Bilinear Transformation

$X(s) \rightarrow \boxed{H(s)} \rightarrow Y(s) = H(s)X(s)$

$\Rightarrow H(s) = \dfrac{Y(s)}{X(s)}$

- **Approximation in the Transform Domain** – the basis for the approximation in the transform domain begins with the standard form

$$\boxed{Y(s) = H(s)X(s)}$$

- Assume that H(s) is a ratio of polynomials in s, an LTI differential equation:

$\frac{1}{s}$ $\times s$

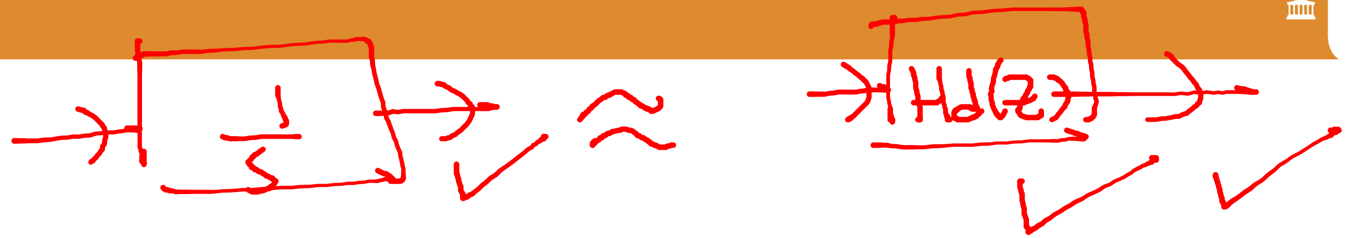
$$H(s) = \frac{a_m s^m + a_{m-1} s^{m-1} + \cdots + a_1 s + a_0}{b_n s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0}$$

A

- Dividing the numerator and denominator by s^n leads to

$$H(s) = \frac{a_m \left(\frac{1}{s}\right)^{n-m} + a_{m-1} \left(\frac{1}{s}\right)^{n-m-1} + \cdots + a_0 \left(\frac{1}{s}\right)^n}{b_n + b_{n-1}\left(\frac{1}{s}\right) + b_{n-2}\left(\frac{1}{s}\right)^2 + \cdots + b_0 \left(\frac{1}{s}\right)^n} = \frac{A\left(\frac{1}{s}\right)}{B\left(\frac{1}{s}\right)}$$

$\rightarrow \boxed{\frac{1}{s}} \rightarrow \; = \; \rightarrow \boxed{\int (\,)\,dt} \rightarrow$

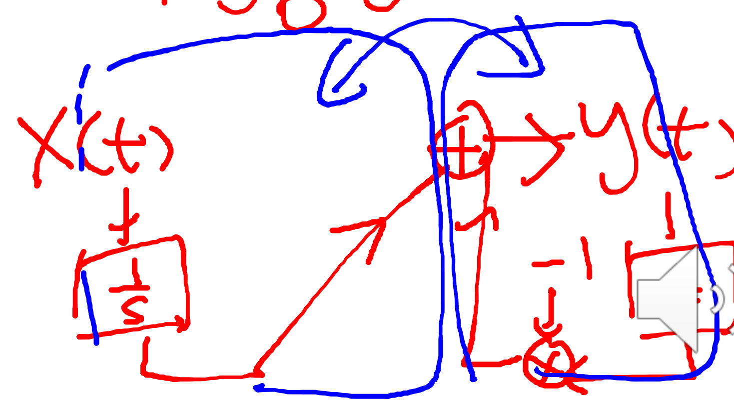


$$H(s) = \frac{1}{s+1} = \frac{s^{-1}}{1+s^{-1}} = \frac{Y(s)}{X(s)}$$

$$s^{-1}X(s) = (1+s^{-1})Y(s)$$

$$\int_0^t x(\tau)\,d\tau = y(t) + \int_0^t y(\tau)\,d\tau$$

$$\Rightarrow y(t) = \int_0^t x(\tau)\,d\tau - \int_0^t y(\tau)\,d\tau$$



$\rightarrow \boxed{\frac{1}{s}} \rightarrow$ replaced by $\rightarrow \boxed{H(z)} \rightarrow$

Bilinear Transformation

$\int (\ )dt$

- Therefore the input-output relationship can be rewritten as

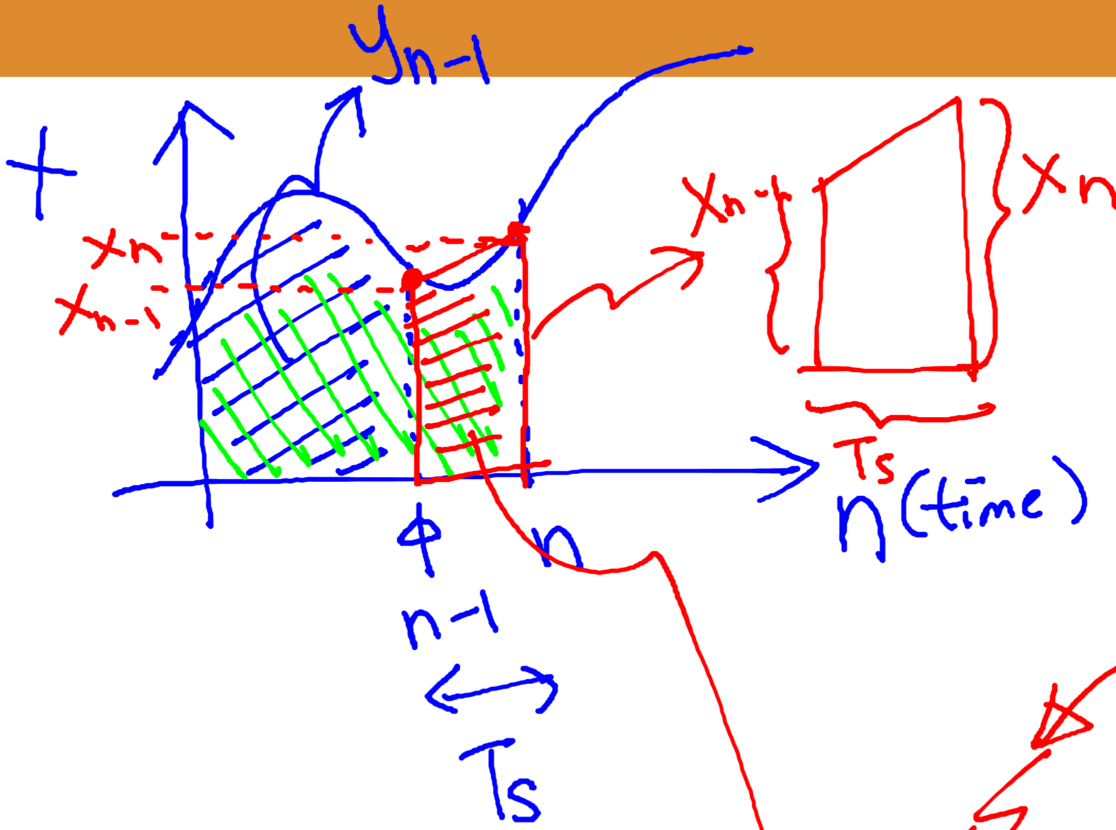
$$\boxed{Y(s)B\left(\frac{1}{s}\right) = X(s)A\left(\frac{1}{s}\right)}$$

- Since $(1/s)^k$ represents k-fold integration, suggesting that one can approximate the integration operation (1/s) by some discrete form

- Approximation in the transform domain means formally to have

$$\boxed{\frac{1}{s} \longrightarrow f(z^{-1})}$$

where z^{-1} is the delay operator. In principle, $f(\cdot)$ could be arbitrary



$$x \rightarrow \boxed{\int^t (\,)\,d\tau} \rightarrow \int_0^t x(\tau)\,d\tau$$

$$\approx$$

$$x \rightarrow \boxed{H_d(z)} \rightarrow y_n \approx \int_0^t x\,d\tau$$

$$H_d(z) = ?$$

$$\Rightarrow y_n = y_{n-1} + \frac{(x_{n-1}+x_n)T_s}{2}$$

$$z \Rightarrow Y(z) = z^{-1}Y(z) + \frac{T_s}{2}\left(z^{-1}X(z) + X(z)\right)$$

$$\Rightarrow Y(z)(1-z^{-1}) = \frac{T_s}{2}X(z)(1+z^{-1})$$

$$\Rightarrow H_d(z) = \frac{Y(z)}{X(z)} = \frac{T_s}{2}\frac{(1+z^{-1})}{(1-z^{-1})}$$

$$x(t) \rightarrow \boxed{\frac{1}{s}} \rightarrow \int_0^t x(\tau)\,d\tau$$

$$\frac{1}{s} \approx \frac{T_s}{2}\,\frac{(1+z^{-1})}{(1-z^{-1})}$$

$$x(n) \rightarrow \boxed{H_d(z)} \rightarrow y(n)$$

$$H_d(z) = \frac{T_s}{2}\left(\frac{1+z^{-1}}{1-z^{-1}}\right)$$

$$H(s) = \frac{s^{-1}}{1+s^{-1}}$$

$$H(z) = \frac{\frac{T_s}{2}\left(\frac{1+z^{-1}}{1-z^{-1}}\right)}{1+\frac{T_s}{2}\left(\frac{1+z^{-1}}{1-z^{-1}}\right)}$$

$$\frac{1}{s} = \frac{T_s}{2}\frac{1+z^{-1}}{1-z^{-1}}$$

Bilinear Transformation

- By far the most commonly used instance of this type of mapping is

$$\boxed{s = \frac{2}{T_s}\frac{1-z^{-1}}{1+z^{-1}}}$$

or

$$\boxed{\frac{1}{s} = \frac{T_s}{2}\frac{1+z^{-1}}{1-z^{-1}}}$$

- To obtain the digital structure corresponding to a filter H(s), we use

$$\boxed{H_d(z) = H\left(\frac{2}{T_s}\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

Bilinear Transformation

- The substitution

$$\boxed{\frac{1}{s} \rightarrow \frac{T_s}{2}\frac{1+z^{-1}}{1-z^{-1}}}$$

need to memorise this formula

corresponds to the trapezoidal rule integration formula which is a second-order implicit integration formula given by

$$y_n = y_{n-1} + \frac{T_s}{2}(x_n + x_{n-1})$$

- By taking the z transform, we get the transfer function

$$H(z) = \frac{Y(z)}{X(z)} = \boxed{\frac{T_s}{2}\frac{1+z^{-1}}{1-z^{-1}}}$$

Bilinear Transformation

Connection between S-plane and Z-plane

- Another interpretation is:

$$\boxed{z = e^{sT_s}}$$

$$= \frac{e^{\frac{sT_s}{2}}}{e^{-\frac{sT_s}{2}}}$$

linearisation two times

$$\approx \frac{1 + \frac{sT_s}{2}}{1 - \frac{sT_s}{2}}$$

$$\Rightarrow z\left(1 - \frac{sT_s}{2}\right) = 1 + \frac{sT_s}{2}$$

$$\Rightarrow \boxed{s = \frac{2}{T_s}\frac{1 - z^{-1}}{1 + z^{-1}}}$$

$T_s \to 0$

$e^{sT_s} = e^{\frac{sT_s}{2}} e^{\frac{sT_s}{2}}$

$e^{x} \approx 1 + x$

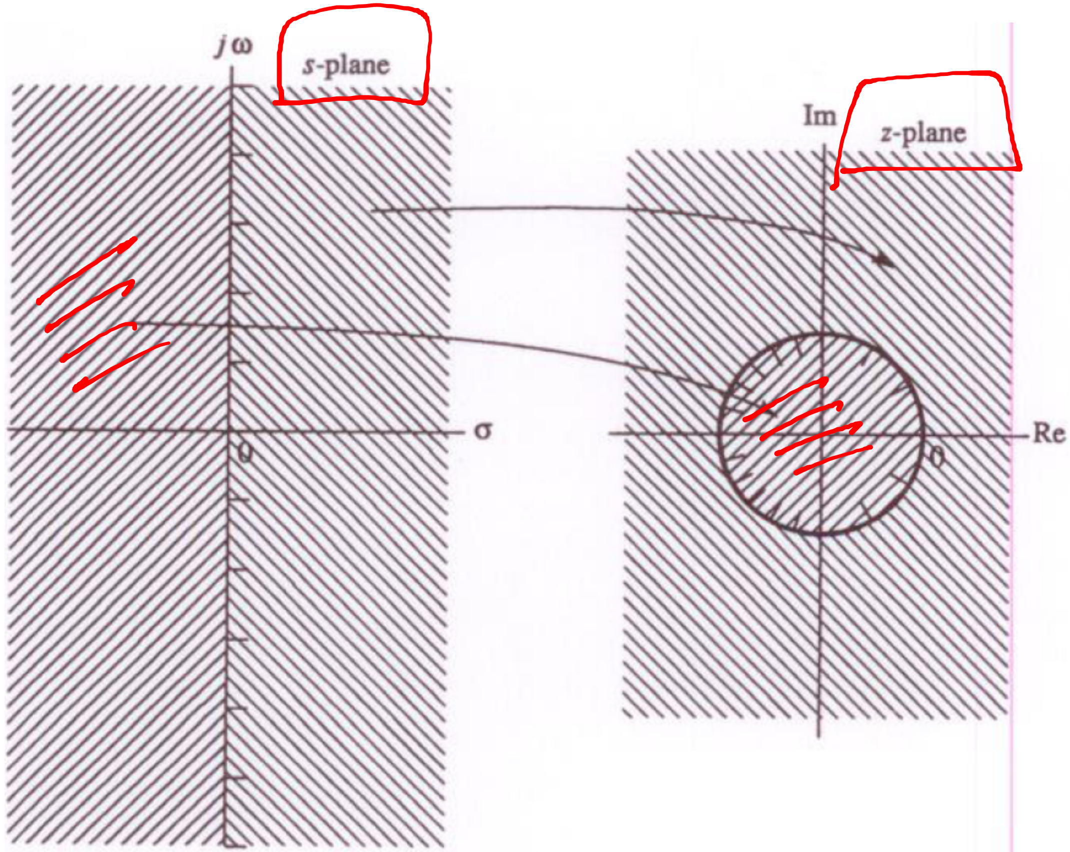
$e^{-x} \approx 1 - x$

for small x

Bilinear Transformation

safe to use ✓

- For σ<0, we find |z|<1 and for σ>0, |z|>1. The entire left half-plane is mapped in the interior of the unit circle and the right-half to the outside



Stable

frequency warping

$$H_d(z) \qquad \boxed{z = e^{j\Omega T_s}} \qquad H_a(s)$$

$$= H_a\left(\frac{2}{T_s}\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$= H_a\left(\frac{2}{T_s}\frac{1-e^{-j\Omega T_s}}{1+e^{-j\Omega T_s}}\right)$$

$$= H_a\left(\frac{2}{T_s}\frac{\cancel{e^{-\frac{j\Omega T_s}{2}}}\left(e^{\frac{j\Omega T_s}{2}}-e^{-\frac{j\Omega T_s}{2}}\right)}{\cancel{e^{-\frac{j\Omega T_s}{2}}}\left(e^{\frac{j\Omega T_s}{2}}+e^{-\frac{j\Omega T_s}{2}}\right)}\right)$$

$$= 2j\sin\frac{\Omega T_s}{2} \qquad = 2\cos\frac{\Omega T_s}{2}$$

$$= H_a\left(j\frac{2}{T_s}\tan\frac{\Omega T_s}{2}\right)$$

original analogue filter in the freq. $\to$ & $H_a(j\boxed{\omega})$

for small Ω

$$\omega \to \frac{2}{T_s}\tan\frac{\Omega T_s}{2}$$

digital filter in the freq. domain

$\tan\theta \simeq \theta$ for small θ

$\omega \neq \Omega$

Bilinear Transformation

- Aliasing errors are eliminated ✓
- BUT there is a nonlinear relationship between the continuous frequency ω and the discrete frequency Ω. With for $s=j\omega$ and $z=e^{j\Omega T_s}$, we get

$$\boxed{j\omega \rightarrow \frac{2}{T_s}\frac{1-e^{-j\Omega T_s}}{1+e^{-j\Omega T_s}}}$$

$\omega \neq \Omega$

or

$$\boxed{\omega \rightarrow \frac{2}{T_s}\tan\frac{\Omega T_s}{2}}$$

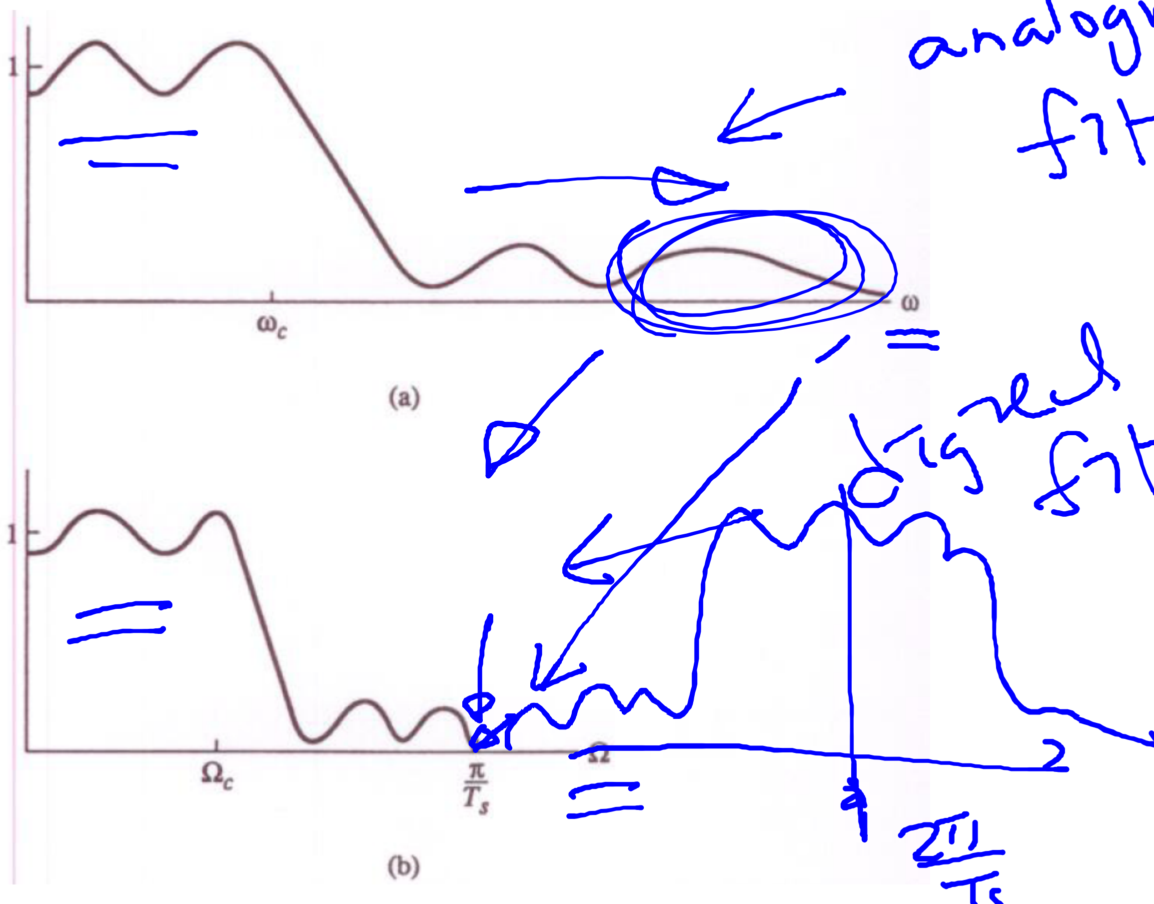
- For small values of Ω, the mapping is almost linear. For most of the frequency scale it is highly nonlinear. Thus, the bilinear transform preserves the amplitude but warps the frequency

$$\omega \rightarrow \frac{2}{T_s}\tan\frac{\Omega T_s}{2}$$

Bilinear Transformation

- Nonlinear frequency warping caused by the bilinear transformation:

No more aliasing because the limited bandwidth

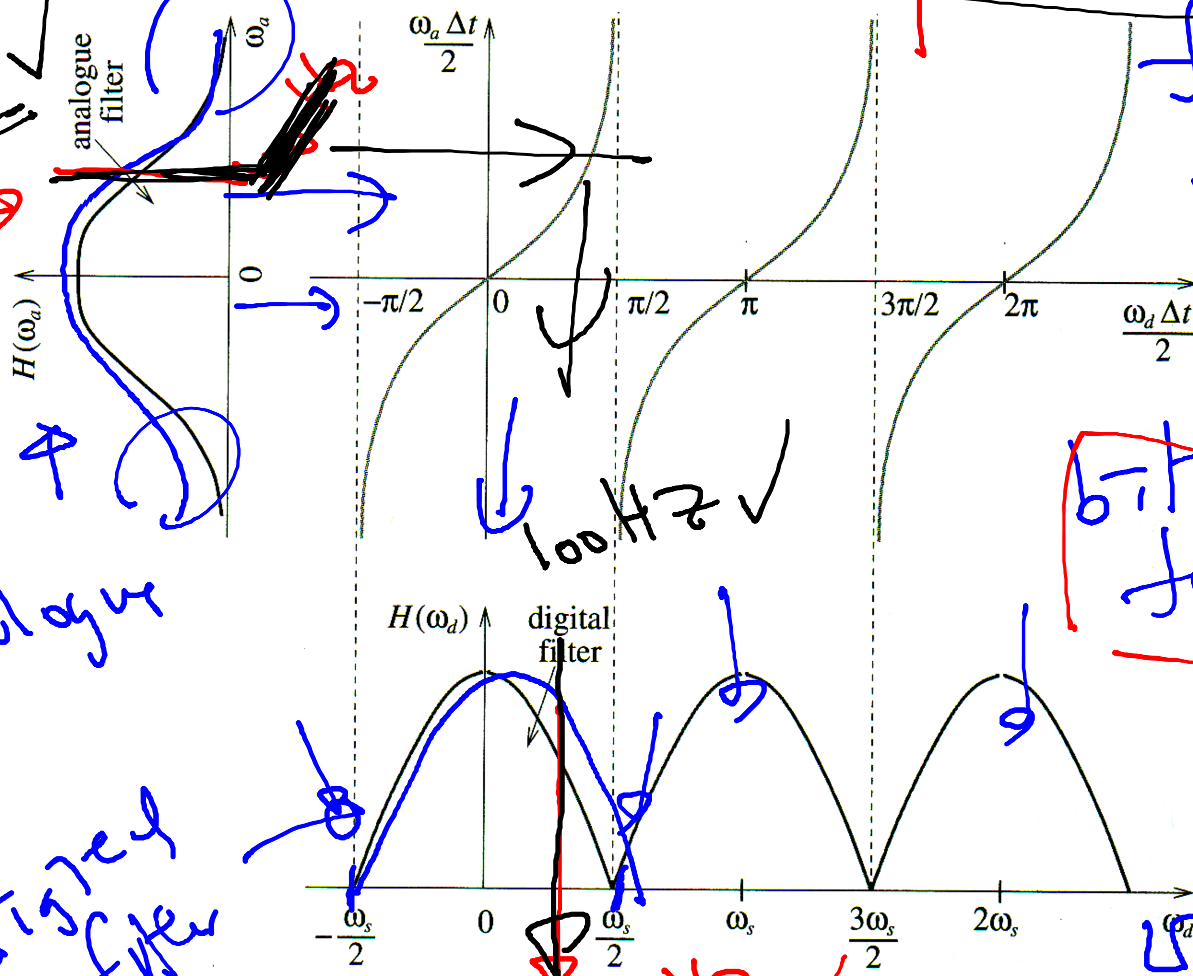


100Hz digital LPF

$$\omega \to \frac{2}{T_s}\tan\frac{\Omega T_s}{2} \qquad 2\pi(100)$$

Bilinear Transformation

ω_a ?



100Hz

frequency warping for bilinear transformation

analogue

digital filter

Ω

Bilinear Transformation

- **Advantages of the Bilinear Transformation:**

1. Mapping of the s plane to the z plane is one-to-one
2. Aliasing is avoided
3. The closed left half of the s plane is mapped onto the unit disk of the z plane

- **Disadvantages of the Bilinear Transformation:**

1. The frequency scale is warped
2. Waveforms are not preserved if T_s is not small enough

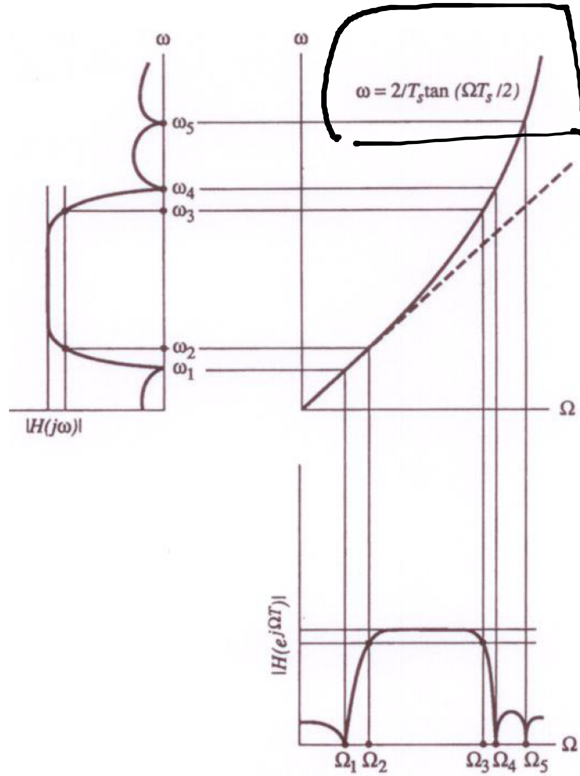
Bilinear Transformation

memorise

- We can find the new set of cutoff frequencies of the continuous filter to compensate the nonlinear frequency warping of the bilinear transform:

$$\boxed{\omega = \frac{2}{T_s}\tan\frac{\Omega T}{2}}$$

$H_a(\omega)$



$H_d(\Omega)$