Convolution

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Summary

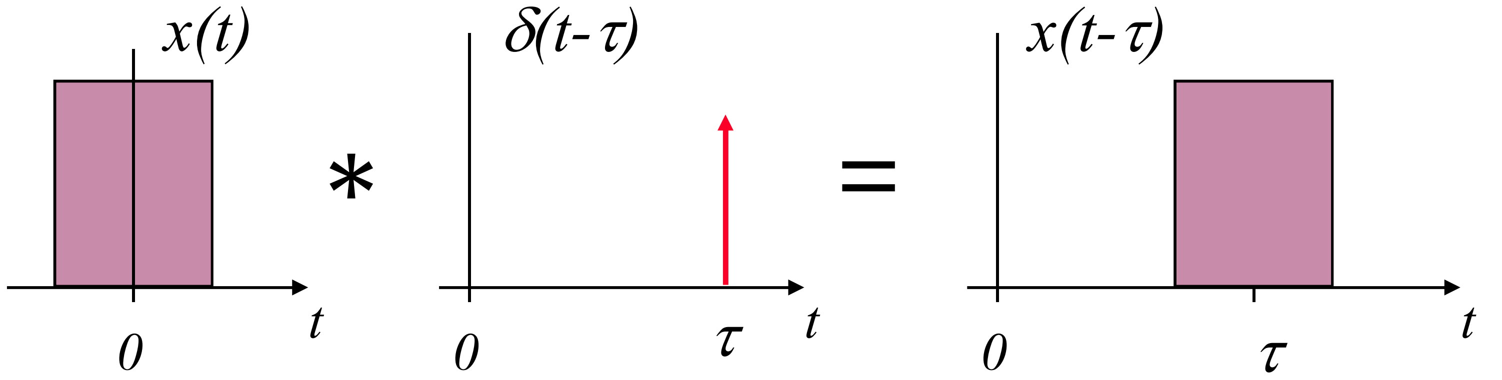
Convolution:

formal definition

- The convolution integral defines a binary operator, indicating how two functions *x(t)* and *y(t)* combine to form the new function *z(t)*
- The convolution operator indicated by $*$ is commutative

$$
\begin{aligned}
z(t) &= x(t) * y(t) \\
&= \int_{-\infty}^{\infty} x(u)\,y(t-u)\,du \\
&= \int_{-\infty}^{\infty} y(u)\,x(t-u)\,du \\
&= y(t) * x(t)
\end{aligned}
$$

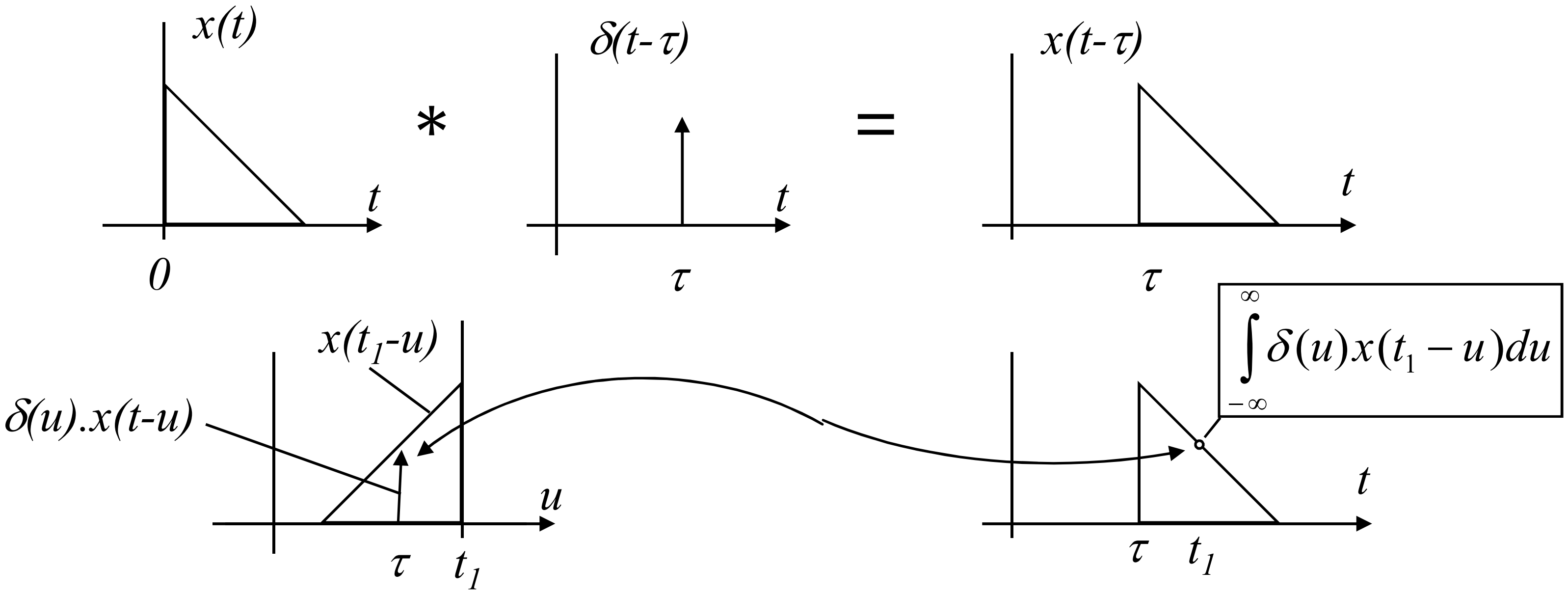
Convolution with a delta function



- Convolving a function *x(t)* located at the origin with a delta function located at $t=\tau$ causes the function to be re-located to $t=\tau$ to produce $x(t-\tau)$

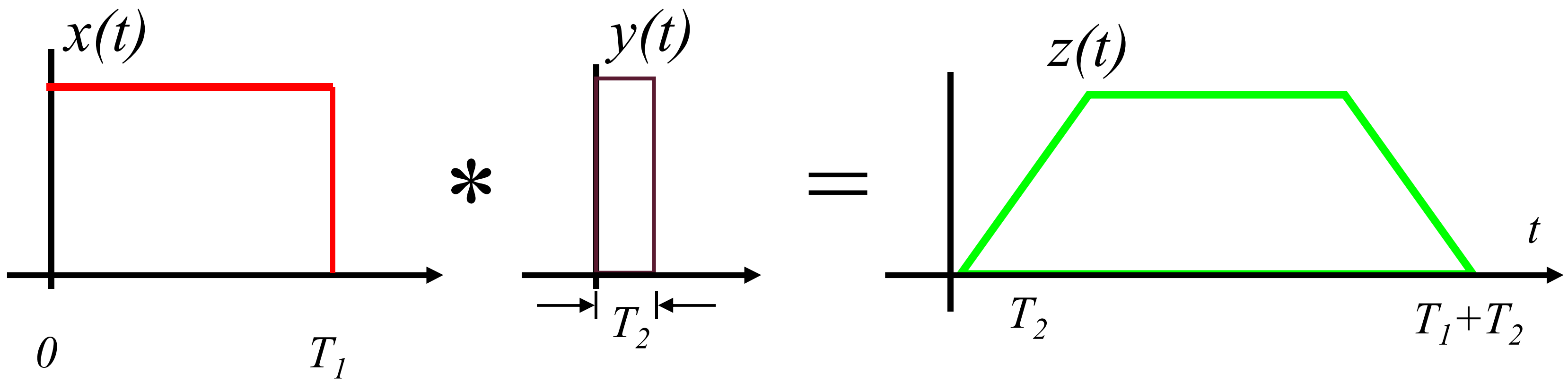
Convolution with a delta function

a graphically perspective



- **Area of delta function, weighted by function value, gives value of convolution $t=t_1$**
- **Considering all values of t, the function x(t) is relocated to the point $t=\tau$**

Graphical convolution more generally



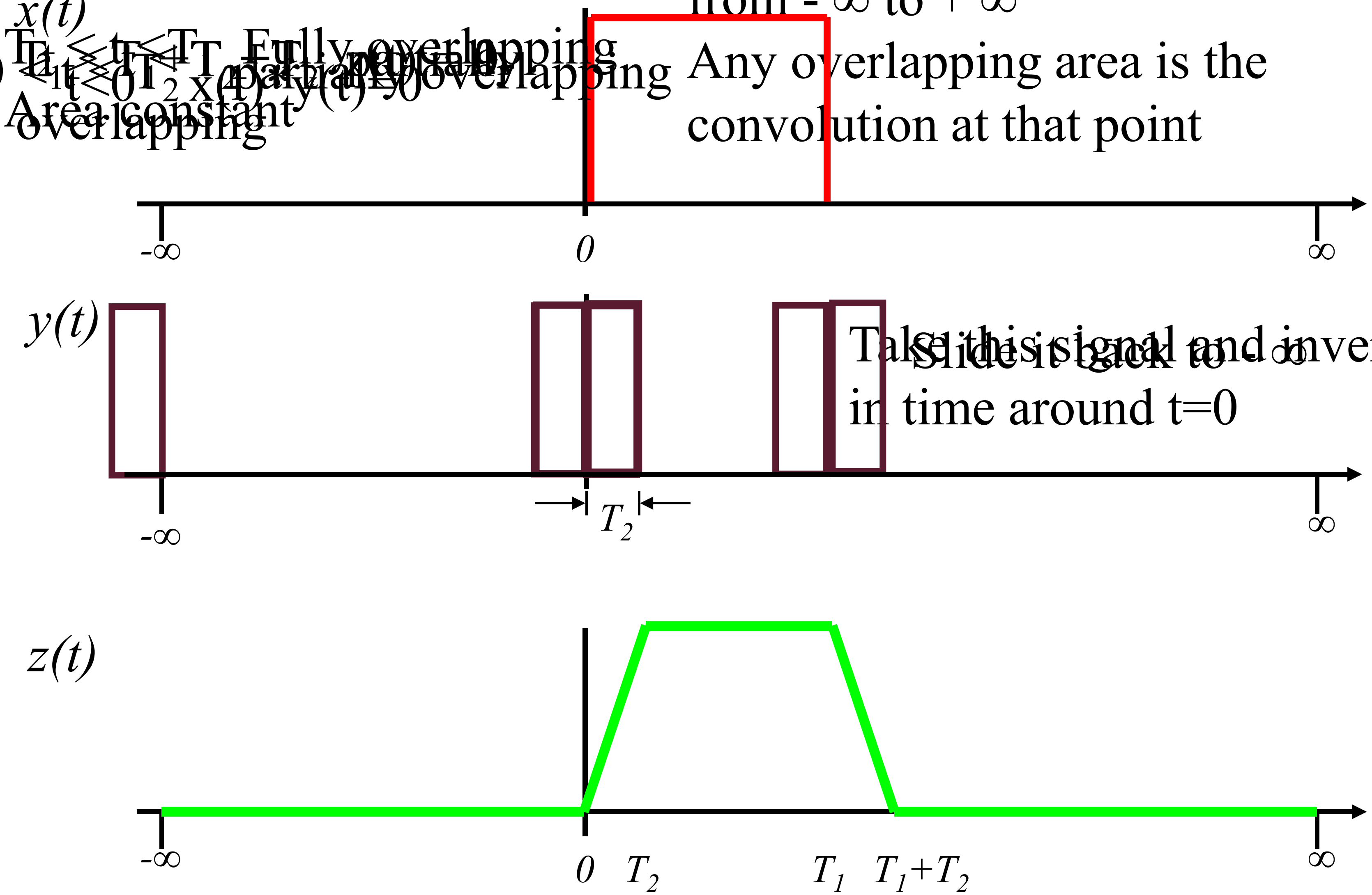
- $x(t)$ of duration T_1 convolves with $y(t)$ of duration T_2 to produce the new functions z(t) of duration T_1+T_2
- This is sometimes referred to informally as the smearing property of convolution

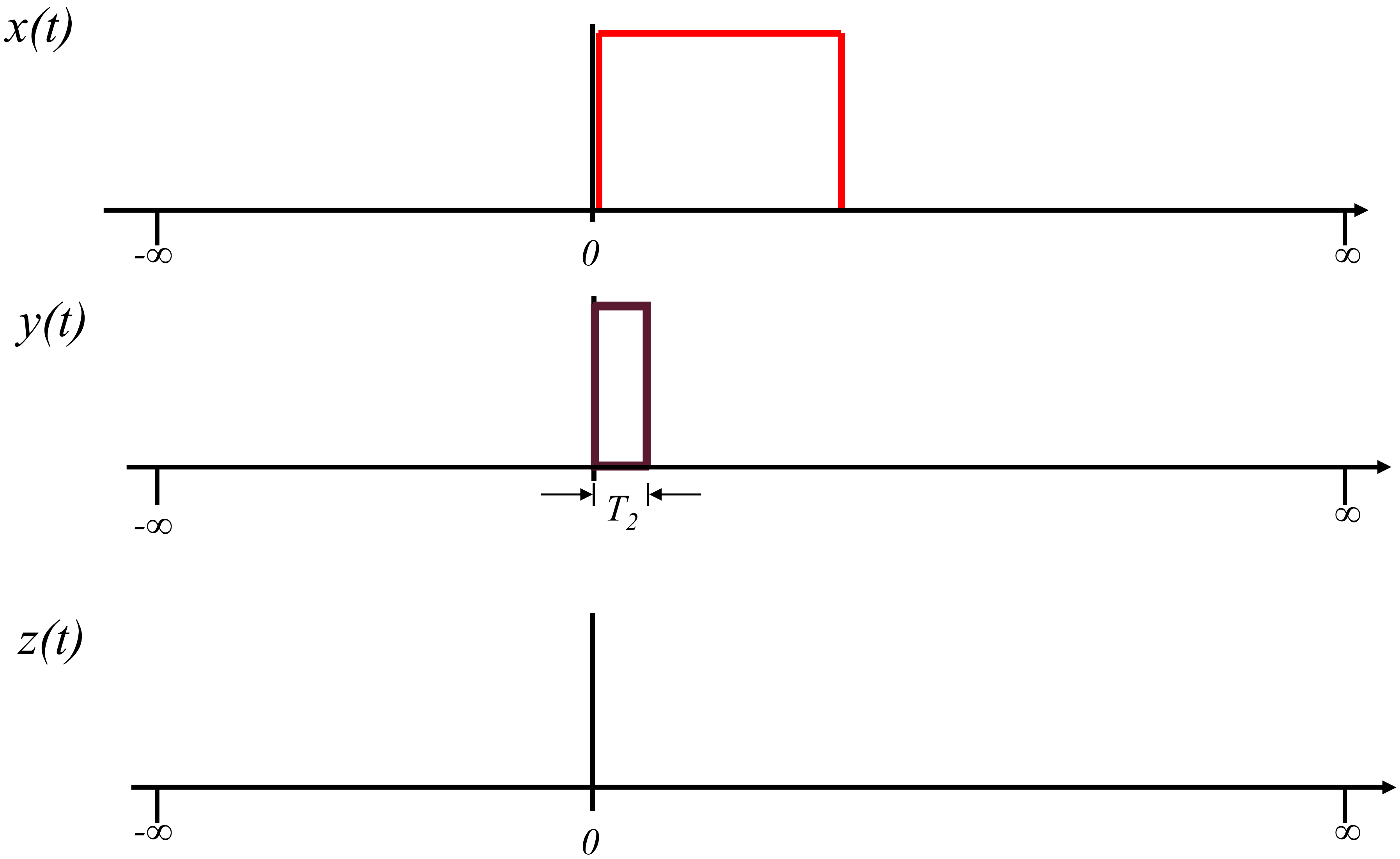
Now slide the function forward from - ∞ to + ∞

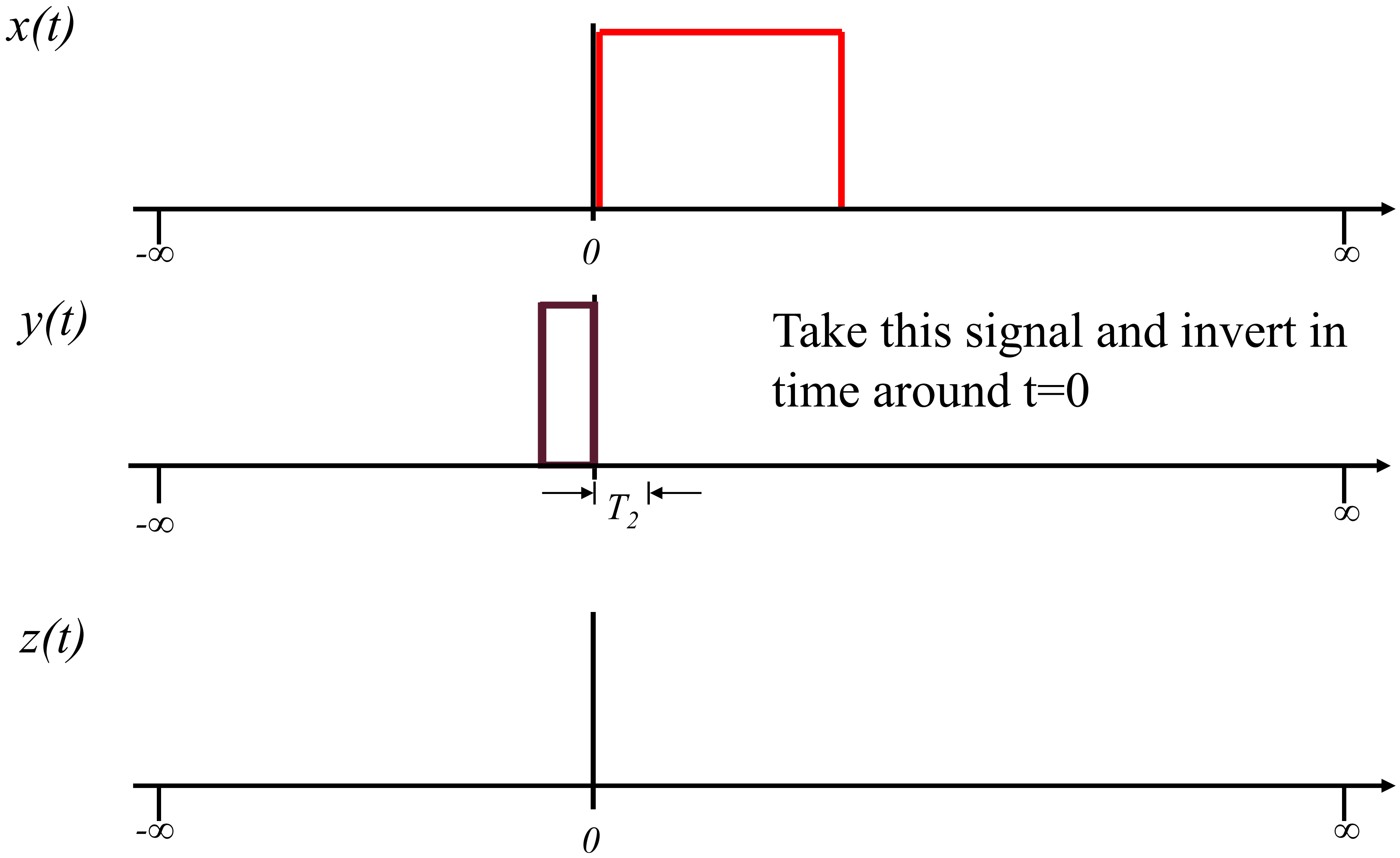
Any overlapping area is the convolution at that point

Take this signal and invert in time around t=0

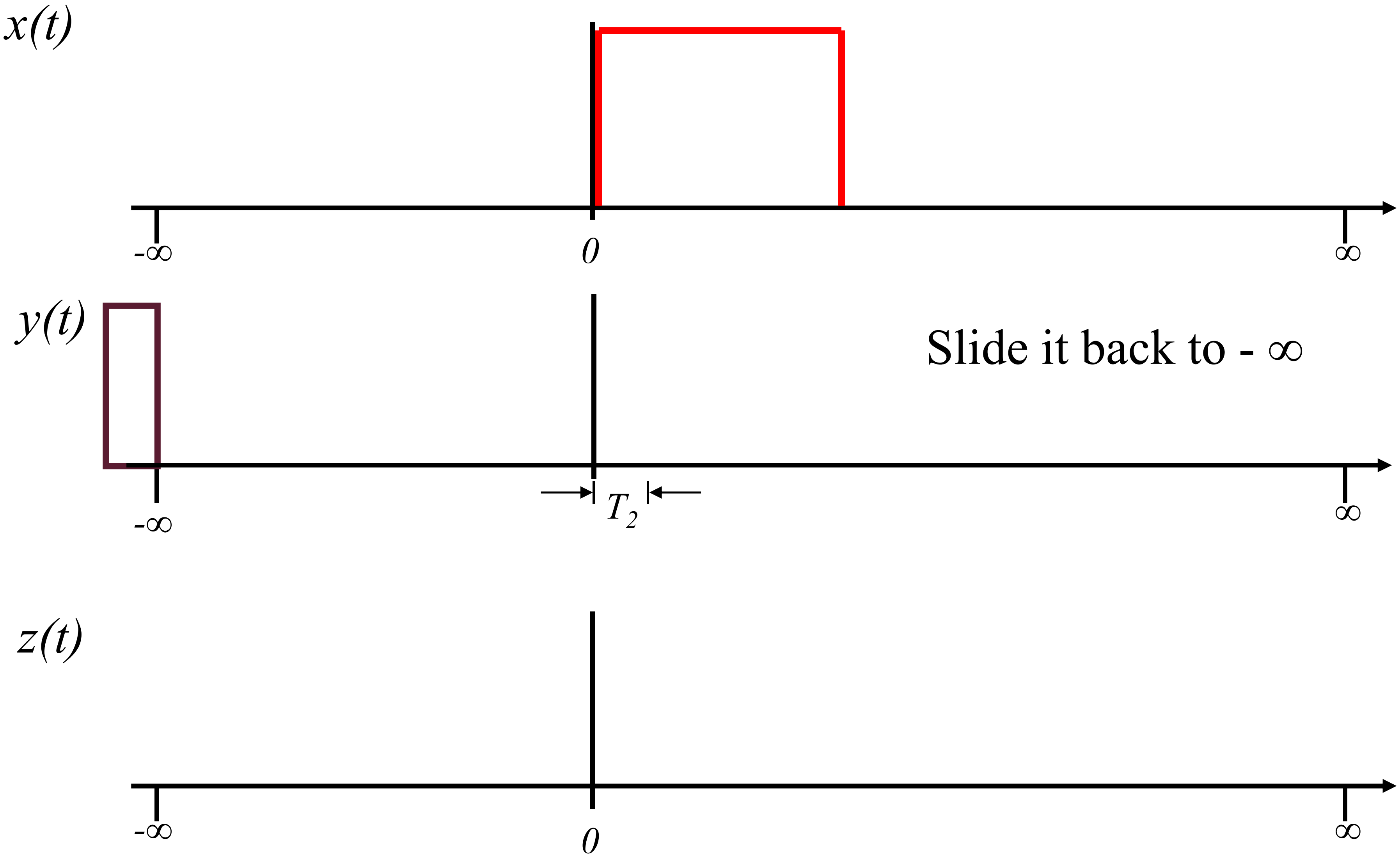
Slide it back to - ∞




x(t)
-∞
0
∞
y(t)
-∞
T2
∞
z(t)
-∞
0
∞



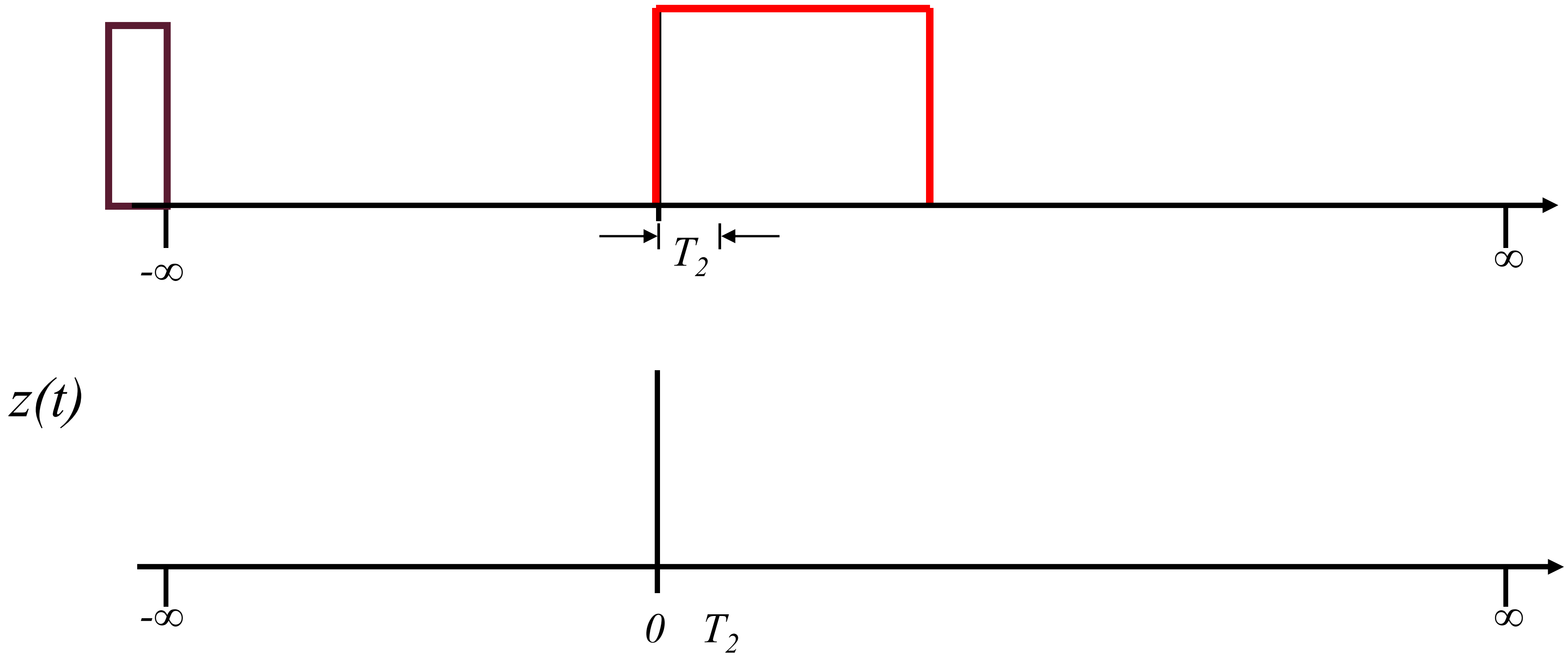
Take this signal and invert in time around t=0



Slide it back to - ∞

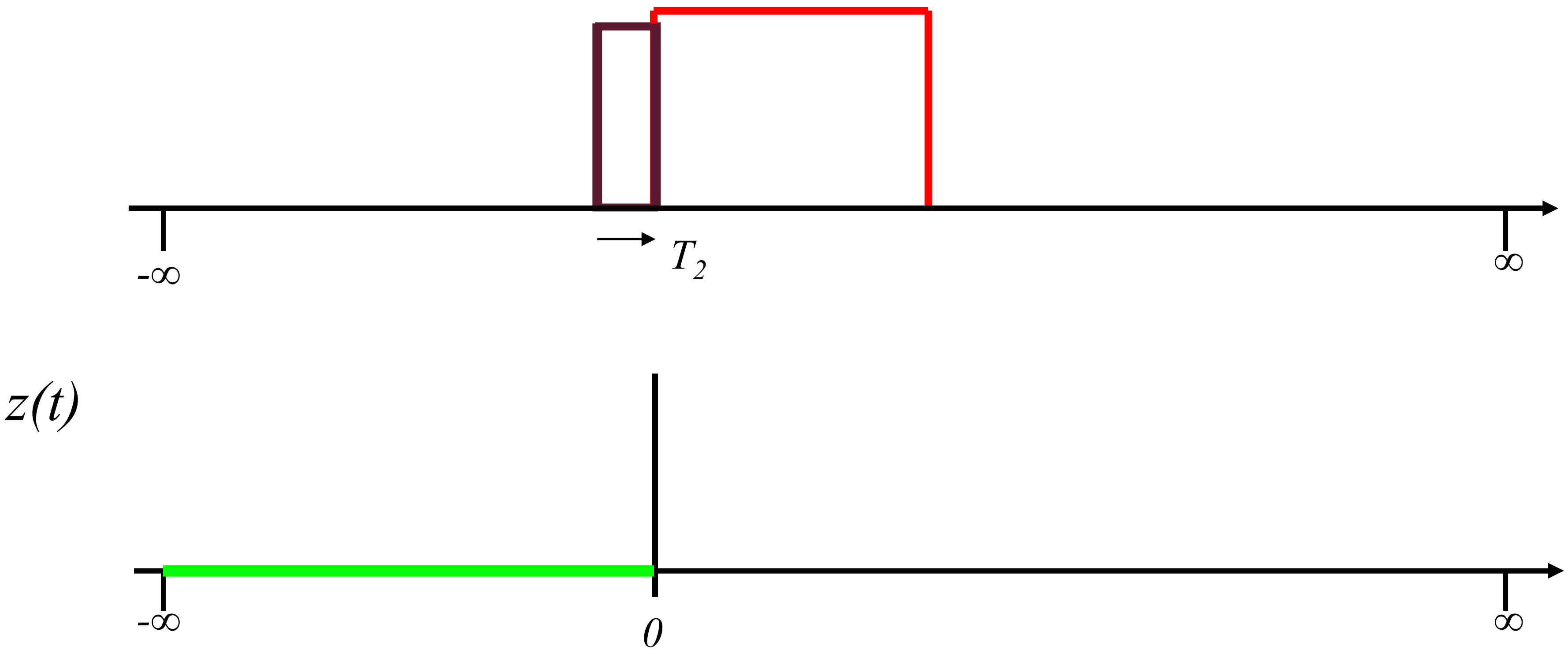
Now slide the function forward from -∞ to + ∞

Any overlapping area is the convolution at that point



$t<0 \quad x(t)*y(t)=0$

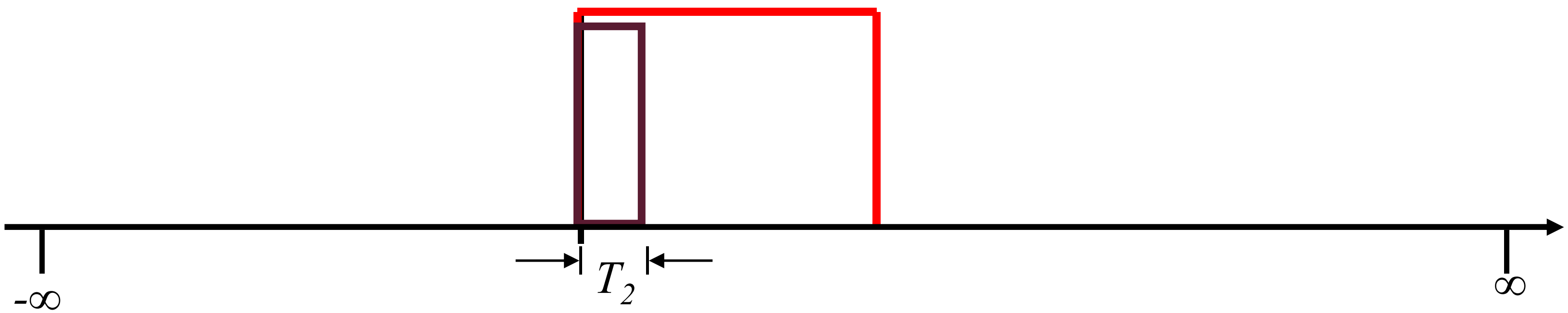
Now slide the function forward from -∞ to + ∞
Any overlapping area is the convolution at that point



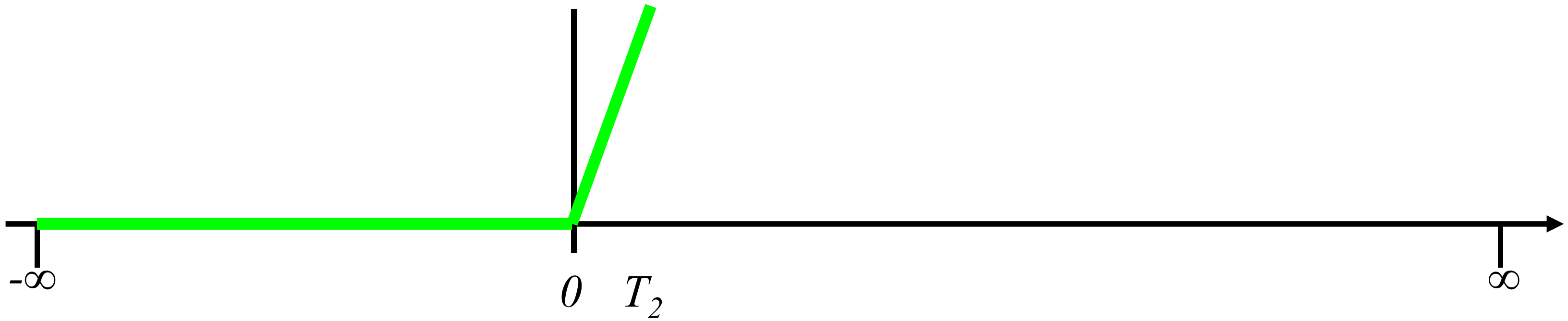
$0 < t < T_2$ partially overlapping

Now slide the function forward from $-\infty$ to $+\infty$

Any overlapping area is the convolution at that point



$z(t)$

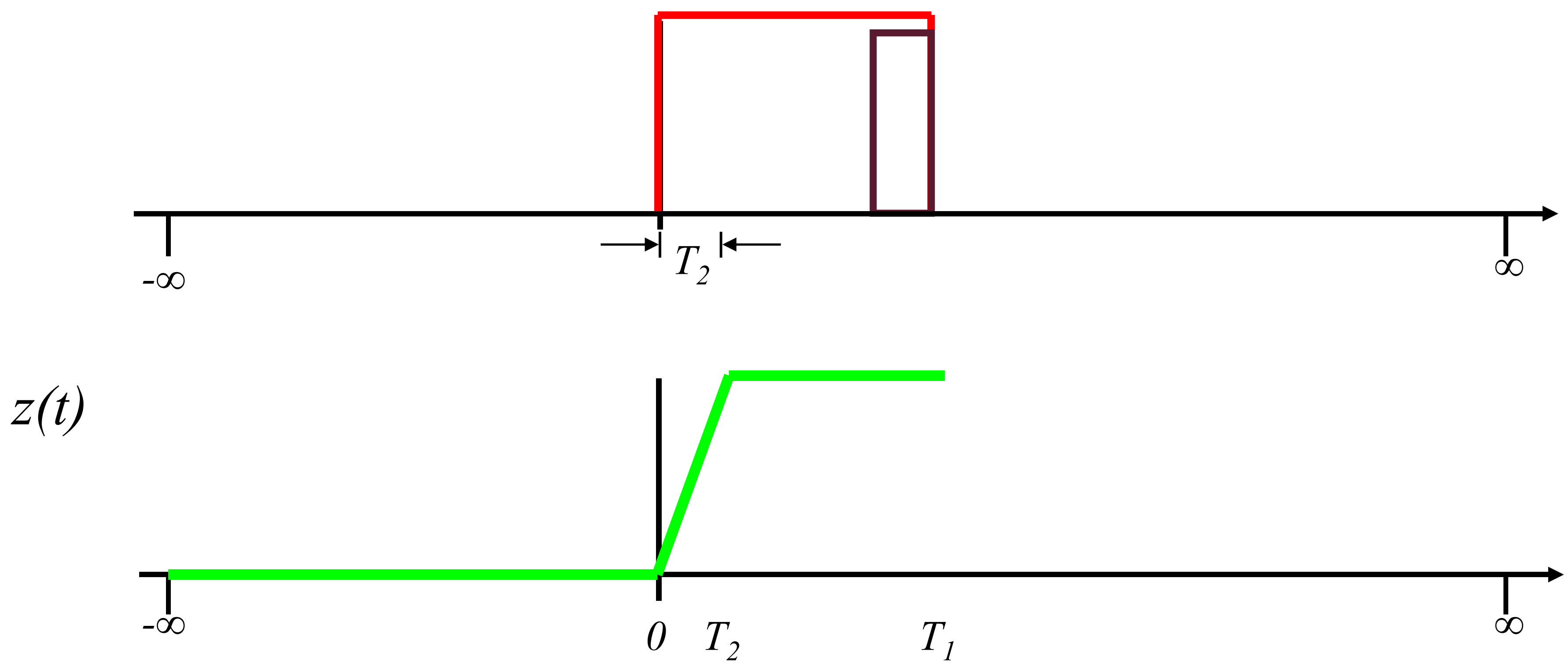


Now slide the function forward from -∞ to + ∞

Any overlapping area is the convolution at that point

$T_2 < t < T_1$ Fully overlapping

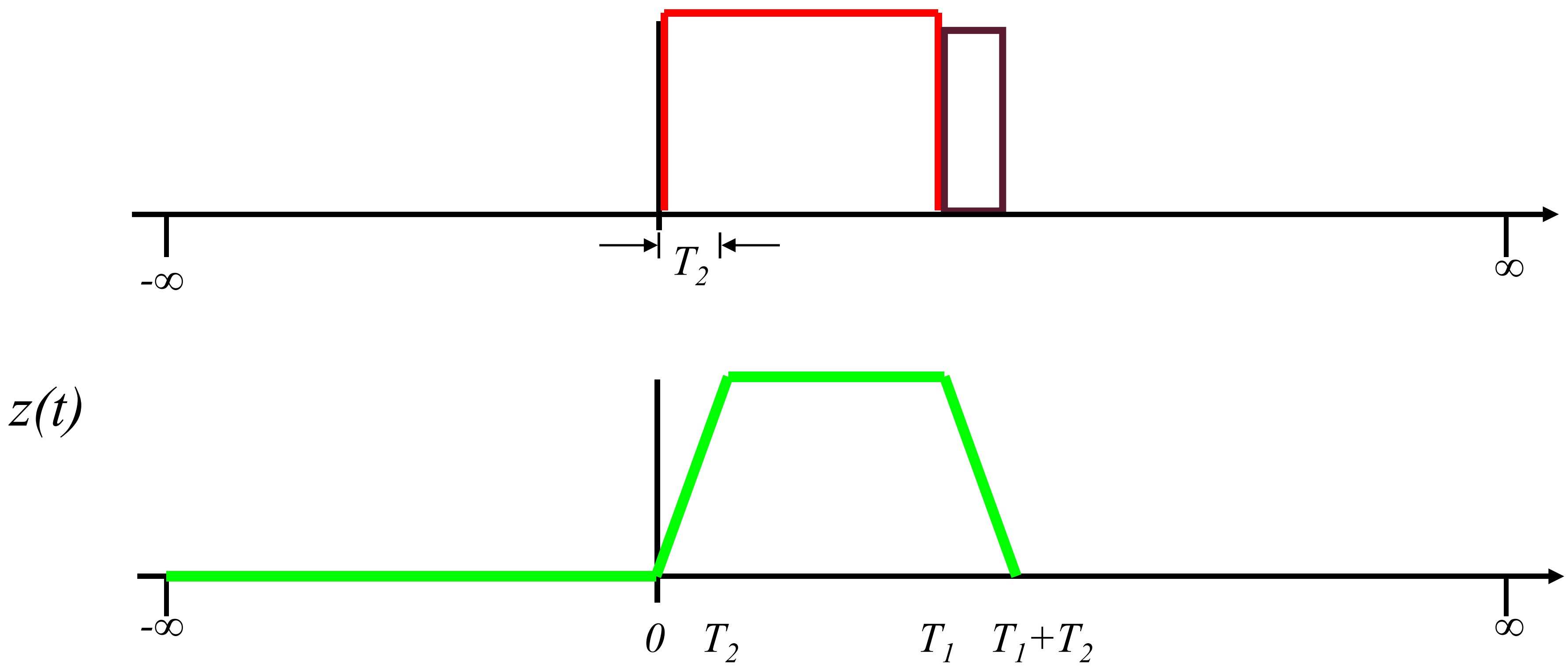
Area constant



$T_1 < t < T_1+T_2$ partially overlapping

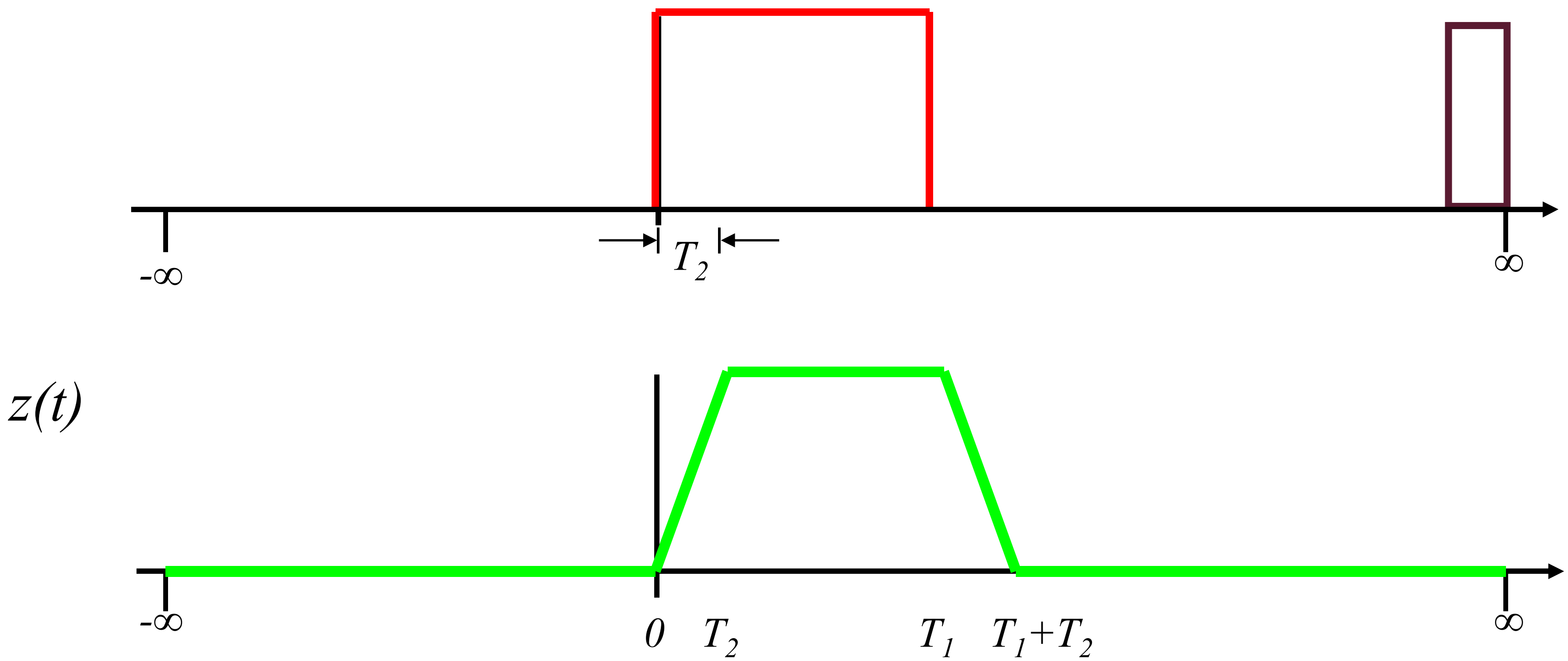
Now slide the function forward from $-\infty$ to $+\infty$

Any overlapping area is the convolution at that point



Now slide the function forward from -∞ to + ∞
Any overlapping area is the convolution at that point

$t > T_1+T_2 \qquad z(t) = 0$



Convolution theorem

A 5 step process

- Replace arguments of the function with a dummy variable
- Reverse one of the functions about the zero, x-axis
- Introduce a variable time shift, ***t***,
- Form the product of the function for every possible value of ***t***
- Calculate the area under the function for every value of ***t***

Convolution theorem

$$y(t) = x(t) * h(t) \Leftrightarrow Y(f) = X(f).H(f)$$

$$z(t) = x(t).h(t) \Leftrightarrow Z(f) = X(f) * H(f)$$

Informally:

- Convolution in the time domain corresponds to multiplication in the frequency domain
- Multiplication in the time domain corresponds to convolution in the frequency domain

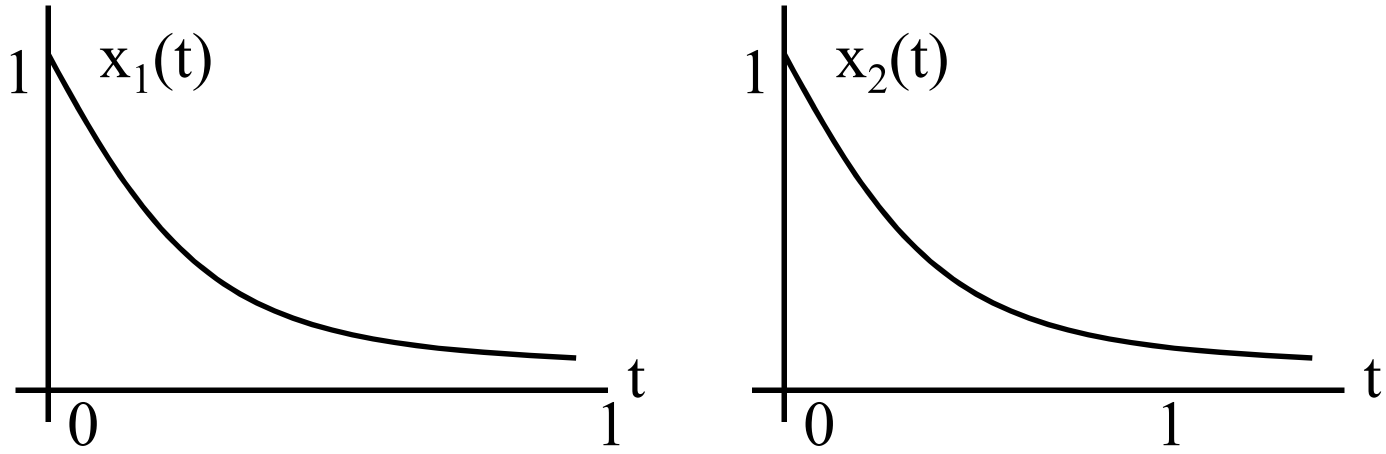
Convolution

Find the convolution of the signals

$$x_1(t) = e^{-\alpha t}u(t)$$

$$x_2(t) = e^{-\beta t}u(t) \qquad \alpha > \beta > 0$$

$$for\ \alpha = 4 \text{ and } \beta = 2$$



Convolution - Answer

$$x(t) = x_1 * x_2 = \int_{-\infty}^{\infty} e^{-\alpha\lambda} u(\lambda) e^{-\beta(t-\lambda)} u(t-\lambda)$$

But

$$u(\lambda)u(t-\lambda) = \begin{cases} 0 & \lambda < 0 \\ 1 & 0 < \lambda < t \\ 0 & \lambda > t \end{cases}$$

Thus

$$x(t) = \begin{cases} 0 & t < 0 \\ \int_0^t e^{-\beta t} e^{-(\alpha-\beta)\lambda} d\lambda = \dfrac{1}{\alpha-\beta}\left(e^{-\beta t} - e^{-\alpha t}\right) & t \geq 0 \end{cases}$$

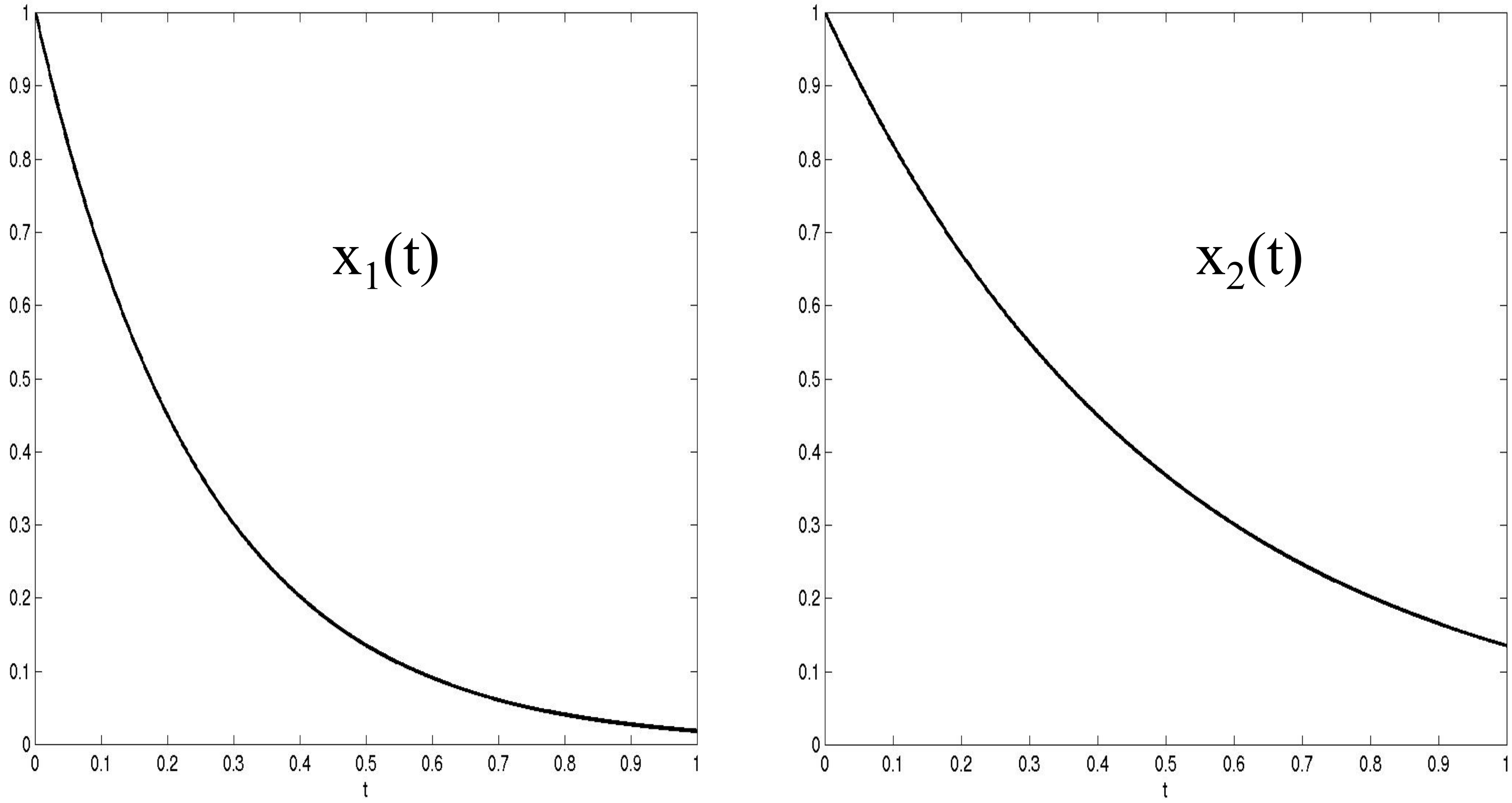
Practice the calculations!

$$\int_0^t e^{-bt} e^{-(a-b)s}\, ds = e^{-bt}\left[\frac{e^{-(a-b)s}}{-(a-b)}\right]_0^t =$$

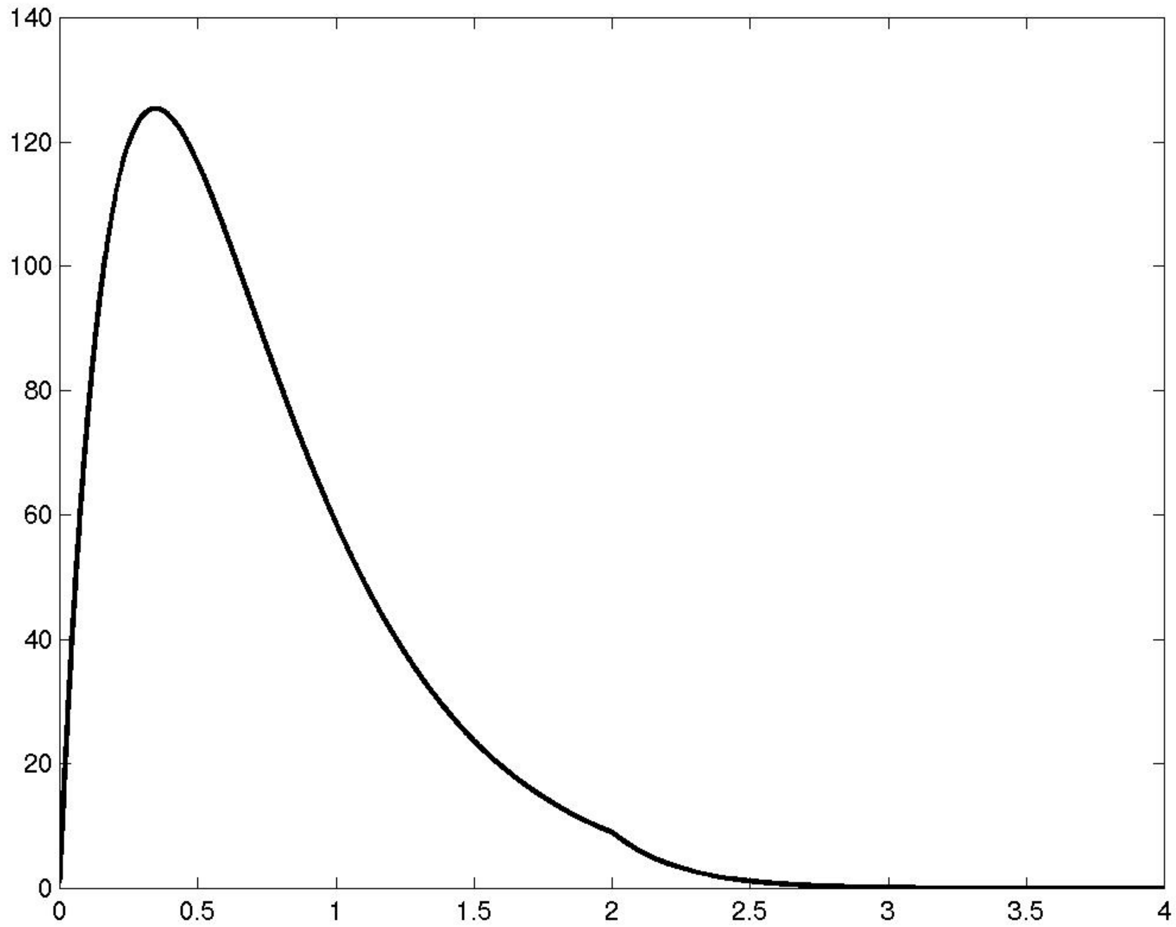
$$= \frac{1}{a-b} e^{-bt}\left(\overbrace{e^{-(a-b)0}}^{1} - e^{-(a-b)t}\right) =$$

$$= \frac{1}{a-b}\left(e^{-bt} - e^{-at}\right) \quad (\text{for } t \geq 0)$$

Convolution - Answer



Convolution - Answer



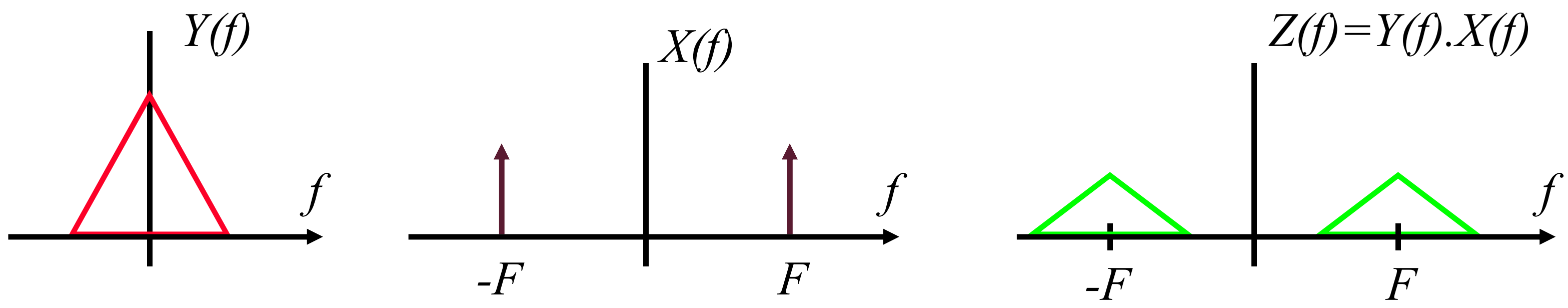
Convolution with impulses

the frequency domain

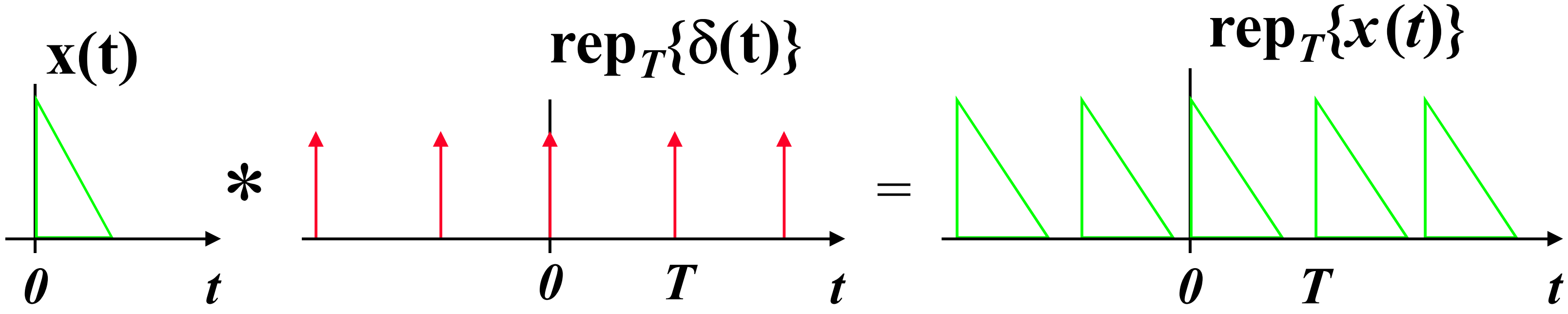
- We have seen that convolving $x(t)$ with $d(t-\tau)$ relocates $x(t)$ to $t=\tau$, as $x(t-\tau)$
- This applies equally in the frequency domain: $X(f)*d(f-F)=X(f-F)$
- As an example, consider a signal $y(t)$ multiplied by a cosine wave $x(t)$

Now $x(t)=cos(2\pi Ft)$ with $X(f)=1/2[\delta(f-F)+\delta(f+F)]$

and y(t).x(t) becomes $1/2[Y(f-F)+Y(f+F)]$



Function replication



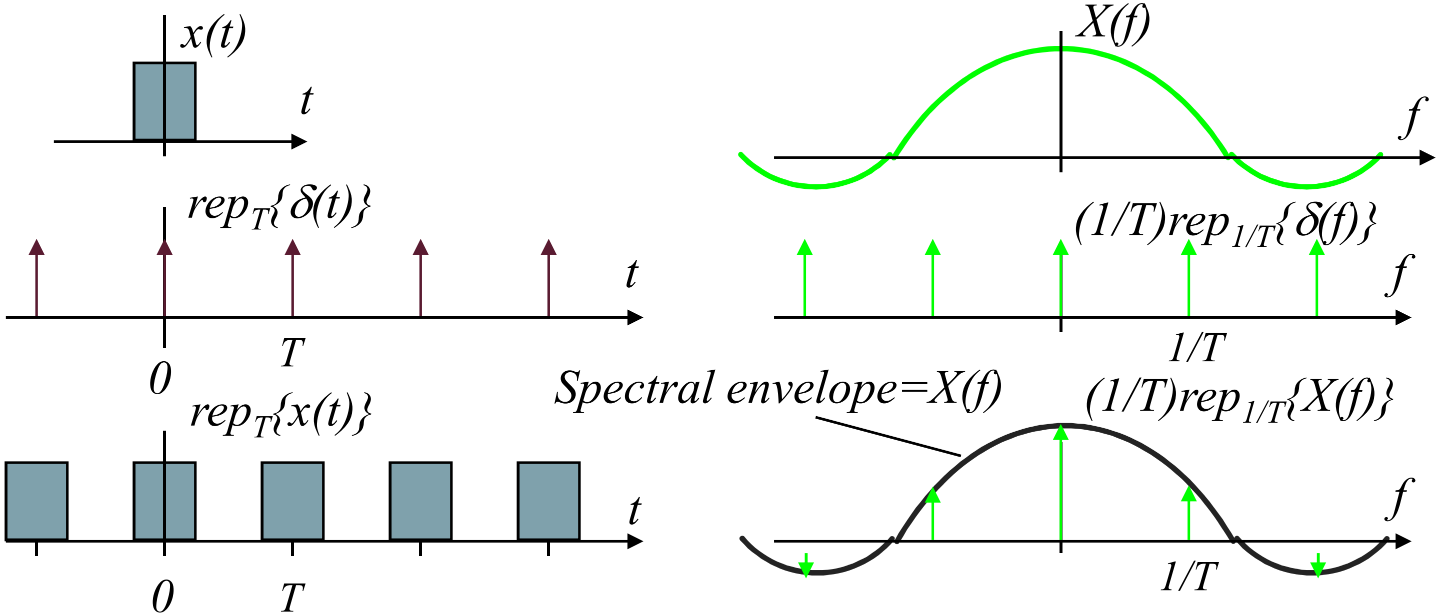
$$x(t) * \mathrm{rep}_T\{\delta(t)\} = \mathrm{rep}_T\{x(t)\}$$

- A periodic signal may be realised as convolution of a finite duration pulse with a rep-delta function
- The same operation may be effected to obtain replication of a spectrum in the frequency domain

Replicated function - Fourier transform

use replicated function with convolution in time to sample the function in frequency

$$rep_T\{x(t)\} = x(t) * rep_T\{\delta(t)\} \Leftrightarrow X(f).\frac{1}{T}\,rep_{\frac{1}{T}}\{\delta(f)\}$$



Summary

- Convolution - binary operator $*$ - combines two functions to produce a third function
- Convolution in the time domain corresponds to multiplication in the frequency domain - and *vice -versa*
- Convolution with a rep-delta function realises a periodic signal