) = \dot{x}_2 = u = 0 \Rightarrow u(t_1) = 0$$

Have 3 eqns for 3 unknowns C_1, D_1, t_1

⇓

$$C_1 = \frac{128}{9}, \quad D_1 = \frac{16}{3}, \quad t_1 = \frac{3}{8}$$

(Note: Here we were able to solve independently of λ. Otherwise adjoin interior point constraint)

$$\left.\begin{matrix} x_1(t) = \frac{128}{54}t^3 - \frac{16}{6}t^2 + t \\ x_2(t) = \frac{128}{68}t^2 - \frac{16}{3}t + t \end{matrix}\right\} 0 \le t \le t_1$$

- Trajectory leaves constraint boundary at t_2

For $t_2 \le t \le 1$

$$x_1 = \frac{1}{6}C_2 t^3 - \frac{1}{2}D_2 t^2 + E_2 t + F_2$$
$$x_2 = \frac{1}{2}C_2 t^2 - D_2 t + E_2$$
$$u = C_2 t - D_2$$
$$x_1(1) = 0$$
$$x_2(1) = -1$$

⇓ Rewrite equations from $t_f = 1$

$$x_1 = \frac{1}{6}C_2'(t-1)^3 - \frac{1}{2}D_2'(t-1)^2 - (t-1)$$
$$x_2 = \frac{1}{2}C_2'(t-1)^2 - D_2'(t-1) - 1$$
$$u = C_2'(t-1) - D_2'$$

- At $t = t_2$: $x_1(t_2) = \frac{1}{8}$, $x_2(t_2) = 0$, $u = 0$

- 3 eqns for 3 unknowns C_2', D_2', t_2

For $t_1 \le t \le t_2$, trajectory lies on constraint boundary

$$x_1(t) = \frac{1}{8}$$
$$x_2(t) = 0$$
$$u = 0$$