Wireless Communications Principles

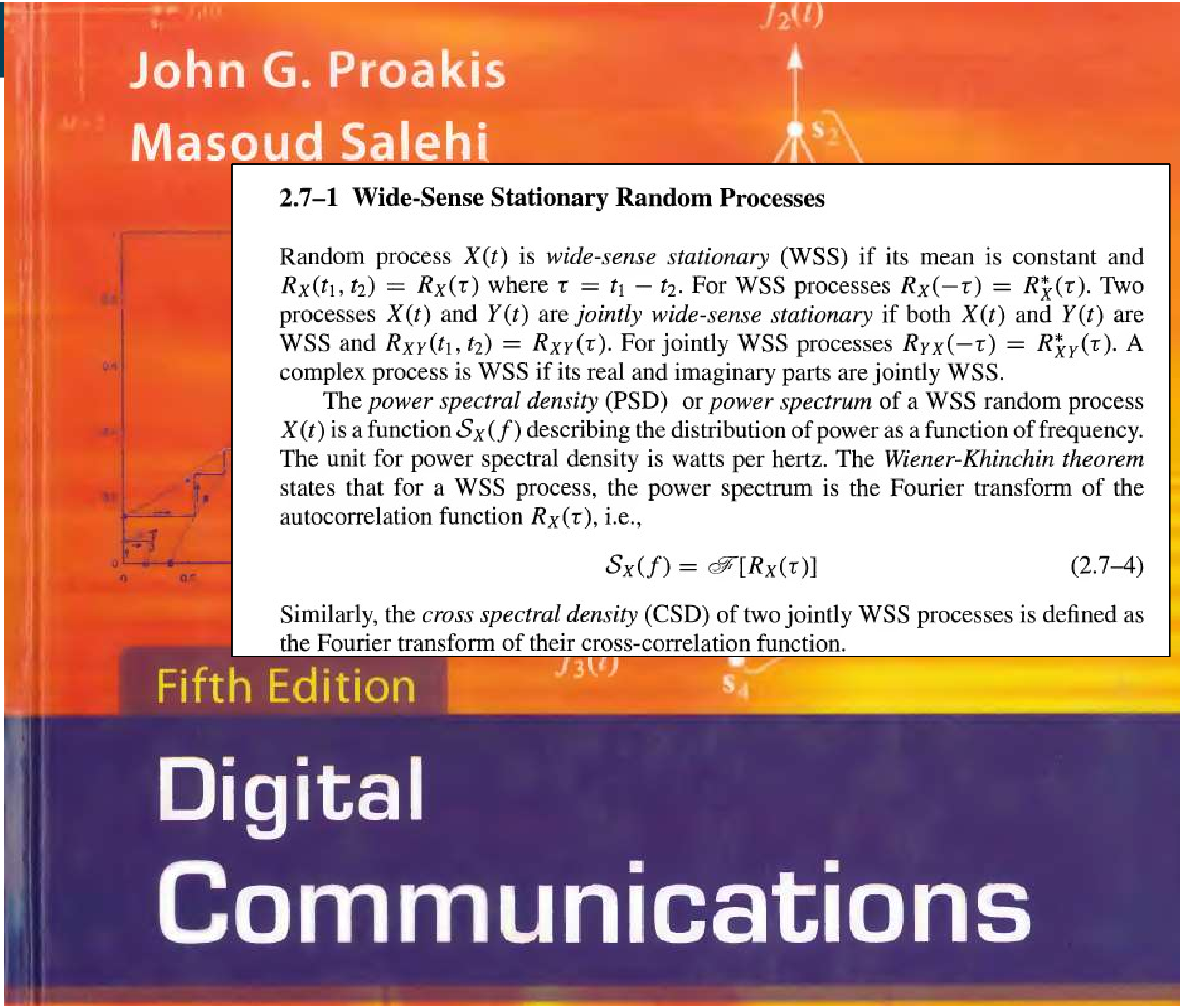
Digital Baseband Transmission
– Spectral Characteristics

Relevant Properties

- Symbol/Bit Rate
- Symbol/Bit Duration
- Energy Conveyed per Symbol
- Energy Conveyed per Bit
- Spectral Characteristics
- Bandwidth Usage

Power Spectral Density and Bandwidth

- We have a finite bandwidth available for modulation
- Different pulse shapes will have different spectral shapes
  - Think of Fourier transforms…
- Some signaling schemes (with temporal correlations) may also change the spectral shape
- Some pulse shapes and signaling schemes will be more bandwidth-efficient than others
- The spectral occupation is described by the **Wiener–Khinchin theorem**



2.7–1 Wide-Sense Stationary Random Processes

Random process $X(t)$ is *wide-sense stationary* (WSS) if its mean is constant and $R_X(t_1, t_2) = R_X(\tau)$ where $\tau = t_1 - t_2$. For WSS processes $R_X(-\tau) = R_X^*(\tau)$. Two processes $X(t)$ and $Y(t)$ are *jointly wide-sense stationary* if both $X(t)$ and $Y(t)$ are WSS and $R_{XY}(t_1, t_2) = R_{XY}(\tau)$. For jointly WSS processes $R_{YX}(-\tau) = R_{XY}^*(\tau)$. A complex process is WSS if its real and imaginary parts are jointly WSS.

The *power spectral density* (PSD) or *power spectrum* of a WSS random process $X(t)$ is a function $\mathcal{S}_X(f)$ describing the distribution of power as a function of frequency. The unit for power spectral density is watts per hertz. The *Wiener-Khinchin theorem* states that for a WSS process, the power spectrum is the Fourier transform of the autocorrelation function $R_X(\tau)$, i.e.,

$$\mathcal{S}_X(f) = \mathscr{F}[R_X(\tau)] \tag{2.7–4}$$

Similarly, the *cross spectral density* (CSD) of two jointly WSS processes is defined as the Fourier transform of their cross-correlation function.

Ergodic and Wide Sense Stationary processes

- Our data sequence might not be described as ergodic
  - That would mean that any sizeable sequence could be equally representative of the whole infinite sequence

- Instead, we describe it as wide sense stationary (or stationary, for simplicity)
  - Its mean is constant (independent of time)
  - And the autocorrelation $R(t_1, t_2)$ depends only on the time difference τ between t_1 and t_2, not on t_1 and t_2 individually

- Even symmetry means
  - $R(\tau) = R^*(-\tau)$ or $R(\tau) = R(-\tau)$ for real-valued processes
  - Maximum at $\tau = 0$, where $R(0)$ is equal to the mean-square value

Power Spectral Density and Bandwidth

- The power spectral density has information about the bandwidth and bandwidth efficiency of the signaling scheme.

- It depicts the distribution of signal power over frequency.

- The question is how to determine the power spectral density of the digital signaling scheme:

$$x(t) = \sum_{k=-\infty}^{\infty} X_k \cdot p(t - kT_s)$$

- Note that this is a cyclo-stationary random process because both the mean and the auto-correlation function are periodic with period T_s.

Power Spectral Density and Bandwidth

- The auto-correlation function is given by:

$$R_x(t+\tau,t) = E\{x(t+\tau)x^*(t)\} = \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty} R_X(k_1-k_2)\,p(t+\tau-k_1T)\,p^*(t-k_2T)$$

- and the average auto-correlation function is given by:

$$\overline{R}_x(\tau) = \frac{1}{T_s}\int_0^{T_s} E\{x(t+\tau)x^*(t)\}\,dt = \frac{1}{T_s}\sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} R_X(k)\,p(t+\tau-kT)\,p^*(t)\,dt$$

- where

$$R_X(k) = R_X(k_1-k_2) = E\{X_{k_1}X_{k_2}^*\}$$

- represents the auto-correlation function of the stationary sequence of symbols $\{X_k\}$.

Power Spectral Density and Bandwidth

- The (average) power spectral density corresponds to the Fourier transform of the (average) auto-correlation function:

$$S_x(f) = \int_{-\infty}^{\infty} \bar{R}_x(\tau) e^{-j2\pi f\tau}\, d\tau = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} R_X(k) \left|P(f)\right|^2 e^{-j2\pi kfT_s} = \frac{1}{T_s} S_X(f) \left|P(f)\right|^2$$

- where $P(f)$ is the Fourier transform of $p(t)$ and $S_X(f)$ represents the power spectral density of the discrete-time stationary random process $\{X_k\}$ i.e.

$$P(f) = \int_{-\infty}^{\infty} p(t) e^{-j2\pi ft}\, dt$$

$$S_X(f) = \sum_{k=-\infty}^{\infty} R_X(k) e^{-j2\pi kfT_s}$$

Power Spectral Density and Bandwidth

- How to obtain $S_X(f)$ the power spectral density of the discrete time random process?
- Need to sum the auto-/cross-correlation for all pulse intervals, $R_X(k) = \mathrm{E}\{X_{i+k} X_i^*\}$

- So, for *k* = 0, $R_X(0) = \mathrm{E}\{X_i X_i^*\}$
- For *k* = 1 etc., $R_X(1) = \mathrm{E}\{X_{i+1} X_i^*\}$
- Which gives:

$$S_X(f) = \mathrm{E}\{X_i X_i^*\} + \mathrm{E}\{X_{i+1}X_i^*\}\, e^{-j2\pi f T_s} + \mathrm{E}\{X_{i-1}X_i^*\}\, e^{j2\pi f T_s} + \ \ldots$$

showing only $k = 0, k = \pm 1$

- If adjacent symbols are uncorrelated and there is no DC content, only *k* = 0 needs to be considered

Spectral Characteristic and Bandwidth : Example I

- Consider a basic pulse shape $p(t)=\pi(t/T_s)$. Assume that

$$x(t)=\sum_{k=-\infty}^{\infty} X_k p(t-kT_s)$$

- where $\{X_k\}$ is a sequence of independent symbols that take the values ±1 with equal probability.
- We know that for such a unit pulse

$$|P(f)|^2 = T_s \operatorname{sinc}^2(fT_s)$$

- Then, for k = 0

$$R_X(0) = E(X_i X_i^*) = \Pr(X_i = 1)\,(1\,.\,1) + \Pr(X_i = -1)\,(-1\,.\,-1) = 1$$

- $R_x(k)$ = 0 for all other k, due to pulses being uncorrelated, and no DC content
- Which results in $S_x(f) = T_s \operatorname{sinc}^2(fT_s)$

Spectral Characteristic and Bandwidth : Example II

- Assume now that

$$x(t) = \sum_{k=-\infty}^{\infty} X_k p(t - kT_s)$$

- where $\{X_k\}$ is a sequence of independent symbols that take the values −3,−1,1,3 with equal probability.
- We assume the same unit pulse.
- Then, for k = 0

$$R_X(0) = E(X_i X_i^*)$$
$$= \Pr(X_i = 3)\,(3\,.\,3) + \Pr(X_i = 1)\,(1\,.\,1) + \Pr(X_i = -1)\,(-1\,.\,-1)$$
$$+ \Pr(X_i = -3)\,(-3\,.\,-3) = 5$$

- $R_X(k)$ = 0 for all other k, due to the pulses being uncorrelated
- Which results in

$$S_x(f) = 5T_s \mathrm{sinc}^2(fT_s)$$

Spectral Characteristic and Bandwidth : Example II