Fluid Dynamics

Lecture 1 – The Navier-Stokes equations

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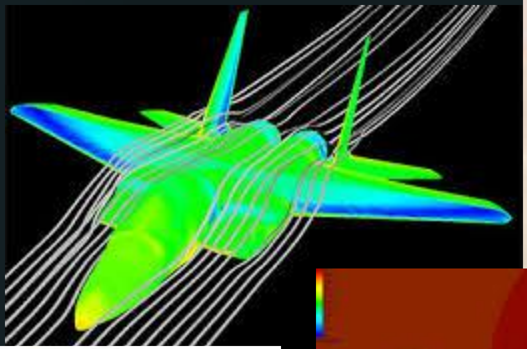
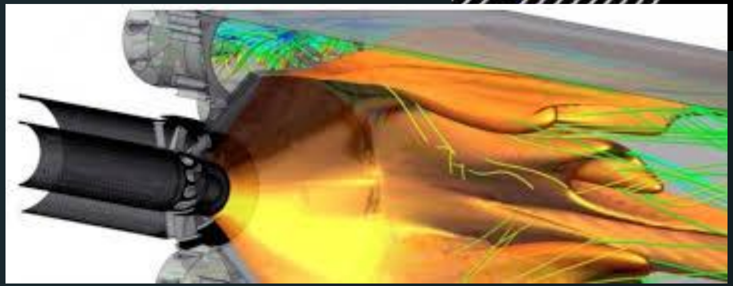
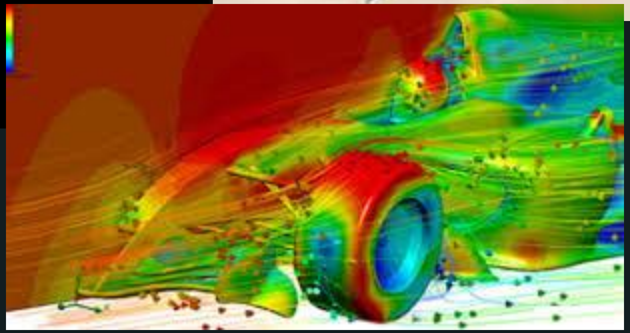
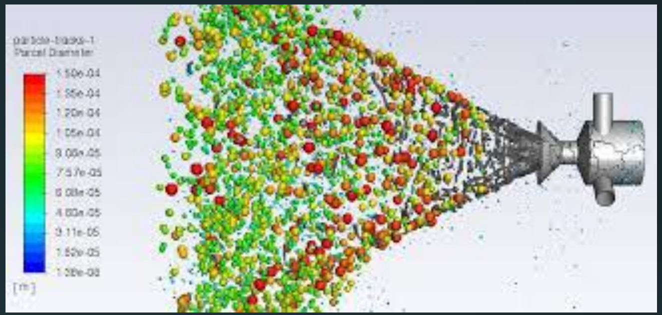
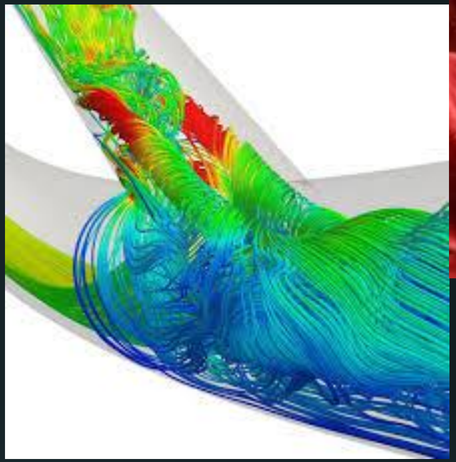
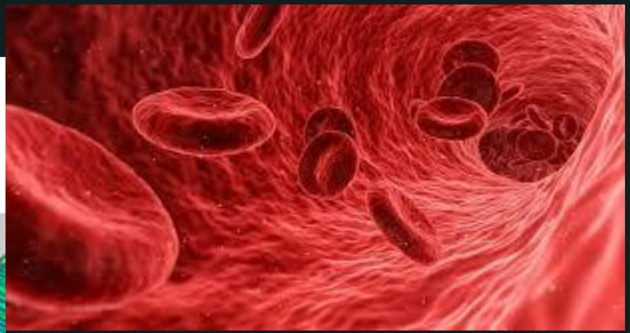


Outline of this lecture

We will derive the general equations of fluid dynamics

- we review operators and important theorems from 3D calculus
- we review differentiation of integrals
- we derive for the compressible case
    - conservation of mass
    - conservation of momentum
    - conservation of energy

- we formulate the incompressible equations
- we formulate 1D inviscid flow

Review of 3D Calculus

$$U = (u(x, y, z), v(x, y, z), w(x, y, z))$$

$$\phi(x, y, z)$$

$$div\, U = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

$$\nabla\, \phi = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}$$

The Divergence Theorem

$$\int_{\Omega} div\, U\, dx = \int_{\partial\Omega} U \cdot n\, ds$$

$$\int_{\Omega} \nabla\, \phi\, dx = \int_{\partial\Omega} \phi\, n\, ds$$

Divergence theorem in 1D

The divergence theorem has discrete analogue.

In 2D if we use staggered grid: u on vertical cell edges, and v on horizontal cell edges, and discretize the divergence operator as shown in figure,

Summing up all cell gives rise to cancellation of all internal contribution- only the boundary term remains. Note the relation to the normal velocity component

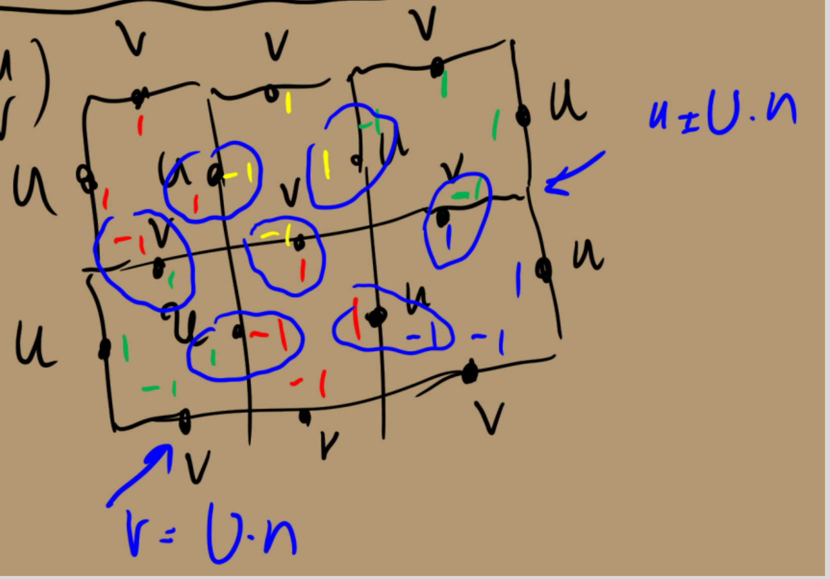
div theorem 1D & 2D

$$\int_a^b \varphi_x \, dx = \varphi(b) - \varphi(a)$$

Discrete Version 1D:

$$\sum_{i=0}^{N-1} (\varphi_{i+1} - \varphi_i) = \varphi_N - \varphi_0$$

2D $\begin{pmatrix} u \\ v \end{pmatrix}$



Review of 3D Calculus - Cont.

The Laplacian

$$\triangle\phi \equiv div\,\nabla\,\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$\int_\Omega div\,\nabla\phi\,dx = \int_{\partial\Omega} \nabla\phi\cdot n\,ds \equiv \int_{\partial\Omega} \frac{\partial\phi}{\partial n}\,ds$$

Gauss-Green Theorem

$$\int_\Omega \phi\,div\,U\,dx = -\int_\Omega \nabla\phi\cdot U\,dx + \int_\Omega \phi U\cdot n\,ds$$

Differentiation of Integrals

$$\frac{d}{dx}\int_a^x f(z)\,dz = f(x) \qquad \frac{d}{dx}\int_x^a f(z)\,dz = -f(x)$$

$$\frac{d}{dt}\int_a^b f(z,t)\,dz = \int_a^b \frac{\partial f}{\partial t}(z,t)\,dz$$

$$I(t,a(t),b(t)) = \int_{a(t)}^{b(t)} f(z,t)\,dz \qquad \frac{DI}{dt} = \frac{\partial I}{\partial t} + \frac{\partial I}{\partial a}\frac{da}{dt} + \frac{\partial I}{\partial b}\frac{db}{dt}$$

$$\frac{dI}{dt} = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t} f(z,t)\,dz + f(b(t),t)\frac{db}{dt} - f(a(t),t)\frac{da}{dt}$$

Differentiation of Integrals

$$I(t, a(t), b(t)) = \int_{a(t)}^{b(t)} f(z,t)\,dz$$

We show the result of differentiating I(t,a(t),b(t)) in a different way.

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(t,s)\,ds \approx$$

$$\frac{1}{\Delta t}\left[\int_{a(t+\Delta t)}^{b(t+\Delta t)} f(t+\Delta t,s)\,ds - \int_{a(t)}^{b(t)} f(t,s)\,ds\right]$$

$$a(t+\Delta t) = a(t) + \Delta t\, a'(t)$$
$$b(t+\Delta t) = b(t) + \Delta t\, b'(t)$$

$$\int_{a+\Delta t a'}^{b+\Delta t b'} = \int_a^b + \int_b^{b+\Delta t b'} - \int_a^{a+\Delta t a'}$$

$$f(t+\Delta t, s) \approx f(t,s) + \Delta t \frac{\partial f}{\partial t}$$

Differentiation of Integrals

$$I(t, a(t), b(t)) = \int_{a(t)}^{b(t)} f(z,t)\, dz$$

Putting the arguments on the right together we get the formula we derived before.

$$\frac{dI}{dt} = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t} f(z,t)\, dz + f(b(t),t)\frac{db}{dt} - f(a(t),t)\frac{da}{dt}$$

$$\int_{a(t+\Delta t)}^{b(t+\Delta t)} f(t+\Delta t, s)\, ds - \int_{a(t)}^{b(t)} f(t,s)\, ds \approx$$

$$\Delta t \int_a^b \frac{\partial f}{\partial t} + \Delta t\, b'(t) f(t,b) - \Delta t\, a'(t) f(t,a)$$

we used $\displaystyle\int_x^{x+\Delta t} f(s)\, ds \approx \Delta t\, f(x)$

$$\frac{d}{dt}\int_{a(t)}^{b(t)} f(t,s)\, ds = \int_a^b \frac{\partial f}{\partial t} + b'(t) f(t,b) - a'(t) f(t,a)$$

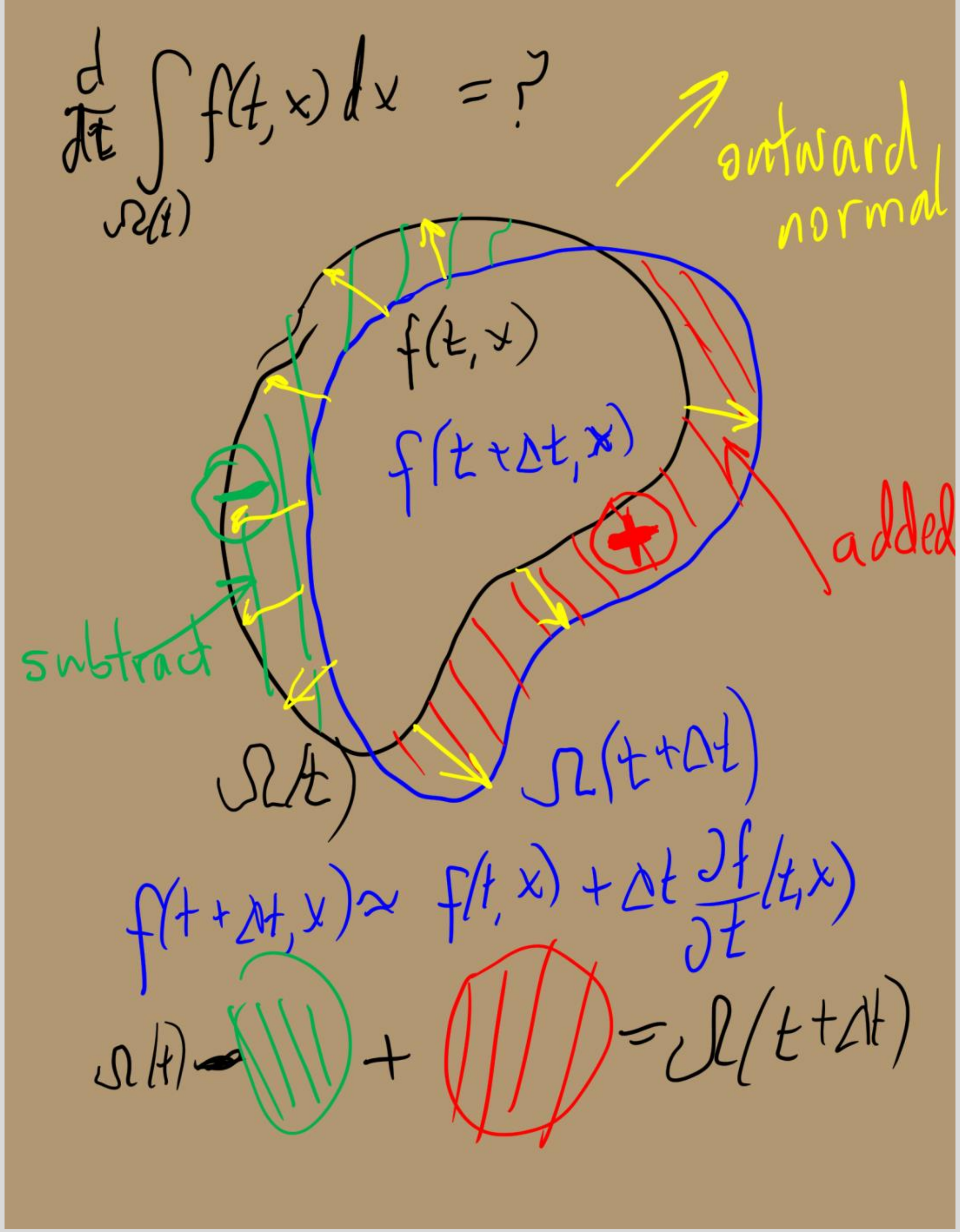
Differentiation – cont.

We now consider the derivative of an integral over a domain which moves in time. The integrand is also a function of time.

Note that as the domain changes, there is additions (red areas) and subtractions (green areas) for the original integral – see figure.

In addition, the function f depends on time so at a later time it can be expanded by Taylor up to first order.

We put all these details together.



Differentiation – cont.

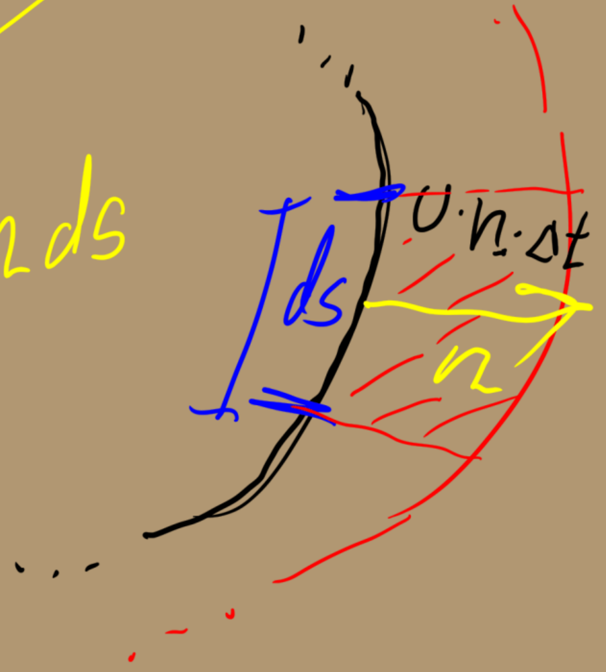
We calculate the integral at a later time,   t + dt

Note that the change due to the movement of the domain is described in terms of the normal velocity of the boundary. This gives rise to a surface integral.

$$\int_{\Omega(t+\Delta t)} f(t+\Delta t, x)\,dx \approx$$

$$\int_{\Omega(t)} \left(f + \Delta t \frac{\partial f}{\partial t}\right) dx + \int f\,dx - \int f\,dx$$

$$\Delta t \oint_{\partial\Omega(t)} U \cdot n \, ds$$



Differentiation – cont.

Putting together the details we get :

$$\frac{d}{dt}\int_{\Omega(t)} f(v,t)\,dv = \int_{\Omega(t)} \frac{\partial f}{\partial t}(v,t)\,dv + \int_{\partial\Omega(t)} f(s,t)\,U(s,t)\cdot n\,ds$$

This formula is the basis for several derivations we do next.

$$\boxed{\frac{d}{dt}\int_{\Omega(t)} f(t,x)\,dt \approx}$$

$$\frac{1}{\Delta t}\left[\int_{\Omega(t+\Delta t)} f(t+\Delta t,x)\,dx - \int_{\Omega(t)} f(t,x)\,dx\right]$$

$$\approx \frac{1}{\cancel{\Delta t}}\left[\int_{\Omega(t)} \left[\cancel{f} + \cancel{\Delta t}\,\frac{\partial f}{\partial t} - \cancel{f}\right] + \cancel{\Delta t}\int_{\partial\Omega(t)} f\,U\cdot n\,ds\right]$$

$$= \boxed{\int_{\Omega(t)} \frac{\partial f}{\partial t}\,dx + \int_{\partial\Omega(t)} f\,U\cdot n\,ds}$$

Differentiation of Integrals - summary

$$\frac{d}{dt}\int_{\Omega(t)} f(v,t)\,dv = \int_{\Omega(t)} \frac{\partial f}{\partial t}(v,t)\,dv + \int_{\partial\Omega(t)} f(s,t)\,U(s,t)\cdot n\,ds$$

Rate of change of integral = rate of change due to changes in f
+
rate of change due to changes in the domain

This mathematical formula, together with physical principles

1) **Conservation of mass**
2) **Newton’s second law**
3) **Conservation of energy**

lead the the Navier-Stokes equations – the general equations of fluid flow

Conservation of mass- The Continuity Equation

Conservation of mass in a volume moving with the flow

$$\frac{d}{dt}\int_{\Omega(t)} \rho(v,t)\,dv = 0$$

Using

$$\frac{d}{dt}\int_{\Omega(t)} f(v,t)\,dv = \int_{\Omega(t)} \frac{\partial f}{\partial t}(v,t)\,dv + \int_{\partial\Omega(t)} f(s,t)\,U(s,t)\cdot n\,ds$$

We get

$$0 = \frac{d}{dt}\int_{\Omega(t)} \rho(v,t)\,dv = \int_{\Omega(t)} \frac{\partial \rho}{\partial t} + \int_{\partial\Omega(t)} \rho(s,t)\,U(s,t)\cdot n\,ds$$

But,

$$\int_{\partial\Omega(t)} \rho(s,t)\,U(s,t)\cdot n\,ds = \int_{\Omega(t)} div\,(\rho U)\,dv$$

Implies,

$$\int_{\Omega(t)} [\frac{\partial \rho}{\partial t} + div\,(\rho U)]\,dv = 0$$

Giving,

$$\frac{\partial \rho}{\partial t} + div\,(\rho U) = 0$$

The Continuity Equation

Conservation of Momentum

Change in momentum is due to surface forces – Newton's second Law

$$\frac{d}{dt}\int_{\Omega(t)} \rho U \, dv = \int_{\partial\Omega(t)} \sigma \, n \, ds$$

The math formula from before gives,

$$\frac{d}{dt}\int_{\Omega(t)} \rho U \, dv = \text{[redacted]} \int_{\Omega(t)} \frac{\partial(\rho U)}{\partial t} + \int_{\partial\Omega(t)} \rho U U \cdot n \, ds$$

Divergence Theorem implies,

$$\int_{\partial\Omega(t)} \rho U U \cdot n \, ds = \int_{\Omega(t)} div \, (\rho U \otimes U) \, dv$$

Surface forces are expressed as

$$\sigma = -pI + \tau$$

Where,

$$\tau_{i,j} = \mu(\partial_i v_j + \partial_j v_i) + \lambda(div \, U)\delta_{i,j}$$

$$2\mu + 3\lambda = 0$$

Summarizing

$$\int_{\Omega(t)} [\frac{\partial(\rho U)}{\partial t} + div \, (\rho U \otimes U + pI - \tau)] = 0$$

Or

$$\frac{\partial(\rho U)}{\partial t} + div \, (\rho U \otimes U + pI) = div \, \tau$$

Momentum Equation

Conservation of Energy

Energy is a sum of internal energy and kinetic energy

$$E = e + \frac{1}{2}|U|^2$$

Change in energy is due to the work of surface forces plus heat flux

$$\frac{d}{dt}\int_{\Omega(t)} \rho E \, dv = \int_{\partial\Omega(t)} [\sigma U + k\nabla T \cdot n] \, ds$$

Divergence Theorem implies,

$$\int_{\partial\Omega(t)} \frac{\partial T}{\partial n} \, ds = \int_{\Omega} div\,(k\nabla T)$$

Like in previous case we conclude

$$\int_{\Omega(t)} \frac{\partial(\rho E)}{\partial t} + div(\rho E U - k\nabla T - \sigma U)\, dv = 0$$

Summarizing

$$\frac{\partial(\rho E)}{\partial t} + div(\rho E U) = div\,(k\nabla T) + div\,(\sigma U)$$

Energy Equation

The Navier-Stokes Equations

The Continuity Equation

$$\frac{\partial \rho}{\partial t} + div\,(\rho U) = 0$$

The Momentum Equation

$$\frac{\partial (\rho U)}{\partial t} + div\,(\rho U \otimes U + pI) = div\,\tau$$

The Energy Equation

$$\frac{\partial (\rho E)}{\partial t} + div(\rho E U) = div\,(k \nabla T) + div\,(\sigma U)$$

These equations need to be supplied with a domain, boundary conditions, initial conditions.

For stationary problems we set time derivative to 0

Non-Dimensional Equations

To understand non dimensionalization of the NS equations we start with two simple models. The advection equation and the heat equation.

Consider the equation,

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

And assume that we want to introduce new variables,

$$u' = u/U \quad t' = t/T \quad x' = x/L$$

It is easily verified that $\frac{\partial}{\partial t} = \frac{\partial}{\partial (Tt')} = \frac{1}{T}\frac{\partial}{\partial t'}$ and $\frac{\partial}{\partial x} = \frac{\partial}{\partial (Lx')} = \frac{1}{L}\frac{\partial}{\partial x'}$

Putting this into our equation gives,

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = \frac{\partial (Uu')}{\partial (Tt')} + a\frac{\partial (Uu')}{\partial (Lx')} = \frac{U}{T}\frac{\partial u'}{\partial t'} + \frac{aU}{L}\frac{\partial u'}{\partial x'}$$

And the equation in terms of the new variables is

$$\frac{\partial u'}{\partial t'} + \frac{aT}{L}\frac{\partial u'}{\partial x'} = 0$$

Note that if we choose T = L, the equation is unchanged!

Non-Dimensional Equations – cont.

Consider the equation, $$\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} = 0$$

And assume that we want to introduce new variables, $u' = u/U \quad t' = t/T \quad x' = x/L$

It is easily verified that $\frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial (Lx')^2} = \frac{1}{L^2}\frac{\partial^2}{\partial x'^2}$ and

Putting this into our equation and using results of the previous slide we have

$$\frac{\partial u'}{\partial t'} - \frac{DT}{L^2}\frac{\partial^2 u'}{\partial x'^2} = 0$$

Note that T = L, does not keep the equation unchanged!!

We will use this analysis for the Navier Stokes equations, and this will give us the non dimensional form Which involves the Reynolds number, Re.

Non-Dimensional NS Equations

In the Navier-Stokes equations we have several variables, and we scale all of them.

Assume that we want to introduce new variables,
Where all scaling parameters are scalars.

$$u = u'/U \quad \rho' = \rho/\rho_0 \quad p' = p/p_0, \quad x' = x/L, \quad t' = t/T$$

Starting with the continuity equation

$$\frac{\partial \rho}{\partial t} + div(\rho u) = 0$$

It is easily verified that

$$\frac{\partial \rho'}{\partial t'} + \frac{UT}{L} div'(\rho' u') = 0$$

Choosing U T = L, will keep the continuity equation unchanged!!

In the momentum equation

$$\frac{\partial(\rho u)}{\partial t} + div(\rho u \otimes u + pI) = div\tau$$

Using the relation $3\lambda + 2\mu = 0$ we get

$$\tau = \mu[(\nabla u) + (\nabla u)^T - \frac{2}{3}(div\, u)I]$$

Non-Dimensional NS Equations - cont

The momentum equation transform to

$$\frac{\rho_0 U}{T}\frac{\partial(\rho' u')}{\partial t'} + \frac{U^2\rho_0}{L}div'(\rho' u' \otimes u' + \frac{p_0}{\rho_0 U^2}p' I) = \frac{U}{L^2}div'\tau'$$

Note that choosing, $p_0 = \rho_0 U^2$

Simplify the pressure term in the new equation. So we will use the scaling of the pressure.

Next, if we choose, $UT = L$ we keep the first order terms unchanged.

Using the last two relations for scaling we have,

$$\frac{\partial(\rho' u')}{\partial t'} + div'(\rho' u' \otimes u' + p' I) = \frac{1}{\rho_0 U L}div'\tau'$$

Introducing the Reynolds number $Re = \frac{\rho_0 U L}{\mu}$ and $\tau = \mu[(\nabla u) + (\nabla u)^T - \frac{2}{3}(div\, u)I]$

We get $$\frac{\partial(\rho' u')}{\partial t'} + div(\rho' u' \otimes u' + p' I) = \frac{1}{Re}div'[(\nabla u') + (\nabla u')^T - \frac{2}{3}(div\, u')I]$$

Incompressible Case

The incompressible model
– assuming the density is constant

In the continuity equation the time derivatives drops,
And the density factors out from the divergence
operator.

The result is a time independent equation.

Incompressible Eq

$\rho(t,x) = \rho_0$ const.

$\frac{\partial \rho}{\partial t} = 0 \qquad div(\rho u) = \rho_0 \, div(u)$

continuity eq

$\frac{\partial \rho}{\partial t} + div(\rho u) = 0 \Longrightarrow$

$\boxed{div\, u = 0}$

In momentum eq $\tau = \mu \Delta u$

since $div\, u = 0$.

Incompressible Case

In the momentum equation :
The density factors out of all operators, so we divide
 by it the whole equation.

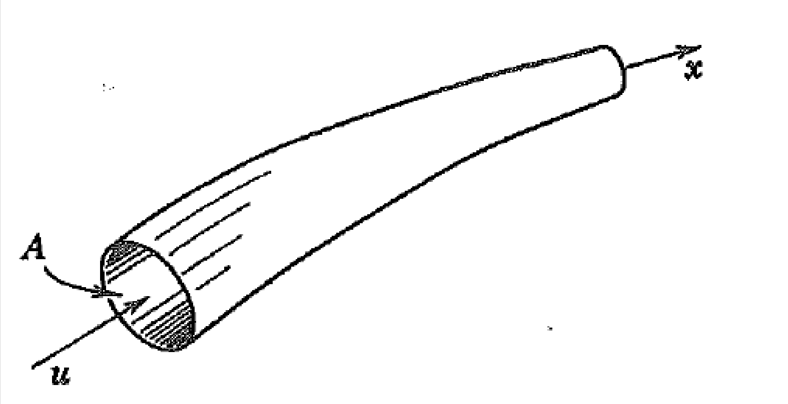
momentum eq: (divide by ρ_0 $\Rightarrow$)

$$\frac{\partial u}{\partial t} + \mathrm{div}\left(u \otimes u + p\,\underline{I}\right) = \nu \Delta u$$

diffent p.

Nozzle Flow – Continuity Eq

Uniform flow in a tube of cross section A(x)

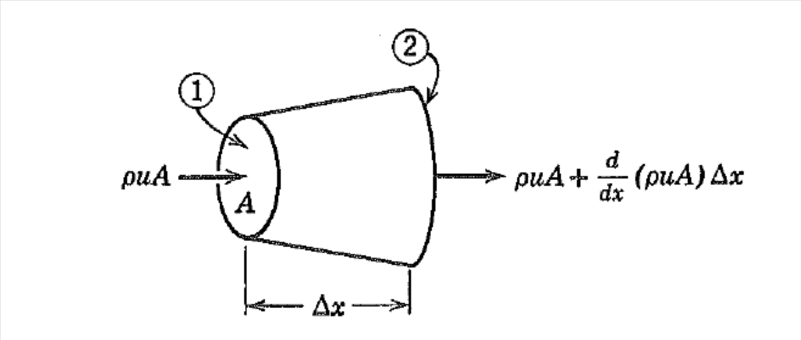


Continuity equation in integral form simplifies to :

$$\rho_1 u_1 A_1 = \rho_2 u_2 A_2$$

The corresponding differential equations is

$$\frac{d}{dx}(\rho u A) = 0$$



Mass of fluid enclosed, $\rho A\ \Delta x$

$$-\frac{\partial}{\partial x}(\rho u A)\,\Delta x = \frac{\partial}{\partial t}(\rho A\ \Delta x)$$

For unsteady flow we keep also the time derivative. In this case,

$$\frac{\partial(\rho A)}{\partial t} + \frac{\partial(\rho u A)}{dx} = 0$$

Nozzle Flow – Momentum Eq

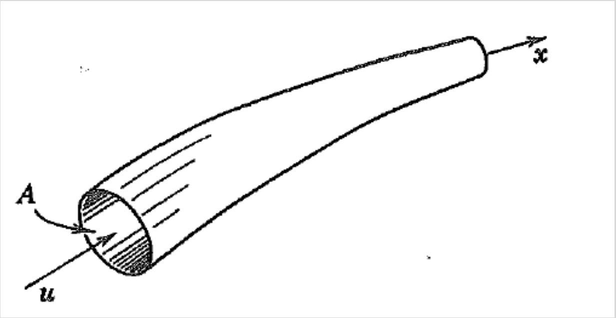
Flow in a tube of cross section A(x)

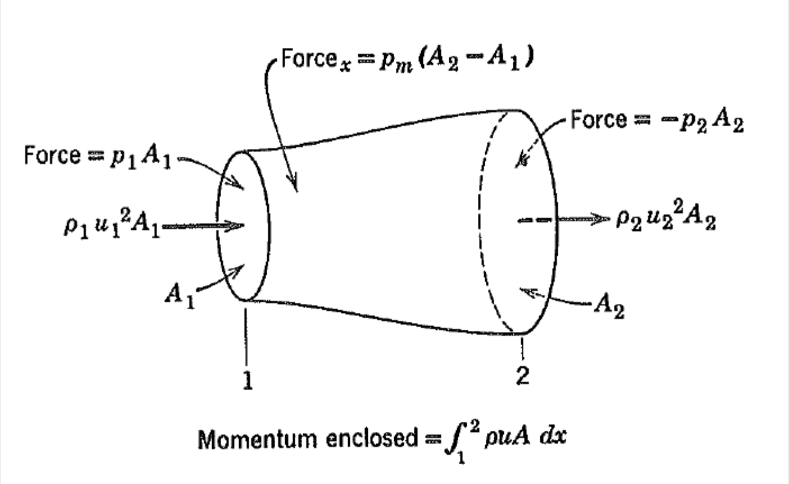
Momentum equation for steady flow in integral form simplifies to :

$$\frac{\partial}{\partial t}\int_1^2 (\rho u A)\,dx + (\rho_2 u_2^2 A_2 - \rho_1 u_1^2 A_1) = (p_1 A_1 - p_2 A_2) + \int_1^2 p\,dA$$

The corresponding differential equations is

$$\frac{\partial}{\partial t}(\rho u A) + \frac{\partial}{\partial x}(\rho u^2 A) = -A\frac{\partial p}{\partial x} = -\frac{\partial}{\partial x}(pA) + p\frac{\partial A}{\partial x}$$





1D Flow – Inviscid Flow

The Euler Equations

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho \\ \rho u \\ \rho E\end{pmatrix}+\frac{\partial}{\partial x}\begin{pmatrix}\rho u \\ \rho u^2+p \\ \rho u(E+p/\rho)\end{pmatrix}=0$$

$$p=(\gamma-1)\rho e$$

$$c^2=\gamma p/\rho$$

Rearranging the equations gives,

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho \\ u \\ p\end{pmatrix}+\begin{pmatrix}u & \rho & 0 \\ 0 & u & 1/\rho \\ 0 & \rho c^2 & u\end{pmatrix}\frac{\partial}{\partial x}\begin{pmatrix}\rho \\ u \\ p\end{pmatrix}=0$$

That is,

$$\frac{\partial \mathbf{f}}{\partial t}+\mathbf{A}\frac{\partial \mathbf{f}}{\partial x}=0$$

1D Flow – Lax-Friedrich Scheme - 1   Order

Write the Euler equations as

$$\frac{\partial \mathbf{f}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} = 0$$

Update the solution by

$$\mathbf{f}_j^{n+1} = \frac{1}{2}\left(\mathbf{f}_{j+1}^n + \mathbf{f}_{j-1}^n\right) - \frac{1}{2}\frac{\Delta t}{h}\left(\mathbf{F}_{j+1}^n - \mathbf{F}_{j-1}^n\right)$$

More Numerical Schemes later

Shocktube problem of
G.A. Sod, JCP 27:1, 1978

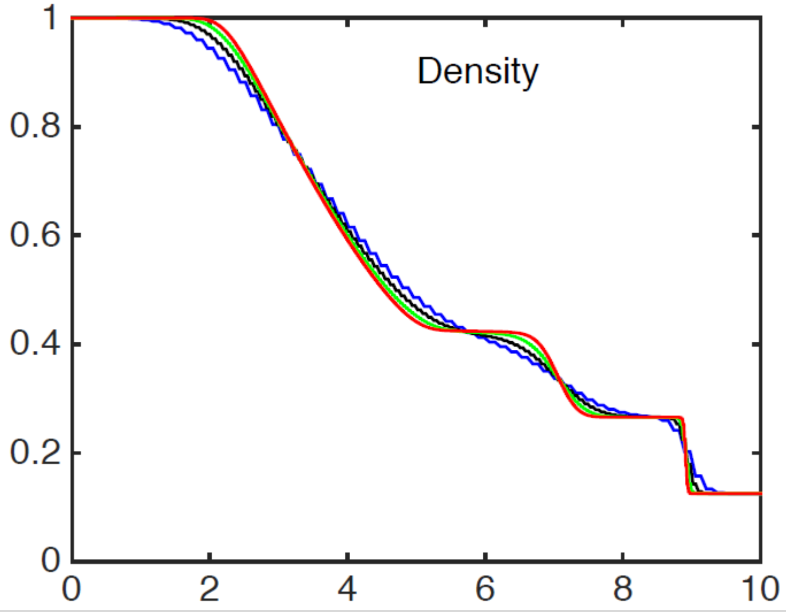
$$p_L = 10^5; \quad \rho_L = 1.0; \quad u_L = 0$$

$$p_R = 10^4; \quad \rho_R = 0.125; \quad u_R = 0$$

Grid size
128
256
512
1024

Final time
0.0071

Results for LF



Potential flow

If vorticity is zero $\nabla \times v = 0$

Then $v = \nabla\phi$

And the continuity equation becomes,

$$\frac{\partial\rho}{\partial t} + div(\rho\nabla\phi) = 0$$

We will derive an equation for the density in terms of the potential in a different lecture. For steady flow
The time derivative disappears and we have the relation

$$\frac{\rho}{\rho_0} = \left(1 - \frac{|\nabla\phi|^2}{2H_0}\right)^{1/(\gamma-1)}$$

This flow model involves a single unknown, the potential which is a scalar function.

We will study this model in details, for subsonic, transonic and supersonic flows.
This model was used for many decades to design airplane, until in the late 80’s and 90’s was replaces with the Euler and the Navier-Stokes equations