REPORT - Homework Assignment 3

Sean Zimmari - SN 6151337

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1 Analytical Formulations

The main steps, necessary for deriving analytical formulations, are explained below:

1.) Calculation of the force in function of z
2.) Derivation of the stress relation from the forces
3.) Derivation of the displacements from the stresses and strains
4.) Computation of the strain and squared strain
5.) Integration of the squared strain to find the strain energy
6.) Deriving strain energy to find the element of the stiffness matrix
7.) Computation of the forces from the work

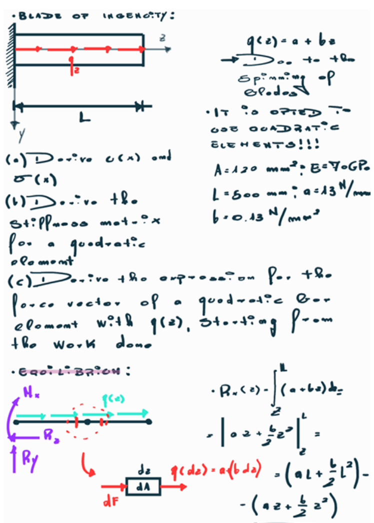

Figure 1: Introduction to the problem

$$T = \frac{b}{2}L^2 + aL - \left(az + \frac{b}{2}z^2\right)$$

$$\rightarrow F(z) = \sigma(z) A \; : \; \sigma(z) = \frac{1}{A}\left[\frac{b}{2}L^2 + aL - \left(az + \frac{b}{2}z^2\right)\right]$$

$$u(z) = -\frac{1}{EA}\left[\frac{b}{6}z^3 + \frac{a}{2}z^2 - \left(aL + \frac{b}{2}L^2\right)z + C\right]$$

BOUNDARY CONDITION

$\rightarrow u_A = 0 \rightarrow C_1 = \emptyset$

$$\begin{cases} u(z) = -\dfrac{1}{EA}\left[\dfrac{b}{6}z^3 + \dfrac{a}{2}z^2 - \left(aL + \dfrac{b}{2}L^2\right)z\right] \\[2ex] \sigma(z) = -\dfrac{1}{A}\left[\dfrac{b}{2}z^2 + az - \left(aL + \dfrac{b}{2}L^2\right)\right] \end{cases}$$

Figure 2: Derivation of analytical displacements and stresses

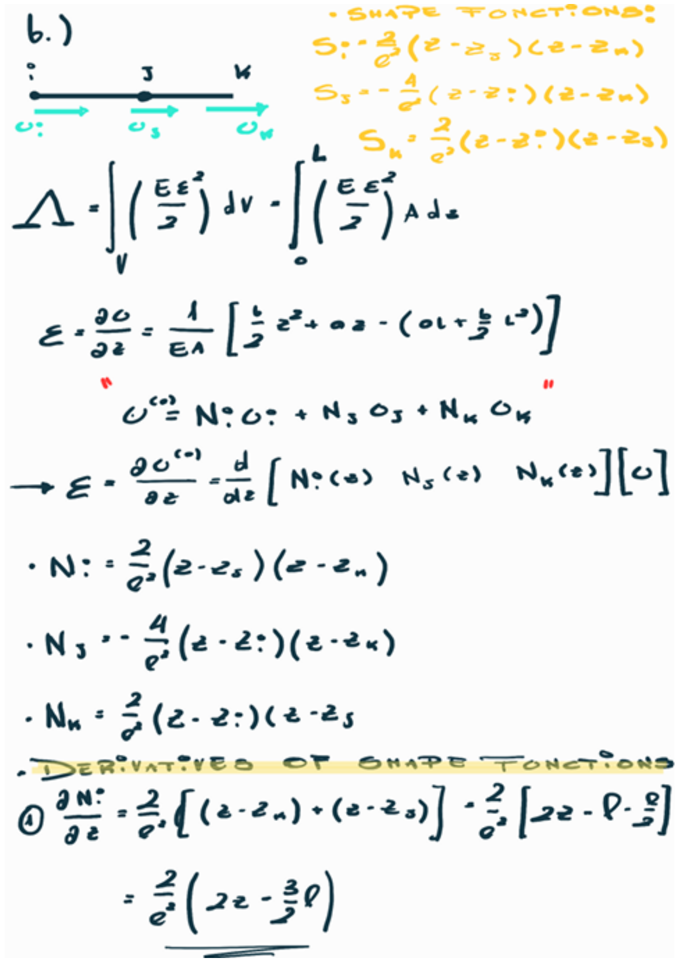

Figure 3: Introduction to point B

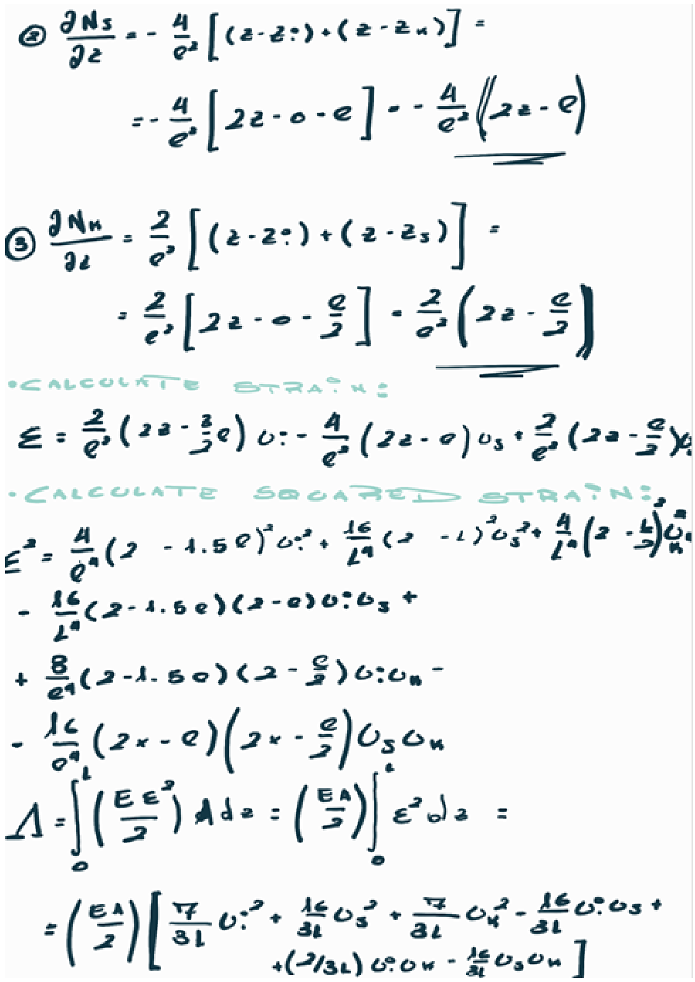

Figure 4: Computation of strain and squared strain

$$① \quad \frac{\partial \Lambda}{\partial u_i} = \frac{EA}{2}\left(\frac{14}{3L}u_i - \frac{16}{3L}u_j + \frac{2}{3L}u_k\right) =$$

$$= \frac{EA}{L}\left(\frac{7}{3}u_i - \frac{8}{3}u_j + \frac{1}{3}u_k\right)$$

$$② \quad \frac{\partial \Lambda}{\partial u_j} = \frac{EA}{2}\left(\frac{32}{3L}u_j - \frac{16}{3L}u_i - \frac{16}{3L}u_k\right) =$$

$$= \frac{EA}{L}\left(-\frac{8}{3}u_i + \frac{16}{3}u_j - \frac{8}{3}u_k\right)$$

$$③ \quad \frac{\partial \Lambda}{\partial u_k} = \frac{EA}{2}\left(\frac{14}{3L}u_k + \frac{2}{3L}u_i - \frac{16}{3L}u_j\right) =$$

$$= \frac{EA}{L}\left(\frac{1}{3}u_i - \frac{8}{3}u_j + \frac{7}{3}u_k\right)$$

$$① \quad \frac{EA}{L}\left(\frac{7}{3}u_i - \frac{8}{3}u_j + \frac{1}{3}u_k\right) - F_i = 0$$

$$② \quad \frac{EA}{L}\left(-\frac{8}{3}u_i + \frac{16}{3}u_j - \frac{8}{3}u_k\right) - F_j = 0$$

$$③ \quad \frac{EA}{L}\left(\frac{1}{3}u_i - \frac{8}{3}u_j + \frac{7}{3}u_k\right) - F_k = 0$$

Figure 5: Derivatives of strain energy and setting of the equations

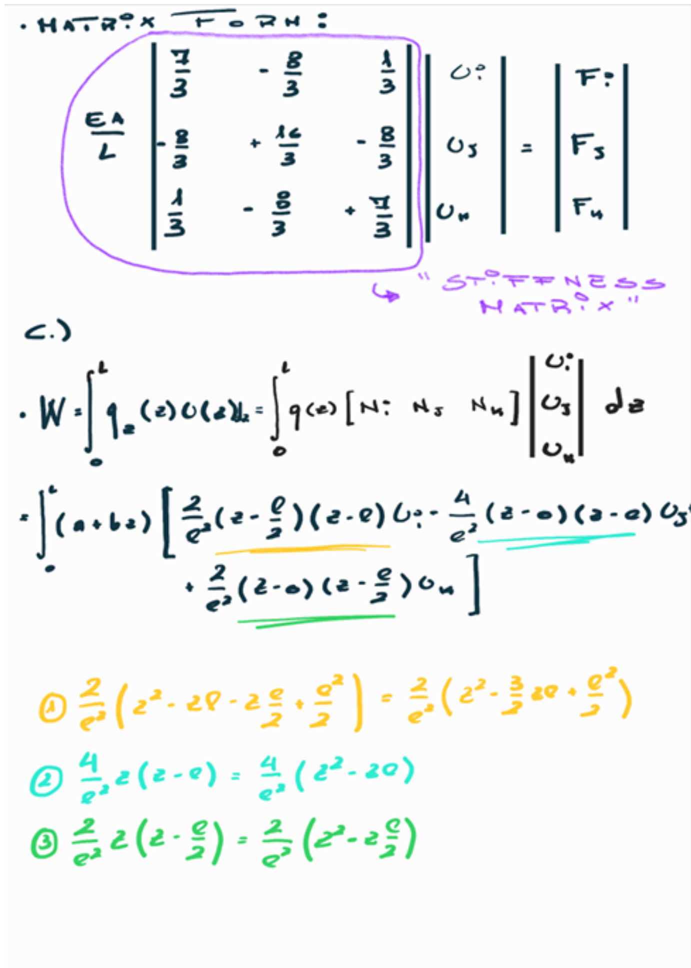

Figure 6: Computation of work

$$\rightarrow W = \int_0^{\ell} (a+bz)\left[\frac{2}{\ell^2}\left(z^2-\frac{3}{2}z\ell+\frac{\ell^2}{2}\right)U_1 - \right.$$

$$\left. - \frac{4}{\ell^2}\left(z^2 - z\ell\right)U_3 + \frac{2}{\ell^2}\left(z^2 - z\frac{\ell}{2}\right)U_4\right] dz$$

$$\rightarrow W = \int_0^{\ell}\left\{\left[\frac{2}{\ell^2}(a+bz)\left(z^2-\frac{3}{2}z\ell+\frac{\ell^2}{2}\right)\right]U_1\right.$$

$$\left. + \left[-\frac{4}{\ell^2}(a+bz)\left(z^2 - z\ell\right)\right]U_3 + \left[\frac{2}{\ell^2}(a+bz)\left(z^2-\frac{z\ell}{2}\right)\right]\right\} dz$$

$$= \int_0^{\ell}\left\{\left[\left(\frac{2}{\ell^2}a + \frac{2}{\ell^2}bz\right)\left(z^2 - \frac{3}{2}z\ell + \frac{\ell^2}{2}\right)\right] + \right.$$

$$\left. + \left[\left(-\frac{4}{\ell^2}a - \frac{4}{\ell^2}bz\right)\left(z^2 - z\ell\right)\right] + \left[\left(\frac{2}{\ell^2}a + \frac{2}{\ell^2}bz\right)\left(z^2 - \frac{z\ell}{2}\right)\right]\right\} dz$$

$$= \int_0^{\ell}\left\{\left[\frac{2}{\ell^2}az^2 - \frac{3z}{\ell}a + a + \frac{2}{\ell^2}bz^3 - \frac{3bz^2}{\ell} + bz\right]U_1 + \right.$$

$$+ \left[\left(-\frac{4az^2}{\ell^2} + \frac{4az}{\ell} - \frac{4bz^3}{\ell^2} + \frac{4bz^2}{\ell}\right)\right]U_3 +$$

$$\left. + \left[\frac{2az^2}{\ell^2} - \frac{za}{\ell} + \frac{2bz^3}{\ell^2} - \frac{bz^2}{\ell}\right]U_4\right\} dz$$

$$= \int_0^{\ell}\left[+\frac{2}{\ell^2}bz^3 + \left(\frac{2a}{\ell^2} - \frac{3b}{\ell}\right)z^2 + \left(-\frac{3a}{\ell} + b\right)z + a\right]U_1 +$$

$$+ \left[\left(-\frac{4b}{\ell^2}z^3 + \left(\frac{4b}{\ell} - \frac{4a}{\ell^2}\right)z^2 + \frac{4az}{\ell}\right]U_3 +$$

$$+ \left[\frac{2bz^3}{\ell^2} + \left(\frac{2a}{\ell^2} - \frac{b}{\ell}\right)z^2 - \frac{a}{\ell}z\right]U_4 \; dz$$

Figure 7: Computation of work 2

$$= \Big[\Big(\frac{b z^4}{2\ell^2} + \left(\frac{2a}{\ell^2} - \frac{3b}{\ell}\right)\frac{z^3}{3} + \left(-\frac{3a}{\ell} + b\right)\frac{z^2}{2} + a z\Big]_0^{\ell} U_i +$$

$$+ \Big[\Big(-\frac{b z^4}{\ell^2} + \left(\frac{4b}{\ell} - \frac{4a}{\ell^2}\right)\frac{z^3}{3} + \frac{2az^2}{\ell}\Big)\Big]_0^{\ell} U_j +$$

$$+ \Big[\Big(\frac{b z^4}{2\ell^2} + \left(\frac{2a}{\ell^2} - \frac{b}{\ell}\right)\frac{z^3}{3} - \frac{a}{\ell}\frac{z^2}{2}\Big]_0^{\ell} U_k =$$

$$= \Big(\cancel{\frac{b\ell^2}{2}} + \frac{2}{3}a\ell - \cancel{b\ell^2} - \frac{3}{2}a\ell + \cancel{\frac{1}{2}b\ell^2} + a\ell\Big) U_i +$$

$$+ \Big(-b\ell^2 + \frac{4}{3}b\ell^2 - \frac{4}{3}a\ell + 2a\ell\Big) U_j$$

$$+ \Big(\frac{b\ell^2}{2} + \frac{2}{3}a\ell - \frac{1}{3}b\ell^2 - \frac{a\ell}{2}\Big) U_k =$$

$$= \Big(\frac{1}{6}a\ell\Big) U_i + \Big(\frac{1}{3}b\ell^2 + \frac{2}{3}a\ell\Big) U_j + \Big(\frac{1}{6}b\ell^2 + \frac{1}{6}a\ell\Big) U_k$$

$$F_i = \frac{\partial W}{\partial U_i} = \frac{a\ell}{6} = 1083.333\ \mathrm{N}$$

$$F_j = \frac{\partial W}{\partial U_j} = \frac{1}{3}b\ell^2 + \frac{2}{3}a\ell = 15166.667\ \mathrm{N}$$

$$F_k = \frac{\partial W}{\partial U_k} = \frac{1}{6}b\ell^2 + \frac{1}{6}a\ell = 6500\ \mathrm{N}$$

Figure 8: Computation of force vector components

2 FEM Solution vs Analytical solution

From the plots, derived with the usage of Python, it can be perfectly understood how the approximated FEM solution converges to the exact solution. Firstly, analytical displacements are well-approximated by the FEM solution, even though the accuracy of the approximation is lower for one element only. On the other hand, when displacements are used for the computation of stresses and strains, the error just propagates itself and, as a result, differences between the solutions are larger. Below the plots of displacements and stresses, there is a representation of the mean stress error, which is representative of the convergence rate, against the amount of elements. It can be seen from these plots that stresses will not converge for 2 elements, unlike displacements, because the error is larger than 1 percent of the distance from the analytical solution

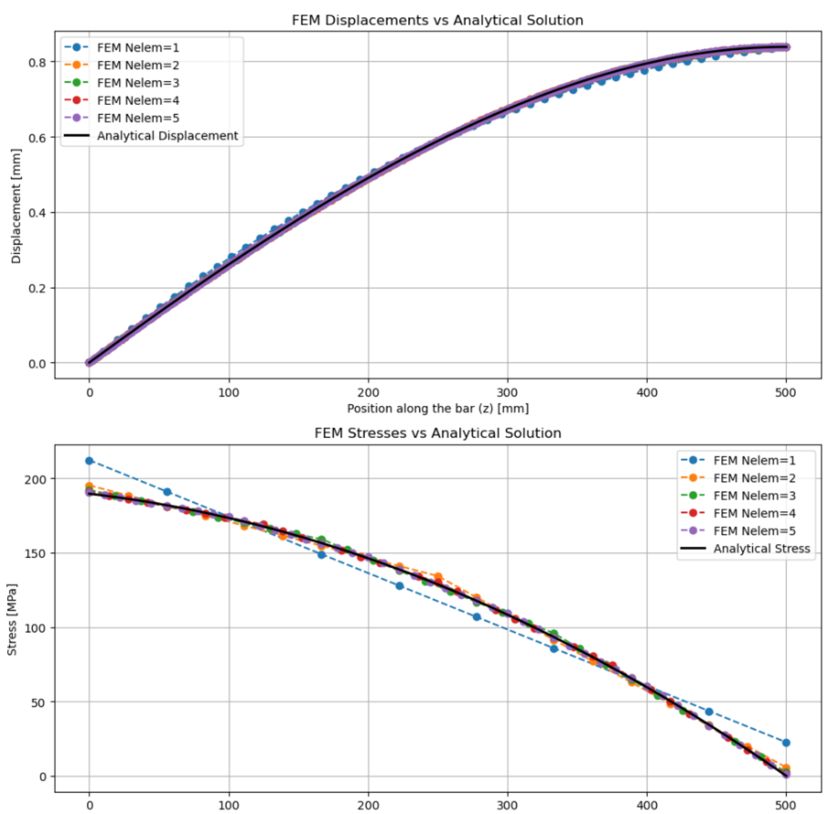

Figure 9: Plots FEM solution (varying the number of elements) vs Analytical solution

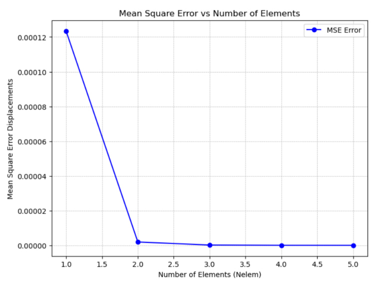

Figure 10: Mean Square Error for displacements

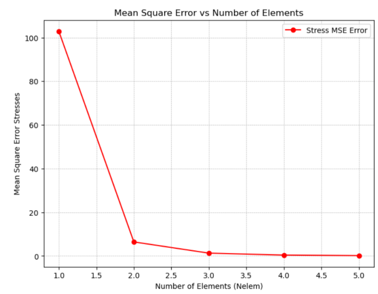

Figure 11: Mean Square Error for stress

3 Linear variation of the cross section of the bar

If the bar would vary linearly across the whole length, evaluating all the parameters only in the mid points of each element **WOULD BE A GOOD OPTION ONLY FOR THE MEAN STRESS** since you can approximate well the mean stress for each element. BUT, it would not be conservative since stresses can be larger than the mean stress for different positions of the element, so FEM solution would not approximate well all the displacement and stress distribution across the bar.

4 Design of Ingenuity system - Steel B23

For this purpose, B23 are used as representative element types for the considered part. They have been modeled in steel, with a square section whose size is 12 mm. Then, mesh has been applied considering 15 elements in order to get a good approximation of the effective results, and of course for this case B23 elements have been chosen during the process. Furthermore, orientations has been assigned in such a way that the x-axis will remain parallel to the bar. At the end, the boundary condition of zero displacements in every direction of the system has been introduced at the first node, due to the interlocking; along with the loads, applied in their respective positions along the bar. The results are the following:

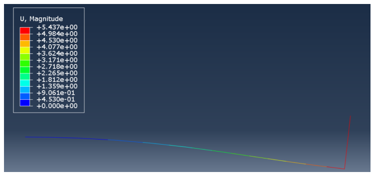

Figure 12: Displacements magnitude

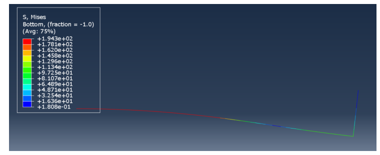

Figure 13: Mises stresses

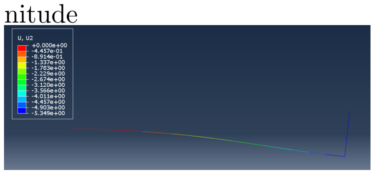

Figure 14: Displacements along y-axis

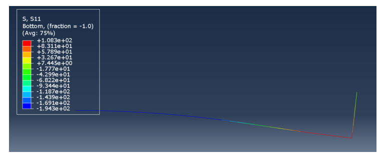

Figure 15: Axial stresses

5 Choice of element type

If complete freedom was ensured for the choice of a specific element type to model the part, I would still choose the **B23 elements**. The motivation of this choice is explained below:

- B23 elements are based on Euler-Bernoulli beam theory, hence they do not allow shear deformation but it is just fine since the examined bar has a dimensional ratio of 1/15; hence the bar is slender and do not have a moderate thickness. Consequently, it is a problem in which shear deformations are negligible as well as torsion and rotations.

- Secondly, a cubical polynomial function will approximate accurately the distributed load

6 Varying the mesh of the element to reach the convergence of axial stress

In order to get to a converged result, the first step to do is to decide which parameter is going to be the considered one for the convergence. In this case, since it is previously said that displacements are well-approximated by the FEM solution, maximum axial stress is going to be the parameter of convergence study. Once the quantity has been chosen, we can proceed with varying the amount of elements in the mesh. At first place, 1 element will be considered and the result derived by this will be the one below:

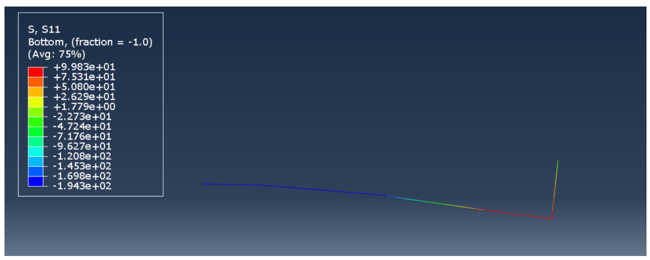

Figure 16: Axial stress values with 1 element

It can be seen that the starting point of the maximum axial stresses is S11 = + 9.983e+01 MPa. If now the number of element is increased, the accuracy of the model will be larger of course and the maximum axial stress will converge to a fixed value. As second case of study, the amount of elements has been increased to 8 and the outputs are:

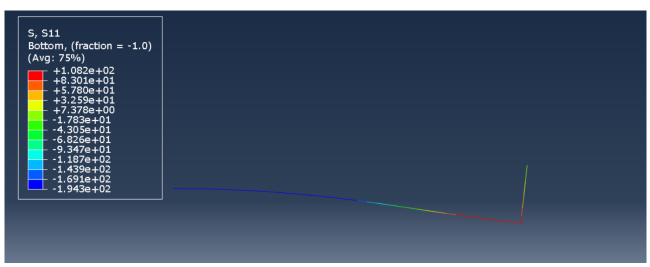

Figure 17: Axial stress values with 8 elements

As a result, the maximum axial stress is increased to +1.082e+02 MPa. As last case of study, we will consider a sufficiently large amount of elements in such a way that the converged value is reached. For this purpose, 15 elements have been chosen and the results were almost equal to the previous case:

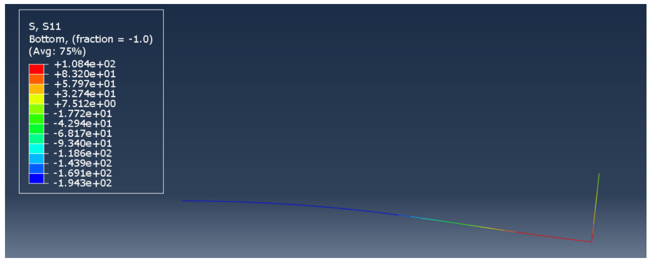

Figure 18: Axial stress values with 15 elements

The maximum result in this case is +1.084e+02 MPa which is very near to the previous case. From this moment, increasing the number of elements will not have an effect on the axial stresses since convergence has been reached. Hence, it can be stated that convergence happens approximately at 10 elements for the maximum axial stress.

7 Re-design the Ingenuity system with different material and cross-sections

The problem asks to re-design the studied system in order to get the **lightest possible solution** that respect the following requirements:

- Max Angle at point F is equal to 5 degrees
- Max displacement in y-direction is 17.5 mm
- No yielding in the structure

For this, two different materials could be assigned:

- **Aluminium** : E = 70 GPa, v=0.31, density = 2700 kg/m3, yield strength 270 MPa
- **Steel** : E = 210 GPa, , v=0.3, density = 8000 kg/m3, yield strength 550 MPa

As well as three different cross-sectional areas:

- **Rectangular** : Both dimensions equal to 12 mm as the previous case
- **Circular** : Diameter of 16 mm
- **Hollow square profile**: Thickness of 3 mm, outer dimension of 16 mm

First analytical evaluations have been made in order to find the lightest structure. Since data include the density of each material along with the dimensions of sections, mass of each element can be easily calculated. Indeed $m = \rho V = \rho A l$, consequently the lightest possible solution has to be made with aluminium since the density is lower. For the cross-sectional area, we have to chose the one with lower area: so, in this case the solid rectangular one seems perfect for our needs. Once analytical evaluations have been made, the element has been simulated on Abaqus and the outputs were the following:

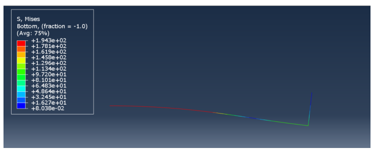

Figure 19: Stresses values

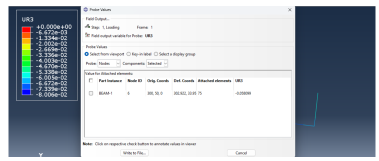

Figure 20: Rotation of point F in radians

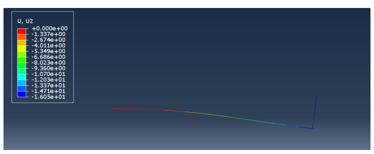

Figure 21: Displacements along y-axis

In the end, the results show that yielding is not reached across the entire structure, since the maximum stress value of 194.3 MPa is below the yielding strength of 270 MPa. Furthermore, rotation at point F is equal to -0.058099 radians, which in degrees will be equal to -3.330516: hence, the upper limit of the rotations is not reached. Lastly, the maximum displacement along y-axis is 16.05 mm which is below the required maximum displacement. Consequently, the lightest solution, respecting all the requirements, is found and have **Solid rectangular cross-section in alumimium for both parts of the model**.