FIR filter at time n

h(n)

FIR Filter Design

- FIR filter is **stable** and all poles are at the origin

always stable, all poles at origin, inside the unit circle. Compared to IIR filter, IIR may be unstable because of feedback

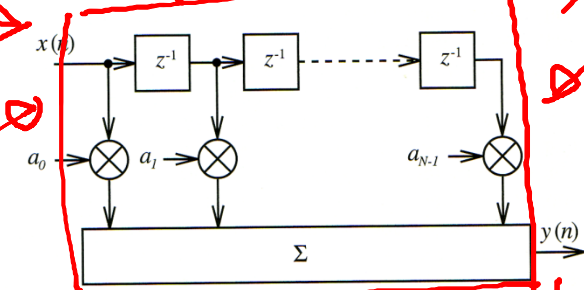
$$y(n) = \sum_{i=0}^{N-1} a_i x(n-i)$$

$$\Rightarrow H(z) = \frac{Y(z)}{X(Z)} = \sum_{i=0}^{N-1} a_i z^{-i}$$

$$\Rightarrow H(z) = \frac{a_0 z^{N-1} + a_1 z^{N-2} + \cdots + a_{N-1} z^0}{z^{N-1}}$$

feedforward path

IIR



h(n)

x feedback from y

- Therefore, H(z) has N-1 poles and zeros. The filter response is controlled by the positions of the zeros, i.e., the a coefficients.

$$y(n) = x(n) * h(n) = \sum_{i=-\infty}^{+\infty} h(i)x(n-i) = \sum_{i=0}^{N-1} h(i)x(n-i)$$

$h(n)$ a_0 a_1 .. a_{N-1} FIR → ①

0 1 .. $N-1$ n

Z-transform → ② $z = e^{j\omega\Delta t}$

FIR Filter Design

$H(\omega)$

sampling interval

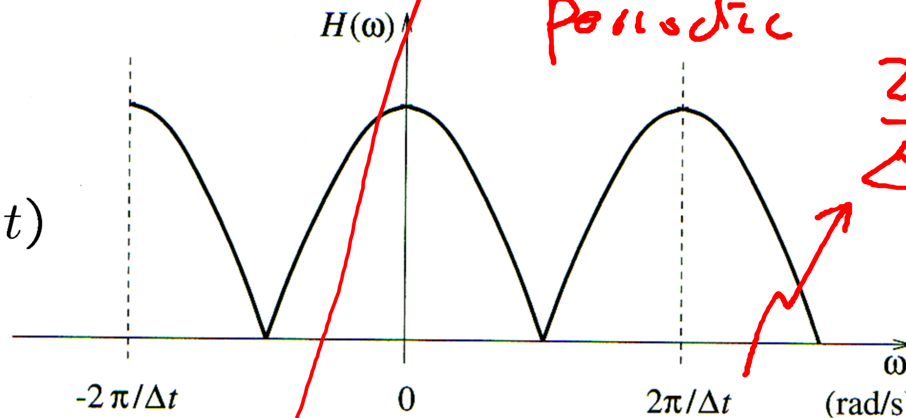
- The frequency response for an FIR filter is obtained by setting $z = e^{j\omega\Delta t}$

frequency response of any digital filter must be periodic

periodic

frequency domain formula

$$H(\omega) = \sum_{i=0}^{N-1} a_n \exp(-j\omega n\Delta t)$$



$\frac{2\pi}{\Delta t}$

- Since this is a periodic function in ω with period 2π/△t where 1/△t is the sampling frequency, we can express H(ω) using Fourier series expansion so that the Fourier coefficients are found as

$$a_n = \frac{\Delta t}{2\pi} \int_0^{\frac{2\pi}{\Delta t}} H(\omega) \exp(jn\omega\Delta t) d\omega$$

why ??

Fourier series expansion ✓ a periodic function

$x(t)$ periodic T

$$x(t) = \sum_{n=-\infty}^{+\infty} a_n e^{-jn\left(\frac{2\pi}{T}\right)t}$$ ✓

✓

$$a_n = \frac{1}{T}\int_0^T x(t)\, e^{jn\frac{2\pi}{T}t}\, dt$$ ✓

$H(\omega)$ period $\frac{2\pi}{\Delta t}$ ✓

$$H(\omega) = \sum_{n=-\infty}^{+\infty} a_n e^{-jn\left(\frac{2\pi}{\left(\frac{2\pi}{\Delta t}\right)}\right)\omega} = \sum_{n=-\infty}^{+\infty} a_n e^{-jn\omega\Delta t}$$

UCL

IIR

$$H(\omega) = \sum_{n=-\infty}^{+\infty} a_n e^{-jn\omega\Delta t}$$

Δt = sampling period

$$a_n = \frac{1}{\left(\frac{2\pi}{\Delta t}\right)} \int_0^{\frac{2\pi}{\Delta t}} H(\omega) e^{jn\omega\Delta t} \, d\omega$$

$a_0, a_1, a_2, \ldots, a_{N-1}$

$$a_n = \frac{\Delta t}{2\pi} \int_0^{\frac{2\pi}{\Delta t}} \boxed{H(\omega)} e^{jn\omega\Delta t} \, d\omega$$

FIR

$$H(\omega) = \sum_{n=0}^{N-1} a_n e^{-jn\omega\Delta t}$$

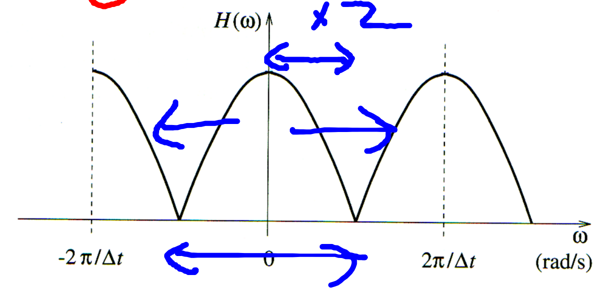
① z

② $z = e^{j\omega\Delta t}$

$$a_n = \frac{\Delta t}{2\pi}\int_0^{\frac{2\pi}{\Delta t}} H(\omega) e^{jn\omega\Delta t} d\omega$$

memorise

FIR Filter Design



- The integration limit can be reduced if H(ω) is even

H(ω) even function

$$a_n = \frac{\Delta t}{2\pi}\int_0^{\frac{2\pi}{\Delta t}} H(\omega)\exp(jn\omega\Delta t)d\omega$$

$$= \frac{\Delta t}{2\pi}\int_0^{\frac{2\pi}{\Delta t}} H(\omega)\cos(n\omega\Delta t)d\omega + j\frac{\Delta t}{2\pi}\int_0^{\frac{2\pi}{\Delta t}} H(\omega)\sin(n\omega\Delta t)d\omega$$

0

odd

$$= \frac{\Delta t}{2\pi}\int_0^{\frac{2\pi}{\Delta t}} H(\omega)\cos(n\omega\Delta t)d\omega$$

$$= \frac{\Delta t}{\pi}\int_0^{\frac{\pi}{\Delta t}} H(\omega)\cos(n\omega\Delta t)d\omega$$

$$\boxed{a_n = \frac{\Delta t}{\pi}\int_0^{\frac{\pi}{\Delta t}} H(\omega)\cos(n\omega\Delta t)d\omega}$$

linear phase response

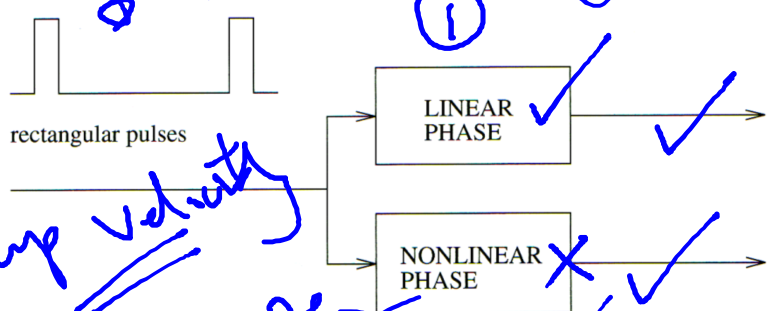
How to ensure that?

Linear and Non-linear Phase Filters

may diff freq components

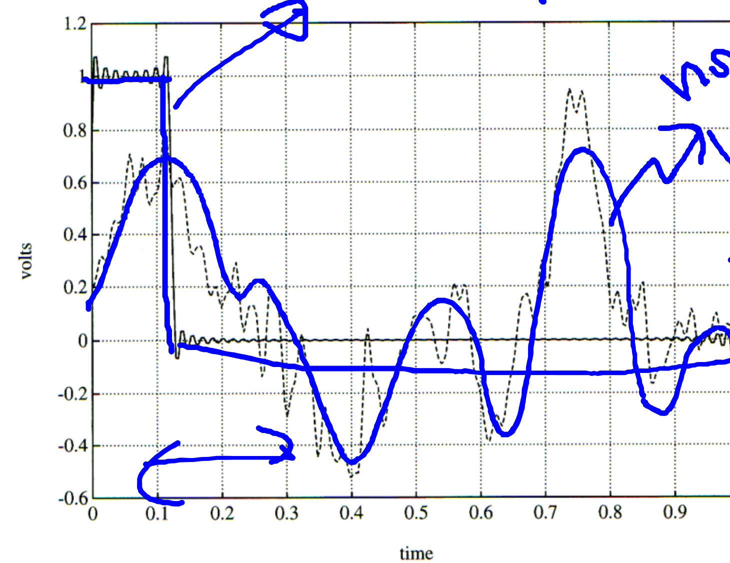
- Linear phase means that all signal inputs are subject to the same delay passing through the filter

linear phase filter



group velocity

phase velocity



non linear phase filter

dispersion

- So, linear phase filters are desirable and important!

In linear phase filter, group velocity and phase velocity are the same. In non-linear filter, group velocity and phase velocity are not the same

$H(\omega)$

What a Linear Phase Filter Looks Like?

nonlinear function

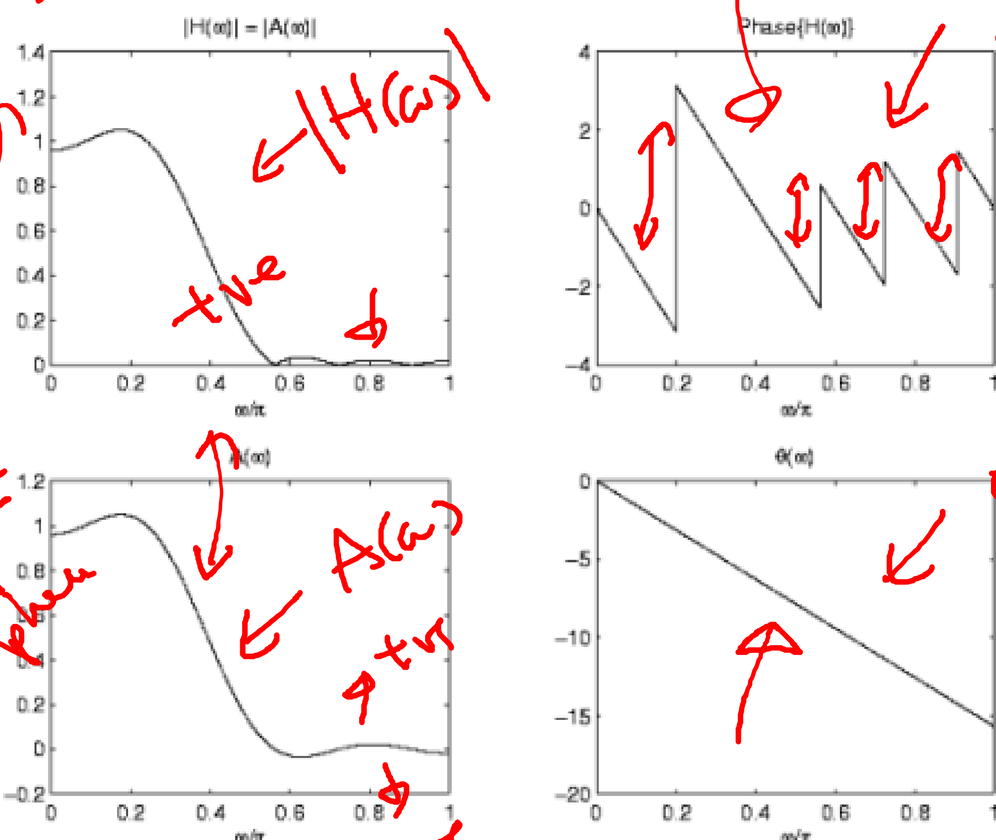
- A linear phase filter is one for which the continuous phase is linear

① magnitude

$H(\omega) = |H(\omega)| e^{j\theta(\omega)}$

② $H(\omega) = A(\omega) e^{j\phi(\omega)}$

amplitude



$\phi(\omega)$ linear in ω

linear

① stable ✓ ② linear phase response

What FIRs have Linear Phase?

symmetric

anti-symmetric

- Linear phase FIRs can be divided into four types:

always use larger number N

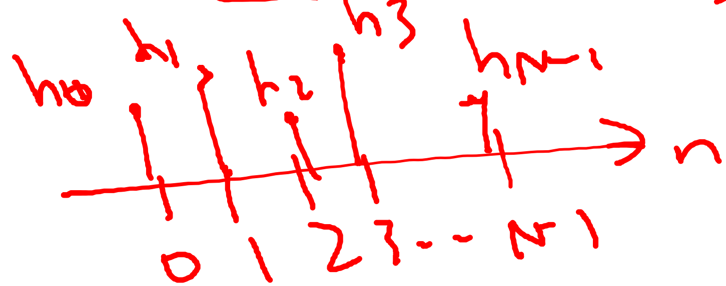
only type 1 filter for convenience

Type	impulse response	
I	symmetric	length is odd
II	symmetric	length is even
III	anti-symmetric	length is odd
IV	anti-symmetric	length is even

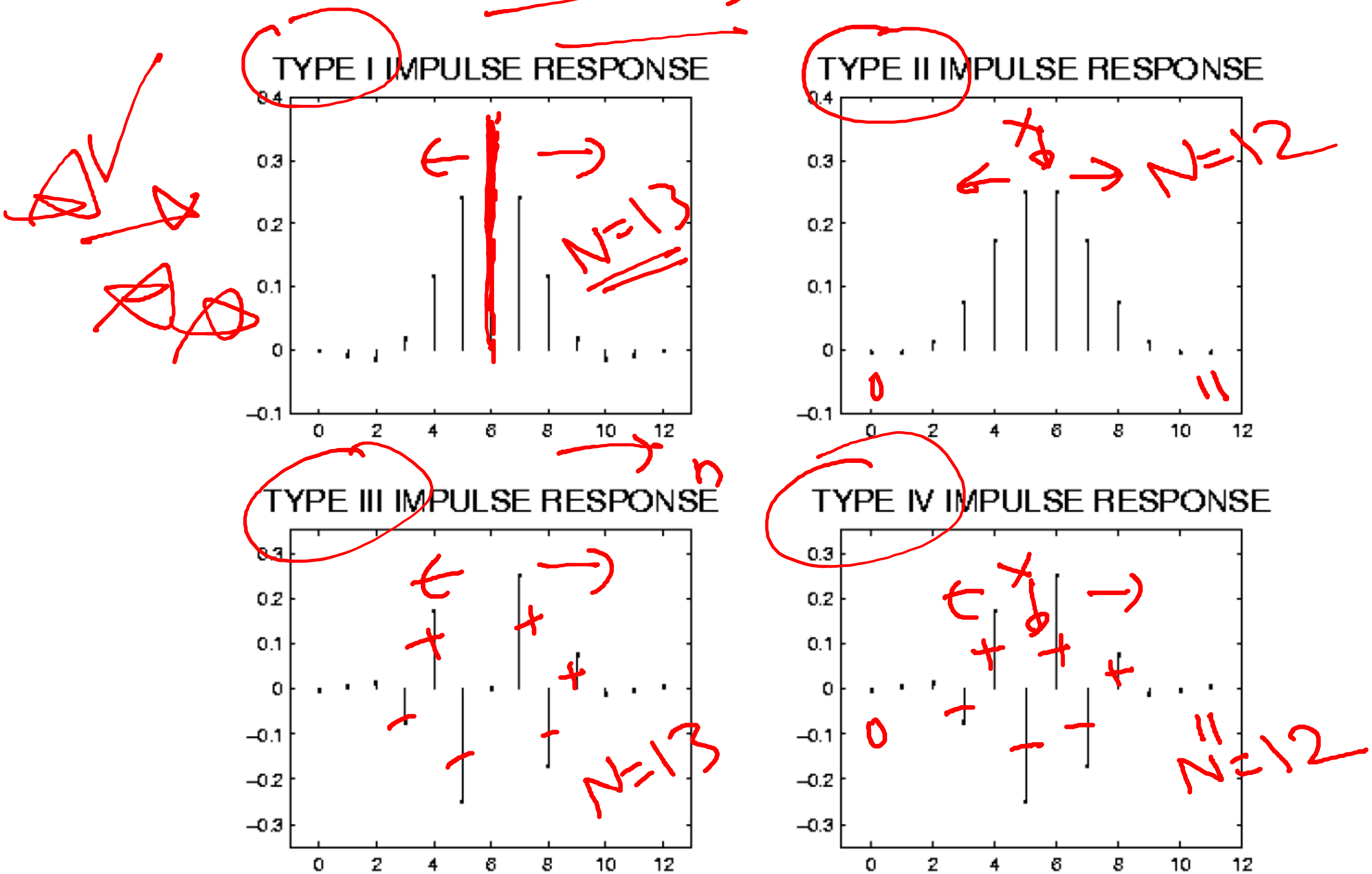
guaranteed to have linear phase response

"I"

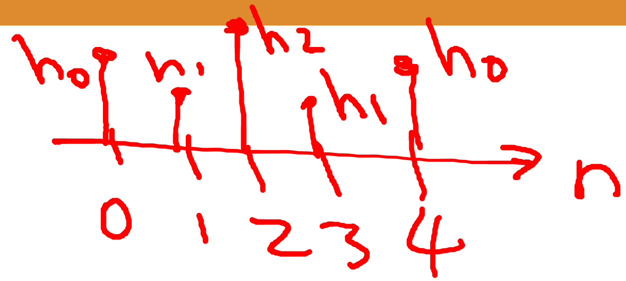
N



The 4 Types Linear Phase FIRs



Type I



H(ω)

① z-transform

② $z = e^{j\omega(1)}$

odd

How It Works?

- Let's see it for Type I odd-length symmetric sequences with N=5

$$H^f(\omega) = h_0 + h_1 e^{(-i)\omega} + h_2 e^{-2i\omega} + h_1 e^{-3i\omega} + h_0 e^{-4i\omega}$$

$$\Rightarrow H^f(\omega) = e^{-2i\omega}\left(h_0 e^{2i\omega} + h_1 e^{i\omega} + h_2 + h_1 e^{(-i)\omega} + h_0 e^{-2i\omega}\right)$$

$$\Rightarrow H^f(\omega) = e^{-2i\omega}\left(h_0(e^{2i\omega} + e^{-2i\omega}) + h_1(e^{i\omega} + e^{(-i)\omega}) + h_2\right)$$

$$\Rightarrow H^f(\omega) = e^{-2i\omega}\boxed{\left(2h_0\cos(2\omega) + 2h_1\cos(\omega) + h_2\right)}$$

A(ω)

- Therefore, the phase response is $\theta(\omega) = -2\omega$

odd

- In general, for a TYPE I FIR filter of length N, we have

$$\theta(\omega) = -M\omega \text{ where } M = \frac{N-1}{2}$$

N=2M+1

$H^f(\omega) = A(\omega)\, e^{j(-2\omega)}$

Linear Phase Filter Design

- **IIR filters**
  - \+ Steep sided frequency responses
  - \+ Low transition bandwidths
  - – Phase response often very non-linear

roll off rate greater

10 delay elements

- **FIR filters** – The FIR filter is achieved by removing the b coefficients from the general IIR filter difference
  - \+ Non-recursive
  - \+ Inherently stable
  - \+ Potentially linear phase responses
  - – Implementation requires more delay and multiply stages

25 delay elements

H(ω) ✓

Designing Linear Phase FIR Filter (TYPE I)

- **Step 1** Given H(ω), a digital filter can be obtained by

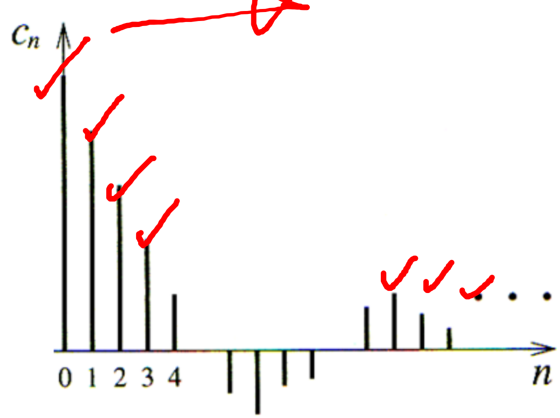
for $n \to 0$

$$\boxed{c_n = \frac{\Delta t}{\pi}\int_0^{\frac{\pi}{\Delta t}} H(\omega)\cos(n\omega\Delta t)\,d\omega}$$

memorise this please

which in general gives an IIR digital filter with taps

$$c_0, c_1, c_2, c_3, c_4, c_5, c_6, \ldots\ldots, c_\infty$$

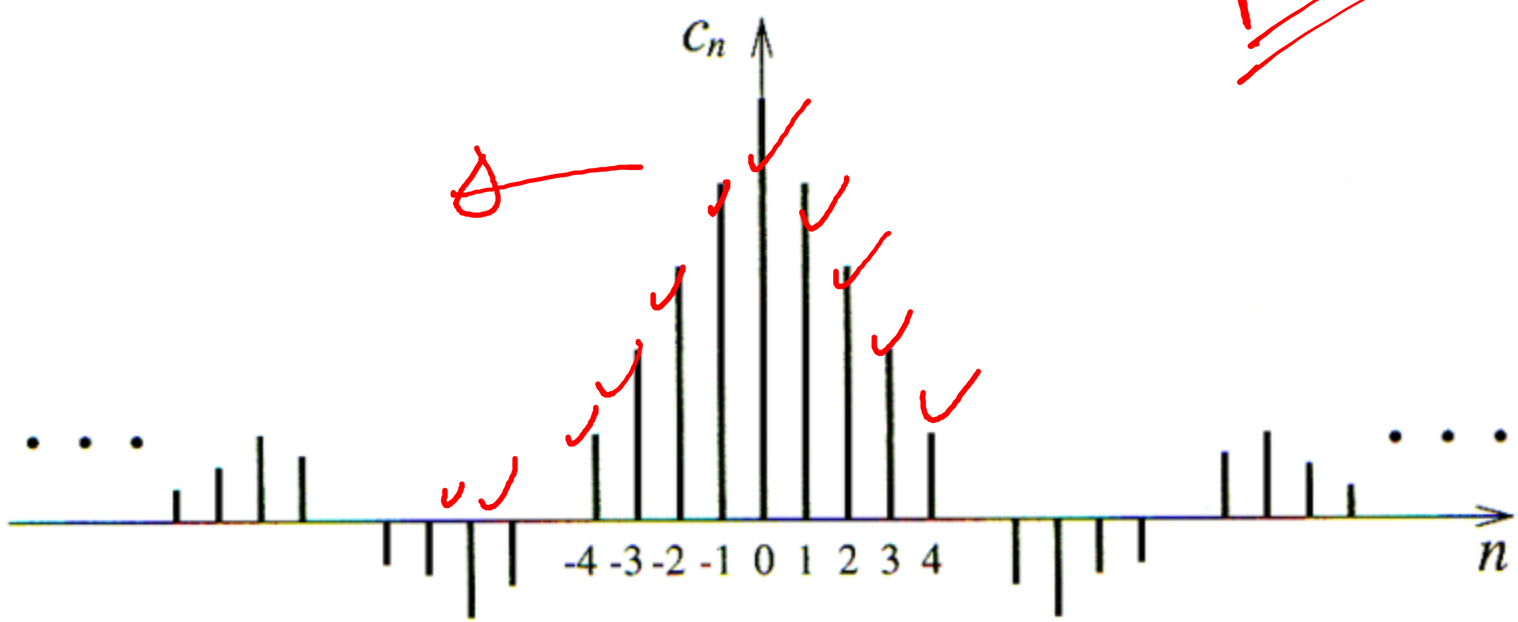


Designing Linear Phase FIR Filter (TYPE I)

- **Step 2** Copy c_n for positive n to the negative n to obtain

$$c_n = c_{-n}, \quad \text{for } n = 1, 2, 3, \ldots$$

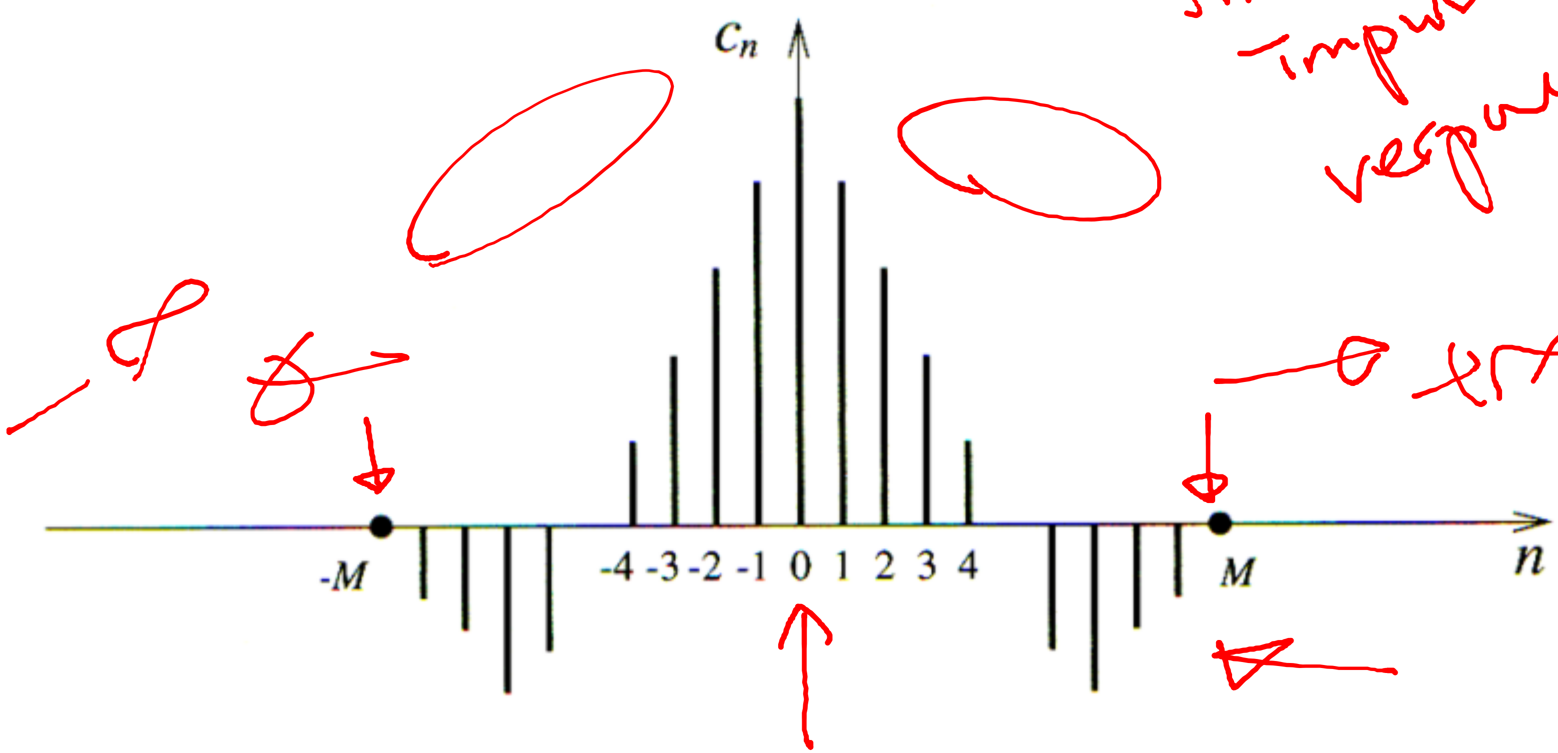
Symmetric



Designing Linear Phase FIR Filter (TYPE I)

- **Step 3** Truncate the sequence to obtain an FIR with length $\boxed{N=2M+1}$

finite impulse response



acausal filter ?

$y \uparrow \boxed{} \rightarrow x$

Designing Linear Phase FIR Filter (TYPE I)

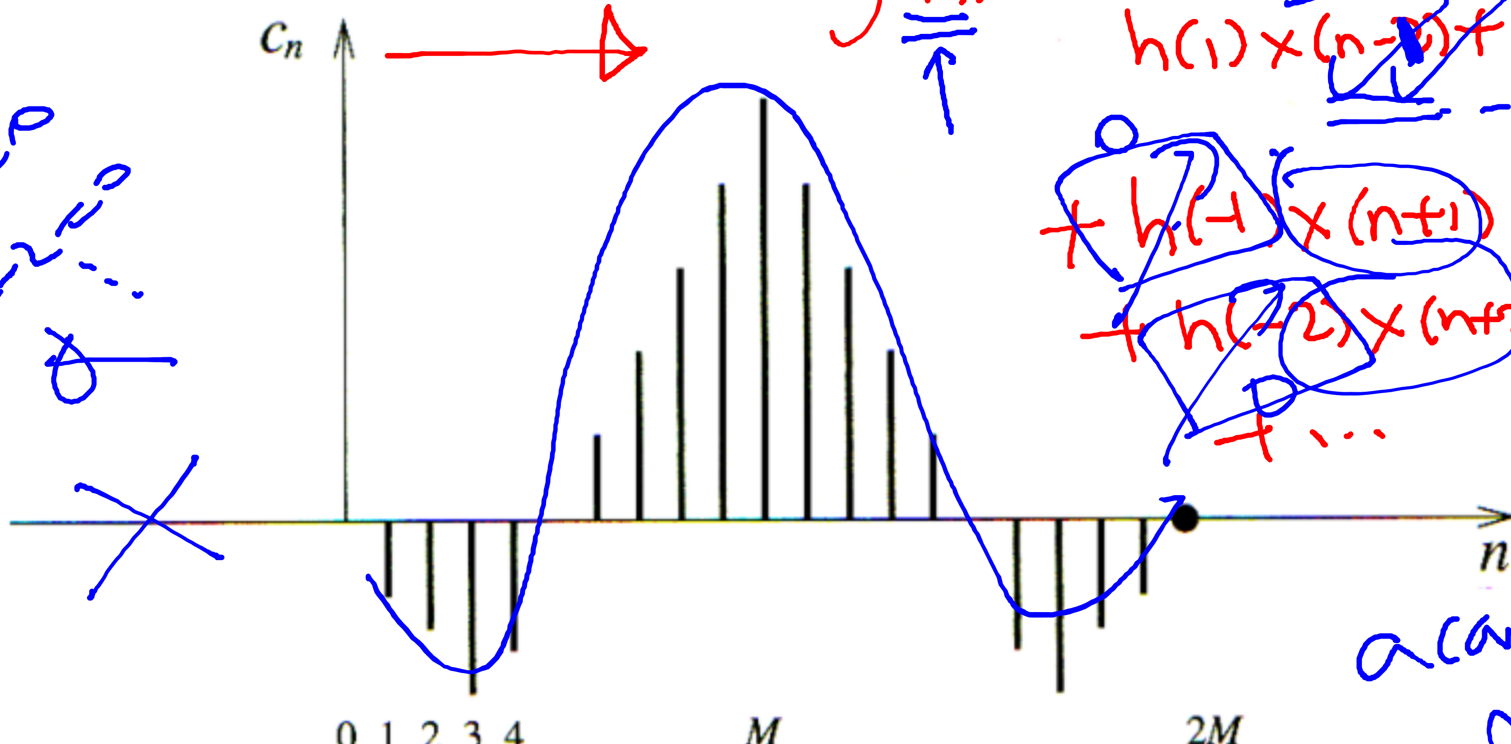
$$y(n) = \sum_{k=-\infty}^{+\infty} h(k)\,x(n-k)$$

- **Step 4** Shift the entire sequence to make it causal

$$y(n) = h(0)x(n) + h(1)x(n-1) + \dots + h(-1)x(n+1) + h(-2)x(n+2) + \dots$$

$c^{-1} = 0$

$c^{-2} = 0 \dots$



acausal filter

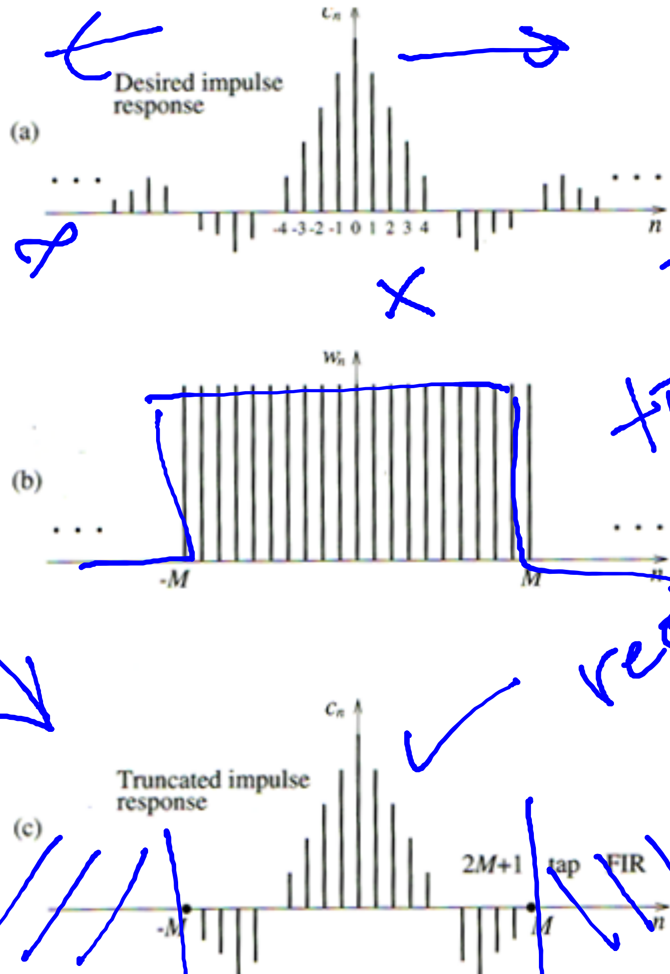
Windowing Effects

$a_n = \frac{\Delta t}{\pi}\int_0^{\frac{\pi}{\Delta t}} H(\omega)\cos n\omega\Delta t \, d\omega$

FIR

symmetric

$-\infty$ $+\infty$



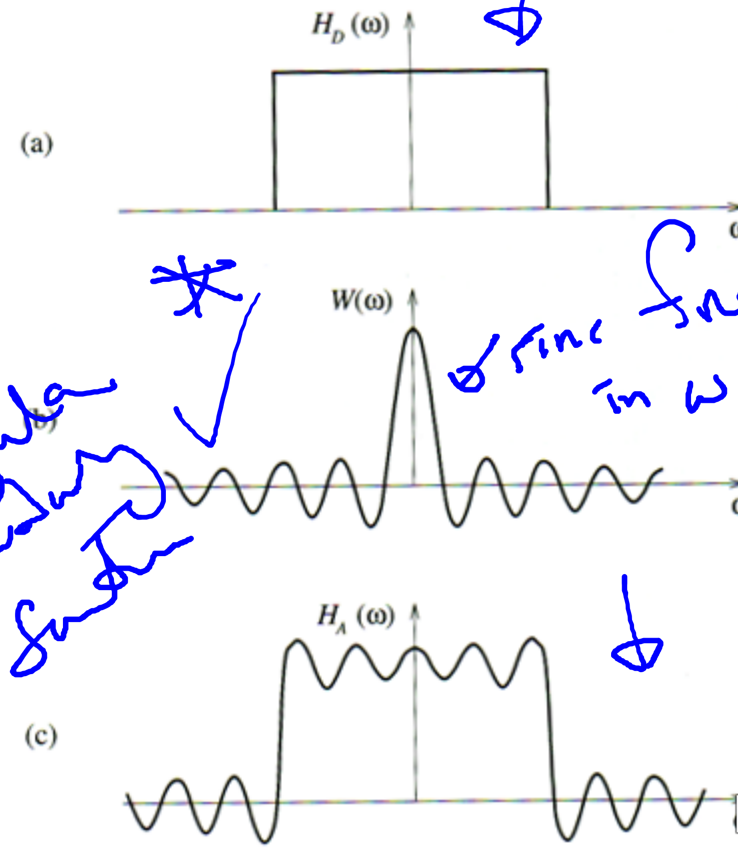
$-\infty$ $+\infty$

×

time

rectangular window function

LPF



*

sinc freq. in ω

Window Filter Design

$$M = \frac{N-1}{2}$$

don't memorise the functions

- Hanning window or raised cosine window

$$w_n = \frac{1}{2}\left[1 + \cos\left(\frac{n\pi}{M}\right)\right]$$

- Hamming window

$$w_n = 0.54 + 0.46\cos\left(\frac{n\pi}{M}\right)$$

- Blackman window

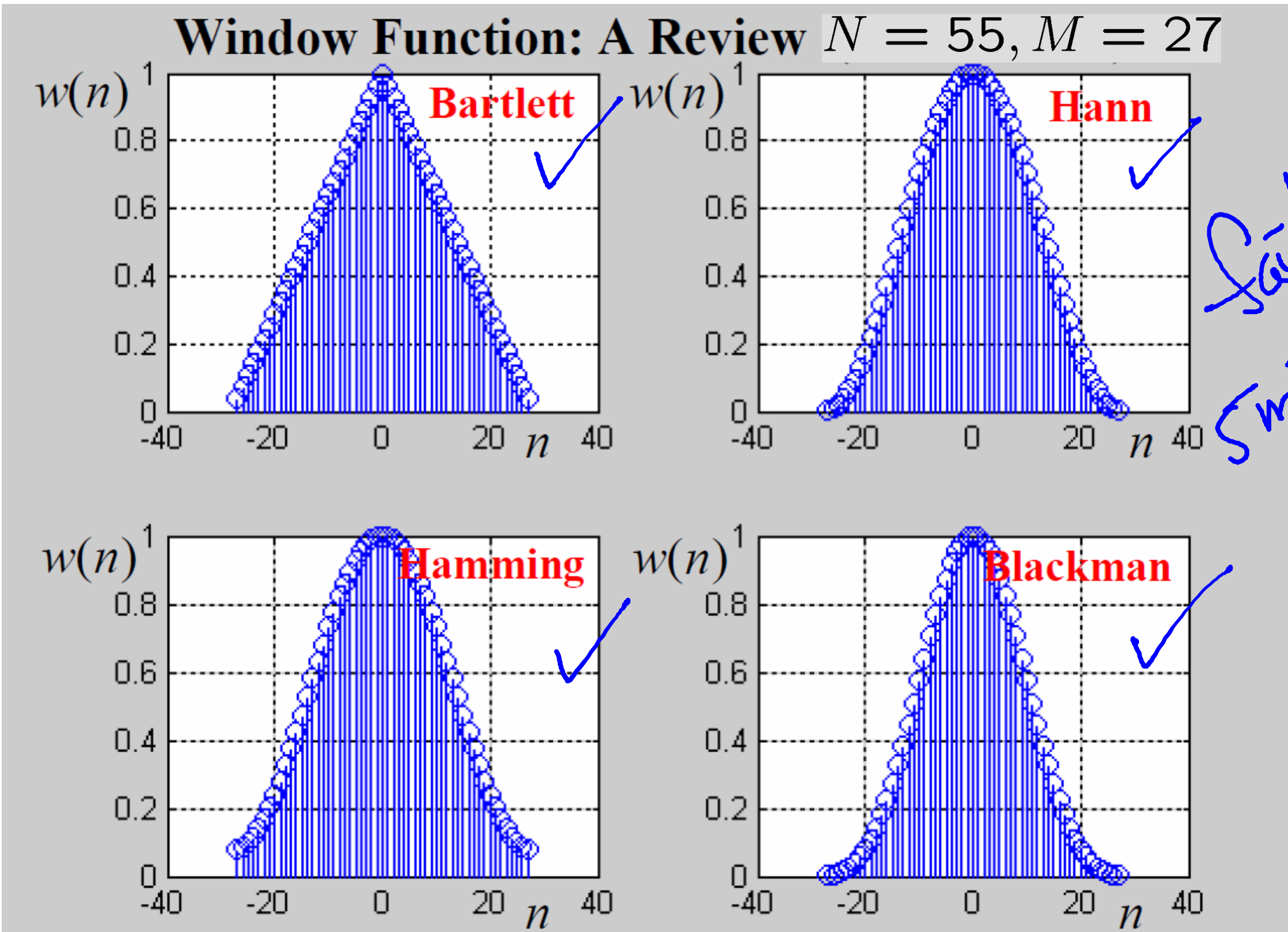
$$w_n = 0.42 + 0.5\cos\left(\frac{n\pi}{M}\right) + 0.08\cos\left(\frac{2n\pi}{M}\right)$$

- Kaiser window

$$w_n = \frac{I_0\left(\beta\sqrt{1-\left(\frac{2n}{M}\right)^2}\right)}{I_0(\beta)}$$

The modified 0-order Bessel function of the first kind

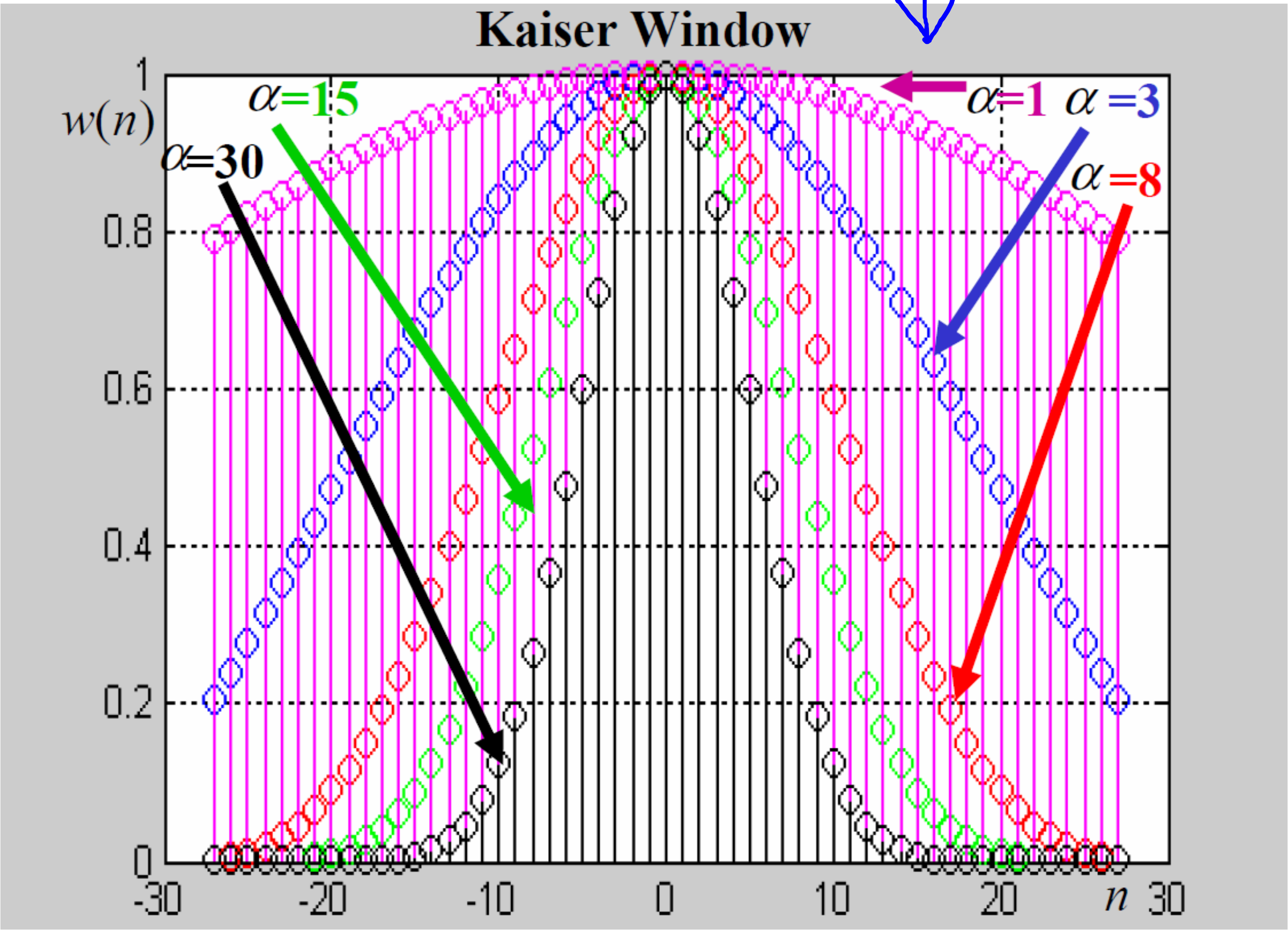
Window Filter Design



all

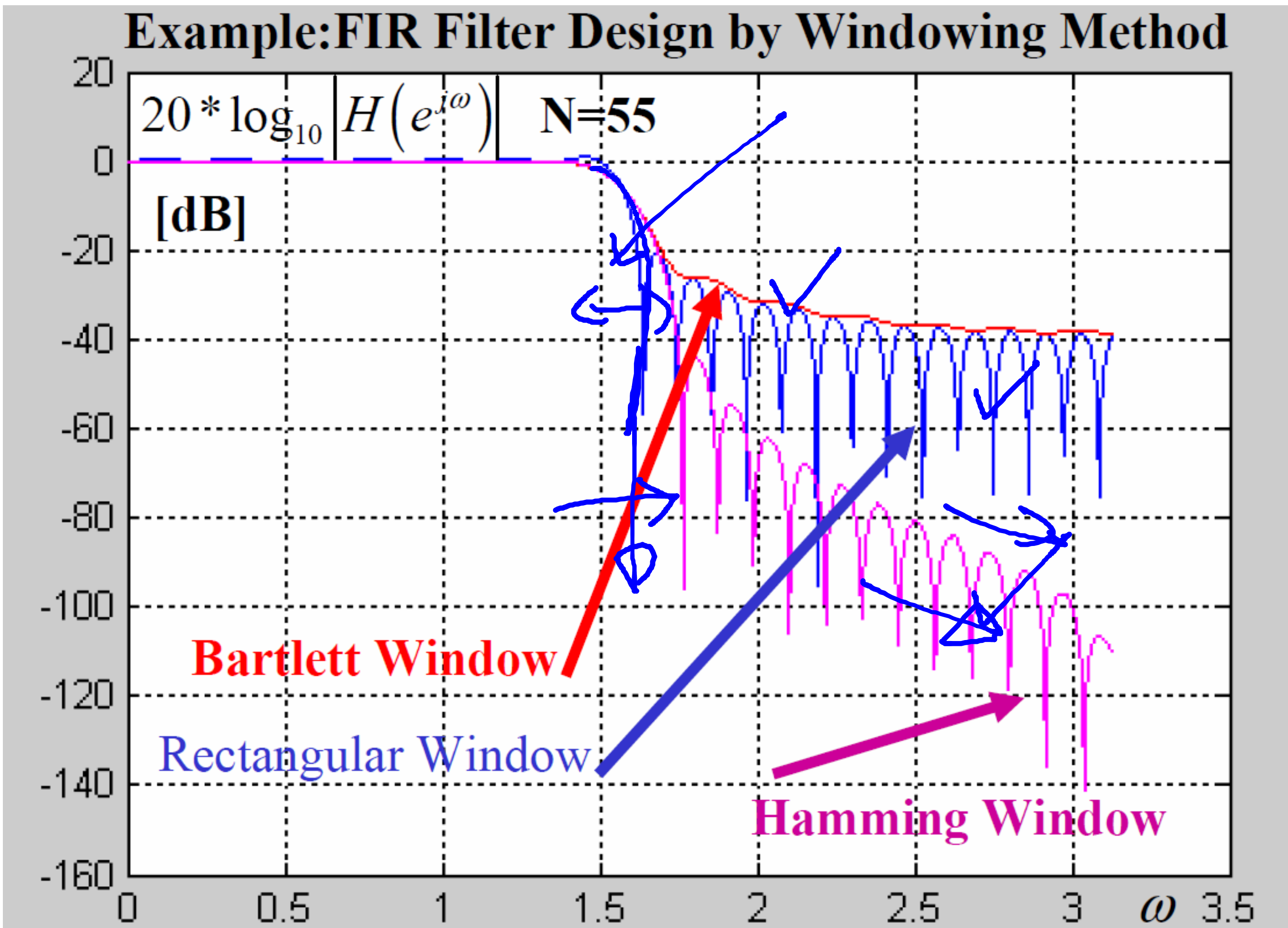
fairly similar

Window Filter Design

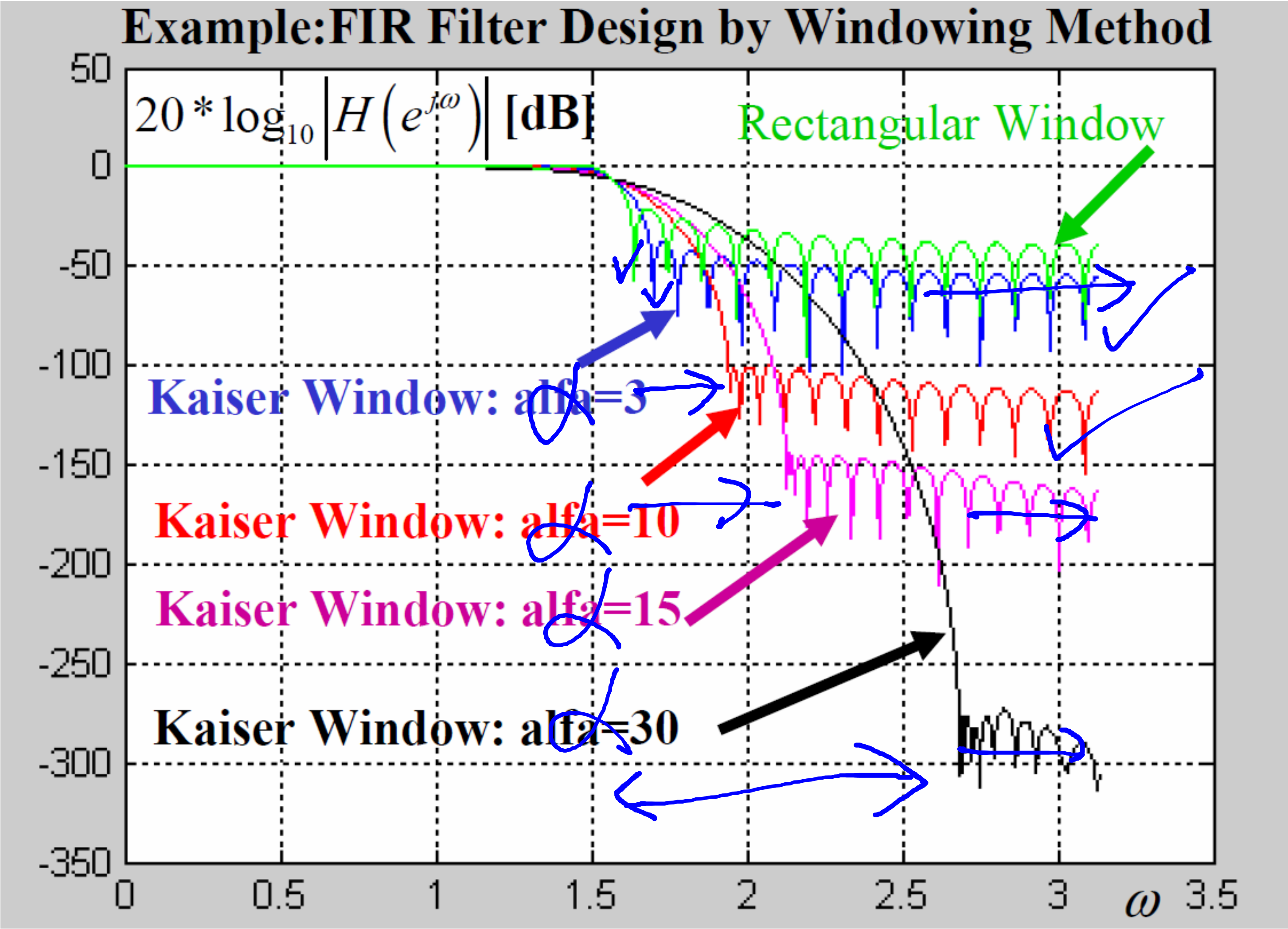


Window Filter Design

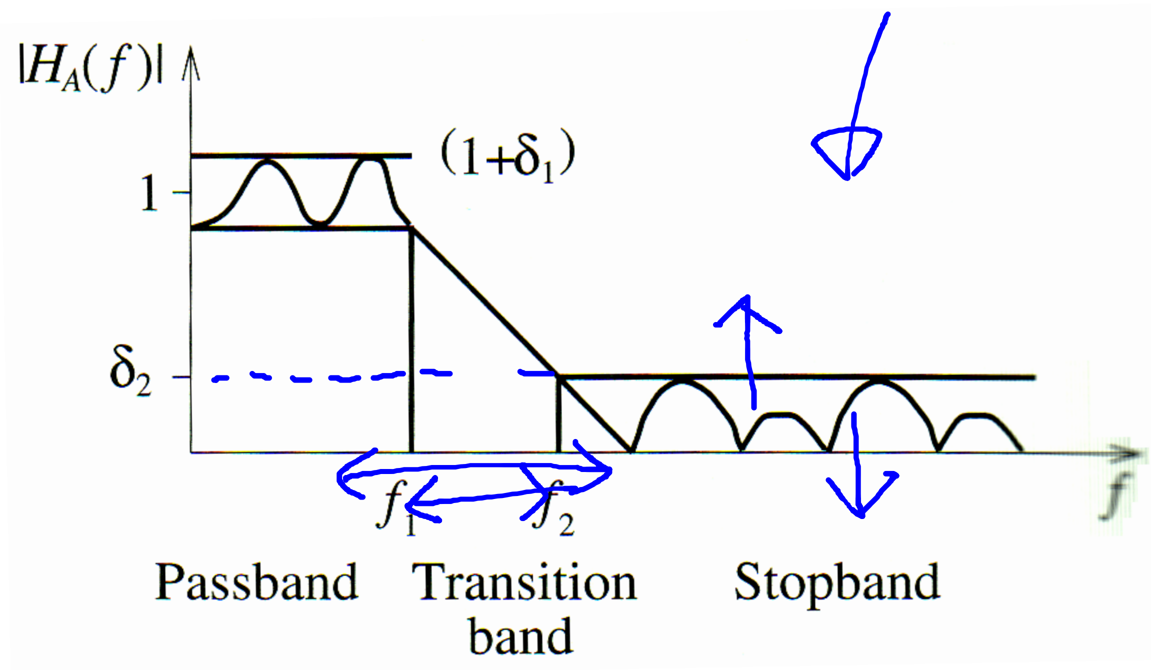
rectangular
transition ✓
stopband ✗



Window Filter Design



Window Filter Design



- Transition band $\boxed{f_2 - f_1 \ \text{(Hz)}}$
- Passband ripple $\boxed{20 \log 10(1 + \delta_1) \ \text{(dB)}}$
- Stopband rejection $\boxed{20 \log 10(\delta_2) \ \text{(dB)}}$

order N $N\Delta t$ → improve the performance of the filter   memorize this table

Window Characteristics

$N, \Delta t$ ↓ $f_2 - f_1$ $20\log_{10}\delta_2$

Window	Transition band (Hz)	Stopband rejection (dB)
✓ Rectangular	$\frac{1}{N\Delta t}$	21 ✓
✓ Hanning	$\frac{3.1}{N\Delta t}$	44 ✓
✓ Hamming	$\frac{3.3}{N\Delta t}$ ✓	53 ✓
✓ Kaiser, β =6	$\frac{4}{N\Delta t}$	63
✓ Blackman	$\frac{5.5}{N\Delta t}$	74
✓ Kaiser, β =9	$\frac{5.7}{N\Delta t}$	90

w_n step 5 $a_n = C_n w_n$

increase order of FIR filter does not change the stop band rejection, but change transition band