

Nobel Prize 2016 in Physics Lecture Note : Kosterlitz - Thouless Transition

Content:

- ① Model
- ② High Temperature behaviour and Low Temperature behaviour
- ③ Topological excitation and vortex
- ④ Renormalization group analysis of Kosterlitz - Thouless Transition
- ⑤ Experiments

① Model : XY model in 2D (Examples : Josephson Junction Array , superfluid)

$$H = - J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j)$$

So the partition function of this model is

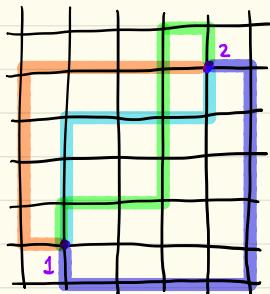
$$Z = \int_0^{2\pi} \prod_i \frac{d\theta_i}{2\pi} \exp \left[\beta J \sum_{\langle ij \rangle} \cos(\theta_i - \theta_j) \right]$$

② High Temperature behaviour : $\beta J \ll 1$, then expand the exponential function :

$$Z = \int_0^{2\pi} \prod_i \frac{d\theta_i}{2\pi} \prod_{\langle ij \rangle} \left[1 + \beta J \cos(\theta_i - \theta_j) + O(J^2) \right]$$

now we are interested in the correlation function of θ_i , we need to calculate:

$$\begin{aligned} \langle \cos(\theta_1 - \theta_2) \rangle &= \frac{1}{Z} \int_0^{2\pi} \prod_i \frac{d\theta_i}{2\pi} \cos(\theta_1 - \theta_2) \prod_{\langle ij \rangle} \left[1 + \beta J \cos(\theta_i - \theta_j) \right] \\ &= \frac{1}{Z} \int_0^{2\pi} \prod_i \frac{d\theta_i}{2\pi} \cos(\theta_1 - \theta_2) \left[1 + \sum_{\langle ij \rangle} \beta J \cos(\theta_i - \theta_j) + \sum_{\substack{\langle i j | l \rangle \\ \langle i' j' | l' \rangle}} (\beta J)^2 \cos(\theta_i - \theta_j) \cos(\theta_{i2} - \theta_{j2}) + \dots \right] \\ &\quad \uparrow \qquad \uparrow \qquad \uparrow \\ &\quad \text{o bond} \qquad \text{1 bond} \qquad \text{2 bonds} \end{aligned}$$



Only the terms which connect point 1 and 2 contribute to the integral:

$$\langle \rangle \propto \int_{\text{I}} \frac{d\theta_1}{2\pi} \cos(\theta_1 - \theta_2) \sum_{j(i=2)} (\beta J)^{m_j} \prod_{j \neq i} \cos(\theta_j - \theta_j)$$

$$= \sum_{\{l\}} \left(\frac{\beta J}{2} \right)^{m_l}$$

in which m_l is length of path l . Factor $\frac{1}{2}$ comes from:

$$\int_0^{2\pi} \frac{d\theta_2}{2\pi} \cos(\theta_1 - \theta_2) \cos(\theta_2 - \theta_3) = \frac{1}{2} \cos(\theta_1 - \theta_3)$$

Thus we find the correlation function has the following form:

$$C(\vec{r}_1 - \vec{r}_2) = \langle \cos(\theta_1 - \theta_2) \rangle \sim \left(\frac{\beta J}{2} \right)^{\frac{|r_1 - r_2|}{a}} = \exp \left(-\frac{|r_1 - r_2|}{\xi} \right)$$

and the correlation length $\xi = \frac{a}{\ln(\frac{2}{\beta J})}$. This is a short range order.

Low temperature behaviour: at low temperature the model can be viewed as a continuous field theory because long wave excitations are dominant.

$$H = \frac{J}{2} \int d^2x \left[(\partial_x \theta)^2 + (\partial_y \theta)^2 \right]$$

The partition function then becomes a "classical" path integral

$$Z = \int D\theta \exp \left\{ -\frac{\beta J}{2} \int d^2x \left[(\partial_x \theta)^2 + (\partial_y \theta)^2 \right] \right\}. \quad (D\theta \propto \prod_x \theta(x))$$

Low temperature correlation function is given by:

$$\langle \cos(\theta_1 - \theta_2) \rangle = \text{Re} \left[\frac{1}{Z} \int D\theta e^{i(\theta_1 - \theta_2)} e^{-\frac{\beta J}{2} \int d^2x (\nabla \theta)^2} \right]$$

it can be written as :

$$C(\vec{r}_i - \vec{r}) = \frac{1}{Z} \int D\theta \exp \left\{ - \int d^2r \left[\frac{\beta J}{2} (\nabla \theta)^2 + \theta J \right] \right\} \text{ in which } J = -i\delta^2(\vec{r} - \vec{r}_i) + i\delta^2(\vec{r} - \vec{r}_s)$$

Gaussian Integral

$$\int d^N x \exp \left[-\frac{1}{2} x^T A x + J^T x \right] = \sqrt{\frac{(2\pi)^N}{\det A}} \exp \left(\frac{1}{2} x^T A^{-1} x \right)$$

By analogy we can get

$$\int D\theta \exp \left\{ - \int d^2r \left[\frac{\beta J}{2} (\nabla \theta)^2 + \theta J \right] \right\} \propto \exp \left[\frac{1}{2} \int d^2x_1 d^2x_2 J(x_1) \phi(x_1 - x_2) J(x_2) \right]$$

The Green function $G(x_1 - x_2)$ is the inverse of $-\beta J \nabla^2$:

$$-\beta J \nabla_{x_1}^2 G(x_1 - x_2) = \delta^2(x_1 - x_2)$$

Green Function of 2D Poisson Equation is a logarithm function of $|x_1 - x_2|$,

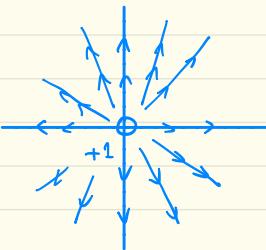
so the correlation function of spin is power-law decay and the result is :

$$C(\vec{r}_1 - \vec{r}_2) \propto \left(\frac{1}{|\vec{r}_1 - \vec{r}_2|} \right)^{\frac{1}{2\pi\beta J}}$$

We find the XY model has a quasi-long range order at low temperature which is different from the high temperature case. Between the two should have a transition point.

③ Topological Excitation, vortex and phase transition temperature.

A Vortex at position (x_0, y_0) with topological charge K looks like:



$$\theta = K \arctan\left(\frac{y - y_0}{x - x_0}\right)$$

The energy of this vortex is :

$$\begin{aligned} E &= \frac{J}{2} \int d^2 r (\nabla \theta)^2 \\ &= \frac{JK^2}{2} \int_a^R 2\pi r dr \frac{1}{(x - x_0)^2 + (y - y_0)^2} \\ &= K^2 \pi J \ln \frac{R}{a} \end{aligned}$$

R is the size of our system, and a is the short-range cutoff (in our lattice model) of Josephson Junction Array, it's the lattice constant ; in the Boson superfluid it's healing length ξ).

$K = \pm 1$ vortex costs the least energy, so we consider about $S = \pm 1$ case.

To create a vortex with charge $K = \pm 1$, the entropy gain should be:

$$S = \ln \Omega = 2 \ln \frac{R}{a}$$

That's because the numbers of available microstates (with ONE vortex whose charge $K = \pm 1$) is proportional to the area of the system.

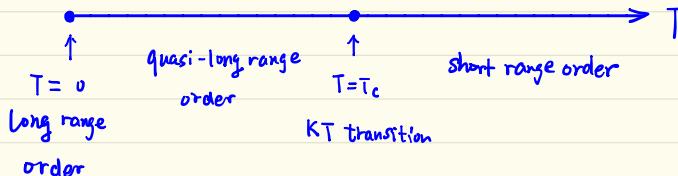
If $E > TS$, then energy cost is dominant, so it's unlikely to be excited.

If $E < TS$, the entropy gain is dominant, so vortices can be excited.

$E = T_c S$ gives the transition temperature :

$$T_c = \frac{\pi J}{2}$$

Conclusion : Phase Diagram of XY model



④ Renormalization group analysis.

Actually the vortex correspond to singularities of field $\vec{\phi}(\vec{r})$. So it can be

devided into two parts : $\vec{\nabla}\theta = \vec{\nabla}\phi + \vec{u}'$, and all of the topological defects are absorbed into this vector field :

$$\oint_{\partial C} \vec{u}' \cdot d\vec{l} = 2\pi \sum_i n_i$$

So it can be written as $\vec{\nabla} \times \vec{u}' = 2\pi \sum_i n_i \delta(\vec{r} - \vec{r}_i) \hat{e}_z$,

that means vortices with charge n_i at position \vec{r}_i .

Consider about the interaction between vortices :

$$E = \sum_i S_i^{(i)} + \frac{J}{2} \int d^2r \left[\vec{\nabla}\phi + \vec{u}' \right]^2$$

$$= \frac{J}{2} \int d^2r (\nabla\phi)^2 - \pi J \sum_{ij} n_i n_j \ln \left(\frac{|\vec{r}_i - \vec{r}_j|}{a} \right)$$

$$E = \frac{J}{2} \int d^2r (\nabla\phi)^2 - 2\pi J \sum_{i < j} n_i n_j \ln \left(\frac{|\vec{r}_i - \vec{r}_j|}{a} \right) + \sum_i S_c^{(i)}$$

↑
 Energy of
spin wave

↑
 interactions between
vortices

↑
 Energy inside
the vortex core.

Example of a field \vec{u}' : the simplest vortex $\Theta = \arctan(\frac{y}{x})$.

$$\nabla\Theta = \nabla\phi + \vec{u}' \quad \text{in which } \nabla\phi = 0, \quad \vec{u}' = -\frac{y}{r^2} \hat{e}_x + \frac{x}{r^2} \hat{e}_y$$

We find that the energy of vortices form a 2D Coulomb gas (without kinetic energy), the partition function is:

$$Z = Z_{SW} \sum_{N=0}^{\infty} \frac{1}{N! N!} \int \prod_{i=1}^{2N} \frac{d^2r_i}{a^2} \exp \left[- \sum_i \beta S_c^{(i)} + 2\pi \beta J \sum_{i < j} n_i n_j \ln \left(\frac{|\vec{r}_i - \vec{r}_j|}{a} \right) \right]$$

Since vortex with charge large than 1 costs too much energy, we should only consider about charges $n_i = \pm 1$, then the partition function becomes:

$$Z = Z_{SW} \sum_{N=0}^{\infty} \frac{y_0^{2N}}{N! N!} \int \prod_{i=1}^{2N} \frac{d^2r_i}{a^2} \exp \left[2\pi \beta J \sum_{i < j} \sigma_i \sigma_j \ln \left(\frac{|\vec{r}_i - \vec{r}_j|}{a} \right) \right]$$

in which the quantity $y_0 = \exp[-\beta S_c^{(1)}]$

Effective Theory. We put two external vortices in the XY model and see how the excited vortices effect the interaction between them.

$$\langle e^{-2\pi\beta J \ln |\vec{r}_i - \vec{r}_j|} \rangle = \frac{(y_0^{\Theta} - \sigma r') + y_0^{\sigma} \int d^2s d^2s' \left(\frac{s^{\Theta} - \Theta s'}{r^{\Theta} - \Theta r'} \right) + O(y_0^4)}{1 + y_0^2 \int d^2s d^2s' (s^{\Theta} - \Theta s') + O(y_0^4)}$$

we have redefined $r \rightarrow \frac{r}{a}$, and $(\Theta - \Theta y) = 2\pi\beta J \sigma_x \sigma_y \ln(|\vec{x} - \vec{y}|)$.

Define the effective constant $(\beta J)_{\text{eff}}$:

$$e^{-2\pi(\beta J)_{\text{eff}} \ln |\vec{r}_i - \vec{r}_j|} = \langle e^{-2\pi \beta J \ln |\vec{r}_i - \vec{r}_j|} \rangle$$

Expand $\langle e^{-2\pi \beta J \ln |\vec{r}_i - \vec{r}_j|} \rangle$ to 2nd order of y_0 and J , we have:

$$(\beta J)_{\text{eff}} = \beta J - 4\pi^3 (\beta J)^2 y_0^2 \int_1^\infty dx \propto x^{3-2\pi \beta J} + O(y_0^4)$$

↑ short range cutoff (lattice constant)

$$(\beta J)_{\text{eff}}^{-1} = (\beta J)^{-1} + 4\pi^3 y_0^2 \int_1^\infty dx \propto x^{3-2\pi \beta J}$$

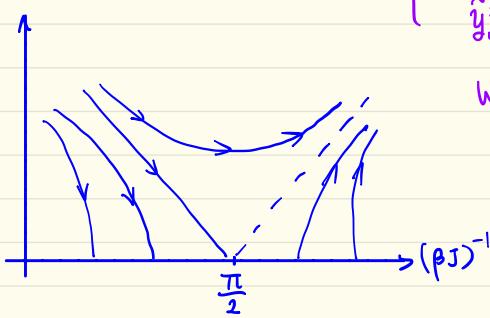
Next is the Renormalization Group progress:

Rescaling : $x \rightarrow x' = x/b$

$$a \rightarrow a' = a/b$$

$$\begin{aligned} (\beta J)_{\text{eff}}^{-1} &= (\beta J)^{-1} + 4\pi^3 y_0^2 \int_1^b dx \propto x^{3-2\pi \beta J} + 4\pi^3 y_0^2 \int_b^\infty dx \propto x^{3-2\pi \beta J} \\ &= (\beta J)^{-1} + 4\pi^3 y_0^2 \int_1^b dx \propto x^{3-2\pi \beta J} + 4\pi^3 y_0^2 \int_1^b dx' x'^{3-2\pi \beta J} b^{4-2\pi \beta J} \\ &= (\tilde{\beta J})^{-1} + 4\pi^3 y_0^2 \int_1^\infty dx' x'^{3-2\pi \beta J} \end{aligned}$$

$$y_0 = e^{-\beta S_c^{(0)}} \quad \left\{ \begin{array}{l} (\tilde{\beta J})^{-1} = (\beta J)^{-1} + 4\pi^3 y_0^2 \int_1^b dx \propto x^{3-2\pi \beta J} \\ \tilde{y}_0^2 = y_0^2 b^{4-2\pi \beta J} \end{array} \right.$$



When $b = e^t \approx 1 + t$, we get the β -function:

$$\frac{d(\beta J)^{-1}}{dt} = 4\pi^3 y_0^2$$

$$\frac{dy_0}{dt} = 2 - \pi \beta J$$