Pre-LAB: OFDM Simulation

N symbols transmitted

Signal Model for ISI Channels

- Let us consider the transmitted data signal

L paths

$$s(t) = \sum_{n=0}^{N-1} s_n f(t - nT)$$

Data symbols at time n
e.g., {+1,–1} for BPSK

T

Symbol duration

Pulse shaping function, e.g., sinc functions

- And we have the ISI channel

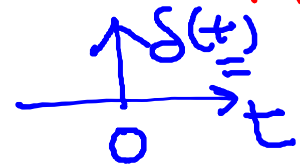
$$h(t) = \sum_{\ell=0}^{L-1} \alpha_\ell \delta(t - \tau_\ell)$$

- The received signal can therefore be expressed as

$$\begin{aligned} y(t) &= s(t) * h(t) + n(t) \\ &= \int_{-\infty}^{\infty} s(x) h(t-x) dx + n(t) \\ &= \int_{-\infty}^{\infty} \sum_{n=0}^{N-1} s_n f(x - nT) \sum_{\ell=0}^{L-1} \alpha_\ell \delta(t - x - \tau_\ell) dx + n(t) \end{aligned}$$

$$y(t)=\int_{-\infty}^{+\infty}\sum_{n=0}^{N-1} s_n f(x-nT)\sum_{l=0}^{L-1}\alpha_l\,\delta(t-x-\tau_l)\,dx + n(t)$$

at $t = mT$, $0, T, 2T, 3T, \dots, (N-1)T$



$$y(mT)=\sum_{n=0}^{N-1}\sum_{l=0}^{L-1} s_n\alpha_l\int_{-\infty}^{+\infty} f(x-nT)\,\delta(mT-x-\tau_l)\,dx + n(mT)$$

$$=\sum_{n=0}^{N-1}\sum_{l=0}^{L-1} s_n\alpha_l f(mT-\tau_l-nT) + n(mT)$$

$x=mT-\tau_l$

$$=\sum_{n=0}^{N-1}\sum_{l=0}^{L-1} s_n\alpha_l f((m-n)T-\tau_l) + n(mT)$$

at $t=0, T, 2T, 3T, \dots, (N-1)T$

$y(t)$ at $t = 0, T, 2T, \ldots, (N-1)T$

$y(0) = y_0 \quad y(T) = y_1 \quad y(2T) = y_2 \quad \ldots \quad y((N-1)T) = y_{N-1}$

Signal Model for ISI Channels

$m, n = 0, 1, 2, \ldots, N-1$

- We are only interested in samples at time t=mT, so we have

$$y_m = \sum_{n=0}^{N-1} s_n \underbrace{\left[\sum_{\ell=0}^{L-1} \alpha_\ell f((m-n)T - \tau_\ell) \right]}_{g_{m,n}} + n_m = \sum_{n=0}^{N-1} s_n g_{m,n} + n_m$$

$$\Rightarrow \begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ \vdots \\ \\ \\ y_{N-1} \end{bmatrix} = \begin{bmatrix} g_0 & 0 & 0 & \cdots & & \cdots & 0 \\ g_1 & g_0 & 0 & 0 & \cdots & & 0 \\ g_2 & g_1 & g_0 & 0 & \ddots & \ddots & 0 \\ g_3 & g_2 & g_1 & g_0 & 0 & \ddots & 0 \\ 0 & g_3 & g_2 & g_1 & g_0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ & & \ddots & \ddots & \ddots & \ddots & \ddots \\ 0 & 0 & \cdots & 0 & g_3 & g_2 & g_1 \; g_0 \end{bmatrix} \begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ \vdots \\ \vdots \\ \\ \\ s_{N-1} \end{bmatrix} + \begin{bmatrix} n_0 \\ n_1 \\ n_2 \\ \vdots \\ \vdots \\ \\ \\ n_{N-1} \end{bmatrix}$$

$g_{0,0} \quad g_{0,1} \quad g_{0,2} \quad g_{0,N-1}$

$g_{1,0} \quad g_{1,2} \quad g_{2,2} \quad g_{2,N-1}$

$g_{N-1,0} \quad g_{N-1,1} \quad g_{N-1,N-1}$

A Toeplitz channel matrix

$$y_m = \sum_{n=1}^{N} s_n g_{m,n} + n_m$$

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_M \end{bmatrix} = \begin{bmatrix} g_{1,1} & g_{1,2} & \cdots & g_{1,N} \\ g_{2,1} & g_{2,2} & \cdots & g_{2,N} \\ \vdots & & \vdots & \\ g_{M,1} & g_{M,2} & \cdots & g_{M,N} \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_N \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_M \end{bmatrix}$$

$\tilde{s}_3 = \dfrac{y_3}{g_0}$ ✗

→ Toeplitz matrix

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_M \end{bmatrix} = \begin{bmatrix} g_0 & 0 & \cdots & 0 \\ g_1 & g_0 & \cdots & 0 \\ g_2 & g_1 & g_0 \cdots & 0 \\ & & \ddots & \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \\ \vdots \\ s_N \end{bmatrix} + \begin{bmatrix} n_1 \\ n_2 \\ \vdots \\ n_M \end{bmatrix}$$

$\tilde{s}_1 = \dfrac{y_1}{g_0}$ ✓

$$\begin{cases} y_1 = g_0 s_1 + n_1 \\ y_2 = g_0 s_2 + g_1 s_1 + n_2 \end{cases}$$

$g_1 s_1$ → Interference

$\tilde{s}_2 = \dfrac{y_2}{g_0}$ ✗

$$y_3 = g_0 s_3 + g_2 s_1 + g_1 s_2 + n_3$$

OFDM = orthogonal frequency division multiplexing

WiFi, 4G, 5G, ⋯

From Toeplitz to Circulant ISI Channels

If

- ISI is a serious problem causing error in detection
- The channel matrix can be made circulant by adding a cyclic prefix at the transmitter side and removing it at the receiver side

Toeplitz

Rx-side cyclic prefix

Tx-side cyclic prefix

$$
\begin{bmatrix} \boxed{\begin{matrix} y_{-3} \\ y_{-2} \\ y_{-1} \end{matrix}} \\ y_0 \\ y_1 \\ y_2 \\ \vdots \\ \vdots \\ \\ y_{N-1} \end{bmatrix}
=
\begin{bmatrix}
g_0 & 0 & 0 & \cdots & & & \cdots & 0 \\
g_1 & g_0 & 0 & 0 & \cdots & & & 0 \\
g_2 & g_1 & g_0 & 0 & \ddots & \ddots & & 0 \\
g_0 & 0 & 0 & \cdots & 0 & g_3 & g_2 & g_1 \\
g_1 & g_0 & 0 & 0 & \cdots & 0 & g_3 & g_2 \\
g_2 & g_1 & g_0 & 0 & \ddots & \ddots & 0 & g_3 \\
g_3 & g_2 & g_1 & g_0 & 0 & \ddots & & 0 \\
0 & g_3 & g_2 & g_1 & g_0 & 0 & & 0 \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\
 & & \ddots & \ddots & \ddots & \ddots & \ddots & \\
0 & 0 & \cdots & 0 & g_3 & g_2 & g_1 & g_0
\end{bmatrix}
\begin{bmatrix} \boxed{\begin{matrix} s_{N-3} \\ s_{N-2} \\ s_{N-1} \end{matrix}} \\ s_0 \\ s_1 \\ s_2 \\ \vdots \\ \vdots \\ \\ s_{N-1} \end{bmatrix}
+
\begin{bmatrix} n_{-3} \\ n_{-2} \\ n_{-1} \\ n_0 \\ n_1 \\ n_2 \\ \vdots \\ \vdots \\ \\ n_{N-1} \end{bmatrix}
$$

A circulant channel matrix

$$\begin{bmatrix} g_{1,1} & \cdots & g_{1,n} \\ \vdots & & \\ g_{r,1} & \cdots & g_{r,n} \end{bmatrix}$$

s_{N-1}, s_{N-2}, s_{N-3}, s_0

$\mathbf{F}\mathbf{G}\mathbf{F}^{-1} = [\phi]$ circulant matrix

$y_1 = h_1 s_1 + n_1$

$y_2 = h_2 s_2 + n_2$

$y_3 = h_3 s_3 + n_3$

OFDM: Converting ISI into Parallel Channels

$F = $ DFT matrix

FS

- After removing the cyclic prefix at the receiver side, we have

$$\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ \vdots \\ \\ y_{N-1} \end{bmatrix} = \underbrace{\begin{bmatrix} g_0 & 0 & 0 & \cdots & 0 & g_3 & g_2 & g_1 \\ g_1 & g_0 & 0 & 0 & \cdots & 0 & g_3 & g_2 \\ g_2 & g_1 & g_0 & 0 & \ddots & \ddots & 0 & g_3 \\ g_3 & g_2 & g_1 & g_0 & 0 & \ddots & & 0 \\ 0 & g_3 & g_2 & g_1 & g_0 & 0 & & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ & & \ddots & \ddots & \ddots & \ddots & \ddots & \\ 0 & 0 & \cdots & 0 & g_3 & g_2 & g_1 & g_0 \end{bmatrix}}_{\mathbf{G}} \begin{bmatrix} s_0 \\ s_1 \\ s_2 \\ \vdots \\ \vdots \\ \\ s_{N-1} \end{bmatrix} + \begin{bmatrix} n_0 \\ n_1 \\ n_2 \\ \vdots \\ \vdots \\ \\ n_{N-1} \end{bmatrix}$$

$Fy = FGS + Fn$

DFT y

- G can be diagonalised by DFT matrices so that

$$\boxed{\mathbf{F}\mathbf{G}\mathbf{F}^{-1} \text{ is } \textit{diagonal} \text{ where } \mathbf{F} = \left[\frac{1}{\sqrt{N}} e^{-\frac{j2\pi}{N}(k-1)(l-1)}\right]_{k,l}}$$

$N \times N$

OFDM: Converting ISI into Parallel Channels

F S

- Therefore, the transmission process of OFDM is

$$\mathbf{s} \xrightarrow{\text{IDFT}} \mathbf{x} = \mathbf{F}^{-1}\mathbf{s} \xrightarrow{\text{add cyclic prefix}} \mathbf{x}'$$

DFT(y) = Fy

- At the receiver side, remove cyclic prefix and then DFT

$$\mathbf{y}' \xrightarrow{\text{remove cyclic prefix}} \mathbf{y} \xrightarrow{\text{DFT}} \tilde{\mathbf{s}} = \mathbf{F}\mathbf{y}$$

$\eta = Fn$

- The result of this is that

$\tilde{s} = DFT(y)$

Fy

$$\tilde{\mathbf{s}} = \begin{bmatrix} H_0 & 0 & \cdots & 0 \\ 0 & H_1 & & \vdots \\ \vdots & & \ddots & \\ 0 & \cdots & & H_{N-1} \end{bmatrix} \mathbf{s} + \boldsymbol{\eta}$$

FGF^{-1}

- Or

$\tilde{s}_n = \frac{\tilde{y}_n}{H_n}$

$$\tilde{s}_n = H_n s_n + \eta_n \quad \text{with no ISI}$$