The Fourier Series and Transform

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Formal definition

Some notation

Illustrative examples

Shifting, scaling and reciprocity

Fourier transforms using tables

Fourier Techniques

- Fourier Series: Determines the spectrum of a *periodic* signal
- Fourier Transform: Determines the spectrum of an *aperiodic* signal and the reverse

Fourier Series

- A periodic signal is one that repeats at equal intervals of T. Formally we can say that:

$$\boxed{v(t) = v(t \pm nT)}$$

First, both formulae are correct. Note that the formula in Week 6 is applied for a function in the time domain and the formula in Week 9 attempts to apply the result for a function in the frequency domain. Secondly, if you look at the Fourier series formula in Week 6 closely, it does NOT matter whether there is a minus sign on the exponent because the summation index runs from -infinity to +infinity.

where n is any integer

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi nt/T}$$

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-j2\pi nt/T} dt$$

Interlude: Fourier Series – Meaning

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi nt/T}$$

$$c_n = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-j2\pi nt/T} dt$$

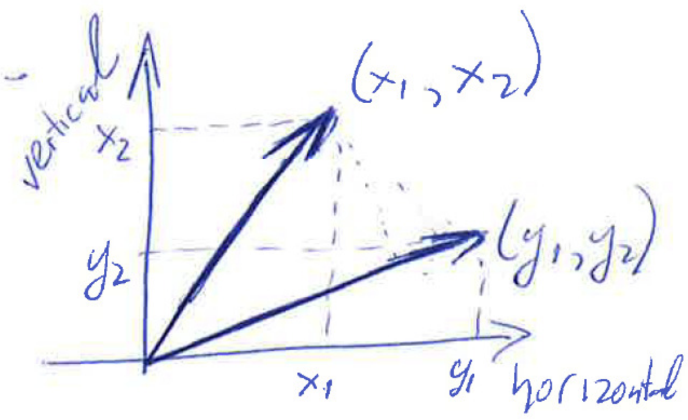
→Infinite sum of weighted phasors of increasing frequency

→the weights c_n are the projection of the base period of the signal onto the Hilbert space formed by the phasors of increasing frequency

Interlude: Spaces and Hilbert Space

Interlude ... on Spaces ...

In $\mathbb{R}^2$: $\mathbf{x} = (x_1, x_2)$
$\mathbf{y} = (y_1, y_2)$



1) $\langle \mathbf{x}, \mathbf{y} \rangle = x_1 y_1 + x_2 y_2$ → inner product

2) $\mathbf{x} \perp \mathbf{y} \Leftrightarrow \langle \mathbf{x}, \mathbf{y} \rangle = 0$ → orthogonality

3) $\|\mathbf{x}\|_2 = \sqrt{x_1^2 + x_2^2} = \langle \mathbf{x}, \mathbf{x} \rangle^{\frac{1}{2}}$ → distance from $(0,0)$

4) Cauchy-Schwarz inequality:
$\langle \mathbf{x}, \mathbf{y} \rangle \leq \|\mathbf{x}\|_2 \|\mathbf{y}\|_2$

Please note: when you see handwriting on the slides, it means you should take notes and ensure you understand the concept/math

Interlude: Spaces and Hilbert Space

recap the video

5) Distance (general)

$$\|\mathbf{x}-\mathbf{y}\|_2 = \sqrt{(x_1-y_1)^2+(x_2-y_2)^2} = d(\mathbf{x},\mathbf{y})$$

All of 1~5 can be generalised to $\mathbb{R}^N$ (N dimensions)... but it starts becoming difficult to visualise things and use intuition as we live in 3 dimensions

Also, generalising to N dimensions allows us to define new basis vectors!!!

An N-dimensional space is defined by N basis vectors that are orthogonal i.e. N vectors $b_0, b_1, \ldots, b_{N-1}$ that are orthogonal to each other. Then they satisfy 1~5.

It can be shown that $b_k^{(n)} = e^{+j\frac{2\pi}{N}nk}$ form such a basis.

Interlude: Fourier Series – Meaning (cont'd)

- Phasors increasing frequency: $e^{j2\pi nt/T}, \; n = 0, 1, \ldots, +\infty$ (we also have negative phasors to form a real signal)

- Hilbert space: A generalization of the Euclidean space, where a space is defined by the notions of: distance, orthogonality and basis vectors

- Because the signal is continuous and periodic, we need a basis with infinite phasors and the projection operation to each phasor becomes an integral!

$$\left\langle x(t)_T, e^{j2\pi nt/T} \right\rangle = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-j2\pi nt/T} \, dt$$

Interlude: Fourier Series – Significance

- <u>Why use phasors as basis functions of the Hilbert space?</u>

  Because the primary part of the solution to ordinary linear differential equations is the linear combination of phasors (they are also called eigenfunctions of such derivatives)

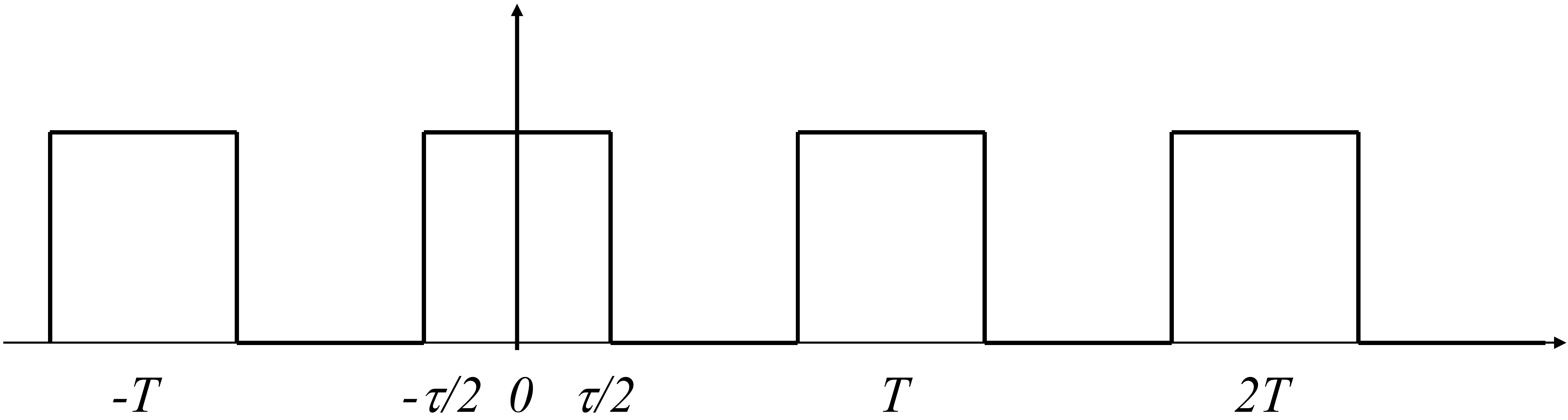
$$a_0 f + a_1 \frac{df}{dt} + a_2 \frac{d^2 f}{dt^2} + \ldots + a_k \frac{d^k f}{dt^k} + b = 0$$

$$f(t) = \sum_i c_i f_i(t), \quad \text{where}$$

$$f_i(t) = \begin{cases} t^{n_i} e^{s_i t} & t \geq 0 \\ 0 & t < 0 \end{cases}, \quad \text{with } n_i \in \mathbb{R},\ s_i \in \mathbb{C}$$

Example

- Consider the periodic rectangular wave



$$\boxed{x(t) = \sum_{\mathrm{k}} \mathrm{rect}\left[\left(t - kT\right) / \tau\right]}$$

Solution

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} \mathrm{rect}(t/\tau)e^{-j2\pi nt/T}dt$$

$$= \frac{1}{T}\left\{\int_{-T/2}^{-\tau/2} 0\cdot e^{-j2\pi nt/T}dt + \int_{-\tau/2}^{\tau/2} 1\cdot e^{-j2\pi nt/T}dt + \int_{\tau/2}^{T/2} 0\cdot e^{-j2\pi nt/T}dt\right\}$$

$$= \frac{1}{T}\int_{-\tau/2}^{\tau/2} e^{-j2\pi nt/T}dt$$

$$\boxed{= \frac{1}{T}\left[\frac{e^{-j2\pi nt/T}}{-j2\pi nt/T}\right]_{-\tau/2}^{\tau/2} = \frac{1}{T}\frac{\sin(\pi n\tau/T)}{\pi n/T} = \frac{\tau}{T}\mathrm{sinc}\left(n\tau/T\right)}$$

Solution

$$\frac{1}{T}\left[\frac{e^{-j2\pi n t/T}}{-j2\pi n/T}\right]_{-\tau/2}^{\tau/2} = \frac{1}{T\pi n/T}\,\frac{e^{-j2\pi\left(\frac{n\tau}{2T}\right)} - e^{j2\pi\left(\frac{n\tau}{2T}\right)}}{-2j} =$$

$$= \frac{1}{T\pi n/T}\,\frac{e^{j2\pi\left(\frac{n\tau}{2T}\right)} - e^{-j2\pi\left(\frac{n\tau}{2T}\right)}}{2j} =$$

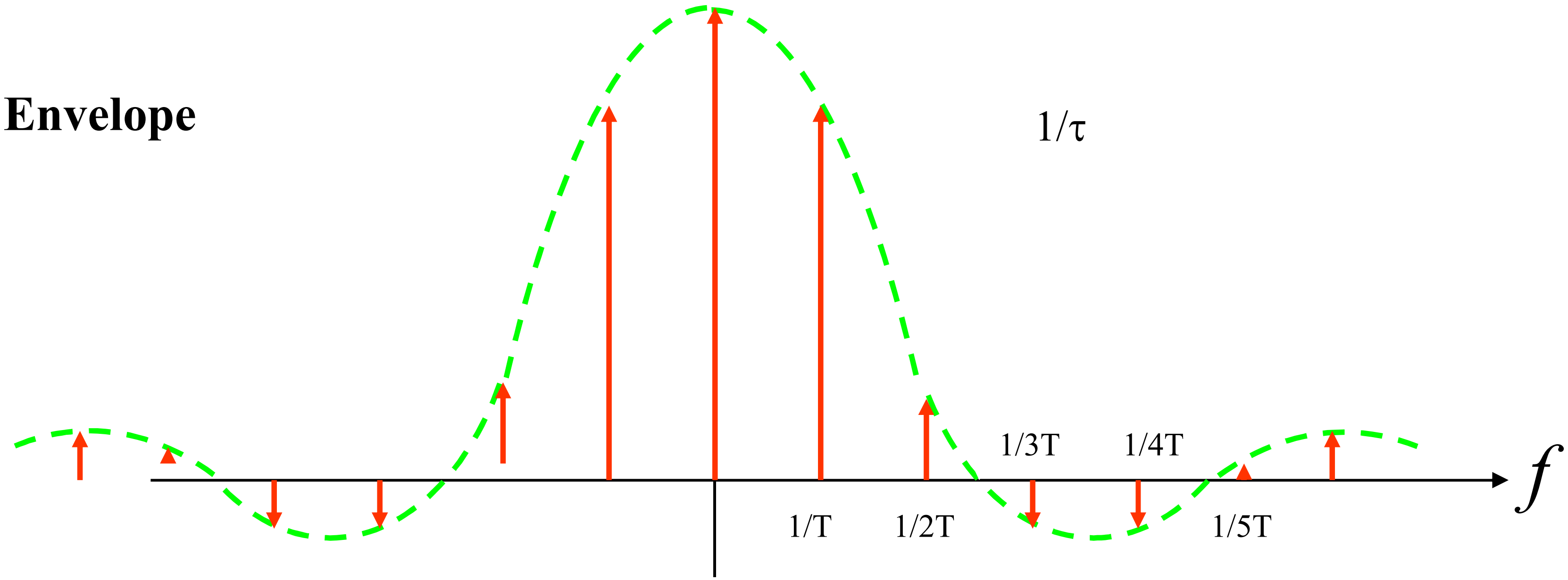
$$= \frac{1}{T\pi n/T}\sin\left(2\pi\left(\frac{n\tau}{2T}\right)\right) = \frac{1}{T}\,\frac{\sin(\pi n\tau/T)}{\pi n/T} =$$

$$= \frac{\tau}{T}\,\frac{\sin(\pi n\tau/T)}{\pi n\tau/T} = \boxed{\frac{\tau}{T}\operatorname{sinc}(n\tau/T)}$$

Solution

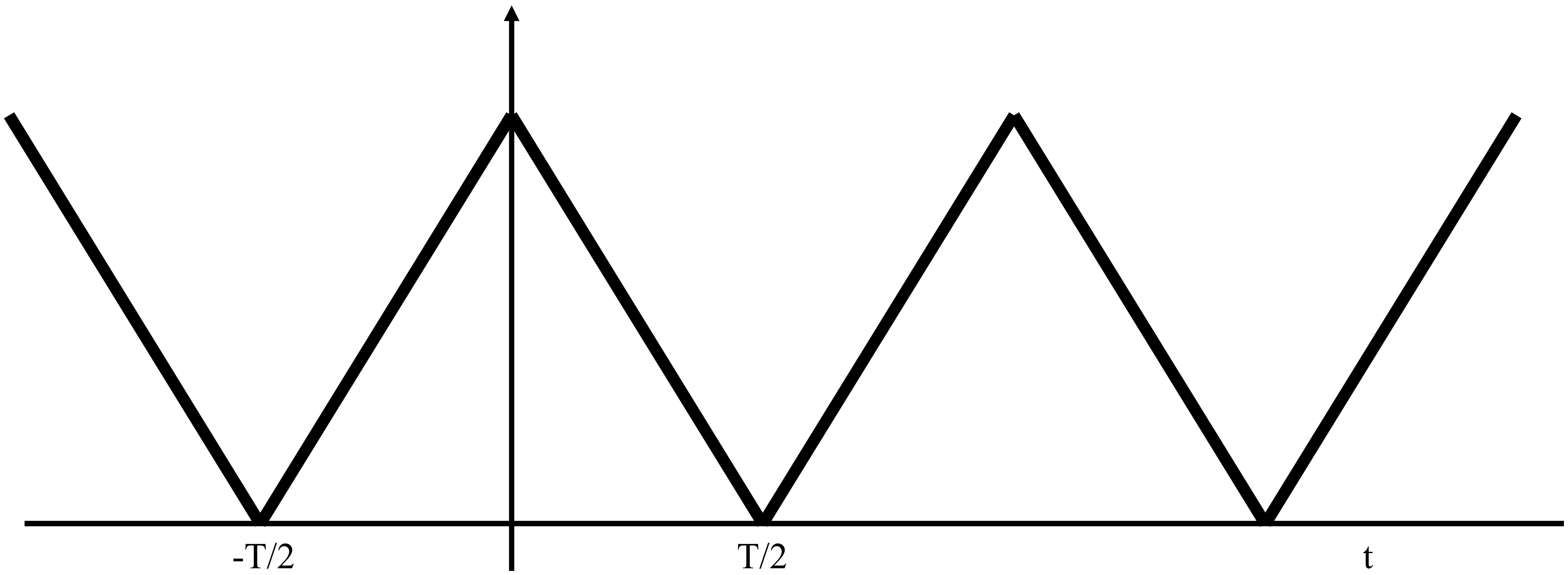
$$\Rightarrow x(t) = \sum_n \frac{\tau}{T} \text{sinc}\left(n\tau / T\right) e^{j2\pi nt/T}$$

Spectral Envelope



Fourier Series - Exercise

- Consider a triangular wave above ground



Fourier Series - Hint

integration by parts

$$x(t) = \begin{cases} V\left(1 + \dfrac{2t}{T}\right) & -T/2 < t < 0 \\ V\left(1 - \dfrac{2t}{T}\right) & 0 < t < T/2 \end{cases}$$

Fourier Series - Answer

$$c_n = \frac{V}{n^2\pi^2}\left(1 - \cos n\pi\right)$$

$$v(t) = \sum_n \frac{V}{n^2\pi^2}\left(1 - \cos n\pi\right)e^{j2\pi nt/T}$$

Please try to work out the answer on your own as a practice calculation

The Fourier transform

- *formal definition*

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$

Forward transform

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{+j2\pi ft} df$$

Inverse transform

- The forward and inverse transforms relate a time signal x(t) and its (Fourier) spectrum X(f)
- Commonly *x(t)* is real and *X(f)* complex, although in general both *x(t)* and *X(f)* may be complex.
- Note the use of *f* as the frequency variable, rather than $\omega=2\pi f$
- Produce a continuous spectrum as they have no well-defined period
- It is closely related to the (double-sided) Laplace transform via

  $\exp(s) \leftrightarrow \exp(j2\pi f)$ with $s = \sigma + j2\pi f$

Some notation

$X(f) = \mathcal{F}\{x(t)\}$ *indicates the (forward) Fourier transformation of* $x(t)$ *to produce* $X(f)$;

$x(t) = \mathcal{F}^{-1}\{X(f)\}$ *indicates the inverse transform (IFFT)*

$x(t) \Leftrightarrow X(f)$ *indicates a Fourier transform pair*

- Note in particular the use of a lower case letter for a time-domain signal and the corresponding upper case letter for its Fourier transform

Symmetry properties

$$1.\quad x(t)\ \ real\ \ and\ \ even \Rightarrow X(f)\ \ real\ \ and\ \ even$$

$$2.\quad x(t)\ \ real\ \ \Rightarrow X(f) Hermitian\ \ symmetric$$

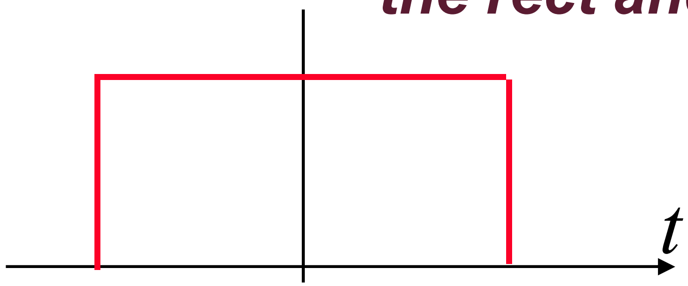
$$i.e.\ \ \mathrm{Re}\{X(f)\}\ \ even;\ \ \mathrm{Im}\{X(f)\}\ \ odd$$

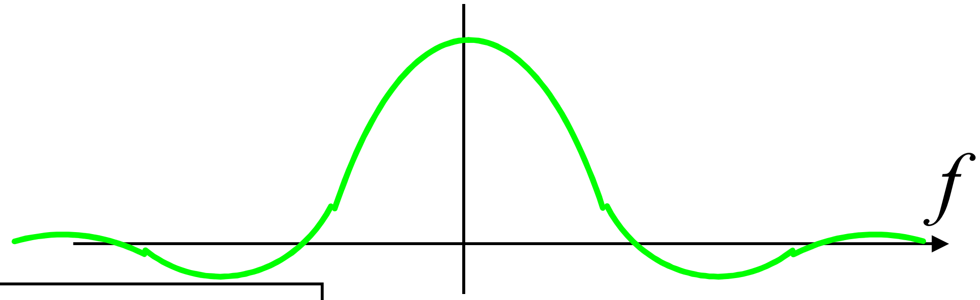
$$\Rightarrow |X(f)|\ \ even;\ \ Arg\{X(f)\}\ \ odd$$

$$3.\ x(t)\ real\ odd \Rightarrow \mathrm{Re}\{\mathrm{X(f)}\}=0\ \ \mathrm{Im}\{\mathrm{X(f)}\}\ odd$$

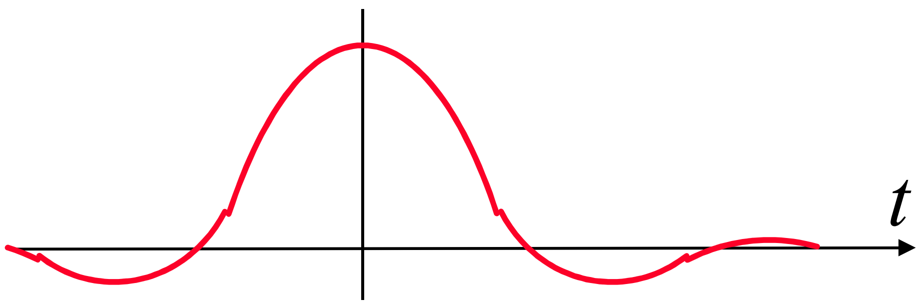
Illustrative examples 1:

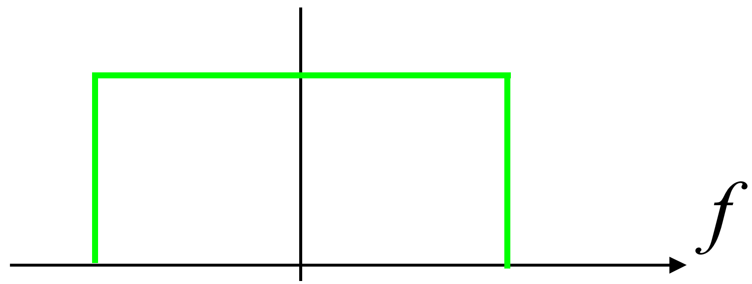
the rect and sinc function





$$\boxed{rect(t) \Leftrightarrow sinc(f)}$$

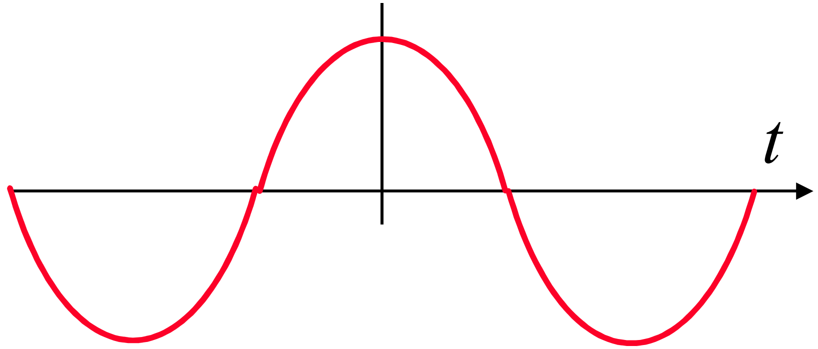


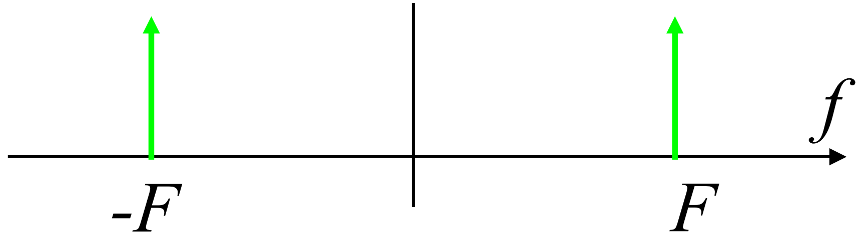


$$\boxed{sinc(t) \Leftrightarrow rect(f)}$$

Illustrative examples 2:

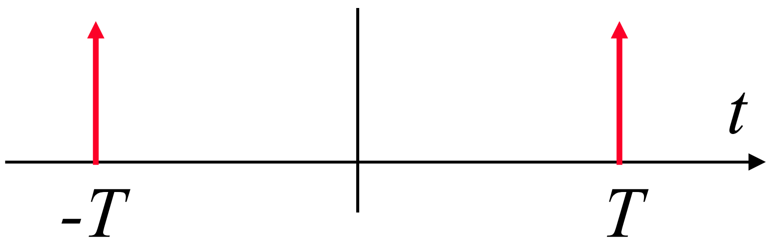
cosine functions

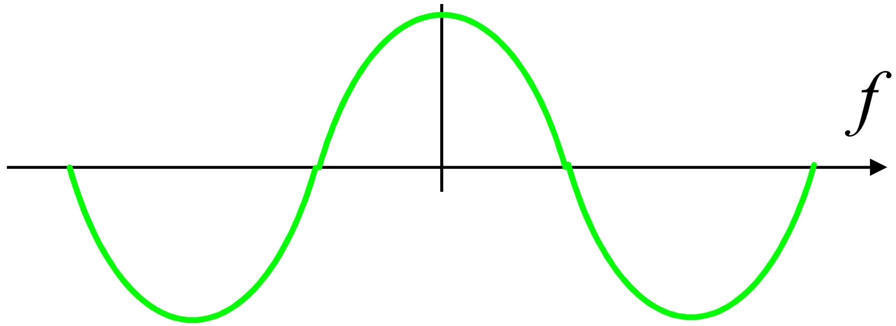




$$\cos(2\pi F t) \Leftrightarrow \frac{1}{2}\delta(f+F) + \frac{1}{2}\delta(f-F)$$

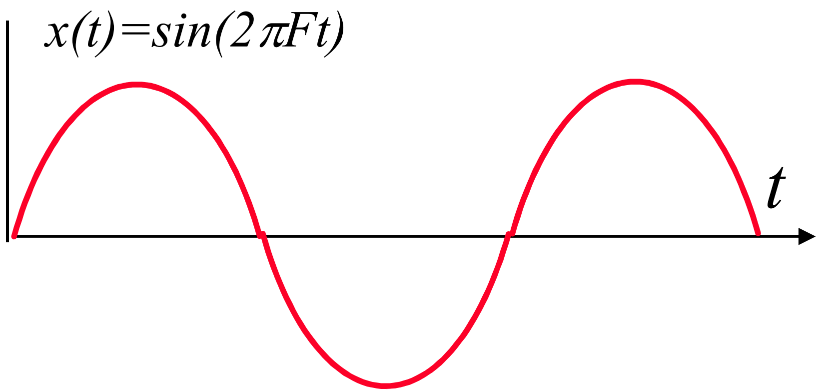
$$\frac{1}{2}\delta(t+T) + \frac{1}{2}\delta(t-T) \Leftrightarrow \cos(2\pi f T)$$

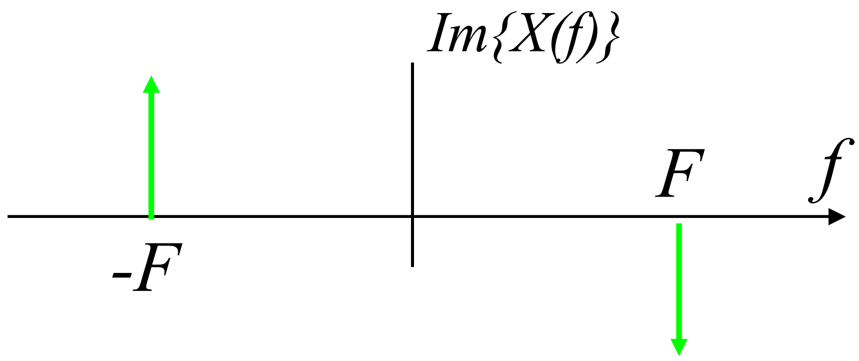




Illustrative examples 3:

sine function





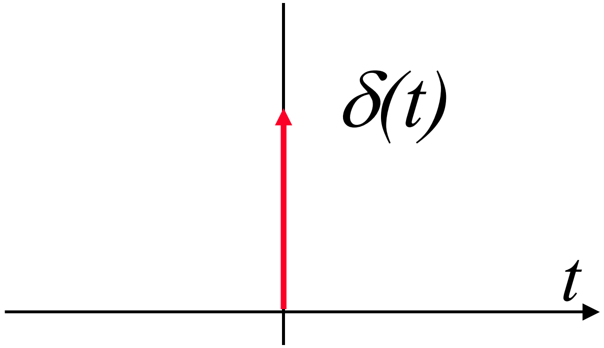
- The sine function is an odd function:

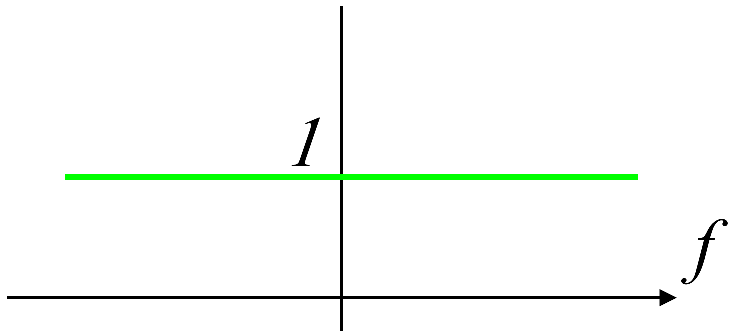
$$x(-t) = -x(t)$$

- The spectrum is purely imaginary

Illustrative examples 4:

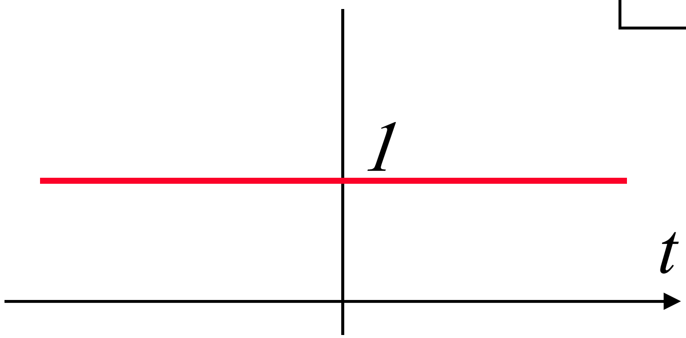
the delta function

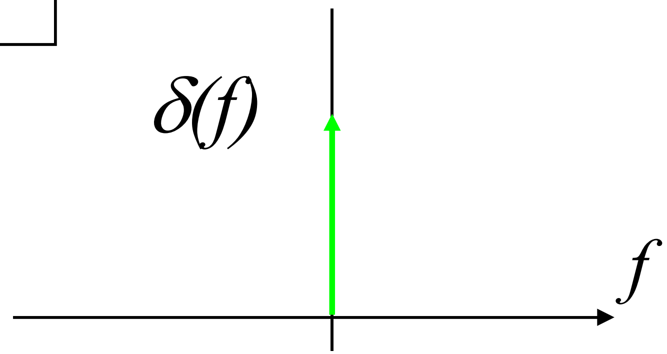




$$\delta(t) \Leftrightarrow 1$$

$$1 \Leftrightarrow \delta(f)$$





Shifting and scaling

$$x(t) \Leftrightarrow X(f)$$

$$x\left(\frac{t}{T}\right) \Leftrightarrow |T|\, X(fT)$$

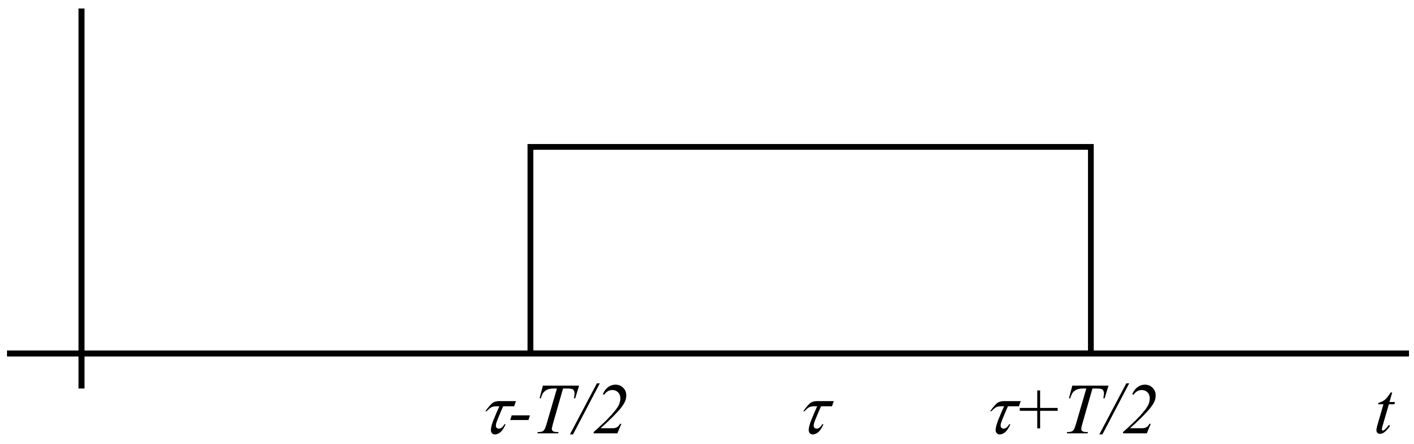
$$x(t-T) \Leftrightarrow X(f)e^{-j2\pi fT}$$

- The results are presented in terms of shifting and scaling a time function but apply equally - via symmetry - to shifting and scaling of frequency domain functions, to within the sign of the exponent.

Fourier Transforms

$$x(t) = A\,\mathrm{rect}\left(\frac{t-\tau}{T}\right)$$

Find $X(f)$



Fourier Transforms - Answer

$$x(t) = A\ \mathrm{rect}\left(\frac{t-\tau}{T}\right)$$

$$X(f) = \int_{-\infty}^{\infty} A\ \mathrm{rect}\left(\frac{t-\tau}{T}\right) e^{-j2\pi f t}\, dt$$

$$X(f) = A \int_{-\tau-T/2}^{\tau-T/2} e^{-j2\pi f t}\, dt$$

$$X(f) = AT\ \mathrm{sinc}\ (fT) e^{-j2\pi f \tau}$$

Practice

$$\int_{\tau-\frac{T}{2}}^{\tau+\frac{T}{2}} e^{-j2\pi ft}\,dt = \left[\frac{e^{-j2\pi ft}}{-j2\pi f}\right]_{\tau-\frac{T}{2}}^{\tau+\frac{T}{2}} =$$

$$= \frac{e^{-j2\pi f\left(\tau+\frac{T}{2}\right)} - e^{-j2\pi f\left(\tau-\frac{T}{2}\right)}}{-j2\pi f} =$$

$$= e^{-j2\pi f\tau}\left[\frac{e^{-j2\pi f\frac{T}{2}} - e^{j2\pi f\frac{T}{2}}}{-j2\pi f}\right] =$$

$$= e^{-j2\pi f\tau}\,\frac{e^{j2\pi\left(f\frac{T}{2}\right)} - e^{-j2\pi\boxed{\left(f\frac{T}{2}\right)}}}{j2\pi f} = e^{-j2\pi f\tau}\,\frac{\sin\left(2\pi\left(f\frac{T}{2}\right)\right)}{\pi f} =$$

$$= T\,e^{-j2\pi f\tau}\,\frac{\sin(\pi fT)}{\pi fT} = T\cdot \mathrm{sinc}(fT)\,e^{-j2\pi f\tau}$$

Fourier Transforms – Answer #2

Properties to use:

$$x\left(\frac{t}{T}\right) \leftrightarrow |T|\,X(fT)$$

$$x(t-T) \leftrightarrow X(f)\,e^{-j2\pi fT}$$

From table of Fourier transforms:

$$\mathrm{rect}(t) \leftrightarrow \mathrm{sinc}(f)$$

$$A\,\mathrm{rect}(t) \leftrightarrow A\,\mathrm{sinc}(f)$$

$$A\,\mathrm{rect}\left(\frac{t}{T}\right) \leftrightarrow AT\,\mathrm{sinc}(fT)$$

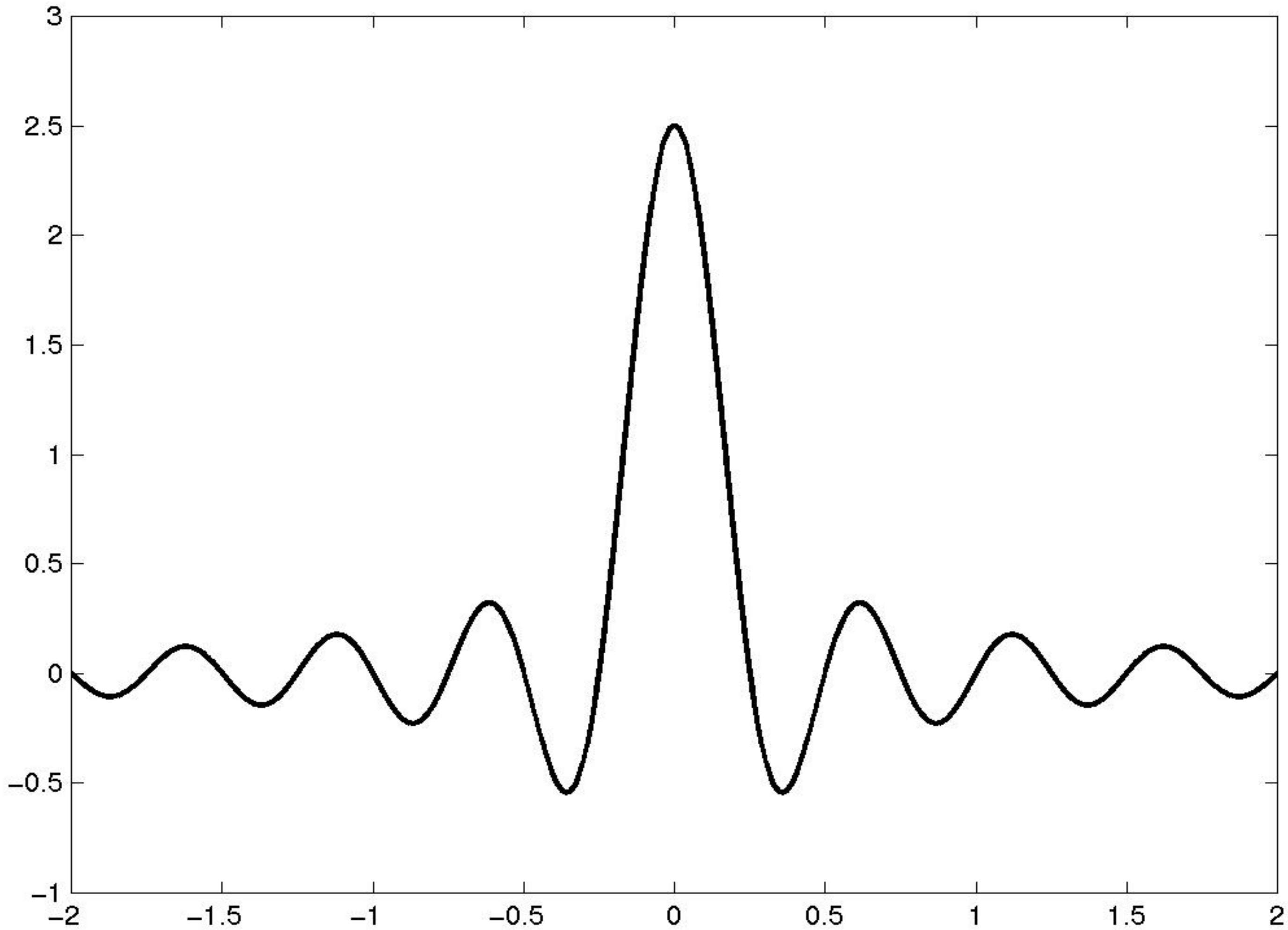
$$A\,\mathrm{rect}\left(\frac{t-\tau}{T}\right) \leftrightarrow AT\,\mathrm{sinc}(fT)\,e^{-j2\pi f\tau}$$

Fourier Transforms - Answer

A=5

T=0.5

τ=2



Reciprocity

$$x(t) \Leftrightarrow X(f) \Rightarrow X(t) \Leftrightarrow x(-f)$$

- The forward and inverse transforms are almost identical in form, which gives the reciprocity relationship indicated above
- This has been enhanced by the use of ***f*** rather than ω as the frequency variable , avoiding an amplitude scaling factor that would otherwise be involved
- For real and even functions, x(t)=x(-t) it simplifies further, as we have seen in examples previously, to:

$$\begin{aligned} &If \quad x(t) = x(-t) \Leftrightarrow X(f) \\ &then \quad X(t) \Leftrightarrow x(f) \end{aligned}$$

Gaussian function

The normalised transform relationship is:

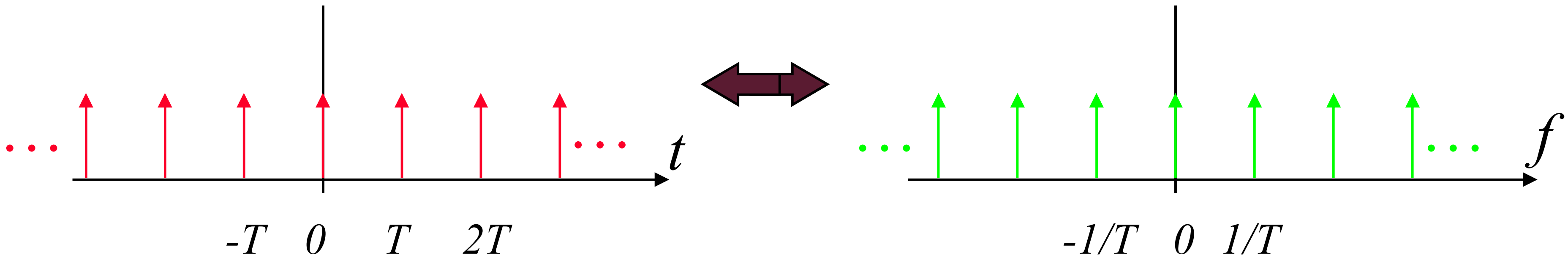
$$e^{-\pi t^2} \Leftrightarrow e^{-\pi f^2}$$

Scaling to width σ_t gives:

$$\frac{1}{\sqrt{2\pi}\,\sigma_t} e^{-\frac{1}{2}\left(\frac{t}{\sigma_t}\right)^2} \Leftrightarrow e^{-2(\pi f \sigma_t)^2}$$

- This provides a good exercise in use of the scaling relation for the Fourier transform - try to show it yourself!

Repeated delta functions



$$rep_T\{\delta(t)\} = \sum_{n=-\infty}^{\infty} \delta(t - nT) \Leftrightarrow \frac{1}{T} rep_{\frac{1}{T}}\{\delta(f)\} = \frac{1}{T}\sum_{n=-\infty}^{\infty} \delta\left(f - \frac{n}{T}\right)$$

- The ‘rep-delta’ function (actually, of course , a *generalised function* since it is composed of delta functions or impulses) is an invaluable building block when dealing with periodic signals

Summary

Properties:

$$x\left(\frac{t}{T}\right) \leftrightarrow |T|\,X(fT)$$

$$x(t-T) \leftrightarrow X(f)e^{-j2\pi fT}$$

$$\mathrm{rep}_T\{\delta(t)\} \leftrightarrow \mathrm{rep}_{\frac{1}{T}}\{\delta(f)\}\cdot\left|\frac{1}{T}\right|$$

$$\mathrm{rep}_T\{\delta(t)\} = \sum_{n=-\infty}^{\infty}\delta(t-nT)$$

$$\mathrm{rep}_{\frac{1}{T}}\{\delta(f)\}\cdot\frac{1}{T} = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{-j2\pi\left(f-\frac{n}{T}\right)}$$

Also: Remember that the Fourier transform preserves the signal energy:

$$\int_{-\infty}^{+\infty}|x(t)|^2dt = \int_{-\infty}^{+\infty}|X(f)|^2df$$

Fourier transforms using tables

- A short table of Fourier transforms and relationships such as shifting and scaling - together with convolution, which we will deal with in a separate presentation - can enable us to construct transforms for more complicated functions.
- Accordingly it is possible to obtain transforms for such functions by breaking them down into recognised results

Transform properties

- Superposition Theorem

$$a_1 x_1(t) + a_2 x_2(t) \leftrightarrow a_1 X_1(f) + a_2 X_2(f)$$

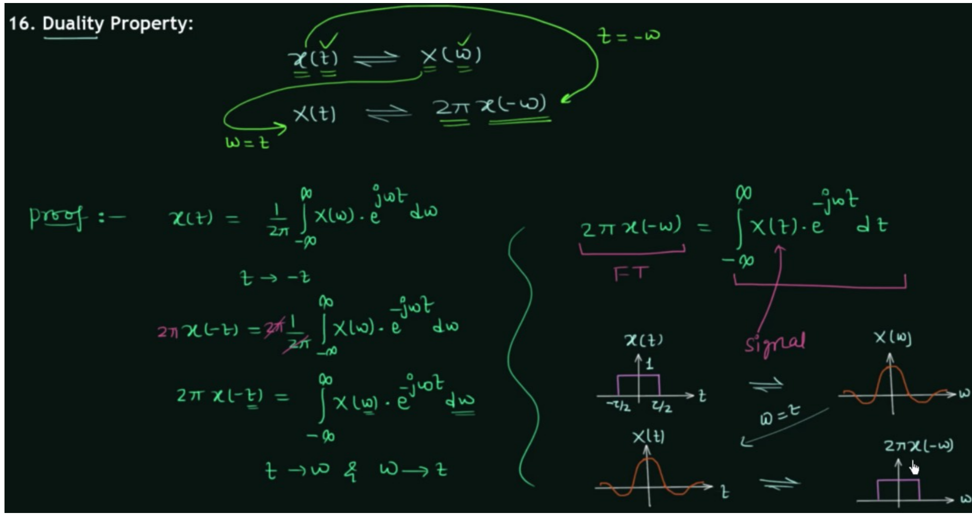
- Time Delay Theorem

$$x(t - t_0) \leftrightarrow X(f) e^{-j2\pi f t_0}$$

- Scale-change theorem

$$x(at) \leftrightarrow \frac{1}{|a|} X\left(\frac{f}{a}\right)$$

Transform properties



- Duality Theorem

$$x(-f) \leftrightarrow X(t)$$

- Frequency translation Theorem

$$x(t)e^{j2\pi f_0 t} \leftrightarrow X(f - f_0)$$

Summary

- The forward and inverse transforms are nearly symmetrical
- Using $f = \omega/2\pi$ as the frequency variable increases the symmetry, the sign of the exponent being the only difference
- Shifting and scaling relationships enable complex problems to be broken down and solved in terms of normalised standard transforms