Nonlinear Systems

Motivation for Simulating Nonlinear Systems

- In principle the performance of linear systems can be tackled by analytical means but the study of nonlinear systems by such means is by and large intractable

- Hence, simulation is generally the appropriate tool for most nonlinear com systems because simulation of such systems is no more difficult than for linear systems given the model

- **Note that** the transform methods for linear systems cannot strictly be applied since superposition does not hold! The nonlinear system, therefore, generally has to be simulated in the *time domain*

Modelling Considerations

- For nonlinear system models, a number of model-type descriptors are:

  1. ***Memoryless models***
  2. ***Models with memory***
  3. Baseband models
  4. Bandpass models
  5. Block (input/output) models
  6. Analytical models
  7. Nonlinear differential equation models

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Modelling Considerations

- The *most significant* categorical distinction is:

  - Zero-memory nonlinearity (ZMNL)
  - Nonlinearity with memory (NLWM)

- The term “memoryless” implies that the output of a device is a function of the input signal at the present instance only!

- Memoryless models are an idealisation: **No** physical device is truly (input) frequency independent. Rather, as the BW of an input signal increases, we can expect filtering effects to become manifest

Memoryless Nonlinearities

- **Memoryless Baseband Nonlinearities** – This model is characterised by a simple functional relationship of the form

$$y(t) = F\left[x(t)\right]$$

- Our definition of a baseband nonlinearity means that x(t) is a baseband signal, meaning that its power (or energy) is spectrally concentrated around zero frequency and this will also be true of y(t)

- Certain nonlinearity can be given in analytical form, e.g., a diode

$$I = I_s\left(e^{\lambda V} - 1\right)$$

- The exponential can be expanded into a power series which could be truncated after a moderate number of terms with acceptable error

Memoryless Nonlinearities

- This suggests that F might also be representable by a power series, or by an orthogonal function expansion, or by a polynomial in x. Thus,

$$\boxed{y(t) = F\left[x(t)\right] \approx \sum_{n=0}^{N} a_n x^n(t)}$$

- The coefficients a_n may be obtained by fitting a polynomial of degree N

- Often, the most efficient and accurate model is to use the raw experimental data themselves in a lookup table using appropriate interpolation between tabulated data points

Memoryless Nonlinearities

- **Estimating the Sampling Rate for Nonlinear Systems** – Given a nonlinear model, the sampling rate in simulation must be increased

- Suppose that the input x(t) is bandlimited to ±B/2 and assume that the polynomial approximation on the previous slide holds. Then, we have

$$\boxed{Y(f) = a_0\delta(f) + \sum_{n=1}^{N} a_n \left[X(f) \overset{n-1}{*} X(f) \right]}$$

(n-1)-fold convolution

- Hence, the term $x^n(t)$ has BW, nB (i.e., ±nB/2)

Memoryless Nonlinearities

- As we know, the sampling rate for any bandlimited system has to be at least twice the BW in order not to introduce aliasing distortion
- To avoid aliasing error for a nonlinear system, it would seem that we must have $f_s > 2NB$. However,
  - For many nonlinear systems the coefficients a_n decrease with n
  - The magnitude of the spectrum of $x^n(t)$ is not uniformly distributed over nB. Indeed, as n increases, we generally expect the spectrum of $x^n(t)$ to be increasingly concentrated around zero frequency
- Hence if the sampling rate is less than 2NB, only the relatively low valued tails of the spectrum are aliased $\rightarrow$ error is small!

Memoryless Bandpass Nonlinearities

- Let us start with $y(t) = F\left[x(t)\right]$

- Consider an input bandpass signal

$$x(t) = A(t)\cos\left[2\pi f_c t + \theta(t)\right] \equiv A(t)\cos\alpha(t)$$

- Considered as a function of α, the nonlinearity output

$$z = F\left(A\cos\alpha\right)$$

is a periodic function and hence can be expanded in a Fourier series

$$z = a_0 + \sum_{k=1}^{\infty}\left(a_k\cos k\alpha + b_k\sin k\alpha\right)$$

- This expression makes explicit the *harmonics* of the carrier. Hence, we can see the possibility of a model for a filter characteristic at each kf_c

Memoryless Bandpass Nonlinearities

- We are usually interested in the *first-zone* output, i.e., at f_c or k=1 term

- For k≥1, the coefficients in the Fourier series are given by

$$\begin{cases} a_k \equiv g_{k1}(A) = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} F(A\cos\alpha)\cos k\alpha\, d\alpha \\ b_k \equiv g_{k2}(A) = \dfrac{1}{\pi}\displaystyle\int_0^{2\pi} F(A\cos\alpha)\sin k\alpha\, d\alpha \end{cases}$$

- For k=1, we can write the first-zone output y(t) as

$$y(t) = g_{11}\left(A(t)\right)\cos\left(2\pi f_c t + \theta(t)\right) + g_{12}\left(A(t)\right)\sin\left(2\pi f_c t + \theta(t)\right)$$

Memoryless Bandpass Nonlinearities

- Clearly, the complex envelope of the first-zone output is given by

$$\tilde{y}(t) = \left[g_{11}\left(A(t)\right) - jg_{12}\left(A(t)\right)\right] e^{j\theta(t)}$$

- The first-zone expression implies that the signal can undergo both amplitude and phase distortions. In most cases g_{12} will be zero and hence the model under consideration leads to only the first term

$$y(t) = g_{11}\left(A(t)\right) \cos\left(2\pi f_c t + \theta(t)\right)$$

- Thus, can produce only amplitude distortion. Similarly, the complex envelop reduces to the first term

$$\tilde{y}(t) = g_{11}\left(A(t)\right) e^{j\theta(t)}$$

which is the form in which it would be implemented in simulation

Memoryless Bandpass Nonlinearities

- **Power Series Model** – It has the nice property of being able to expose clearly the effect that it has on an input signal. Again, we have

$$y(t) = F\left[x(t)\right] \approx \sum_{n=0}^{N} a_n x^n(t)$$

- Expressing the input signal in terms of the complex envelope

$$\boxed{x(t) = \mathrm{Re}\left\{\tilde{x}(t)e^{j2\pi f_c t}\right\} = \frac{1}{2}\left[\tilde{x}(t)e^{j2\pi f_c t} + \tilde{x}^*(t)e^{-j2\pi f_c t}\right]}$$

- Using the binomial expansion for $x^n(t)$, we obtain

$$\boxed{x^n(t) = \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}\left[\tilde{x}(t)\right]^k\left[\tilde{x}^*(t)\right]^{n-k} e^{j2\pi f_c(2k-n)t}}$$

Memoryless Bandpass Nonlinearities

- For the first-zone output we see that only terms where n is odd and 2k-n=1 can contribute. The first-zone contribution is then

$$\boxed{\frac{1}{2^n}\binom{n}{\frac{n+1}{2}}|\tilde{x}^*(t)|^{n-1}\tilde{x}(t) \quad \text{for odd } n}$$

- Summing all odd n, the complex envelope of the first-zone of y(t) is

$$\boxed{\tilde{y}(t) = \tilde{x}(t)\sum_{m=0}^{\frac{N-1}{2}} \frac{a_{2m+1}}{2^{2m}}\binom{2m+1}{m+1}|\tilde{x}^*(t)|^{2m}}$$

Memoryless Bandpass Nonlinearities

- The bandpass output signal can now be written as

$$\begin{aligned} y(t) &= \mathrm{Re}\left[\tilde{y}(t)e^{j2\pi f_c t}\right] \\ &= \left[\sum_{m=0}^{\frac{N-1}{2}} \frac{a_{2m+1}}{2^{2m}} \binom{2m+1}{m+1} |\tilde{x}^*(t)|^{2m+1}\right] \cos\left(2\pi f_c t + \theta(t)\right) \\ &= g_{11}\left(A(t)\right) \cos\left(2\pi f_c t + \theta(t)\right) \end{aligned}$$

Nonlinearities with Memory (NLWM)

- The memoryless models represent fairly accurately a variety of devices driven by narrowband inputs

- The suitability of a memoryless model for narrowband signals stems from the fact that many RF amplifiers have indeed a wide BW and over any relatively small portion of the band the transfer characteristic does look nearly frequency independent

- However, when we send “wideband” signals by which we mean that the BW of the signal is comparable to the inherent BW of the device, then we are bound to encounter some frequency-dependent behaviour

❑ **Analytical models** such as the Volterra series are, under some circumstances, able to represent exactly the functioning of a NLWM

Volterra Series Modelling

- The Volterra series approach to modelling is appealing due to its generality and its relatively intuitive decomposition of the response of a nonlinear system into that due to an equivalent linear filter plus additional terms produced by nonlinear behaviour

- A Volterra series is described by

$$\boxed{y(t) = \sum_{n=0}^{\infty} y_n(t)}$$

where the term of order n is given by an n-fold convolution

$$\boxed{y_n(t) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} h_n(\tau_1, \ldots, \tau_n) x(t-\tau_1) \cdots x(t-\tau_n) d\tau_1 \cdots d\tau_n}$$

Volterra Series Modelling

- Volterra series expansion provides an input-output relationship for nonlinear time-invariant continuous systems
- The functions h_0, $h_1(t)$, $h_2(t_1,t_2)$, … are called the ***Volterra kernels***
- The 0-th order term accounts for the response to a dc input
- The 1^{st} order kernel is the IR of a linear system
- The higher-order kernels are higher-order IRs for nonlinear behaviours
- The computation necessary to produce an output sample from the n-th kernel box is that for n=1, raised to the n-th power, hence prohibitive

Volterra Series Modelling

- Block diagram interpretation of the Volterra series representation