Modelling data networks – Tutorial lecture

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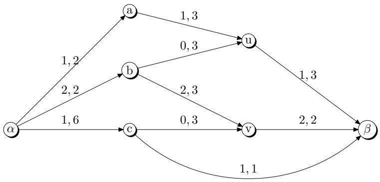


Basic revision

- Recall the following notions:
  1. Arrival process – process governing arrivals to the queue
  2. Server process – process governing the time it takes to serve the head of queue
  3. Number of servers – number of server processes
- The queues are given in Kendal's notation. Examples:
  1. M/D/1 – Poisson (Markovian/Memoryless) Arrival, single deterministic server.
  2. D/G/3 – Deterministic (regular) arrival, three General servers

The special nature of the Poisson process

- The Poisson process is in many ways the simplest stochastic process of all.
- This is why the Poisson process is so commonly used.
- Imagine your system has the following properties:
  - The number of arrivals does not depend on the number of arrivals so far.
  - No two arrivals occur at exactly the same instant in time.
  - The number of arrivals in time period τ depends only on the length of τ.
- The Poisson process is the only process satisfying these conditions (see notes for proof).

Some remarkable things about Poisson processes

- The mean number of arrivals in a period τ is $\lambda\tau$ (see notes).
- If two Poisson processes arrive together with rates λ_1 and λ_2 the arrival process is a Poisson process with rate $\lambda_1 + \lambda_2$.
- In fact this is a general result for n Poisson processes.
- If you randomly "sample" a Poisson process – e.g. pick arrivals with probability p, the sampled process is Poisson, rate $p\lambda$.
- This makes Poisson processes easy to deal with.
- Many things in computer networks really are Poisson processes (e.g. people logging onto a computer or requesting web pages).

Little's theorem

Little's theorem

Let N be the average number of customers in a queue. Let λ be the average rate of arrivals. Let T be the average time spent queuing. Then we have

$$N = \lambda T.$$

- In fact this simple theorem hides much complexity.
- It is only true under certain conditions.

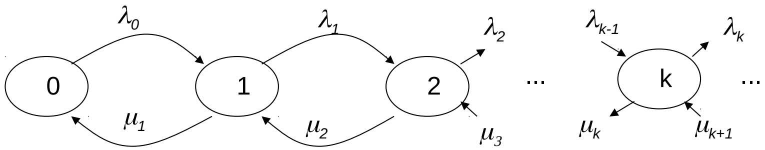
The Birth-Death process

The Birth–Death process

The birth–death process is a queue with a population which increases or decreases with rates which depend only on k the population at the time. Many queues can be modelled this way.

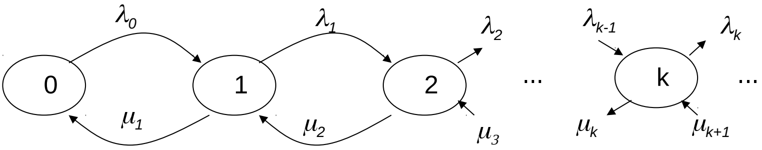
- Think of it as a queue – state 0 has no people. Arrivals are a Poisson process, rate λ_0.
- State k has births (arrivals) at rate λ_k but deaths (departures) at rate μ_k.

Starting the Birth–Death process



- Here we can see the arrivals and departures as a Markov chain.
- The state represents the number of people in the queue.
- An M/M/1 system would be modelled by $\lambda_k = \lambda$ for all k and $\mu_k = \mu$ for all k.
- Should model this as a continuous time Markov chain.
- Could pretend that μ_k and λ_k are the arrival probabilities in some small δt so small that $1 - \delta t(\mu_k + \lambda_k) > 0$ and $\delta t(\mu_k) < 1$ and $\delta t(\lambda_k) < 1$.

Birth–Death process – Transition matrix



Continuous time version:

$$Q = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & \dots \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 & \dots \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & \dots \\ 0 & 0 & \mu_3 & -(\lambda_3+\mu_3) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Comparing discrete time and continous time versions

$$Q = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & \dots \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 & \dots \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & \dots \\ 0 & 0 & \mu_3 & -(\lambda_3+\mu_3) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$\boldsymbol{\pi} Q = 0$$

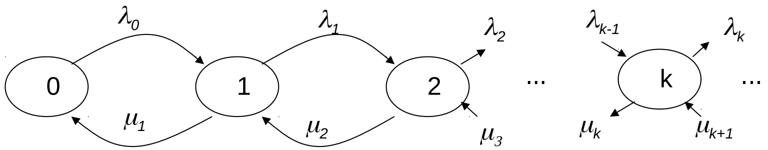
$$P = \begin{bmatrix} 1-\delta t\lambda_0 & \delta t\lambda_0 & 0 & 0 & \dots \\ \delta t\mu_1 & 1-\delta t(\lambda_1+\mu_1) & \delta t\lambda_1 & 0 & \dots \\ 0 & \delta t\mu_2 & 1-\delta t(\lambda_2+\mu_2) & \delta t\lambda_2 & \dots \\ 0 & 0 & \delta t\mu_3 & 1-\delta t(\lambda_3+\mu_3) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

$$\boldsymbol{\pi} P = P$$

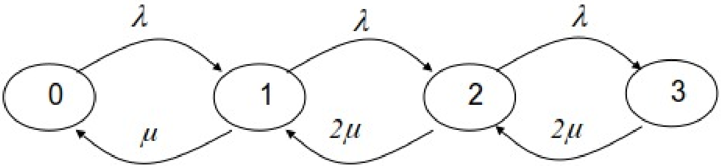
Worked queuing theory example

- The M/M/2/3 queue – the final 3 means at most 3 customers.
- This queue has two servers.
- If two or more customers are in the queue then both servers are active.
- Let λ be the arrival rate (birth rate).
- Let μ be the service rate of a single queue (death rate when 1 customer in queue).
- When both queues are active, therefore, the total service rate is 2μ – why? Remember why Poisson processes are useful.

M/M/2/3 queue as birth death process



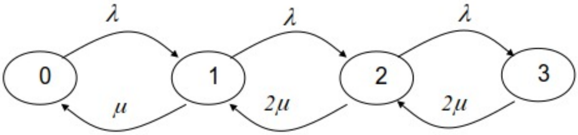
Above is the general birth death process – below specific for $M/M/2/3$.



Solving Continuous Time Markov chains

- Want to get the equilibrium probabilities for the chain π_i.
- The same procedure is always used.
- Get the balance equations for each state.
- The balance equation for state i says that the sum of the inputs is the sum of the outputs.
- That is $\sum_{j \neq i} \pi_j q_{ji} = \sum_{j \neq i} \pi_i q_{ij}$.
- In addition we need that the probabilities sum to one $\sum_i \pi_i = 1$.
- The balance equations contain one redundant equation – due to the stochastic nature of the matrix.

M/M/2/3 queue equations



- Balance equations – left hand side input – right hand side output.
- Balance state 0: $\mu\pi_1 = \lambda\pi_0$
- Balance state 1: $\lambda\pi_0 + 2\mu\pi_2 = (\lambda + \mu)\pi_1$.
- Balance state 2: $\lambda\pi_1 + 2\mu\pi_3 = (\lambda + 2\mu)\pi_2$.
- Balance state 3: $\lambda\pi_2 = 2\mu\pi_3$.
- Probability sum: $1 = \pi_0 + \pi_1 + \pi_2 + \pi_3$.

M/M/2/3 queue equations

- Balance state 0: $\mu\pi_1 = \lambda\pi_0$
- Balance state 1: $\lambda\pi_0 = (\lambda + \mu)\pi_1 - 2\mu\pi_2$.
- Balance state 2: $\lambda\pi_1 = (\lambda + 2\mu)\pi_2 - 2\mu\pi_3$.
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- From (0) $\pi_1 = \frac{\lambda}{\mu}\pi_0$

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- Balance state 3: $\lambda\pi_2 = 2\mu\pi_3$.
- From (0) $\pi_1 = \frac{\lambda}{\mu}\pi_0$
- From (1) $\pi_2 = \frac{1}{2\mu}\left[(\lambda + \mu)\pi_1 - \lambda\pi_0\right] = \frac{\lambda^2}{2\mu^2}\pi_0$.

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- From (3) $\pi_3 = \frac{\lambda\pi_2}{2\mu} = \frac{\lambda^3}{4\mu^3}\pi_0$.

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- From probability sum $1 = \pi_0 + \frac{\lambda}{\mu}\pi_0 + \frac{\lambda^2}{2\mu^2}\pi_0 + \frac{\lambda^3}{4\mu^3}\pi_0$.

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- From probability sum $1 = \pi_0 + \frac{\lambda}{\mu}\pi_0 + \frac{\lambda^2}{2\mu^2}\pi_0 + \frac{\lambda^3}{4\mu^3}\pi_0$.
- Therefore $\pi_0 = \frac{1}{1+\frac{\lambda}{\mu}+\frac{\lambda^2}{2\mu^2}+\frac{\lambda^3}{4\mu^3}} = \frac{4\mu^3}{4\mu^3+4\lambda\mu^2+2\lambda^2\mu+\lambda^3}$

M/M/2/3 queue solution

- From π_0 we can get π_1, π_2 and π_3.
- Similarly the mean queue length $N = \pi_1 + 2\pi_2 + 3\pi_3$.
- Was there a quicker way to get the answer?
- Yes: in a previous lecture we showed for B-D chain

$$\pi_0 = \frac{1}{1 + \sum_{k=1}^{\infty} \prod_{i=1}^{k} \frac{\lambda_{i-1}}{\mu_i}}.$$

- Here $\lambda_i = \lambda$ for all i, $\mu_1 = \mu$ and $\mu_2 = \mu_3 = 2\mu$.
- Therefore we have (as before)

$$\pi_0 = \frac{1}{1 + \frac{\lambda}{\mu} + \frac{\lambda^2}{2\mu^2} + \frac{\lambda^3}{4\mu^3}}$$

M/M/2/3 in terms of utilisation

- Utilisation in this case is $\rho = \lambda/2\mu$ – the system is fully utilised with both servers working.
- The equilibrium probabilities can now be written more simply.
- $\pi_0 = \frac{1}{1+2(\rho+\rho^2+\rho^3)}$
- $\pi_1 = \frac{2\rho}{1+2(\rho+\rho^2+\rho^3)}$
- $\pi_2 = \frac{2\rho^2}{1+2(\rho+\rho^2+\rho^3)}$
- $\pi_3 = \frac{2\rho^3}{1+2(\rho+\rho^2+\rho^3)}$
- $N = \pi_1 + 2\pi_2 + 3\pi_3 = \frac{2\rho+4\rho^2+6\rho^3}{1+2(\rho+\rho^2+\rho^3)}$

Example exam question. 2017 exam

Consider a buffer that can hold only three packets of data. In a time period of one microsecond the following happens in order.

- Zero, one, two or three packets arrive at the buffer with probabilities a_0, a_1, a_2 and a_3. This implies $a_0 + a_1 + a_2 + a_3 = 1$.
- If the buffer has any packets remaining or any new packets arrive then exactly one packet is processed and removed from the buffer.
- Any arriving packets are added to the buffer. If the buffer contains more than three packets then the excess packets are discarded.

In Kendall's notation what is the type of the queue?

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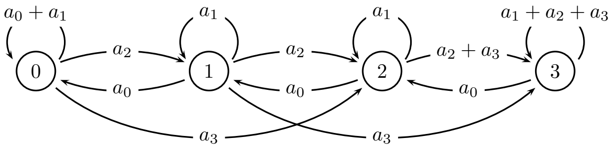
In Kendall's notation what is the type of the queue? It is a G/D/1/3 queue G/D/1 is also an acceptable answer.

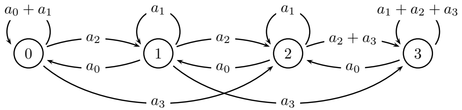
Example exam question. 2017 exam

ii) Draw the resulting chain



Example exam question. 2017 exam

iii) Write down the transition matrix P for the chain and the four balance equations (30%)



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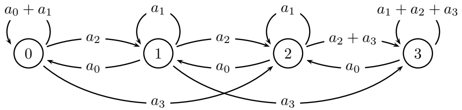

$$P = \begin{bmatrix} a_0 + a_1 & a_2 & a_3 & 0 \\ a_0 & a_1 & a_2 & a_3 \\ 0 & a_0 & a_1 & a_2 + a_3 \\ 0 & 0 & a_0 & a_1 + a_2 + a_3 \end{bmatrix}$$

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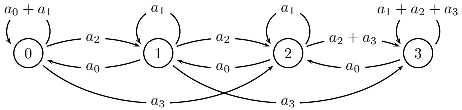

$$P = \begin{bmatrix} a_0 + a_1 & a_2 & a_3 & 0 \\ a_0 & a_1 & a_2 & a_3 \\ 0 & a_0 & a_1 & a_2 + a_3 \\ 0 & 0 & a_0 & a_1 + a_2 + a_3 \end{bmatrix}$$

The balance equations are:

$$\pi_0 = (a_0 + a_1)\pi_0 + a_0\pi_1$$
$$\pi_1 = a_2\pi_0 + a_1\pi_1 + a_0\pi_2$$
$$\pi_2 = a_3\pi_0 + a_2\pi_1 + a_1\pi_2 + a_0\pi_3$$
$$\pi_3 = a_3\pi_1 + (a_2 + a_3)\pi_2 + (a_1 + a_2 + a_3)\pi_3$$

Example exam question. 2017 exam

iii) Solve the system to get the equilibrium states $\pi_0, \pi_1, \pi_2, \pi_3$ (the equilibrium probabilities of the buffer having 0, 1, 2 or 3 packets in it) given $a_0 = a_1 = a_2 = a_3 = 1/4$.

The balance equations are:

$$\pi_0 = (a_0 + a_1)\pi_0 + a_0\pi_1 = 1/2\pi_0 + 1/4\pi_1$$

$$\pi_1 = a_2\pi_0 + a_1\pi_1 + a_0\pi_2 = 1/4\pi_0 + 1/4\pi_1 + 1/4\pi_2$$

$$\pi_2 = a_3\pi_0 + a_2\pi_1 + a_1\pi_2 + a_0\pi_3 = 1/4\pi_0 + 1/4\pi_1 + 1/4\pi_2 + 1/4\pi_3$$

$$\pi_3 = a_3\pi_1 + (a_2 + a_3)\pi_2 + (a_1 + a_2 + a_3)\pi_3 = 1/4\pi_1 + 1/2\pi_2 + 3/4\pi_3$$

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$$\pi_3 = a_3\pi_1 + (a_2 + a_3)\pi_2 + (a_1 + a_2 + a_3)\pi_3 = 1/4\pi_1 + 1/2\pi_2 + 3/4\pi_3$$

Simultaneous equations and $\pi_0 + \pi_1 + \pi_2 + \pi_3 = 1$ give:

$$\pi_0 = 1/20$$
$$\pi_1 = 2/20 = 1/10$$
$$\pi_2 = 5/20 = 1/4$$
$$\pi_3 = 12/20 = 3/5$$

Example exam question. 2017 exam

v) Imagine the system did not have a limited buffer. Write down the mean arrival rate of the system in packets per microsecond in terms of a_0, a_1, a_2 and a_3. Write down a condition that means the queue will not grow forever.

Example exam question. 2017 exam

v) Imagine the system did not have a limited buffer. Write down the mean arrival rate of the system in packets per microsecond in terms of a_0, a_1, a_2 and a_3. Write down a condition that means the queue will not grow forever. The mean arrival rate is
$a_1 + 2a_2 + 3a_3$.

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v) Imagine the system did not have a limited buffer. Write down the mean arrival rate of the system in packets per microsecond in terms of a_0, a_1, a_2 and a_3. Write down a condition that means the queue will not grow forever. The mean arrival rate is
$a_1 + 2a_2 + 3a_3$.
The queue will not grow forever if $a_1 + 2a_2 + 3a_3 < 1$ (full marks for less than or equal to here).

Queuing theory summary

- Kendal's notation – M/M/1, M/D/2 ...
- Little's theorem connects number in queue, waiting time and arrival rate.
- Queues with Poisson arrivals and service times can be viewed in terms of Markov chains as a birth death process.
- The Markov chain can be formed as usual with the state of the chain indicating the number in the queue.
- The equilibrium probabilities give the proportion of the time the queue has that length.
- π_0 is the proportion of the time the queue is empty.
- π_n is the proportion of the time the queue has exactly n customers.
- $\sum_{i=0}^{i=\infty} i\pi_i$ is the mean length of the queue.