# *Communication Systems Modelling*

## Introduction and overview

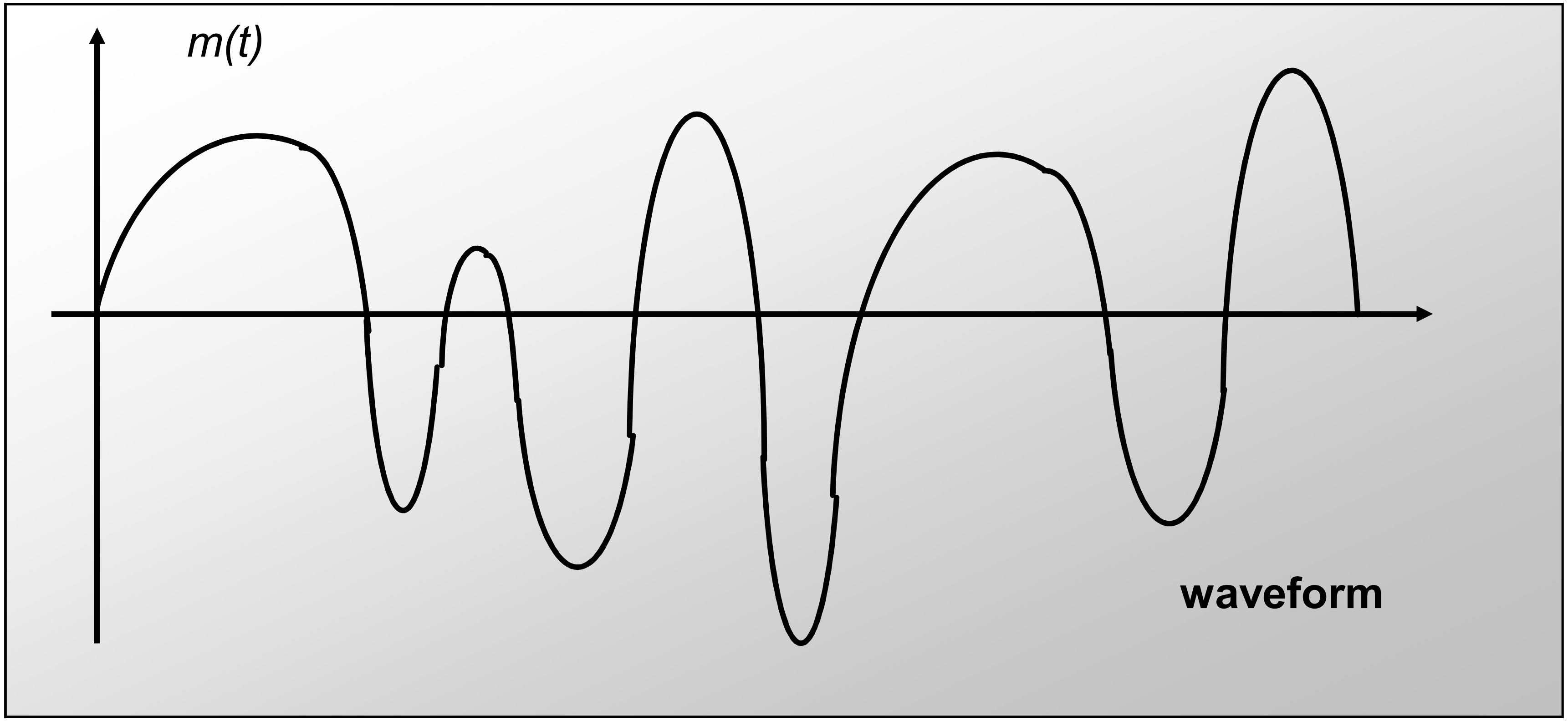
**Prof Yiannis Andreopoulos**

- **Signals, Fourier transform and convolution**
- **Sampling theory and the FFT and Z transforms,**
- **Signal design and analysis**

## Signal Types

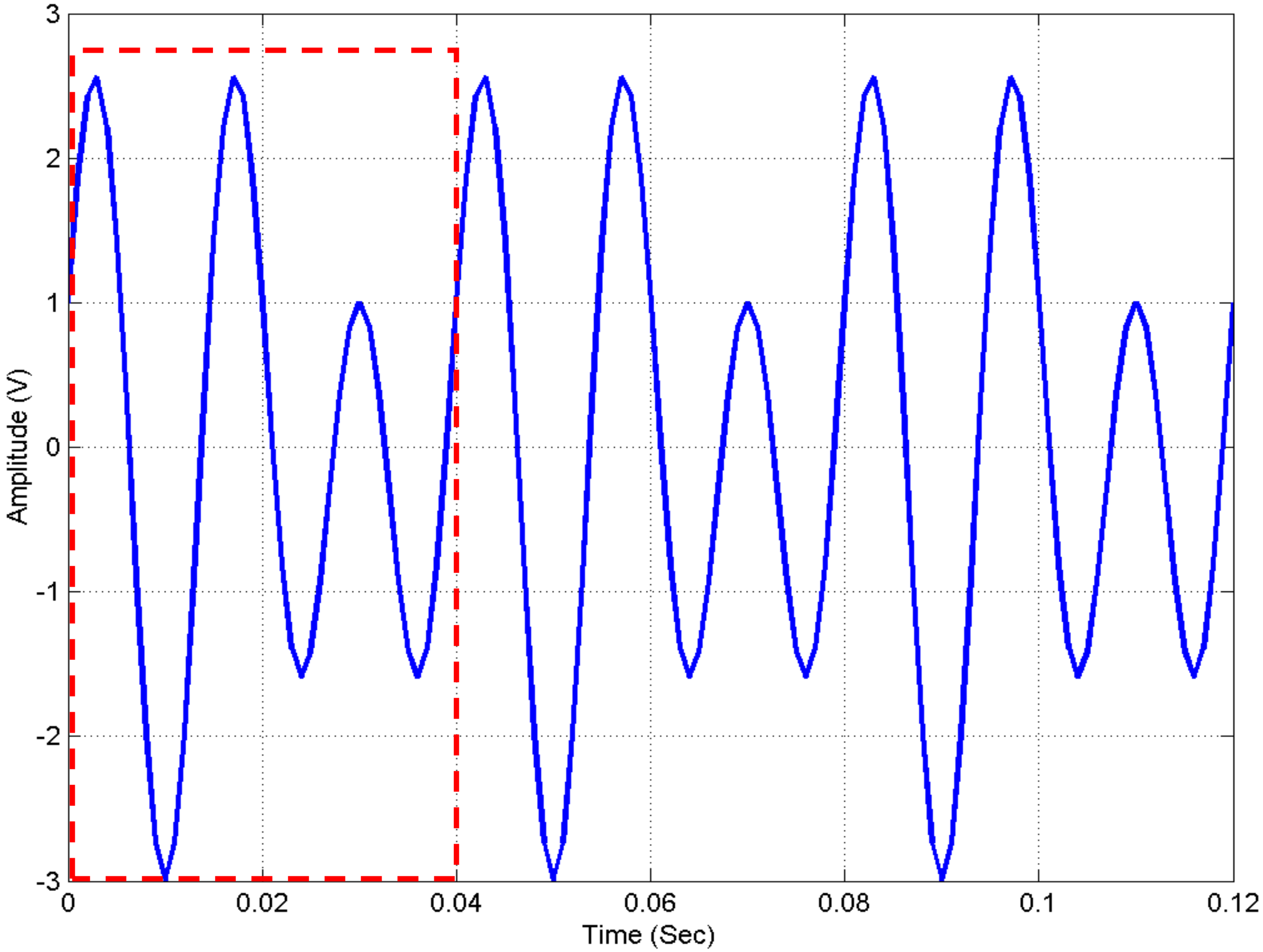
- Random – Deterministic
- Discrete Time – Continuous Time
- Discrete Amplitude – Continuous Amplitude
- Lowpass – Bandpass
- Periodic or non Periodic

# Random

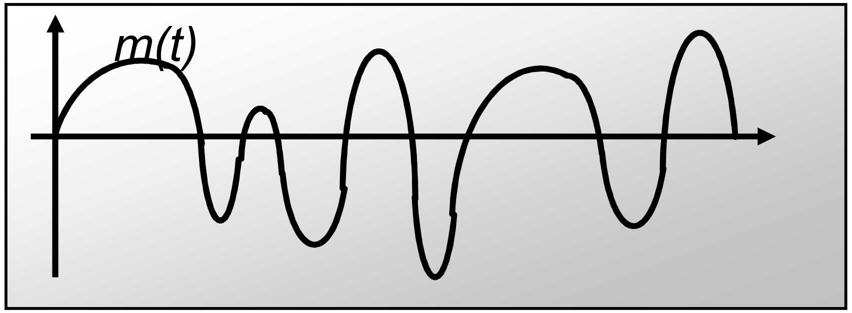


## Deterministic

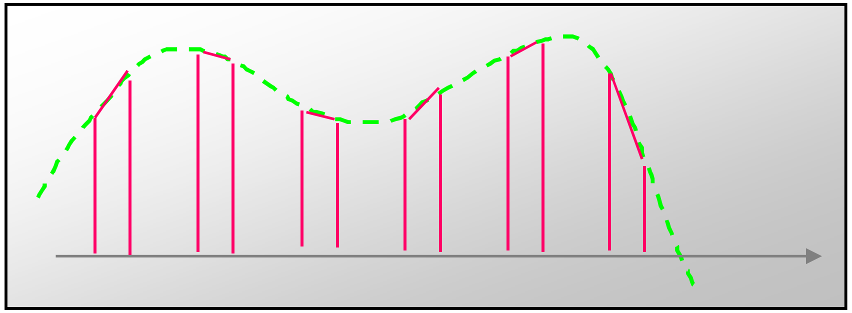
$$v(t) = \cos(2\pi 50t) + 2\cos(2\pi 75t - \pi/2)$$



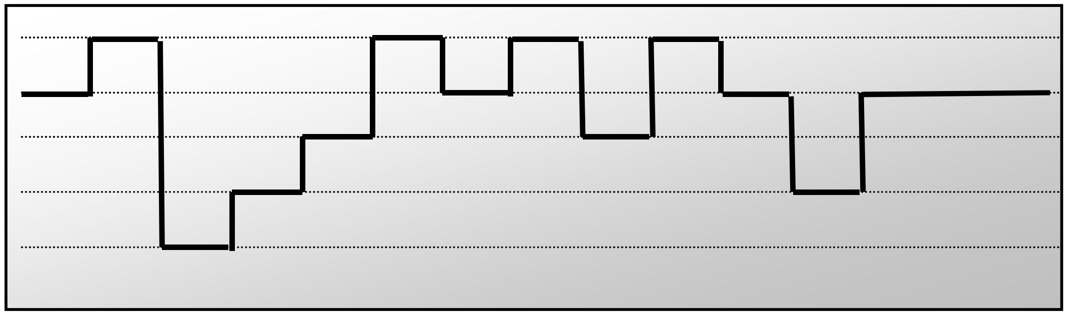
# Discrete - Continuous




- Continuous Time and Amplitude

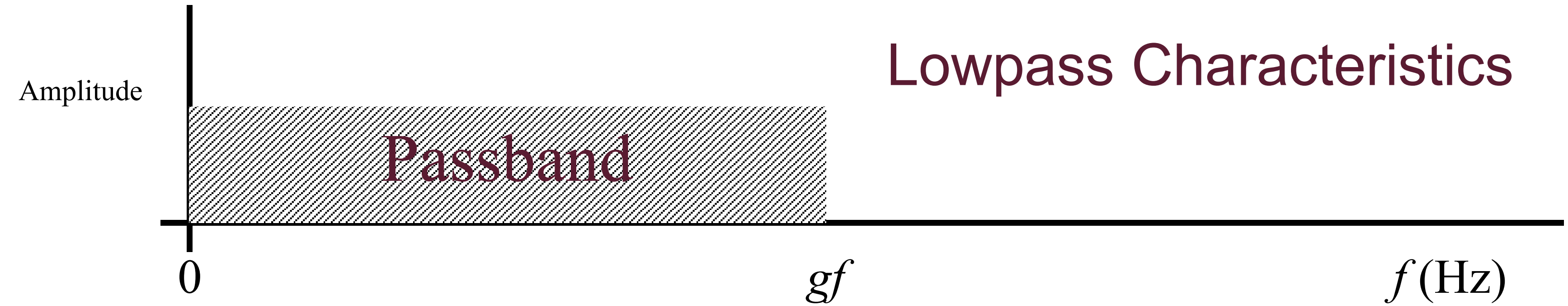


- Discrete Time, continuous Amplitude – PAM signal



- Discrete Time, and Amplitude – Multi-level digital

# Passband

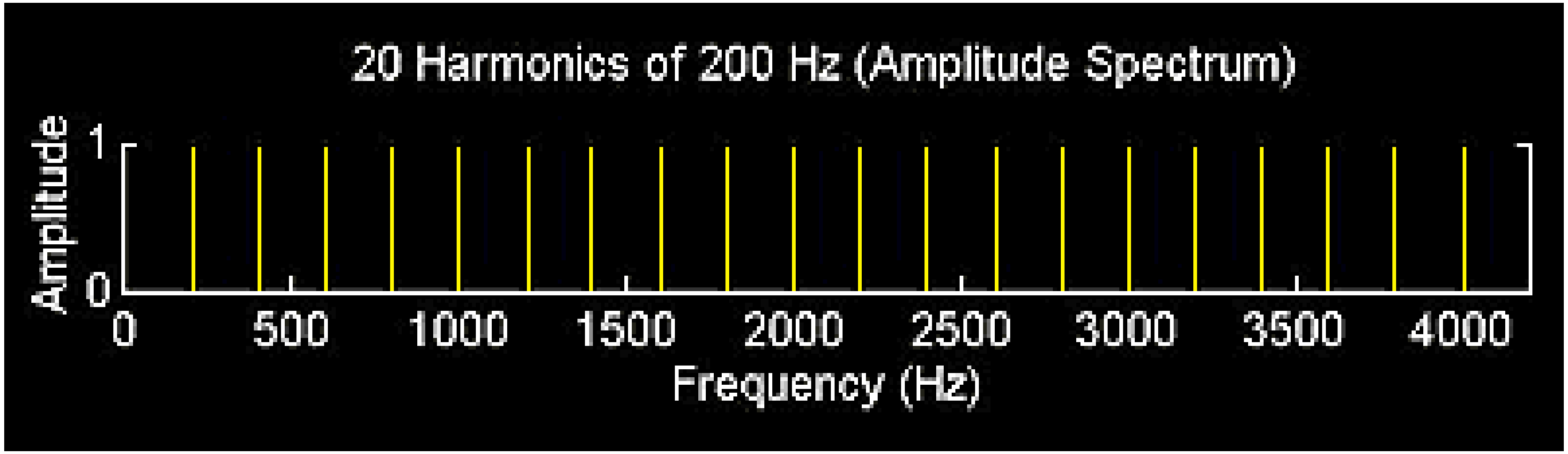


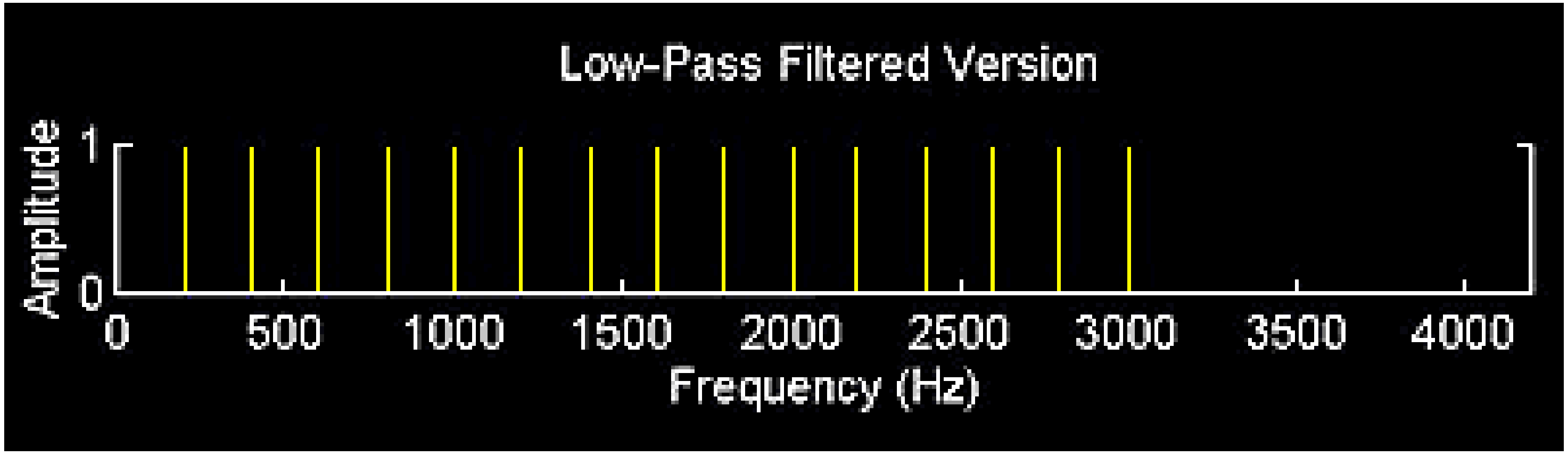
Lowpass Characteristics
Amplitude
Passband
0
gf
f (Hz)


Bandpass Characteristics
Amplitude
Passband
0
mf
nf
f (Hz)

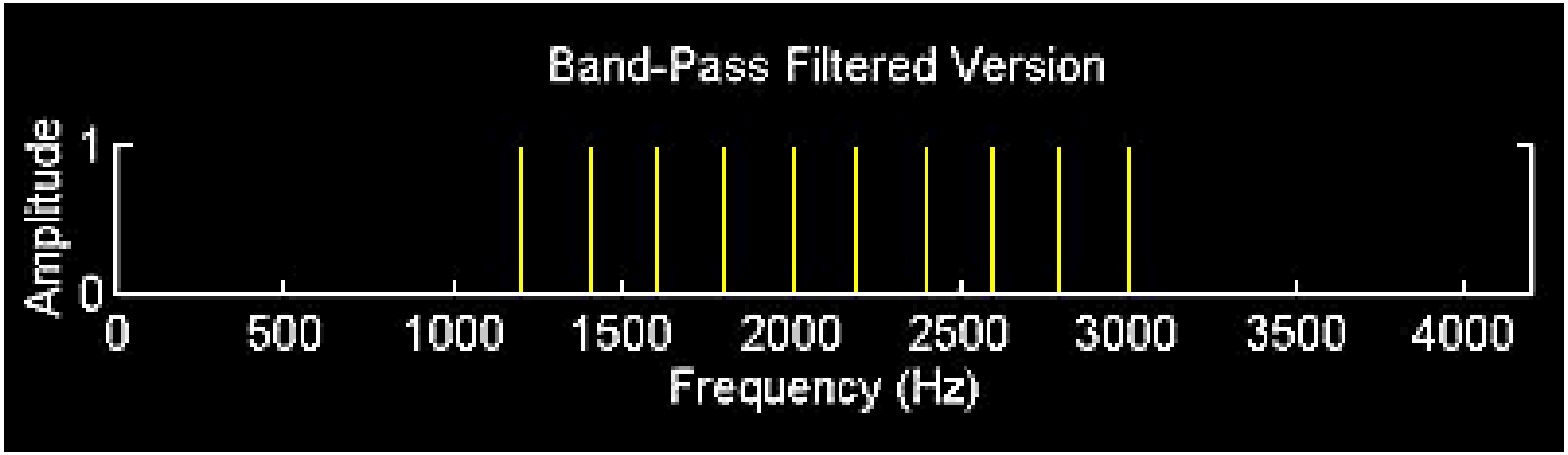

Highpass Characteristics
Amplitude
Passband
0
hf
f (Hz)

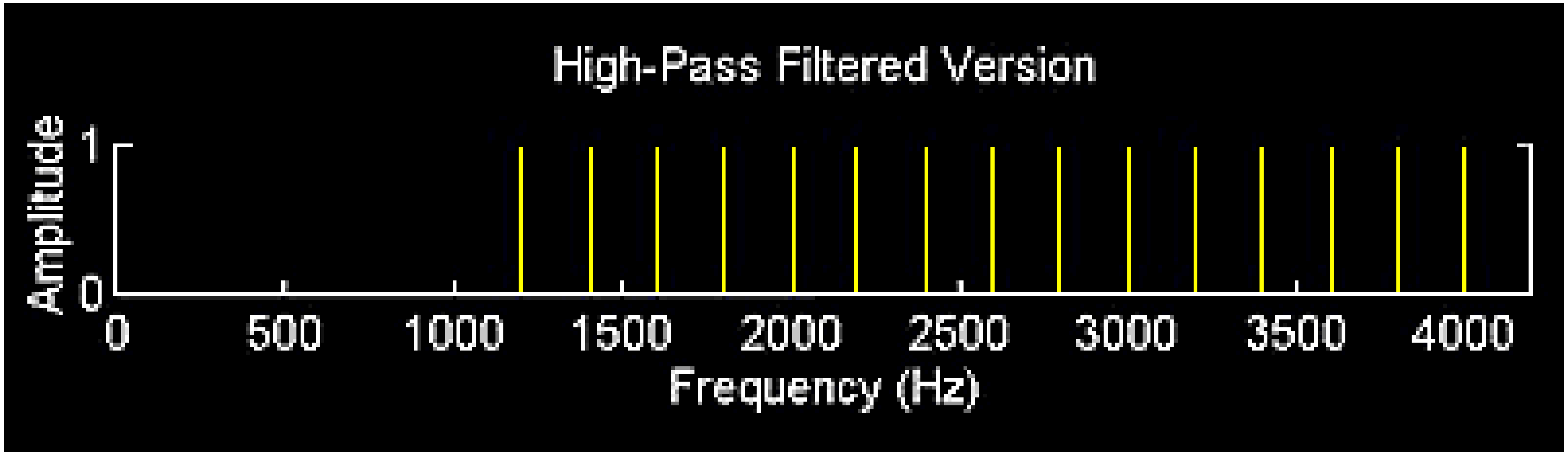
# Sound


20 Harmonics of 200 Hz (Amplitude Spectrum)
Amplitude
1
0
0
500
1000
1500
2000
2500
3000
3500
4000
Frequency (Hz)


Low-Pass Filtered Version
Amplitude
1
0
0
500
1000
1500
2000
2500
3000
3500
4000
Frequency (Hz)

# Sound


Band-Pass Filtered Version
Amplitude
1
0
0
500
1000
1500
2000
2500
3000
3500
4000
Frequency (Hz)


High-Pass Filtered Version
Amplitude
1
0
0
500
1000
1500
2000
2500
3000
3500
4000
Frequency (Hz)

# *Signals*

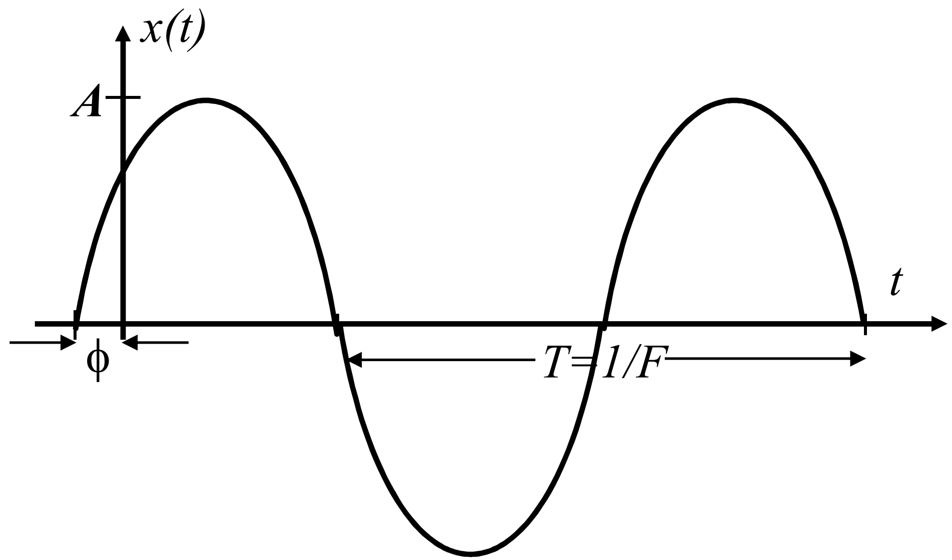
## *in the time and frequency domains*

- Sinusoid
- Complex exponential
- ‘Negative’ frequency
- Rect pulse
- Dirac delta function
- Sinc function
- Raised cosine family
- Gaussian pulse
- Sums of sinusoidal waves

# Sinusoidal wave

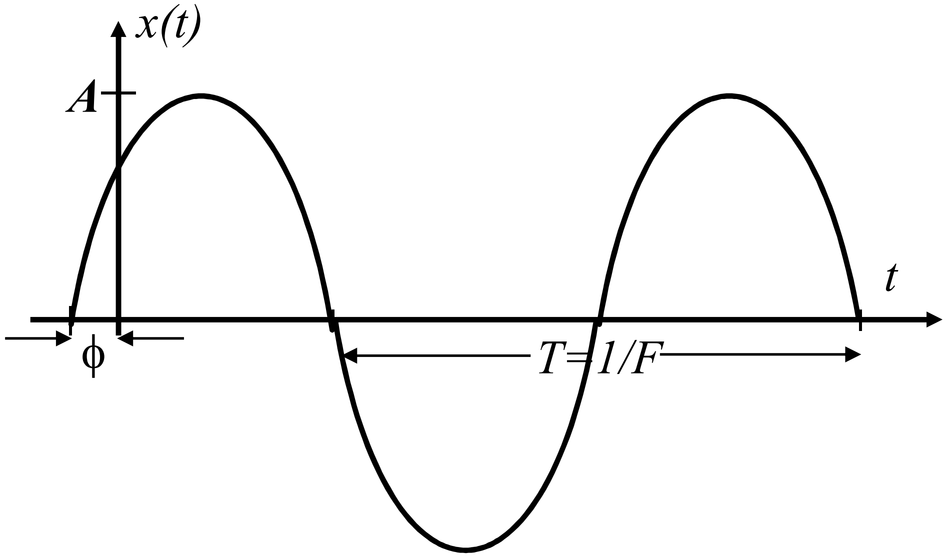
$$\boxed{x(t) = A\sin(2\pi F t + \phi)}$$

- Amplitude *A*
- Frequency *F*
- Period *T*
- Phase $\phi$



# Periodic Signal, if and only if

$$\boxed{x(t+T) = x(t), \quad -\infty < t < +\infty}$$




- Radian Frequency:

  $$\omega = 2\pi F$$

- Fundamental Period:

  $$T = 1/F = 2\pi/\omega$$

# Mathematical Baseline

Period $T = \frac{1}{F} = \frac{2\pi}{\omega} \Rightarrow \omega = 2\pi F$

$\omega \rightarrow$ rad $\Rightarrow \frac{2\pi}{T} = 2\pi F$

$x(t) = A\cos(2\pi F t + \varphi) = A\cos(\omega t + \varphi) = A\cos\left(\frac{2\pi}{T} t + \varphi\right)$

Also, from trigonometry: $\sin(\omega t) = \cos\left(\omega t - \frac{\pi}{2}\right)$

And, Euler's formula: $e^{\pm j\theta} = \cos\theta \pm j\sin\theta$

$$\frac{e^{j2\pi F t} + e^{-j2\pi F t}}{2} = \frac{\cos(2\pi F t) + j\sin(2\pi F t) + \cos(2\pi F t) - j\sin(2\pi F t)}{2} = \cos(2\pi F t)$$

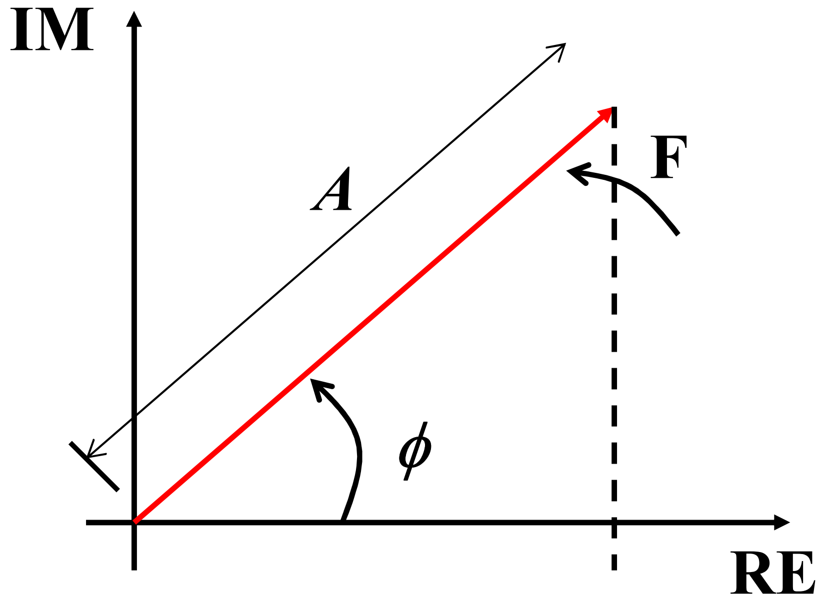
$$\frac{e^{j2\pi F t} - e^{-j2\pi F t}}{2j} = \dots = \sin(2\pi F t)$$

**Please note:** when you see handwriting on the slides, it means you should take notes and ensure you understand the concept/math

# Phasor Representation

$$\boxed{e^{\pm j(\omega t+\phi)} = \cos(\omega t+\phi) \pm j\sin(\omega t+\phi)}$$

- We use *Euler's theorem* to give the phasor representation of the sinusoid
- We can write any sinusoid as the real part of a complex exponential:



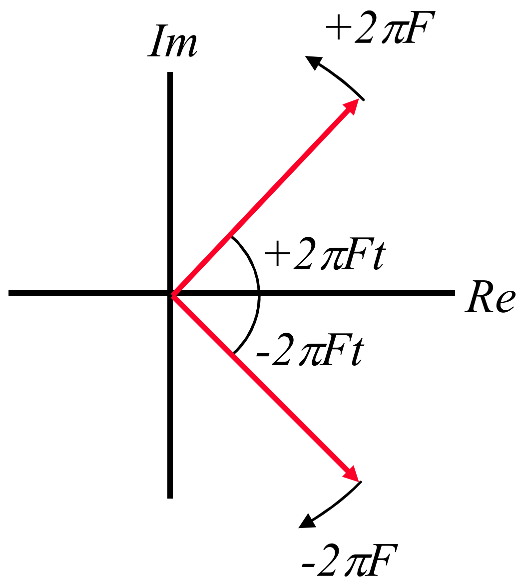

$$A\cos(\omega t+\phi) = A\mathrm{Re}\left[e^{j(\omega t+\phi)}\right]$$
$$= \mathrm{Re}\left[Ae^{j\phi}e^{j\omega t}\right]$$

# Complex exponential form

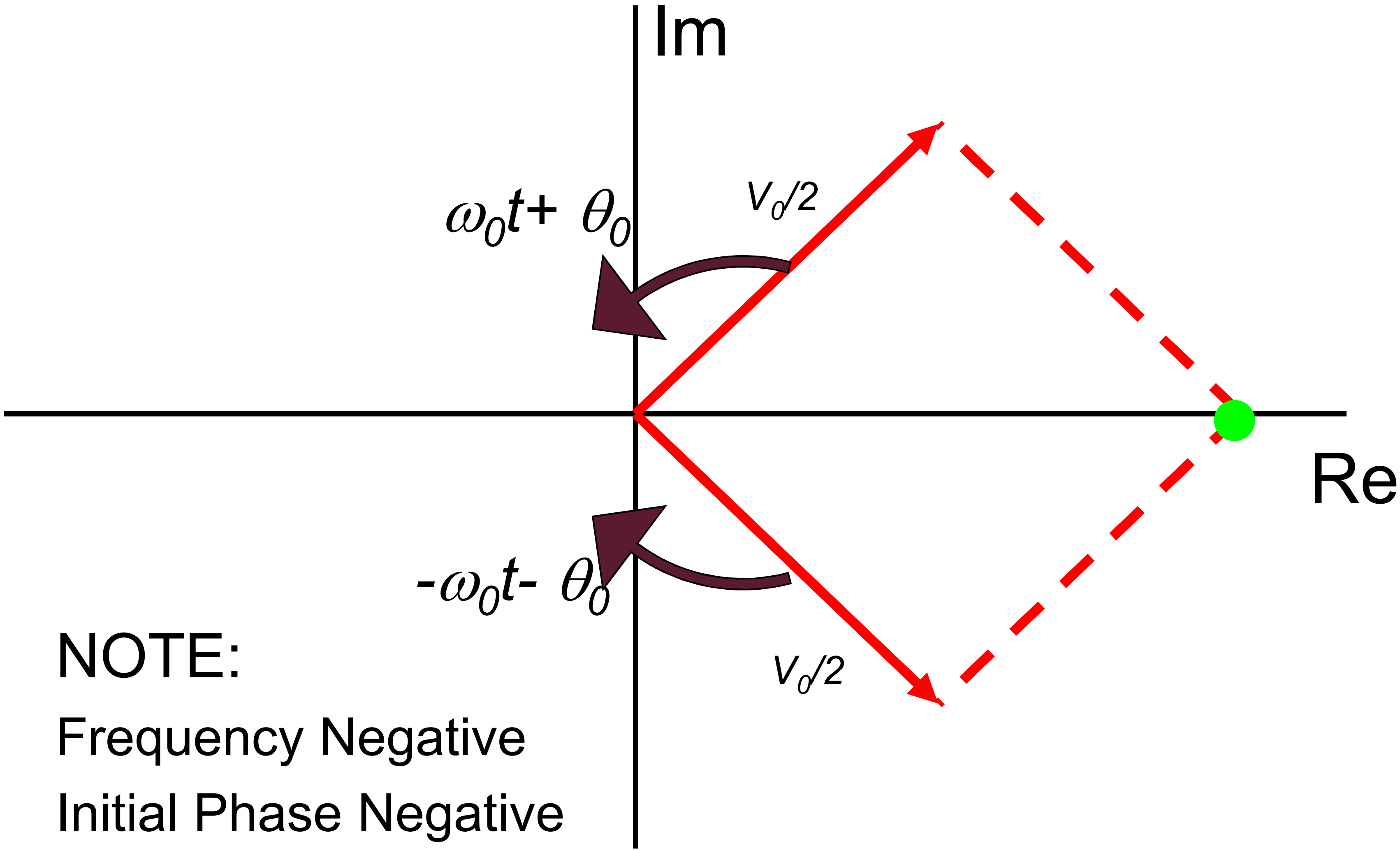
$$V_0 \cos(2\pi Ft) = \frac{V_0 e^{j2\pi Ft} + V_0 e^{-j2\pi Ft}}{2}$$

$$V_0 \sin(2\pi Ft) = \frac{V_0 e^{j2\pi Ft} - V_0 e^{-j2\pi Ft}}{2j}$$

**A cosine wave (or sine wave) corresponds to two contra-rotating phasors and may be considered as the sum of two complex exponential functions, one represents a *positive* frequency term at +*F*, and the other a *negative* frequency term at -*F***



# Complex Exponential Form

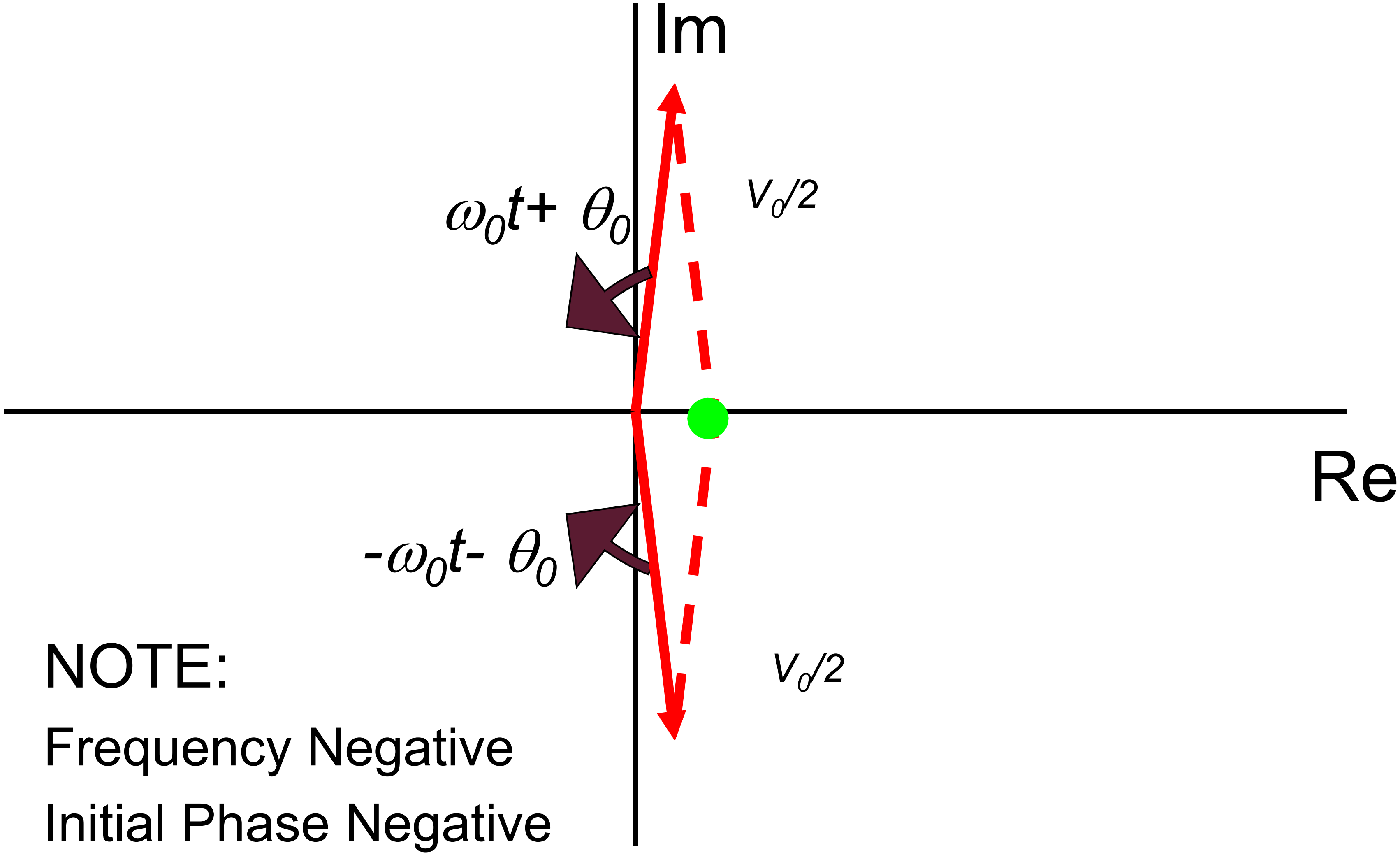



NOTE:

Frequency Negative

Initial Phase Negative

# Complex Exponential Form

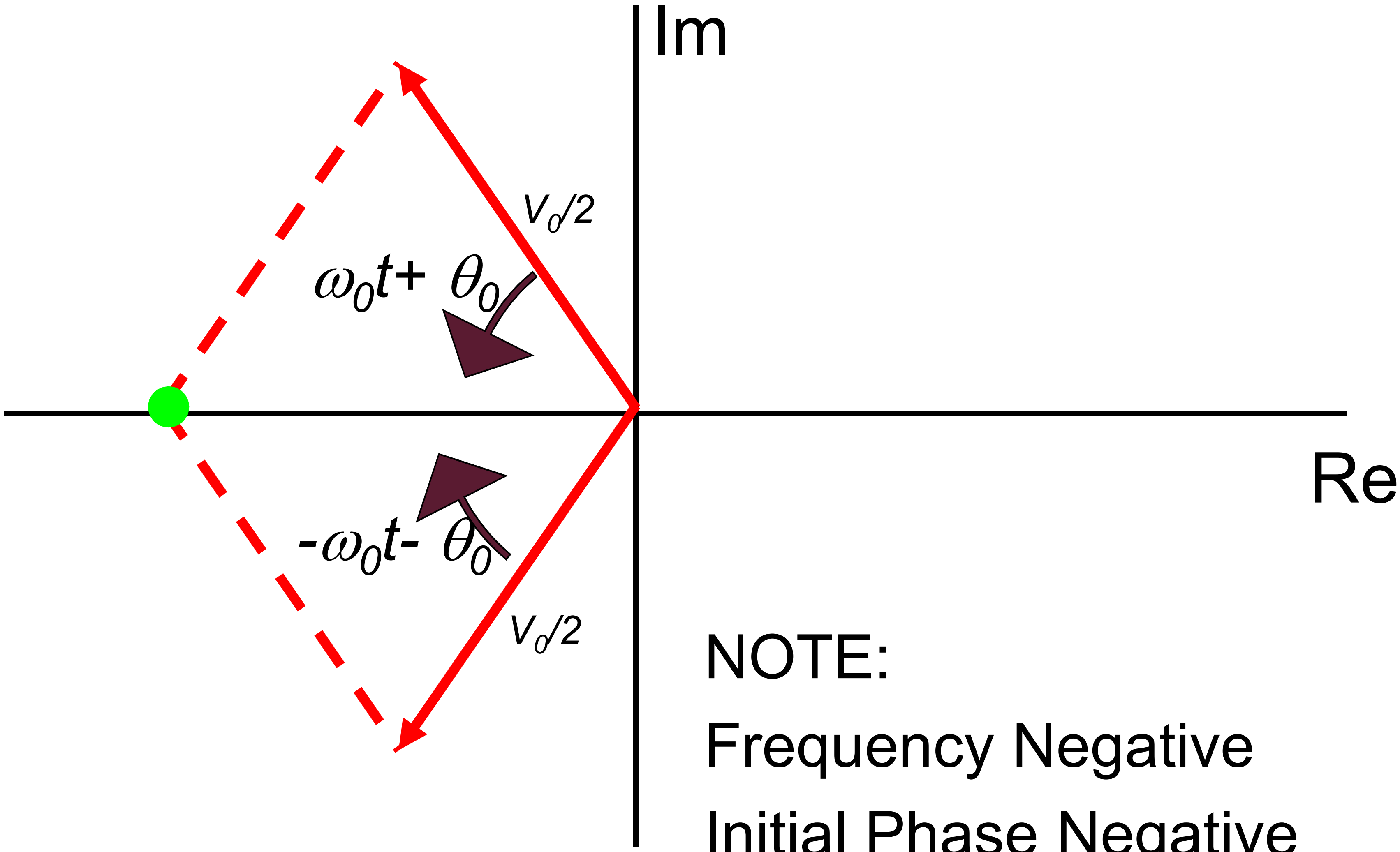



NOTE:

Frequency Negative

Initial Phase Negative

# Complex Exponential Form

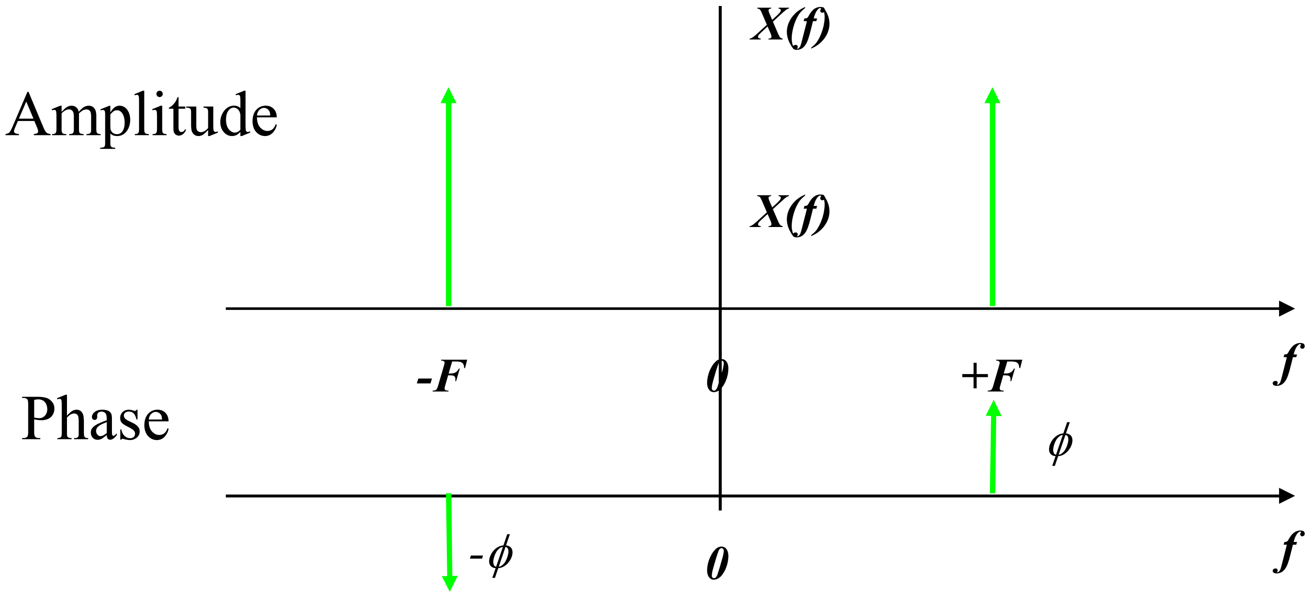



NOTE:

Frequency Negative

Initial Phase Negative

# *Spectrum of a cosine wave*




- The bilateral frequency domain representation contains *positive* and *negative* frequency terms
- In general these are *complex*, with *real and imaginary parts*, or equivalently with *amplitude and phase components*
- For *real* time signals positive and negative frequency components exist in matching, *complex conjugate* pairs of terms

# Signals

Steps: 1. turn sine to cosine
2. use Euler's equation
3. simplify the equation

- Sketch the double sided spectrum of the following signal:

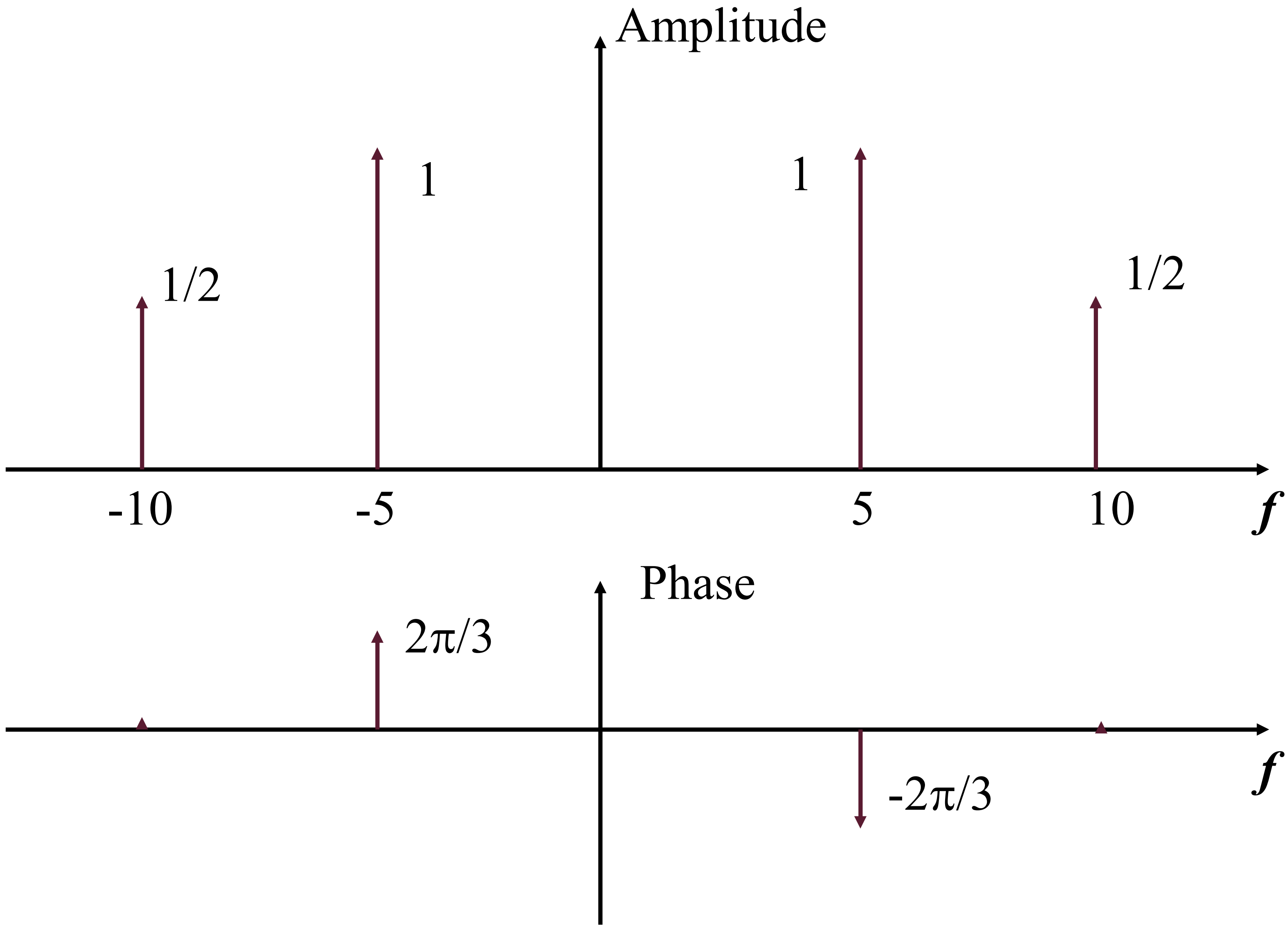
$$x(t) = 2\sin\left(10\pi t - \frac{1}{6}\pi\right) + \cos(20\pi t)$$

# Signals - Answer

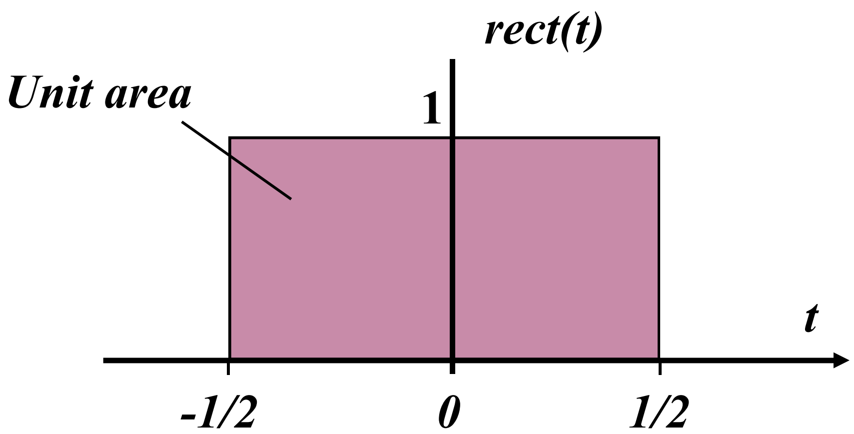
$$x(t) = 2\cos(10\pi t - 2\pi / 3) + \cos(20\pi t)$$

$$x(t) = \mathrm{Re}\left[ 2e^{j(10\pi t - 2\pi / 3)} + e^{j(20\pi)} \right]$$

$$x(t) = e^{j(10\pi t - 2\pi / 3)} + e^{-j(10\pi t - 2\pi / 3)} + \frac{1}{2} e^{j(20\pi t)} + \frac{1}{2} e^{-j(20\pi t)}$$


Amplitude
1/2
1
1
1/2
-10
-5
5
10
f
Phase
2π/3
-2π/3
f

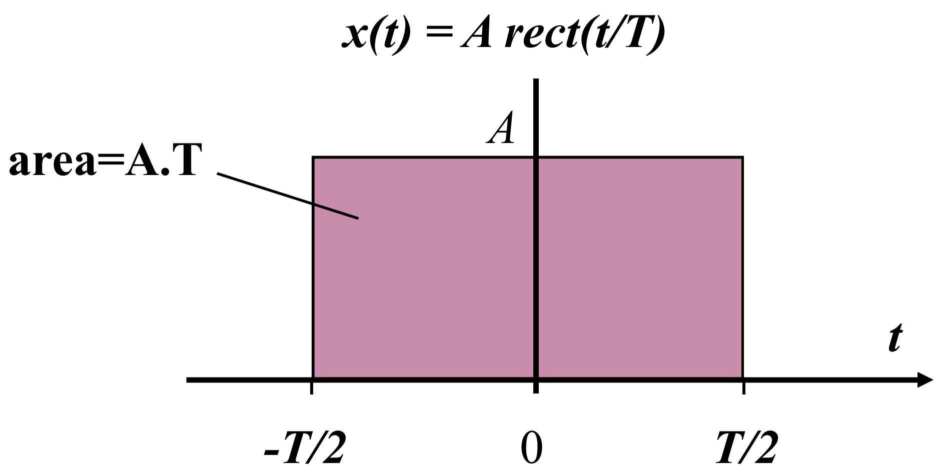
# Rect function




$$rect\,(t) = 1 \quad |t| \le \frac{1}{2}, \quad 0 \quad elsewhere$$

- The rect function is an important building block for the analytic description of digital signals in the time domain

# Rect function - scaled

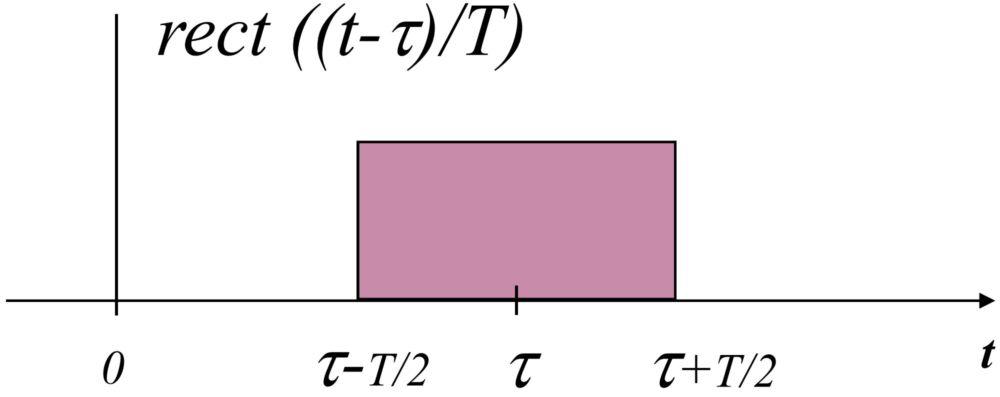



$$A.rect\left(\frac{t}{T}\right) = A, \quad |t| \le \frac{T}{2}, \quad 0 \quad elsewhere$$

- The rect function has been scaled in amplitude, to A, and in time duration or width, to T.

# Translation and scaling

### *the Rect function shifted and time scaled*

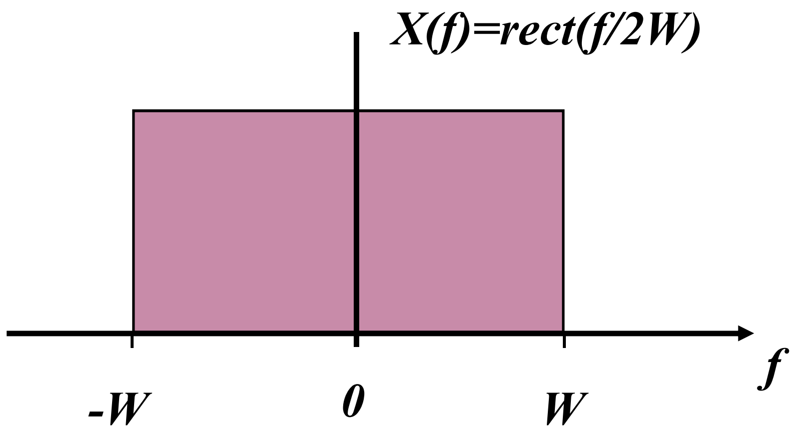



$$rect\left(\frac{t-\tau}{T}\right)=1,\quad \tau-\frac{T}{2}\le t\le \tau+\frac{T}{2}$$

- **The time scaled and shifted rect function has width T and is centred on $t=\tau$.**

# *Frequency domain rect function*

### *- an ideal low-pass spectrum*

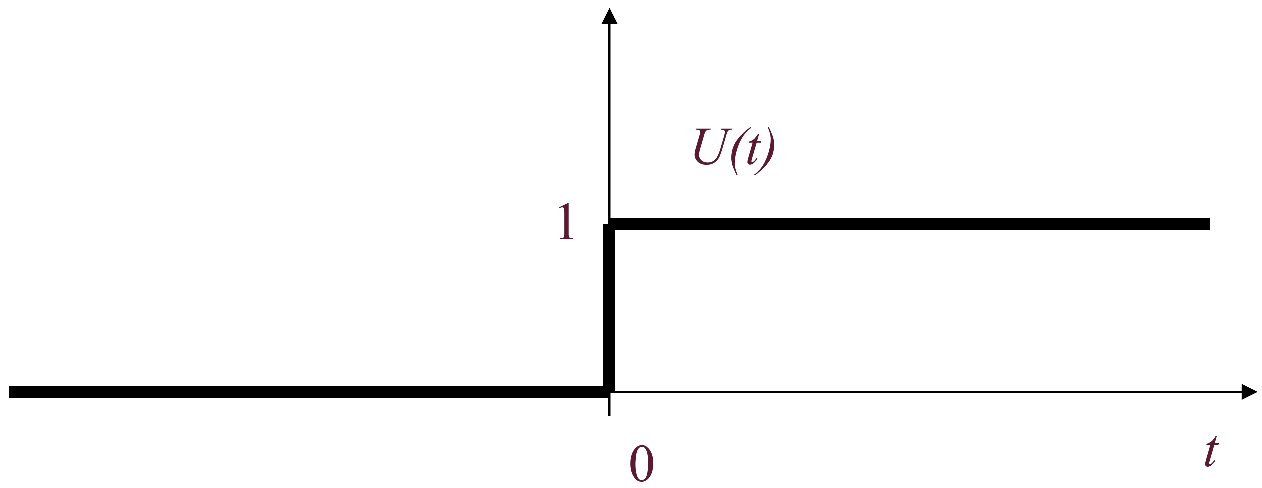



- This is an ideal low-pass function of bandwidth $W$, strictly bandlimited to $|\,f\,|<W$

## Unit Step Function

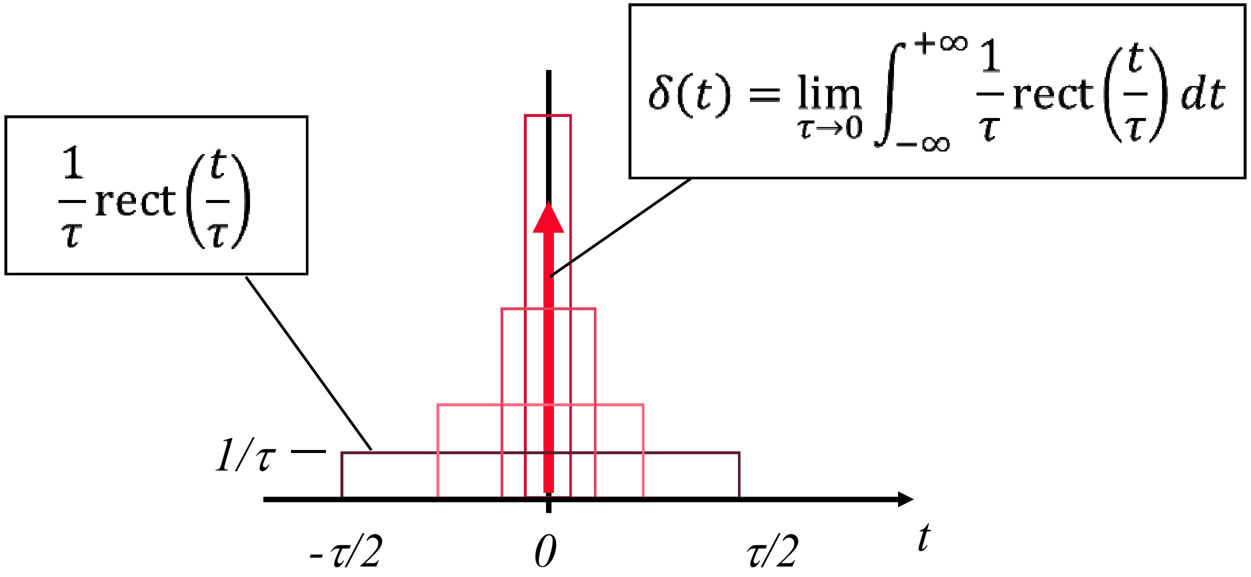
- *U(t)* is the unit step function
- This term denotes a function as follows

$$U(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$$



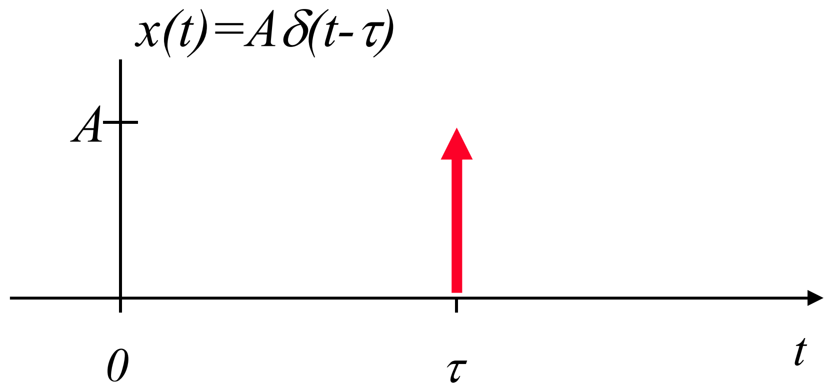
# Dirac delta function

***as limit of a sequence of rect functions***




- With the $1/\tau$ amplitude factor and width $\tau$ each rect function $\tau$ approaches $0$ the amplitude increases, with the area remaining constant; the limit function is the Dirac delta function of unit weight

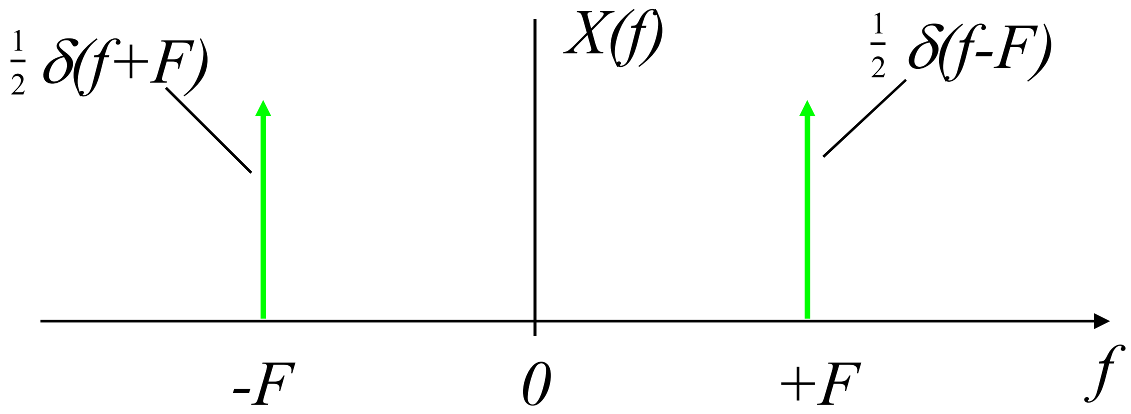
# *Time-shifted and amplitude scaled delta function*




- Note that the $-\tau$ in the function argument induces a shift to the right, in the *positive* direction
- The amplitude factor *A* means that rather than unit area the delta function has area *A*

# *Frequency domain delta functions*

### *- spectrum of a cosine wave*

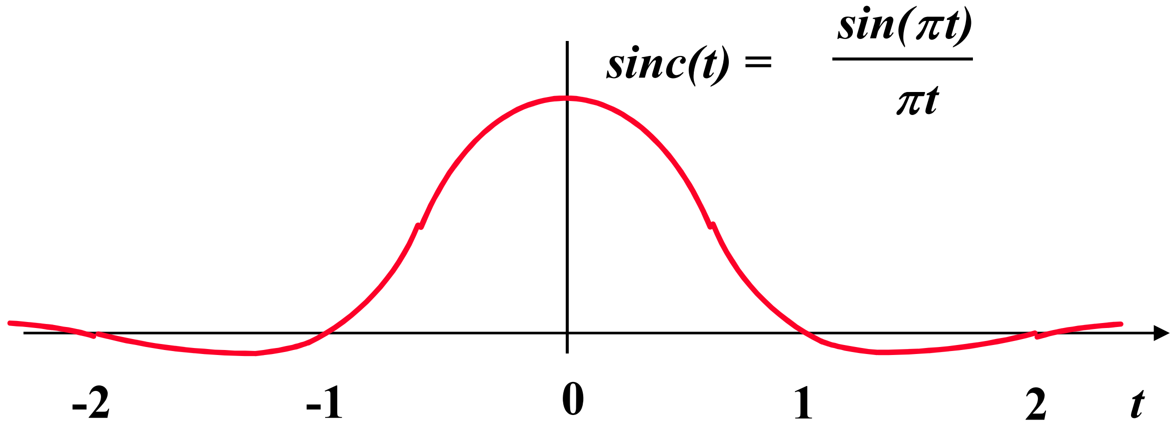



$$x(t) = \cos(2\pi F t)$$

$$X(f) = \tfrac{1}{2}\,\delta(f - F) + \tfrac{1}{2}\,\delta(f + F)$$

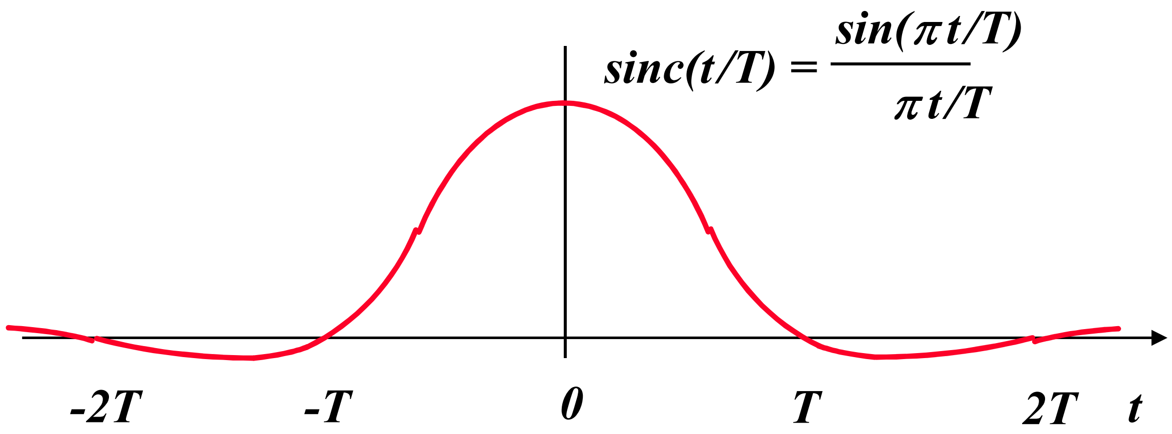
- The spectrum of a cosine wave, seen previously, may be represented analytically as the sum of two frequency domain delta functions

# Sinc function



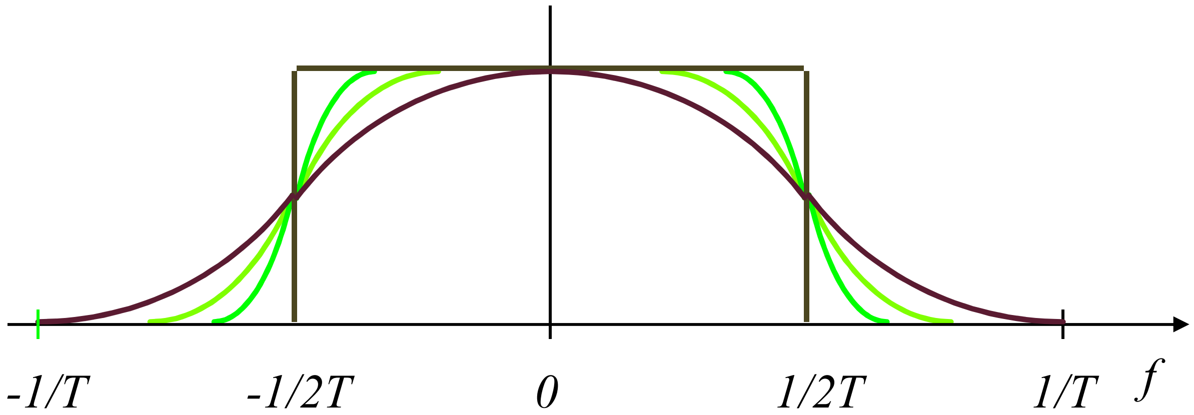

- The sinc function has zeros at non-negative integer values of the argument
- It decays asymptotically at the rate 1/|t|

# Scaled sinc function




- The sinc function scaled on the time axis such that the zeros occur at multiples of |T|

# Raised cosine functions




- Introducing a cosinusoidal transition to the ‘steps’ of a rect function gives a ‘raised cosine’ function
- The 100% raised cosine function has finite support (i.e. is non-zero over twice the interval of the original rect function) and corresponds to a period of a cosine function ‘raised up’ so that it is non-negative
- Other members of the raised cosine family are intermediate between the rect function and the 100% raised cosine function

# Raised cosine functions: Mathematical definition

$$x(t) = \mathrm{sinc}\left(\frac{t}{T}\right)\frac{\cos\left(\frac{\pi\beta t}{T}\right)}{1-\frac{4\beta^2 t^2}{T^2}}$$

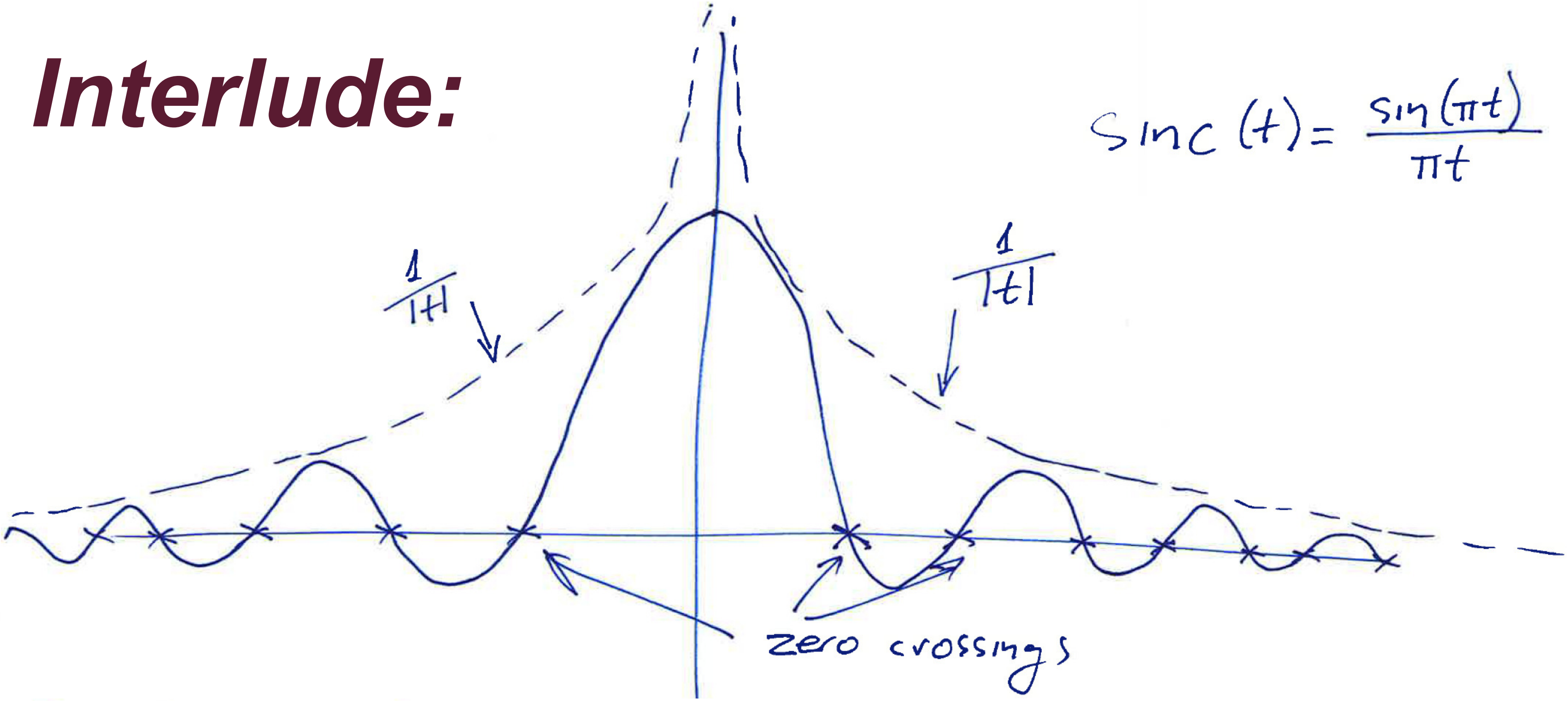
$\beta$ : "roll-off" factor

$\beta \to 0 \Rightarrow$ approaches a rect() function

$\beta \to 1 \Rightarrow$ 100% raised cosine

x(t) = sinc(t/T) which is rect function in frequency

# Interlude:



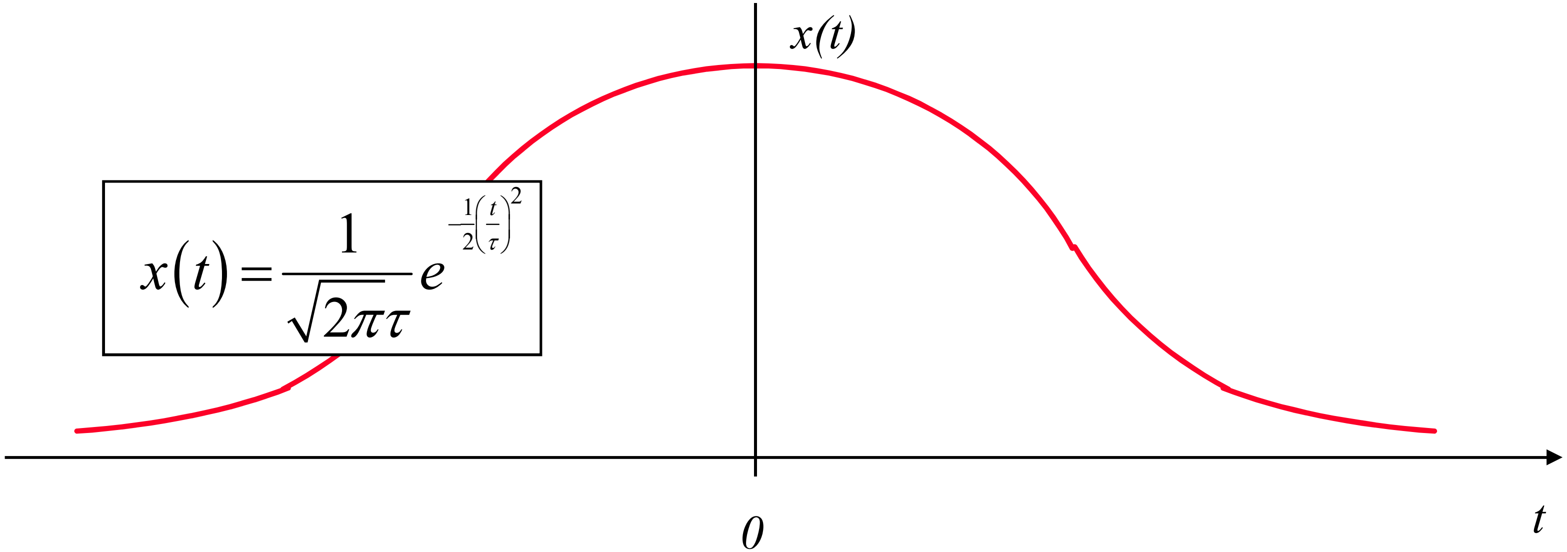

$$\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$

Raised cosine function:

$$X(f) = \mathrm{rect}\left(\frac{f}{2/T}\right) \frac{1}{2}\left[1 + \cos(\pi f T)\right]$$ in frequency

where: $\mathrm{rect}\left(\frac{f}{2/T}\right) = 1$ if $|f| \leq \frac{2/T}{2} = \frac{1}{T}$, $\emptyset$ elsewhere

# Gaussian pulse

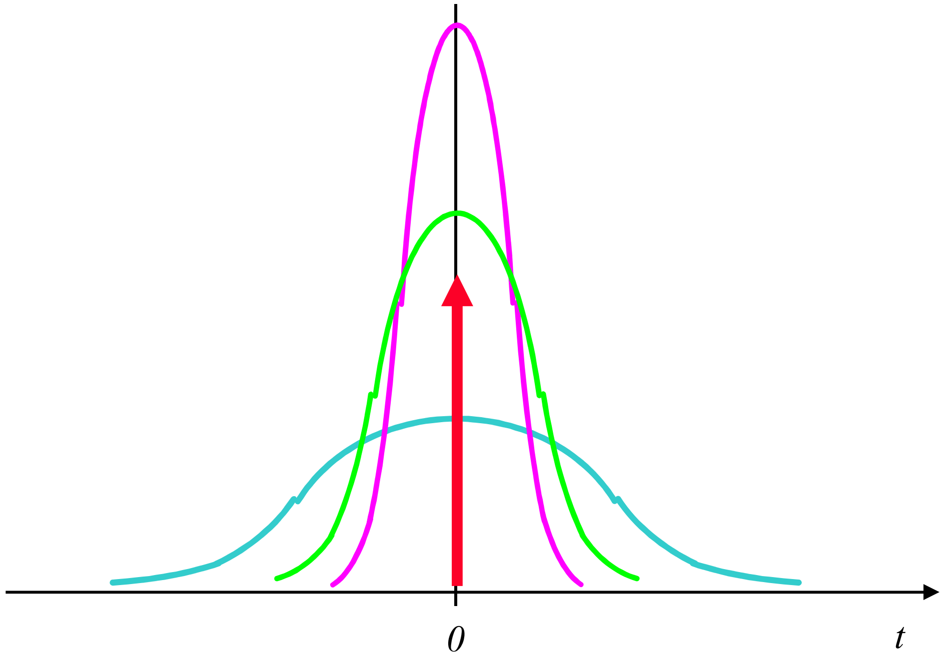



$$x(t) = \frac{1}{\sqrt{2\pi\tau}} e^{-\frac{1}{2}\left(\frac{t}{\tau}\right)^2}$$

- The Gaussian function is commonly encountered when dealing with random processes but is used also as a signal model, both in the time domain - for pulses - and in the frequency domain - for spectra
- Here $\tau$ corresponds to the rms width of the pulse, which means that much of the pulse area lies in the interval ***t*=±**$\tau$

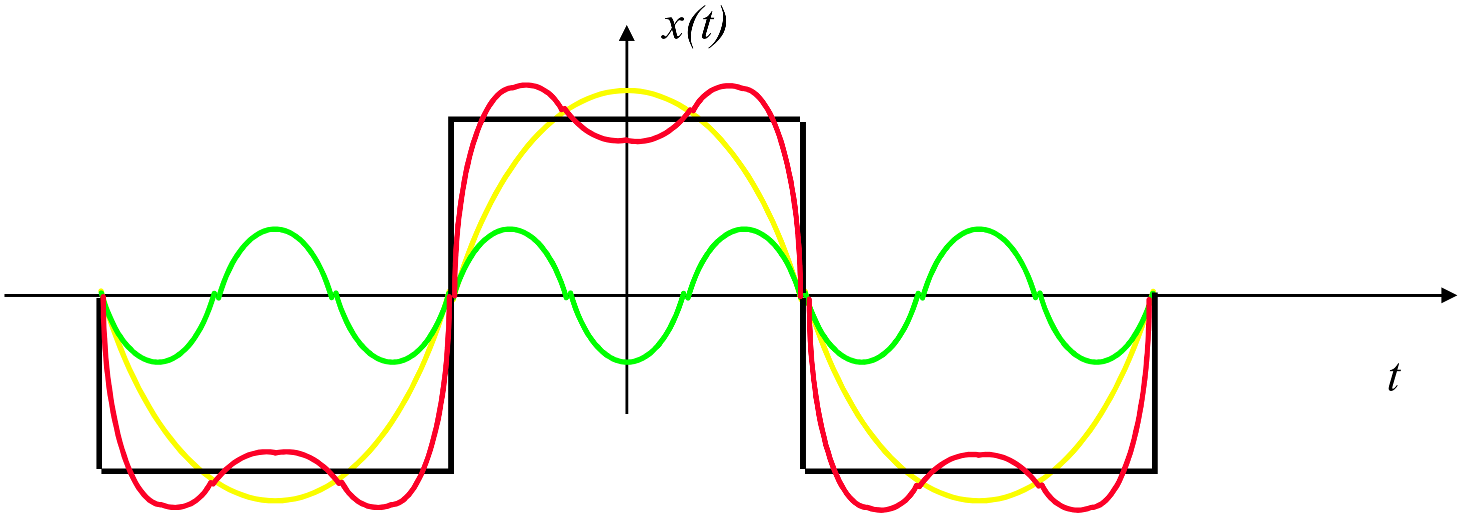
# The Delta function

### *limit of a sequence of Gaussian functions*




- As $\tau$ tends to ***0*** the amplitude at the origin of the Gaussian function increases, with the area remaining constant at unity
- The limit function is thus the *generalised function* - the Dirac delta function

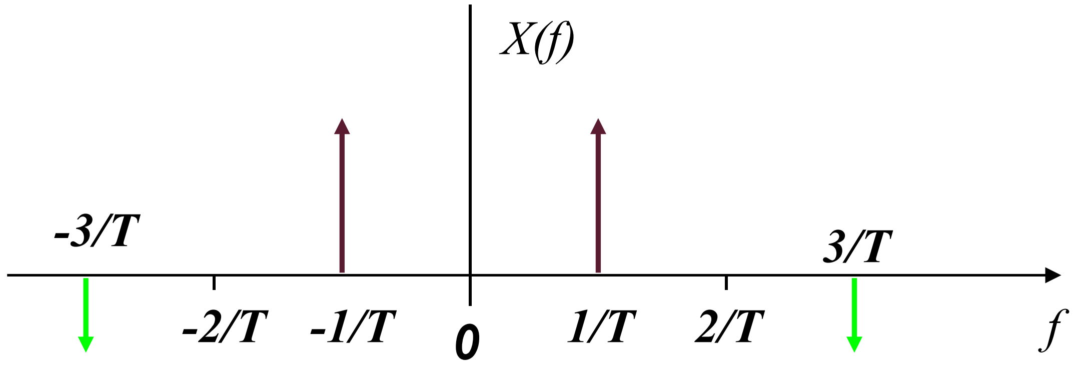
# Sums of sinusoidal signals




- A square wave may be synthesised as a sum of sinusoidal components comprising the fundamental component and all odd harmonics, with amplitudes decreasing with harmonic number
- Here we show the sum of just the fundamental and the third harmonic; the emergence of a square wave is clear even from this most limited sum!

# Sum of sinusoids

## - a frequency domain view



# Summary

- Sinewaves
- Complex exponential
- ‘Negative’ frequency
- Rect pulse
- Dirac delta function
- Sinc function
- Raised cosine family
- Gaussian pulse
- Sums of sinusoidal waves