Introduction to Control and Robotic Systems
Pitch-Plane Rocket Stabilization Project
EE 386 - 01

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1 Introduction

The main objective of this project was to derive a way to stabilize the angle of a rocket flying vertically. The problem of stabilizing a flying rocket is important because the flight path of an unstable rocket is unpredictable. The unpredictable nature of the unstable rocket's flight path makes it very difficult for the rocket to reach its planned destination.

This problem was solved by stabilizing the angle of attack, denoted by α, using a mathematical model of the rotational motion of the rocket. This mathematical model was a function of the angle of thrust, δ. The controller would stabilize the rocket by vectoring the thrust until the rocket's angle of attack reached a value of zero. Two possible controllers were derived to solve this problem. The first was a Proportional + Derivative, or PD, controller. The second was a Proportional + Integral + Derivative, or PID, controller.

2 Mathematical Model

2.1 Functional Diagram

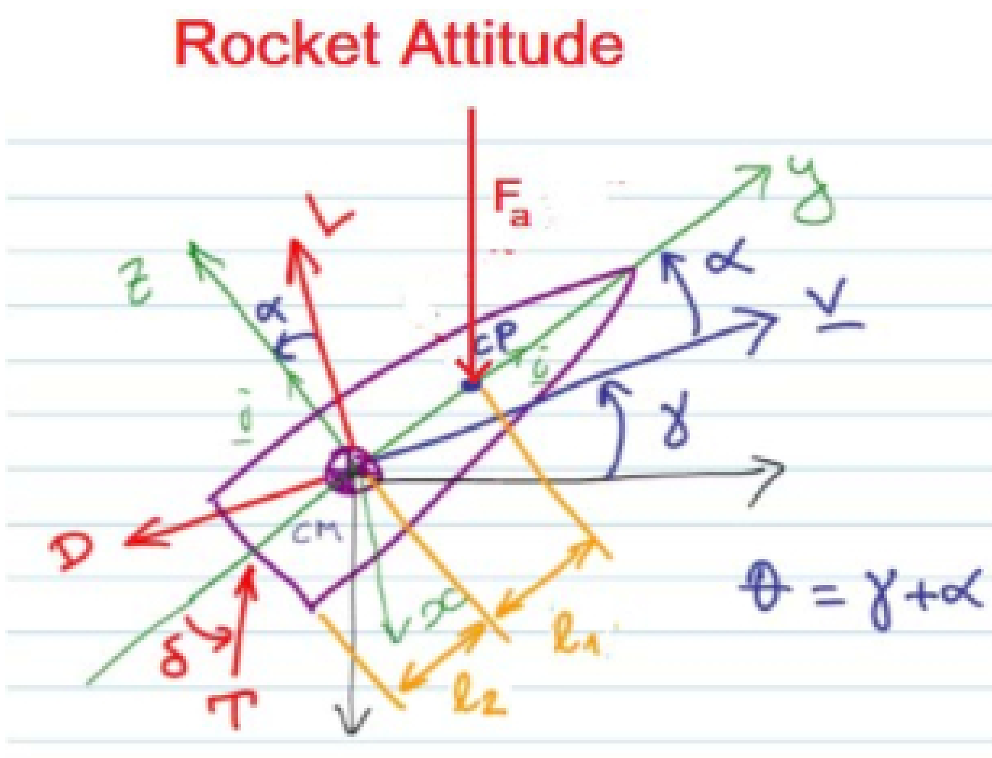

Figure 1: Functional diagram of the forces action on the rocket

In Figure 1, the following variables are defined. γ is the flight path, from the local horizon to velocity. α is the angle of attack, from local horizon to roll axis. $\theta = \gamma + \alpha$ is the rocket's pitch angle [2].

2.2 Derivation of Math Model

For this derivation, the following have been assumed. First, the rocket is in vertical flight, $\gamma = 90°$. Second, the rocket engine thrust is constant, $T =$ constant. Finally, the force on the rocket is constant, $F_a =$ constant. The goal of the math model is to find a $\delta = \delta(\alpha, \dot{\alpha})$ such that $\lim_{t\to\infty} \alpha(t) = 0$ The rocket's rotational motion can be described with Newton's Second Law for Rotional Motion

$$J\epsilon = \Sigma_i \rightarrow J\ddot{\alpha} = Tl_2 sin(\delta) + F_a l_1 sin(\alpha) \tag{1}$$

Since the above equation is non-linear, the function has to be linearized for ease of computation. Starting with the Taylor series [1]

$$f(x, y) \approx f(x_{eq}, y_{eq}) + \frac{\partial f(x_{eq}, y_{eq})}{\partial x}(x - x_{eq}) + \frac{\partial f(x_{eq}, y_{eq})}{\partial y}(y - y_{eq}) \tag{2}$$

We find the following two solutions

$$sin(\delta) \approx sin(0) + cos(0)(\delta - 0) = \delta, \text{ if } |\delta| \leq \frac{\pi}{6} \tag{3}$$

$$sin(\alpha) \approx sin(0) + cos(0)(\alpha - 0) = \delta, \text{ if } |\alpha| \leq \frac{\pi}{6} \tag{4}$$

Substituting equations 3 and 4 into equation 1

$$\ddot{\alpha} = \frac{Tl_2}{J}\delta + \frac{F_a l_1}{J}\alpha \tag{5}$$

$$\ddot{\alpha} = b_1\delta + a_1\alpha \tag{6}$$

This substitution leads to a linear model for the rotational motion of the rocket.

$$\ddot{\alpha} - a_1\alpha = b_1\delta \tag{7}$$

2.3 Stability Analysis of Uncompensated Angle Of Attack dynamics

Starting with the linear model for the rotational motion of the rocket.

$$\ddot{\alpha} - a_1\alpha = b_1\delta, \delta \equiv 0 \tag{8}$$

This leads to the following characteristic equation.

$$\lambda^2 - a_1 = 0 \tag{9}$$

Solving the characteristic equation finds the roots

$$\lambda_1 = \sqrt{a_1} > 0, \lambda_2 = -\sqrt{a_1} < 0 \tag{10}$$

The general solution of equation 10 is

$$\alpha(t) = c_1 e^{\sqrt{a_1}} + c_2 e^{-\sqrt{a_1}} \tag{11}$$

Checking the BIBO Stability of equation 11, we find that the system is unstable.

$$\lim_{t\to\infty} \alpha(t) = \infty \tag{12}$$

3 Controller Design

3.1 PD Controller Design

$$\delta = k_1 e_a + k_2 \dot{e_a} = -k_1\alpha - k_2\dot{\alpha} \tag{13}$$

Substituting equation 13 into equation 7, the compensated system dynamics take a form of

$$\ddot{\alpha} - a_1\alpha = b_1(-k_1\alpha - k_2\dot{\alpha}) \tag{14}$$

$$\ddot{\alpha} + b_1k_2\dot{\alpha} + (b_1k_1 - a_1)\alpha = 0 \tag{15}$$

This leads to the following characteristic equation.

$$\lambda^2 + r_2\lambda + r_1 = 0 \tag{16}$$

$$\lambda_{1,2} = \frac{-r_2 \pm \sqrt{a^2 - 4b}}{2} \tag{17}$$

$$\alpha(t) = c_1e^{\lambda_1 t} + c_2e^{\lambda_2 t} \tag{18}$$

The system is asymptotically stable iff $r_1, r_2 > 0$

$$\ddot{\alpha} + r_2\dot{\alpha} + r_1\alpha = 0 \tag{19}$$

$$\ddot{\alpha} + 2\zeta\omega_n\dot{\alpha} + \omega_n^2\alpha = 0 \tag{20}$$

Using $t_s = 2$ seconds and $\zeta = 0.7$, the natural frequency, ω_n was found to be 2.857 rad/s. Next the gains of the PD controller were computed using the below equations.

$$k_1 = \frac{a_1 + \omega_n^2}{b1}, k_2 = \frac{2\zeta\omega_n}{b_1} \tag{21}$$

For a settling time of 2 seconds, a damping factor of 0.7, and a natural frequency of 2.857 rad/s, the following gain values were found

$$k_1 = 0.940$$

$$k_2 = 0.288$$

3.2 PD Controller with Wind Disturbance Design

Starting with the perturbed rocket dynamics

$$\ddot{\alpha} - a_1\alpha = b_1\delta + \omega(t) \tag{22}$$

$$\delta = -k_1\alpha - k_2\dot{\alpha} \tag{23}$$

Substituting equation 23 into equation 22

$$\ddot{\alpha} - a_1\alpha = b_1(-k_1\alpha - k_2\dot{\alpha}) + \omega(t) \tag{24}$$

$$\ddot{\alpha} - b_1k_2\dot{\alpha} - (b_1k_1 - a_1)\alpha = \omega(t) \tag{25}$$

This leads to the following characteristic equation.

$$\ddot{\alpha} - r_2\dot{\alpha} - r_1\alpha = \omega(t), r_1, r_2 > 0 \tag{26}$$

This nonhomogenous differential equation will be BIBO stable, but $\lim\limits_{t\to\infty} \alpha(t) \neq 0$

3.3 PID Controller Design

Starting again with the perturbed rocket dynamics

$$\ddot{\alpha} - a_1\alpha = b_1\delta + \omega(t) \tag{27}$$

Substituting into the general form of a PID controller

$$\delta = k_1 e_\alpha + k_2 \dot{e_\alpha} + k_3 \int e_\alpha dt, e_\alpha = \alpha_c - \alpha \tag{28}$$

$$\delta = -k_1\alpha - k_2\dot{\alpha} - k_3 \int \alpha dt \tag{29}$$

results in the equations

$$\ddot{\alpha} - b_1 a_1 \dot{\alpha} + (b_1 k_1 - a_1)\alpha + b_1 k_3 \int \alpha dt = \omega(t) \tag{30}$$

$$\delta = -k_1\alpha - k_2\dot{\alpha} - k_3 \int \alpha dt \tag{31}$$

simplifying

$$\alpha^{(3)} + b_1 k_2 \ddot{\alpha} - (b_1 k_1 - a_1)\dot{\alpha} + b_1 k_3 \alpha = 0 \tag{32}$$

$$\alpha^{(3)} + r_3\ddot{\alpha} - r_2\dot{\alpha} + r_1\alpha = 0 \tag{33}$$

The above equation is stable for

$$r_1, r_2, r_3 > 0, r_2 r_3 > r_1 \tag{34}$$

The natural frequency can be found with the following equation

$$\omega_n \approx \frac{10}{t_s} \tag{35}$$

With a settling time, t_s of 2 seconds, the natural frequency is 5 rad/s Substituting ω_n into the following equation

$$\alpha^{(3)} + 1.75\omega_n\ddot{\alpha} - 2.15\omega_n^2\dot{\alpha} + \omega_n^3\alpha = 0 \tag{36}$$

Using the above equation, the controller gains, k_1, k_2, and k_3, can be computed.

$$k_1 = \frac{2.15\omega_n^2 + a_1}{b_1}, k_2 = \frac{1.75\omega_n}{b_1}, k_3 = \frac{\omega_n^3}{b_1} \tag{37}$$

For a settling time of 2 seconds and a natural frequency of 5 rad/s, the following gain values were found

$$k_1 = 4.190$$

$$k_2 = 0.625$$

$$k_3 = 8.92$$

4 Simulations

Using the previously derived mathematical models and MATLAB, the rocket dynamics and controllers were simulated. Using this information and tools, four simulations were performed. The first simulation performed was of the uncompensated dynamics of the rocket. The second simulation was a PD controller designed to stabilize the rocket's angle of attack. The third simulation was the same PD controller used in the second simulation, but a wind disturbance was introduced in this simulation. The final simulation was a PID controller designed to stabilize the rocket's angle of attack with a wind disturbance.

4.1 Simulation of Uncompensated Plant

The following plot shows the simulation of the uncompensated rocket dynamics. From the plot, it is clear that without a controller, the rocket's angle of attack quickly goes to infinity and therefore is unstable.

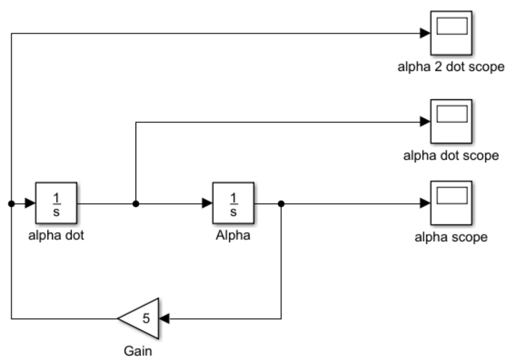

Figure 2: Math flow diagram of the uncompensated plant

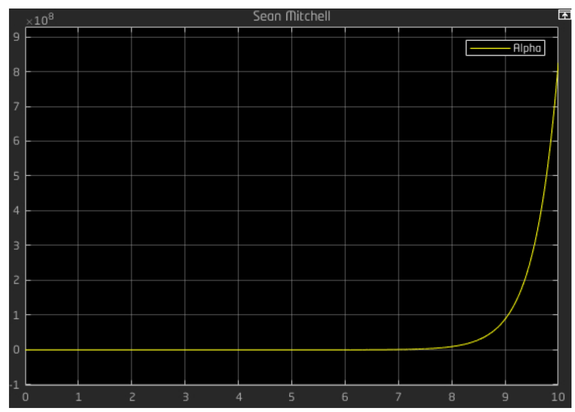

Figure 3: Simulation of $\alpha(t)$ (uncompensated plant)

4.2 Simulation of PD Controller

The following four plots show the simulation of the PD controller. From these plots, it is clear the PD controller is able to successfully drive the angle of attack, γ, to zero.

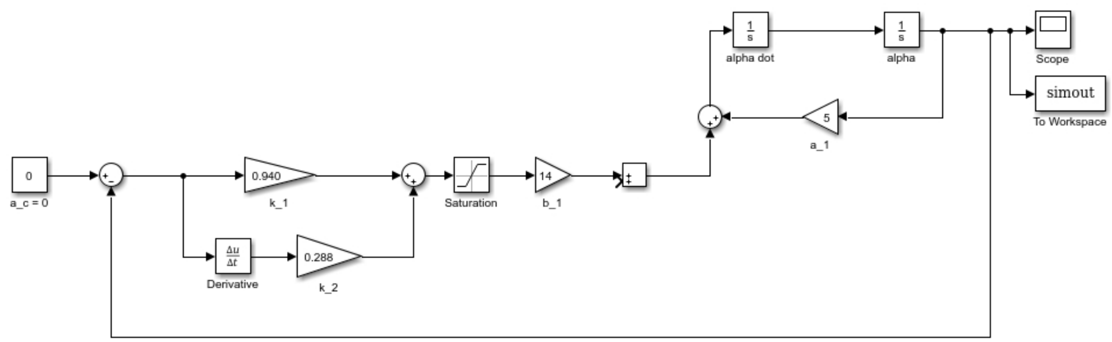

Figure 4: Math flow diagram of the PD controller

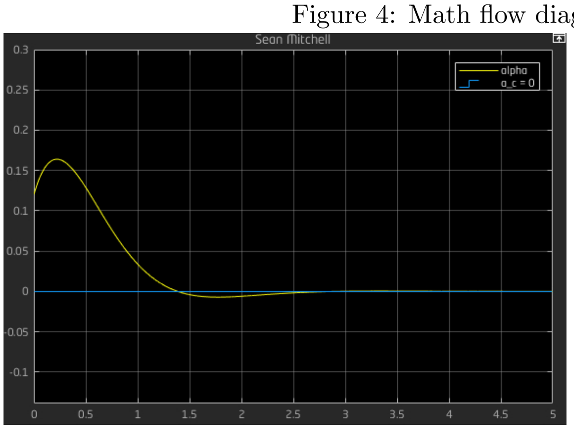

Figure 5: Simulation of $\alpha(t)$ (PD controller)

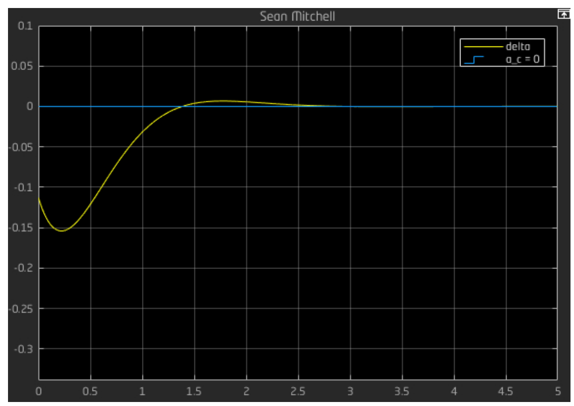

Figure 6: Simulation of $\delta(t)$ (PD controller)

4.3 Simulation of PD Controller with Wind Disturbance

The following four plots show the simulation of the PD controller with a wind disturbance. From these plots, it is clear the PD controller is able to successfully drive the angle of attack, γ, to zero until the wind disturbance is introduced. Once the wind disturbance is introduced, the PD controller keeps γ stable, but not at a value of zero.

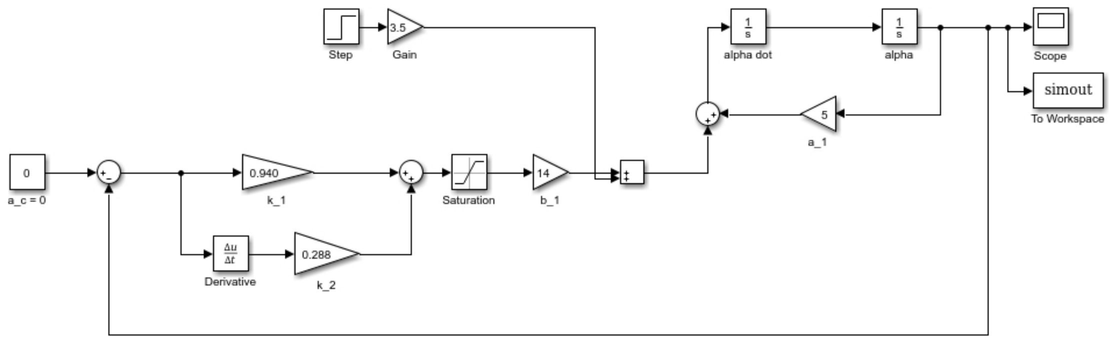

Figure 7: Math flow diagram of the PD controller with wind disturbance

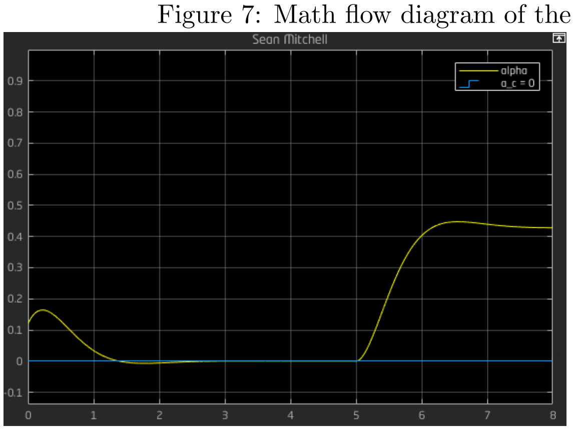

Figure 8: Simulation of $\alpha(t)$ (PD controller with wind disturbance)

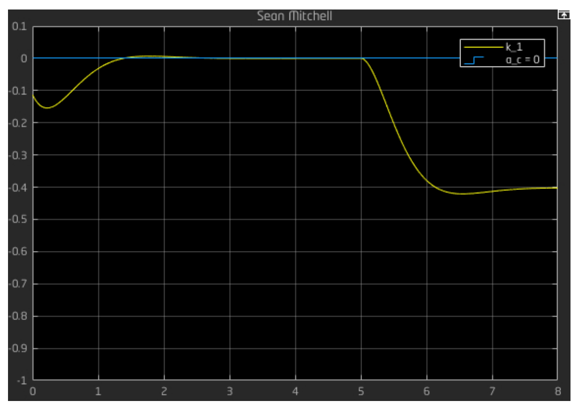

Figure 9: Simulation of $\delta(t)$ (PD controller with wind disturbance)

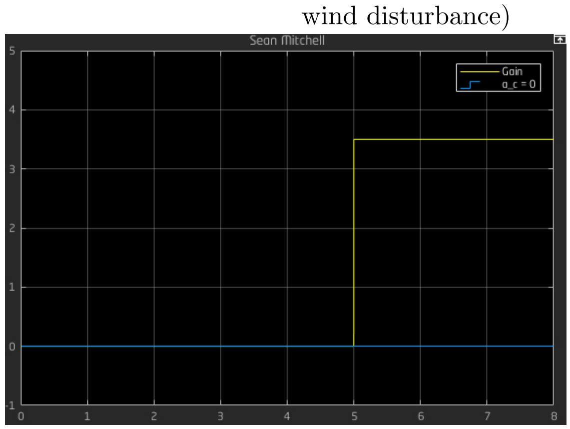

Figure 10: Simulation of $\omega(t)$ (PD controller with wind disturbance)

4.4 Simulation of PID Controller with Wind Disturbance

The following four plots show the simulation of the PID controller with a wind disturbance. From these plots, it is clear the PID controller is able to successfully drive the angle of attack, γ, to zero. Once the wind disturbance is introduced, the PID controller not only keeps γ stable, but drives γ back to zero even when the wind disturbance has been introduced.

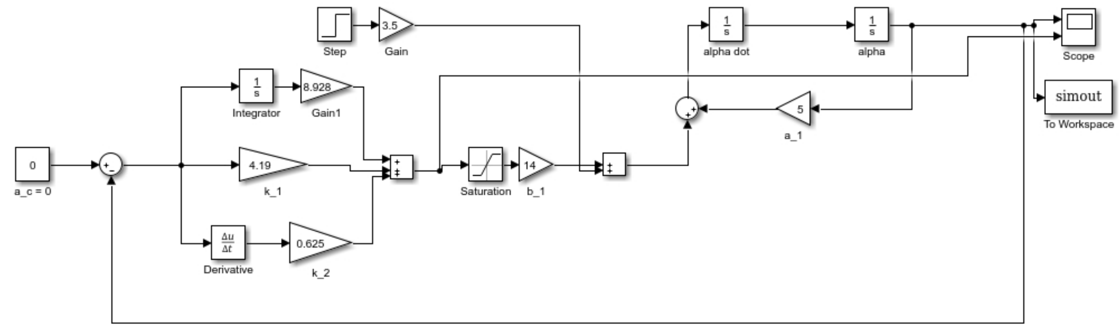

Figure 11: Math flow diagram of the PID controller with wind disturbance

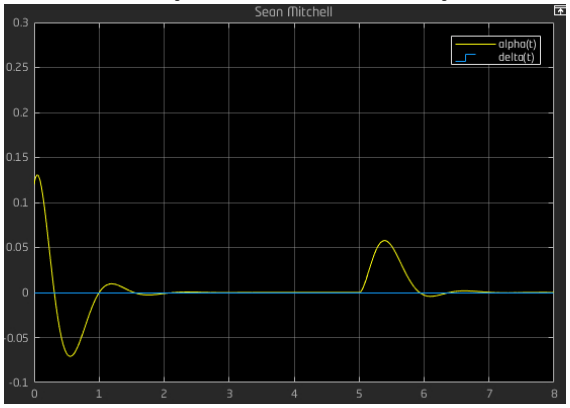

Figure 12: Simulation of $\alpha(t)$ (PID controller with wind disturbance)

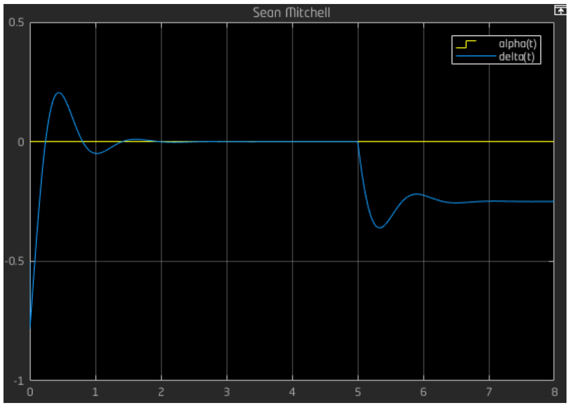

Figure 13: Simulation of $\delta(t)$ (PID controller with wind disturbance)

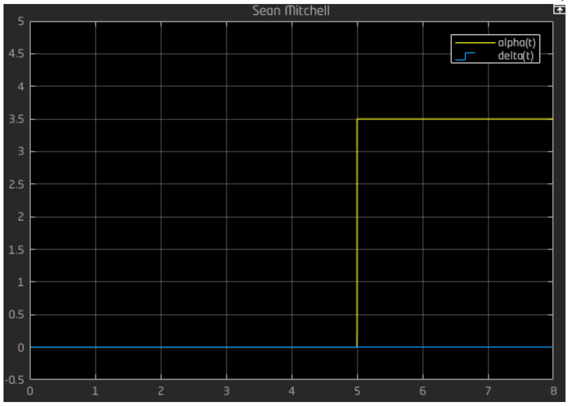

Figure 14: Simulation of $\omega(t)$ (PID controller with wind disturbance)

5 Conclusion

In conclusion, the goal of this project was to derive a way to stabilize the angle of attack of a rocket flying vertically. This goal was accomplished using two different controllers. The first controller designed was a PD controller. This controller stabilized the rocket's angle of attack when there was no wind disturbance. When a wind disturbance was introduced, the PD controller was unable to drive the angle of attack to zero. Next, a PID controller was successfully designed to set the angle of attack to zero with a wind disturbance.

Both the PD and PID controller were capable of successfully controlling the rocket's angle of attack. The PD controller could drive the angle attack to zero, when there was no wind disturbance. When there was a wind disturbance, the rocket's angle of attack was stable, but not at a value of zero. In comparison, a PID controller could drive the angle of attack to zero in the presence of a wind disturbance. However, this additional level of control is harder to achieve. The PID controller is more complicated to design and build when compared to the PD controller.

6 Appendix

6.1 References

[1] R. Dorf and R. Bishop, Modern Control Systems. Boston, MA: Pearson, 2016, pp. 59.

[2] Y. Shtessel, Class Lecture, Topic: "Simple Rocket Stabilization Problem." EE386, College of Engineering, Universty of Alabama in Huntsville, Huntsville, AL, Jan. 10, 2018.