Memoryless Bandpass Nonlinearities

$\alpha(t) = 2\pi f_c t + \theta(t)$

- Let us start with $y(t) = F\left[x(t)\right]$

$k\alpha =$

kf_c

- Consider an input bandpass signal

$$x(t) = A(t)\cos\left[2\pi f_c t + \theta(t)\right] \equiv A(t)\cos\alpha(t)$$

- Considered as a function of α, the nonlinearity output

$$z = F\left(A\cos\alpha\right)$$

is a periodic function and hence can be expanded in a Fourier series

$$z = a_0 + \sum_{k=1}^{\infty}\left(a_k\cos k\alpha + b_k\sin k\alpha\right)$$

- This expression makes explicit the *harmonics* of the carrier. Hence, we can see the possibility of a model for a filter characteristic at each kf

$x(t) = A\cos\alpha$ θ or f_c → F → z, α at dly = harmonics $f_c, 2f_c, \ldots$

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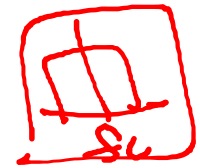
- We are usually interested in the *first-zone* output, i.e., at f_c or k=1 term



- For k≥1, the coefficients in the Fourier series are given by

$$
\begin{cases}
a_k \equiv g_{k1}(A) = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} F(A\cos\alpha)\cos k\alpha\, d\alpha \\
b_k \equiv g_{k2}(A) = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} F(A\cos\alpha)\sin k\alpha\, d\alpha
\end{cases}
$$

- For k=1, we can write the first-zone output y(t) as

$$
y(t) = g_{11}\left(A(t)\right)\cos\left(2\pi f_c t + \theta(t)\right) + g_{12}\left(A(t)\right)\sin\left(2\pi f_c t + \theta(t)\right)
$$

at f_c: $y(t) = g_{11}(A)\cos(2\pi f_c t + \theta) + g_{12}(A)\sin(2\pi f_c t + \theta)$

$y(t) = \mathrm{Re}\{\tilde{y}(t)\, e^{j2\pi f_c t}\}$

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lowpass equivalent

$\tilde{y}(t) = g_{11}(A)\, e^{j\theta}$

- Clearly, the complex envelope of the first-zone output is given by

$$\tilde{y}(t) = \left[g_{11}\left(A(t)\right) - j g_{12}\left(A(t)\right)\right] e^{j\theta(t)}$$

- The first-zone expression implies that the signal can undergo both amplitude and phase distortions. In most cases g_{12} will be zero and hence the model under consideration leads to only the first term

$$y(t) = g_{11}\left(A(t)\right) \cos\left(2\pi f_c t + \theta(t)\right)$$

- Thus, can produce only amplitude distortion. Similarly, the complex envelop reduces to the first term

$$\boxed{\tilde{y}(t) = g_{11}\left(A(t)\right) e^{j\theta(t)}}$$

which is the form in which it would be implemented in simulation

$F(\;) = \sum_{n=0}^{N} a_n x^n(t)$

the nonlinearity

$y(t)$?

Memoryless Bandpass Nonlinearities

- **Power Series Model** – It has the nice property of being able to expose clearly the effect that it has on an input signal. Again, we have

$$y(t) = F\left[x(t)\right] \approx \sum_{n=0}^{N} a_n x^n(t)$$

- Expressing the input signal in terms of the complex envelope

x_c

$$\boxed{x(t) = \mathrm{Re}\left\{\tilde{x}(t) e^{j2\pi f_c t}\right\} = \frac{1}{2}\left[\tilde{x}(t) e^{j2\pi f_c t} + \tilde{x}^*(t) e^{-j2\pi f_c t}\right]}$$

- Using the binomial expansion for $x^n(t)$, we obtain

$$\boxed{x^n(t) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k} \left[\tilde{x}(t)\right]^k \left[\tilde{x}^*(t)\right]^{n-k} e^{j2\pi f_c (2k-n)t}}$$

$\frac{1}{2}(a + a^*) = \mathrm{Re}\{a\}$

$$x(t) = \mathrm{Re}\{\tilde{x}(t)e^{j2\pi f_c t}\} = \frac{1}{2}\left[\tilde{x}(t)e^{j2\pi f_c t} + \tilde{x}^*(t)e^{-j2\pi f_c t}\right]$$

$$aa^* = |a|^2$$

$$x^n(t) = \frac{1}{2^n}\left[\tilde{x}e^{j2\pi f_c t} + \tilde{x}^* e^{-j2\pi f_c t}\right]^n$$

$$= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\left(\tilde{x}e^{j2\pi f_c t}\right)^k\left(\tilde{x}^* e^{-j2\pi f_c t}\right)^{n-k}$$

$$= \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\underbrace{\tilde{x}^k\tilde{x}^{*\,n-k}}_{=|\tilde{x}|^{2k}(\tilde{x}^*)^{n-2k}}\ e^{j2\pi k f_c t - j2\pi f_c (n-k)t} = e^{-j2\pi f_c(2k-n)t}$$

$$\tilde{x}^k\tilde{x}^{*\,n-k} = \tilde{x}^k\frac{\tilde{x}^{*k}}{\tilde{x}^{*k}}\tilde{x}^{*\,n-k} = |\tilde{x}|^{2k}\tilde{x}^{*\,n-2k}$$

$$x^n(t) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}|\tilde{x}|^{2k}(\tilde{x}^*)^{n-2k}\, e^{j2\pi f_c(2k-n)t}$$

$$f_c \text{ of } x^n(t) = \frac{1}{2^n}\binom{n}{\frac{n+1}{2}}|\tilde{x}|^{n+1}(\tilde{x}^*)^{-1}\, e^{j2\pi f_c t}$$

$$2k-n=1$$

$$f_c$$

$$= \frac{1}{2^n}\binom{n}{\frac{n+1}{2}}\frac{|\tilde{x}|^{n+1}\tilde{x}}{\tilde{x}^*\tilde{x}} = \boxed{\frac{1}{2^n}\binom{n}{\frac{n+1}{2}}|\tilde{x}|^{n-1}\tilde{x}}$$

$$\tilde{x}^*\tilde{x} = |\tilde{x}|^2$$

$$\Rightarrow \tilde{y}(t) = \sum_{n=0}^{N}\frac{a_n}{2^n}\binom{n}{\frac{n+1}{2}}|\tilde{x}|^{n-1}\tilde{x}$$

$$2k-n=2$$

$$\Rightarrow k=\frac{n+1}{2}$$

$$y(t) = \sum_{n=0}^{N} a_n x^n(t)$$

f_c

$$\tilde{y}(t) = \sum_{n=0}^{N} \frac{a_n}{2^n}\binom{n}{\frac{n+1}{2}}|\tilde{x}|^{n-1}\tilde{x}$$

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$$\tilde{y}(t) = \tilde{x}(t)\sum_{n=0} \frac{a_n}{2^n}\binom{n}{\frac{n+1}{2}}|\tilde{x}|^{n-1}$$

- For the first-zone output we see that only terms where n is odd and 2k-n=1 can contribute. The first-zone contribution is then

n+1/2 must be an integer, so the n must be odd

$$\boxed{\frac{1}{2^n}\binom{n}{\frac{n+1}{2}}|\tilde{x}^*(t)|^{n-1}\tilde{x}(t) \quad \text{for odd } n}$$

$\underline{\underline{x^n(t)}}$

nonlinearity

- Summing all odd n, the complex envelope of the first-zone of y(t) is

$$\boxed{\tilde{y}(t) = \tilde{x}(t)\sum_{m=0}^{\frac{N-1}{2}} \frac{a_{2m+1}}{2^{2m}}\binom{2m+1}{m+1}|\tilde{x}^*(t)|^{2m}}$$

odd values of n

$n=2m+1$

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- The bandpass output signal can now be written as

$$
\begin{aligned}
y(t) &= \mathrm{Re}\left[\tilde{y}(t)e^{j2\pi f_c t}\right] \\
&= \left[\sum_{m=0}^{\frac{N-1}{2}} \frac{a_{2m+1}}{2^{2m}} \binom{2m+1}{m+1} |\tilde{x}^*(t)|^{2m+1}\right] \cos\left(2\pi f_c t + \theta(t)\right) \\
&= \boxed{g_{11}\left(A(t)\right)} \cos\left(2\pi f_c t + \theta(t)\right)
\end{aligned}
$$

$g_{11}(A(t))$