Fluid Dynamics

Lecture III – The Full Potential Equation

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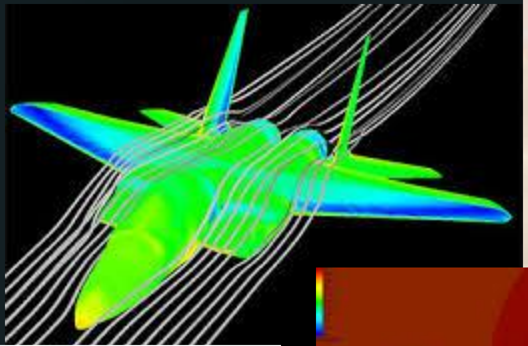
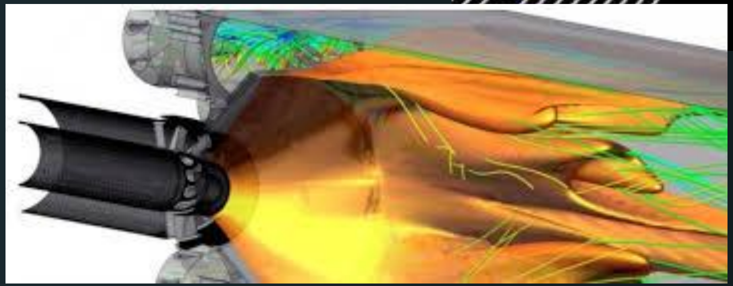
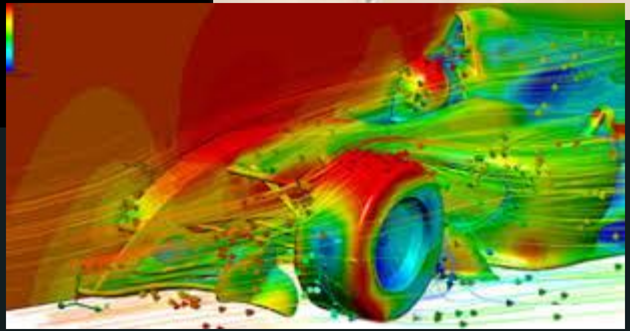
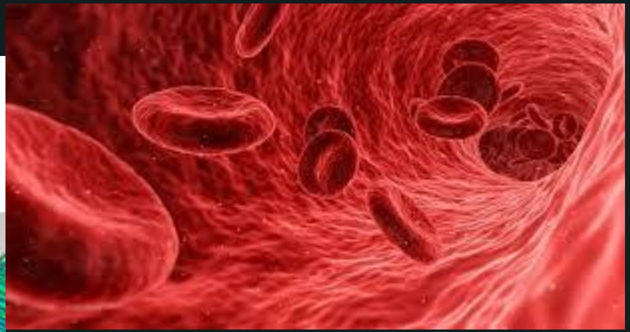
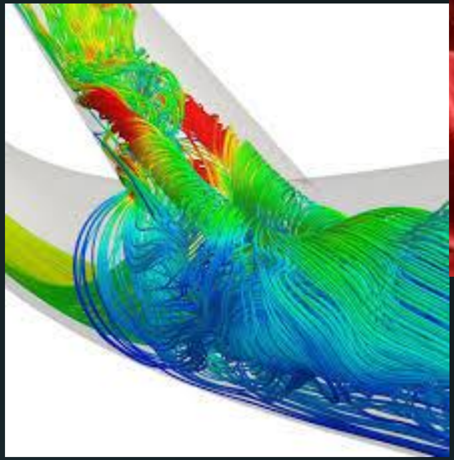
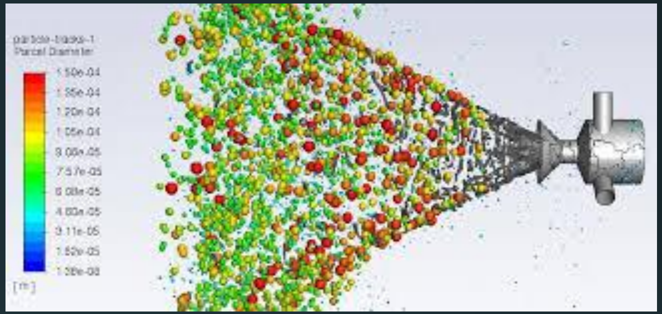












Outline

Today we focus on the 2D/3D full potential equation and its approximations

To understand it we start with simplified models that capture the physics of the full model.

-The Laplace equation. Boundary conditions and Discretization
- The Full Potential equation
- Small Disturbance Approximation equation
- Subsonic flows. Airfoil problem. Lift, drag.

The Poisson Equation

The Laplacian is ubiquitous in mathematical physics
It plays an important role in subsonic flow

$$\triangle\phi \equiv div\,\nabla\,\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

We need to approximate it using a numerical scheme.

We start with 1D model to understand discretization techniques.

We distinguish three approaches to approximate the equation

- Finite elements - requires extensive study
- Finite differences – easy but is limited to simple domains
- Finite volumes - easy and can handle arbitrary domain

We focus on the last two approaches.

Approximation using finite differences

First order derivative, using calculus

$$\frac{1}{h}[f(x+h) - f(x)] \approx f'(x) + O(h)$$

the error term O(h) can be derived using Taylor expansion

A better approximation for first derivative is

$$\frac{1}{2h}[f(x+h) - f(x-h)] \approx f'(x) + O(h^2)$$

Second derivative are approximated as

$$\frac{1}{h^2}[f(x+h) - 2f(x) + f(x-h)] \approx f''(x) + O(h^2)$$

with second order accuracy

Now for our problem in 1D $\Phi_{xx} = f$ $\Phi(0) = a, \Phi(1) = b$

We get

$$\Phi^h_{j+1} - 2\Phi^h_j + \Phi^h_{j-1} = h^2 f_j \qquad j = 1, \ldots, n-1$$

$$\Phi^h_0 = a, \Phi^h_n = b$$

Think of it as $\Phi^h_j \approx \Phi(x_j)$ and as $h \to 0$ we obtain $\Phi^h_{j(x)} \to \Phi(x), \qquad j(x) = [x/h]$

The Matrix

$$\Phi_{xx} = f$$

$$\Phi(0) = a, \Phi(1) = b$$

$$\Phi^h_{j+1} - 2\Phi^h_j + \Phi^h_{j-1} = h^2 f_j \qquad j = 1, \ldots, n-1$$

$$\Phi^h_0 = a, \Phi^h_n = b$$

$$\Phi^h_j \approx \Phi(x_j)$$

$$h \to 0$$

$$\Phi^h_{j(x)} \to \Phi(x), \qquad j(x) = [x/h]$$

The Matrix 1D Vertex scheme

$$\varphi_{i-1} - 2\varphi_i + \varphi_{i+1} = h^2 f_i \qquad \varphi_1 = a, \varphi_N = b$$

$$\begin{bmatrix} 1 & 0 & & & & \\ 1 & -2 & 1 & & 0 & \\ & 1 & -2 & 1 & & \\ & & \ddots & \ddots & \ddots & \\ & 0 & & 1 & -2 & 1 \\ & & & & & 1 \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \\ \varphi_{N-1} \\ \varphi_N \end{bmatrix} = \begin{bmatrix} a \\ h^2 f_1 \\ h^2 f_2 \\ \vdots \\ \\ b \end{bmatrix}$$

tridiagonal system.

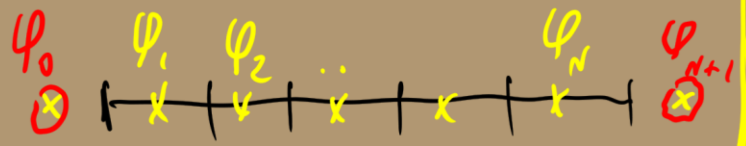
What about Neumann B.C.?

The Matrix - cont

In cell centered scheme the interior equations remain the same. Only the boundary condition implementation changes.

Cell centered schemes are natural to use when Neumann BC is involved.

$$\varphi_{xx} = f \qquad \varphi(0)=a \; ; \; \frac{\partial \varphi}{\partial x}(1) = b$$



Cell centered Scheme

ghost variables

Use it if we have Neumann B.C.

$$\varphi(0)=a \Rightarrow \frac{1}{2}(\varphi_0+\varphi_1)=a$$

$$\frac{\partial \varphi}{\partial n}(1)=b \Rightarrow \frac{1}{h}(\varphi_{N+1}-\varphi_N)=b$$

(*) The eq $\varphi_0 - 2\varphi_1 + \varphi_2 = h^2 f_1$ becomes

$$(2a-\varphi_1)-2\varphi_1+\varphi_2=h^2 f_1$$

$$\boxed{-3\varphi_1+\varphi_2=h^2 f_1 - 2a}$$ first eq

(**) The eq $\varphi_{N+1} - 2\varphi_N + \varphi_{N-1} = h^2 f_N$ becomes

$$hb+\varphi_N-2\varphi_N+\varphi_{N-1}=h^2 f_N$$

$$\boxed{-\varphi_N+\varphi_{N-1}=h^2 f_N - hb}$$ last eq

The Matrix - cont

In cell centered scheme the interior equations remain the same. Only the boundary condition implementation changes.

Cell centered schemes are natural to use when Neumann BC is involved.

The resulting matrix is:

$$\begin{bmatrix} -3 & 1 & & & 0 \\ 1 & -2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ 0 & & 1 & -2 & 1 \\ & & & 1 & -1 \end{bmatrix} \begin{bmatrix} \varphi_1 \\ \\ \vdots \\ \\ \varphi_N \end{bmatrix} = \begin{bmatrix} h^2 f_1 - 2a \\ h^2 f_2 \\ h^2 f_3 \\ \vdots \\ h^2 f_{N-1} \\ h^2 f_N - hb \end{bmatrix}$$

How do we use this in a Python code?

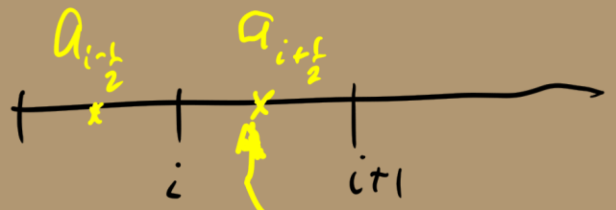
- sparse matrix representation only 3 diagonals.

The Matrix - cont

In a non-constant coefficient problem we need to discretize the coefficient I na way that we do not loose accuracy. The coefficient may be needed at places where first derivatives are calculated.

Non Constant Coefficient Problem 1D

$$(a(x)u_x)_x = f(x) \qquad [0,1]$$



Vertex Scheme

u_i at nodes

$$\frac{u_{i+1}-u_i}{h} \approx u_x(\cdot)$$

$$a_{i+\frac{1}{2}} = \frac{1}{2}(a_i + a_{i+1})$$

$$a_{i+\frac{1}{2}}(u_{i+1}-u_i) - a_{i-\frac{1}{2}}(u_i - u_{i-1}) = h^2 f_i$$

$$a_{i-\frac{1}{2}}u_{i-1} - (a_{i-\frac{1}{2}} + a_{i+\frac{1}{2}})u_i + a_{i+\frac{1}{2}}u_{i+1} = h^2 f_i$$

The matrix?

The Matrix - cont

In a non-constant coefficient problem we need to discretize the coefficient I na way that we do not loose accuracy. The coefficient may be needed at places where first derivatives are calculated.

B.C.

$$\begin{bmatrix} 1 & \cdot & & & \\ \frac{a_1+a_2}{2} & -\frac{a_1+2a_2+a_3}{2} & \frac{a_2+a_3}{2} & & \\ & \ddots & \ddots & \ddots & \\ & & & & 1 \end{bmatrix}$$

B.C

i-th row:

$$\frac{a_{i-1}+a_i}{2} \qquad -\frac{a_{i-1}+2a_i+a_{i+1}}{2} \qquad \frac{a_i+a_{i+1}}{2}$$

We will need this for the full potential eq later:

$$(\rho\varphi_x)_x = 0$$

Finite Differences Approximation in 2D

The 2D Poisson equation is

$$\Phi_{xx} + \Phi_{yy} = f$$

We use double indexing to represent 2D and we get similar to the 1D case, $\Phi^h_{i,j} \approx \Phi(x,y) \quad h \to 0$

The discretization for the x and y directions are

$$\Phi_{yy} \approx \frac{1}{h^2}\left[\Phi^h_{i,j+1} - 2\Phi^h_{i,j} + \Phi^h_{i,j-1}\right]$$

$$\Phi_{xx} \approx \frac{1}{h^2}\left[\Phi^h_{i+1,j} - 2\Phi^h_{i,j} + \Phi^h_{i-1,j}\right]$$

And the discretization of the Poisson equation in 2D becomes

$$\frac{1}{h^2}(\Phi^h_{i+1,j} - 2\Phi^h_{i,j} + \Phi^h_{i-1,j}) + \frac{1}{h^2}(\Phi^h_{i,j+1} - 2\Phi^h_{i,j} + \Phi^h_{i,j-1}) = f_{i,j}$$

What about boundary ocnditions?

Cell Centered vs Vertex Schemes

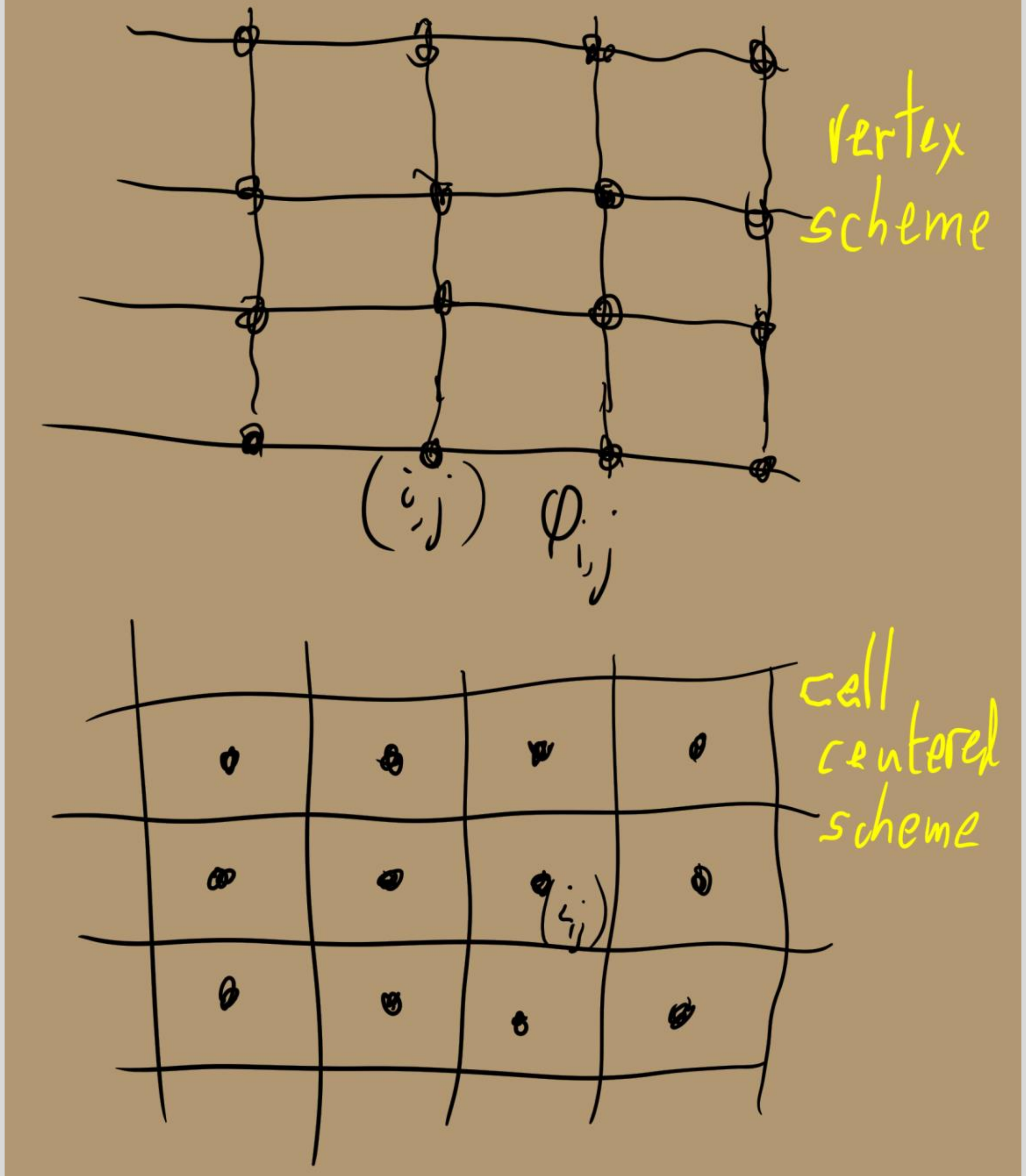
The interior equations look the same for both Cell centered and vertex schemes.

$$\Phi_{xx} + \Phi_{yy} = f$$

$$\frac{1}{h^2}(\Phi^h_{i+1,j} - 2\Phi^h_{i,j} + \Phi^h_{i-1,j}) + \frac{1}{h^2}(\Phi^h_{i,j+1} - 2\Phi^h_{i,j} + \Phi^h_{i,j-1}) = f_{i,j}$$

$$\Phi^h_{i,j} \approx \Phi(x,y) \quad h \to 0$$

But B.C. implementation is different



Vertex Schemes - BC

Dirichlet BC is straightforward. We just specify the BC value

Neumann BC is more involved.

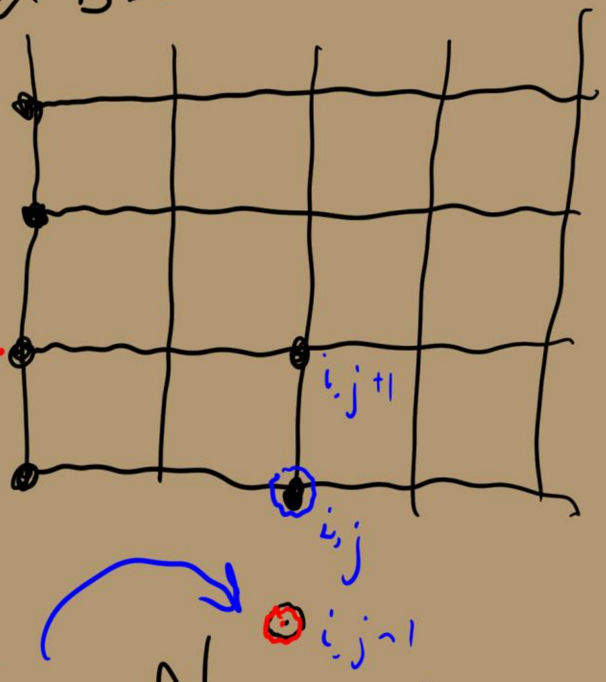
We introduce a ghost point (red)
We discretize the interior equation at the boundary point
And the Neumann BC at that point.

The ghost point (red) has a ghost variable (red) that we eliminate. The resulting equation is used at the boundary.
No ghost variables is used explicitly in calculation

Dirichlet B.C.

$\varphi\big|_{\partial} = g$

$\varphi_{i,j} = g$



ghost point

does not exist in calculation

Neumann B.C.

$$\frac{\partial \varphi}{\partial n} = h$$

at (i,j) use eq $\frac{1}{h^2}\begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix}\varphi = f$

& $\frac{1}{2h}\left[\varphi_{i,j+1} - \varphi_{i,j-1}\right] = h$ & $\boxed{\text{eliminate } \varphi_{i,j-1}}$

Vertex Scheme –Neumann BC

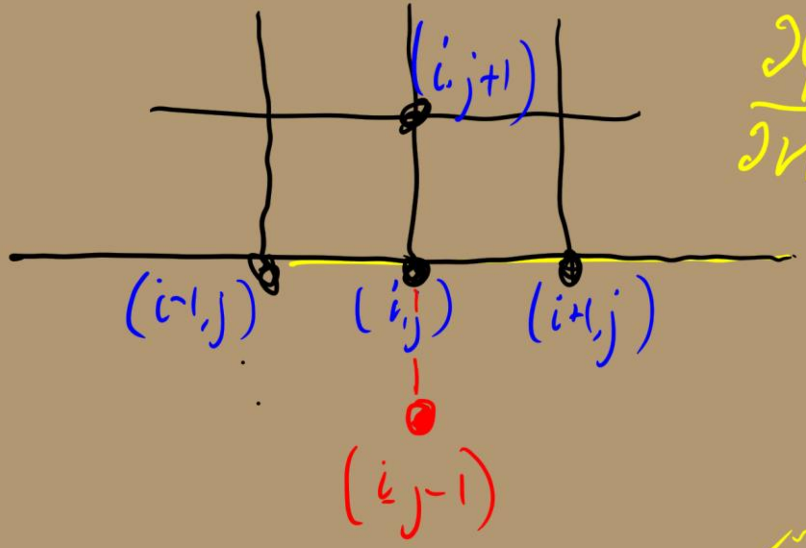
Dirichlet BC is straightforward. We just specify the BC value

Neumann BC is more involved.

We introduce a ghost point (red)
We discretize the interior equation at the boundary point
And the Neumann BC at that point.

The ghost point (red) has a ghost variable (red) that we eliminate. The resulting equation is used at the boundary.
No ghost variables is used explicitly in calculation

Actual calculation is shown on the right.



$$\frac{\partial \varphi}{\partial n} = g$$

$$\frac{1}{h^2}\left[-4\varphi_{i,j} + \varphi_{i+1,j} + \varphi_{i-1,j} + \varphi_{i,j+1} + \varphi_{i,j-1}\right] = f_{i,j}$$

$$\frac{1}{2h}\left[\varphi_{i,j+1} - \varphi_{i,j-1}\right] = g_{i,j} \quad (**)$$

Eliminate $\varphi_{i,j-1}$ using (**) & substitute in (*)

$$\varphi_{i,j-1} = \varphi_{i,j+1} - 2h g_{ij}$$

Cell-Centered Scheme
Neumann BC

Neumann BC is done in a similar way to vertex scheme

We introduce a ghost point (red)
We discretize the interior equation at the boundary point
And the Neumann BC at that point.

The ghost point (red) has a ghost variable (red) that we eliminate. The resulting equation is used at the boundary.
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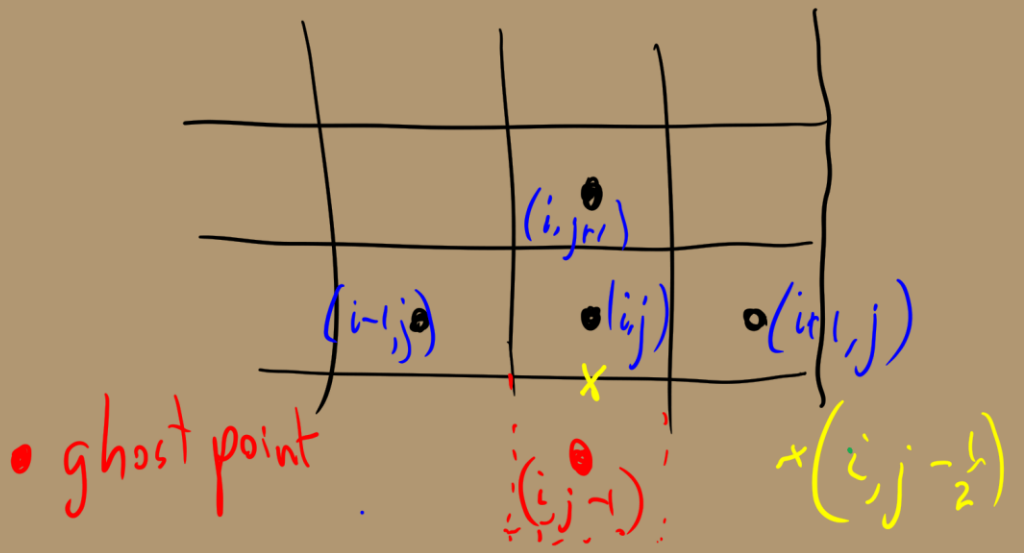


$$\frac{1}{h^2}\left[-4\varphi_{i,j}+\varphi_{i+1,j}+\varphi_{i-1,j}+\varphi_{i,j+1}+\varphi_{i,j-1}\right]=f_{i,j} \qquad (**)$$

$$\frac{1}{2h}\left[\varphi_{i,j}-\varphi_{i,j-1}\right]=g_{i,j-\frac{1}{2}}$$

$$\varphi_{i,j-1}=\varphi_{i,j}-2h\,g_{i,j-\frac{1}{2}}$$

(**) with $\varphi_{i,j-1}$ eliminated is the correct eq.

Cell-Centered Scheme
Dirichlet BC

Dirichlet BC is done in a similar way to Neumann BC

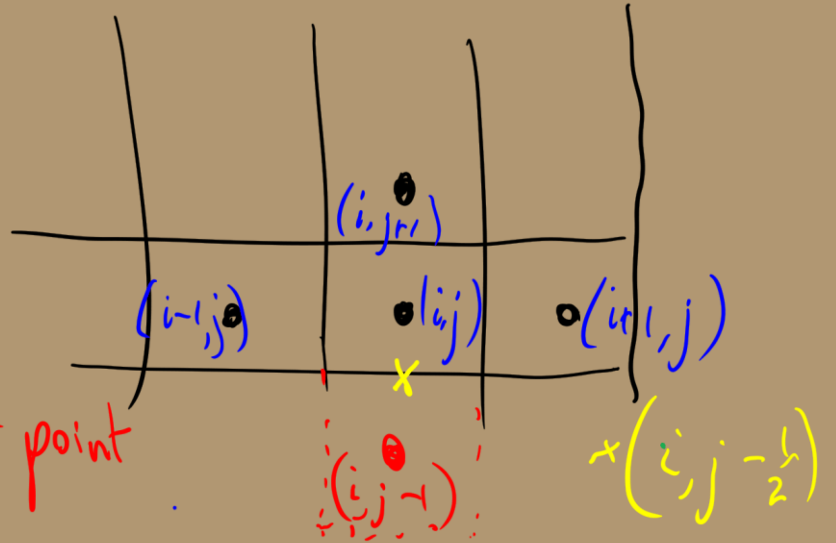
We introduce a ghost point (red)
We discretize the interior equation at the boundary point
And the Dirichlet BC at that point.

The ghost point (red) has a ghost variable (red) that we eliminate. The resulting equation is used at the boundary.
No ghost variables is used explicitly in calculation

Actual calculation is shown on the right.

B.C.

$\varphi = g$



• ghost point

$$\frac{1}{h^2}\left[-4\varphi_{i,j} + \varphi_{i+1,j} + \varphi_{i-1,j} + \varphi_{i,j+1} + \varphi_{i,j-1}\right] = f_{ij} \quad (**)$$

$$\frac{1}{2}\left[\varphi_{i,j} + \varphi_{i,j-1}\right] = g_{i,j-\frac{1}{2}}$$

$$\varphi_{i,j-1} = -\varphi_{i,j} + 2g_{i,j-\frac{1}{2}}$$

(**) with $\varphi_{i,j-1}$ eliminated is the correct eq.

Finite Volume Scheme

We use the integral form of the conservation law.
See figure.

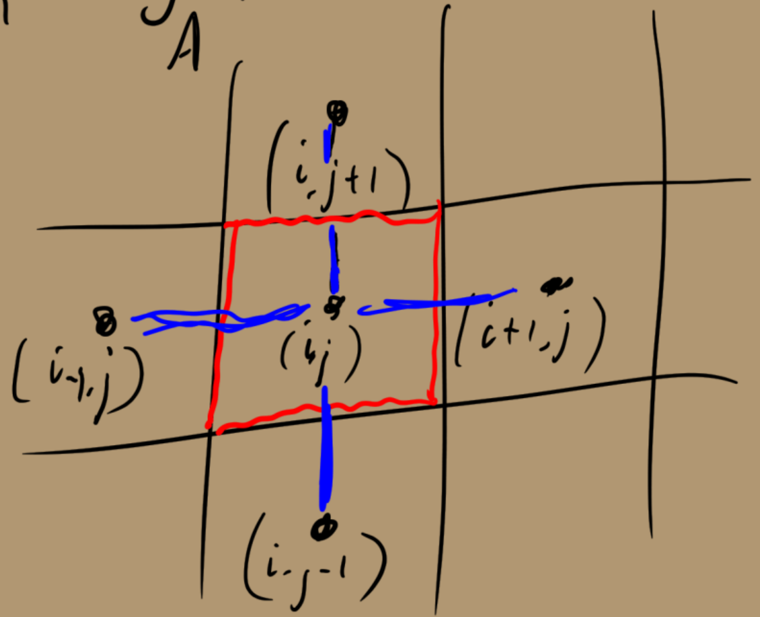
Finite Volume Discretization

$$\Delta u = f$$

$$\int_A \Delta u \, dx = \int_A f \, dx$$

$$\|$$

$$\int_{\partial A} \frac{\partial u}{\partial n} \, ds = \int_A f \, dx$$



Finite Volume Scheme - cont

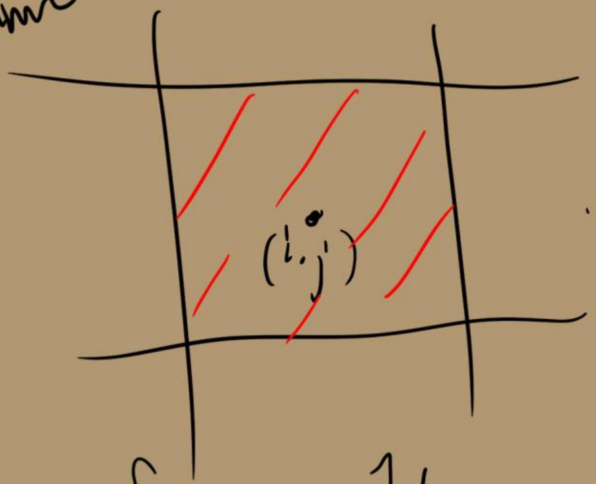
We approximate the line (surface in 3D) integral
By a sum. We use the usual approximation for normal derivative.

The right-hand side is approximated by a simple cell-centered integration rule.

For a rectangular grid this is identical to th eprevious equations we derive.

What about Neumann BC?

Finite Volume, cont.



$$\int \frac{\partial \varphi}{\partial n} ds \approx \frac{1}{h}[\varphi_{i+1,j} - \varphi_{i,j}]h + \frac{1}{h}[\varphi_{i,j+1} - \varphi_{i,j}]h - \frac{1}{h}[\varphi_{i,j} - \varphi_{i-1,j}]h - \frac{1}{h}[\varphi_{i,j} - \varphi_{i,j-1}]h$$

$$\int f dx \approx h^2 f_{ij}$$

Eq is the same as before.
B.C. is different slightly

Finite Volume Scheme - cont

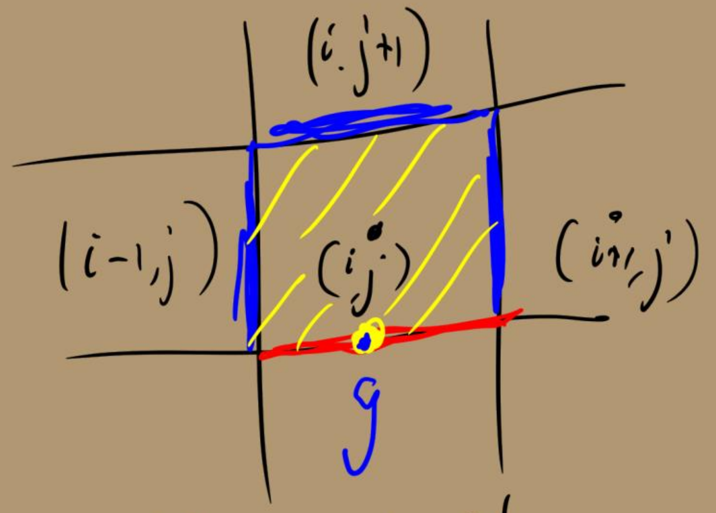
Neumann BC is implemented by using it in the line (surface) Integral

Note that we used here a midpoint rule for integrating the RHS on boundary.

Implementing a Dirichlet BC is done like in the cell centered Scheme

Finite Vol. Neumann B.C.

$$\frac{\partial \varphi}{\partial n} = g$$



$$\int_{\square} \frac{\partial \varphi}{\partial n} ds \approx \frac{1}{h}\left[\varphi_{i+1,j} - \varphi_{i,j}\right]h + $$

$$- \frac{1}{h}\left[\varphi_{i,j} - \varphi_{i-1,j}\right]h + $$

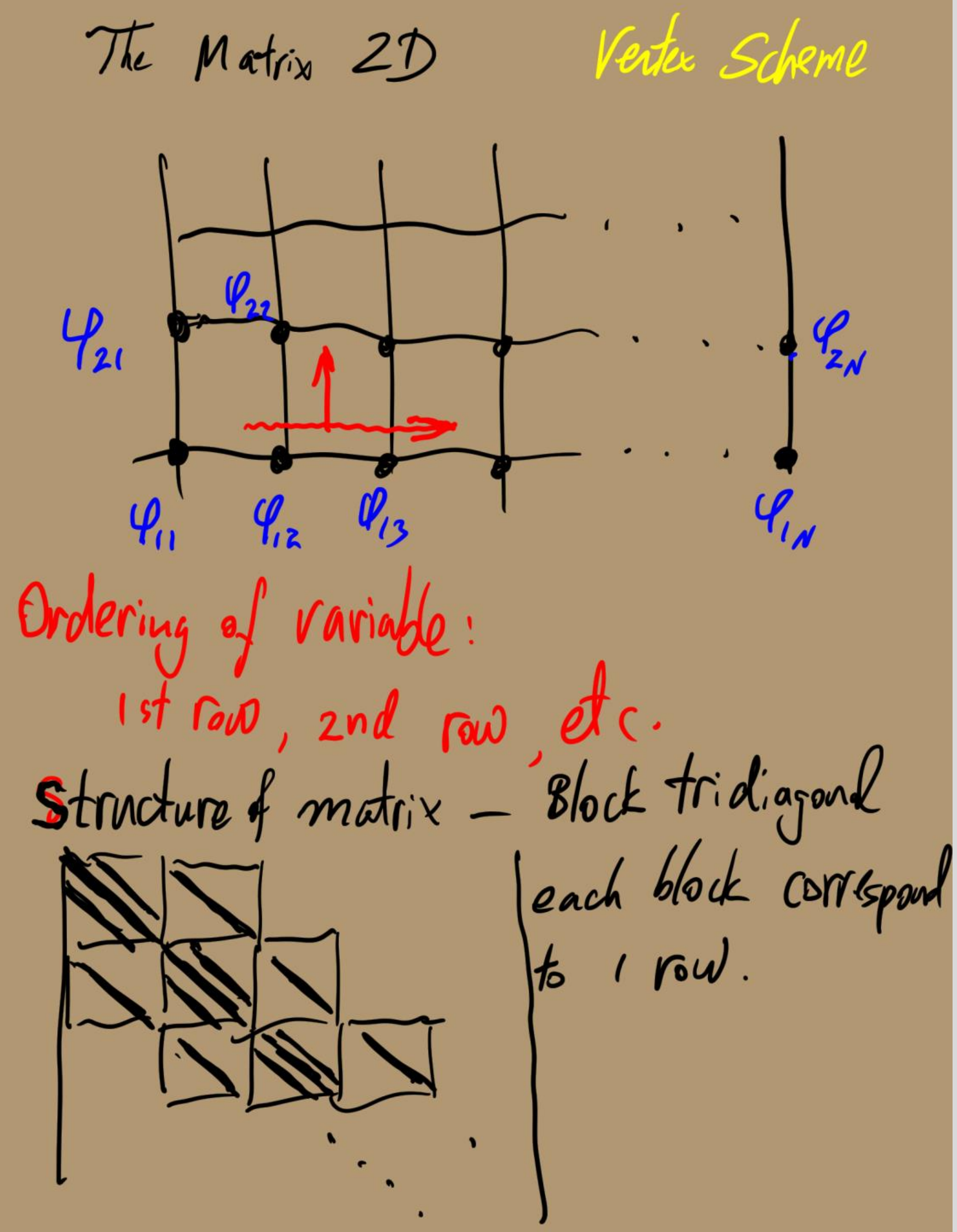
$$\frac{1}{h}\left[\varphi_{i,j+1} - \varphi_{i,j}\right]h + $$

$$+ h g_{i,j+\frac{1}{2}}$$

The Matrices – 2D

In two dimensions or above we need to order the unknowns in a long vector. A natural way for doing it is by rows of the lattice where the unknowns are defines.

The resulting matrix has a block structure as shown
The blocks are at most tridiagonal.



The Matrices 2D - cont

In two dimensions or above we need to order the unknowns in a long vector. A natural way for doing it is by rows of the lattice where the unknowns are defines.

The resulting matrix has a block structure as shown
The blocks are at most tridiagonal.

Diagonal Block for

$$\varphi_{xx} + \varphi_{yy} = 0$$

$$-4\varphi_{i,j} + \varphi_{i+1,j} + \varphi_{i-1,j} + \varphi_{i,j+1} + \varphi_{i,j-1}$$

in diagonal block

upper diagonal block.

lower diagonal blocks

$$\begin{bmatrix} -4 & 1 & 0 \\ 1 & -4 & 1 \\ 0 & 1 & -4 & 1 \\ & & & \ddots \end{bmatrix}$$

diagonal blocks

$$\begin{bmatrix} 1 & & 0 \\ & 1 & \\ 0 & & 1 \\ & & & \ddots \end{bmatrix}$$

upper & lower diagonal blocks

The Matrices 2D - cont

In two dimensions or above we need to order the unknowns in a long vector. A natural way for doing it is by rows of the lattice where the unknowns are defines.

The resulting matrix has a block structure as shown
The blocks are at most tridiagonal.

The B.C.

$\varphi|_\partial = g \Rightarrow \varphi_{ij} = g$ at all points (i,j) on boundary. This may affect diagonal blocks & upper/lower blocks.

bottom boundary.
1st row.

instead of $1 \ -4 \ 1$ on that row we have 1 on diag. of block.

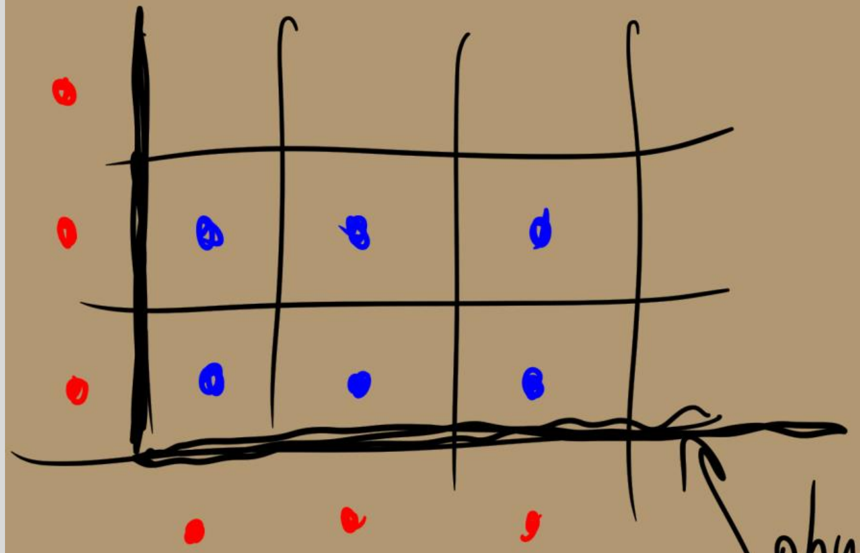
Can you think of all other situations?

The Matrices 2D - cont

In two dimensions or above we need to order the unknowns in a long vector. A natural way for doing it is by rows of the lattice where the unknowns are defines.

Blocks that correspond to rows that involve boundary conditions are affected. Use elimination of ghost variables if relevant, according to the arguments made for 1D case.

2D case Cell Centered Scheme



[1 -4 1] blocks as before

modification similar to 1D case near boundaries.

Can you do this?

The Full Potential Equation

Under the assumption that vorticity is zero, $\nabla \times v = 0$

tvelocity field is the gradient of a scalar valued function $v = \nabla\phi$

This will help simplify the fluid dynamics equations.

We start with the continuity equation $$\frac{\partial \rho}{\partial t} + div(\rho\nabla\phi) = 0$$

For steady flow this simplifies to $div(\rho\nabla\phi) = 0$ with wall B.C. $v_n = \frac{\partial\phi}{\partial n} = 0$

Under the condition of constant entropy it is possible to derive the relation $$\frac{\rho}{\rho_0} = \left(1 - \frac{|\nabla\phi|^2}{2H_0} - \frac{\partial_t\phi}{H_0}\right)^{1/(\gamma-1)}$$

Where ρ_0 is the stagnation (U=0) density. H_0 Is the stagnation Enthalpy. This results in a single scalar equation that describes the fluid flow.

The Full Potential Equation – non conservative

If we expand derivatives of the density we can show that the following holds,

$$\frac{1}{c^2}[\phi_{tt}+\partial_t(|\nabla\phi|^2)] = (1-M_x^2)\phi_{xx}+(1-M_y^2)\phi_{yy}+(1-M_z^2)\phi_{zz}-2M_xM_y\phi_{xy}-2M_xM_z\phi_{xz}-2M_yM_z\phi_{yz}$$

$$M_x = \frac{\phi_x}{c} \quad M_y = \frac{\phi_y}{c} \quad M_z = \frac{\phi_z}{c} \quad M^2 = \frac{|\nabla\phi|^2}{c^2}$$

$$c^2 = (\gamma-1)\left[H_0 - \frac{1}{2}|\nabla\phi|^2 - \phi_t\right]$$

M is the Mach number and c is the speed of sound.

This is the time dependent full potential equation in non-conservative form. This form reveals more about the equation than its conservative counterpart. Later we see that M<1 and M>1 need different discretization.

For now we focus on M < 1, the subsonic case.

The Full Potential Equation – non conservative

If we expand derivatives of the density, we can show that the following holds,

$$\frac{1}{c^2}[\phi_{tt}+\partial_t(|\nabla\phi|^2)] = (1-M_x^2)\phi_{xx}+(1-M_y^2)\phi_{yy}+(1-M_z^2)\phi_{zz}-2M_xM_y\phi_{xy}-2M_xM_z\phi_{xz}-2M_yM_z\phi_{yz}$$

$$M_x = \frac{\phi_x}{c} \quad M_y = \frac{\phi_y}{c} \quad M_z = \frac{\phi_z}{c} \quad M^2 = \frac{|\nabla\phi|^2}{c^2}$$

$$c^2 = (\gamma-1)\left[H_0 - \frac{1}{2}|\nabla\phi|^2 - \phi_t\right]$$

M is the Mach number and c is the speed of sound.

This is the time dependent full potential equation in non-conservative form.

For now, we focus on M < 1, the subsonic case.

The Steady Full Potential Equation – non conservative

In many applications we are interested in steady flow and therefore omit all time derivatives,

$$\frac{\rho}{\rho_0} = \left(1 - \frac{|\nabla\phi|^2}{2H_0}\right)^{1/(\gamma-1)} \qquad c^2 = (\gamma - 1)\left[H_0 - \frac{1}{2}|\nabla\phi|^2\right]$$

$$(1-M_x^2)\phi_{xx}+(1-M_y^2)\phi_{yy}+(1-M_z^2)\phi_{zz}-2M_xM_y\phi_{xy}-2M_xM_z\phi_{xz}-2M_yM_z\phi_{yz} = 0$$

$$M_x = \frac{\phi_x}{c} \quad M_y = \frac{\phi_y}{c} \quad M_z = \frac{\phi_z}{c} \quad M^2 = \frac{|\nabla\phi|^2}{c^2}$$

with wall BC $v_n = \dfrac{\partial\phi}{\partial n} = 0$

This is the steady full potential equation in non-conservative form. This form reveals more about the equation than its conservative counterpart. Later we see that M<1 and M>1 need different discretization.

For now, we focus on M < 1, the subsonic case. We will deal with M > 1 in the next lecture.

The Small Disturbance Approximation

A simplification of the full potential equations is done in the case of thin obstacles, such as thin airfoils. We will restrict the discussion to 2D.

Since the obstacle is “small” it’s effect on the flow is small and we consider perturbation to uniform flow with velocity of magnitude U_∞ in the x-direction. The potential has a representation

$$\phi = U_\infty(x + \Phi)$$

Where velocities are recovered from the potential using the formulas

$$u = U_\infty(1 + \Phi_x)$$

$$v = U_\infty \Phi_y$$

The equation becomes $(1 - M_x^2)\Phi_{xx} + \Phi_{yy} = 0$ which can be simplified further to $(1 - M_\infty^2)\Phi_{xx} + \Phi_{yy} = 0$

The wall BC becomes $v = (U_\infty + u) f'(x) \approx U_\infty f'(x)$ where f(x) is the shape of the airfoil.

Numerical Approximation

The discretization of the full potential equations is done in conservation form, starting from the integral formulation, using the divergence theorem.

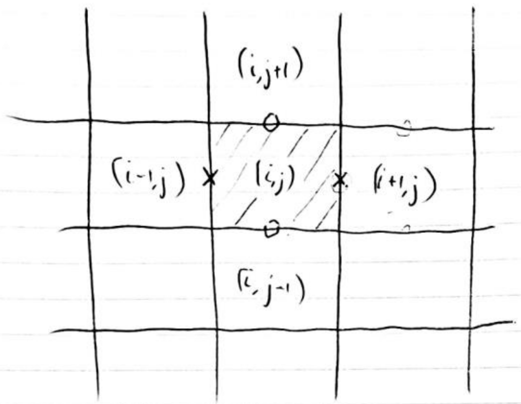
$$\int_{\Omega} div\, \rho\nabla\phi\, dx = \int_{\partial\Omega} \rho\nabla\phi \cdot n\, ds \equiv \int_{\partial\Omega} \rho\frac{\partial\phi}{\partial n}\, ds$$

In 1D the discretization takes the form

$$\rho^h_{i+1/2}(\phi^h_{i+1} - \phi^h_i) - \rho^h_{i-1/2}(\phi^h_i - \phi^h_{i-1}) = 0$$

$$\frac{\partial\Phi}{\partial n} \approx \frac{1}{h}[\Phi_{1,j} - \Phi_{0,j}] = 0$$

Subsonic Case $(M<1)$



φ is approximated at cell centers.

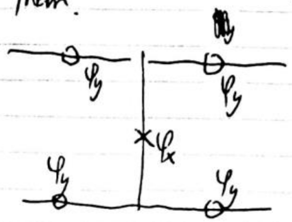
φ_x is approx at x - vertical sides of cell

φ_y o - horizontal sides of cell

$$(\rho\varphi_x)_x + (\rho\varphi_y)_y = 0$$

We need ρ at places where φ_x & φ_y are defined. $\rho = \rho(\nabla\varphi)$

Since φ_x & φ_y are defined in different points we need to interpolate them.



φ_y at o

φ_x at x

$$(\varphi_y)^x = \frac{1}{4}\sum(\varphi_y)^o$$

Numerical Approximation

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$$\int_\Omega div\, \rho\nabla\phi\, dx = \int_{\partial\Omega} \rho\nabla\phi\cdot n\, ds \equiv \int_{\partial\Omega}\rho\frac{\partial\phi}{\partial n}\, ds$$

Subsonic Case $(M<1)$

The FP eq discretized as,

$$\rho_{i+\frac{1}{2},j}(\varphi_{i+1,j}-\varphi_{i-1,j}) - \rho_{i-\frac{1}{2},j}(\varphi_{ij}-\varphi_{i-1,j}) + \rho_{i,j+\frac{1}{2}}(\varphi_{i,j+1}-\varphi_{i,j}) - \rho_{i,j-\frac{1}{2}}(\varphi_{ij}-\varphi_{i,j-1}) = 0$$

$$\rho = \rho(|\nabla\varphi|^2)$$

$$\nabla\varphi = (\varphi_x, \varphi_y)$$

$$(\nabla\varphi)_{i+\frac{1}{2},j} = \left(\frac{\varphi_{i+1,j}-\varphi_{ij}}{h}, \frac{1}{4}\left[\frac{\varphi_{i,j+1}-\varphi_{ij}}{h} + \frac{\varphi_{i,j}-\varphi_{i,j-1}}{h} + \frac{\varphi_{i+1,j+1}-\varphi_{i+1,j}}{h} + \frac{\varphi_{i+1,j}-\varphi_{i+1,j-1}}{h}\right]\right)$$

Similarly for $(\nabla\varphi)_{i,j+\frac{1}{2}}$

We use iterative method to solve the eq. ~~def~~ ~~denote it~~

$$\rho^n_{i+\frac{1}{2},j}(\varphi^{n+1}_{i+1,j}-\varphi^{n+1}_{ij}) - \rho^n_{i-\frac{1}{2},j}(\varphi^{n+1}_{ij}-\varphi^{n+1}_{i-1,j}) + \rho^n_{i,j+\frac{1}{2}}(\varphi^{n+1}_{i,j+1}-\varphi^{n+1}_{i,j}) - \rho^n_{i,j-\frac{1}{2}}(\varphi^{n+1}_{ij}-\varphi^{n+1}_{i,j-1}) = 0$$

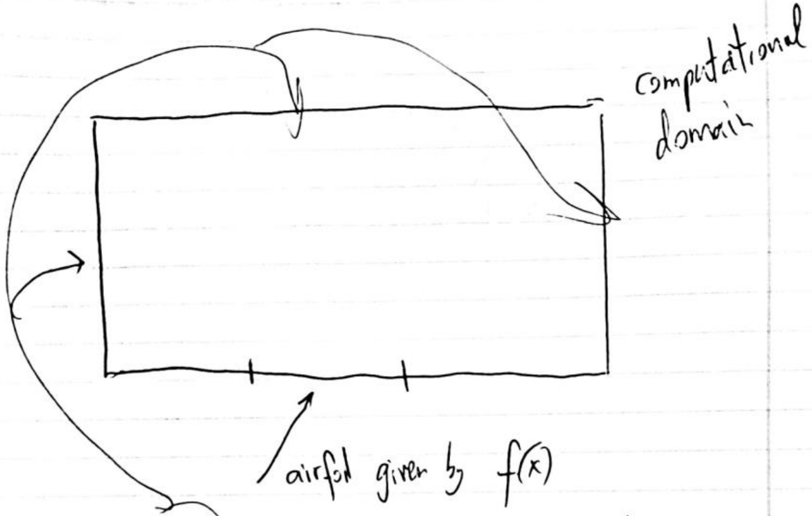
$$\rho^n = \rho((\nabla\varphi)^n)$$

Numerical Approximation

The discretization of the full potential equations is done in conservation form, starting from the integral formulation, using the divergence theorem.

$$\int_{\Omega} div\, \rho \nabla \phi \, dx = \int_{\partial\Omega} \rho \nabla \phi \cdot n \, ds \equiv \int_{\partial\Omega} \rho \frac{\partial \phi}{\partial n} \, ds$$

Airfoil Calculation



φ at farfield $\sim U_\infty \cdot x$ we specify it

$\varphi_{ij} = U_\infty \cdot x_i$

at bottom we specify $\frac{\partial \varphi}{\partial n} = \boxed{-\frac{\partial \varphi}{\partial y} = U_\infty f'(x)}$

where no airfoil exists we set $f=0$.

$\Rightarrow$ Dirichet B.C. on left, right & top

Neumann ~~cond~~ BC on bottom.

Numerical Approximation - FP

Notice that we had several places where approximations were used.

If we want to describe the obstacles (bodies) in full details, we must approximate the geometry properly and use non-rectangular grids!

But if the objects are airfoils then we may use rectangular grids with the small disturbance approximation (SDA) for the wall boundary condition.

If we want to simplify the non-linearity of the equation we may use the non-conservation form
And replace the Mach number by its value at infinity (far field).

GO and EXPLORE !!!