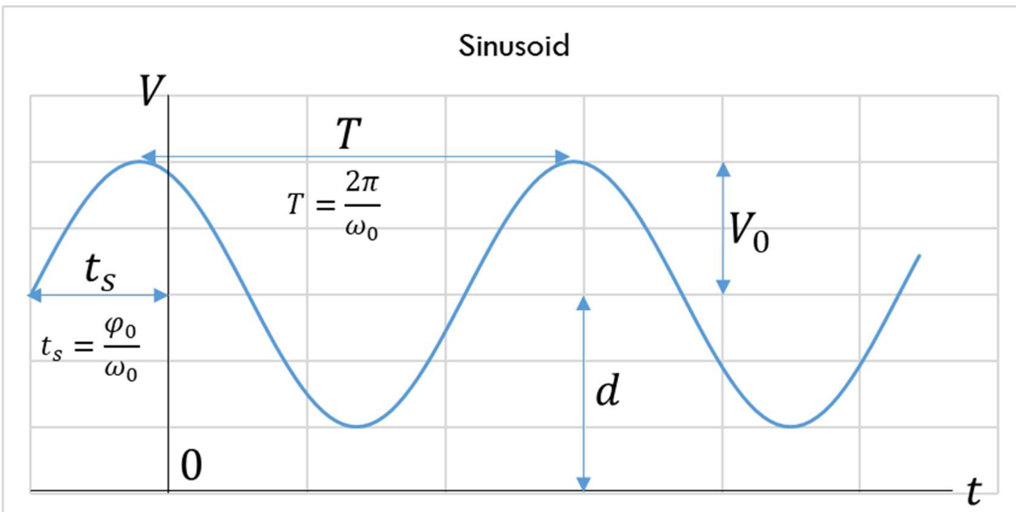
Lecture 1 – Sinusoids

By the end of this lecture, you should be able to:

- Describe a signal in frequency.
- Describe, verbally, an instrument's sound wave with regards to sinusoids.
- See why complex representation is often more convenient than trigonometric.

Sinusoids are fundamental to communication systems, so you should know them intimately. This reading has some properties that you should be familiar with before the lecture, as well as some new perspectives on sinusoids.

A sinusoid is a periodic function, meaning that it repeats itself after a certain amount of time called the **period**. The following graph shows an arbitrary sinusoid that could be seen on an oscilloscope:



T is the period in seconds [s]

ω_0 is the **angular frequency** in radians per second [rads^{-1}]

V_0 is the maximum **amplitude**, which is often in volts [V] for signals

d is the **mean** of the sinusoid (also called DC bias, same units as V_0 [V])

t_s is the time shift if this were a sine wave (as opposed to cosine)

φ_0 is the **initial phase** in radians [rad]

V is the voltage axis [V]

t is the time axis [s]

The equation of this sinusoid looks like this:

$$V(t) = V_0 \sin(\omega_0 t + \varphi_0) + d$$

Remember that sine and cosine are related by the following formulae:

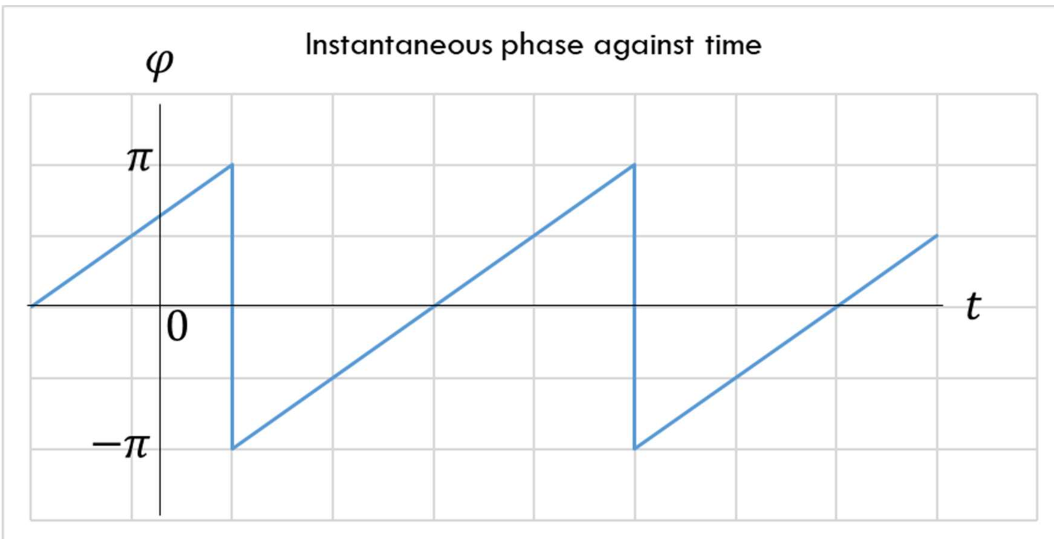
$$\sin t = \cos\left(t - \frac{\pi}{2}\right)$$

$$\cos t = \sin\left(t + \frac{\pi}{2}\right)$$

This means that we could also write the above sinusoid as:

$$V(t) = V_0 \cos\left(\omega_0 t + \varphi_0 - \frac{\pi}{2}\right) + d$$

The terms in the parentheses are known as the **instantaneous phase** φ of the sinusoid, which shouldn't be confused with the *initial* phase φ_0. Here's a graph of instantaneous phase against time for the sinusoid above (for the sine, not the cosine):



This is quite simple, but this concept will become more significant in the future (hint – if the period decreased, what would happen to the gradient?). **Note that instantaneous phase is defined only on the interval $[-\pi, \pi]$;** any value outside that range becomes redundant (and will be marked wrong on the exam!).

We tend not to use angular frequency when describing sinusoids, and just use frequency f_0, measured in hertz [Hz], which is given by:

$$f_0 = \frac{1}{T}$$

For example, in the UK, the voltage measured from the mains oscillates at 50 Hz, i.e., it goes through 50 periods or cycles in one second. Frequency is related to angular frequency by 2π:

$$\omega_0 = 2\pi f_0$$

You may be wondering where the equations shown in the graph come from. Time shift can be derived by setting the sinusoid equation equal to zero, substituting t_s for t, and solving for t_s. To derive period, consider that angular frequency is defined as the radians ‘travelled’ per second. A sinusoid ‘travels’ 2π radians per cycle, and since the length of each cycle is defined as one period, the angular frequency is given by $\omega_0 = {}^{2\pi}/_{T}$. Rearrange for T.

Also related to sinusoids is complex representation. From Euler’s identity (derived from power series), we can write the complex exponential as a complex sum of cosine and sine:

$$e^{j\theta} = \cos\theta + j\sin\theta$$

This is an extremely important identity that often makes electrical engineering much easier. You’ll be seeing this quite frequently in signals and systems. Here are some derivations that may (definitely will) be helpful:

$$e^{-j\theta} = \cos(-\theta) + j\sin(-\theta)$$

$$e^{-j\theta} = \cos\theta - j\sin\theta$$

$$e^{j\theta} - j\sin\theta = \cos\theta$$

$$e^{-j\theta} - \cos\theta = -j\sin\theta$$

$$e^{j\theta} + (e^{-j\theta} - \cos\theta) = \cos\theta$$

$$e^{j\theta} + e^{-j} \ = 2\cos\theta$$

$$\cos\theta = \frac{1}{2}(e^{j\theta} + e^{-j\theta})$$

$$e^{j\theta} - \cos\theta = j\sin\theta$$

$$e^{-j\theta} + j\sin\theta = \cos\theta$$

$$e^{j\theta} - (e^{-j\theta} + j\sin\theta) = j\sin\theta$$

$$e^{j\theta} - e^{-j\theta} = 2j\sin\theta$$

$$\sin\theta = \frac{1}{2j}(e^{j\theta} - e^{-j\theta})$$