Introduction to queuing theory

Queu(e)ing theory

Queu(e)ing theory is the branch of mathematics devoted to how objects (packets in a network, people in a bank, processes in a CPU etc etc) join and leave queues.

- Queuing is the traditional British spelling but now queueing is probably more common.
- The first papers about queuing theory were published by Erlang who was studying the Danish telephone system.
- Queuing theory involves the study of Markov chains.

Leonard Kleinrock (1934–)

Little's theorem

Little's theorem

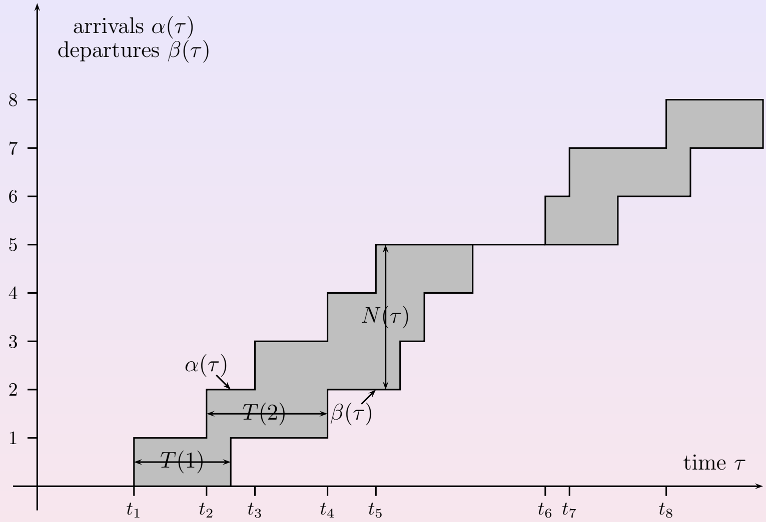
Let N be the average number of customers in a queue. Let λ be the average rate of arrivals. Let T be the average time spent queuing. Then we have

$$N = \lambda T.$$

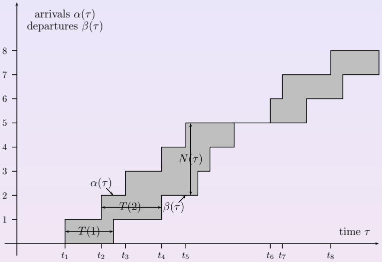
- In fact this simple theorem hides much complexity.
- It is only true under certain conditions.

John Little (1928–)

Little's theorem illustration



Little's theorem requirements



1. The limit $\lambda = \lim_{t\to\infty} \alpha(t)/t$ exists
2. The limit $\delta = \lim_{t\to\infty} \beta(t)/t$ exists
3. The limit $T = \lim_{t\to\infty} \sum_{i=1}^{\alpha(t)} \frac{T(i)}{\alpha(t)}$ exists
4. $\delta = \lambda$

Little's theorem example

- You are building a website and want to know how big a server you need.
- You believe your website will attract 24,000 visitors a day – 1,000 visitors an hour.
- You believe the average visitor will spend 6 minutes on the website.
- How many visitors does your server need to cope with?
- $\lambda = 1{,}000$ per hour, $T = 0.1$ hours.
- From $N = \lambda T$, $N = 100$, the average number of visitors at a time is 100.
- But because arrival is in "peaks" better plan for a peak hour.

Queuing theory notation

- Queuing theory uses a particular notation (Kendall's notation) to describe a queuing system.
- The arrival process describes the distribution of the interarrival times.
  - M – memoryless (Exponential) – a Poisson process.
  - D – deterministic – equally spaced.
  - G – general (no specific distribution).
  - Also Ph (phase), EK (Erlangian)
- The service time distribution determines how long it will take to process an item in the queue.
- The number of servers describes how many servers deal with the queue.
- For example $M/D/1$ is a Poisson input to a single queue which processes in constant time.

Agner Krarup Erlang (1878 – 1929)

Test your understanding

Little's theorem: True or False

Assuming the conditions for Little's Theorem hold:

- ☐ Doubling the average time spent in the queue doubles the average length of the queue.
- ☐ If we know mean queue N and departure rate δ we can calculate the mean time spent in queue T.
- ☐ Packets arrive at 500 packets per microsecond. They spend an average of 10 microseconds in the queue. If Little's theorem holds then N the average length of the queue is 50.

Test your understanding

Little's theorem: True or False

Assuming the conditions for Little's Theorem hold:

- ☑ Doubling the average time spent in the queue doubles the average length of the queue.
- ☑ If we know mean queue length N and departure rate δ we can calculate the mean time spent in queue T.
- ☒ Packets arrive at 500 packets per microsecond. They spend an average of 10 microseconds in the queue. Therefore N the average length of the queue is 50.

- Since $N = \lambda T$, if T is twice as large then N is twice as large.
- Little's theorem holds if $\lambda = \delta$ (arrivals match departures) hence $N = \delta T$ or $T = N/\delta$.
- False. $N = \lambda T$ and both have the same unit so $N = 500.10 = 5,000$.

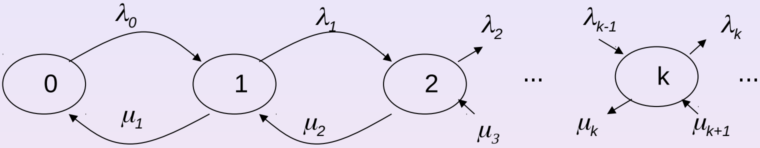
The Birth-Death process

The Birth–Death process

The birth–death process is a queue with a population which increases or decreases with rates which depend only on k the population at the time. Many queues can be modelled this way.

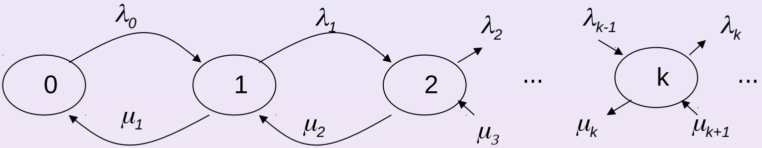
- Think of it as a queue – state 0 has no people. Arrivals are a Poisson process, rate λ_0.
- State k has births (arrivals) at rate λ_k but deaths (departures) at rate μ_k.

Starting the Birth–Death process



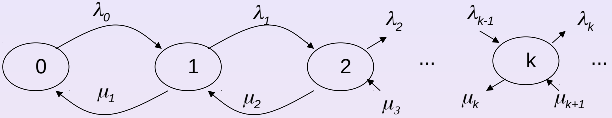
- Here we can see the arrivals and departures as a Markov chain.
- The state represents the number of people in the queue.
- An M/M/1 system would be modelled by $\lambda_k = \lambda$ for all k and $\mu_k = \mu$ for all k.
- We can model this as a continuous time Markov chain.

Birth–Death process – Transition matrix



$$\mathbf{Q} = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & \dots \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 & \dots \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & \dots \\ 0 & 0 & \mu_3 & -(\lambda_3+\mu_3) & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Birth–Death process – Balance equations



Balance equation for state 0

$\mu_1 \pi_1 = \lambda_0 \pi_0$

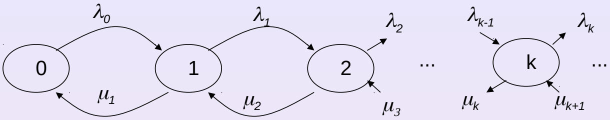
and for state k with $k > 0$ $\lambda_{k-1}\pi_{k-1} + \mu_{k+1}\pi_{k+1} = (\lambda_k + \mu_k)\pi_k$.

Rearrange to get: $\pi_1 = \lambda_0 \pi_0 / \mu_1$.

For state 1 $\lambda_0 \pi_0 + \mu_2 \pi_2 = \lambda_1 \pi_1 + \mu_1 \pi_1$,

Rearrange to get: $\pi_2 = \frac{\lambda_1 \lambda_0 \pi_0}{\mu_2 \mu_1}$.

Birth–Death process – Balance equations



We have:

$$\pi_1 = \lambda_0 \pi_0 / \mu_1.$$

$$\pi_2 = \frac{\lambda_1 \lambda_0 \pi_0}{\mu_2 \mu_1}.$$

Can show that in general:

$$\pi_k = \pi_0 \prod_{i=1}^{k} \frac{\lambda_{i-1}}{\mu_i}.$$

Birth–Death process – Balance equations

- Given $\pi_k = \pi_0 \prod_{i=1}^{k} \frac{\lambda_{i-1}}{\mu_i}$. now solve with $\sum_k \pi_k = 1$.
- This is complicated, the full solution is given in the notes.
- $$\pi_0 = \frac{1}{1 + \sum_{k=1}^{\infty} \prod_{i=1}^{k} \frac{\lambda_{i-1}}{\mu_i}}.$$
- This may not seem to help much – but we also have an equation for π_k in terms of π_0.
- Given μ_k and λ_k all the π_k can be worked out and hence the average queue length.
- To get further and finally solve M/M/1 we need the concept of utilisation.

Utilisation

Utilisation

Utilisation (utilization if you are American) ρ is given by the equation

$$\rho = \frac{\lambda}{\mu},$$

where λ is the mean arrival rate and μ is the maximum possible service rate of the system (when all servers are working).

- Utilisation is a good measure of the "fullness" of the system.
- A system at low utilisation is likely to be empty much of the time.
- Utilisation can also be thought of as the proportion of time the system is "busy".

Solving the M/M/1

- Finally we are ready to solve M/M/1
- Substituting in $\pi_k = \prod_{i=1}^{k} \frac{\lambda_{i-1}}{\mu_i} \pi_0 = \rho^k \pi_0$.
- Also $\pi_0 = \frac{1}{1+\sum_{k=1}^{\infty} \rho^k} = \frac{1}{1+\rho/(1-\rho)} = 1 - \rho$.
- Hence for M/M/1 $\pi_k = \rho^k (1 - \rho)$.
- The mean queue length is $E[Q] = \sum_{k=1}^{\infty} k\pi_k = \sum_{k=1}^{\infty} k\rho^k (1 - \rho)$.
- A neat trick gives us the answer.

A neat trick to solve M/M/1 expected queue

$E[Q] = \sum_{i=0}^{\infty} i(1-\rho)\rho^i$

$E[Q] = (1-\rho)\rho \sum_{i=0}^{\infty} i\rho^{i-1}$

$E[Q] = (1-\rho)\rho \sum_{i=0}^{\infty} \frac{d\rho^i}{d\rho}$

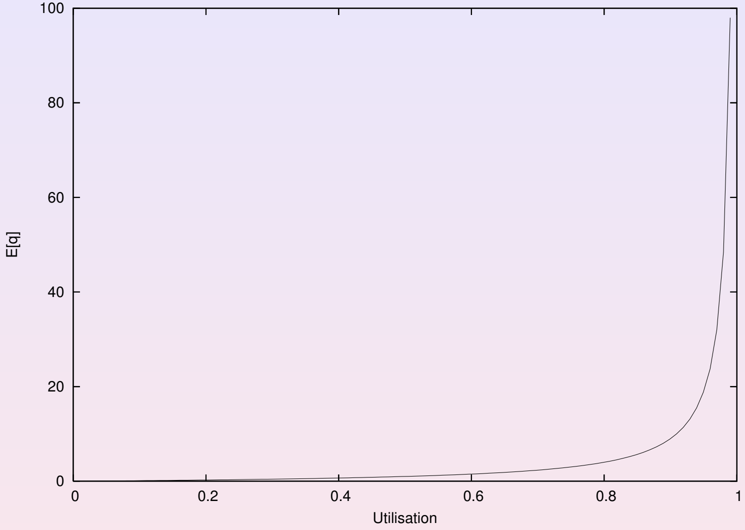
$E[Q] = (1-\rho)\rho \frac{d}{d\rho} \sum_{i=0}^{\infty} \rho^i$

$E[Q] = (1-\rho)\rho \frac{d}{d\rho} \frac{1}{1-\rho}$

$E[Q] = (1-\rho)\rho \frac{1}{(1-\rho)^2}$

$E[Q] = \frac{\rho}{1-\rho}$. At last! the solution for M/M/1.

Queue size versus utilisation for M/M/1



Queuing theory summary

- For the M/M/1 queue we have $E[Q] = \rho/(1-\rho)$.
- As the utilisation goes to 1 the queue length goes to infinity.
- The mean waiting time can be found from Little's theorem $N = E[Q]$.
- Therefore $T = \frac{\rho}{\lambda(1-\rho)} = \frac{1/\mu}{1-\rho} = \frac{1}{\mu-\lambda}$.
- If we wanted an M/M/k queue (k servers) this is $\lambda_i = \lambda$ for all i and $\mu_i = i\mu$ for $i < k$ and $\mu_k = k\mu$.
- If we wanted an M/M/k/l queue (maximum l people in the queue) we would say $\lambda_k = 0$ for $k \geq l$.
- So it can be seen that we have solved a lot more in this lecture than simply M/M/1

Test your understanding

Birth death processes

What are the λ_i and μ_i parameters for the following birth death processes assuming the "base" birth and death rates are λ and μ.

- The M/M/n process (where n is the number of servers).
- The M/M/n/m process (where n is the number of servers and m is the maximum number in the system).

Test your understanding

Birth death processes

What are the λ_i and μ_i parameters for the following birth death processes assuming the "base" birth and death rates are λ and μ.

- The M/M/n process (where n is the number of servers).
- The M/M/n/m process (where n is the number of servers and m is the maximum number in the system).

- $\lambda_i = \lambda$ for $i \geq 0$. $\mu_0 = 0$, $\mu_i = i\mu$ for $0 < i < n$ and $\mu_i = n\mu$ for $i \geq n$
- As above but $\lambda_i = 0$ for $i > m$ (and $\mu_i = 0$ for $i > m + 1$ but it does not matter this as system does not reach these states).

Queuing theory summary

- This lecture can only scratch the surface of queuing theory.
- Little's Theorem relates queue size, arrivals and mean queuing time $N = \lambda T$.
- The birth-death process is a very general way to look at queues of arrivals where arrivals and departures are related to Poisson processes.
- The birth-death process can be completely solved and the probability of every queue length calculated in terms of λ_k and μ_k.
- Utilisation is a measure of the fullness of the system $\rho = \lambda/\mu$.
- M/M/1, M/M/k and M/M/k/l queues can be solved as birth-death processes.