Fluid Dynamics

Lecture V – Review

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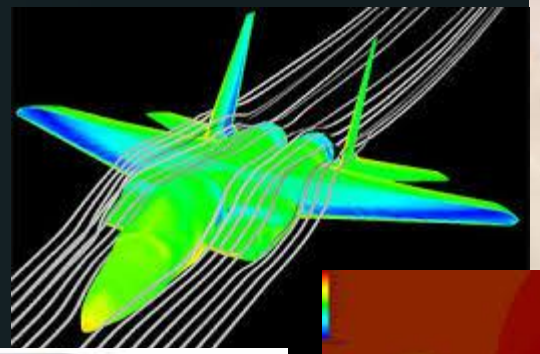
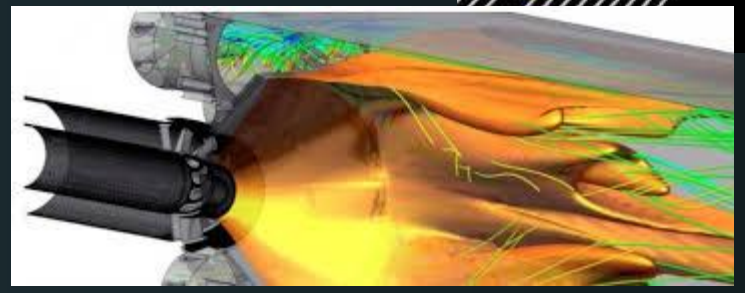
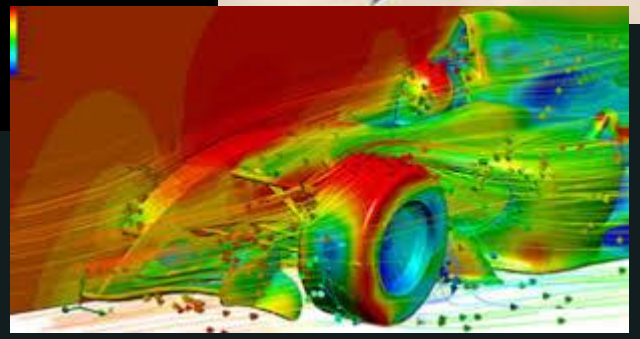
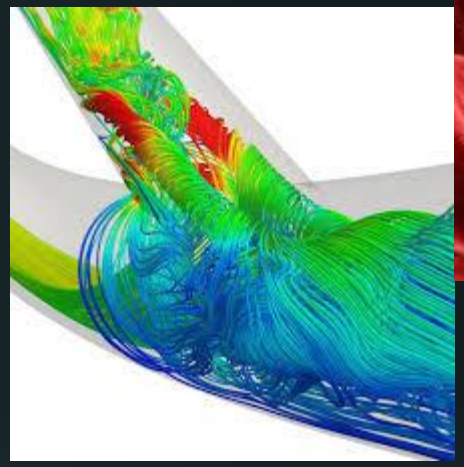
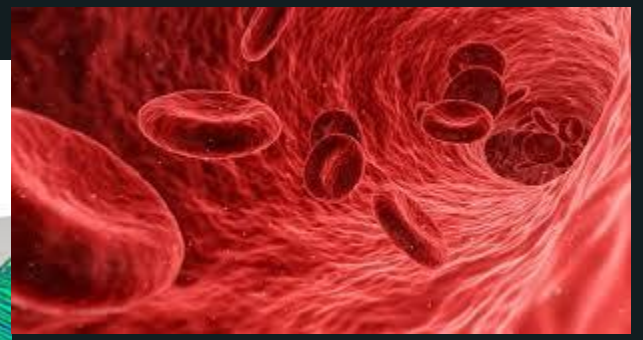
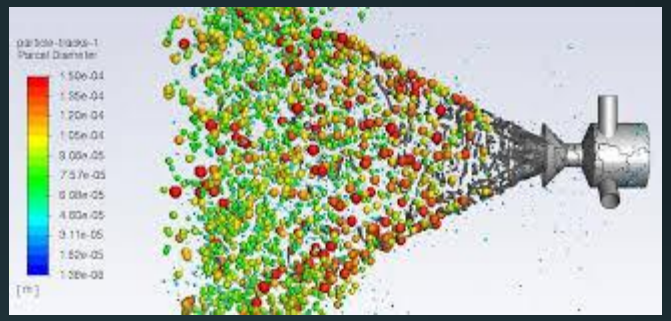












Derivation

Review of 3D Calculus

$$U = (u(x,y,z), v(x,y,z), w(x,y,z))$$

$$\phi(x,y,z)$$

$$div\, U = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

$$\nabla \phi = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}$$

The Divergence Theorem

$$\int_{\Omega} div\, U\, dx = \int_{\partial\Omega} U \cdot n\, ds$$

$$\int_{\Omega} \nabla \phi\, dx = \int_{\partial\Omega} \phi\, n\, ds$$

Differentiation – cont.

Putting together the details we get :

$$\frac{d}{dt}\int_{\Omega(t)} f(v,t)\,dv = \int_{\Omega(t)} \frac{\partial f}{\partial t}(v,t)\,dv + \int_{\partial\Omega(t)} f(s,t)\,U(s,t)\cdot n\,ds$$

This formula is the basis for several derivations we do next.

$$\boxed{\frac{d}{dt}\int_{\Omega(t)} f(t,x)\,dx \approx}$$

$$\frac{1}{\Delta t}\left[\int_{\Omega(t+\Delta t)} f(t+\Delta t,x)\,dx - \int_{\Omega(t)} f(t,x)\,dx\right]$$

$$\approx \frac{1}{\cancel{\Delta t}}\left[\int_{\Omega(t)} \left[\cancel{f} + \cancel{\Delta t}\frac{\partial f}{\partial t} - \cancel{f}\right] + \cancel{\Delta t}\int_{\partial\Omega(t)} f\,U\cdot n\,ds\right]$$

$$= \boxed{\int_{\Omega(t)} \frac{\partial f}{\partial t}\,dx + \int_{\partial\Omega(t)} f\,U\cdot n\,ds}$$

Differentiation of Integrals - summary

$$\frac{d}{dt}\int_{\Omega(t)} f(v,t)\,dv = \int_{\Omega(t)} \frac{\partial f}{\partial t}(v,t)\,dv + \int_{\partial\Omega(t)} f(s,t)\,U(s,t)\cdot n\,ds$$

Rate of change of integral = rate of change due to changes in f
+
rate of change due to changes in the domain

This mathematical formula, together with physical principles

1) **Conservation of mass**
2) **Newton's second law**
3) **Conservation of energy**

lead the the Navier-Stokes equations – the general equations of fluid flow

Conservation of mass- The Continuity Equation

Conservation of mass in a volume moving with the flow

$$\frac{d}{dt}\int_{\Omega(t)} \rho(v,t)\,dv = 0$$

Using

$$\frac{d}{dt}\int_{\Omega(t)} f(v,t)\,dv = \int_{\Omega(t)} \frac{\partial f}{\partial t}(v,t)\,dv + \int_{\partial\Omega(t)} f(s,t)\,U(s,t)\cdot n\,ds$$

We get

$$0 = \frac{d}{dt}\int_{\Omega(t)} \rho(v,t)\,dv = \int_{\Omega(t)} \frac{\partial \rho}{\partial t} + \int_{\partial\Omega(t)} \rho(s,t)\,U(s,t)\cdot n\,ds$$

But,

$$\int_{\partial\Omega(t)} \rho(s,t)\,U(s,t)\cdot n\,ds = \int_{\Omega(t)} div\,(\rho U)\,dv$$

Implies,

$$\int_{\Omega(t)} [\frac{\partial \rho}{\partial t} + div\,(\rho U)]\,dv = 0$$

Giving,

$$\frac{\partial \rho}{\partial t} + div\,(\rho U) = 0$$

The Continuity Equation

Conservation of Momentum

Change in momentum is due to surface forces – Newton's second Law

$$\frac{d}{dt}\int_{\Omega(t)} \rho U \, dv = \int_{\partial\Omega(t)} \sigma \, n \, ds$$

The math formula from before gives,

$$\frac{d}{dt}\int_{\Omega(t)} \rho U \, dv = \int_{\Omega(t)} \frac{\partial(\rho U)}{\partial t} + \int_{\partial\Omega(t)} \rho U U \cdot n \, ds$$

Divergence Theorem implies,

$$\int_{\partial\Omega(t)} \rho U U \cdot n \, ds = \int_{\Omega(t)} div\,(\rho U \otimes U)\, dv$$

Surface forces are expressed as

$$\sigma = -pI + \tau$$

Where,

$$\tau_{i,j} = \mu(\partial_i v_j + \partial_j v_i) + \lambda(div\, U)\delta_{i,j}$$

$$2\mu + 3\lambda = 0$$

Summarizing

$$\int_{\Omega(t)} [\frac{\partial(\rho U)}{\partial t} + div\,(\rho U \otimes U + pI - \tau)] = 0$$

Or

$$\frac{\partial(\rho U)}{\partial t} + div\,(\rho U \otimes U + pI) = div\,\tau$$

Momentum Equation

Conservation of Energy

Energy is a sum of internal energy and kinetic energy

$$E = e + \frac{1}{2}|U|^2$$

Change in energy is due to the work of surface forces plus heat flux

$$\frac{d}{dt}\int_{\Omega(t)} \rho E \, dv = \int_{\partial\Omega(t)} [\sigma U + k\nabla T \cdot n] \, ds$$

Divergence Theorem implies,

$$\int_{\partial\Omega(t)} \frac{\partial T}{\partial n} \, ds = \int_{\Omega} div\,(k\nabla T)$$

Like in previous case we conclude

$$\int_{\Omega(t)} \frac{\partial(\rho E)}{\partial t} + div(\rho E U - k\nabla T - \sigma U) \, dv = 0$$

Summarizing

$$\frac{\partial(\rho E)}{\partial t} + div(\rho E U) = div\,(k\nabla T) + div\,(\sigma U)$$

Energy Equation

The Navier-Stokes Equations

The Continuity Equation

$$\frac{\partial \rho}{\partial t} + div\,(\rho U) = 0$$

The Momentum Equation

$$\frac{\partial (\rho U)}{\partial t} + div\,(\rho U \otimes U + pI) = div\,\tau$$

The Energy Equation

$$\frac{\partial (\rho E)}{\partial t} + div(\rho E U) = div\,(k \nabla T) + div\,(\sigma U)$$

These equations need to be supplied with a domain, boundary conditions, initial conditions.

For stationary problems we set time derivative to 0

Non-Dimensional NS Equations

In the Navier-Stokes equations we have several variables, and we scale all of them.

Assume that we want to introduce new variables,
Where all scaling parameters are scalars.

$$u = u'/U \quad \rho' = \rho/\rho_0 \quad p' = p/p_0, \quad x' = x/L, \quad t' = t/T$$

Starting with the continuity equation

$$\frac{\partial \rho}{\partial t} + div(\rho u) = 0$$

It is easily verified that

$$\frac{\partial \rho'}{\partial t'} + \frac{UT}{L} div'(\rho' u') = 0$$

Choosing U T = L, will keep the continuity equation unchanged!!

In the momentum equation

$$\frac{\partial(\rho u)}{\partial t} + div(\rho u \otimes u + pI) = div\tau$$

Using the relation $3\lambda + 2\mu = 0$ we get

$$\tau = \mu[(\nabla u) + (\nabla u)^T - \frac{2}{3}(div\, u)I]$$

Non-Dimensional NS Equations - cont

The momentum equation transform to

$$\frac{\rho_0 U}{T}\frac{\partial(\rho u')}{\partial t'} + \frac{U^2\rho_0}{L} div'(\rho u' \otimes u' + \frac{p_0}{\rho_0 U^2} p' I) = \frac{U}{L^2} div' \tau'$$

Note that choosing, $p_0 = \rho_0 U^2$

Simplify the pressure term in the new equation. So we will use the scaling of the pressure.

Next, if we choose, $UT = L$ we keep the first order terms unchanged.

Using the last two relations for scaling we have,

$$\frac{\partial(\rho' u')}{\partial t'} + div'(\rho' u' \otimes u' + p' I) = \frac{1}{\rho_0 U L} div' \tau'$$

Introducing the Reynolds number $Re = \frac{\rho_0 U L}{\mu}$ and $\tau = \mu[(\nabla u) + (\nabla u)^T - \frac{2}{3}(div\, u) I]$

We get

$$\frac{\partial(\rho' u')}{\partial t'} + div(\rho' u' \otimes u' + p' I) = \frac{1}{Re} div'[(\nabla u') + (\nabla u')^T - \frac{2}{3}(div\, u') I]$$

Inviscid Flows

Shocks in Systems

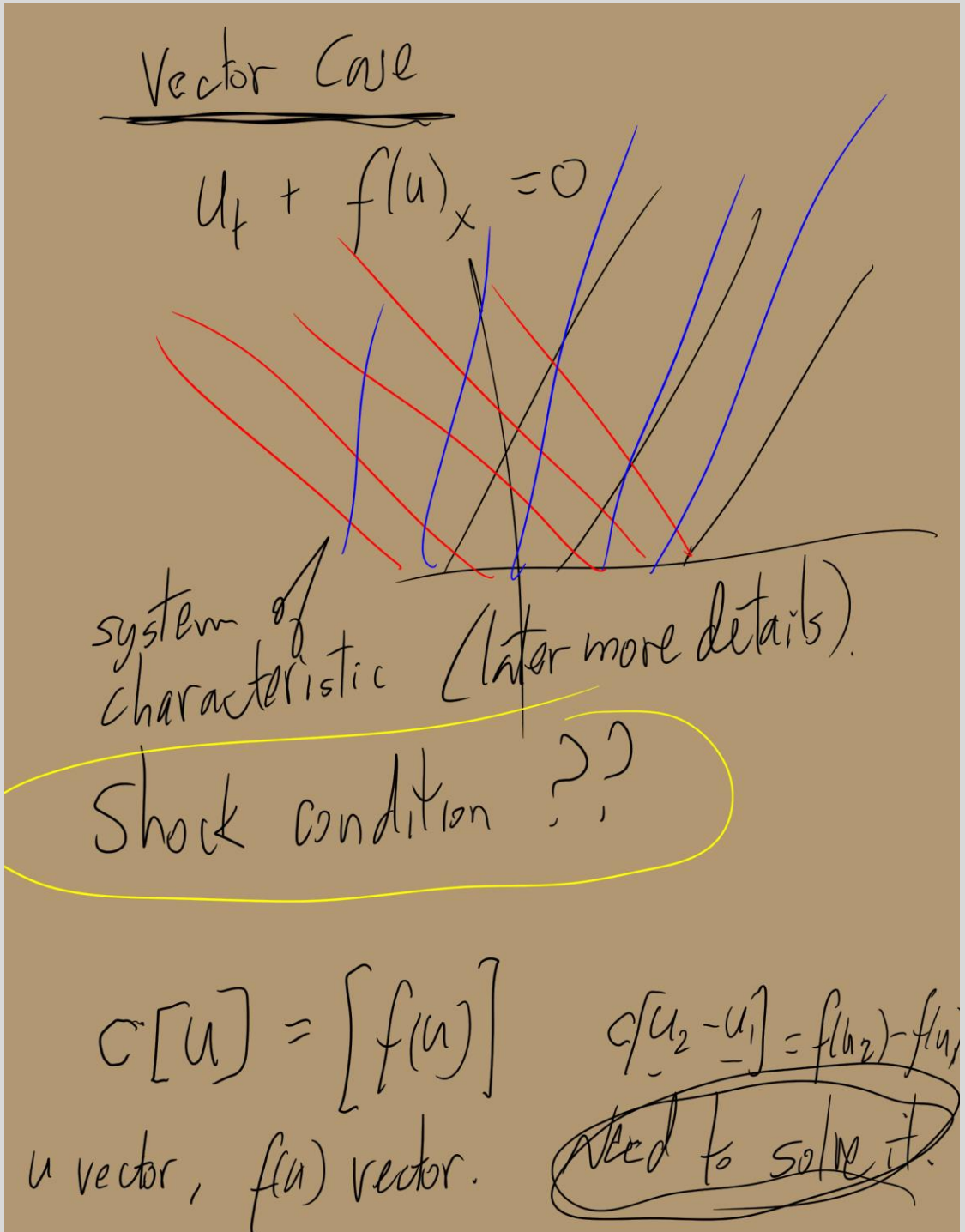
The jump condition for general 1D equation is shown to satisfy

$$c[u] = [f(u)]$$

Where c is the shock speed, [u] is the jump in u, and [f(u)] is the jump in f(u)

But to find c we need to solve a set of equations setting u to have two vector values across the shock and correspondingly f(u) is nonlinear vector valued expression in terms of these values.

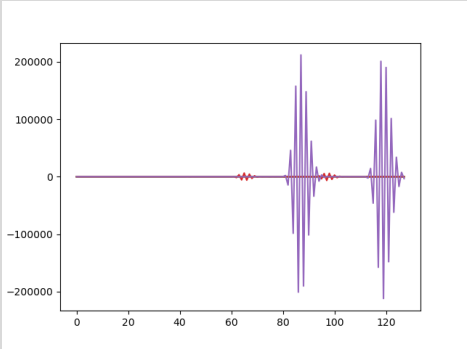
Note that the analysis of shocks was done in conservation Form of the equations!!!

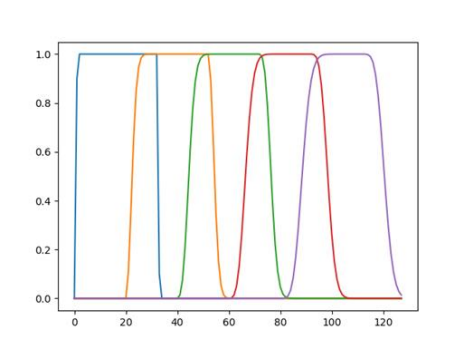


Understanding Stability

Von Neumann Analysis

A simple analysis for constant coefficient problems that gives us information about stability or blowup





Von-Neumann Analysis.

u_j^n — n ← time step.

j ← space index

In general

$$\underbrace{u^{n+1}}_{\text{sol at new time}} = L \underbrace{u^n}_{\text{sol at prev. time.}} \qquad L\text{-matrix}$$

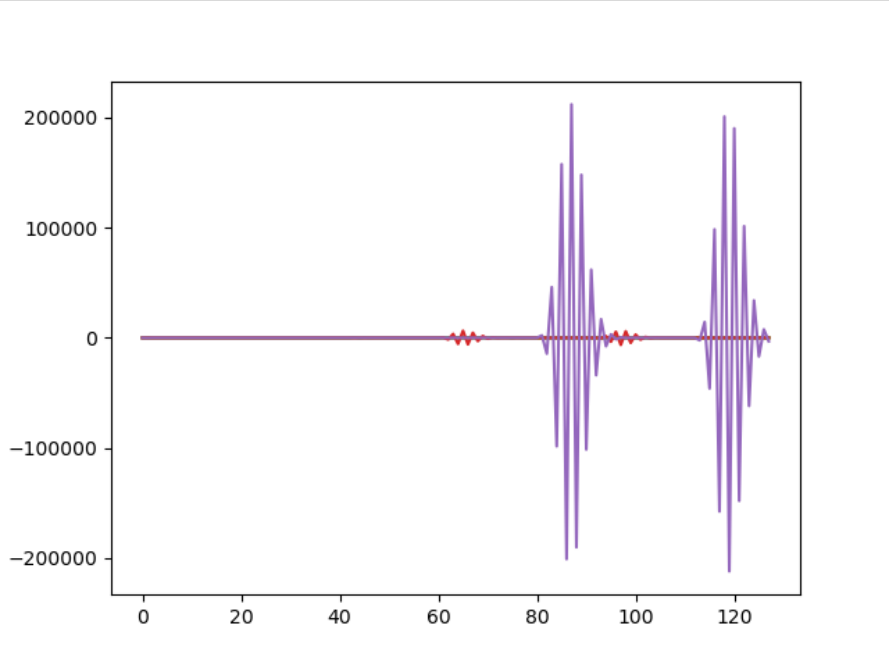
Assume $u_j^n = e^{i\theta j}$ $\qquad |\theta| \le \pi$

$$u_j^{n+1} = A e^{i\theta j}$$

plug into eq & calc A.

Von Neumann Analysis

A simple analysis explaining the blowup we observed numerically for a > 0.



Example

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\,\frac{u_{j+1}^n - u_j^n}{\Delta x} = 0$$

$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{\Delta x}\left(u_{j+1}^n - u_j^n\right)$$

$$u_j^n = e^{i\theta j} \qquad u_j^{n+1} = A e^{i\theta j}$$

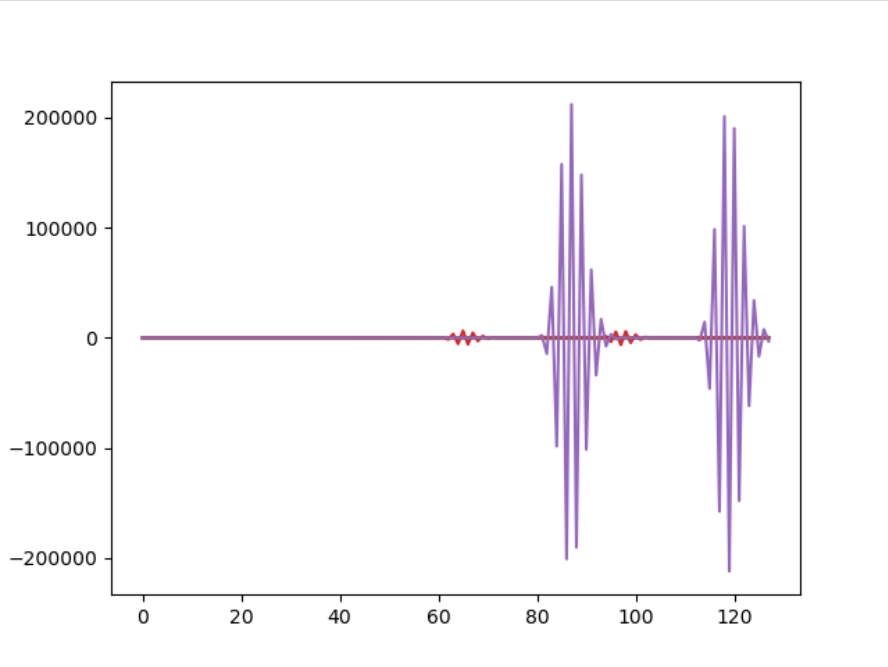
$$A\cancel{e^{i\theta j}} = \left(1 - \frac{a\Delta t}{\Delta x}\left(e^{i\theta} - 1\right)\right)\cancel{e^{i\theta j}}$$

$$A = \left(1 + \frac{a\Delta t}{\Delta x} - \frac{a\Delta t}{\Delta x}e^{i\theta}\right)$$

set $\lambda = a\Delta t / \Delta x$

Von Neumann Analysis - cont

A simple analysis explaining the blowup we observed numerically if a > 0.
For a<0, it is OK.



$$A = \left(1 + \lambda - \lambda\cos\theta - i\lambda\sin\theta\right)$$

$$\left(1 + \lambda(1 - \cos\theta) - i\lambda\sin\theta\right)$$

$$= \left(1 + 2\lambda\sin^2\frac{\theta}{2} - i\lambda\sin\theta\right)$$

$$|A|^2 = \left(1 + 2\lambda\sin^2\frac{\theta}{2}\right)^2 + \lambda^2\sin^2\theta$$

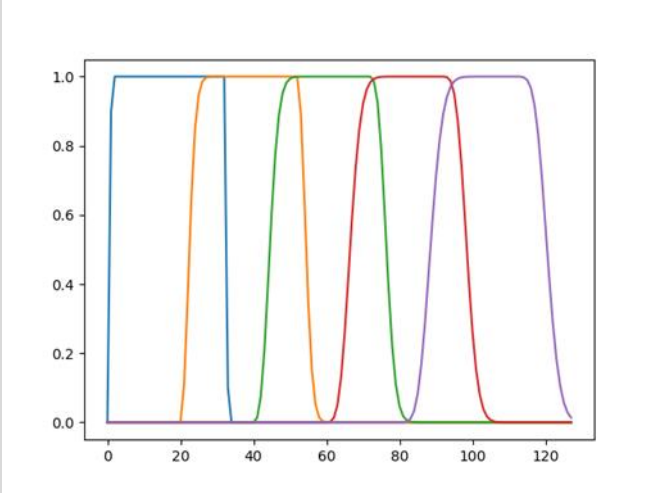
for $\theta = \pi$ $\quad |A(\pi)|^2 = (1 + 2\lambda)^2$

at each time sol amplitude grow by this factor.

$\boxed{\text{blow up of sol}}$

Stability Analysis

A simple analysis explaining good behavior of our upwind scheme



Stability

Want $|A(\theta)| \leq 1$ stable

or $|A(\theta)| \leq 1 + \alpha \Delta t$

But NOT $\boxed{|A(\theta)| \geq \beta_0 > 1}$ β_0 ind. of Δt.

unstable

The vectorial case

We consider the case where u(x,t) is a vector and A is a constant matrix with distinct real eigenvalues.

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = 0$$

To understand the solution of such systems we introduce the diagonalization of A. Let S be a matrix whose columns are the eigenvectors of A, then we have

$$AS = S\Gamma$$ or equivalently $$S^{-1}AS = \Gamma$$

Where Γ is a diagonal matrix with the eigenvalues of A on the diagonal.
Multiply the time dependent eq by S^{-1} we get

$$S^{-1}\frac{\partial u}{\partial t} + S^{-1}A\frac{\partial u}{\partial x} = 0$$ or $$S^{-1}\frac{\partial u}{\partial t} + S^{-1}ASS^{-1}\frac{\partial u}{\partial x} = 0$$

Introducing $$\delta w = S^{-1}\delta u$$ gives $$\frac{\partial w}{\partial t} + \Gamma\frac{\partial w}{\partial x} = 0$$ which are 3 independent equations

$$\frac{\partial w_j}{\partial t} + \lambda_j\frac{\partial w_j}{\partial x} = 0$$

The w's are called characteristic variables.

1D Euler Equations – conservative form

The conservative unknowns are

$$U = \begin{pmatrix} \rho \\ \rho u \\ \rho E \end{pmatrix} = \begin{pmatrix} \rho \\ m \\ \epsilon \end{pmatrix}$$

And the corresponding flux vector is

$$F = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u H \end{pmatrix} = \begin{pmatrix} m \\ m^2/\rho + p \\ m(\epsilon + p) \end{pmatrix}$$

And the equation is written in conservation form as

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0$$

Or as

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} = 0$$

where,

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -(3-\gamma)u^2/2 & (3-\gamma) & \gamma - 1 \\ (\gamma-1)u^3 - \gamma u E & \gamma E - 3\frac{\gamma-1}{2}u^2 & \gamma u \end{pmatrix}$$

This equation uses the conservative variables.

1D Euler Equations – non-conservative form

The primitive variables are

$$U = \begin{pmatrix} \rho \\ u \\ p \end{pmatrix} = \begin{pmatrix} \rho \\ m/\rho \\ (\gamma - 1)(\epsilon - \frac{m^2}{2\rho}) \end{pmatrix}$$

And the corresponding equations are

$$\frac{\partial \rho}{\partial t} + u\rho_x + \rho u_x = 0$$

$$\frac{\partial u}{\partial t} + uu_x + \frac{1}{\rho} p_x = 0$$

$$\frac{\partial p}{\partial t} + up_x + \rho c^2 u_x = 0$$

Or

$$\frac{\partial}{\partial t} \begin{pmatrix} \rho \\ u \\ p \end{pmatrix} + \begin{pmatrix} u & \rho & 0 \\ 0 & u & \frac{1}{\rho} \\ 0 & \rho c^2 & u \end{pmatrix} \frac{\partial}{\partial x} \begin{pmatrix} \rho \\ u \\ p \end{pmatrix} = 0$$

In short $\frac{\partial U}{\partial t} + A \frac{\partial U}{\partial x} = 0$ where the matrix A is given by

$$A = \begin{pmatrix} u & \rho & 0 \\ 0 & u & \frac{1}{\rho} \\ 0 & \rho c^2 & u \end{pmatrix}$$

1D Euler Equations – characteristic variables

We will perform now a diagonalization procedure to discover the characteristic variables. We need the eigenvalues and eigenvectors of the matrix A. This is done as follows

$$\det \begin{pmatrix} u-\lambda & \rho & 0 \\ 0 & u-\lambda & \frac{1}{\rho} \\ 0 & \rho c^2 & u-\lambda \end{pmatrix} = 0$$

giving

$$det = (u-\lambda)[(u-\lambda)^2 - c^2] = 0$$

There are 3 roots (eigenvalues)

$$\lambda_1 = u$$
$$\lambda_2 = u + c$$
$$\lambda_3 = u - c$$

and eigenvectors

$$l_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad l_2 = \begin{pmatrix} \frac{\rho}{2c} \\ \frac{1}{2} \\ \frac{\rho c}{2} \end{pmatrix} \quad l_3 = \begin{pmatrix} -\frac{\rho}{2c} \\ \frac{1}{2} \\ -\frac{\rho c}{2} \end{pmatrix}$$

The characteristic variables are obtained form the inverse matrix and gives,

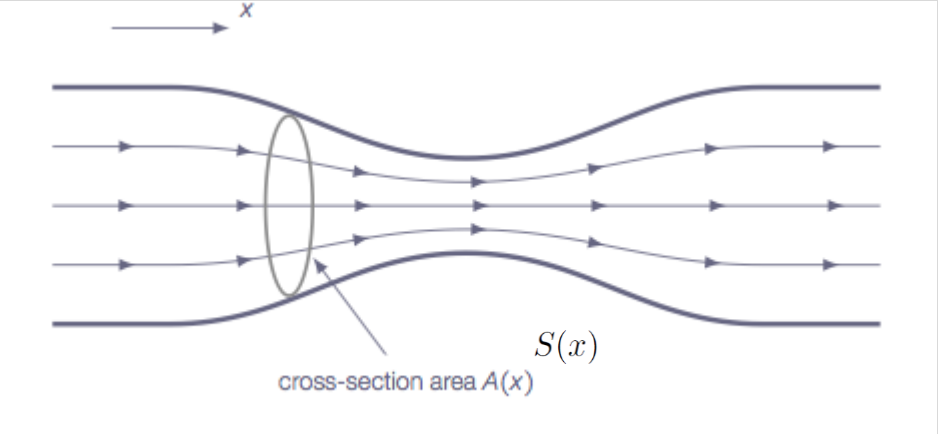
$$\delta w_1 = \delta\rho - \frac{1}{c^2}\delta p$$
$$\delta w_2 = \delta u + \frac{1}{\rho c}\delta p$$
$$\delta w_3 = \delta u - \frac{1}{\rho c}\delta p$$

and the equations for characteristic variables become

$$\frac{\partial}{\partial t}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} + \begin{pmatrix} u & 0 & 0 \\ 0 & u+c & 0 \\ 0 & 0 & u-c \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = 0$$

Nozzle flow

Cross section of nozzle is $S(x)$



The equations are

$$\frac{\partial(\rho S)}{\partial t} + \frac{\partial(\rho u S)}{\partial x} = 0$$

$$\frac{\partial(\rho u S)}{\partial t} + \frac{\partial(\rho u^2 + p)S}{\partial x} = p\frac{dS}{dx}$$

$$\frac{\partial(\rho E S)}{\partial t} + \frac{\partial(\rho u H S)}{\partial x} = 0$$

In primitive variables

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = -\frac{\rho u}{S}\frac{dS}{dx}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$$

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + \rho c^2\frac{\partial u}{\partial x} = -\frac{\rho u c^2}{S}\frac{dS}{dx}$$

RHS

$$\bar{Q} = \begin{pmatrix} -\rho u \\ 0 \\ -\rho c^2 u \end{pmatrix} \frac{1}{S}\frac{dS}{dx}$$

Note that these are the 1D Euler equations with a source term depending on the nozzle shape.

Finite Differences Approximation in 2D

The 2D Poisson equation is

$$\Phi_{xx} + \Phi_{yy} = f$$

We use double indexing to represent 2D and we get similar to the 1D case,

$$\Phi^h_{i,j} \approx \Phi(x, y) \quad h \to 0$$

The discretization for the x and y directions are

$$\Phi_{yy} \approx \frac{1}{h^2}\left[\Phi^h_{i,j+1} - 2\Phi^h_{i,j} + \Phi^h_{i,j-1}\right]$$

$$\Phi_{xx} \approx \frac{1}{h^2}\left[\Phi^h_{i+1,j} - 2\Phi^h_{i,j} + \Phi^h_{i-1,j}\right]$$

And the discretization of the Poisson equation in 2D becomes

$$\frac{1}{h^2}(\Phi^h_{i+1,j} - 2\Phi^h_{i,j} + \Phi^h_{i-1,j}) + \frac{1}{h^2}(\Phi^h_{i,j+1} - 2\Phi^h_{i,j} + \Phi^h_{i,j-1}) = f_{i,j}$$

What about boundary ocnditions?

Vertex Schemes - BC

Dirichlet BC is straightforward. We just specify the BC value

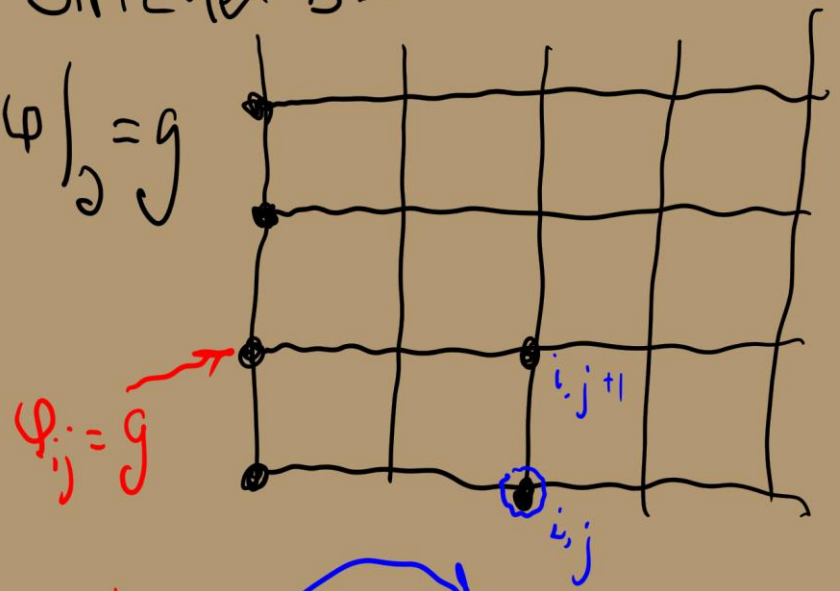
Neumann BC is more involved.

We introduce a ghost point (red)
We discretize the interior equation at the boundary point
And the Neumann BC at that point.

The ghost point (red) has a ghost variable (red) that we eliminate. The resulting equation is used at the boundary.
No ghost variables is used explicitly in calculation

Dirichlet B.C.

$\varphi|_\partial = g$



ghost point does not exist in calculation

Neumann B.C.

$$\frac{\partial \varphi}{\partial n} = h$$

at (i,j) use eq $\frac{1}{h^2}\begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix}\varphi = f$

& $\frac{1}{2h}[\varphi_{i,j+1} - \varphi_{i,j-1}] = h$ & $\boxed{\text{eliminate } \varphi_{i,j-1}}$

Vertex Scheme –Neumann BC

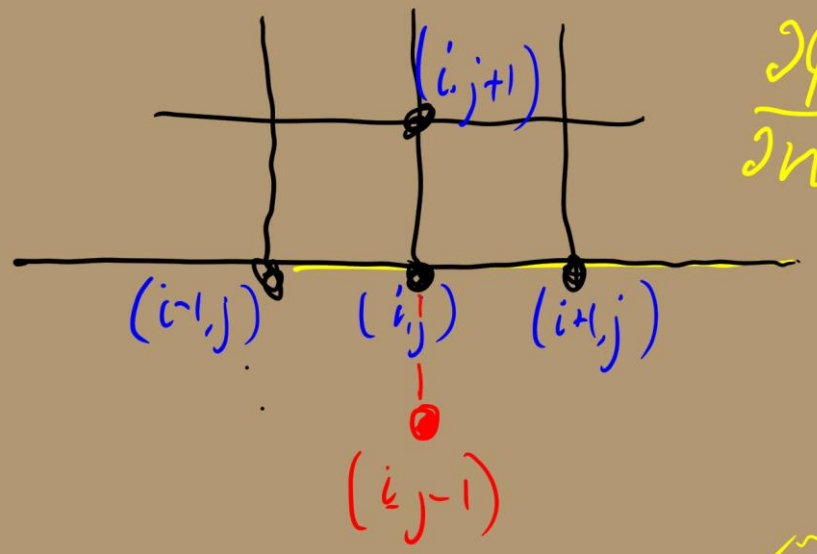
Dirichlet BC is straightforward. We just specify the BC value

Neumann BC is more involved.

We introduce a ghost point (red)
We discretize the interior equation at the boundary point
And the Neumann BC at that point.

The ghost point (red) has a ghost variable (red) that we eliminate. The resulting equation is used at the boundary.
No ghost variables is used explicitly in calculation

Actual calculation is shown on the right.



$$\frac{\partial \varphi}{\partial n} = g$$

$$\frac{1}{h^2}\left[-4\varphi_{i,j} + \varphi_{i+1,j} + \varphi_{i-1,j} + \varphi_{i,j+1} + \varphi_{i,j-1}\right] = f_{i,j}$$

$$\frac{1}{2h}\left[\varphi_{i,j+1} - \varphi_{i,j-1}\right] = g_{i,j} \quad (**)$$

Eliminate $\varphi_{i,j-1}$ using $(**)$ & substitute in $(*)$

$$\varphi_{i,j-1} = \varphi_{i,j+1} - 2h\,g_{i,j}$$

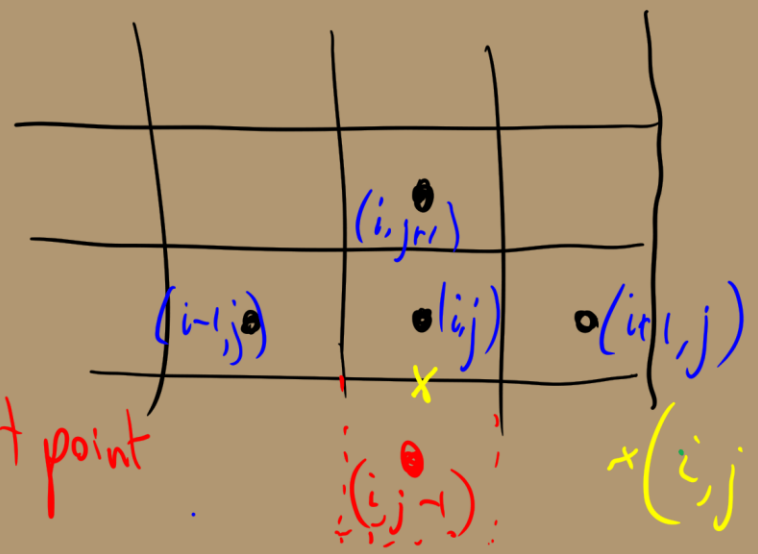
Cell-Centered Scheme Neumann BC

Neumann BC is done in a similar way to vertex scheme

We introduce a ghost point (red)
We discretize the interior equation at the boundary point
And the Neumann BC at that point.

The ghost point (red) has a ghost variable (red) that we eliminate. The resulting equation is used at the boundary.
No ghost variables is used explicitly in calculation

Actual calculation is shown on the right.



$$\frac{1}{h^2}\left[-4\varphi_{ij} + \varphi_{i+1,j} + \varphi_{i-1,j} + \varphi_{i,j+1} + \varphi_{i,j-1}\right] = f_{ij} \qquad (**)$$

$$\frac{1}{h}\left[\varphi_{i,j} - \varphi_{i,j-1}\right] = g_{i,j-\frac{1}{2}}$$

$$\varphi_{i,j-1} = \varphi_{i,j} - h\, g_{i,j-\frac{1}{2}}$$

(**) with $\varphi_{i,j-1}$ eliminated is the correct eq.

Cell-Centered Scheme Dirichlet BC

Dirichlet BC is done in a similar way to Neumann BC

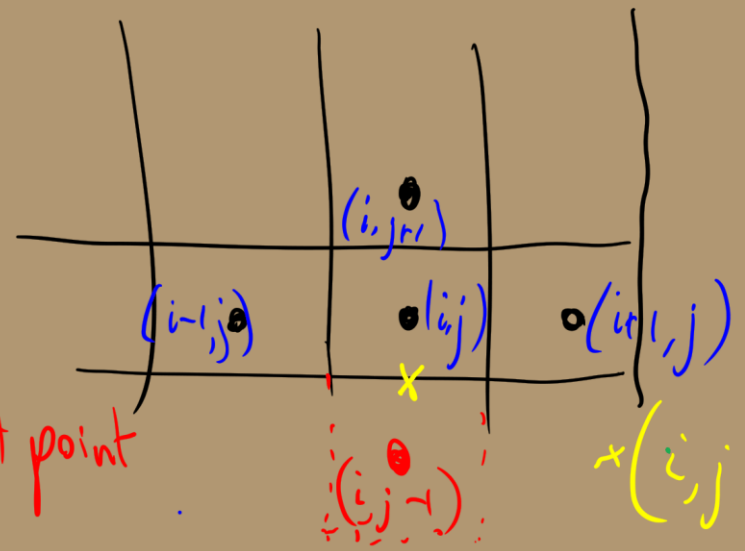
We introduce a ghost point (red)
We discretize the interior equation at the boundary point
And the Dirichlet BC at that point.

The ghost point (red) has a ghost variable (red) that we eliminate. The resulting equation is used at the boundary.
No ghost variables is used explicitly in calculation

Actual calculation is shown on the right.

B.C.

$\varphi = g$



$$\frac{1}{h^2}\left[-4\varphi_{ij} + \varphi_{i+1,j} + \varphi_{i-1,j} + \varphi_{i,j+1} + \varphi_{i,j-1}\right] = f_{ij} \qquad (**)$$

$$\frac{1}{2}\left[\varphi_{i,j+1} + \varphi_{i,j-1}\right] = g_{i,j-\frac{1}{2}}$$

$$\varphi_{i,j-1} = -\varphi_{i,j+1} + 2\, g_{i,j-\frac{1}{2}}$$

(**) with $\varphi_{i,j-1}$ eliminated is the correct eq.

The Steady Full Potential Equation – non conservative

In many applications we are interested in steady flow and therefore omit all time derivatives,

$$\frac{\rho}{\rho_0} = \left(1 - \frac{|\nabla\phi|^2}{2H_0}\right)^{1/(\gamma-1)}$$

$$c^2 = (\gamma - 1)\left[H_0 - \frac{1}{2}|\nabla\phi|^2\right]$$

$$(1-M_x^2)\phi_{xx}+(1-M_y^2)\phi_{yy}+(1-M_z^2)\phi_{zz}-2M_xM_y\phi_{xy}-2M_xM_z\phi_{xz}-2M_yM_z\phi_{yz} = 0$$

$$M_x = \frac{\phi_x}{c} \quad M_y = \frac{\phi_y}{c} \quad M_z = \frac{\phi_z}{c} \quad M^2 = \frac{|\nabla\phi|^2}{c^2}$$

with wall BC

$$v_n = \frac{\partial\phi}{\partial n} = 0$$

This is the steady full potential equation in non-conservative form. This form reveals more about the equation than its conservative counterpart. Later we see that M<1 and M>1 need different discretization.

For now, we focus on M < 1, the subsonic case. We will deal with M > 1 in the next lecture.

The Small Disturbance Approximation

A simplification of the full potential equations is done in the case of thin obstacles, such as thin airfoils. We will restrict the discussion to 2D.

Since the obstacle is “small” it’s effect on the flow is small and we consider perturbation to uniform flow with velocity of magnitude U_∞ in the x-direction. The potential has a representation

$$\phi = U_\infty(x + \Phi)$$

Where velocities are recovered from the potential using the formulas

$$u = U_\infty(1 + \Phi_x)$$
$$v = U_\infty \Phi_y$$

The equation becomes $(1 - M_x^2)\Phi_{xx} + \Phi_{yy} = 0$ which can be simplified further to $(1 - M_\infty^2)\Phi_{xx} + \Phi_{yy} = 0$

The wall BC becomes $v = (U_\infty + u)f'(x) \approx U_\infty f'(x)$ where f(x) is the shape of the airfoil.

Supersonic Flows

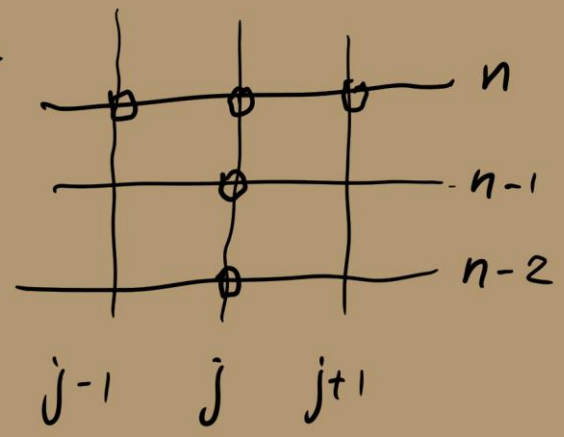
The Wave Equation - cont

We want a scheme that is stable for all

$$\frac{c\,\Delta t}{\Delta x}.$$

$$\frac{\partial^2 u}{\partial t^2} \sim \frac{u_j^n - 2u_j^{n-1} + u_j^{n-2}}{\Delta t^2}$$

$$\frac{\partial^2 u}{\partial x^2} \sim \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}$$



$$\frac{u_j^n - 2u_j^{n-1} + u_j^{n-2}}{\Delta t^2} - c^2\,\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}$$

$$s = c^2\Delta t^2/\Delta x^2$$

$$\boxed{-s\,u_{j+1}^n + (2s+1)u_j^n - s\,u_{j-1}^n = 2u_j^{n-1} - u_j^{n-2}}$$

We show now that this is stable all all values of $s>0$.

The Small Disturbance Eq

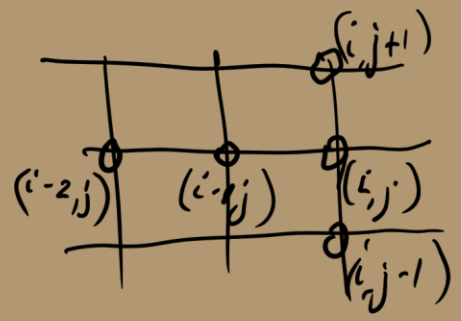
SDE

$$(1-M_\infty^2)\varphi_{xx} + \varphi_{yy} = 0$$

$M>1$

x is like t in wave eq.

y " x " " "



$$(1-M^2)(\varphi_{ij} - 2\varphi_{i-1,j} + \varphi_{i-2,j}) +$$
$$(\varphi_{i,j+1} - 2\varphi_{ij} + \varphi_{i,j-1}) = 0$$

assume $\Delta x = \Delta y$

for $M<1$ we need a different schm.

$$(1-M^2)(\varphi_{i+1,j} - 2\varphi_{ij} + \varphi_{i-1,j}) +$$
$$(\varphi_{i,j+1} - 2\varphi_{ij} + \varphi_{i,j-1}) = 0$$

The Small Disturbance Eq

SDE - Scheme

$$\underbrace{\left[1-M_\infty^2-(\gamma+1)M_\infty^2\varphi_x\right]}\varphi_{xx}+\varphi_{yy}=0$$

$$1-M^2=K$$

$$K_{ij}=1-M_\infty^2-(\gamma+1)M_\infty^2\frac{\varphi_{i+1,j}-\varphi_{i-1,j}}{2\Delta x}$$

for $M<1$, $\Delta x=\Delta y$ we have

$$K_{ij}\left(\varphi_{i+1,j}-2\varphi_{ij}+\varphi_{i-1,j}\right)+\left(\varphi_{i,j+1}-2\varphi_{ij}+\varphi_{i,j-1}\right)=0$$

for $M>1$

$$K_{i-1,j}\left(\varphi_{ij}-2\varphi_{i-1,j}+\varphi_{i-2,j}\right)+\left(\varphi_{i,j+1}-2\varphi_{ij}+\varphi_{i,j-1}\right)=0$$

Combine as:

$$\mu_{ij}K_{i-1,j}\left(\varphi_{ij}-2\varphi_{i-1,j}+\varphi_{i-2,j}\right)+\left(\varphi_{i,j+1}-2\varphi_{ij}+\varphi_{i,j-1}\right)+$$

$$(1-\mu_{ij})K_{ij}\left(\varphi_{i+1,j}-2\varphi_{ij}+\varphi_{i-1,j}\right)=0$$

$\mu=0$ for $M<1$

$\mu=1$ for $M\geq 1$

Non-conservative form!

The Full Potential Eq

Full Potential Equation for $M>1$

$$(\rho\varphi_x)_x + (\rho\varphi_y)_y = 0$$

artificial compressibility idea:

$$(\bar{\rho}\varphi_x)_x + (\bar{\bar{\rho}}\varphi_y)_y = 0$$

$$\bar{\rho} = \rho - \mu\rho_x \Delta x$$

$$\bar{\bar{\rho}} = \rho - \mu\rho_y \Delta y$$

since we need $\rho_{i\pm\frac{1}{2},j}$ $\rho_{i,j\pm\frac{1}{2}}$ we have to shift ρ based on the flow direction.

If $u_{i+\frac{1}{2},j} > 0$

$$\bar{\rho}_{i+\frac{1}{2},j} = \rho_{i+\frac{1}{2},j} - \mu_{ij}(\rho_{i+\frac{1}{2},j} - \rho_{i-\frac{1}{2},j})$$

The Full Potential Eq

if $u_{i+\frac{1}{2},j} < 0$

$$\overline{\rho}_{i+\frac{1}{2},j} = \rho_{i+\frac{1}{2},j} + \mu_{i+1,j}\left(\rho_{i+\frac{3}{2},j} - \rho_{i+\frac{1}{2},j}\right)$$

if $v_{i,j+\frac{1}{2}} > 0$

$$\overline{\overline{\rho}}_{i,j+\frac{1}{2}} = \rho_{i,j+\frac{1}{2}} - \mu_{i,j}\left(\rho_{i,j+\frac{3}{2}} - \rho_{i,j-\frac{1}{2}}\right)$$

if $v_{i,j-\frac{1}{2}} < 0$

$$\overline{\overline{\rho}}_{i,j+\frac{1}{2}} = \rho_{i,j+\frac{1}{2}} + \mu_{i,j+1}\left(\rho_{i,j+\frac{3}{2}} - \rho_{i,j+\frac{1}{2}}\right)$$

$$\mu = \max\left[\left(0, 1 - \frac{M_c^2}{M^2}\right)\right] \cdot C M^2$$

$M_c = 0.95$, $1 < C < 2$

Final Project

Comparison of nozzle flow using the small disturbance equation with the 1D Euler equation for nozzle Geometry.

More details on Ed.