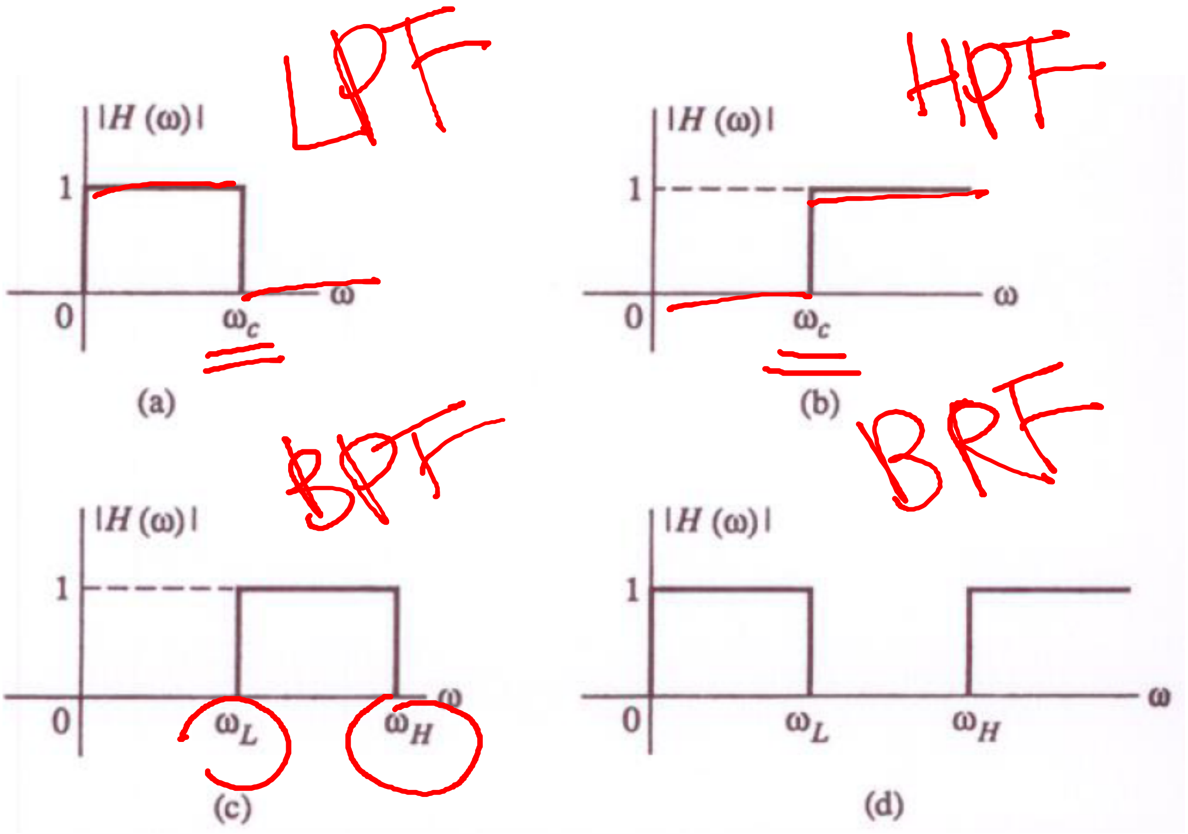
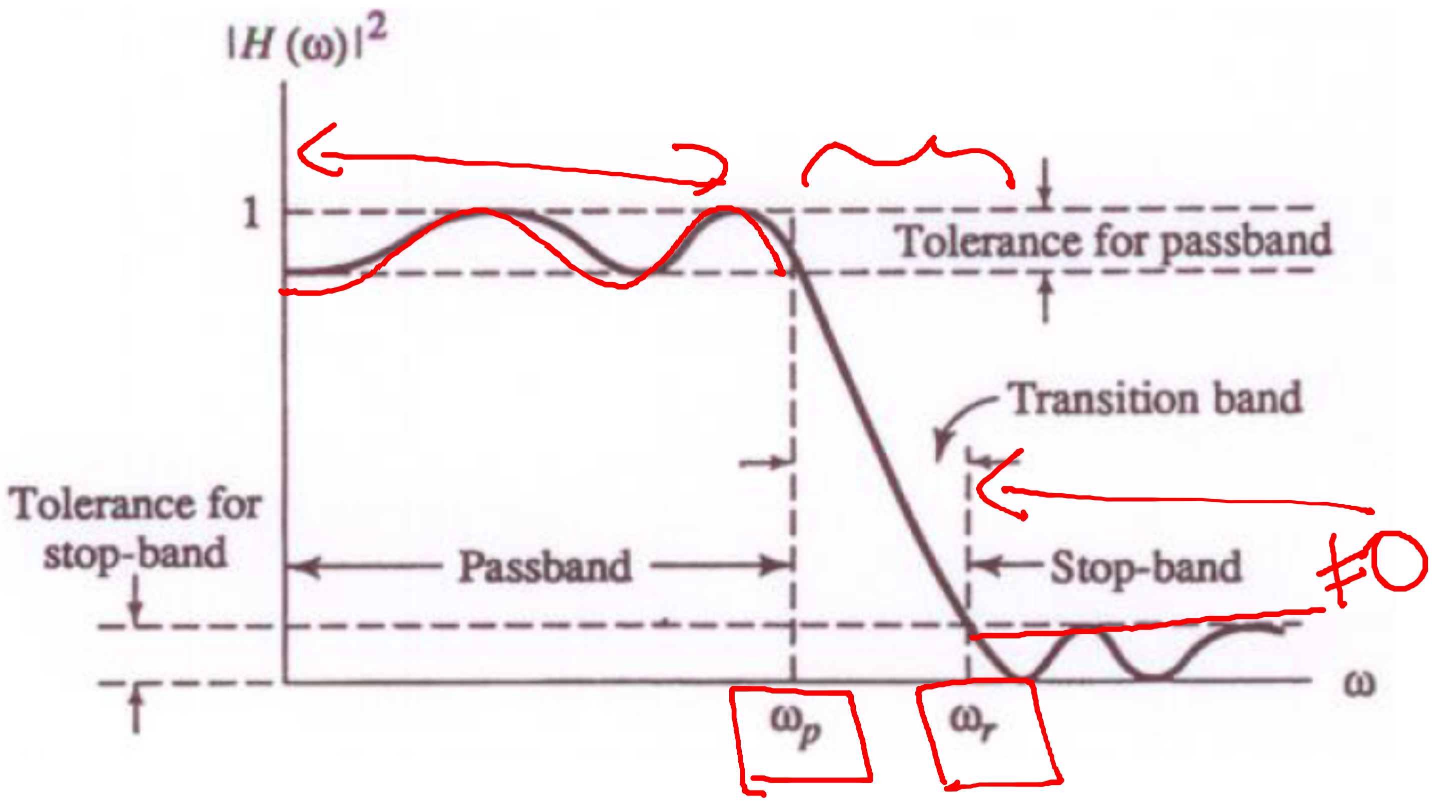
LTI Filters: Continuous Classical Filters

- The basic ideal filters in the frequency domain include:



LTI Filters: Continuous Classical Filters

- Practical filters are different and have more parameters:



$S = \sigma + j\omega$ $\quad S = j\omega$ $\quad P = j\omega \Rightarrow \omega = \frac{P}{j}$

$1 + \omega^{2n} = 0$ $\quad \boxed{1 + \left(\frac{P}{j}\right)^{2n} = 0}$ $\quad P$ ✓

LTI Filters: Low Pass Filters

we set cutoff frequency to be 1, which means it is normalized

- **Butterworth filters** – they are maximally flat (no ripples) in the pass band and roll off towards 0 in the stop band

at $\omega = 1$, $|H(\omega)| = \frac{1}{\sqrt{2}}$

$$\boxed{|H(\omega)| = \frac{1}{\sqrt{1+\omega^{2n}}}}$$ ✓

for small ω, $|H(\omega)| \approx 1$

for large ω $|H(\omega)| \approx 0$

- The parameter n is referred to as the order of the filter and ω_b is the 3-dB bandwidth. As $n \to \infty$, $|H(\omega)|^2$ approaches the ideal LPF
- It can be shown that the s-domain transfer function is of the form

$|H(\omega)| \to H(S)$?

$$H(s) = \frac{\omega_b^n}{\prod_{k=1}^{n}(s - \boxed{s_k})}$$ ✓

$H(S) = K \frac{\prod (S - Z_k)}{\prod_k (S - P_k)}$

where $s_k = \omega_b \exp(j\pi(2k+n-1)/(2n))$ are the poles on a semicircle

Butterworth Filters

$(s - p_i)(s - p_i^*)$

- In general, the transfer function is given by

$$H(s) = \begin{cases} \dfrac{1}{\prod_{k=1}^{\frac{n}{2}} \left(s^2 - 2s\cos\left(\frac{2k+n-1}{2n}\pi\right) + 1\right)}, & \text{even } n, \\ \dfrac{1}{(s+1)\prod_{k=1}^{\frac{n-1}{2}} \left(s^2 - 2s\cos\left(\frac{2k+n-1}{2n}\pi\right) + 1\right)}, & \text{odd } n, \end{cases}$$

$n=3$

$|H(\omega)|$

n	Butterworth Polynomials in factored form
1	$s+1$
2	$s^2+\sqrt{2}s+1$
3	$(s^2+s+1)(s+1)$
4	$(s^2+0.76536s+1)(s^2+1.84776s+1)$
5	$(s+1)(s^2+0.168s+1)(s^2+0.168s+1)$
6	$(s^2+0.518s+1)(s^2+1.414s+1)(s^2+1.932s+1)$

$H(s) = \dfrac{1}{(s^2+s+1)(s+1)}$

normalised LPF

Φ

$H(s)$

① $|H(\omega)| = \frac{1}{\sqrt{2}}$ $20\log_{10}|H(\omega)| = -3dB$

$\boxed{G \downarrow \text{ for } \omega}$

No ripples

Butterworth Filters

②

- The filter roll-off rate is approximately 6n dB/octave or 20n dB/decade

$G = 20\log_{10}\frac{1}{\sqrt{1+\omega^{2n}}} \approx 20\log_{10}\frac{1}{\sqrt{\omega^{2n}}}$

- Can you work out why?

$G = -10\log_{10}\omega^{2n} = -20n\log_{10}\omega$ ✓

$G_{at\,2\omega} = -20n\log_{10}2\omega$

$\Delta G = -20n\log_{10}2 \approx -6n\,dB$

③

- Also, why is it maximally flat?

$G = \frac{1}{\sqrt{1+\omega^{2n}}}$ $\frac{\partial G}{\partial \omega} < 0$

$\frac{\partial G}{\partial \omega} = \boxed{\left(-\frac{1}{2}\right)}\left(1+\omega^{2n}\right)^{-\frac{3}{2}} 2n\omega^{2n-1}$

< 0 > 0

$G_{at\,10\omega} = -20n\log_{10}10\omega$

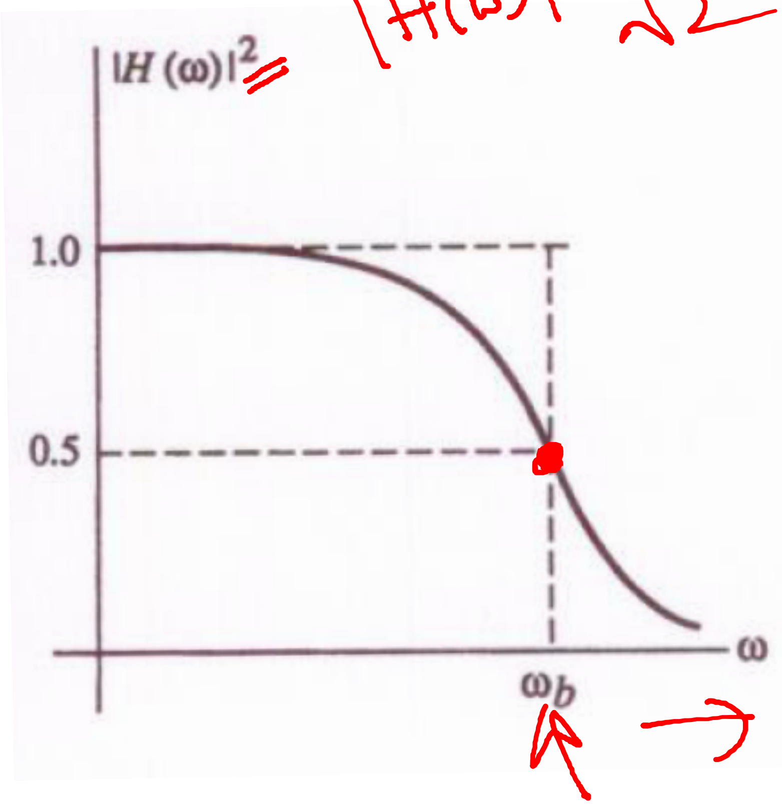
$\Delta G = -20n\log_{10}10 = -20n\,dB$

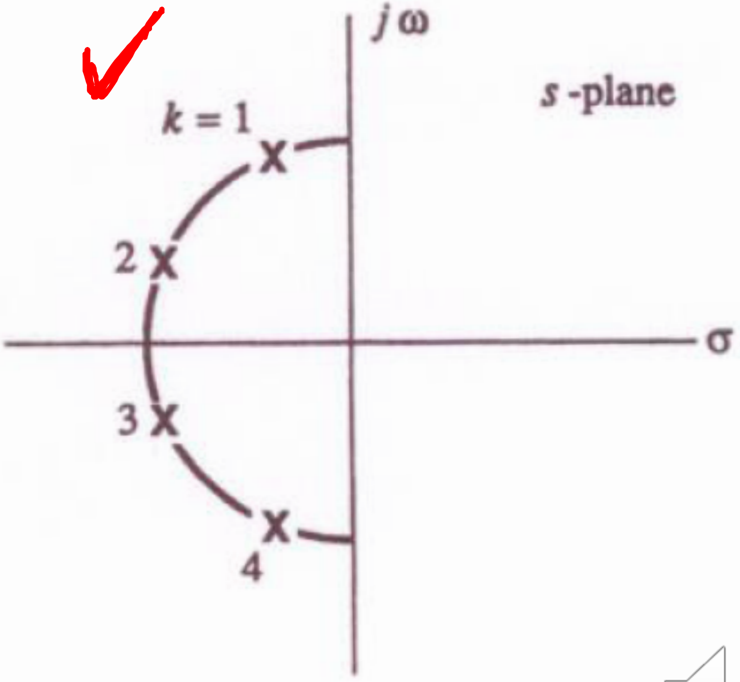
LTI Filters: Continuous Classical Filters

- Properties of a Butterworth filter:

$|H(\omega)| = \frac{1}{\sqrt{2}}$

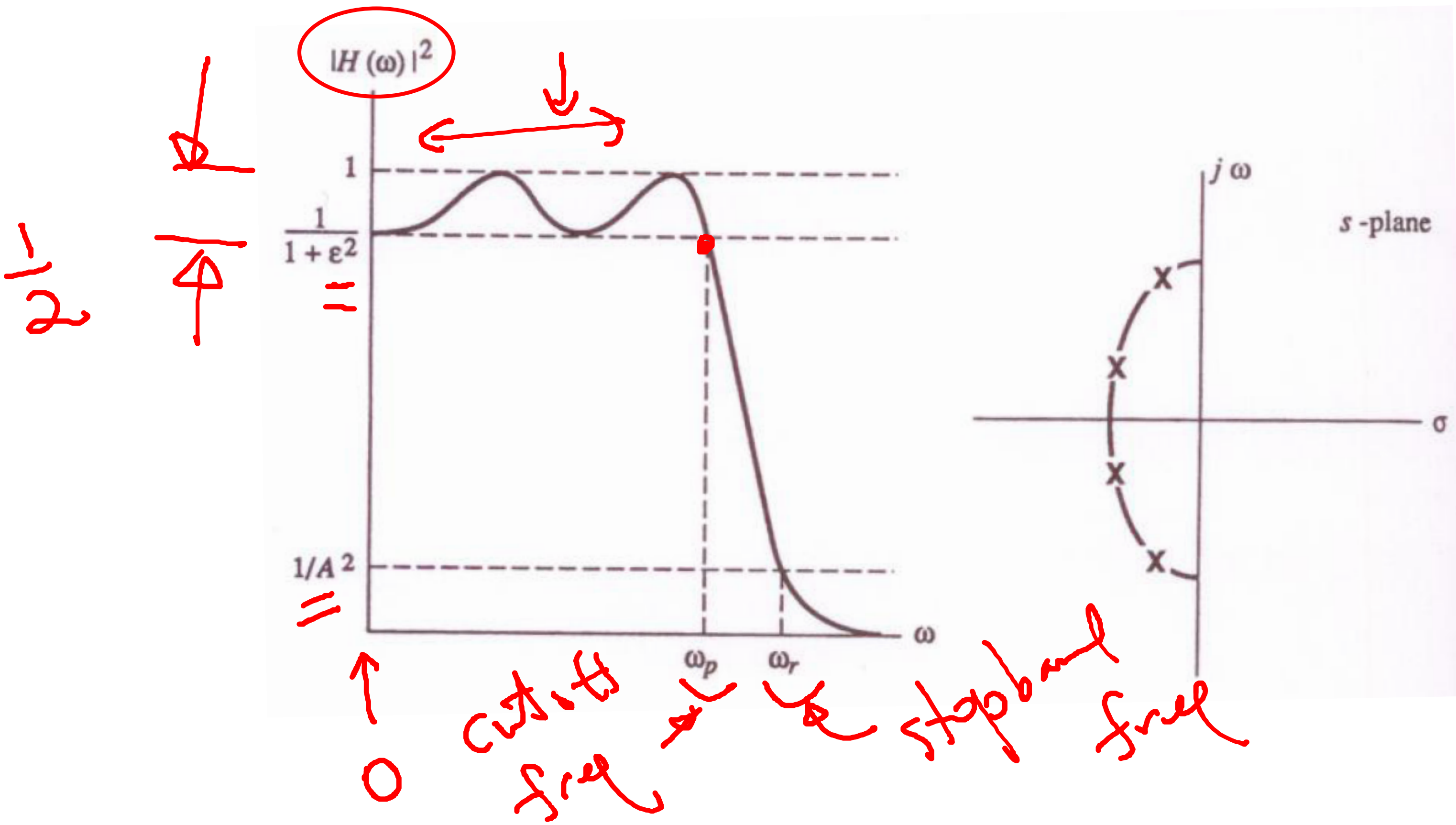
pole locations





Chebyshev Filters

- **Chebyshev filters** – it is characterised by the property that the magnitude error is equi-ripple in the passband



Butterworth: 6n dB/octave

20n dB/decade

Chebyshev Filters

- The frequency response of the Chebyshev filter is

$$\boxed{|H(\omega)|^2 = \frac{1}{1+\varepsilon^2 C_n^2\left(\frac{\omega}{\omega_p}\right)}}$$

Chebyshev polynomial

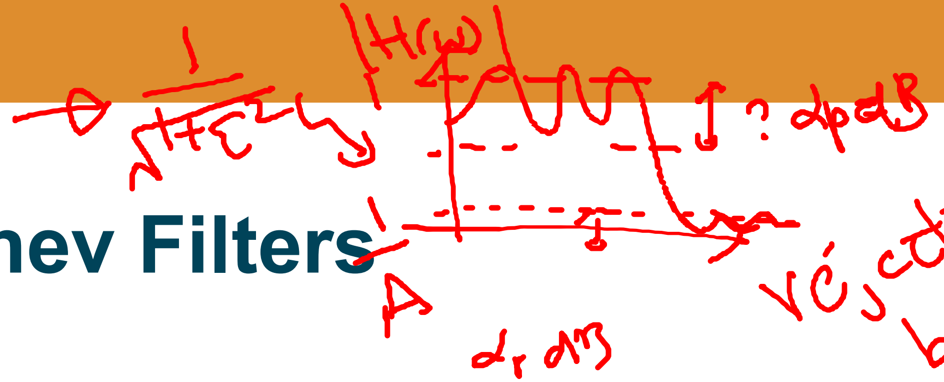
where $C_n(\omega)$ is the n-th order Chebyshev polynomial

w is not normalized

$$C_n(\omega) = \begin{cases} \boxed{\cos(n\cos^{-1}\omega), \quad |\omega| \le 1,} \\ \cosh(n\cosh^{-1}\omega), \quad \omega > 1, \\ (-1)^n\cosh(n\cosh^{-1}(-\omega)), \quad \omega < -1, \end{cases}$$

- The Chebyshev filter is an optimal all-pole filter

- The roll-off rate is much faster than that of the Butterworth filter



Chebyshev Filters

- The coefficients of the biquadratic sections, normalised to the passband edge frequency ω_p, are tabulated for the filter order n and the passband ripple parameter $\alpha_p=10\log(1+\varepsilon^2)$ [dB]

ω_r, ω_p

α_r, α_p

- The necessary filter order n can be obtained from

X memorise this formula

$$\boxed{n = \frac{\cosh^{-1}\sqrt{\dfrac{10^{\frac{\alpha_r}{10}}-1}{10^{\frac{\alpha_p}{10}}-1}}}{\cosh^{-1}\left(\dfrac{\omega_r}{\omega_p}\right)}}$$

where the stopband attenuation is given as $\alpha_r=20\log A$ [dB]

$H(s)$ ✓ $|H(\omega)| \rightarrow$ find the poles

Chebyshev polynomials (0.5dB Ripple)

$\boxed{\varepsilon}$

$$H(s) = \frac{k}{\prod_{e}(s - p_e)}$$

n	*Chebyshev polynomials*
1	$s + 2.863$
2	$s^2 + 1.426s + 1.516$
3	$(s^2 + 0.626s + 1.142)(s + 0.626)$
4	$(s^2 + 0.351s + 1.064)(s^2 + 0.846s + 0.356)$
5	$(s^2 + 0.224s + 1.036)(s^2 + 0.586s + 0.477)(s + 0.362)$
6	$(s^2 + 0.155s + 1.023)(s^2 + 0.424s + 0.590)(s^2 + 0.580s + 0.157)$

normalised cut-off freq.

$\omega_p = 1 \text{ rad s}^{-1}$

when:

$\alpha_p = 0.5$ dB

$n=2$

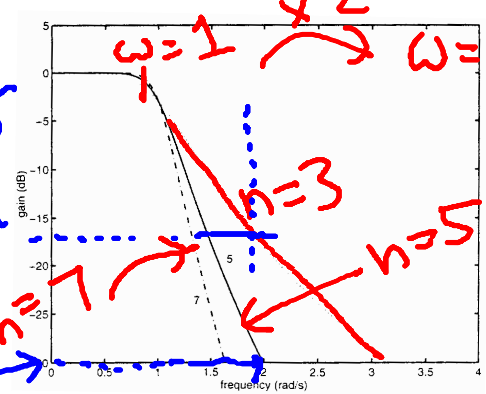
$$H(s) = \frac{1}{s^2 + 1.426s + 1.516}$$

$H(\omega) = |H(\omega)| e^{j\angle H(\omega)}$

Amplitude+Phase for Butterworth Filters

- The frequency response for a Butterworth filter with n=3,5,7
- The filter roll off rate into the stop band is approximately 6n dB/octave

magnitude $= |H(\omega)|$



6n dB/octave

n=3 → 18 dB

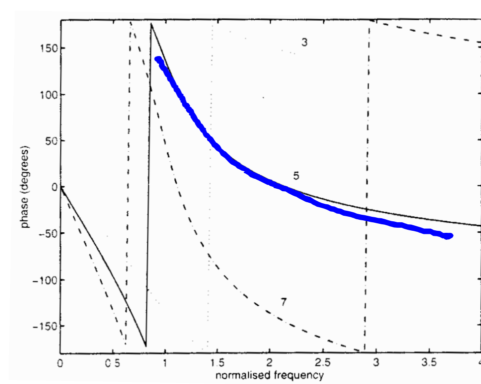
n=5 → 30 dB

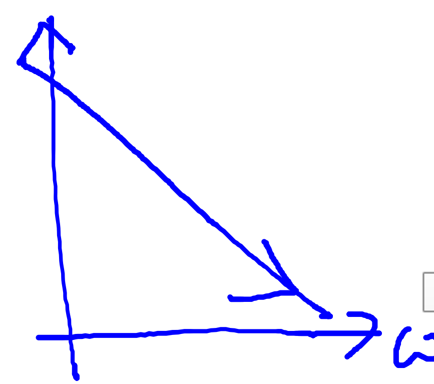
n=7 → 42 dB

- The phase response for a Butterworth filter with n=3,5,7

Linear Phase response is exptected

phase $= \angle H(\omega)$





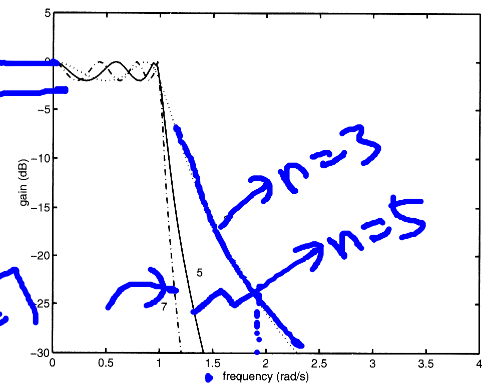
IRR does not have linear phase response

Amplitude+Phase for Chebyshev Filters

n= ✓ ✓ ✓

- Frequency response for a 2dB ripple Chebyshev filters of order 3,5, 7

2dB



n=3

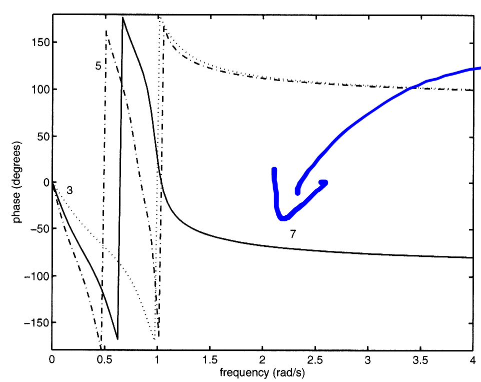
n=5

n=7

Steeper attenuation for the transition band

- Phase response for a 2dB ripple Chebyshev filters of order 3,5 and 7

Phase response



Phase responses of both filters are not linear, which means they are not perfect

$H(s) = \frac{1}{s+1}$ normalised cutoff freq.

$n=1$

LPF → LPF, HPF, BPF, BRF

Frequency Transformations

- LP to LP at ω_c

$$s \rightarrow \frac{s}{\omega_c}$$

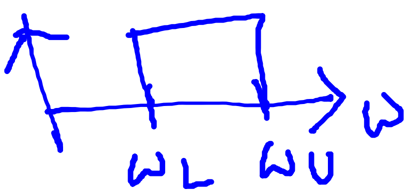
$$H(s) = \frac{1}{\frac{s}{\omega_c} + 1}$$

- LP to HP at ω_c

$$s \rightarrow \frac{\omega_c}{s}$$

$$H(s) = \frac{1}{\frac{\omega_c}{s} + 1}$$

- LP to BP with ω_L and ω_U



$$s \rightarrow \frac{s^2 + \omega_L \omega_U}{s(\omega_U - \omega_L)}$$

$$H(s) = \frac{1}{\frac{s^2 + \omega_L \omega_U}{s(\omega_U - \omega_L)} + 1}$$

- LP to BR with ω_L and ω_U

$$s \rightarrow \frac{s(\omega_U - \omega_L)}{s^2 + \omega_L \omega_U}$$

Just replace s

Rarely used in this module which focuses on digital filter rather than analogue filter, since there are direct transformation for digital filter like these equation