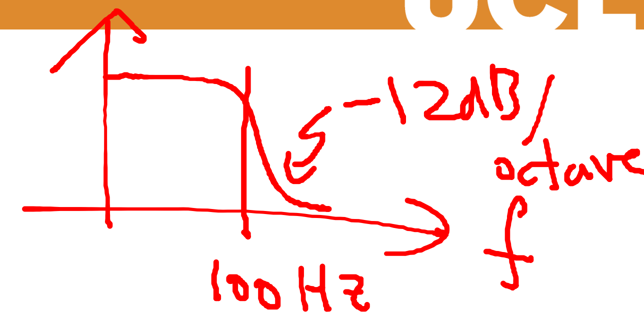
$f_s = 800Hz$

Example Digital LPF Design



- We need to filter the ambient noise corrupting a sinusoidal signal. The signal is sampled at 800 samples/s and its maximum frequency is 100Hz. Design a LPF without ripple and a roll-off rate of –12dB/octave

→ Butterworth filter (−6n dB/octave)

1. Determine the transfer function of the normalised filter ✓
2. Calculate the analogue filter cut-off frequency prior to warping ✓
3. Apply the bilinear transformation ✓
4. Calculate y(n) by taking the inverse z-transform ✓
5. Draw the physical realisation and justify your choice ✓

$f_s = 800\,Hz \qquad f_c = 100\,Hz$

$-12\,dB/\text{octave}$

(1) Choose Butterworth filter (no ripples)

$-6n\,dB/\text{octave} \Rightarrow -12\,dB$

$\Rightarrow n = 2$ ✓

$$H(s) = \frac{1}{s^2 + \sqrt{2}s + 1} \quad \text{(normalised LPF)}$$

(2) $$\omega_a = \frac{2}{T_s} \tan \frac{\Omega T_s}{2}$$

$$\omega_a = \frac{2}{(1/800)} \tan \frac{2\pi(100)\left(\frac{1}{800}\right)}{2} = 663 \text{ rad}\,s^{-1}$$

$$H(s) = \frac{1}{\left(\frac{s}{663}\right)^2 + \sqrt{2}\left(\frac{s}{663}\right) + 1}$$

LPF

$$\boxed{s \rightarrow \frac{s}{\omega_c}}$$

③

$$H(z) = \frac{1}{\left[\left(\dfrac{\frac{2}{1/800}\,\frac{1-z^{-1}}{1+z^{-1}}}{663}\right)^2 + \sqrt{2}\,\dfrac{\frac{2}{1/800}\,\frac{1-z^{-1}}{1+z^{-1}}}{663} + 1\right]}$$

$$s = \frac{2}{T_s}\,\frac{1-z^{-1}}{1+z^{-1}}$$

$$T_s = \frac{1}{800}$$

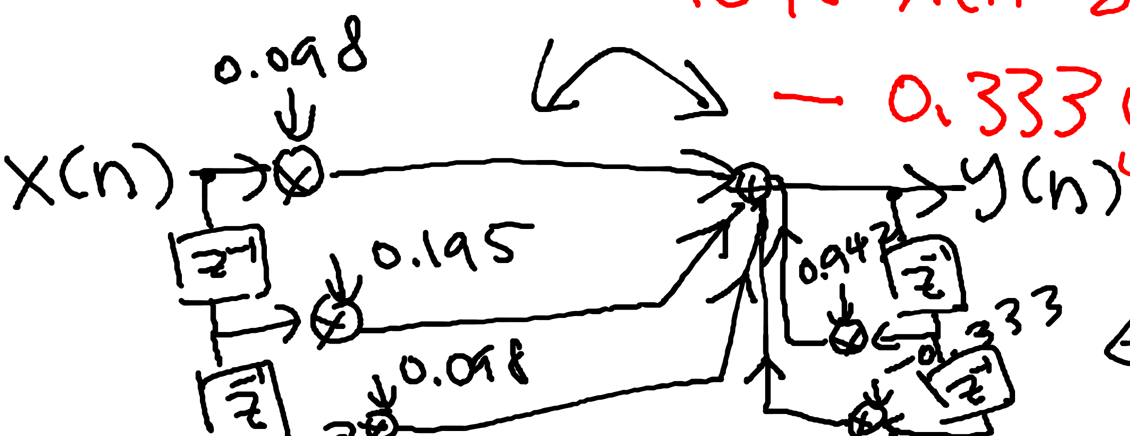
$$H(z) = \frac{0.098 + 0.195\,z^{-1} + 0.098\,z^{-2}}{1 - 0.942\,z^{-1} + 0.333\,z^{-2}}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{0.098 + 0.195 z^{-1} + 0.098 z^{-2}}{1 - 0.942 z^{-1} + 0.333 z^{-2}}$$

$$y(n) - 0.942\, y(n-1) + 0.333\, y(n-2)$$

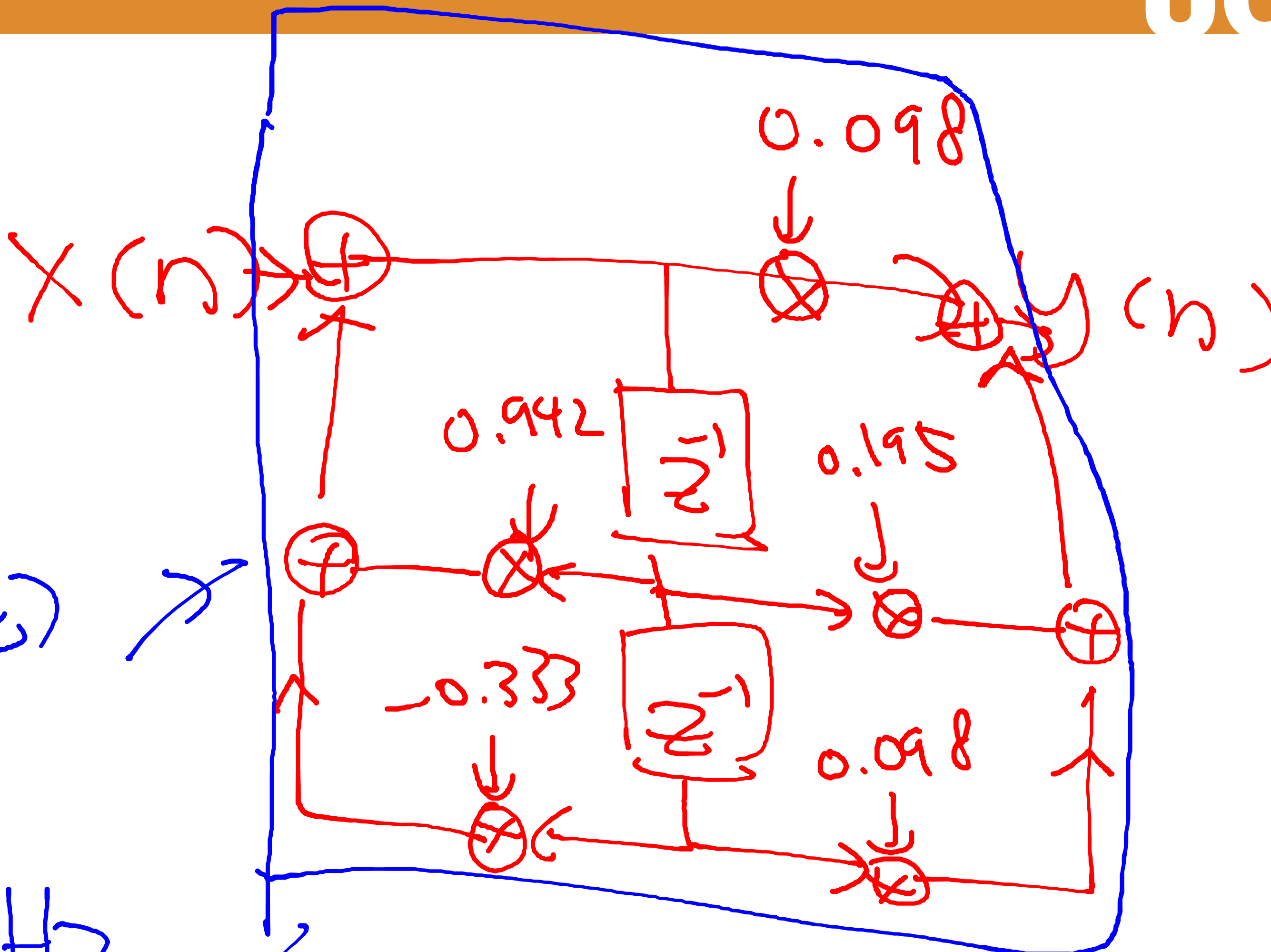
$$= 0.098\, x(n) + 0.195\, x(n-1) + 0.098\, x(n-2)$$

$$\Rightarrow y(n) = 0.098\, x(n) + 0.195\, x(n-1) + 0.098\, x(n-2) + 0.942\, y(n-1) - 0.333\, y(n-2)$$



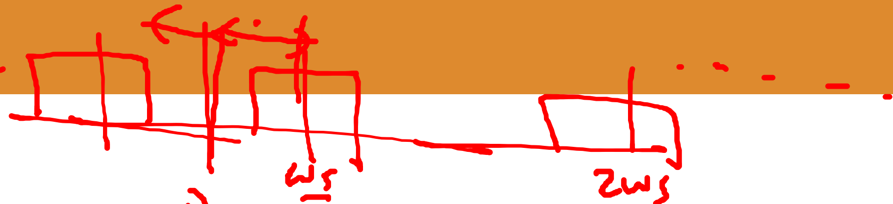
← Canonical form

$H_d(z)$



100 Hz

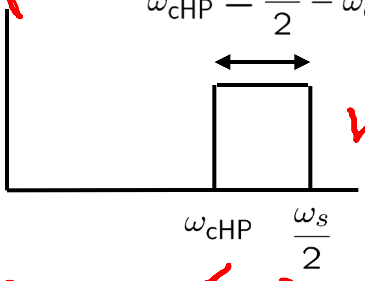
periodic



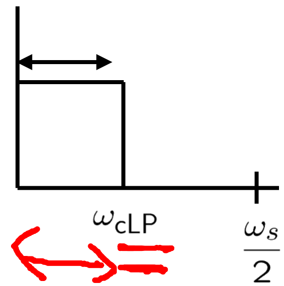
Frequency Transformations for Digital Filters

BPF

$\omega_{cHP} = \frac{\omega_s}{2} - \omega_{cLP}$



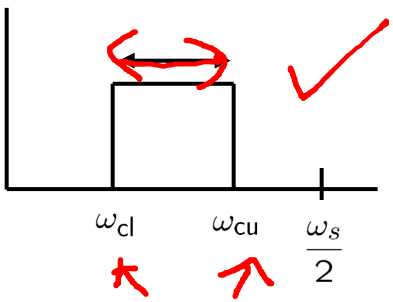
LPF

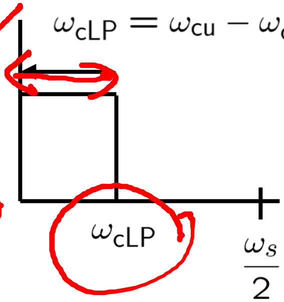


$$\boxed{H_{HP}(z) = H_{LP}(z)\big|_{z\to -z}}$$

BPF

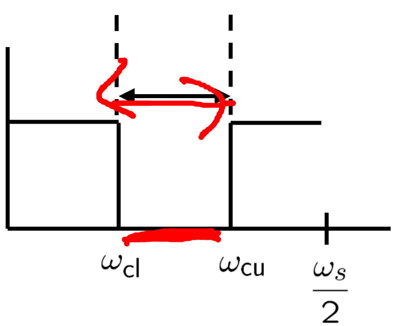
$\omega_{cLP} = \omega_{cu} - \omega_{cl}$

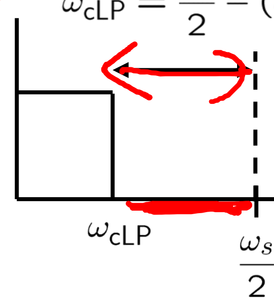




$$\boxed{z_{LP}^{-1} = -\frac{z^{-1}(z^{-1}-\alpha)}{1-\alpha z^{-1}}}$$

$\omega_{cLP} = \frac{\omega_s}{2} - (\omega_{cu} - \omega_{cl})$





$$\boxed{z_{LP}^{-1} = \frac{z^{-1}(z^{-1}-\alpha)}{1-\alpha z^{-1}}}$$

$$\boxed{\alpha = \frac{\cos\left(\frac{\pi(\omega_{cu}+\omega_{cl})}{\omega_s}\right)}{\cos\left(\frac{\pi(\omega_{cu}-\omega_{cl})}{\omega_s}\right)}}$$

Example LPF, HPF, BPF Designs

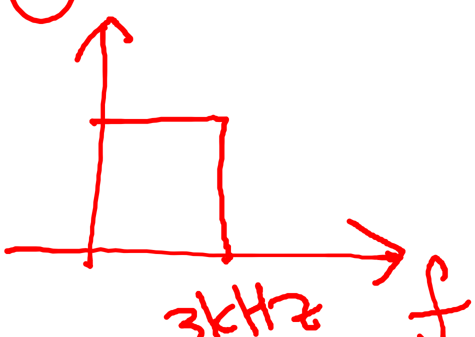
$f_s = 18\text{kHz}$

- A signal sampled at 18kHz is to be filtered using three IIR filters in parallel. The first filter is low pass with a cut-off at 3kHz. The second is a high pass with a cut-off at 6KHz. The third is a band pass with cut-offs at 3 and 6 kHz. Using a second order Chebyshev analogue prototype determine the transfer function of all three filters.

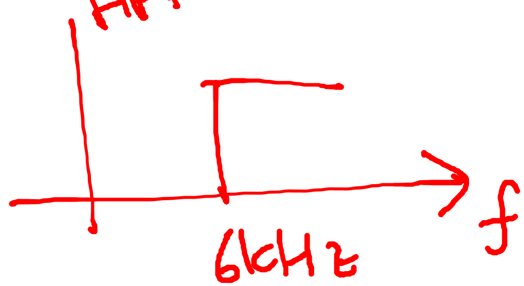
- The 2nd-order Chebyshev filter transfer function with a 0.5 dB ripple is:

$$\boxed{H(s)} = \frac{1}{s^2 + 1.426s + 1.516}$$

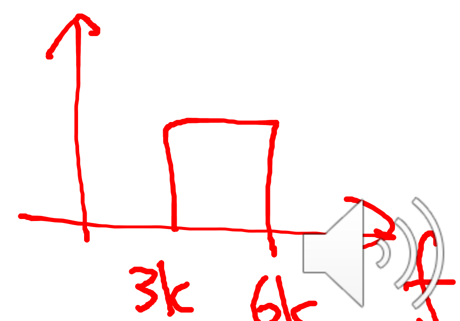
1 LPF



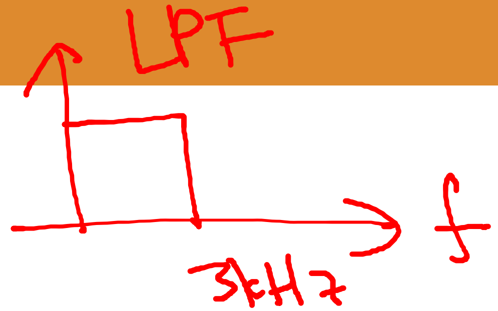
2 HPF



3



bilinear transformation



$$\omega = \frac{2}{T_s}\tan\frac{\Omega T_s}{2} = \frac{2}{(1/18k)}\tan\frac{2\pi(3k)\left(\frac{1}{18k}\right)}{2}$$

$$s = \frac{2}{T_s}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$2\pi(3308) = 20785 \text{ rad } s^{-1}$$

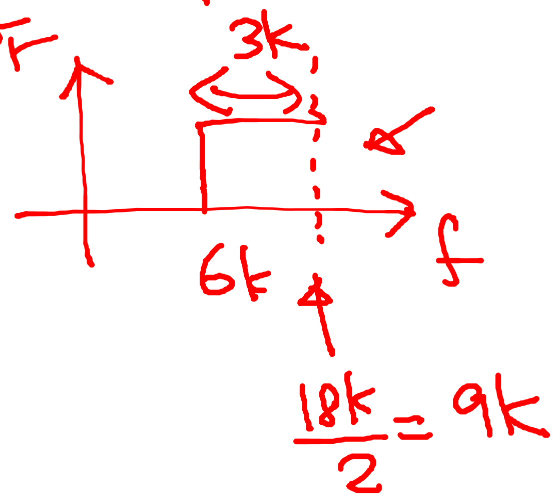
$$H(s) = \frac{1}{\left(\frac{s}{20785}\right)^2 + 1.426\left(\frac{s}{20785}\right) + 1.516}$$

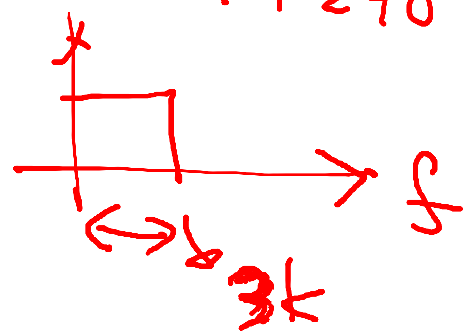
$$\Rightarrow H(z)_{LPF} = \frac{1}{\left[\frac{\left(\frac{2}{18k}\right)\frac{1-z^{-1}}{1+z^{-1}}}{20785}\right]^2 + 1.426\left(\frac{\frac{2}{18k}\frac{1-z^{-1}}{1+z^{-1}}}{20785}\right) + 1.516}$$

$$\Rightarrow H_{LPF}(z) = \frac{0.2409 + 0.4098 z^{-1} + 0.2049 z^{-2}}{1 - 0.4248 z^{-1} + 0.293 z^{-2}}$$

HPF

②

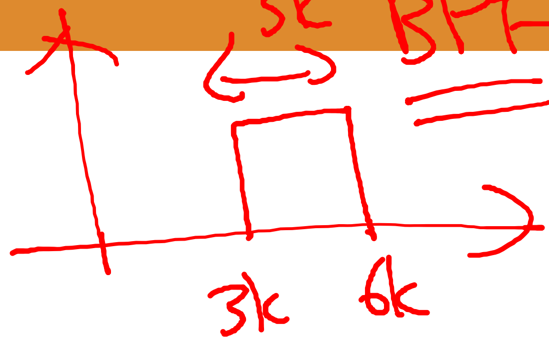


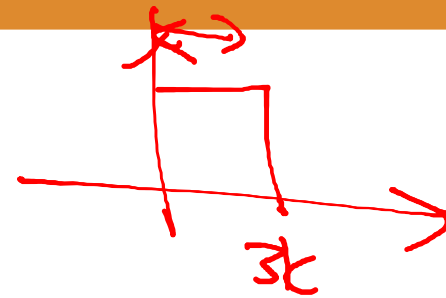


$$H_{HPF}(z) = H_{LPF}(-z)$$

$$= \frac{0.2409 - 0.4098 z^{-1} + 0.2049 z^{-2}}{1 + 0.4248 z^{-1} + 0.293 z^{-2}}$$

③ 3k BPF





$$z^{-1} = \frac{-z^{-1}\left(z^{-1} - \cancel{\alpha}^{\,0}\right)}{1 - \cancel{\alpha z^{-1}}^{\,0}} = -z^{-2}$$

$$\alpha = \frac{\cos\left[\frac{\pi(3k+6k)}{18k}\right]}{\cos\left[\frac{\pi(6k-3k)}{18k}\right]} = \frac{\cos\frac{\pi}{2}}{\cos\frac{\pi}{6}} = \emptyset$$

$$H_{BPF}(z) = H_{LPF}(-z^{-2})$$

$$= \frac{0.2409 + 0.4098(-z^{-2}) + 0.2049\, z^{-4}}{1 - 0.4248(-z^{-2}) + 0.293\, z^{-4}}$$

$$= \frac{0.2409 - 0.4098\, z^{-2} + 0.2049\, z^{-4}}{1 + 0.4248\, z^{-2} + 0.293\, z^{-4}} \quad \#\#$$