Filter Design Methods

$h(t)$ $\xrightarrow{\text{sampling}}$ $h(n)$

- Most digital filter designs attempt to imitate a realisation based on an analogue equivalent

1

- **Impulse invariant:** This produces an H(z) whose impulse response is identical to the sampled version of h(t).

2

- **Matched z-transform**: poles and zeros of H(s) are directly mapped into poles and zeros of H(z)

- **Bilinear transformation:** The whole left half of the s plane is mapped into the unit circle in the z-plane in one go. i.e., the whole |H(ω)| from 0 to infinite frequency is telescoped into the range $\omega=0$ to $\omega_s/2$ where ω_s is the sample rate

$h(t) \leftarrow t = nT_s$

$h(n) = h(nT_s)$

Impulse Invariant Method

① $x(t) \rightarrow \boxed{h(t)} \rightarrow y(t)$ ②

- It is the simplest approach

$$\boxed{h(n) = T_s h(nT_s)}$$

$x(n) \rightarrow \boxed{h(n)} \rightarrow y(n)$

- This is motivated by the approximation in the time domain to obtain

$t = kT_s$

$$y(kT_s) \approx \sum_{n=-\infty}^{\infty} T_s h(nT_s) x((k-n)T_s)$$

- That is,

$y(t) = \int_{-\infty}^{+\infty} h(\tau) x(t-\tau) \boxed{d\tau}$ $\quad \boxed{T_s}$

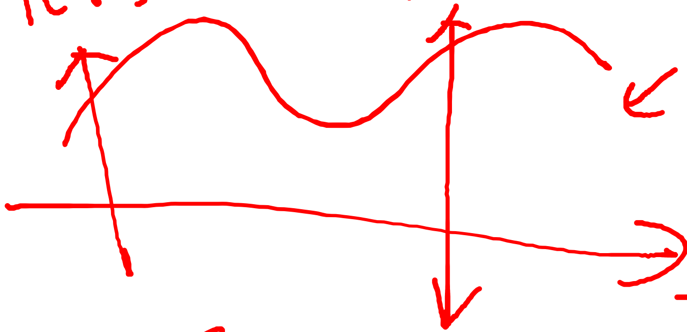
$$y(k) = \sum_{n=-\infty}^{\infty} h(n)x(k-n) \quad \text{or} \quad y_k = \sum_{n=-\infty}^{\infty} h_n x_{k-n}$$

- However, there is aliasing

$$H_p(f) = \sum_k H(f + kf_s) \quad \text{where} \quad f_s = T_s^{-1}$$

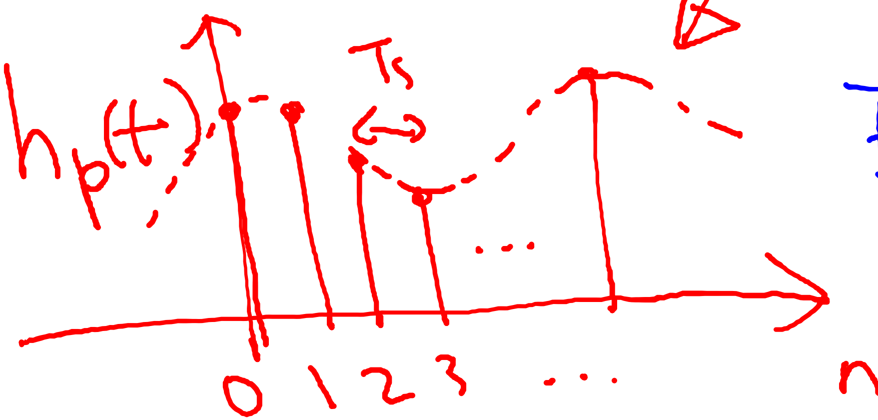
$$h(t) \xrightarrow{\mathcal{F}} H(f)$$

$$\sum_{k=-\infty}^{+\infty} \delta(t-kT_s)$$



$\times$

$$h_p(t) \xrightarrow{\mathcal{F}} H_p(f)$$



$$\mathcal{F}\left\{\sum_{k=-\infty}^{+\infty} \delta(t-kT_s)\right\}$$

$\rightarrow$ periodic

Fourier series expansion T_s

$x(t)$ with period T

$$\omega = \frac{2\pi}{T}$$

$$x(t) = \sum_{n=-\infty}^{+\infty} a_n e^{-j2\pi k\omega t}$$

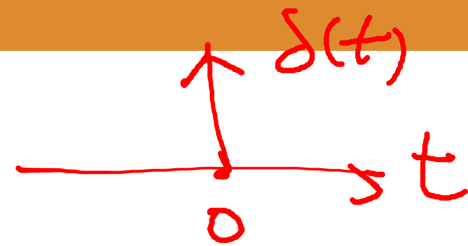
$$a_n = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-j2\pi k\omega t}\, dt$$

$$h_p(t) = h(t) \sum_{k=-\infty}^{+\infty} \delta(t-kT_s)$$

$$H_p(f) = H(f) * \mathcal{F}\left\{\sum_{k=-\infty}^{+\infty} \delta(t-kT_s)\right\}$$

$$x(t) = \sum_{k=-\infty}^{+\infty} \delta(t - kT_s)$$

$$= \sum_{n=-\infty}^{+\infty} a_n e^{\frac{j2\pi n t}{T_s}}$$



$$a_n = \frac{1}{T_s}\int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} \underbrace{\sum_{k=-\infty}^{+\infty} \delta(t - kT_s)}_{=\delta(t)} \; e^{-\frac{j2\pi n 0}{T_s}} \, dt$$

$$a_n = \frac{1}{T_s}\int_{-\frac{T_s}{2}}^{\frac{T_s}{2}} \delta(t)(1)\, dt$$

$$= \frac{1}{T_s}$$

$$\sum_{k=-\infty}^{+\infty} \delta(t - kT_s) = \sum_{n=-\infty}^{+\infty} \frac{1}{T_s} e^{\frac{j2\pi n t}{T_s}}$$

$$\mathcal{F}\left\{\sum_{k=-\infty}^{+\infty} \delta(t - kT_s)\right\}$$

$$= \mathcal{F}\left\{\frac{1}{T_s}\sum_{n} e^{\frac{j2\pi n t}{T_s}}\right\}$$

$$= \frac{1}{T_s}\sum_{n=-\infty}^{+\infty} \delta\left(f - \frac{n}{T_s}\right)$$

$$H_p(f) = H(f) * \mathcal{F}\left\{ \sum_{k=-\infty}^{+\infty} \delta(t - kT_s) \right\}$$

$$= H(f) * \frac{1}{T_s} \sum_{n=-\infty}^{+\infty} \delta\left(f - \frac{n}{T_s}\right)$$

$$= \frac{1}{T_s} \sum_{n=-\infty}^{+\infty} H\left(f - \frac{n}{T_s}\right)$$

$$T_s h_p(t) = T_s h(nT_s)$$

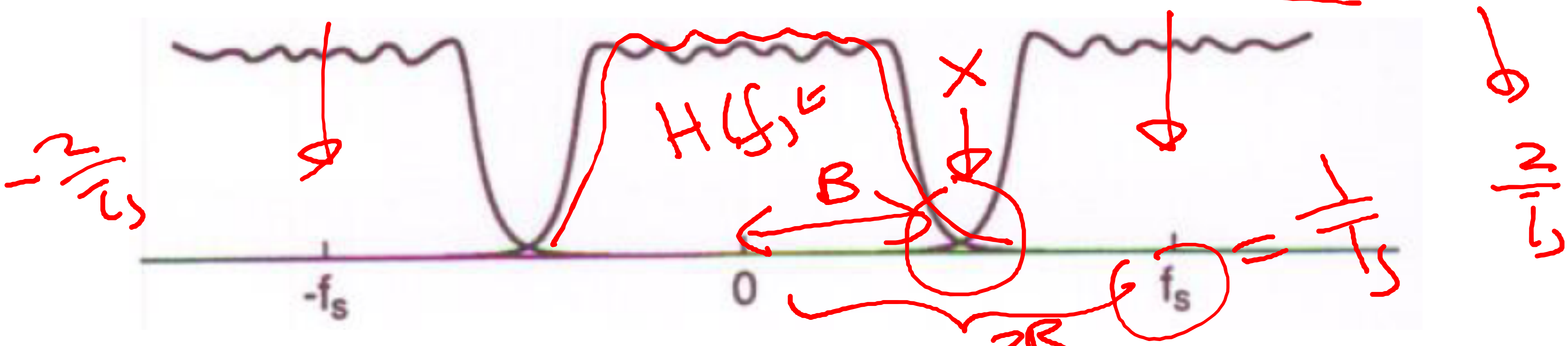
Impulse invariant method:

$$H_p(f) = \sum_{n=-\infty}^{+\infty} H\left(f - \frac{n}{T_s}\right)$$

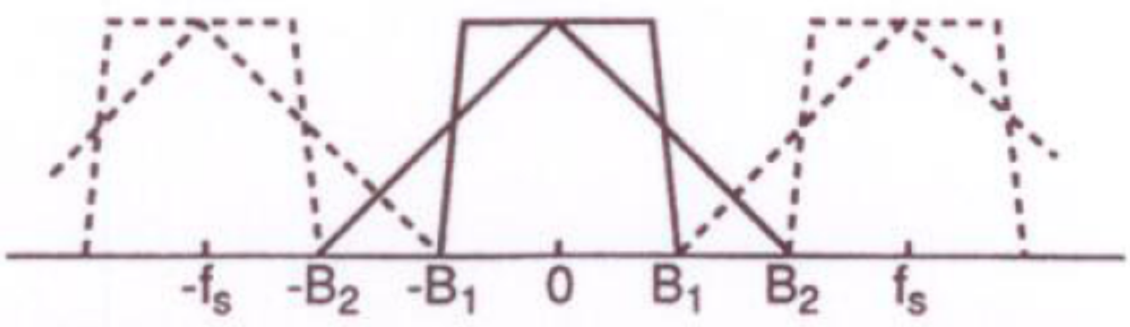
Impulse Invariant Method

$$H_p(f) = \sum_{n=-\infty}^{+\infty} H\left(f - \frac{n}{T_s}\right)$$

- The IR must be sampled sufficiently rapidly to control aliasing error



- If h(t) is bandlimited with BW B_1 and x(t) with BW B_2, then y(k) is computed without error if $f_s \geq 2B$, where $B = \max(B_1, B_2)$



Impulse Invariant Method

$h(t) \rightarrow H(s) = \int_{-\infty}^{+\infty} h(\tau) e^{-s\tau} d\tau$

sampling interval

- Impulse invariant method can also be interpreted as having a mapping

$h(n) = T_s h(nT_s)$

$\rightarrow H(z) = \sum_{n=-\infty}^{+\infty} T_s h(nT_s) z^{-n}$

$$\boxed{z = e^{sT_s}}$$

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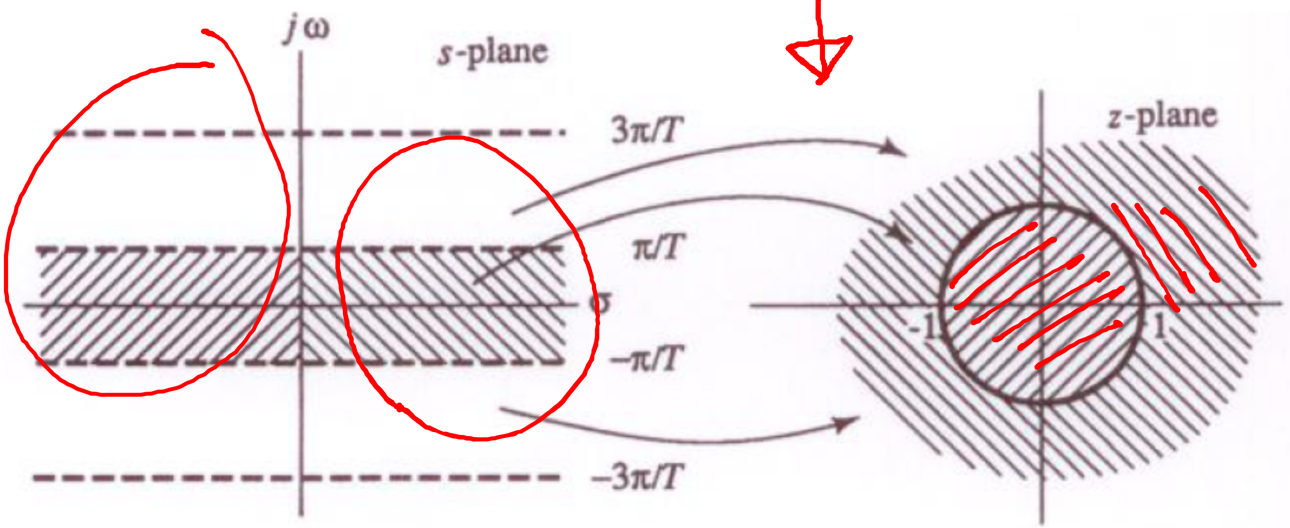
- The mapping between s-domain and z-domain will be exact if $T_s \rightarrow 0$

- The mapping from the s plane to the z plane defined by $z=e^{sT}$



Matched z-Transform

s-domain

$z = e^{sT_s}$

z-domain

z_k

$e^{z_k T_s}$

$e^{p_k T_s}$

p_k

- The **Basic principle** is:
  - Mapping the poles and zeros of $H_a(s)$ (from the *s*-plane)
  - directly into poles and zeros of $H_d(z)$ (in the *z*-plane).

Transfer function of the analogue filter

$$H_a(s) = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k} = \frac{\prod_{k=1}^{M}(s - z_k)}{\prod_{k=1}^{N}(s - p_k)}$$

Transfer function of the digital filter

$$H_d(z) = \frac{\prod_{k=1}^{M}(1 - e^{z_k T_s} z^{-1})}{\prod_{k=1}^{N}(1 - e^{p_k T_s} z^{-1})}$$

Sampling period

$H(s) = \frac{1}{s-1}$

$\boxed{H(z) = \frac{1}{1 - e^{T_s} z^{-1}}}$

$s - a \to 1 - e^{aT_s} z^{-1}$

Matched z-Transform

- Comparing the analogue and digital filters, it can be seen that the digital filter can be obtained using the mapping relation:

$$\boxed{s - a \longrightarrow 1 - e^{aT_s} z^{-1}}$$

From s-domain to z-domain

- To preserve the frequency response characteristics of an analogue filter, the sampling interval in the matched *z*-transform must be selected to yield the pole and zero locations at the equivalent positions in the *z*-plane

- Thus, aliasing must be avoided by selecting T_s sufficiently small

Matched z-Transform

$s-a \rightarrow 1-e^{aT_s}z^{-1}$

$\boxed{z=e^{sT_s}}$

- To demonstrate how it works, consider

$$H_c(s)=\sum_{i=1}^{m}\frac{A_i}{s-s_i}$$

$s-s_i$

- The IR of the filter is

$\mathcal{L}^{-1}$

$$\boxed{h_c(t)=\sum_{i=1}^{m}A_i e^{s_i t}u(t)}$$

Compare these two!

sampling + z-transform

- The z-transform of the IR is

$$H_d(z)=\sum_{n=0}^{\infty}h_c(nT_s)z^{-n}$$

which gives

$1-e^{s_iT_s}z^{-1}$

$$H_d(z)=\sum_{n=0}^{\infty}\sum_{i=1}^{m}A_i\left(e^{s_iT_s}z^{-1}\right)^n=\sum_{i=1}^{m}\frac{A_i}{1-e^{s_iT_s}z^{-1}}$$