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Optical Signal-to-Noise Ratio and the Q-Factor in Fiber-Optic Communication Systems

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1 Introduction

The ratio of signal power to noise power at the receiver of a fiber-optic communication system has a direct impact on the system performance. Many electrical engineers are familiar with signal-to-noise ratio (SNR) concepts when referring to electrical signal and noise powers, but have less familiarity with the equivalent optical signal and noise powers. The purpose of this application note is to show the relationship between the electrical and optical signal-to-noise ratio (SNR), and then introduce the Q-factor.

While the principles outlined in this application note may be applied to many types of systems, the scope of the discussion is limited to binary digital communications over optical fiber. Within this scope, there are only two possible symbols that can be transmitted, where these symbols represent a binary one or a binary zero. Thus, the symbol rate and the bit rate are equivalent.

2 Signal Power

The power in an arbitrary electrical waveform can be defined as the voltage multiplied by the current, which is written mathematically as:

$$P_E(t) = v(t)i(t) \tag{1}$$

Using ohm's law, we can substitute *v(t) = i(t)R*, or alternately *i(t) = v(t)/R*, into equation (1) to get:

$$P_E(t) = v^2(t)/R = i^2(t)R \tag{2}$$

where R = voltage/current is the resistance in ohms. In binary digital communications, the signal is limited to two discrete levels. Based on this, we can represent the electrical signal power at any given time by either:

$$S_{EL} = \frac{v_L^2}{R} = i_L^2 R \text{ or } S_{EH} = \frac{v_H^2}{R} = i_H^2 R \tag{3}$$

where S_E represents electrical signal power and the subscripts L and H represent the low or high power, voltage, or current levels associated with a binary zero or one respectively.

Now we will repeat the above derivations for the case of optical signals using electromagnetic vector notation. Using this notation, the power in an optical signal can be defined as the magnitude of the vector cross product of the electric and magnetic fields, which can be written and simplified as follows:

$$P_O(t) = \left|\vec{E}(t) \times \vec{H}(t)\right| = \left|\vec{E}(t)\right| \frac{\left|\vec{E}(t)\right|}{\eta} = \frac{\left|\vec{E}(t)\right|^2}{\eta} \tag{4}$$

where the notation | X | represents the magnitude of X, and $\eta = \sqrt{\frac{\mu}{\varepsilon}}$ is the optical impedance of the fiber (μ = permeability and ε = permittivity). Recognizing that there are only two discrete power levels leads to the optical equivalent of equation (3), i.e.,

$$S_{OL} = \frac{\left|\vec{E}_L\right|^2}{\eta} \text{ and } S_{OH} = \frac{\left|\vec{E}_H\right|^2}{\eta} \tag{5}$$

where S_O represents optical signal power and the subscripts L and H represent the low and high power or low and high electric field strengths associated with a binary zero or one respectively.

3 Noise Power

Noise can be defined as any unwanted or interfering "signal" other than the one that is intended or expected. The various types of noise and their sources are beyond the scope of this application note. For purposes of illustration we will model the noise power as random, normally distributed, zero mean, and additive (the most common type of noise).

The random nature of the noise means that the instantaneous value of the noise amplitude is unpredictable. Thus, instead of classifying the noise in terms of its actual value at any given time, we use statistical averages and probabilities. We will classify the noise amplitude in terms of its root-mean-square (rms) average, which is commonly given the symbol σ. The noise power is similarly expressed in terms of its mean-square average (equivalent to the statistical variance), which is given the symbol σ^2. In general, the noises associated with the high and low signal levels in binary optical digital communications each have a different value.

The mean-square average electrical and optical noise powers can be computed mathematically using the following equations:

$$N_E = \tfrac{1}{T}\int_0^T i_N^2(t)\, R\, dt = \sigma_i^2 R$$

$$= \tfrac{1}{T}\int_0^T v_N^2(t)\frac{1}{R}\, dt = \sigma_v^2 \frac{1}{R} \qquad (6)$$

$$N_o = \tfrac{1}{T}\int_0^T \left|\vec{E}_N(t)\right|^2 \frac{1}{\eta}\, dt = \sigma_o^2 \frac{1}{\eta} \qquad (7)$$

where N is the noise power, T is the integration period, σ^2 is the mean-square average power, and the subscript N signifies that the associated current, voltage or electric field is classified as noise.

4 Signal Plus Noise

Addition of the signal and noise *amplitudes* versus addition of the signal and noise *powers* can sometimes cause mathematical confusion. For example, if the combined signal and noise amplitude is written as $i(t) = (i_H + \sigma_i)$, then the power would seemingly be $i^2(t)R = (i_H + \sigma_i)^2 R$, which, when multiplied out, is equal to $(i_H^2 + 2 i_H \sigma_i + \sigma_i^2) R$. But, addition of the results of equations (3) and (6) gives $S_E + N_E = (i_H^2 + \sigma_i^2) R$. So, why is there a difference in the two results?

The answer lies in the fact that when we add the signal (a constant) and the noise (an average value), we compute the result as an average. (We don't need to know the value of the signal plus noise at every instant of time–we only care about the average value.) In the average, the cross term $2 i_H \sigma_i$ is equal to zero. The reason for this is that the probability density function (pdf) of the noise was defined as zero mean and normally distributed. Since this pdf is symmetric about the mean, multiplication by a constant will not change the mean, which will remain zero, i.e., in the average, the result will always be zero.

5 Signal-to-Noise Ratio (SNR)

Knowledge of the ratio of the signal power to the noise power (signal-to-noise ratio or SNR) is important because it is directly related to the bit error ratio (BER) in digital communication systems, and the BER is a major indicator of the quality of the overall system.

Drawing from the results of the preceding sections, we can mathematically express the electrical SNR as

$$SNR_E = \frac{S_E}{N_E} = \frac{v^2/R}{\sigma_v^2/R} = \frac{v^2}{\sigma_v^2}$$

$$= \frac{i^2 R}{\sigma_i^2 R} = \frac{i^2}{\sigma_i^2} \qquad (8)$$

Similarly, the optical SNR is

$$SNR_o = \frac{S_o}{N_o} = \frac{\left|\vec{E}\right|^2 \frac{1}{\eta}}{\sigma_o^2 \frac{1}{\eta}} = \frac{\left|\vec{E}\right|^2}{\sigma_o^2} \qquad (9)$$

In practice, optical powers are rarely measured directly. Instead, the optical power is converted to a proportional electric current using a device such as a PIN photodiode, and then the current is measured. The ratio between the output current and the incident optical power is called the responsivity (mathematically represented using the symbol $\mathcal{R}$), which has the units of Amperes per Watt (A/W). It is important to note that the conversion between optical power (units of Watts) and electrical current (units of Amperes) essentially results in a square root operation. In other words, as we recall from equation (2), electrical power is related to the square of the voltage or current, and, as we recall from equation (5) optical power is related to the square of the magnitude of the electric field. The result is that the conversion between optical signal or noise power, (S_o or N_o – both related to $|\vec{E}|^2$) and electrical current results in what is essentially a square root relationship, i.e.,

$$i_{signal} = S_o \mathcal{R} \text{ and } i_{noise} = N_o \mathcal{R} \qquad (10)$$

Also, the optical SNR, when converted to an electrical SNR, is equal to the square root of the equivalent electrical SNR. This is illustrated mathematically by combining equations (8) and (10) as follows:

$$\sqrt{SNR_E} = \sqrt{\frac{S_E}{N_E}} = \frac{i_{signal}}{\sigma_i} = \frac{S_o \mathcal{R}}{N_o \mathcal{R}} = SNR_o \quad (11)$$

6 The Q-Factor

As discussed previously, there are only two possible signal levels in binary digital communication systems and each of these signal levels may have a different average noise associated with it. This means that there are essentially two discrete signal-to-noise ratios, which are associated with the two possible signal levels. In order to calculate the overall probability of bit error, we must account for both of the signal-to-noise ratios. In this section we will show that the two SNRs can be combined into a single quantity – providing a convenient measure of overall system quality – called the Q-factor.

In the following discussion, we will assume the signals are electrical voltages, but, as demonstrated in the previous sections, the concepts can easily be extended to electrical current signals or optical signals.

To begin this discussion, we consider the decision circuit in a fiber-optic receiver, which simply compares the sampled voltage, v(t), to a reference value, γ, called the decision threshold. If v(t) is greater than γ, it indicates that a binary one was sent, whereas if v(t) is less than γ, it indicates that a binary zero was sent. Assuming perfect synchronization between the bit stream and the bit clock, the major obstacle to making the correct decision is noise added to the received data.

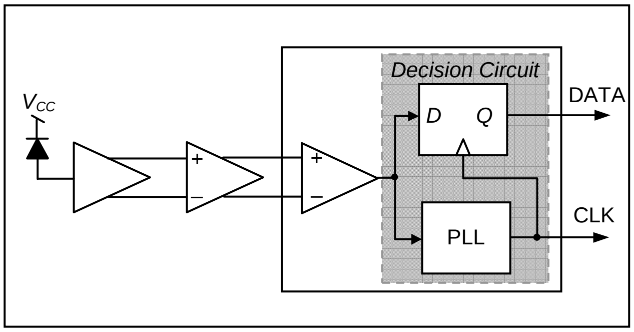

Figure 1. Block-diagram of a fiber-optic receiver

If we assume that additive white Gaussian noise (AWGN) is the dominant cause of erroneous decisions, then we can calculate the statistical probability of making such a decision. The probability density function for v(t) with AWGN can be written mathematically using the Gaussian probability density function (pdf) as follows:

$$PROB[v(t), \sigma_x] = \frac{1}{\sqrt{2\pi\sigma_x^2}} e^{-\frac{1}{2}\left(\frac{v(t)-v_s}{\sigma_x}\right)^2} \quad (12)$$

where v_S is the voltage sent by the transmitter (the mean value of the density function), v(t) is the sampled voltage value in the receiver at time t, and σ is the standard deviation of the noise. Equation (12) is illustrated in Figure 2.

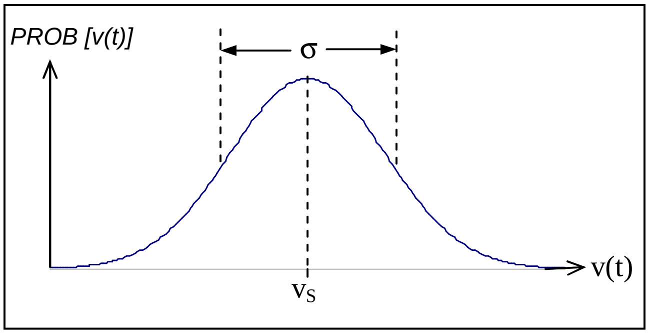

Figure 2. AWGN probability density function

If we assume that v_S can take on one of two voltage levels, which we will call v_L and v_H, then the probability of making an erroneous decision in the receiver is:

$$P[\varepsilon] = P[v(t) > \gamma \mid v_S = v_L]\, P[v_S = v_L] + P[v(t) < \gamma \mid v_S = v_H]\, P[v_S = v_H] \qquad (13)$$

where $P[\varepsilon]$ is the probability of error and $P[x \mid y]$ represents the conditional probability of x given y. If we further assume an equal probability of sending v_L versus v_H (50% mark density), then $P[v_S = v_L] = P[v_S = v_H] = 0.5$. Using this assumption, equation (13) can be reduced to:

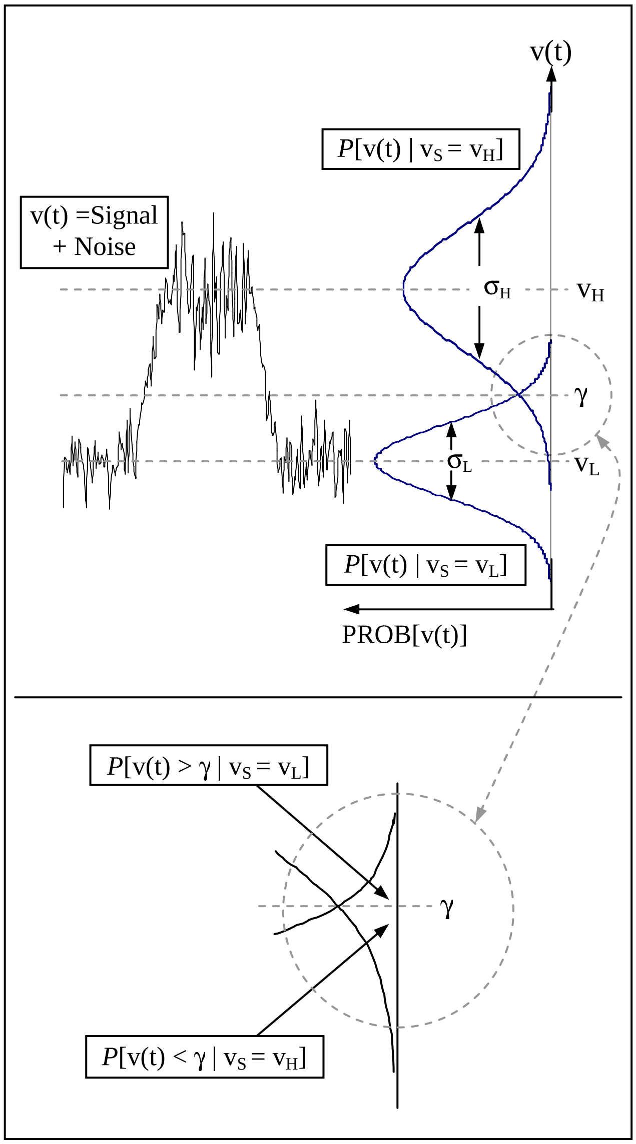

Figure 3. Probability of error for binary signaling

$$\begin{aligned} P[\varepsilon] &= P[v(t) > \gamma \mid v_S = v_L] \times 0.5 \\ &\quad + P[v(t) < \gamma \mid v_S = v_H] \times 0.5 \\ &= \frac{1}{2}\int_{-\infty}^{\gamma} PROB\,[v(t), \sigma_L]\, dt \\ &\quad + \frac{1}{2}\int_{\gamma}^{\infty} PROB\,[v(t), \sigma_H]\, dt \end{aligned} \qquad (14)$$

where $PROB[v(t),\sigma_x]$ is defined in equation (12). This result is illustrated in Figure 3.

From Figure 3 and equations (13) and (14) we can conclude that the probability of error is equal to the area under the tails of the density functions that extend beyond the threshold, γ. This area, and thus the bit error ratio (BER), is determined by two factors: (1) the standard deviations of the noise (σ_L and σ_H) and (2) the voltage difference between v_L and v_H.

It is important to note that for the special case when $\sigma_L = \sigma_H$, the threshold is halfway between the low and high levels (i.e., $\gamma = (v_H - v_L)/2$). But, for the more general case when $\sigma_L \neq \sigma_H$, the optimum threshold for minimum BER will be higher or lower than $(v_H - v_L)/2$.

In order to solve equation (14) we need a practical way to compute the result of the integrated Gaussian pdf ($PROB[v(t),\sigma_x]$) that is defined in equation (12). Since there is no known closed form solution to this integral, it must be evaluated numerically. To maintain compatibility with existing numerical solutions, equation (12) can be re-written in its equivalent standardized (zero mean and standard deviation of one) form. In order to convert to the standardized form, we use the well known $z = (x - \mu)/\sigma$ substitution, where $x = v(t)$ and $\mu = v_S$ in equation (12). For example, we start with

$$\int_{\gamma}^{\infty} PROB\,[x, \sigma]\, dx = \int_{\gamma}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

(from equations (12) and (14))

and then, substituting $z = \dfrac{x-\mu}{\sigma}$ (so that $x = z\sigma - \mu$ and $dx = \sigma\, dz$) results in:

$$\frac{1}{\sqrt{2\pi}\sigma}\int_{x=z\sigma+\mu=\gamma}^{\infty} e^{\left(\frac{-z^2}{2}\right)}\sigma\, dz\,,$$

which is defined as the error function

$$Er(z)=\frac{1}{\sqrt{2\pi}}\int_{z=\gamma}^{\infty} e^{\left(\frac{-z^2}{2}\right)} dz \qquad (15)$$

(Note that there are a number of variations of this function published in the literature.) The error function gives the area under the tail of the Gaussian pdf (mean = v_S and standard deviation = σ_x) between v(t) and infinity. This form of the error function is useful because numerical solutions are available in both tabulated form[1] and as built-in functions within many software utilities (e.g., Er(x) = 1-NORMSDIST(x) in Microsoft Excel). In terms of Er(z), equation (14) can be rewritten as[2]:

$$P[\varepsilon]=\frac{1}{2}Er\left[\frac{v_H-\gamma}{\sigma_H}\right]+\frac{1}{2}Er\left[\frac{\gamma-v_L}{\sigma_L}\right] \qquad (16)$$

It is interesting to note that the arguments of the error functions in equation (16) represent the square root of the signal power divided by the square root of the noise power, which, we recall from equation (11), is equivalent to the optical signal-to-noise ratio. Thus, equation (16) can be rewritten as follows:

$$P[\varepsilon]=\frac{1}{2}Er\left[SNR_{OH}\right]+\frac{1}{2}Er\left[SNR_{OL}\right] \qquad (17)$$

where SNR_{OH} and SNR_{OL} are the optical SNRs for the high and low levels.

The optimum threshold level, γ_{opt}, is defined as the threshold level that results in the lowest probability of bit error. Further, setting the optimum threshold level also results in the same probability of bit error when a high signal is transmitted as when a low signal is transmitted. This means that for the special condition of γ_{opt}, $SNR_{OH} = SNR_{OL}$, which leads to the following definition of the Q-factor[3]:

$$Q\equiv\frac{v_H-\gamma_{opt}}{\sigma_H}=\frac{\gamma_{opt}-v_L}{\sigma_L} \qquad (18)$$

By substituting the definition of Q from equation (18) into equation (16) we find that, when $\gamma = \gamma_{opt}$

$$P[\varepsilon]=\frac{1}{2}Er\left[Q\right]+\frac{1}{2}Er\left[Q\right]=Er[Q] \qquad (19)$$

Next, we solve equation (18) for γ_{opt} to get

$$\gamma_{opt}=\frac{v_H\sigma_L+v_L\sigma_H}{\sigma_L+\sigma_H} \qquad (20)$$

and then substitute this expression for γ_{opt} back into equation (18) to get

$$Q=\frac{v_H-v_L}{\sigma_L+\sigma_H} \qquad (21)$$

It should be noted that multiplying the individual terms in equation (21) by resistance, impedance, or responsivity will convert the expression for Q to equivalent terms of current or optical power, i.e.,

$$Q=\frac{v_H-v_L}{\sigma_L+\sigma_H}=\frac{i_H-i_L}{\sigma_L+\sigma_H}=\frac{P_{OH}-P_{OL}}{\sigma_L+\sigma_H} \qquad (22)$$

Finally, we can substitute equation (21) into the result from equation (19) to get

$$P[\varepsilon]=Er[Q]=Er\left[\frac{v_H-v_L}{\sigma_L+\sigma_H}\right] \qquad (23)$$

7 Conclusions

The Q-factor defined in equations (18) and (21) represents the optical signal-to-noise ratio for a binary optical communication system. It combines the separate SNRs associated with the high and low levels into overall system SNR. The form of the Q-factor given in equation (21) simplifies both the measurement of SNR and the calculation of the theoretical BER due to additive random noise.

For example, measurement of the Q-factor can be performed with the vertical histogram function on many communications oscilloscopes. This can be done by displaying a portion of the data pattern and alternately applying the vertical histogram to the high (one) level and the low (zero) level. The oscilloscope histogram function will estimate the mean (v_H or v_L) and the standard deviation (σ_H or σ_L), which can then be used directly to compute the Q-factor.

The Q-factor is also useful as an intuitive figure of merit that is directly tied to the BER. For example, the BER can be improved by either (1) increasing the difference between the high and low levels in the numerator of the Q-factor, or (2) decreasing the noise terms in the denominator of the Q-factor.

Finally, the Q-factor allows simplified analysis of system performance. The most direct measure of system performance is the BER, but calculation of the BER requires evaluation of the cumulative normal distribution integral. Since this integral has no closed form solution, evaluation requires numerical integration or the use of tabulated values. A much simpler method of analyzing system performance is to optimize the Q-factor, knowing that this will result in optimized BER.

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