Fluid Dynamics

Lecture II – Inviscid 1D equations

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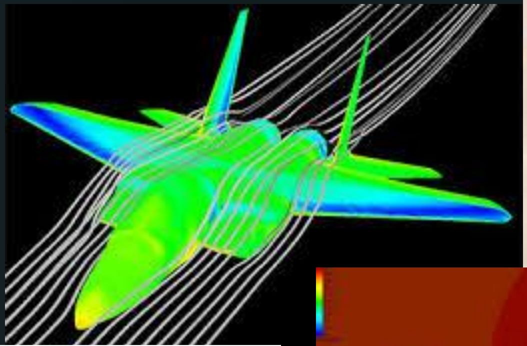
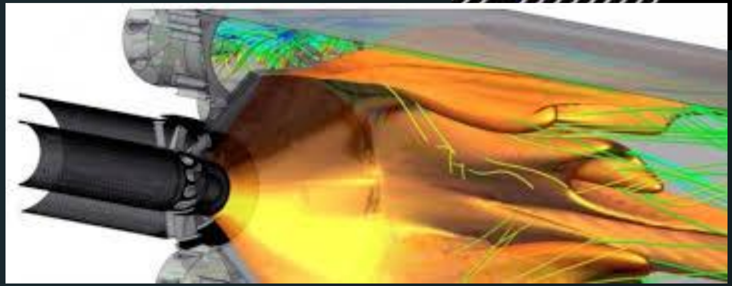
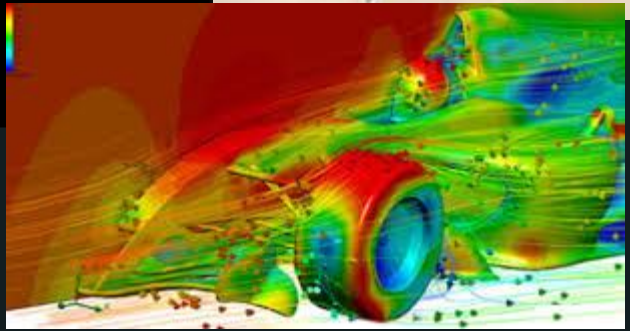
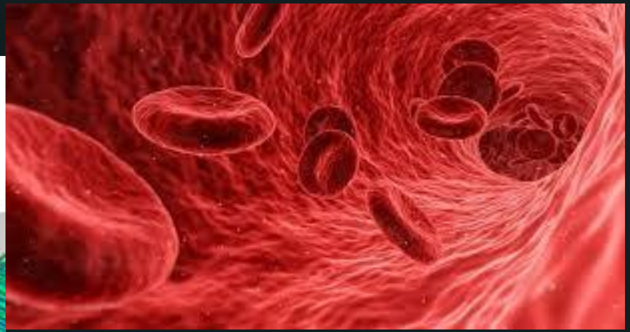
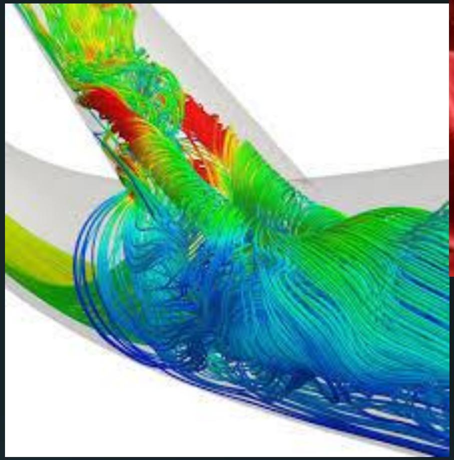
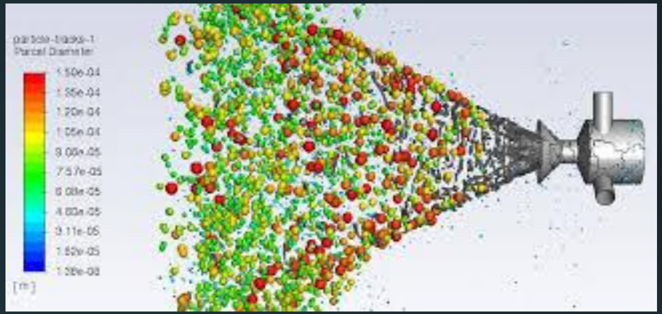












Outline

Today we focus on the **1D inviscid (Euler) equations**.

To understand it we start with simplified model problems that capture the physics of the full model.

-The advection equation and numerical schemes
- Stability analysis – Von Nuemann
- The Burgers equations, shocks and numerical schemes
- Diagonalization of systems of time dependent equations in oe space dimension
- Characteristics for the 1D Euler Equations
- Schemes for 1D Euler equations

The Convection Equation

1D convection equation $$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

Consider the problem t>0, and x in the whole space. Let the initial condition be g(x). Then the solution for all times is g(x – at). Check it!!

Note that this is a moving wave whose shape is g(x) and its speed is a (in the positive direction).

We want to develop numerical schemes to solve this equation as a preparation for more complex problems involving vectors valued functions u. We will start with a scheme that makes sense physically.

The wave is coming from the left so we will bias the scheme to the left.

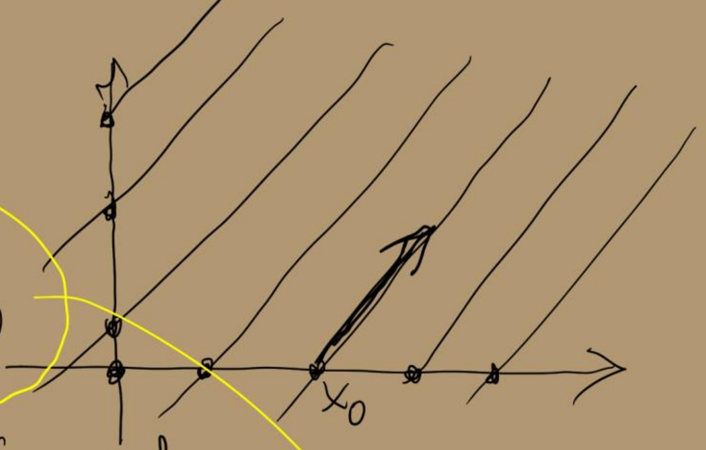
How to build numerical schemes?

The Convection Eq - cont

The solution propagates along lines, called characteristics.

See analysis on right.

slope a



$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

Let $\xi(t)$ satisfy $\frac{d\xi}{dt} = a$ $\xi(0) = x_0$

$\xi(t)$ a point moving at speed a.

$$u(t, \xi(t)) = u(0, \xi(0))$$

u is const along these lines called characteristics.

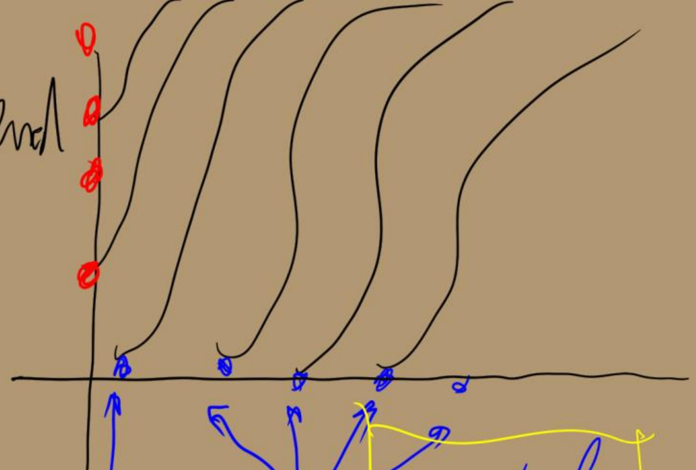
$$\frac{d}{dt}u(t,\xi(t)) = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{d\xi}{dt} =$$

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\cdot a = 0$$

The Convection Eq –

Initial Conditions (IC) Boundary Condition (BC)

a can depend on time.



Need to specify $u(0,x)$ – Initial Condition

$u(0,x) = g(x)$

But solution also propagate along thes lines from RED points in figure.

these are $u(t,0) = h(t)$ – Boundary Condition

To solve the problem

We need $g(x)$ & $h(t)$.

The Convection Eq –

Initial Conditions (IC) Boundary Condition (BC)

Concept of inflow and outflow

Based on U.N

-U velocity vector

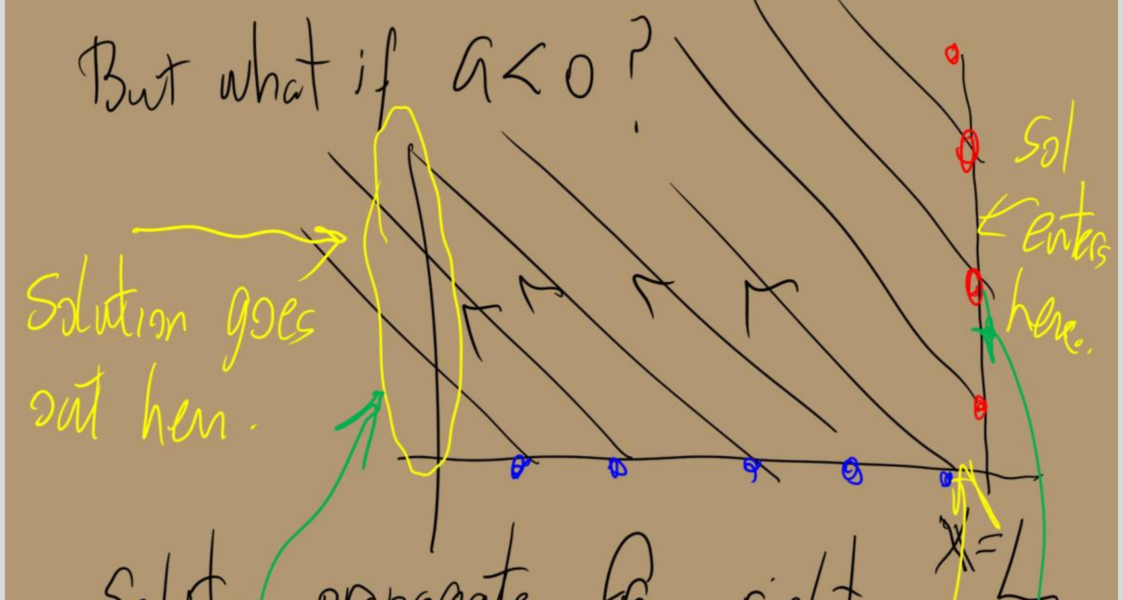
- N outward normal

U.N < 0   Inflow

U.N > 0   Outlflow

U.N = 0   Wall

But what if $a<0$?



Solution propagates from right to left. NO BC is needed at $x=0$!!

But we need B.C. at $x=L$

outflow

Inflow

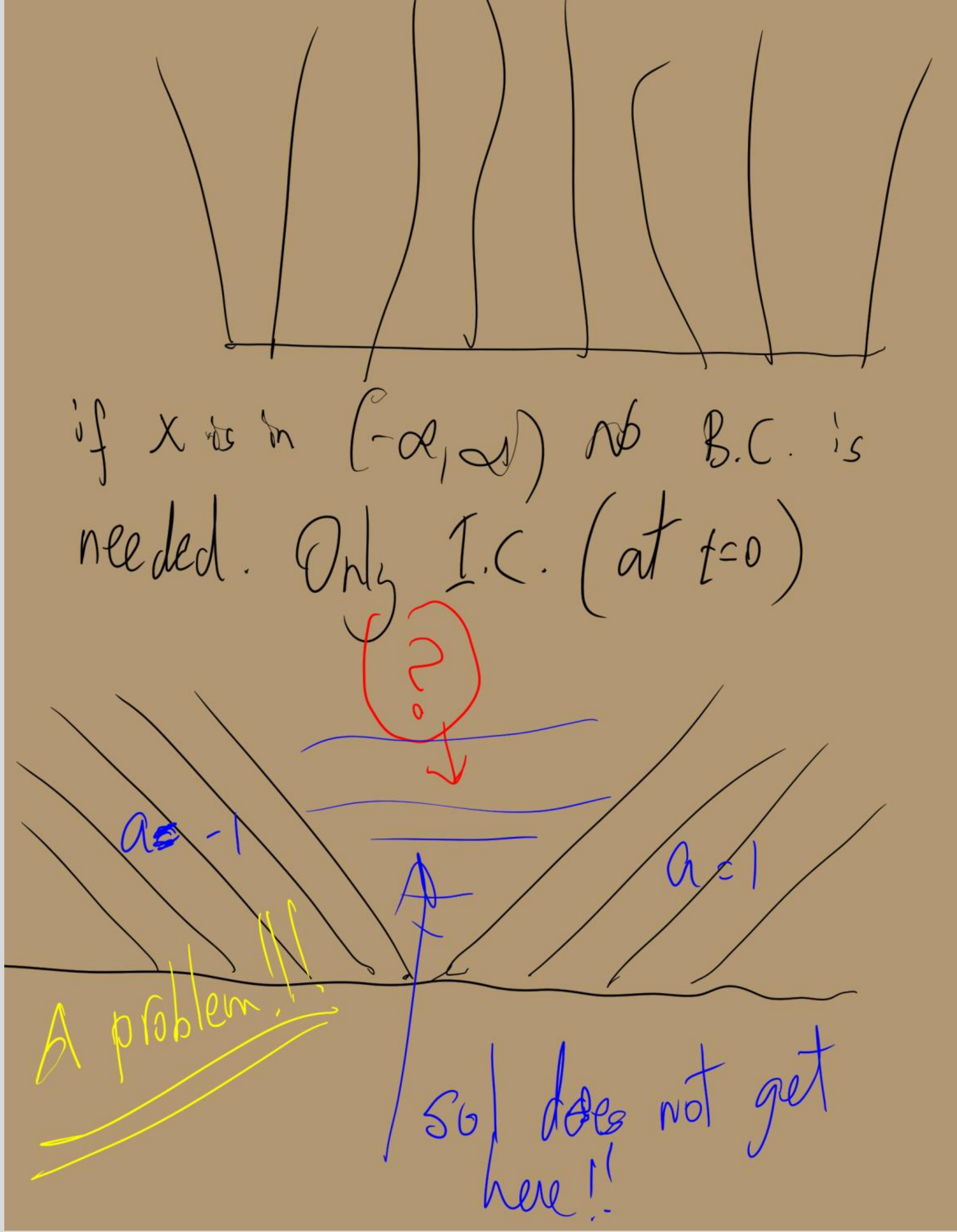
No B.C. at outflow for diff. eq.

B.C. at inflow

The Convection Eq – cont

If the problem is in the whole space in x, no need to specify BC

What if characteristic do not cover the whole space?
- There is a region in which there is ambiguity of the solution

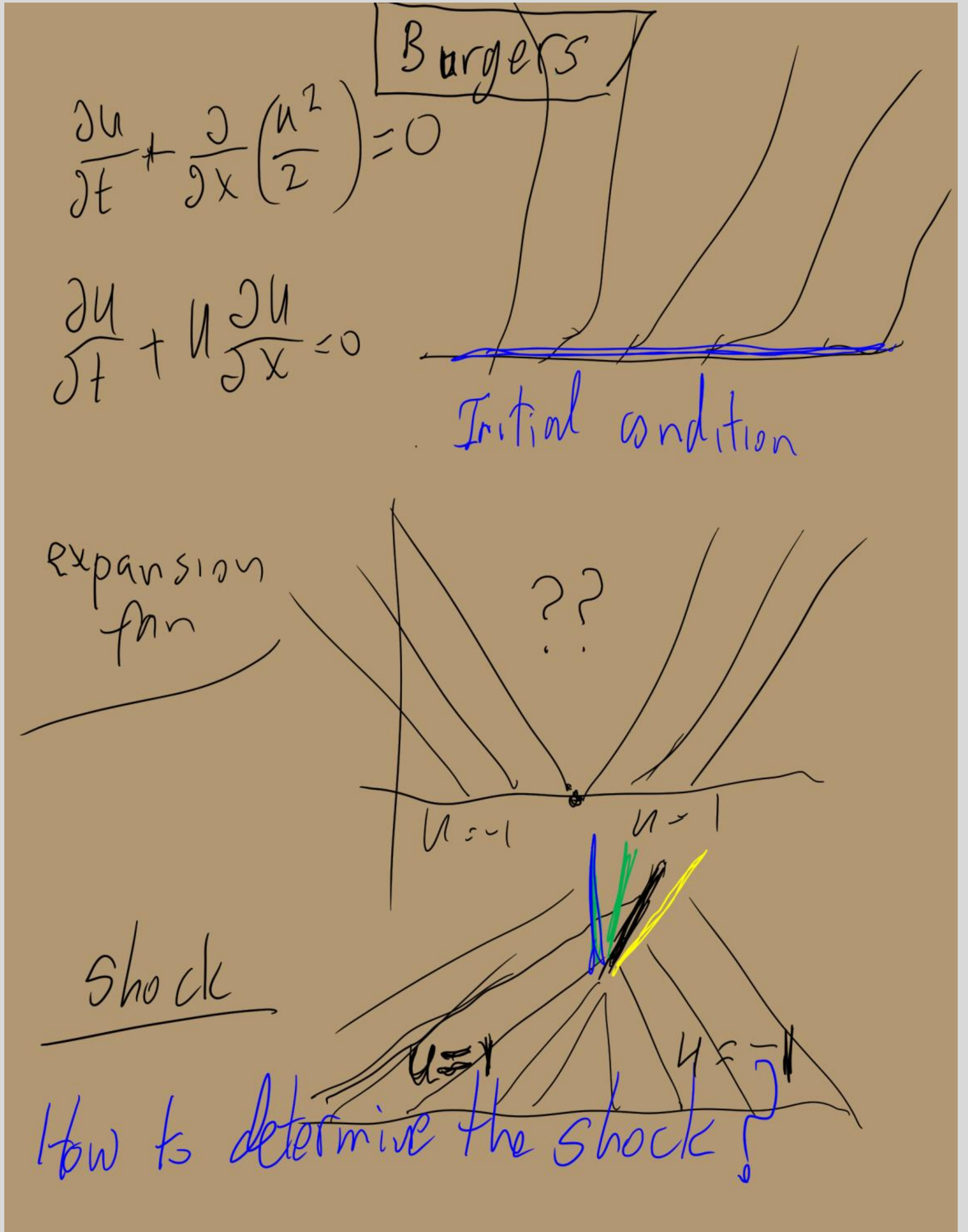


Burgers Equation

Here the speed of the wave depends on the solution as it evolves over time. This model allows us to understand Important properties of the 1D Euler equations.

New things to consider.

1) What if the characteristics do not fill up space?
   Expansion fan
2) What if characteristics meet? Non-uniqueness
   Shocks



Burgers Equation - shocks

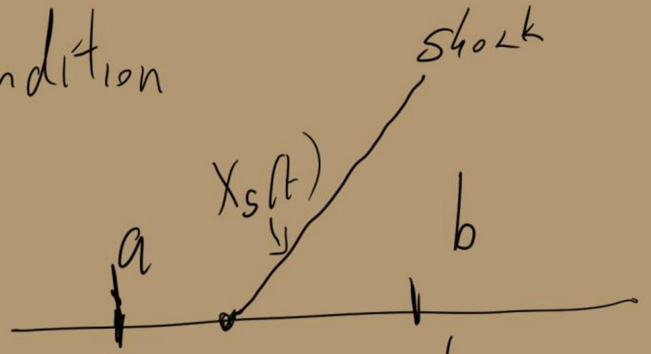
Calculation of the shock speed done by integrating the equation across a shock on an infinitesimal region containing it.

The equation must be in conservation form in this analysis

We consider a general 1D scalar conservation law.
u is the unknown function
f(u) is called the flux

We integrate the equation in x, in an infinitesimal region around the shock. See figure on right

Shock condition



$$\frac{\partial u}{\partial t} + (f(u))_x = 0 \qquad \frac{dx_s}{dt} = c$$

shock speed

$$\int_a^b \left[\frac{\partial u}{\partial t} + f(u)_x\right] dx = 0$$

$$\frac{\partial}{\partial t}\int_a^b u\,dx + \int_a^b (f(u))_x\,dx = 0$$

$$\frac{\partial}{\partial t}\int_a^b u\,dx + f(u(b)) - f(u(a)) = 0$$

$$\int_a^b u\,dx = \int_a^{X_s(t)} u\,dx + \int_{X_s(t)}^b u\,dx$$

Shocks - cont.

Let [z] denotes the jump in quantity z across the shock

The jump condition for general scalar equation is shown to satisfy

$$c[u] = [f(u)]$$

Where c is the shock speed, [u] is the jump in u, and [f(u)] is the jump in f(u)

Note that for Burgers eq we get c = (u1+u2)/2

What is we have a system of equation, like the Euler 1D?

$$\frac{\partial}{\partial t}\int_a^{x_s(t)} u\,dx + \frac{\partial}{\partial t}\int_{x_s(t)}^{b} u\,dx =$$

$$u(x_s^-(t))\frac{dx_s}{dt} - u(x_s^+(t))\frac{dx_s}{dt} =$$

$$c\left[u(x_s^-(t)) - u(x_s^+(t))\right]$$

$$\boxed{c[u] = [f(u)]}$$

[ ]: jump

for Burgers: $f(u) = u^2/2$

$$c(u_2 - u_1) = \frac{1}{2}(u_2^2 - u_1^2)$$

$$\boxed{c = (u_1+u_2)/2}$$ shock speed

Shocks in Systems

The jump condition for general 1D equation is shown to satisfy

c[u] = [f(u)]

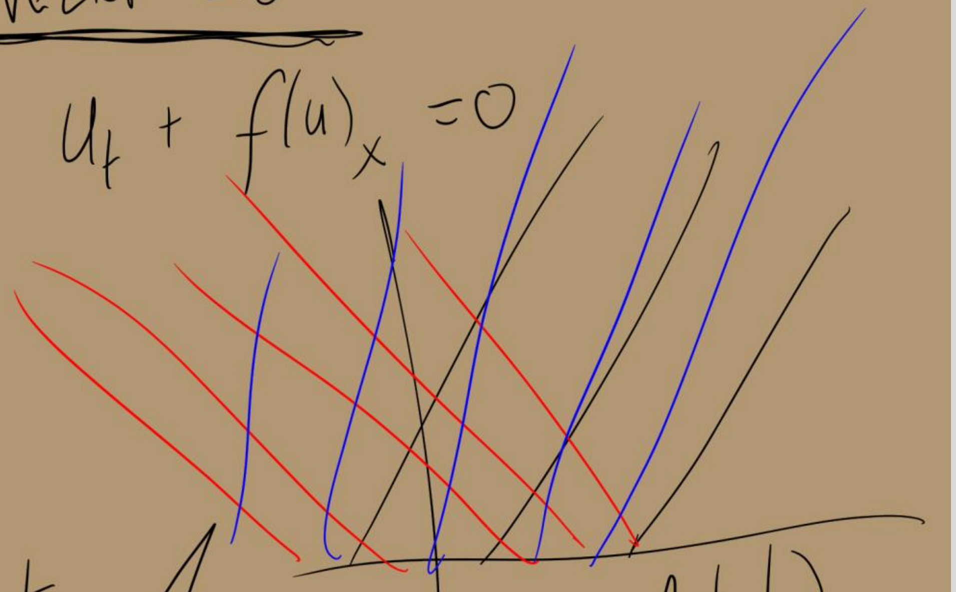
Where c is the shock speed, [u] is the jump in u, and [f(u)] is the jump in f(u)

But to find c we need to solve a set of equations setting u to have two vector values across the shock and correspondingly f(u) is nonlinear vector valued expression in terms of these values.

Note that the analysis of shocks was done in conservation Form of the equations!!!

Vector Case

$$u_t + f(u)_x = 0$$



system of characteristic (later more details)

Shock condition ??

$$c[u] = [f(u)] \qquad c[u_2 - u_1] = f(u_2) - f(u_1)$$

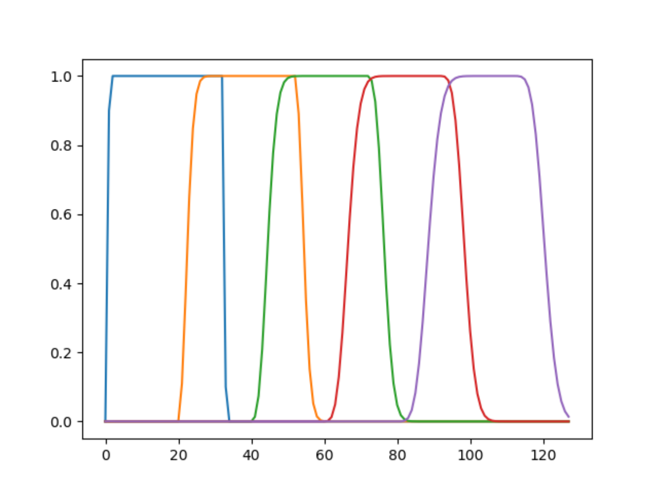
u vector, f(u) vector. Need to solve it.

Upwind Scheme – convection in 1D

We will solve the problem in t>0, x>0. This means that we also have to give a condition at x=0 (a boundary condition) in addition to the one at t=0 (initial condition).
The numerical scheme should be consistent with this physical information – the direction from which the wave is coming.

```
import matplotlib.pyplot as plt
import numpy as np
dt = 0.01
dx =  0.01
T = 0.9
a = 0.1
L = 128
u_prev = np.zeros(L)
u = np.zeros(L)
u_prev[0:int(L/4)] = 1.0
t = 0.0
plt.plot(u_prev)
fig = plt.figure()
```

```
k = 0
while t < T:
    for i in range(L-1):
          u[i+1] = u_prev[i+1] - dt*a *  ( u_prev[i+1]- u_prev[i])/dx
    if k % 20 == 0:
        plt.plot(u)
    for j in range(L):
        u_prev[j] = u[j]
    t = t + dt
    k = k + 1
plt.show()
```



Instability – violating the physics

If we violate the physical information that the wave moves to th right. Say we change a to -1. **Solution develops oscillations and blows up.**

```
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    for j in range(L):
        u_prev[j] = u[j]
    t = t + dt
    k = k + 1
plt.show()
```
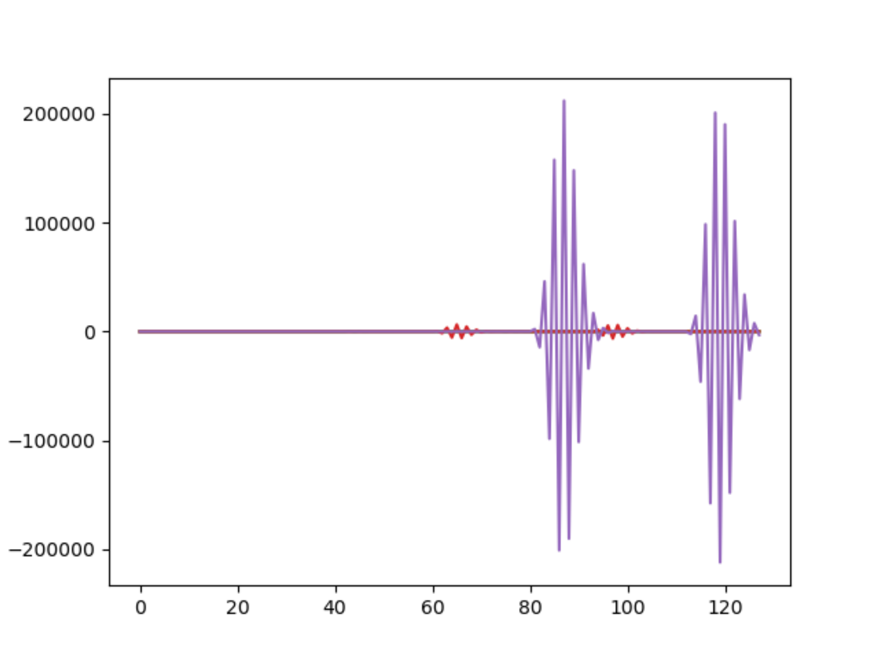
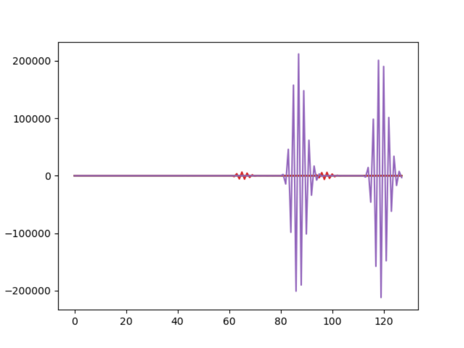
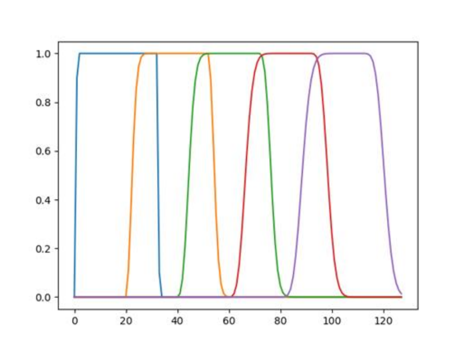



Understanding Stability

Von Neumann Analysis

A simple analysis for constant coefficient problems that gives us information about stability or blowup





Von-Neumann Analysis.

u_j^n

n ← time step.

j ← space index

In general

$$\underbrace{u^{n+1}}_{\text{sol at new time}} = L \underbrace{u^n}_{\text{sol at prev. time.}} \qquad L\text{-matrix}$$

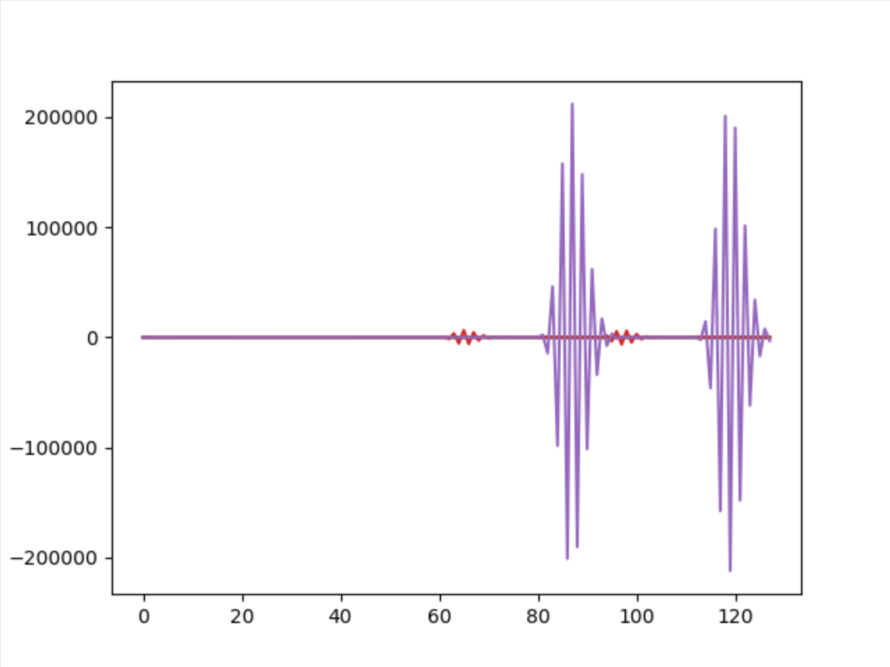
Assume $u_j^n = e^{i\theta j}$ $\quad |\theta| \le \pi$

$$u_j^{n+1} = A e^{i\theta j}$$

plug into eq & calc A.

Von Neumann Analysis

A simple analysis explaining the blowup we observed numerically for a > 0.



Example

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + a\,\frac{u_{j+1}^n - u_j^n}{\Delta x} = 0$$

$$u_j^{n+1} = u_j^n - \frac{a\Delta t}{\Delta x}\left(u_{j+1}^n \; u_j^n\right)$$

$$u_j^n = e^{i\theta j} \qquad u_j^{n+1} = Ae^{i\theta j}$$

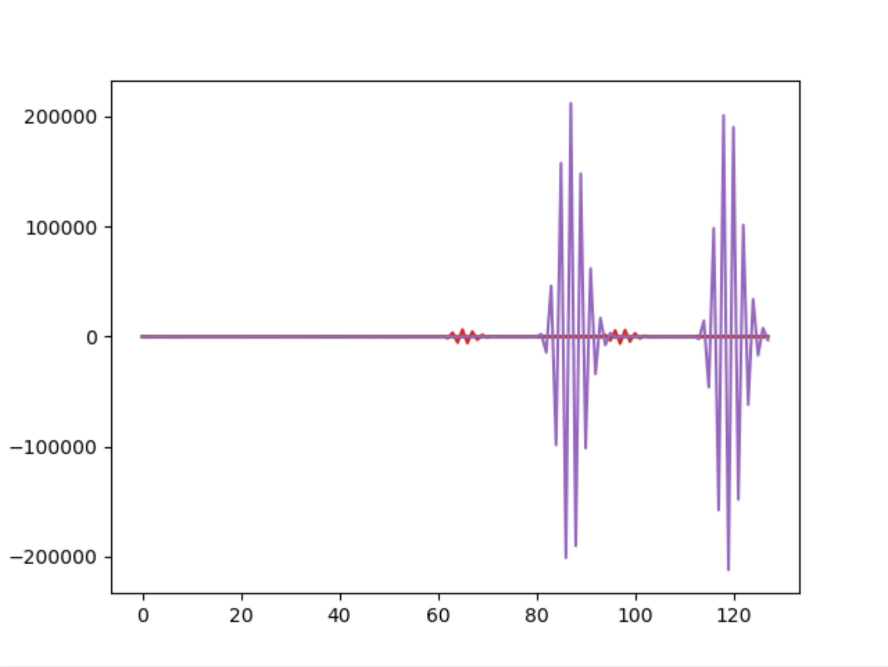
$$A\cancel{e^{i\theta j}} = \left(1 - \frac{a\Delta t}{\Delta x}\left(e^{i\theta} - 1\right)\right)\cancel{e^{i\theta j}}$$

$$A = \left(1 + \frac{a\Delta t}{\Delta x} - \frac{a\Delta t}{\Delta x}e^{i\theta}\right)$$

set $\lambda = a\Delta t/\Delta x$

Von Neumann Analysis - cont

A simple analysis explaining the blowup we observed numerically if a > 0.
For a<0, it is OK.



$$A = \left(1 + \lambda - \lambda\cos\theta - i\lambda\sin\theta\right)$$

$$\left(1 + \lambda(1 - \cos\theta) - i\lambda\sin\theta\right)$$

$$= \left(1 + 2\lambda\sin^2\frac{\theta}{2} - i\lambda\sin\theta\right)$$

$$|A| = \left(1 + 2\lambda\sin^2\frac{\theta}{2}\right)^2 + \lambda^2\sin^2\theta$$

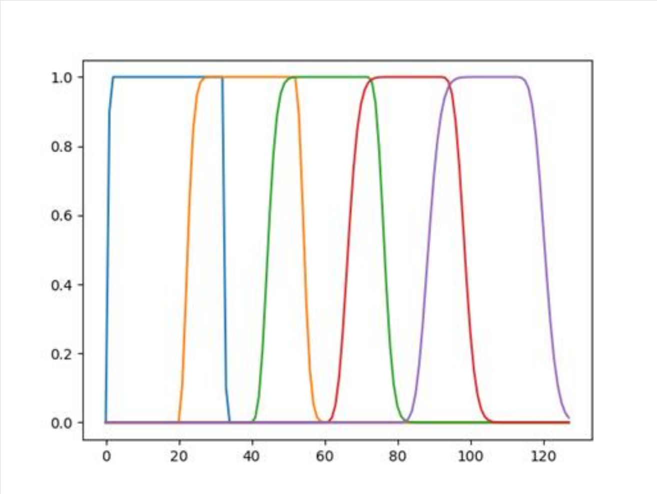
for $\theta = \pi$ $\quad |A(\pi)| = (1 + 2\lambda)^2$

at each time sol amplitude grow by this factor.

$\boxed{\text{blow up of sol}}$

Stability Analysis

A simple analysis explaining good behavior of our upwind scheme



Stability

Want $|A(\theta)| \le 1$ stable

or $|A(\theta)| \le 1 + \alpha \Delta t$

But NOT $\boxed{|A(\theta)| \ge \beta_0 > 1}$ β_0 ind. of Δt.

unstable

Physical vs Numerical B.C.

In solving the advection equation, with a>0, in t>0, x>0, we need to specify a **physical B.C**. at x = 0. When developing different numerical schemes, sometimes we also need to supply a **numerical boundary condition**, at the outflow boundary

The reason for this is that the equation used to defined values of u at the new time cannot be applied at the outlfow boundary since it uses points outside the domain.

The idea is then to supply a different condition for points at the ouflow boundary, that are consistent with the equation. One choice is to extrapolate the solution from the inside. Another is to use an approximation of the equation but in a way that does not involve points outside the domain – e.g. a purely upwind scheme.

Nemerical schemes produce many time **spurious behavior** in the solution and stability of the numerical boundary condition also has to be done. This is beyond the scope of this short course. There is an adaptation of the Von-Neumann analysis to understand what happens at boundaries, especially reflection of waves that may have large amplitude and they are spurious.

The vectorial case

We consider the case where u(x,t) is a vector and A is a constant matrix with distinct real eigenvalues.

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = 0$$

To understand the solution of such systems we introduce the diagonalization of A. Let S be a matrix whose columns are the eigenvectors of A, then we have

$$AS = S\Gamma \quad \text{or equivalently} \quad S^{-1}AS = \Gamma$$

Where Γ is a diagonal matrix with the eigenvalues of A on the diagonal.
Multiply the time dependent eq by S^{-1} we get

$$S^{-1}\frac{\partial u}{\partial t} + S^{-1}A\frac{\partial u}{\partial x} = 0 \quad \text{or} \quad S^{-1}\frac{\partial u}{\partial t} + S^{-1}ASS^{-1}\frac{\partial u}{\partial x} = 0$$

Introducing $\delta w = S^{-1}\delta u$ gives $\frac{\partial w}{\partial t} + \Gamma\frac{\partial w}{\partial x} = 0$ which are 3 independent equations

$$\frac{\partial w_j}{\partial t} + \lambda_j\frac{\partial w_j}{\partial x} = 0$$

The w’s are called characteristic variables.

Numerical Schemes for Systems

From the analysis of the vectorial case, we see that each of the w's might need a different scheme.

- Positive eigenvalue: wave goes from left to right
- Negative eigenvalue: wave goes from right to left.

The wave propagation direction also tells us where we need to give physical B.C.

- Positive eigenvalue: B.C. on the left boundary
- Negative eigenvalue: B.C. on the right boundary

Another issue is the proper treatment of shocks.

- Shocks are correctly described only in **conservation form**
- But waves are properly described in **characteristic form**

- How do we put these two together? More on this later.

1D Euler Equations – conservative form

The conservative unknowns are

$$U = \begin{pmatrix} \rho \\ \rho u \\ \rho E \end{pmatrix} = \begin{pmatrix} \rho \\ m \\ \epsilon \end{pmatrix}$$

And the corresponding flux vector is

$$F = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u H \end{pmatrix} = \begin{pmatrix} m \\ m^2/\rho + p \\ m(\epsilon + p) \end{pmatrix}$$

And the equation is written in conservation form as

$$\frac{\partial U}{\partial t} + \frac{\partial F}{\partial x} = 0$$

Or as

$$\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} = 0$$

where,

$$A = \begin{pmatrix} 0 & 1 & 0 \\ -(3-\gamma)u^2/2 & (3-\gamma) & \gamma - 1 \\ (\gamma-1)u^3 - \gamma u E & \gamma E - 3\frac{\gamma-1}{2}u^2 & \gamma u \end{pmatrix}$$

This equation uses the conservative variables.

1D Euler Equations – non-conservative form

The primitive variables are
$$U = \begin{pmatrix} \rho \\ u \\ p \end{pmatrix} = \begin{pmatrix} \rho \\ m/\rho \\ (\gamma - 1)(\epsilon - \frac{m^2}{2\rho}) \end{pmatrix}$$

And the corresponding equations are
$$\frac{\partial \rho}{\partial t} + u\rho_x + \rho u_x = 0$$
$$\frac{\partial u}{\partial t} + uu_x + \frac{1}{\rho}p_x = 0$$
$$\frac{\partial p}{\partial t} + up_x + \rho c^2 u_x = 0$$

Or
$$\frac{\partial}{\partial t}\begin{pmatrix} \rho \\ u \\ p \end{pmatrix} + \begin{pmatrix} u & \rho & 0 \\ 0 & u & \frac{1}{\rho} \\ 0 & \rho c^2 & u \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} \rho \\ u \\ p \end{pmatrix} = 0$$

In short $\frac{\partial U}{\partial t} + A\frac{\partial U}{\partial x} = 0$ where the matrix A is given by
$$A = \begin{pmatrix} u & \rho & 0 \\ 0 & u & \frac{1}{\rho} \\ 0 & \rho c^2 & u \end{pmatrix}$$

1D Euler Equations – characteristic variables

We will perform now a diagonalization procedure to discover the characteristic variables. We need the eigenvalues and eigenvectors of the matrix A. This is done as follows

$$\det \begin{pmatrix} u-\lambda & \rho & 0 \\ 0 & u-\lambda & \frac{1}{\rho} \\ 0 & \rho c^2 & u-\lambda \end{pmatrix} = 0$$

giving

$$det = (u-\lambda)[(u-\lambda)^2 - c^2] = 0$$

There are 3 roots (eigenvalues)

$$\lambda_1 = u \qquad \lambda_2 = u+c \qquad \lambda_3 = u-c$$

and eigenvectors

$$l_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad l_2 = \begin{pmatrix} \frac{\rho}{2c} \\ \frac{1}{2} \\ \frac{\rho c}{2} \end{pmatrix} \quad l_3 = \begin{pmatrix} -\frac{\rho}{2c} \\ \frac{1}{2} \\ -\frac{\rho c}{2} \end{pmatrix}$$

The characteristic variables are obtained form the inverse matrix and gives,

$$\delta w_1 = \delta\rho - \frac{1}{c^2}\delta p$$

$$\delta w_2 = \delta u + \frac{1}{\rho c}\delta p$$

$$\delta w_3 = \delta u - \frac{1}{\rho c}\delta p$$

and the equations for characteristic variables become

$$\frac{\partial}{\partial t}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} + \begin{pmatrix} u & 0 & 0 \\ 0 & u+c & 0 \\ 0 & 0 & u-c \end{pmatrix} \frac{\partial}{\partial x}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = 0$$

Nozzle flow

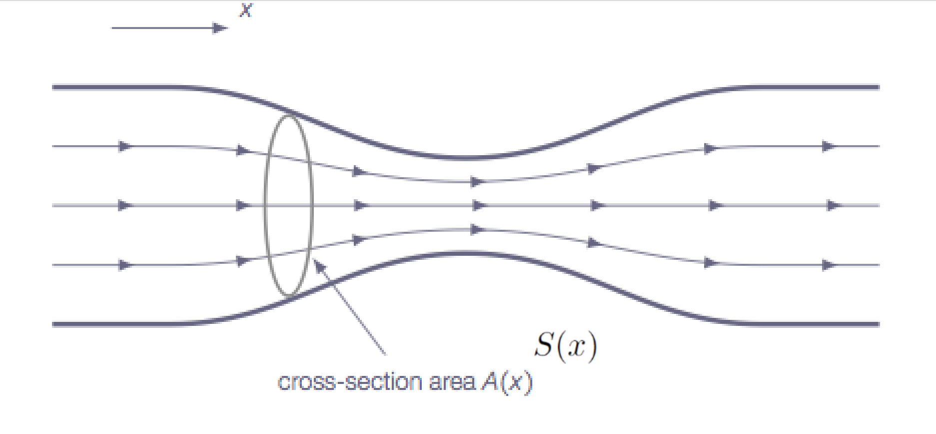
Cross section of nozzle is $S(x)$

The equations are

$$\frac{\partial(\rho S)}{\partial t} + \frac{\partial(\rho u S)}{\partial x} = 0$$

$$\frac{\partial(\rho u S)}{\partial t} + \frac{\partial(\rho u^2 + p)S}{\partial x} = p\frac{dS}{dx}$$

$$\frac{\partial(\rho E S)}{\partial t} + \frac{\partial(\rho u H S)}{\partial x} = 0$$



In primitive variables

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = -\frac{\rho u}{S}\frac{dS}{dx}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$$

$$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + \rho c^2\frac{\partial u}{\partial x} = -\frac{\rho u c^2}{S}\frac{dS}{dx}$$

RHS

$$\bar{Q} = \begin{pmatrix} -\rho u \\ 0 \\ -\rho c^2 u \end{pmatrix} \frac{1}{S}\frac{dS}{dx}$$

Note that these are the 1D Euler equations with a source term depending on the nozzle shape.

Nozzle flow – characteristic variables

The equations are

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}_t + \begin{pmatrix} u & 0 & 0 \\ 0 & u+c & 0 \\ 0 & 0 & u-c \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}_x = \begin{pmatrix} 0 \\ -uc \\ uc \end{pmatrix} \frac{1}{S}\frac{dS}{dx}$$

Using the expressions for the characteristic variables

$$\delta w_1 = \delta\rho - \frac{1}{c^2}\delta p$$

$$\delta w_2 = \delta u + \frac{1}{\rho c}\delta p$$

$$\delta w_3 = \delta u - \frac{1}{\rho c}\delta p$$

we get

$$d^0\rho - \frac{1}{c^2} d^0 p = 0 \qquad d^0 = \partial_t + u\partial_x$$

$$d^+u + \frac{1}{\rho c} d^+ p = -\frac{uc}{S}\frac{dS}{dx} \qquad d^+ = \partial_t + (u+c)\partial_x$$

$$d^-u - \frac{1}{\rho c} d^- p = \frac{uc}{S}\frac{dS}{dx} \qquad d^- = \partial_t + (u-c)\partial_x$$

Schemes

We start with the simplest equation –the advection eq.

We try a natural scheme (1) to use and analyze its stabil
It turns out to be unstable

We introduce a modification (2). It is stable.
We interpret the new scheme as (1) plus a special term.
We call this term artificial viscosity.

We continue with this approach. Central differencing plus artificial viscosity

$u_t + a u_x = 0$

(1) central $\quad \dfrac{u_j^{n+1} - u_j^n}{\Delta t} + a\,\dfrac{u_{j+1}^n - u_{j-1}^n}{2\Delta x} = 0$

stability analysis: $\quad \dfrac{a\Delta t}{\Delta x} = \lambda$

$$|A(\theta)| = \left|1 + \frac{a\Delta t}{\cancel{2}\Delta x}\cancel{2}\, i\sin\theta\right|$$

$1 + \lambda^2\sin^2\theta > 1$ Unstable.

$j-1 \quad j \quad j+1$

(2) $\quad \dfrac{u_j^{n+1} - \frac{1}{2}\left(u_{j+1}^n + u_{j-1}^n\right)}{\Delta t} + \dfrac{a}{2\Delta x}\left(u_{j+1}^n - u_{j-1}^n\right) = 0$

~~by~~ $|A(\theta)| = |\cos\theta + \lambda i\sin\theta| =$

$\cos^2\theta + \lambda^2\sin^2\theta$

$\boxed{\text{if } \lambda < 1 \quad \text{O.k}}$

The difference of the scheme ~~[illegible]~~ is

$$\frac{1}{\Delta t}\left[-\frac{1}{2}\left(u_{j+1} + u_{j-1}\right) + u_j\right] = \frac{1}{2\Delta t}\left[2u_j - u_{j+1} - u_{j-1}\right] \approx -\frac{\Delta x^2}{2\Delta t} u_{xx}$$

This brings artificial viscosity concept.

Schemes - cont

Upwind schemes can be viewed also as central schemes plus Artificial viscosity.

We find the general formula, for both a>0 and a<0.

Take 2nd order accurate scheme which may be unstable & add artificial viscosity.

scheme 3:

(*) $a>0$ $\quad \frac{u_j^{n+1}-u_j^n}{\Delta t} + a\frac{u_j^n-u_{j-1}^n}{\Delta x} = 0$

$|A(\theta)|<1$ stable (check it!)

(**) If $a<0$ $\quad \frac{u_j^{n+1}-u_j^n}{\Delta t} + a\frac{u_{j+1}^n-u_j^n}{\Delta x} = 0$

can combine them as

$$\frac{u_j^{n+1}-u_j^n}{\Delta t} + a\frac{u_{j+1}^n-u_{j-1}^n}{2\Delta x} = \frac{|a|}{2}\Delta x\,\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{\Delta x^2}$$

If $a>0$ since $|a|=a$ it reduces to (*)
$a<0$ $\quad |a|=-a$ . . (**).

we can add a more general viscosity term

$$\beta\frac{|a|\Delta x}{2}\left(\frac{u_{j+1}^n-2u_j^n+u_{j-1}^n}{\Delta x^2}\right) \text{ with } \beta\geq\frac{1}{2}$$

Schemes – Burgers Equation

We follow what we have found for the advection equation.

We use central differencing for the conservation law and Artificial viscosity term inspired by the advection eq.

However, the scheme is not in conservation form. The artificial Viscosity breaks it down.

We will fix it later.

We then consider 1D systems of conservation laws. We add A viscosity which is based on the largest eigenvalue in absolute value. Also here we have lost the conservation and we fix it later.

Burgers Eq

$$u_t + \left(\frac{u^2}{2}\right)_x = 0$$

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{1}{2}\frac{(u_{j+1}^n)^2 - (u_{j-1}^n)^2}{2\Delta x} = \beta \frac{|u_j|\Delta x}{2} \frac{u_{j+1} - 2u_j + u_{j-1}}{\Delta x^2}$$

System 1D

$$u_t + f(u)_x = 0 \iff u_t + A(u)u_x = 0$$

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} + \frac{f(u_{j+1}^n) - f(u_{j-1}^n)}{2\Delta x} = \beta|\lambda(u)|\Delta x \frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2}$$

when $|\lambda(u)|$ is the largest in magnitude eigvalue of $\frac{\partial f}{\partial u}$

But this is far from being optimal

small λ get too much viscosity

Schemes – systems cont

We can add artificial viscosity based on the speeds present in the problem. We do it using the diagonalization of A(U).

Also here we need to modify it to have a conservative scheme s
properly.

$AS = S\Gamma \qquad S^{-1}AS = \Gamma$

Let $|\Gamma|$ be the diagonal matrix with $|\lambda_j|$ on diagonal. Then define

$$|A| = S^{-1}|\Gamma|S.$$

& use artificial viscosity

$$\frac{\beta}{\Delta x}|A(U)|\left(U_{j+1} - 2U_j + U_{j-1}\right)$$

$\beta \geq \frac{1}{2}$

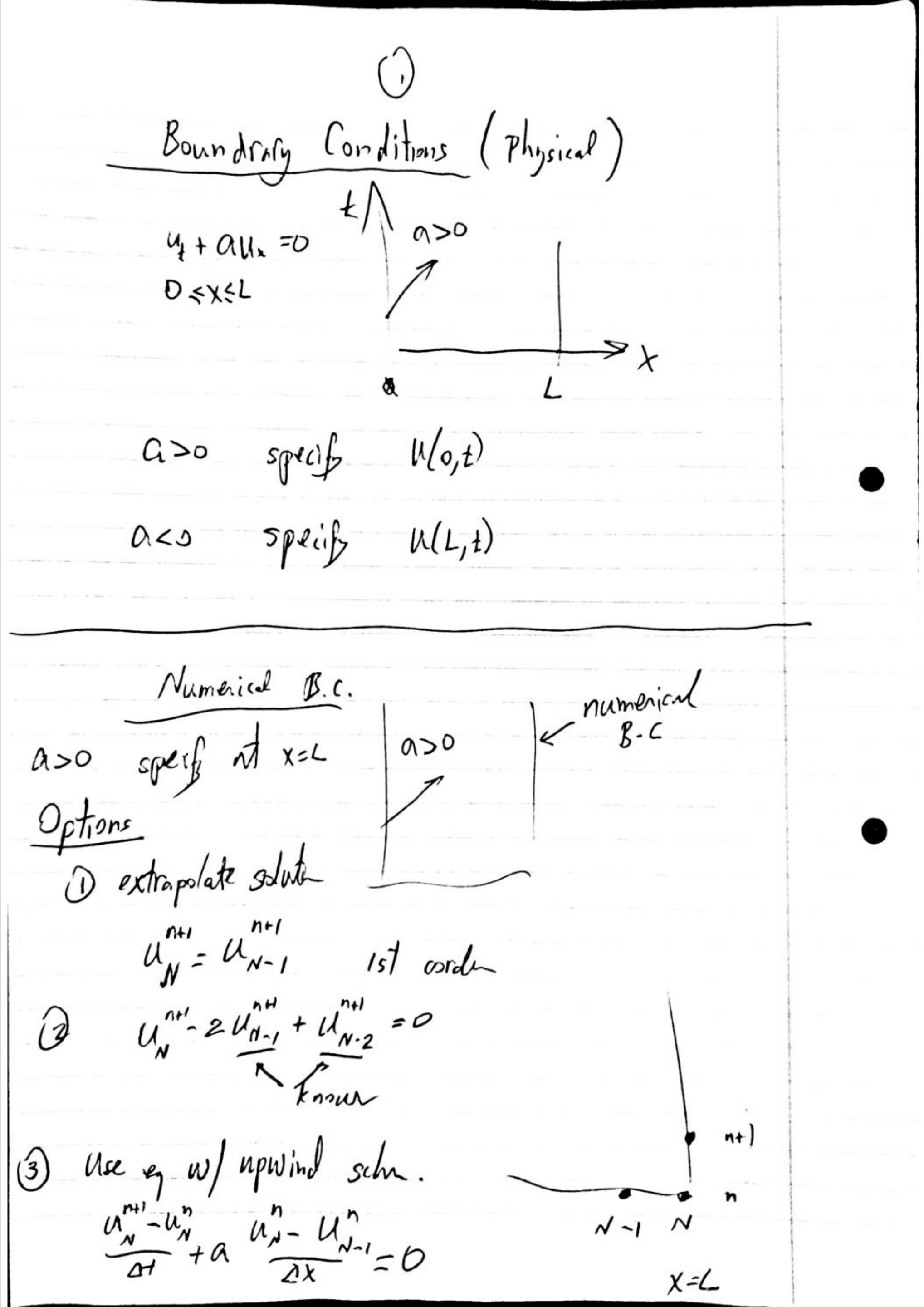
Boundary Conditions

Physical and numerical

The first thing to understand is where to impose physical B.C.
This depends on the waves.

At inflow we need to specify a physical BC
At outflow we need to specify numerical BC.

Physical BC are done by specifying the function at the boundary.
Numerical BC can be done in several ways.
See (1)-(3) as examples. See if you can come up with more.



(1)

Boundary Conditions (Physical)

$u_t + a u_x = 0$

$0 \le x \le L$

$a>0$ specify $u(0,t)$

$a<0$ specify $u(L,t)$

Numerical B.C.

$a>0$ specify at $x=L$

Options

(1) extrapolate solution

$u_N^{n+1} = u_{N-1}^{n+1}$ 1st order

(2) $u_N^{n+1} - 2\underline{u_{N-1}^{n+1}} + \underline{u_{N-2}^{n+1}} = 0$

known

(3) Use eq w/ upwind scheme.

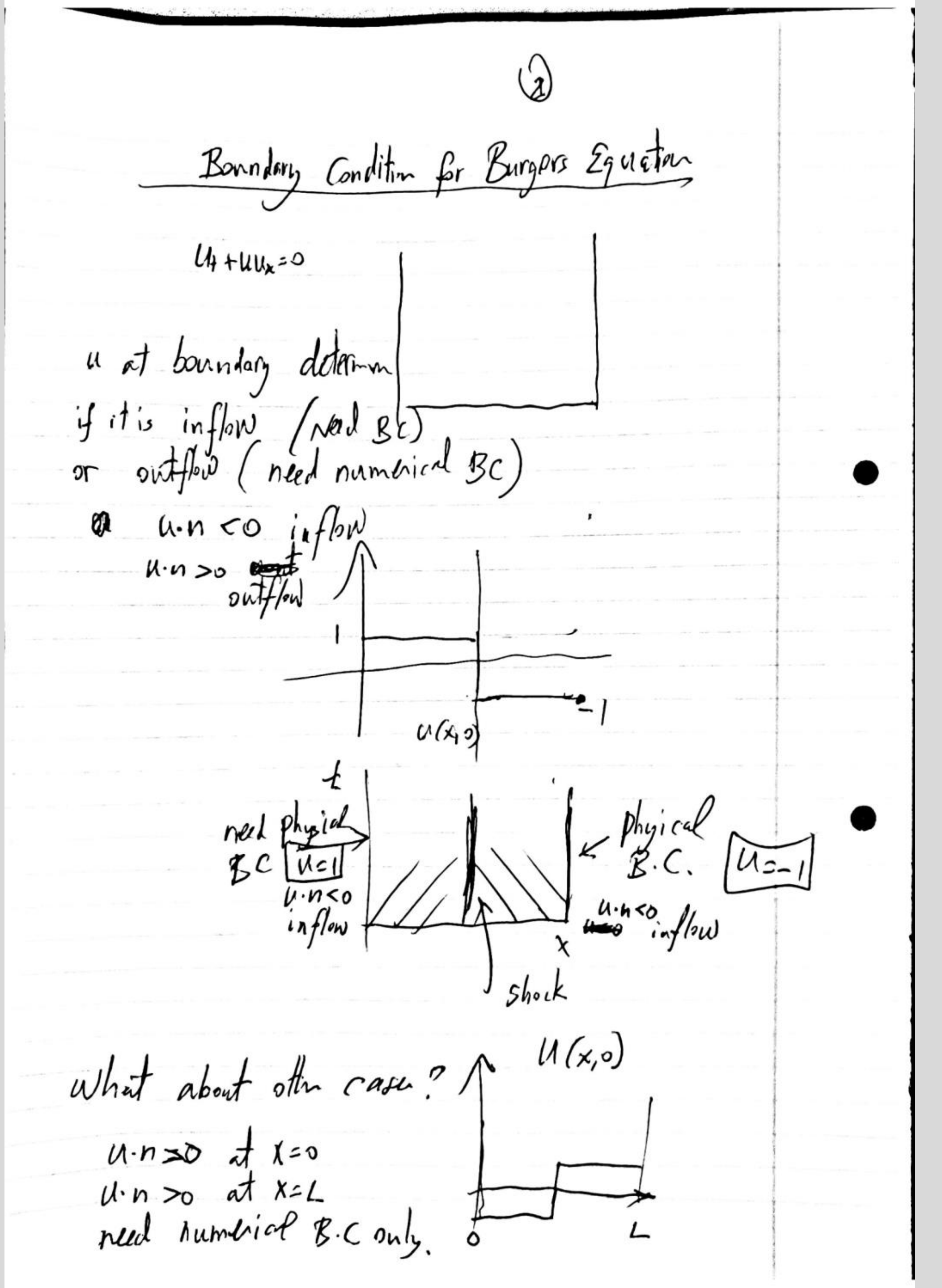
$$\frac{u_N^{n+1}-u_N^n}{\Delta t} + a\,\frac{u_N^n - u_{N-1}^n}{\Delta x} = 0$$

B.C – Burgers equation

Physical and numerical

Depending on the sign of u, we may need to give physical BC
Or not.

If not, then we need to specify numerical BC
See figure.



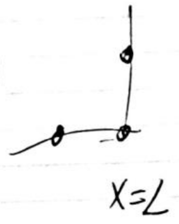
B.C – cont.

Physical and numerical

For systems of equations, the **number** of physical BC needed depends on the eigenvalues.

3

1) extrapolate 1st order
2) 2nd order
3) use eq. (same as prev. case $u_t + a u_x = 0$)



System of Conservation Laws

$$u_t + f(u)_x = 0$$

$$u_t + A(u) u_x = 0$$

e.values of $A \rightarrow \lambda_1, \lambda_2, \ldots$

$$\frac{\partial w_j}{\partial t} + \lambda_j \frac{\partial w_j}{\partial x} = 0$$

for each $\lambda_j > 0$ need a physical BC at $x=0$ & numerical for $\lambda_j < 0$.

at $x=L$ for each $\lambda_j > 0$ need numerical BC
for each $\lambda_j < 0$ need physical B.C.

B.C – 1D Euler

Physical and numerical

The eigenvalues for the 1D Euler are
u, u+c, u-c. (we assume u> 0)

For subsonic flows, u < c and there are 2 positive eigenvalues and one negative. In this case we need 2 physical BC at inflow. one physical BC at outflow.

For supersonic flow u > c, all eigenvalues are positive and we need 3 physical BC at inflow and none at outflow.

In places where we do not specify physical BC, we specify numerical BC, in a way that ateach boundary we have three conditions, some may be physical and the other numerical depending on the situation

2

1D Euler.

e. values are $u, u+c, u-c$

$u>0$

if $|u|<c$ subsonic, $u+c, u>0$
$u-c<0$

Need 2 conditions at inflow

At outflow need 1 condition

~~Example~~ If $|u|>c$ supersonic
need 3 conditions at inflow ($x=0$)
no physical conditions at outflow

How to implement characteristic B.C.

~~example~~ subsonic $u-c<0$ at $x=0$.

$$\delta W_3 = \delta u - \frac{1}{\rho c}\delta p$$

1 Simplest B.C. extrapolate W_3 from inside.

$(W_3)_0 = (W_3)_1$ $\quad (W_3)_0 - (W_3)_1 = 0$

B.C – 1D Euler

Physical and numerical

Implementing numerical BC is best done using characteristic Variables as shown here.

5

$$u_0 - u_1 + \frac{1}{(\rho c)_0}(p_0 - p_1) = 0$$

If we specify ρ, p at inflow $\rho_0\ p_0$ know
$u_0 = ?$ we solve the above for u_0

2 Using the eq for w_3

$$\frac{\partial w_3}{\partial t} + (u-c)\frac{\partial w_3}{\partial x} = 0$$

$$\frac{(w_3)_0^{n+1} - w_0^n}{\Delta t} + (u_1 - c_1)\frac{(w_3)_2^n - (w_3)_1^n}{\Delta x} = 0$$

$$\frac{1}{\Delta t}\left[u_0^{n+1} - u_0^n + \frac{1}{(\rho c)_1}\left(p_0^{n+1} - p_0^n\right)\right] + (u_1 - c_1)\frac{u_2^n - u_1^n + \frac{1}{(\rho c)_1}(p_2^n - p_1^n)}{\Delta x} = 0$$

Solve for u_0^{n+1} if ρ, p are speified at inflow.

~~Same~~ for

Similarly for other cases

Key pt: $\delta w_j = \dots$ in terms of primitive vars