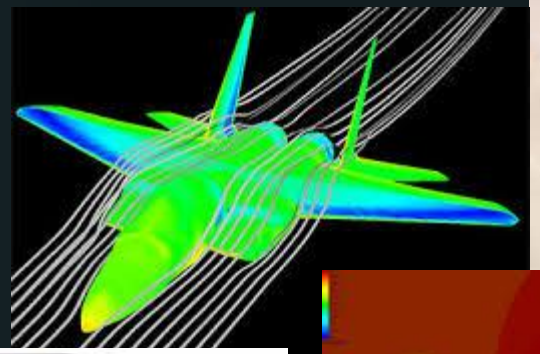
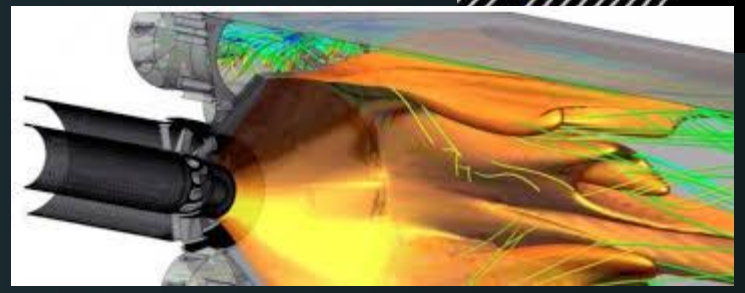
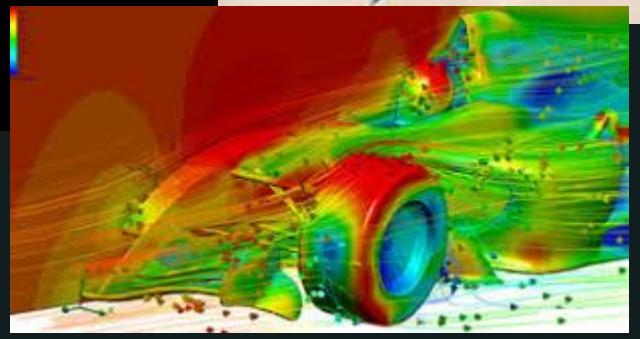
Fluid Dynamics

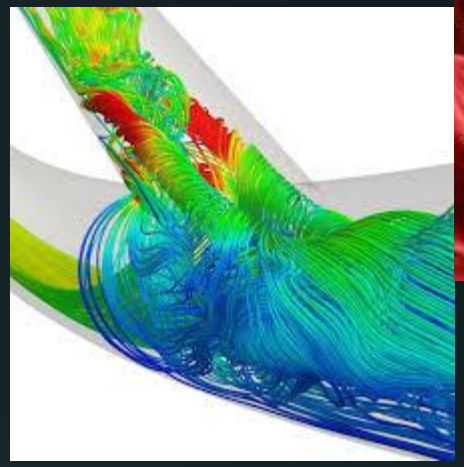
Lecture IV – The Full Potential Equation: Transonic Case

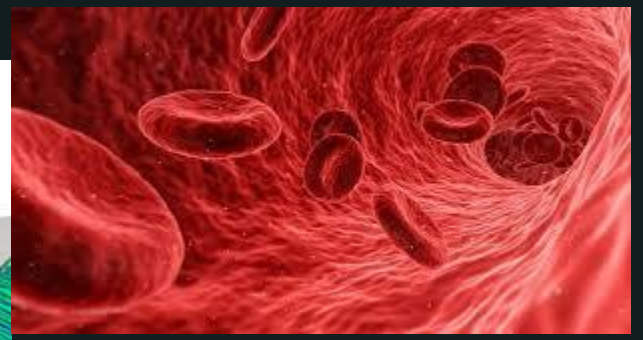
Shlomo Ta'asan
Professor Emeritus
Carnegie Mellon University

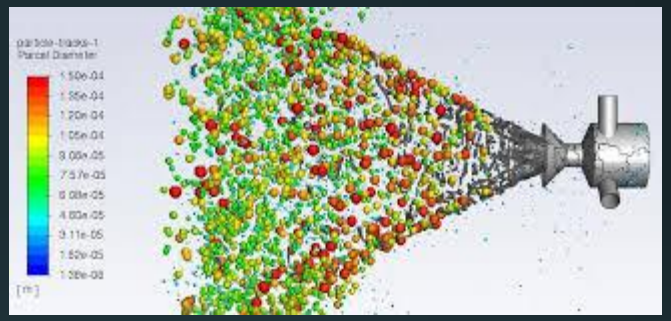












Outline

Today we focus on the 2D/3D full potential equation and its approximations for transonic flows.

To understand it we start with simplified models that capture the physics of the full model.

-The Wave Equation and Its Discretization
- Small Disturbance Approximation equation and its relation to the wave equation
- Transonic flows and discretization
- Model problems – flow around airfoil, channel flow.

The Wave Equation

The wave equation plays an important role in supersonic and transonic flows

$$\frac{\partial^2 u}{\partial t^2} - c^2 \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right] = 0$$

We need to approximate it using a numerical scheme.

Since we will be dealing with 2D flows it is enough to consider one space dimension. The extension of the numerical scheme to the 3D flow is trivial.

We will focus on the finite differences approach

In the full potential equation or in its approximation by the small disturbance equation the flow direction correspond to the time dimension in the wave equation. So, although we consider time independent full potential equation, we understand it in terms of the wave equation above.

Approximation using finite differences - review

First order derivative, using calculus

$$\frac{1}{h}[f(x+h) - f(x)] \approx f'(x) + O(h)$$

the error term O(h) can be derived using Taylor expansion

A better approximation for first derivative is

$$\frac{1}{2h}[f(x+h) - f(x-h)] \approx f'(x) + O(h^2)$$

Second derivative are approximated as

$$\frac{1}{h^2}[f(x+h) - 2f(x) + f(x-h)] \approx f''(x) + O(h^2)$$

with second order accuracy

The Wave Equation

The Wave Equation

$$\frac{\partial^2 u}{\partial t^2} - c^2 \frac{\partial^2 u}{\partial x^2} = 0$$

finite differences

$$u_j^n \approx u(t,x) \qquad t = n\Delta t;\ x = j\Delta x$$

$$\frac{u_j^{n+1} - 2u_j^n + u_j^{n-1}}{\Delta t^2} - c^2 \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} = 0$$

For stability we look for sol of the form $u_j^n = A^n e^{i\theta j}$ $\quad |\theta| \leq \pi$

stability means $|A| \leq 1$.

The Wave Equation - cont

plug this into eq, we get

$$\cancel{A^n}\,\frac{A-2+1/A}{\Delta t^2}\,\cancel{e^{i\theta j}} - c^2\,\frac{e^{i\theta}-2+e^{-i\theta}}{\Delta x^2}\,\cancel{A^n e^{i\theta j}} = 0$$

let $S = \frac{c^2 \Delta t^2}{\Delta x^2}$.

$$A + \frac{1}{A} - 2 - S\left(2\cos\theta - 2\right) = 0$$

$$A + \frac{1}{A} = 2 + 2S\left(\cos\theta - 1\right) \equiv p$$

$p \le 2$

$$A + \frac{1}{A} = p \qquad A^2 - pA + 1 = 0$$

$$A_\pm = \frac{p \pm \sqrt{p^2-4}}{2}$$

instab, $|A_\pm|$

if $\boxed{p^2 - 4 > 0} \Rightarrow p < -2$ (since $p \le 2$)

$\Rightarrow$ 2 roots but $A_- = \frac{p - \sqrt{p^2-4}}{2} < \frac{p}{2} < \frac{-2}{2} = -1$

The Wave Equation - cont

If $p^2-4 \le 0 \Rightarrow -2 \le p \le 2$ roots are complex

$$A_\pm = \frac{p}{2} \pm \frac{i}{2}\sqrt{4-p^2} \qquad |A_\pm| = 1$$

<u>stability</u> $\Leftrightarrow p^2-4 \le 0$. i.e. $-2 \le p \le 2$

$$-2 \le 2 + 2s(\cos\theta - 1) \le 2$$

always true

worse case is for $\cos\theta = -1$

$$\Downarrow$$

$$-2 \le 2 - 4s \Rightarrow \boxed{s \le 1}$$

<u>Summary</u>: Scheme is stable iff

$$\frac{c^2 \Delta t^2}{\Delta x^2} \le 1$$

The Wave Equation - cont

Back to fluids

$$(1-M^2)\varphi_{xx} + \varphi_{yy} = 0$$

$M > 1$

This is like wave eq

$$\varphi_{xx} - \underbrace{\frac{1}{M^2-1}}_{c^2}\varphi_{yy} = 0$$

for flow at transonic region we also have $M \approx 1$. $c^2 \sim \infty$ cannot satisfy

$$\frac{c^2 \Delta t^2}{\Delta x^2} \leq 1$$

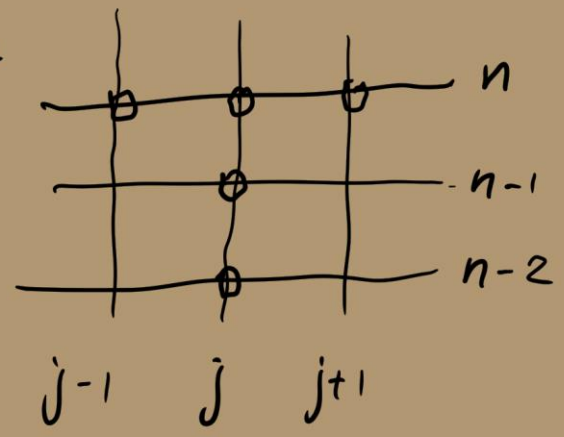
The Wave Equation - cont

We want a scheme that is stable for all

$$\frac{c\,\Delta t}{\Delta x}.$$

$$\frac{\partial^2 u}{\partial t^2} \sim \frac{u_j^n - 2u_j^{n-1} + u_j^{n-2}}{\Delta t^2}$$

$$\frac{\partial^2 u}{\partial x^2} \sim \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}$$



$$\frac{u_j^n - 2u_j^{n-1} + u_j^{n-2}}{\Delta t^2} - c^2\,\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}$$

$$s = c^2\Delta t^2/\Delta x^2$$

$$\boxed{-s\,u_{j+1}^n + (2s+1)u_j^n - s\,u_{j-1}^n = 2u_j^{n-1} - u_j^{n-2}}$$

We show now that this is stable all all values of $s>0$.

The Wave Equation - cont

For stability analysis we check $u_j^n \sim A^n e^{i\theta j}$

$$-s e^{i\theta} + (2s+1) - s e^{-i\theta} = \frac{2}{A} - \frac{1}{A^2}$$

$$A^2\left[2s+1-2s\cos\theta\right] - 2A + 1 = 0$$

$$A^2\left[1+4s\sin^2\frac{\theta}{2}\right] - 2A + 1 = 0$$

$$A_\pm = \frac{2 \pm \sqrt{\cancel{4} - 4\left(\cancel{1} + 4s\sin^2\frac{\theta}{2}\right)}}{2\left(1+4s\sin^2\frac{\theta}{2}\right)} =$$

$$\frac{1 \pm i2\sqrt{s}\left|\sin\frac{\theta}{2}\right|}{\left(1+4s\sin^2\frac{\theta}{2}\right)} = \frac{1+ia}{1+a^2}$$

$|A_\pm| = 1$ stable for all $s > 0$.

The Small Disturbance Eq

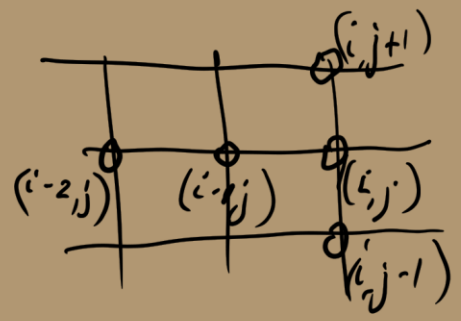
SDE

$$(1-M_\infty^2)\varphi_{xx} + \varphi_{yy} = 0$$

$M>1$

x is like t in wave eq.

y " x " " "



$$(1-M^2)(\varphi_{ij} - 2\varphi_{i-1,j} + \varphi_{i-2,j}) +$$
$$(\varphi_{i,j+1} - 2\varphi_{ij} + \varphi_{i,j-1}) = 0$$

assume $\Delta x = \Delta y$

for $M<1$ we need a different schm.

$$(1-M^2)(\varphi_{i+1,j} - 2\varphi_{ij} + \varphi_{i-1,j}) +$$
$$(\varphi_{i,j+1} - 2\varphi_{ij} + \varphi_{i,j-1}) = 0$$

The Small Disturbance Eq

SDE - Scheme

$$\underbrace{\left[1-M_\infty^2-(\gamma+1)M_\infty^2\varphi_x\right]}\varphi_{xx}+\varphi_{yy}=0$$

$$1-M^2=K$$

$$K_{ij}=1+M_\infty^2-(\gamma+1)M_\infty^2\frac{\varphi_{i+1,j}-\varphi_{i-1,j}}{2\Delta x}$$

for $M<1$, $\Delta x=\Delta y$ we have

$$K_{ij}\left(\varphi_{i+1,j}-2\varphi_{ij}+\varphi_{i-1,j}\right)+\left(\varphi_{i,j+1}-2\varphi_{ij}+\varphi_{i,j-1}\right)=0$$

for $M>1$

$$K_{i-1,j}\left(\varphi_{ij}-2\varphi_{i-1,j}+\varphi_{i-2,j}\right)+\left(\varphi_{i,j+1}-2\varphi_{ij}+\varphi_{i,j-1}\right)=0$$

Combine as:

$$\mu_{ij}K_{i-1,j}\left(\varphi_{ij}-\varphi_{i-1,j}+\varphi_{i-2,j}\right)+\left(\varphi_{i,j+1}-2\varphi_{ij}+\varphi_{i,j-1}\right)+$$

$$(1-\mu_{ij})K_{ij}\left(\varphi_{i+1,j}-2\varphi_{ij}+\varphi_{i-1,j}\right)=0$$

$\mu=0$ for $M<1$

$\mu=1$ for $M\geq 1$

Non-conservative form!

The Full Potential Eq

Full Potential Equation for $M>1$

$$(\rho\varphi_x)_x + (\rho\varphi_y)_y = 0$$

artificial compressibility idea:

$$(\bar{\rho}\varphi_x)_x + (\bar{\bar{\rho}}\varphi_y)_y = 0$$

$$\bar{\rho} = \rho - \mu\rho_x \Delta x$$

$$\bar{\bar{\rho}} = \rho - \mu\rho_y \Delta y$$

since we need $\rho_{i\pm\frac{1}{2},j}$ $\rho_{i,j\pm\frac{1}{2}}$ we have to shift ρ based on the flow direction.

If $u_{i+\frac{1}{2},j} > 0$

$$\bar{\rho}_{i+\frac{1}{2},j} = \rho_{i+\frac{1}{2},j} - \mu_{ij}(\rho_{i+\frac{1}{2},j} - \rho_{i-\frac{1}{2},j})$$

The Full Potential Eq

if $u_{i+\frac{1}{2},j} < 0$

$$\overline{\rho}_{i+\frac{1}{2},j} = \rho_{i+\frac{1}{2},j} + \mu_{i+1,j}\left(\rho_{i+\frac{3}{2},j} - \rho_{i+\frac{1}{2},j}\right)$$

if $v_{i,j+\frac{1}{2}} > 0$

$$\overline{\overline{\rho}}_{i,j+\frac{1}{2}} = \rho_{i,j+\frac{1}{2}} - \mu_{i,j}\left(\rho_{i,j+\frac{3}{2}} - \rho_{i,j-\frac{1}{2}}\right)$$

if $v_{i,j-\frac{1}{2}} < 0$

$$\overline{\overline{\rho}}_{i,j+\frac{1}{2}} = \rho_{i,j+\frac{1}{2}} + \mu_{i,j+1}\left(\rho_{i,j+\frac{3}{2}} - \rho_{i,j+\frac{1}{2}}\right)$$

$$\mu = \max\left[\left(0, 1 - \frac{M_c^2}{M^2}\right)\right] \cdot C M^2$$

$M_c = 0.95$, $1 < C < 2$

Examples

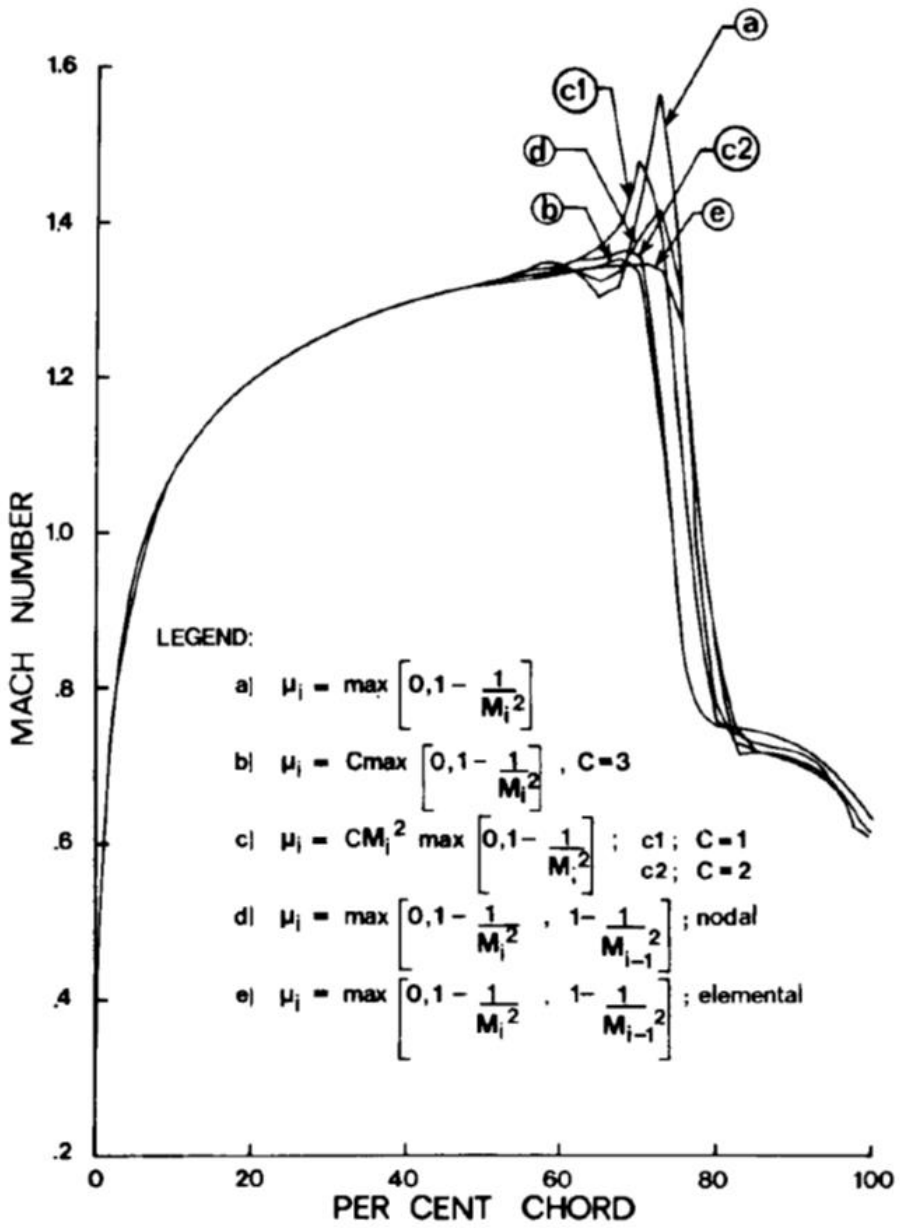


Figure 15.1.5 Comparison of various options for the switching function as applied to a finite element computation with artificial compressibility by Habashi and Hafez (1981). (Courtesy M. Hafez, University of California, Davis, USA)

Examples

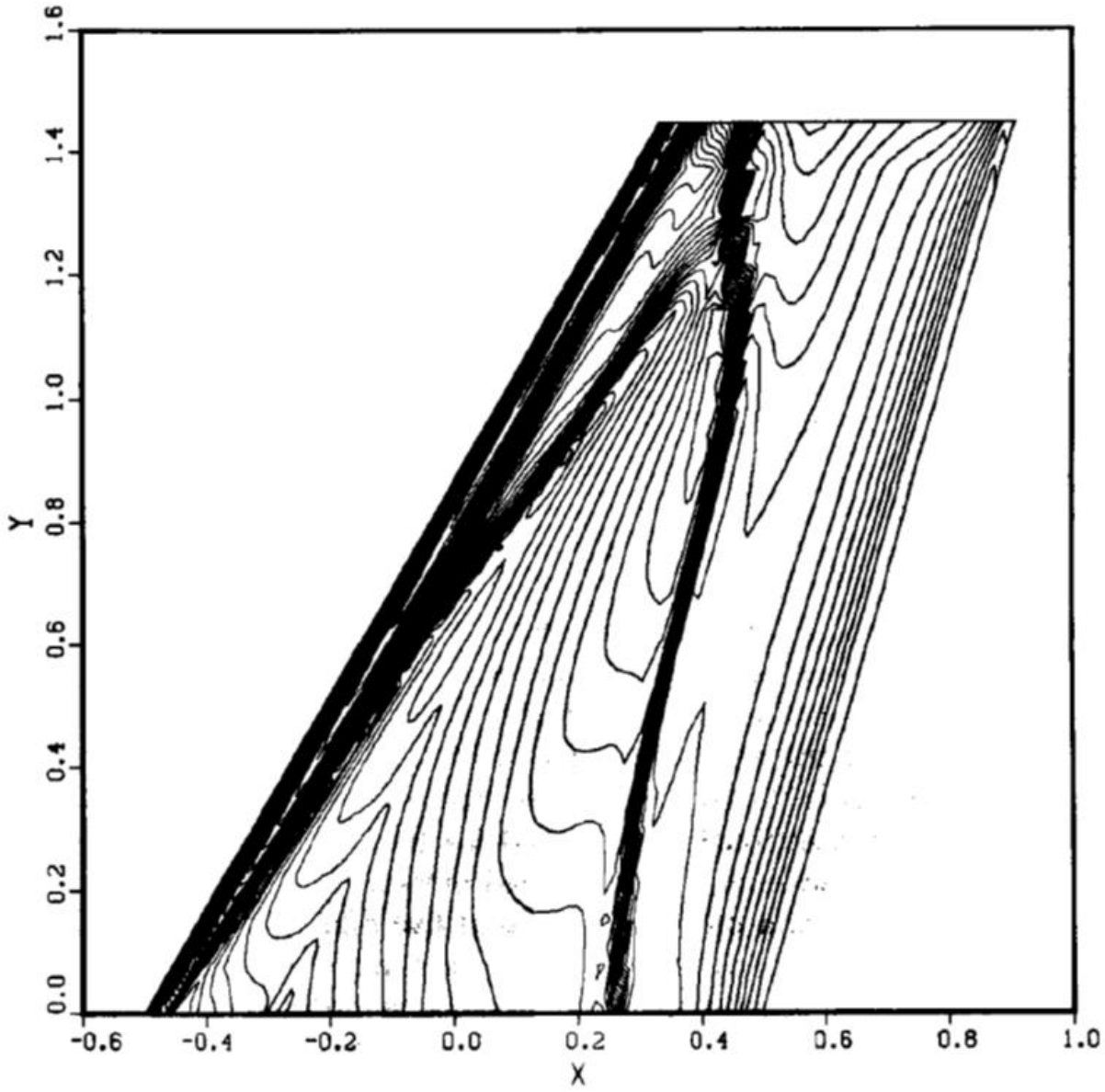


Figure 15.2.4 Mach number distribution on the upper surface of the ONERA M6 wing at a free-stream Mach number of 0.84 and 3.0° incidence, with a 225 × 30 × 30 mesh. (Courtesy T. Holst, NASA Ames Research Center, USA)

Examples

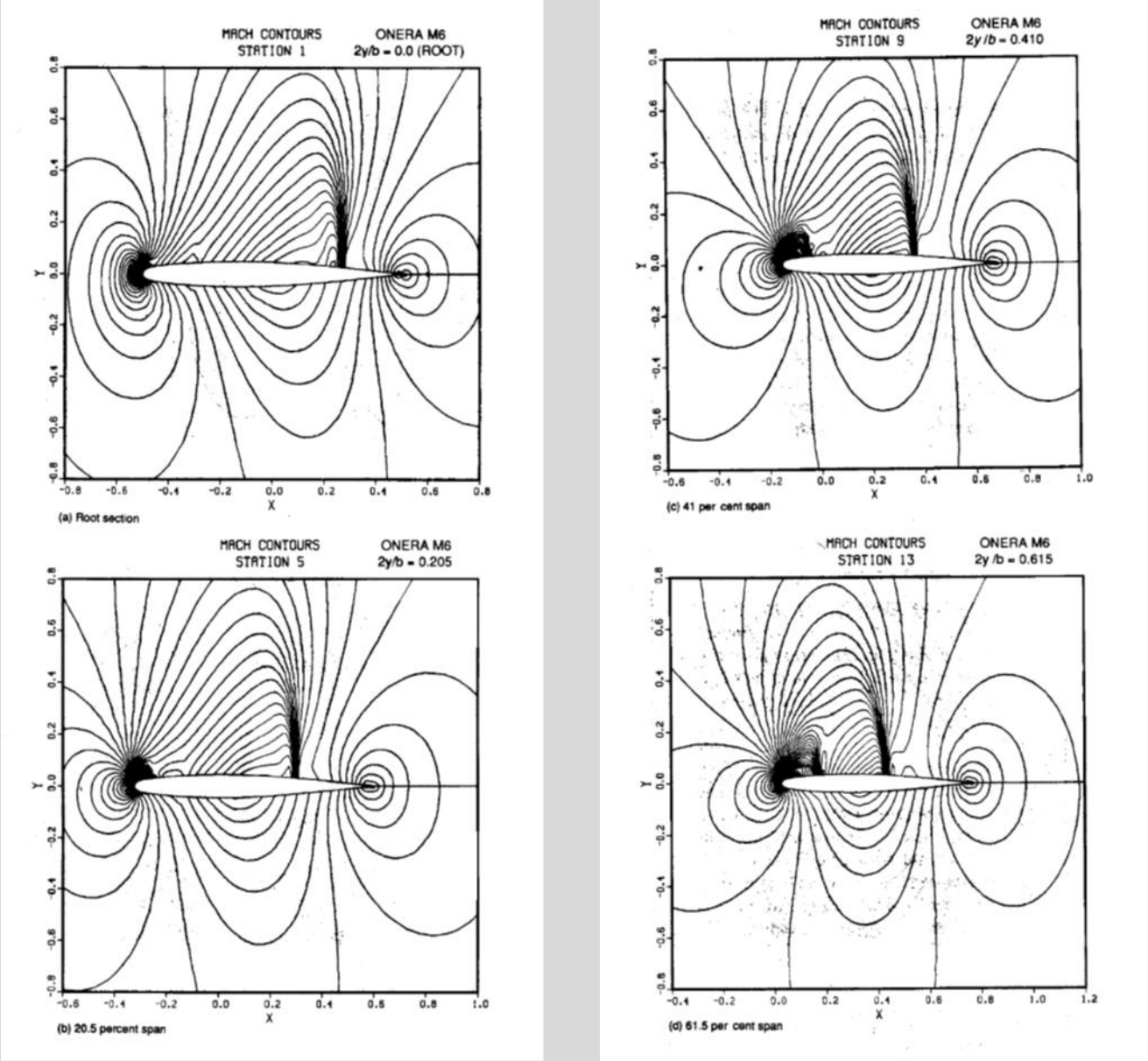


Figure 15.2.5 Mach number cross-sectional plots at five spanwise stations of the ONERA M6 wing at a free-stream Mach number of 0.84 and 3.0° incidence with a 225 × 30 × 30 grid. (Courtesy T. Holst, NASA Ames Research Center, USA)

Examples

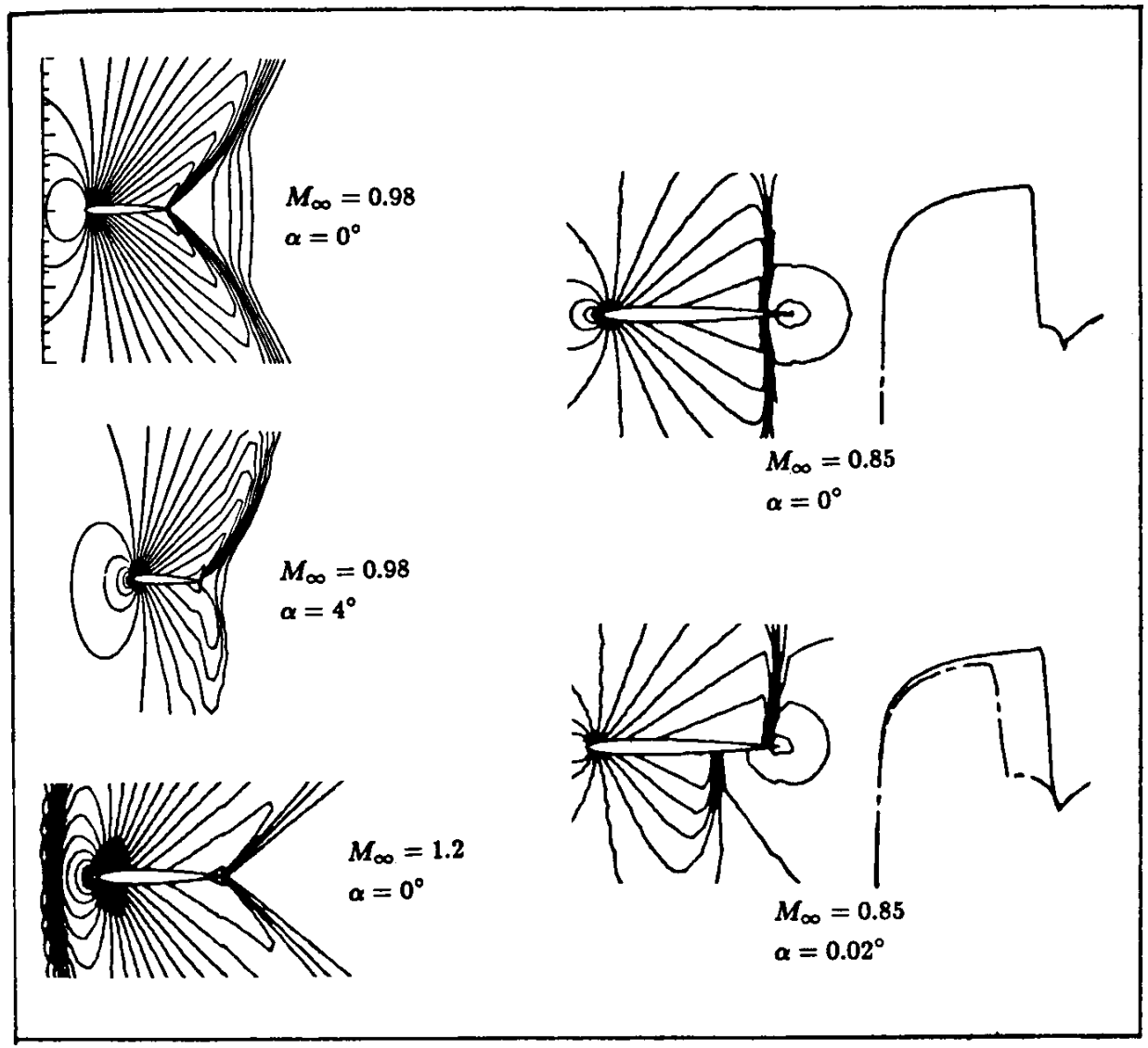
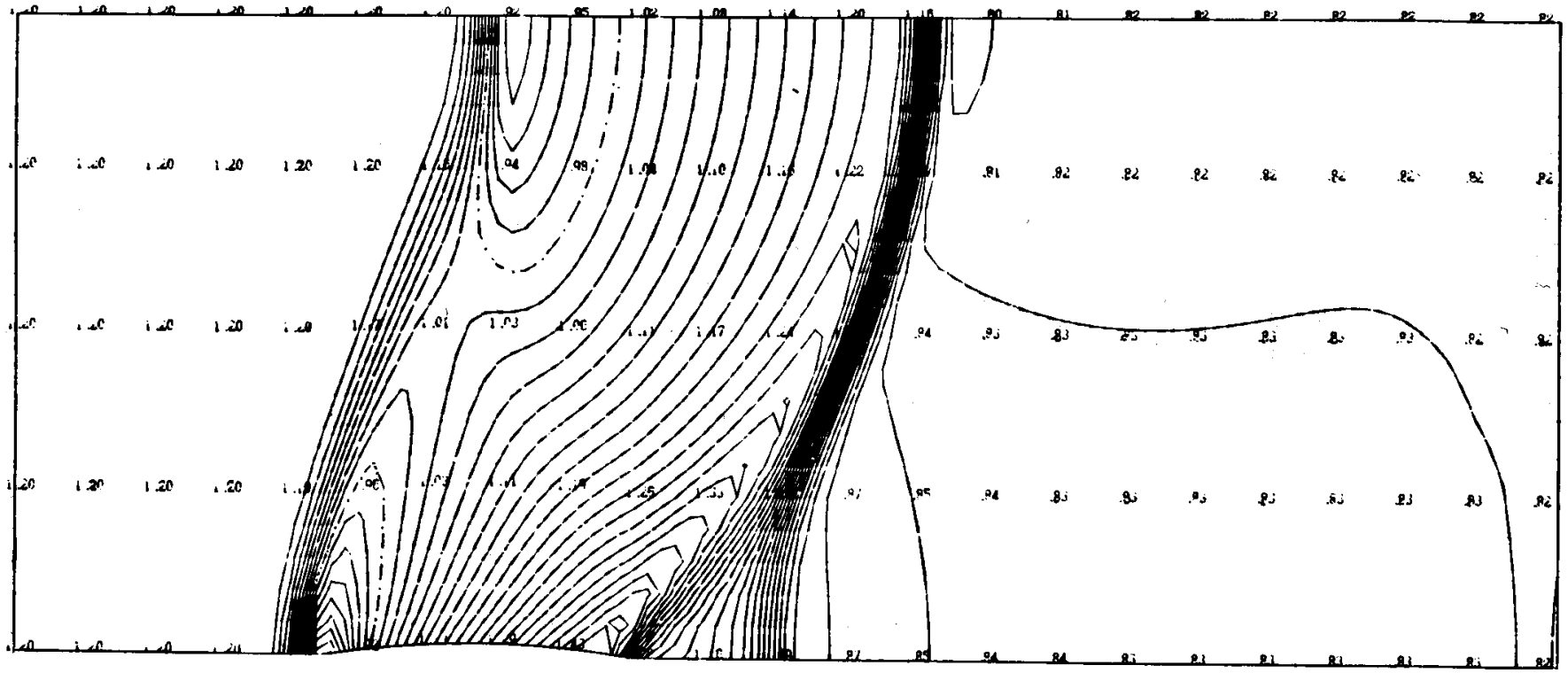


Figure 15.1.1 Typical transonic flow configurations for an isolated airfoil. (From Shankar *et al.*, 1985)

Examples



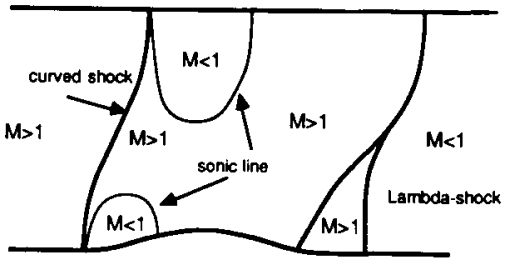


Figure 15.1.2 Typical transonic channel flow with multiple shock configuration; isentropic potential computation. (From Deconinck, 1983)

The Full Potential Equation - review

Under the assumption that vorticity is zero, $\nabla \times v = 0$

tvelocity field is the gradient of a scalar valued function $v = \nabla \phi$

This will help simplify the fluid dynamics equations.

We start with the continuity equation $\frac{\partial \rho}{\partial t} + div(\rho \nabla \phi) = 0$

For steady flow this simplifies to $div(\rho \nabla \phi) = 0$ with wall B.C. $v_n = \frac{\partial \phi}{\partial n} = 0$

Under the condition of constant entropy it is possible to derive the relation $\frac{\rho}{\rho_0} = \left(1 - \frac{|\nabla \phi|^2}{2H_0} - \frac{\partial_t \phi}{H_0}\right)^{1/(\gamma-1)}$

Where ρ_0 is the stagnation (U=0) density. H_0 Is the stagnation Enthalpy. This results in a single scalar equation that describes the fluid flow.

The Steady Full Potential Equation – non conservative

In many applications we are interested in steady flow and therefore omit all time derivatives,

$$\frac{\rho}{\rho_0} = \left(1 - \frac{|\nabla\phi|^2}{2H_0}\right)^{1/(\gamma-1)}$$

$$c^2 = (\gamma - 1)\left[H_0 - \frac{1}{2}|\nabla\phi|^2\right]$$

$$(1-M_x^2)\phi_{xx}+(1-M_y^2)\phi_{yy}+(1-M_z^2)\phi_{zz}-2M_xM_y\phi_{xy}-2M_xM_z\phi_{xz}-2M_yM_z\phi_{yz} = 0$$

$$M_x = \frac{\phi_x}{c} \quad M_y = \frac{\phi_y}{c} \quad M_z = \frac{\phi_z}{c} \quad M^2 = \frac{|\nabla\phi|^2}{c^2}$$

with wall BC

$$v_n = \frac{\partial\phi}{\partial n} = 0$$

This is the steady full potential equation in non-conservative form. This form reveals more about the equation than its conservative counterpart. Later we see that M<1 and M>1 need different discretization.

We discussed M > 1 and its discretization issues.

The Small Disturbance Approximation

A simplification of the full potential equations is done in the case of thin obstacles, such as thin airfoils. We will restrict the discussion to 2D.

Since the obstacle is “small” it’s effect on the flow is small and we consider perturbation to uniform flow with velocity of magnitude U_∞ in the x-direction. The potential has a representation

$$\phi = U_\infty(x + \Phi)$$

Where velocities are recovered from the potential using the formulas

$$u = U_\infty(1 + \Phi_x)$$
$$v = U_\infty \Phi_y$$

The equation becomes $(1 - M_x^2)\Phi_{xx} + \Phi_{yy} = 0$ which can be simplified further to $(1 - M_\infty^2)\Phi_{xx} + \Phi_{yy} = 0$

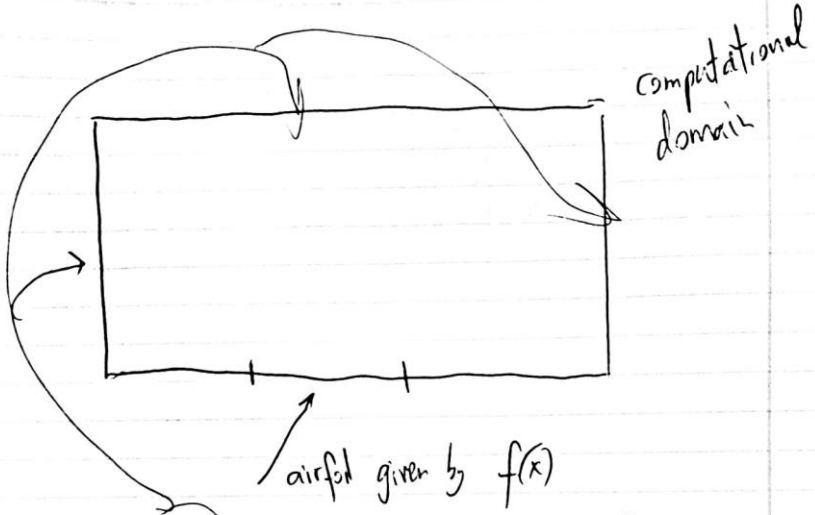
The wall BC becomes $v = (U_\infty + u)f'(x) \approx U_\infty f'(x)$ where f(x) is the shape of the airfoil.

Numerical Approximation

The discretization of the full potential equations is done in conservation form, starting from the integral formulation, using the divergence theorem.

$$\int_{\Omega} div\, \rho \nabla \phi \, dx = \int_{\partial\Omega} \rho \nabla \phi \cdot n \, ds \equiv \int_{\partial\Omega} \rho \frac{\partial \phi}{\partial n} \, ds$$

Airfoil Calculation



φ at far field $\sim U_\infty \cdot x$ we specify it

$\varphi_{ij} = U_\infty \cdot x_i$

at bottom we specify $\frac{\partial \varphi}{\partial n} = \boxed{-\frac{\partial \varphi}{\partial y} = U_\infty f'(x)}$

where no airfoil exists we set $f=0$.

$\Rightarrow$ Dirichlet B.C. on left, right & top

Neumann BC on bottom.

Simulation of NS vs. Experiment

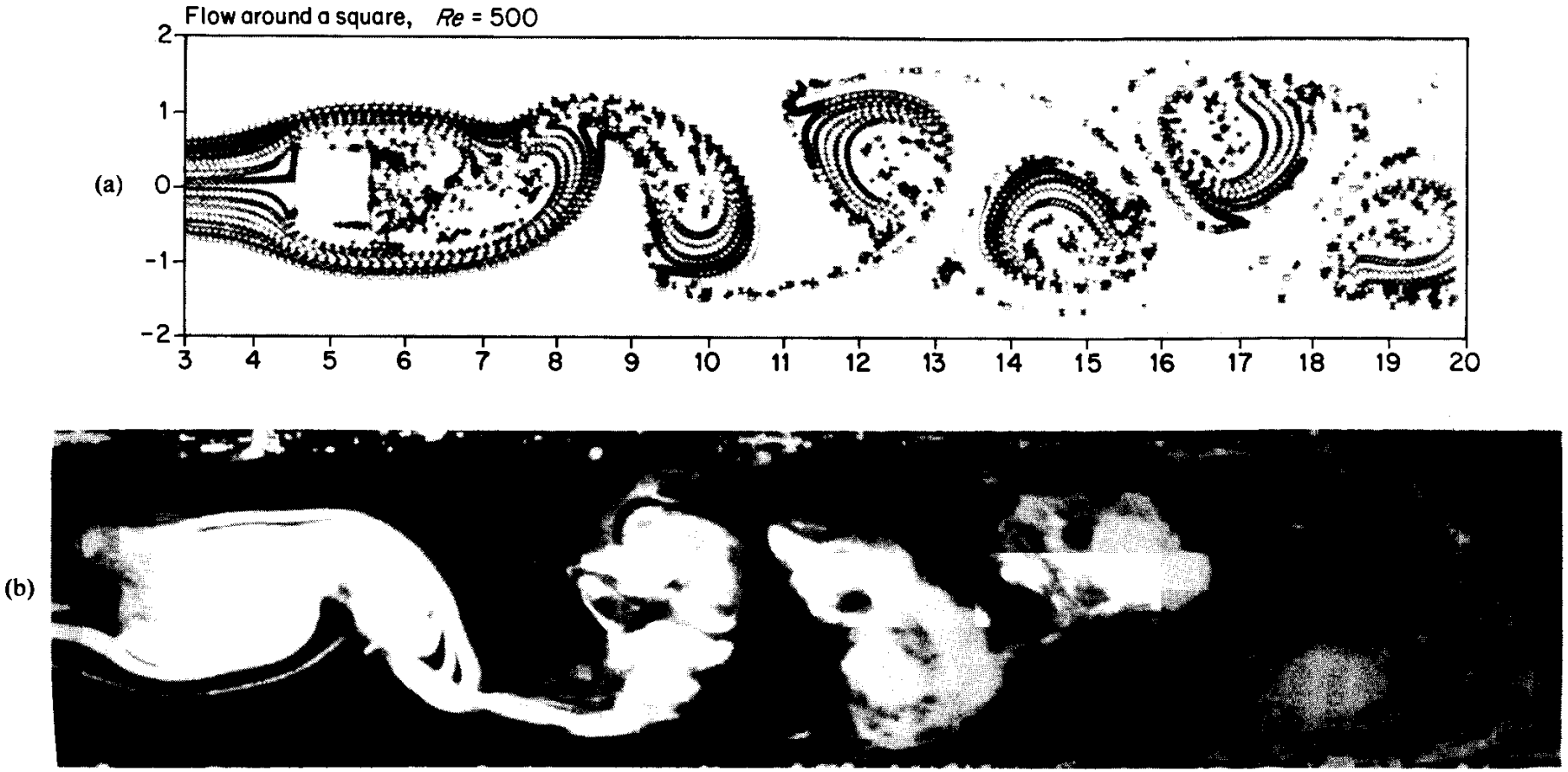
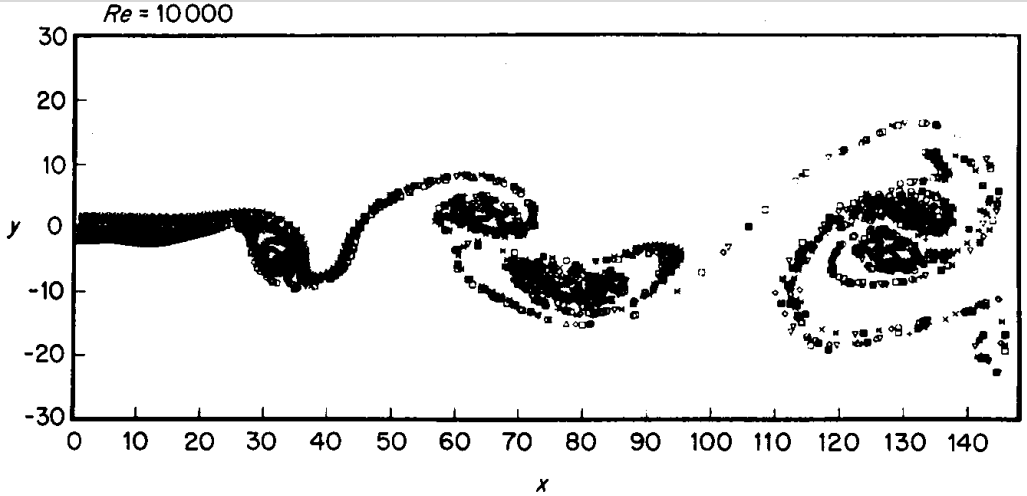
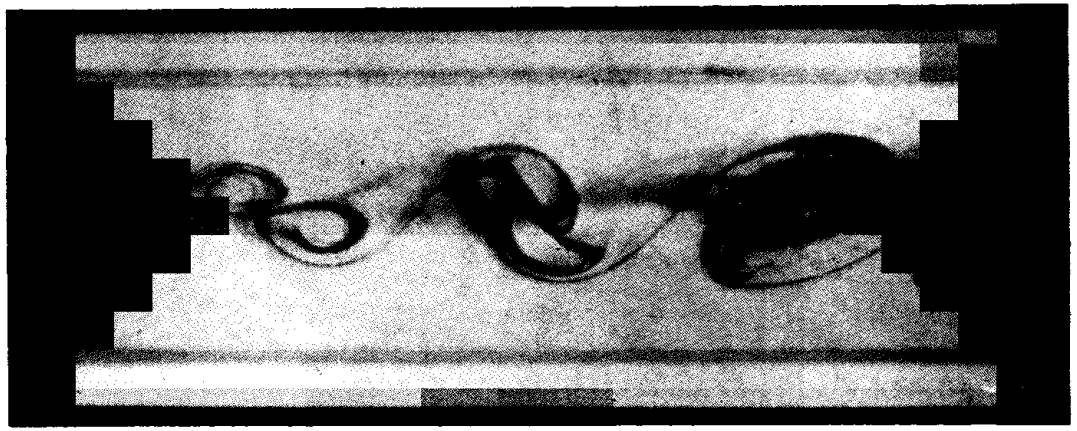


Figure 2.1.7 (a) Computed flow around a square cylinder at $Re = 500$; (b) comparison with a visualization at $Re = 550$. (Reproduced by permission of the American Institute of Physics from Davis *et al*, 1984.)

Simulation NS vs. Experiment



(a)



(b)

Figure 2.1.8 (a) Computation of the coherent structures in a mixing layer at $Re = 10\,000$; (b) comparison with visualizations. (Courtesy R. W. Davis, National Bureau of Standards)

Simulation vs. Experiment

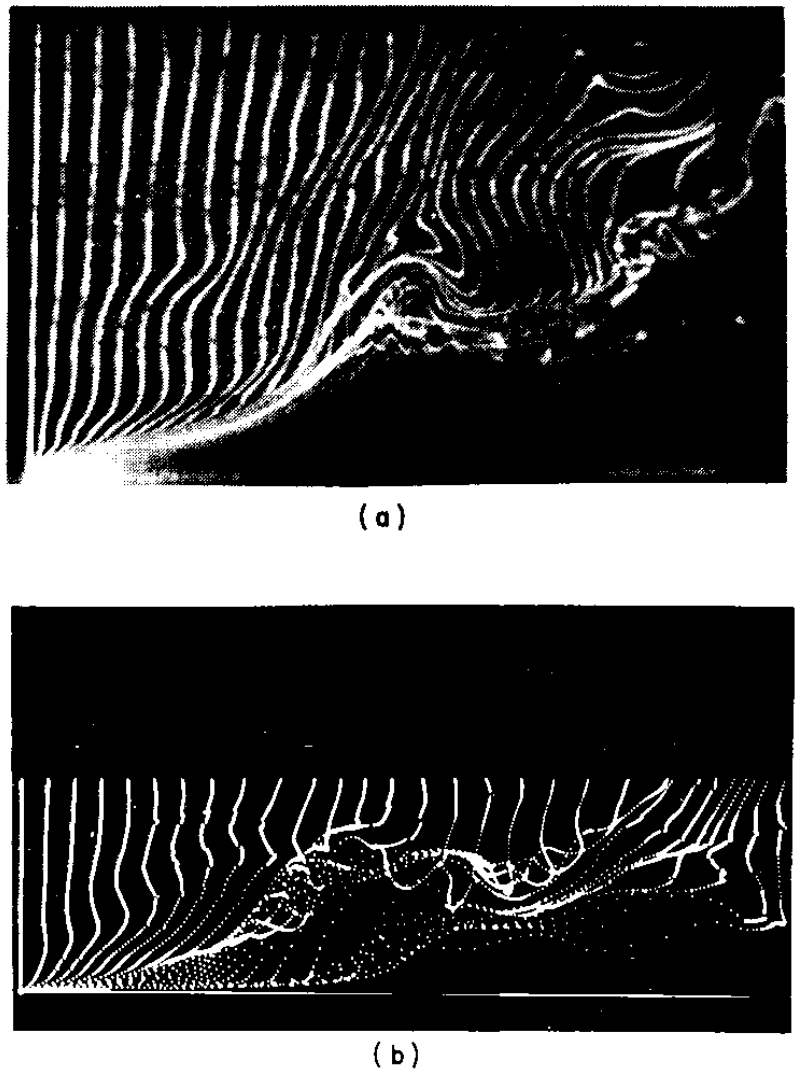


Figure 2.1.9 Time history of particles (hydrogen bubbles) generated along a line normal to the wall in a turbulent boundary layer, showing a coherent, bursting structure. (a) Experiments; (b) computations. (Reproduced by permission of AIAA from Moin, 1984)

Simulation vs. Experiment

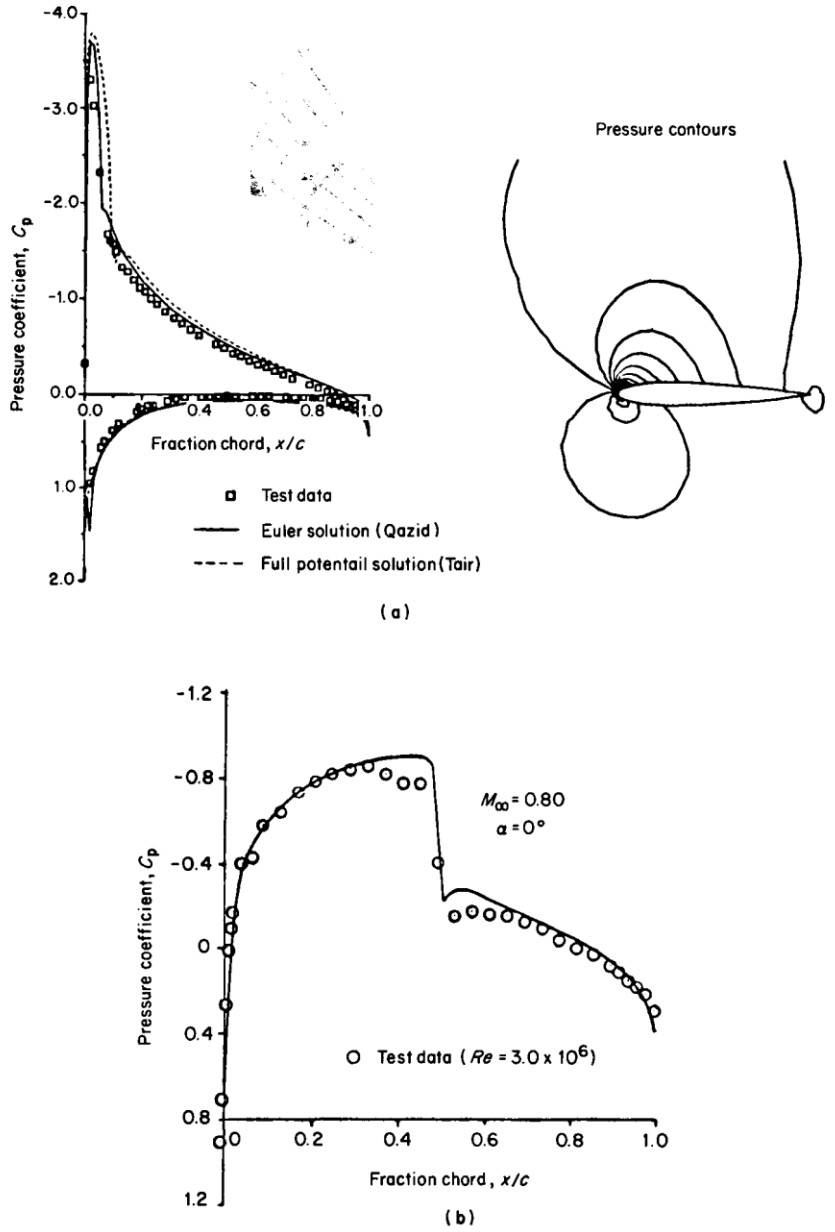


Figure 2.7.4 Comparison of inviscid surface pressure distribution with experimental data on the NACA 0012 airfoil. (a) $M_\infty = 0.50$, 7° incidence (note the comparison with the potential model); (b) $M_\infty = 0.80$, 0° incidence. (Courtesy A. Verhoff, McDonnell Aircraft Co., USA)

Simulation – Euler Eq

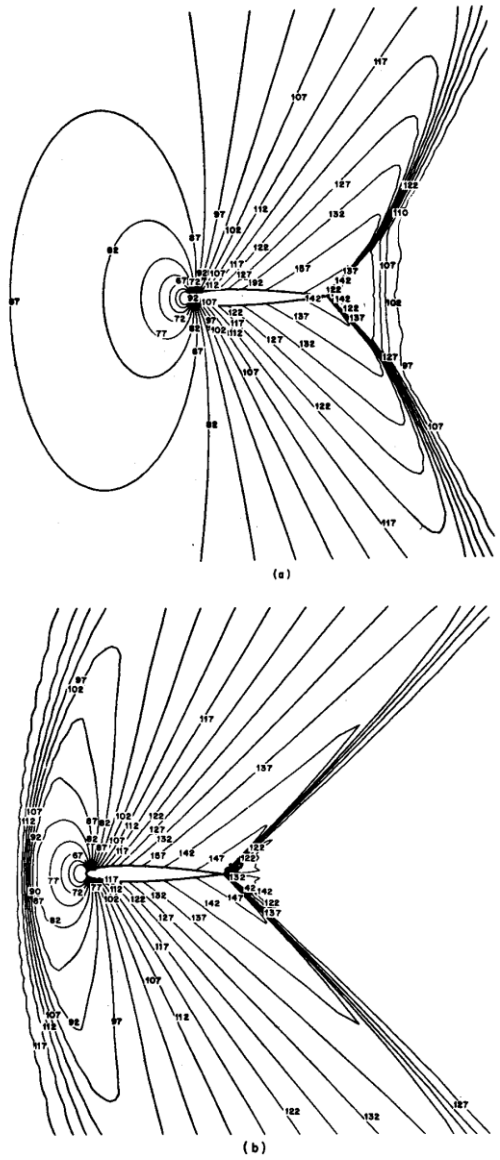


Figure 2.7.3 IsoMach contours for the NACA 0012 profile at incident Mach numbers $M_\infty = 0.95$ and $M_\infty = 1.2$. (From AGARD AR 211 Report, 1985)

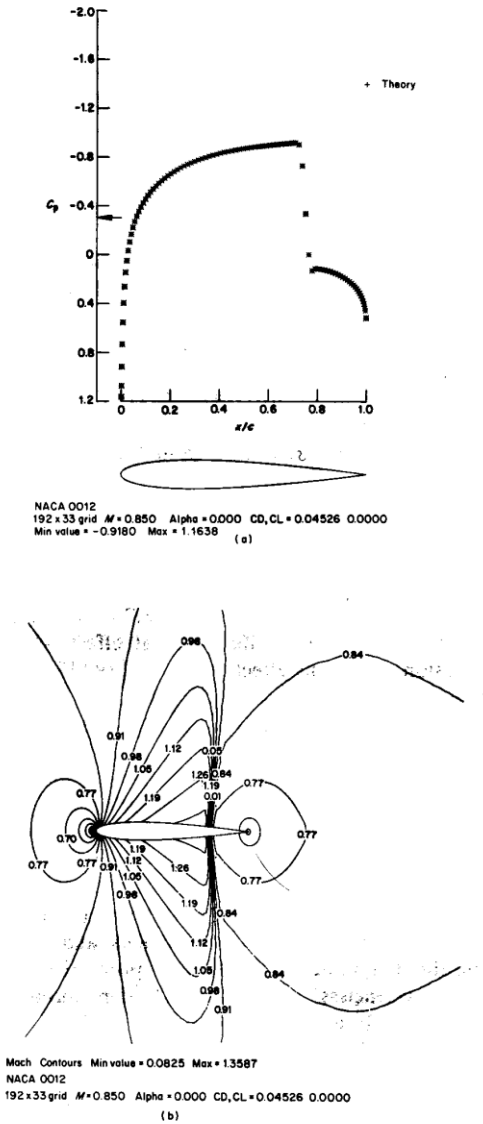


Figure 2.7.2 Pressure and Mach number distribution on a NACA 0012 profile at $M_\infty = 0.85$ and 0° incidence. (a) Surface pressure distribution; (b) isoMach contours. (Courtesy M. Salas, NASA Langley Research Center, USA)

Non-Uniqueness for steady FP equation

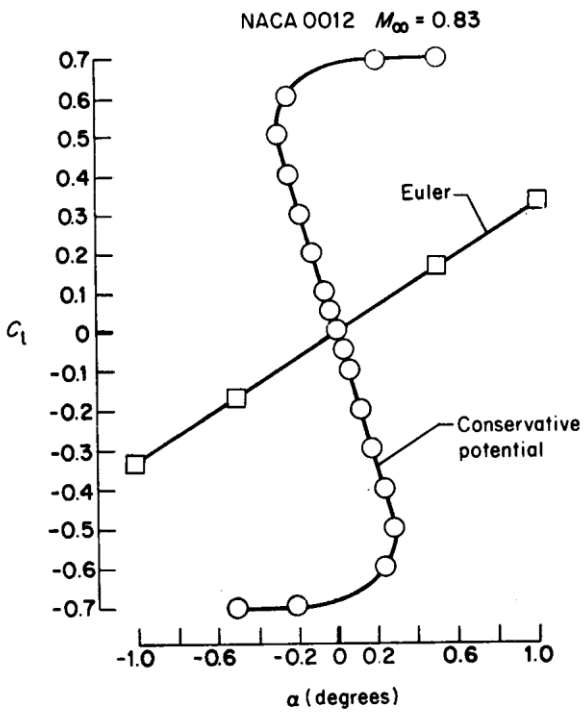


Figure 2.9.16 Computed variation of lift coefficient with incidence angle in the region of non-unique solutions for the potential flow around a NACA 0012 airfoil at $M_\infty = 0.83$. For comparison, the results from an Euler calculation are indicated. (Courtesy M. Salas, NASA Langley Research Center, USA)

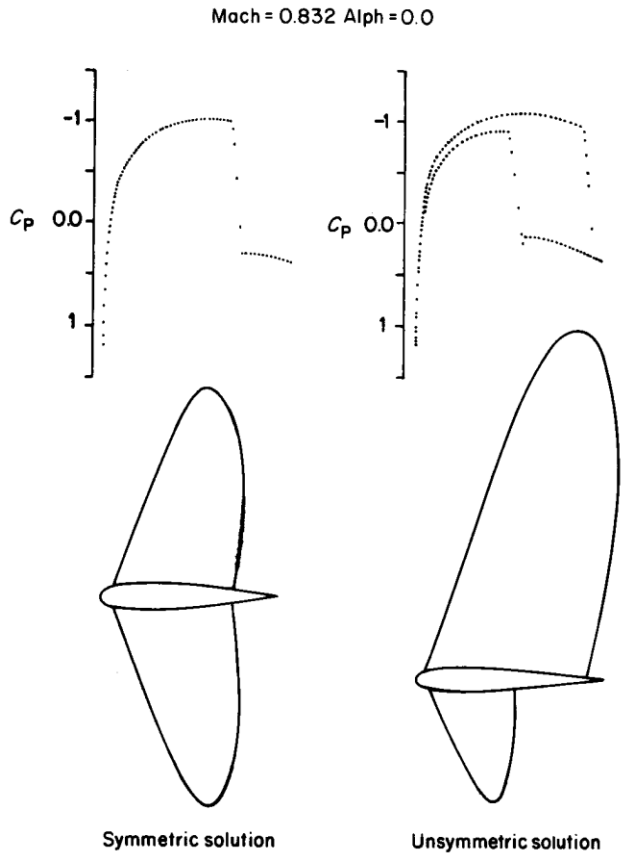


Figure 2.9.15 Non-unique solutions of the conservative potential flow models for a symmetrical NACA 0012 airfoil. (Reproduced by permission of AIAA from Steinhoff and Jameson, 1982)

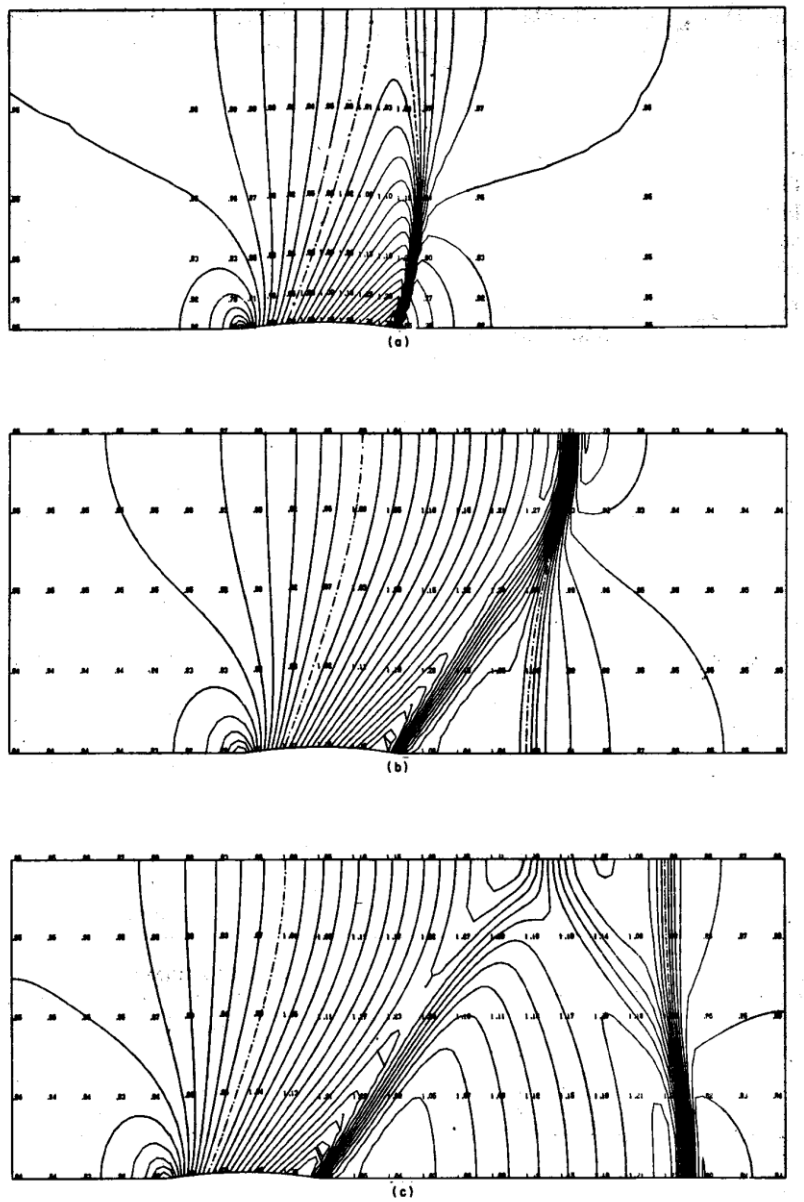


Figure 2.9.14 Isomach lines for choked channel flow. Potential solutions with different shock positions. (Reproduced with permission from Deconinck, 1983.)

Comparison of Solutions – Euler and FP

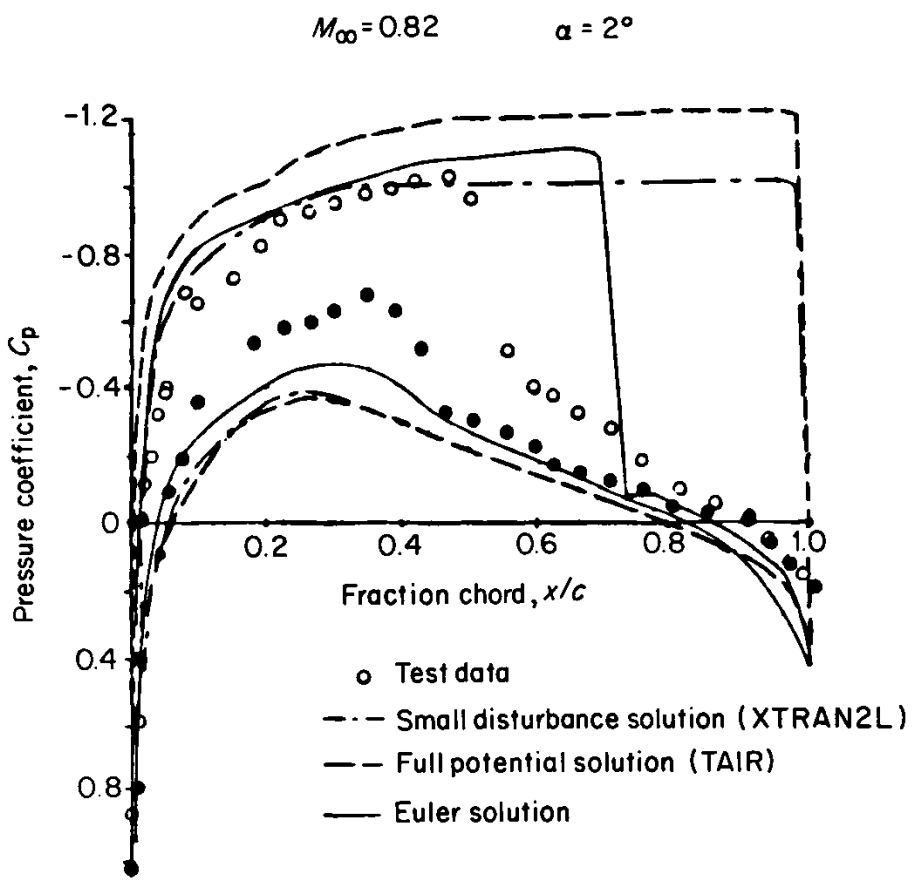
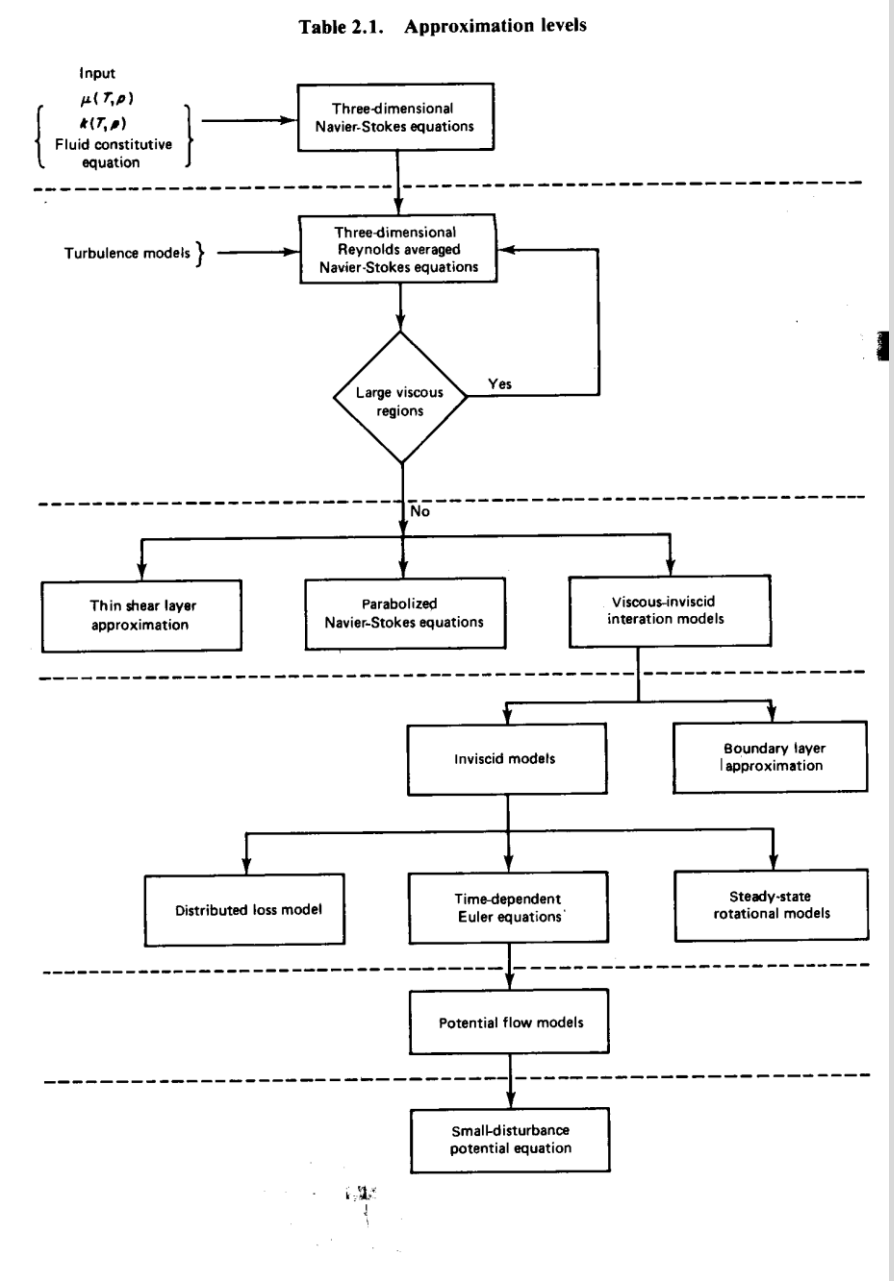


Figure 2.10.1 Comparison of calculations with small disturbance, potential Euler models for the NACA 0012 airfoil at $M_\infty = 0.82$ and 2° incidence and experimental data. (Courtesy A. Verhoff, McDonnell Aircraft Co., USA)

Approximation Levels - summary

We have studies only a small portion of the approximation
Of the Navier-Stokes equation.

In each of these levels there are physical and numerical issues
To deal with.



Comparison of Models

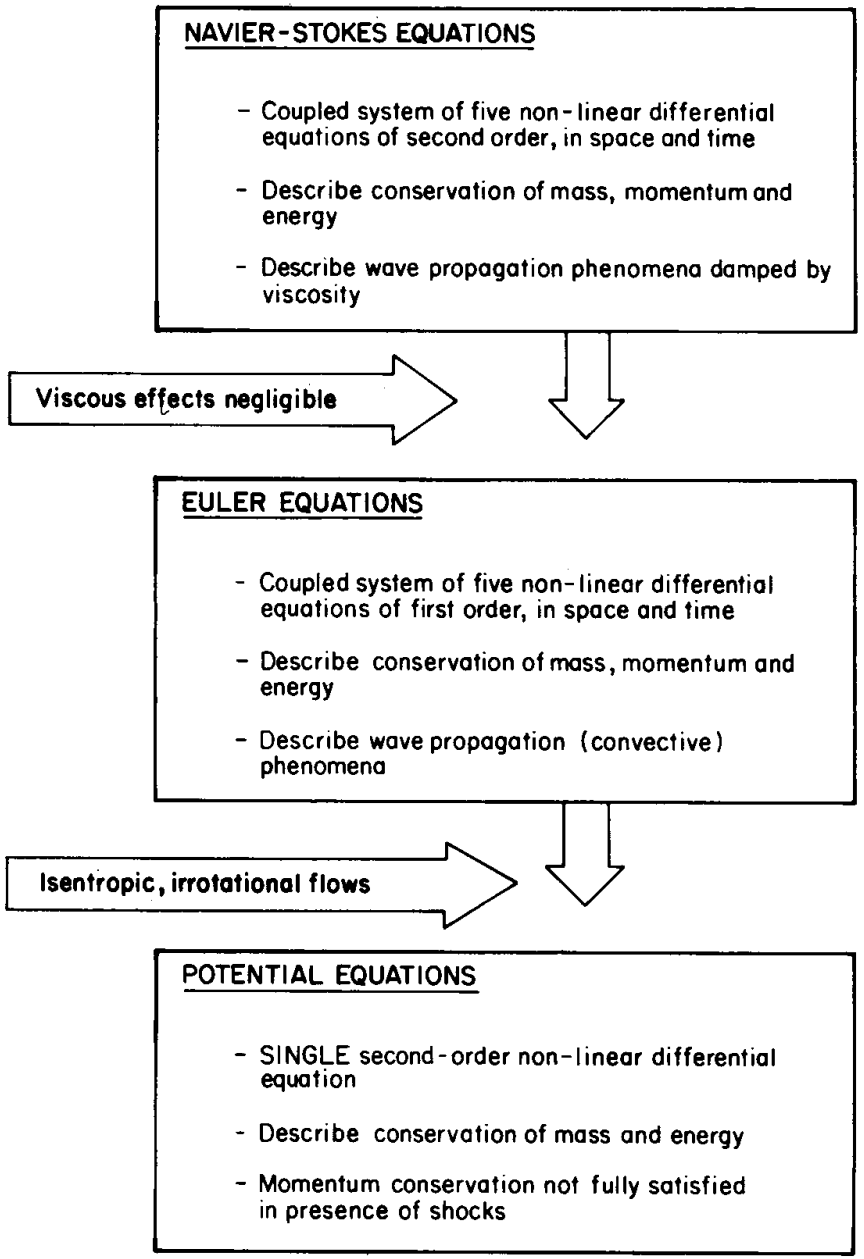


Figure 2.10.2 Classification of various flow models

Final Project

Comparison of nozzle flow using the small disturbance equation with the 1D Euler equation for nozzle Geometry.

More details on Ed.