Exponential Growth and Decay

Goal: understand the applications of exponential change & how calc is used to model real world phenomena.

Need to Know: Newton's Law of Cooling $T = T_s + Ae^{-kt}$

Velocity with Resistance $V = V_0 e^{-(\frac{k}{M})t}$

Continuous Compounding Interest $y = A_0 e^{kt}$

Interest Compounded k-times a year $y = A_0 (1+\frac{r}{k})^{kt}$

Exponential Change (quantity that Inc/Dec proportional to present amount population, money, Radioactive elements)

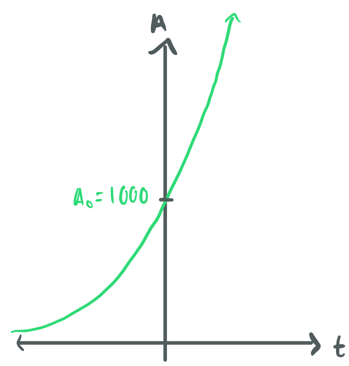
Finance Suppose you deposit \$1000 in an account that pays 10% annual interest. How much \$ will you have 3 years later if interest is:

ⓐ Compounded Yearly

$$1000 \xrightarrow[1\text{yr}]{10\%} 1100 \xrightarrow[2\text{yr}]{10\%} 1210 \xrightarrow[3\text{yr}]{10\%} \boxed{\$1331}$$

$\cdot(1+0.1)$ $\cdot(1+0.1)$ $\cdot(1+0.1)$

$\cdot(1+0.1)^3$



lets generalize this and create a formula →

$$A(4) = 1000(1+0.1)^4$$
$$A(t) = A_0(1+r)^t$$

where r - annual interest Rate

$A(0) = A_0$ - initial \$ amount

But **WAIT!** We're missing something. Bank's don't always compound just once per year.

(b) Compounded Quarterly

lets first compare \$ after 1 yr. Remember it's still 10% annual interest

$$
\begin{tikzcd}[column sep=huge]
1000 \arrow[r, "\frac{10\%}{4}", "\frac{1}{4}\text{ yr}"'] \arrow[r, bend right=50, "\cdot\left(1+\frac{0.1}{4}\right)"'] \arrow[rrrr, bend right=40, "\cdot\left(1+\frac{0.1}{4}\right)^4"'] & 1025 \arrow[r, "\frac{10\%}{4}", "\frac{2}{4}\text{ yr}"'] \arrow[r, bend right=50, "\cdot\left(1+\frac{0.1}{4}\right)"'] & 1050.62 \arrow[r, "\frac{10\%}{4}", "\frac{3}{4}\text{ yr}"'] \arrow[r, bend right=50, "\cdot\left(1+\frac{0.1}{4}\right)"'] & 1076.89 \arrow[r, "\frac{10\%}{4}", "\frac{4}{4}\text{ yr}"'] \arrow[r, bend right=50, "\cdot\left(1+\frac{0.1}{4}\right)"'] & 1103.81
\end{tikzcd}
$$

Formulating this we get: $A(1) = 1000\left(1+\frac{0.1}{4}\right)^{4\cdot 1}$ Remember this is just after 1yr

$$\boxed{A(t) = A_0\left(1+\frac{r}{k}\right)^{kt}}$$

where
- A_0 - initial value
- r - fixed annual interest rate
- k - number of times interest is compounded in a year
- t - years

(c) Compounded continuously

Think of your money as a population. How did we model a continuously growing population?

$A(t) = A_0 e^{kt}$ $\qquad$ $A(8) = 1000e^{(0.1)(3)} = \boxed{\$1349.85}$

Ex: 250 dollars is invested at 5% interest compounded continuously

(a) Find A(t)

(b) How much will be in the account after 6 years?

Chemistry Find the half-life of a decaying radioactive substance modeled by $y = y_0 e^{-kt}$

Atoms sometimes emit mass as Radiation. What mass remains reforms to make a different element. This is called Radioactive Decay. Radioactive elements decay at a rate proportional to the number of nuclei present (exponentially). Half-life is the time required for half of the initial nuclei to decay.

$$\cancel{y_0} e^{-kt} = \frac{1}{2} \cancel{y_0} \qquad \text{Solve for } t$$

$$e^{-kt} = \frac{1}{2} \Rightarrow -kt = \ln\left(\frac{1}{2}\right) \Rightarrow kt = \ln(2) \qquad \boxed{\ln\left(\frac{a}{b}\right) = -\ln\left(\frac{b}{a}\right)}$$

$$\Rightarrow \boxed{t = \frac{\ln(2)}{k}}$$ It's constant

This was used to determine the age of our Earth, and to date certain historical events (Historical Mathematics). Scientist modeled the decay of uranium to determine the Earth is 4.5 billion Years Old.

☆ half-life will always $= \frac{\ln(2)}{k}$

Newton's Law of Cooling

(the rate at which an object's Temp is changed by surrounding Temp T_s

boiled egg, Metal Rod in coffee)

$$\frac{dT}{dt} = -k(T - T_s)$$

$$\Rightarrow \int \frac{1}{T - T_s} dT = \int -k\, dt \Rightarrow \ln|T - T_s| = -kt + c$$

$$\Rightarrow |T - T_s| = A e^{-kt}$$

If $T \geq T_s$ $\quad T - T_s = A e^{-kt}$

☆ object is cooling down $\Rightarrow \boxed{T(t) = T_s + A e^{-kt}}$

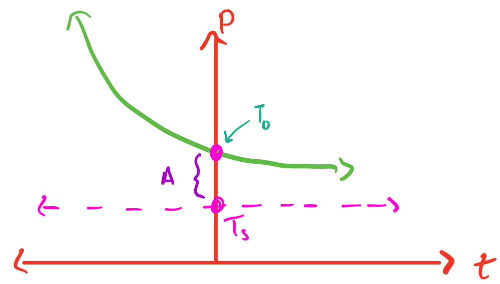
$T < T_s$ $\quad T_s - T = A e^{-kt}$

☆ object is heating up $\Rightarrow \boxed{T(t) = T_s - A e^{-kt}}$

Similare to exponential decay but instead of $\to 0$ it goes to surrounding temp

$$\boxed{T = T_s + Ae^{-kt}}$$

$$\boxed{T - T_s = (T_0 - T_s)e^{-kt}}$$



Since $A = T_0 - T_s$

ex) We place boiling water 100°C outside where it is 20°C
After 2min the water is 70°C
What will the temperature of the water be after 10 mins?

$T(0) = 100 \quad T_s = 20 \quad T(2) = 70 \quad$ Find $T(10) = ?$

(a) Find A_0

$T = 20 + Ae^{-kt}$ At $t = 0$, $T = 100°F$ $\Rightarrow$ $100 = 20 + A\cancel{e^0}^{1}$ $\Rightarrow$ $\underline{A = 80}$

Also, $A = T_0 - T_s$

(b) Find k

$T = 20 + 80e^{-kt}$ At $t = 2$, $T = 70°F$ $\Rightarrow$ $70 = 20 + 80e^{-k\cdot 2}$ $\Rightarrow$ $\frac{50}{80} = e^{-k\cdot 2}$ $\Rightarrow$ $\underline{-\frac{1}{2}\ln\left(\frac{5}{8}\right) = k}$

$$\boxed{T = 20 + 80e^{\left(\frac{t}{2}\ln\left(\frac{5}{8}\right)\right)}}$$

(c) $T = 20 + 80e^{\left(\frac{10}{2}\ln\left(\frac{5}{8}\right)\right)}$ $\Rightarrow$ $\boxed{T = 27.6°F}$

Velocity With Resistance (wind, friction

Resistance $\propto$ Velocity)

Yep, turns out Resistance $\propto$ Velocity. Think of sticking your hand out of the window of your car on a highway vs. on a neighborhood street. Slower $\rightarrow$ less resistance.

$$F = ma = m\frac{dv}{dt}$$

m - mass

$v'(x) =$ acceleration

$$m\frac{dv}{dt} = -kv$$

K - proportionality constant

Negative because coasting to a stop

Thus, acceleration < 0

$$\frac{dv}{dt} = \frac{-K}{M}v$$

$$\int \frac{1}{v}\,dv = \int -\frac{K}{M}\,dt \Rightarrow \boxed{V(t) = V_0 e^{-\left(\frac{K}{M}\right)t}}$$

Skating Ex For a 50 kg skater, $K = 2.5$ m/sec

(a) How long will it take the skater to coast from 7 m/sec to 1 m/sec?

We know $V_0 = 7$ m/s $V_f = 1$ m/s

$$1 = 7e^{-\left(\frac{2.5}{50}\right)t}$$ Solve for t

$$V(t) = 7e^{-\left(\frac{2.5}{50}\right)t}$$

$$\ln\left(\frac{1}{7}\right) = -\frac{1}{20}t$$

$$\Rightarrow t = 20\ln(7) = \boxed{38.9\,s}$$

(b) How far will the skater coast before coming to a complete stop?

Recall: Displacement is area under velocity curve. Distance is denoted S

$$S = \int 7e^{-\frac{t}{20}}\,dt = 7\int e^{-\frac{t}{20}}\,dt = 7\left[-20e^{-\frac{t}{20}} + c\right] \Rightarrow \underbrace{S = -140e^{-\frac{t}{20}} + C}$$

We know: when $s=0,\ t=0$. Thus, $0 = -140 + C \Rightarrow \underbrace{C = 140}$

$$\Rightarrow \boxed{S(t) = -140e^{-\frac{t}{20}} + 140} \qquad \lim_{t\to\infty}\left[-140e^{-\frac{t}{20}} + 140\right] = \boxed{140\,m}$$

Partial Fraction Decomposition

We are very familiar with doing:

$$\frac{2}{x-2} + \frac{3}{x+1} \longrightarrow \frac{2(x+1) + 3(x-2)}{(x-2)(x+1)} = \frac{5x-4}{(x-2)(x+1)}$$

But how can we Reverse this? [Enter Partial Fraction Decomposition]

$$\frac{5x-4}{(x-2)(x+1)} = \frac{A}{(x-2)} + \frac{B}{(x+1)} = \frac{A(x+1) + B(x-2)}{(x-2)(x+1)}$$

Thus, $5x-4 = A(x+1) + B(x-2)$

$$5x-4 = Ax + A + Bx - 2B$$

You need to pair up your variables to match the form of numerator

$$5x-4 = (A+B)x + (A-2B)$$

Substitute the roots in to solve for A, B

let $x=0$: $-4 = A - 2B \Rightarrow A = 2B - 4$

$x=-1$: $-9 = \cancel{-A} - B + \cancel{A} - 2B$

$-9 = -3B \Rightarrow B = 3$

$\Rightarrow A = 2$

Plugging back in we get our final form of

$$\frac{5x-4}{(x-2)(x+1)} = \frac{2}{(x-2)} + \frac{3}{(x+1)}$$

Practice Problems (Textbook) # 9, 11, 20, 21, 24, 30

(12) $y(t) = A_0 e^{kt}$ $y(3) = 10000$ $y(5) = 40{,}000$ Find A_0

$A_0 = y(0)$

$40{,}000 = 10{,}000\, e^{2k}$

$4 = e^{2k}$

$k = \frac{\ln(4)}{2}$

$10{,}000 = A_0\, e^{\left(\frac{\ln(4)}{2}\right)\cdot 3}$

$A_0 = \dfrac{10{,}000}{e^{\left(\frac{3}{2}\cdot \ln(4)\right)}} = \boxed{1250}$

(18) $T_s = 65°F$ $T(10) = 35$ $T(20) = 50$ Find T_0

$T(t) = T_s + A_0 e^{kt}$

$T(t) = 65 + A_0 e^{kt}$

$\left.\begin{array}{l} 35 = 65 + A_0 e^{10k} \\ 50 = 65 + A_0 e^{20k} \end{array}\right\}$ $-30 = A_0 e^{10k}$

$\dfrac{\ln\left(\frac{-30}{A_0}\right)}{10} = k \Rightarrow 50 = 65 + A_0 e^{20\cdot\frac{\ln\left(\frac{-30}{A_0}\right)}{10}}$

$\boxed{e^{a\cdot\ln(x)} = e^{\ln(x^a)}}$

$\Rightarrow \frac{-15}{A_0} = e^{2\cdot\ln\left(\frac{-30}{A_0}\right)} \Rightarrow \frac{-15}{A_0} = e^{\ln\left(\left(\frac{-30}{A_0}\right)^2\right)}$

$\Rightarrow \frac{-15}{A_0} = \left(\frac{-30}{A_0}\right)^2 \Rightarrow \frac{-15}{A_0} = \frac{30^2}{A_0^2}$

$\Rightarrow -15A_0^2 = 30^2\cdot A_0$

$\Rightarrow A_0 = \frac{30^2}{-15} = -60$

$A_0 = T_0 - T_s$

$-60 = T_0 - 65$

$\boxed{T_0 = 5°}$

(14) $90° \to 60°$ in 10min in a Room @ $20°C = T_s$

$\left.\begin{array}{l} T(t) = T_s + A_0 e^{-kt} \\ A_0 = T_0 - T_s = 90 - 20 = 70 \end{array}\right\} \Rightarrow 60 = 20 + 70e^{-k\cdot 10}$

$\frac{4}{7} = e^{-10k} \Rightarrow \dfrac{\ln\left(\frac{4}{7}\right)}{-10} = k$

$\Rightarrow \boxed{T(t) = 20 + 70e^{\left(\frac{\ln(\frac{4}{7})}{-10}\right)\cdot t}}$

(a) How long to go from $60° \to 35°C$?

$A_0 = T_0 - T_s = 60 - 20 = 40$

$35 = 20 + 40e^{\left(\frac{\ln(\frac{4}{7})}{-10}\right)\cdot t}$ Solve for t

$\ln\left(\frac{15}{40}\right) = \left(\dfrac{\ln\left(\frac{4}{7}\right)}{-10}\right)\cdot t$

$$t = \frac{\ln\left(\frac{15}{40}\right)}{\left(\frac{\ln\left(\frac{4}{7}\right)}{-10}\right)} = \ln\left(\frac{15}{40}\right) \cdot \frac{-10}{\ln\left(\frac{4}{7}\right)} = \boxed{17.52 \text{ mins}}$$

(b) $T_s = -15$, $90° \to 35°$ ✱ We use the same k Thankfully

$A_0 = T_0 - T_s = 90 + 15 = 105$

$$35 = -15 + 105e^{-kt}$$

Solve for t

$$\frac{\ln\left(\frac{50}{105}\right)}{-k} = t = \frac{\ln\left(\frac{50}{105}\right)}{\left(\frac{\ln\left(\frac{4}{7}\right)}{10}\right)} = \boxed{13.25 \text{ mins}}$$

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(23) Recall: $V(t) = V_0 e^{-\left(\frac{k}{M}\right)t}$ $M = 73\text{kg}$ $V_0 = 9 \text{ m/s}$ $k = 3.9 \text{ kg/s}$

(a) How far until $V_f = 0$?

$$v(t) = 9e^{-\left(\frac{3.9}{73}\right)t}$$

$$S = \int v(t)\,dt = \int V_0 e^{-\left(\frac{k}{M}\right)t}\,dt = \left[-\frac{M}{k} V_0 e^{-\left(\frac{k}{M}\right)t} + C\right]$$

$$\Rightarrow S(t) = -\left(\frac{73}{3.9}\right) \cdot 9 \cdot e^{-\left(\frac{3.9}{73}\right)t} + C \qquad 0 = -\left(\frac{73}{3.9}\right) \cdot 9 + C \Rightarrow \underline{C = \left(\frac{73}{3.9}\right) \cdot 9}$$

$$\Rightarrow S(t) = -\left(\frac{73}{3.9}\right) \cdot 9 \cdot e^{-\left(\frac{3.9}{73}\right)t} + \left(\frac{73}{3.9}\right) \cdot 9$$

$$\lim_{t \to \infty} \left[S(t)\right] = \left(\frac{73}{3.9}\right) \cdot 9 = \boxed{168.4 \text{ m}}$$

(b) How long until $V_f = 1 \text{ m/s}$?

$$v(t) = 9e^{-\left(\frac{3.9}{73}\right)t}$$

$$1 = 9e^{-\left(\frac{3.9}{73}\right)t} \Rightarrow \ln(9) \cdot \frac{73}{3.9} = t = \boxed{41 \text{ s}}$$

$$\int \frac{2x}{x^2+x-20}\,dx \Rightarrow \int \frac{2x}{(x+5)(x-4)}\,dx$$

$$\frac{A}{x+5} + \frac{B}{x-4} = 2x$$

$$A(x-4) + B(x+5) = 2x$$

$$Ax - 4A + Bx + 5B = 2x$$

$$(A+B)x + (5B-4A) = 2x$$

$x=0$: $\quad 5B = 4A \qquad A = \frac{5B}{4}$

$x=4$: $\quad \cancel{4A} + 4B + 5B - \cancel{4A} = 8$

$$9B = 8$$

$$B = \boxed{\frac{8}{9}}, \quad A = \frac{40}{36} = \frac{20}{18} = \boxed{\frac{10}{9}}$$

$$\frac{A}{x+5} + \frac{B}{x-4} = 2x$$

$$\int \frac{10}{9}\frac{1}{x+5} + \frac{8}{9}\cdot\frac{1}{x-4}\,dx \Rightarrow \boxed{\frac{10}{9}\ln|x+5| + \frac{8}{9}\ln|x-4| + C}$$