Nobel Prize 2016 in Physics Lecture Note : Kosterlitz - Thouless Transition

Continent : ① Model

② High Temperature behaviour and Low Temperature behaviour

③ Topological excitation and vortex

④ Renormalization group analysis of Kosterlitz - Thouless Transition

⑤ Experiments

① Model : XY model in 2D (Examples : Josephson Junction Array , superfluid)

$$H = -J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)$$

So the partition function of this model is

$$Z = \int_0^{2\pi} \prod_i \frac{d\theta_i}{2\pi} \exp\left[\beta J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)\right]$$

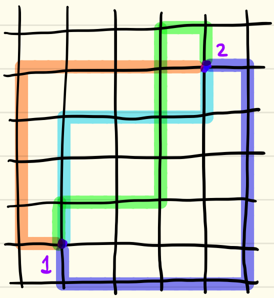
② High Temperature behaviour : $\beta J \ll 1$, then expand the exponential function :

$$Z = \int_0^{2\pi} \prod_i \frac{d\theta_i}{2\pi} \prod_{\langle ij \rangle} \left[1 + \beta J \cos(\theta_i - \theta_j) + O(J^2)\right]$$

now we are interested in the correlation function of θ_i, we need to calculate:

$$\langle \cos(\theta_1 - \theta_2) \rangle = \frac{1}{Z} \int_0^{2\pi} \prod_i \frac{d\theta_i}{2\pi} \cos(\theta_1 - \theta_2) \prod_{\langle ij \rangle} \left[1 + \beta J \cos(\theta_i - \theta_j)\right]$$

$$= \frac{1}{Z} \int_0^{2\pi} \prod_i \frac{d\theta_i}{2\pi} \cos(\theta_1 - \theta_2) \Big[\underbrace{1}_{\text{0 bond}} + \underbrace{\sum_{\langle ij \rangle} \beta J \cos(\theta_i - \theta_j)}_{\text{1 bond}} + \underbrace{\sum_{\substack{\langle i_1 j_1 \rangle \\ \langle i_2 j_2 \rangle}} (\beta J)^2 \cos(\theta_{i_1} - \theta_{j_1}) \cos(\theta_{i_2} - \theta_{j_2})}_{\text{2 bonds}} + \dots\Big]$$



Only the terms which connected point 1 and 2 contribute to the integral:

$$\langle\ \rangle \propto \int \prod_i \frac{d\theta_i}{2\pi} \cos(\theta_1 - \theta_2) \sum_{l(1\to 2)} (\beta J)^{m_l} \prod_{\{l\}} \cos(\theta_i - \theta_j)$$

$$= \sum_{\{l\}} \left(\frac{\beta J}{2}\right)^{m_l}$$

in which m_l is length of path l. Factor $\frac{1}{2}$ comes from:

$$\int_0^{2\pi} \frac{d\theta_2}{2\pi} \cos(\theta_1 - \theta_2)\cos(\theta_2 - \theta_3) = \frac{1}{2}\cos(\theta_1 - \theta_3)$$

Thus we find the correlation function has the following form:

$$C(\vec{r}_1 - \vec{r}_2) = \langle \cos(\theta_1 - \theta_2)\rangle \sim \left(\frac{\beta J}{2}\right)^{\frac{|r_1 - r_2|}{a}} = \exp\left(-\frac{|r_1 - r_2|}{\xi}\right)$$

and the correlation length $\xi = \dfrac{a}{\ln\left(\frac{2}{\beta J}\right)}$. This is a short range order.

Low temperature behaviour: at low temperature the model can be viewed as a continuous field theory because long wave excitations are dominant.

$$H = \frac{J}{2}\int d^2x \left[(\partial_x \theta)^2 + (\partial_y \theta)^2\right]$$

The partition function then becomes a "classical" path integral

$$Z = \int D\theta \exp\left\{-\frac{\beta J}{2}\int d^2x \left[(\partial_x\theta)^2 + (\partial_y\theta)^2\right]\right\}. \quad \left(D\theta \propto \prod_x \theta(x)\right)$$

low temperature correlation function is given by:

$$\langle \cos(\theta_1 - \theta_2)\rangle = \mathrm{Re}\left[\frac{1}{Z}\int D\theta\, e^{i(\theta_1 - \theta_2)} e^{-\frac{\beta J}{2}\int d^2x (\nabla\theta)^2}\right]$$

it can be written as:

$$C(\vec{r}_1-\vec{r}_2) = \frac{1}{Z}\int D\theta \exp\left\{-\int d^2r\left[\frac{\beta J}{2}(\nabla\theta)^2+\theta J\right]\right\}$$ in which $J=-i\delta^2(\vec{r}-\vec{r}_1)+i\delta^2(\vec{r}-\vec{r}_2)$

Gaussian Integral

$$\int d^N x \exp\left[-\frac{1}{2}x^T A x + J^T x\right] = \sqrt{\frac{(2\pi)^N}{\det A}}\exp\left(\frac{1}{2}x^T A^{-1} x\right)$$

By analogy we can get

$$\int D\theta \exp\left\{-\int d^2r\left[\frac{\beta J}{2}(\nabla\theta)^2+\theta J\right]\right\} \propto \exp\left[\frac{1}{2}\int d^2x_1 d^2x_2 J(x_1) G(x_1-x_2) J(x_2)\right]$$

The Green function $G(x_1-x_2)$ is the inverse of $-\beta J\nabla^2$:

$$-\beta J\nabla_{x_1}^2 G(x_1-x_2) = \delta^2(x_1-x_2)$$

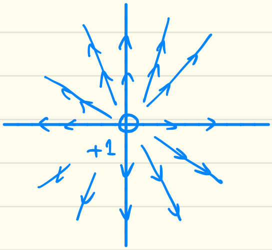
Green Function of 2D Poisson Equation is a logarithm function of $|\vec{x}_1-\vec{x}_2|$, so the correlation function of spin is power-law decay and the result is:

$$C(\vec{r}_1-\vec{r}_2) \propto \left(\frac{1}{|\vec{r}_1-\vec{r}_2|}\right)^{\frac{1}{2\pi\beta J}}$$

We find the XY model has a quasi-long range order at low temperature which is different from the high temperature case. Between the two should have a transition point.

③ Topological Excitation, vortex and phase transition temperature.

A vortex at position (x_0, y_0) with topological charge k looks like:



$$\theta = k \arctan\left(\frac{y-y_0}{x-x_0}\right)$$

The energy of this vertex is:

$$E = \frac{J}{2}\int d^2 r \, (\nabla\theta)^2$$

$$= \frac{Jk^2}{2}\int_a^R 2\pi r dr \frac{1}{(x-x_0)^2 + (y-y_0)^2}$$

$$= k^2 \pi J \ln\frac{R}{a}$$

R is the size of our system, and a is the short-range cutoff (in our lattice model of Josephson Junction Array, it's the lattice constant; in the Boson superfluid it's healing length ξ).

$k = \pm 1$ vortex costs the least energy, so we consider about $S = \pm 1$ case.

To create a vortex with charge $k = \pm 1$, the entropy gain should be:

$$S = \ln\Omega = 2\ln\frac{R}{a}$$

That's because the numbers of available microstates (with ONE vortex whose charge $k = \pm 1$) is proportional to the area of the system.

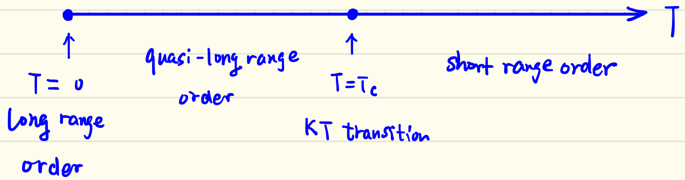
If $E > TS$, then energy cost is dominant, so it's unlikely to be excited.

If $E < TS$, the entropy gain is dominant, so vortices can be excited.

$E = T_c S$ gives the transition temperature :

$$T_c = \frac{\pi J}{2}$$

Conclusion : Phase Diagram of XY model



④ Renormalization group analysis.

Actually the vortex correspond to singularities of field $\theta(\vec{r})$. So it can be devided into to parts : $\vec{\nabla}\theta = \nabla\phi + \vec{u}'$, and all of the topological defects are absorbed into this vector field :

$$\oint_{\partial C} \vec{u}' \cdot \vec{dl} = 2\pi \sum_{i \in C} n_i$$

So it can be written as $\vec{\nabla} \times \vec{u}' = 2\pi \sum_i n_i \delta(\vec{r} - \vec{r}_i)\hat{e}_z$,

that means vortices with charge n_i at position $\vec{r}_i$.

Consider about the interaction between vortices:

$$E = \frac{J}{2}\int d^2r \left[\vec{\nabla}\phi + \vec{u}'\right]^2$$

$$= \frac{J}{2}\int d^2r (\nabla\phi)^2 - \pi J \sum_{ij} n_i n_j \ln\left(\frac{|\vec{r}_i - \vec{r}_j|}{a}\right)$$

$$= \frac{J}{2}\int d^2r (\nabla\phi)^2 - 2\pi J \sum_{i<j} n_i n_j \ln\left(\frac{\vec{r}_i - \vec{r}_j}{a}\right) + \sum_i S_c^{(i)}$$

$$E = \frac{J}{2}\int d^2r (\nabla\phi)^2 - 2\pi J \sum_{i<j} n_i n_j \ln\left(\frac{|\vec{r}_i - \vec{r}_j|}{a}\right) + \sum_i S_c^{(i)}$$

$\uparrow$ Energy of spin wave

$\uparrow$ interactions between vortices

$\uparrow$ Energy inside the vortex core.

Example of a field $\vec{u}'$: the simplest vortex $\theta = \arctan\left(\frac{y}{x}\right)$.

$$\nabla\theta = \nabla\phi + \vec{u}' \quad \text{in which } \nabla\phi = 0, \quad \vec{u}' = -\frac{y}{r^2}\hat{e}_x + \frac{x}{r^2}\hat{e}_y$$

We find that these vortices form a Coulomb gas (without the kinetic energy term) and the partition function can be written as:

$$Z = Z_{sw} \sum_{N=0}^{\infty} \frac{1}{N!N!} \int \prod_{i=1}^{2N} \frac{d^2r_i}{a^2} \exp\left[-\sum_i \beta S_c^{(i)} + 2\pi\beta J \sum_{i<j} n_i n_j \ln\left(\frac{\vec{r}_i - \vec{r}_j}{a}\right)\right]$$

Since vortex with charge large than 1 costs too much energy, we should only consider about charges $n_i = \pm 1$, then the partition function becomes:

$$Z = Z_{sw} \sum_{N=0}^{\infty} \frac{y_0^{2N}}{N!N!} \int \prod_{i=1}^{2N} \frac{d^2r_i}{a^2} \exp\left[2\pi\beta J \sum_{i<j} \sigma_i \sigma_j \ln\left(\frac{\vec{r}_i - \vec{r}_j}{a}\right)\right]$$

in which the quantity $y_0 = \exp[-\beta S_c^{(1)}]$

Effective Theory. We put two external vortices in the XY model and see how the excited vortices effect the interaction between them.

$$\left\langle e^{-2\pi\beta J \ln|\vec{r}_1 - \vec{r}_2|} \right\rangle = \frac{(r\oplus - \ominus r') + y_0^2 \int d^2s\, d^2s' \left(\begin{smallmatrix} s\oplus - \ominus s' \\ \times \\ r\oplus - \ominus r' \end{smallmatrix}\right) + O(y_0^4)}{1 + y_0^2 \int d^2s\, d^2s' (s\oplus - \ominus s') + O(y_0^4)}$$

we have redefined $r \to \frac{r}{a}$, and $(x\oplus - \ominus y) = 2\pi\beta J \sigma_x \sigma_y \ln(|\vec{x} - \vec{y}|)$.

Define the effective constant $(\beta J)_{eff}$:

$$e^{-2\pi(\beta J)_{eff}\ln|\vec{r}_1-\vec{r}_2|} = \left\langle e^{-2\pi\beta J\ln|\vec{r}_1-\vec{r}_2|}\right\rangle$$

Expand $\left\langle e^{-2\pi\beta J\ln|\vec{r}_1-\vec{r}_2|}\right\rangle$ to 2^{nd} order of y_0 and J, we have:

$$(\beta J)_{eff} = \beta J - 4\pi^3(\beta J)^2 y_0^2 \int_1^{\infty} dx\, x^{3-2\pi\beta J} + \mathcal{O}(y_0^4)$$

short range cutoff (lattice constant)

$$(\beta J)_{eff}^{-1} = (\beta J)^{-1} + 4\pi^3 y_0^2 \int_1^{\infty} dx\, x^{3-2\pi\beta J}$$

Next is the Renormalization Group progress:

Rescaling :

$$x \to x' = x/b$$

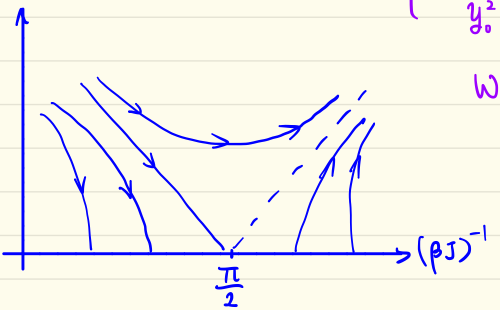
$$a \to a' = a/b$$

$$(\beta J)_{eff}^{-1} = (\beta J)^{-1} + 4\pi^3 y_0^2 \int_1^{b} dx\, x^{3-2\pi\beta J} + 4\pi^3 y_0^2 \int_b^{\infty} dx\, x^{3-2\pi\beta J}$$

$$= (\beta J)^{-1} + 4\pi^3 y_0^2 \int_1^{b} dx\, x^{3-2\pi\beta J} + 4\pi^3 y_0^2 \int_1^{\infty} dx'\, x'^{3-2\pi}\, b^{4-2\pi\beta J}$$

$$= (\tilde{\beta J})^{-1} + 4\pi^3 \tilde{y}_0^2 \int_1^{\infty} dx'\, x^{3-2\pi\beta J}$$

$y_0 = e^{-\beta S_c^{(1)}}$



$$\begin{cases} (\tilde{\beta J})^{-1} = (\beta J)^{-1} + 4\pi^3 y_0^2 \int_1^{b} dx\, x^{3-2\pi\beta J} \\ \tilde{y}_0^2 = y_0^2\, b^{4-2\pi\beta J} \end{cases}$$

When $b = e^{l} \simeq 1 + l$, we get the β-function:

$$\frac{d(\beta J)^{-1}}{dl} = 4\pi^3 y_0^2$$

$$\frac{dy_0}{dl} = (2-\pi\beta J)\, y_0$$