Practice Exercises for Parameter Optimization

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1 Inequality Problem

Problem

$$\min_{\boldsymbol{u}} J = x^2 - u^2 \tag{1.1}$$

Subject to:

$$g = x^2 + u^2 - 4 \leq 0 \tag{1.2}$$

Solution

Without prior knowledge, the inequality can be either inactive (solution not on constraint boundary) or active (constraint on constraint boundary). To ensure that the minimum is obtained, both should be considered.

Case 1: Constraint Inactive

If the constraint is inactive, the problem can be treated as an unconstrained problem. Therefore, stationary points exist where

$$J_{\boldsymbol{x}} = 2x = 0\,; \tag{1.3}$$

$$J_{\boldsymbol{u}} = -2u = 0\,. \tag{1.4}$$

The only resulting candidate for a minimum is $(x, u) = (0, 0)$ where $J = 0$. The feasibility of this constraint should be checked using Eq. (1.2):

$$g\left(0, 0\right) = 0^2 + 0^2 - 4 = -4 \leq 0\,. \tag{1.5}$$

Candidate solution is feasible.

We can check if the candidate is a local minimum, maximum, or saddle-point using second-order derivatives. Defining $\boldsymbol{y} = [x, u]^T$,

$$J_{\boldsymbol{yy}} = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}\,. \tag{1.6}$$

With eigenvalues 2 and -2, $J_{\boldsymbol{yy}}$ is neither positive or negative semi-definite, which means the candidate solution is a saddle-point and not a minimum.

Case 2: Constraint Active

If the constraint is active, the problem can be solved as an equality constraint problem. Therefore, $f = g$ is adjoined to the cost function to yield

$$J' = x^2 - u^2 + \lambda\left(x^2 + u^2 - 4\right)\,. \tag{1.7}$$

The resulting first order necessary conditions are

$$J'_x = 2x\,(1+\lambda) = 0\,; \tag{1.8}$$
$$J'_u = 2u\,(-1+\lambda) = 0\,; \tag{1.9}$$
$$J'_\lambda = f = x^2 + u^2 - 4 = 0\,. \tag{1.10}$$

Solving Eq. (1.10) for x results in

$$x = \pm\sqrt{4-u^2}\,. \tag{1.11}$$

Substituting Eq. (1.11) into Eq. (1.8) yields

$$2\sqrt{4-u^2}\,(1+\lambda) = 0\,, \tag{1.12}$$

so either $u = \pm 2$ or $\lambda = -1$.

If $u = \pm 2$, then $\lambda = 1$ from Eq. (1.9), and $x = 0$ from Eq. (1.11). The resulting cost function value is $J\,(0, \pm 2) = -4$.

If $\lambda = -1$, then $u = 0$ from Eq. (1.9), and $x = \pm 2$ from Eq. (1.11). The resulting cost function value is $J\,(\pm 2, 0) = 4$.

Noting that the solution must lie on the closed, bounded, continuous loop defined by $f = x^2 + u^2 - 4 = 0$ and the cost values of the four candidate solutions, this problem has two minimums at $(x, u) = (0, \pm 2)$.

These solutions are confirmed by consulting the corresponding contour plot in Fig. 1.

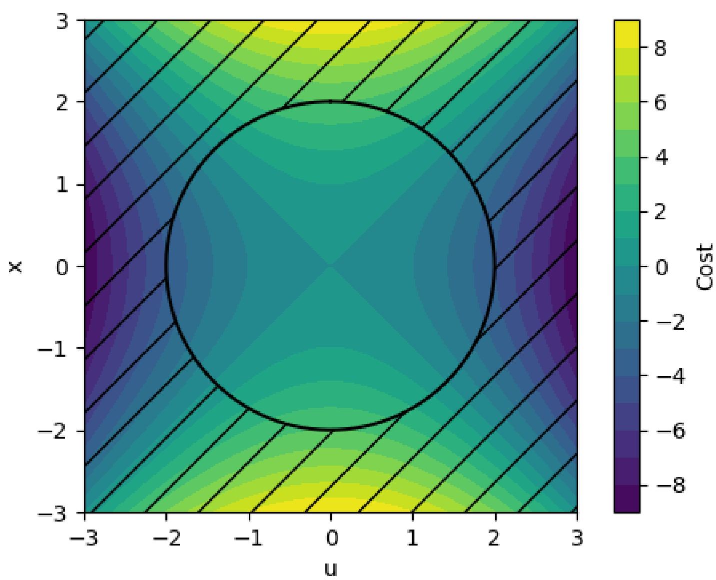

Figure 1: Contour plot of problem

2 Maximum Steady Rate of Climb for Aircraft[1]

Problem

This problem aims to find the maximum steady rate of climb of am aircraft. The rate of climb is $v \sin\gamma$, so the cost function is

$$\min J = -v \sin\gamma \tag{2.1}$$

For a steady climb, acceleration, thus force, in both directions must be zero:

$$\boldsymbol{f}(v,\gamma,\alpha) = \begin{bmatrix} T(v)\cos(\alpha+\epsilon) - D(v,\alpha) - mg\sin\gamma \\ T(v)\sin(\alpha+\epsilon) + L(v,\alpha) - mg\cos\gamma \end{bmatrix} = 0 \tag{2.2}$$

Where:

$$\begin{aligned} v &= \text{velocity} \\ \gamma &= \text{flight path angle} \\ \alpha &= \text{angle of attack} \\ m &= \text{mass} \\ g &= \text{gravity} \\ \epsilon &= \text{angle between thrust axis and zero-lift axis} \\ T(\alpha) &= \text{thrust} \\ L(v,\alpha) &= \text{lift} \\ D(v,\alpha) &= \text{drag} \end{aligned}$$

Derive first order necessary conditions to minimize Eq. (2.1) at a given altitude.

Solution

First adjoin the constraints, Eq. (2.2), to the cost function, Eq. (2.1).

$$\begin{aligned} J' &= J + \boldsymbol{\lambda}^T \boldsymbol{f} \\ &= -v\sin\gamma + \lambda_1 \left(T(v)\cos\alpha + \epsilon - D(v,\alpha) - mg\sin\gamma\right) \\ &\quad + \lambda_2 \left(T(v)\sin\alpha + \epsilon + L(v,\alpha) - mg\cos\gamma\right) \end{aligned} \tag{2.3}$$

Necessary conditions are obtained from taking the partial derivative of Eq. (2.3) with respect to parameters v, γ, and α

$$J'_v = -\sin\gamma + \lambda_1 \left[\frac{\partial T}{\partial v}\cos(\alpha+\epsilon) - \frac{\partial D}{\partial v}\right] + \lambda_2 \left[\frac{\partial T}{\partial v}\sin(\alpha+\epsilon) - \frac{\partial L}{\partial v}\right] = 0 \tag{2.4}$$

$$J'_\gamma = -v\cos\gamma - \lambda_1 mg\cos\gamma + \lambda_2 mg\sin\gamma = 0 \tag{2.5}$$

$$J'_\alpha = \lambda_1 \left[-T\sin(\alpha+\epsilon) - \frac{\partial D}{\partial \alpha}\right] + \lambda_2 \left[T\cos(\alpha+\epsilon) + \frac{\partial L}{\partial \alpha}\right] = 0 \tag{2.6}$$

The two constraint equations (Eq. (2.2)) in addition to the three above give the five equations to solve for the five unknowns. A solution can then be found numerically given functions for T, L, and D.

References

[1] A. E. Bryson and Y.-C. Ho, *Applied optimal control: optimization, estimation, and control.* Washington : New York: Hemisphere Pub. Corp. ; distributed by Halsted Press, rev. printing ed., 1975.