

1 Coordinate systems

1.1 Cylindrical

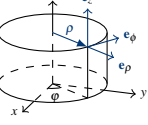
Basis vectors

$$\mathbf{e}_\rho = \left| \frac{\partial \mathbf{r}}{\partial \rho} \right|^{-1} \frac{\partial \mathbf{r}}{\partial \rho} = (\cos \varphi, \sin \varphi, 0),$$
$$\mathbf{e}_\varphi = \left| \frac{\partial \mathbf{r}}{\partial \varphi} \right|^{-1} \frac{\partial \mathbf{r}}{\partial \varphi} = (-\sin \varphi, -\cos \varphi, 0),$$
$$\mathbf{e}_z = \left| \frac{\partial \mathbf{r}}{\partial z} \right|^{-1} \frac{\partial \mathbf{r}}{\partial z} = (0, 0, 1).$$

Velocity vectors

$$\dot{\mathbf{r}}_\rho = \dot{\varphi} \mathbf{e}_\varphi, \quad \dot{\mathbf{r}}_\varphi = -\dot{\rho} \mathbf{e}_\rho, \quad \dot{\mathbf{r}}_z = 0.$$
$$v_\rho = \dot{\rho}, \quad v_\varphi = \rho \dot{\varphi}, \quad v_z = \dot{z}.$$

Acceleration vectors

$$a_\rho = \ddot{\rho} - \rho \dot{\varphi}^2, \quad a_\varphi = \rho \ddot{\varphi} + 2\dot{\rho}\dot{\varphi}, \quad a_z = \ddot{z}.$$


1.2 Spherical

Basis vectors

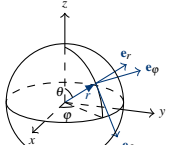
$$\mathbf{e}_r = \left| \frac{\partial \mathbf{r}}{\partial r} \right|^{-1} \frac{\partial \mathbf{r}}{\partial r} = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta) = \frac{\mathbf{r}}{r},$$
$$\mathbf{e}_\theta = \left| \frac{\partial \mathbf{r}}{\partial \theta} \right|^{-1} \frac{\partial \mathbf{r}}{\partial \theta} = (\cos \theta \cos \varphi, \cos \theta \sin \varphi, -\sin \theta),$$
$$\mathbf{e}_\varphi = \left| \frac{\partial \mathbf{r}}{\partial \varphi} \right|^{-1} \frac{\partial \mathbf{r}}{\partial \varphi} = (\sin \theta \sin \varphi, \sin \theta \cos \varphi, 0).$$

Velocity vectors

$$\dot{\mathbf{r}}_r = \frac{\partial \mathbf{r}}{\partial r} \dot{r} + \frac{\partial \mathbf{r}}{\partial \theta} \dot{\theta} + \frac{\partial \mathbf{r}}{\partial \varphi} \dot{\varphi} = \dot{r} \mathbf{e}_r + \dot{\theta} \mathbf{e}_\theta + \sin \theta \dot{\varphi} \mathbf{e}_\varphi,$$
$$\dot{\mathbf{r}}_\theta = \frac{\partial \mathbf{r}}{\partial \theta} \dot{\theta} + \frac{\partial \mathbf{r}}{\partial \varphi} \dot{\varphi} = -\dot{\theta} \mathbf{e}_\theta + \cos \theta \dot{\varphi} \mathbf{e}_\varphi,$$
$$\dot{\mathbf{r}}_\varphi = \frac{\partial \mathbf{r}}{\partial \varphi} \dot{\varphi} = -(\cos \varphi, \sin \varphi, 0) \dot{\varphi} = -\dot{\varphi} (\sin \theta \mathbf{e}_r + \cos \theta \mathbf{e}_\theta).$$

$v_r = \dot{r}, \quad v_\theta = r \dot{\theta}, \quad v_\varphi = r \sin \theta \dot{\varphi}.$

Acceleration vectors

$$a_r = \ddot{r} - r \dot{\theta}^2 - r \sin^2 \theta \dot{\varphi}^2, \quad a_\theta = r \ddot{\theta} + 2\dot{r}\dot{\theta} - r \sin \theta \cos \theta \dot{\varphi}^2,$$
$$a_\varphi = r \sin \theta \cos \theta + 2 \sin \theta \dot{\theta} \dot{\varphi} + 2r \cos \theta \dot{\varphi}^2.$$


2 Newtonian Mechanics

2.1 Newton's Laws

1. In the absence of external forces, the momentum of a particle remains constant.

2. If an external force \mathbf{F} acts on a particle, the rate of variation of its momentum is given by $\dot{\mathbf{p}} = \mathbf{F} \Leftrightarrow \mathbf{F} = m\dot{\mathbf{v}}$ (\Leftarrow eqn. of motion of a particle).

2.2 Conservation laws

Basic definitions

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\mathbf{r} \times \dot{\mathbf{r}}, \quad \dot{\mathbf{L}} = \mathbf{N} = \mathbf{r} \times \mathbf{F} = 0 \Leftrightarrow \mathbf{F} \parallel \mathbf{r}.$$

A force is central $\Leftrightarrow \mathbf{F} = g(\mathbf{r}, \mathbf{r})\hat{\mathbf{r}}$. Let $T = \frac{1}{2} m \dot{\mathbf{r}}^2$, then $\frac{dT}{dt} = m\dot{\mathbf{r}} \cdot \dot{\mathbf{F}}$. Kinetic energy is conserved $\Leftrightarrow \mathbf{F} \perp \dot{\mathbf{r}}, \forall t$.

A force is conservative $\Leftrightarrow \mathbf{F}(\mathbf{r}) = -\sum_{i=1}^3 \frac{\partial V(\mathbf{r})}{\partial x_i} \mathbf{e}_i \equiv -\frac{\partial V(\mathbf{r})}{\partial \mathbf{r}}$.

If \mathbf{F} conservative, then $E = T + V$ conserved.

A force is irrotational $\Leftrightarrow \exists V(\mathbf{r}, t)$ s.t. $\mathbf{F}(\mathbf{r}, t) = -\frac{\partial V(\mathbf{r}, t)}{\partial \mathbf{r}} \Leftrightarrow \frac{d\mathbf{F}}{dt} = \frac{\partial V}{\partial \mathbf{r}}$.

Gravitational and Electrostatic forces

$$\mathbf{F} = f(r)\mathbf{e}_r,$$
$$f(r) = -\frac{GMm}{r^2}, V(r) = -\frac{GMm}{r} \Rightarrow \mathbf{a} = \mathbf{F} = -\frac{GM}{r^3} \mathbf{r} \text{ independent of } m.$$
$$f(r) = k \frac{q_1 q_2}{r^2}, V(r) = k \frac{qQ}{r}.$$

In general, for forces between a particle and a continuous distribution of mass/charge:

$$\mathbf{F}(\mathbf{r}) = -Gm \int \rho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} d^3\mathbf{r}' = -m \frac{\partial \Phi(\mathbf{r})}{\partial \mathbf{r}}, \quad \Phi(\mathbf{r}) = -G \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}'$$

$$\mathbf{F}(\mathbf{r}) = -kq \int \rho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} d^3\mathbf{r}' = -q \frac{\partial \Phi(\mathbf{r})}{\partial \mathbf{r}}, \quad \Phi(\mathbf{r}) = -k \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|} d^3\mathbf{r}'$$

Electromagnetic force

$$\mathbf{F}(\mathbf{r}, \mathbf{t}) = q(\mathbf{E}(\mathbf{r}, \mathbf{t}) + \mathbf{t} \times \mathbf{B}(\mathbf{r}, \mathbf{t})), \quad T + q\Phi(\mathbf{r}) \equiv \text{"electromechanical energy"}.$$

2.3 Motion of a particle in a one-dimensional potential

Law of conservation of energy: $\frac{1}{2} m \dot{x}^2 + V(x) = E \Rightarrow \dot{x}(m\dot{x} - F(x)) = 0$ (if $\dot{x} \neq 0 \Rightarrow m\dot{x} - F(x) = 0$).

Said ODE is an **autonomous system** (i.e. it's of the form $\dot{x} = f(x, \dot{x})$), and it's thus invariant under transformations $t \rightarrow t + c$ and $t \rightarrow -t$.

Equilibrium, and trajectory

Equilibria are the points $x_0 \in \mathbb{R}$ s.t. $x(t) = x_0$ is a solution of the EOM $\Rightarrow x(t) = 0 \Rightarrow F(x(t)) = F(x_0) = 0$.

The **accessible region** for a given E is the set $\{x \in \mathbb{R} \mid V(x) \leq E\}$, which, in general, are disjoint union of closed intervals.

The endpoints of the intervals, if not equilibriums, are the **turning points** of the trajectories: x_i turning point $\Leftrightarrow V(x_i) = E, V'(x_i) \neq 0$.

In general, the relation between time and displacement is given as:

$$t - t_0 = \pm \sqrt{\frac{m}{2}} \int_{x_0}^x \frac{dx}{\sqrt{E - V(x)}} \Rightarrow t = \sqrt{\frac{m}{2}} \int_{x_0}^x \frac{dx}{\sqrt{E - V(x)}} \equiv \theta(x)$$

Interval with two turning points:

The time for the particle to go from x_0 to x_1 is $\frac{\tau}{2}$, being:

$$\tau = 2\theta(x_1) = \sqrt{2m} \int_{x_0}^{x_1} \frac{dx}{\sqrt{E - V(x)}}.$$

The function $\theta(x)$ is mon. increasing for $x \in (x_0, x_1)$, therefore θ is invertible and we can write x in terms of t :

$$x(t) = \theta^{-1}(t), 0 \leq t \leq \frac{\tau}{2}, \quad x(t) = \theta^{-1}(\tau - t), \frac{\tau}{2} \leq t \leq \tau.$$

Semi-infinite interval with turning point at one side $[x_0, \infty)$:

For a particle with initial position $x(0) = x_0$ and moving towards infinity, the time is:

$$t_{\infty} = \theta(\infty) = \sqrt{\frac{m}{2}} \int_{x_0}^{\infty} \frac{dx}{\sqrt{E - V(x)}}, \text{ it can be convergent or divergent.}$$

x can be expressed as:

$$x(t) = \theta^{-1}(t), 0 \leq t \leq t_{\infty}, \quad x(t) = \theta^{-1}(-t), -t_{\infty} \leq t \leq 0.$$

Accessible region is \mathbb{R} :

The time for the particle to reach infinity from an initial position x_0 is:

$$t_{\pm\infty} = \theta(\pm\infty) = \sqrt{\frac{m}{2}} \int_{x_0}^{\pm\infty} \frac{dx}{\sqrt{E - V(x)}}$$

$$x(t) = \theta^{-1}(t), t_{\infty} \leq t \leq t_{\infty}, \quad x(t) = \theta^{-1}(-t), -t_{\infty} \leq t \leq -t_{\infty}.$$

Stability of equilibria, oscillation approximation

An equilibrium is stable \Leftrightarrow it is a *relative minimum* of the potential.

For an open ball centered at a stable eq. $(x_{eq} - \varepsilon, x_{eq} + \varepsilon)$, the Taylor expansion of V is of the form:

$$V(x) = V(x_{eq}) + \frac{1}{2} V''(x_{eq})(x - x_{eq})^2 + O(x^3)$$

$$\text{Then the motion of a particle within said open ball can be approximated to:}$$
$$\ddot{x} = \frac{F(x)}{m} = -\frac{V'(x)}{m} \simeq -\frac{V''(x_{eq})}{m}(x - x_{eq}) \Rightarrow \ddot{x} + \omega^2 x = 0$$

with $\xi = x - x_{eq}, \omega \equiv \sqrt{\frac{V''(x_{eq})}{m}}$.

The solution of this ODE is: $\xi = A \cos(\omega t + \alpha), A \in (0, |x - x_{eq}|)$, and the period of the oscillation is $\tau \simeq \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{V''(x_{eq})}}$.

2.4 Dynamics of a system of particles

The law of motion of N elements with masses m_i , positions \mathbf{r}_i is:

$$m_i \ddot{\mathbf{r}}_i = \sum_{j=1}^N \mathbf{F}_{ij} + \mathbf{F}_i^e, \quad i = 1, \dots, N,$$

where \mathbf{F}_{ij} are the forces exerted by j on i , and \mathbf{F}_i^e are the total external force exerted on i . The solution of this set of N equations is unique if given the initial conditions

$$\mathbf{r}_i(t_0) = \mathbf{r}_{i0}, \quad \dot{\mathbf{r}}_i(t_0) = \mathbf{v}_{i0}, \quad i \in \{1, \dots, N\}.$$

Center of mass

$$\mathbf{F}^e = \left(\sum_{i=1}^N \mathbf{F}_i^e \right), \quad \sum_{i=1}^N m_i \ddot{\mathbf{r}}_i = \mathbf{F}^e, \quad \mathbf{R} = M^{-1} \sum_{i=1}^N m_i \mathbf{r}_i, \quad M \ddot{\mathbf{R}} = \mathbf{F}^e.$$

Conservation

Linear momentum:

$$\mathbf{P} = \sum_{i=1}^N m_i \dot{\mathbf{r}}_i = M \dot{\mathbf{R}} \Rightarrow \dot{\mathbf{P}} = \mathbf{F}^e \Rightarrow \mathbf{P} \text{ conserved} \Leftrightarrow \mathbf{F}^e = 0.$$

Angular momentum:

$$\mathbf{L} = M \mathbf{R} \times \dot{\mathbf{R}} + \sum_{i=1}^N m_i \mathbf{r}_i' \times \dot{\mathbf{r}}_i' = \mathbf{L}_{CM} + \sum_{i=1}^N m_i \mathbf{r}_i' \times \dot{\mathbf{r}}_i', \quad \mathbf{r}_i = \mathbf{R} + \mathbf{r}_i',$$

$$\dot{\mathbf{L}} = \sum_{i=1}^N \mathbf{r}_i \times \mathbf{F}_i^e \equiv \mathbf{N}^e \Rightarrow \mathbf{L} \text{ conserved} \Leftrightarrow \mathbf{N}^e = 0$$

Kinetic energy, potential, and total mechanical energy:

$$T = \frac{1}{2} \sum_{i=1}^N m_i \dot{\mathbf{r}}_i^2 = \frac{1}{2} M \dot{\mathbf{R}}^2 + \frac{1}{2} \sum_{i=1}^N m_i (\mathbf{r}_i')^2.$$

If the forces acting on the system

is *conservative*, in other words, if $\mathbf{F}_i = \mathbf{F}_i^e + \sum_{j=1}^N \mathbf{F}_{ij} = -\frac{\partial V}{\partial \mathbf{r}_i}, \quad i = 1, \dots, N$;

then the total mechanical energy $E = T + V(\mathbf{r}_1, \dots, \mathbf{r}_N)$ of the system is conserved.

The potential of the system has the form: $V = \sum_{i=1}^N V_i(\mathbf{r}_i) + \sum_{1 \leq i < j \leq N} V_{ij}(\mathbf{r}_i, \mathbf{r}_j)$.

Defining $V_{ij}(\mathbf{r}_i, \mathbf{r}_j) = U_{ij}(\mathbf{r}_i - \mathbf{r}_j) \Rightarrow V = \sum_{i=1}^N V_i(\mathbf{r}_i) + \sum_{1 \leq i < j \leq N} U_{ij}(\mathbf{r}_i - \mathbf{r}_j)$.

3 Central forces

3.1 Basic definitions

Conditions for a central force

$$\mathbf{F}_{12} = f(|\mathbf{r}_1 - \mathbf{r}_2|) \frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|} \Rightarrow \mathbf{F}(\mathbf{r}) = \mu \mathbf{r} = f(r) \frac{\mathbf{r}}{r}, \text{ in such case the following}$$

$$\text{equations hold: } \ddot{r} - r\dot{\theta}^2 = \frac{1}{\mu} \ddot{r}, \quad r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$$

3.2 Conserved quantities

$$\mathbf{L} = \mu \mathbf{r} \times \dot{\mathbf{r}} = \mu r^2 \dot{\theta} \mathbf{e}_\theta, |\mathbf{L}| = L_z = \dot{\theta} r^2 \mu > 0$$

$$E = \frac{1}{2} \mu \dot{\mathbf{r}}^2 + U(r) = \frac{1}{2} \mu (\dot{r}^2 + \dot{\theta}^2 r^2) + V\left(\frac{1}{r}\right)$$

Law of areas

$$A(\theta) = \frac{1}{2} \int_{\theta_0}^{\theta} r^2(\alpha) d\alpha,$$

$$\dot{A} = \frac{dA}{d\theta} \dot{\theta} = \frac{1}{2} r^2 \dot{\theta} = \frac{L_z}{2\mu} \in \mathbb{R}$$

3.3 Important equations

Binet's eqn.

$$f(r) = -\frac{L_z^2}{\mu r^2} \left(u'' + u \right)$$

Other equations

$$U(r) = V(r) + \frac{L_z^2}{2\mu r^2}, \quad -\frac{\partial}{\partial r} \left(\frac{L_z^2}{2\mu r^2} \right) = \frac{\mu v_\theta^2}{r} \mathbf{e}_r$$

$$t = \pm \sqrt{\frac{\mu}{2}} \int_{r_1}^{r_2} \frac{dr}{r} = \frac{\mu}{L_z} \int_{r_1}^{r_2} r^2(\theta) d\theta,$$

$$\theta = \frac{L_z}{\mu} \int_{r_1}^{r_2} \frac{dr}{r^2} = \pm \frac{L_z}{2\mu} \int_{r_1}^{r_2} \frac{du}{\sqrt{E - U\left(\frac{1}{u}\right)}},$$

$$\tau_r = \sqrt{2\mu} \int_{r_1}^{r_2} \frac{dr}{r}, \quad \Delta \theta = \sqrt{\frac{2\mu}{L_z}} \int_{r_1}^{r_2} \frac{1}{r^2} = \frac{1}{\sqrt{E - U\left(\frac{1}{r}\right)}},$$

An orbit is periodic $\Leftrightarrow \Delta \theta = q\pi, q \in \mathbb{Q}$.

$$A_r = \pi r^2 = \pi a b = \dot{A} \cdot \tau = \frac{L_z}{2\mu} \tau,$$

$$v = \frac{L_z}{\mu} \sqrt{u^2 + u'^2}$$

3.4 Kepler's potential

Force field and Binet's eqn.

$$\mathbf{F}(\mathbf{r}) = -\frac{\mu k}{r^2} \mathbf{e}_r, V(r) = -\frac{k}{r}, \mathbf{F} = -GM \frac{\mathbf{r}}{r^3}, u'' + u = \frac{\mu k}{L_z^2}$$

Equation of trajectory and its classifications

$$u = \frac{\mu k}{L_z^2} (1 + e \cos(\theta - \theta_0)), \quad e, \theta_0 \in \mathbb{R}.$$

$$r = \frac{L_z^2}{1 + e \cos(\theta)}, \quad \alpha \equiv \frac{L_z^2}{\mu k}, \quad e = \sqrt{1 + \frac{2EL_z^2}{\mu k^2}} \geq 0$$

$$\begin{cases} e > 1 & \Leftrightarrow e > 0 & \Rightarrow \text{hyperbola} \\ e = 1 & \Leftrightarrow E = 0 & \Rightarrow \text{parabola} \end{cases}$$

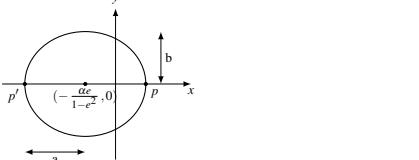
$$\begin{cases} e \in (0, 1) & \Leftrightarrow E \in \left(-\frac{\mu k^2}{2L_z^2}, 0 \right) & \Rightarrow \text{ellipse} \\ e = 0 & \Leftrightarrow E = -\frac{\mu k^2}{2L_z^2} & \Rightarrow \text{circle} \end{cases}$$

3.5 Planetary motion

Study of ellipses

$$\mathbf{C} \equiv \{ (1 - e^2)(x + \frac{a\epsilon}{1 - e^2})^2 + y^2 = \frac{a^2}{1 - e^2} \}$$

$$a = \frac{a}{1 - e^2}, \quad b = -\frac{a}{\sqrt{1 - e^2}}, \quad c = \epsilon a = \epsilon a, \quad e = \epsilon,$$



$$E = -\frac{k}{2a}, \quad (r) = \left(1 + \frac{c}{a} \right) a,$$

$$\tau = 2\pi \sqrt{\frac{\mu}{k}} \frac{a^3/2}{k} = \pi k \sqrt{\frac{\mu}{2}} |E|^{-3/2} = \frac{2\pi a^{3/2}}{\sqrt{GM_\odot}} \simeq \frac{2\pi a^{3/2}}{\sqrt{GM_\odot}}$$

$$p = \frac{a}{1 + e} = a(1 - e), \quad p' = \frac{a}{1 - e} = a(1 + e).$$

$$v^2 = \frac{k^2}{L_z^2} (1 + e^2 + 2e \cos \theta) \Rightarrow \begin{cases} v_p = \frac{k}{L_z} (1 + e) = \sqrt{\frac{k}{\mu p} (1 + e)} \\ v_{p'} = \frac{k}{L_z} (1 - e) = \sqrt{\frac{k}{\mu p'} (1 - e)} \end{cases}$$

4 Lagrangian & Hamiltonian

4.1 Basic Definitions

$$F[y] = \int_{x_1}^{x_2} dx, \text{ where } F \text{ is functional, and } f \text{ is density.}$$

4.2 Important equations and theorems

Euler-Lagrange equations:

$$\frac{d}{dt} \frac{\partial f}{\partial \dot{y}} - \frac{\partial f}{\partial y} = 0; \quad \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0. \quad \text{If } \frac{\partial f}{\partial t} = 0 \Rightarrow y' \frac{\partial f}{\partial y'} - f \text{ is constant. If}$$

$$\frac{\partial f}{\partial t} = 0 \Rightarrow \frac{\partial f}{\partial t} \text{ is constant. Two densities differing by a total derivative have the same E-L eqn.}$$

Hamilton's principle for N particles:

$$L(\mathbf{r}_1, \dots, \mathbf{r}_N, \mathbf{t}; \dot{\mathbf{r}}_1, \dots, \dot{\mathbf{r}}_N) = \frac{1}{2} \sum_i m_i \dot{\mathbf{r}}_i^2 - V(\mathbf{r}_1, \dots, \mathbf{r}_N).$$

$$\text{If } \frac{\partial L}{\partial t} = 0 \Rightarrow \sum_i \dot{\mathbf{r}}_i \cdot \frac{\partial L}{\partial \dot{\mathbf{r}}_i} - L = T + V \text{ conserved.}$$

$$\text{If } \frac{\partial L}{\partial \mathbf{r}_i} = 0 \Rightarrow \frac{\partial L}{\partial \mathbf{r}_i} \text{ is conserved.}$$

Canonical momenta:

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \Rightarrow \dot{p}_i = \frac{\partial L}{\partial q_i}; \quad \text{If } p_i = 0 \text{ then the coordinate } q_i \text{ is cyclic or ignorable.}$$

Hamiltonian:

$$H = \sum_i p_i \dot{q}_i - L = \sum_i p_i \dot{q}_i - L = T + V (\Leftarrow \text{If natural}).$$

Hamilton's canonical eqns.:

$$\dot{q}_i = \frac{\partial H}{\partial p_i}(\mathbf{r}, \mathbf{q}, \mathbf{p}); \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}(\mathbf{r}, \mathbf{q}, \mathbf{p})$$

Jacobi-Poisson Theorem

If f and g are two first integrals of H 's canonical equations, so is $\{f, g\}$.

4.3 Procedures

Find the EOM of N particles with / holonomic constraints:

1. Introduce $3N - l = n$ generalized coordinates $(q_1, \dots, q_n) = \mathbf{q}$ to the non-constraint manifold $\{\Phi_1, \dots, \Phi_l(\mathbf{r}, \mathbf{q}) = 0\}$.

2. Express $T = \frac{1}{2} \sum_i m_i \dot{\mathbf{r}}_i^2$ and V in terms of $(\mathbf{r}, \mathbf{q}, \dot{\mathbf{q}}) \Rightarrow$

$$T \equiv T(\mathbf{r}, \mathbf{q}, \dot{\mathbf{q}}); \quad V \equiv V(\mathbf{r}, \mathbf{q}).$$

$$\text{3. E-L equations of } L = T - V \text{ are the } n \text{ equations of motion in coordinates } q_i \Rightarrow \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0, \quad i = 1, \dots, n.$$

4. Constraint forces on i -th particle are expressed as:

$$\mathbf{F}^c = m_i \ddot{\mathbf{r}}_i + \frac{\partial V}{\partial \mathbf{r}_i}, \quad i = 1, \dots, N.$$

Write Hamilton's canonical equations:

1. Calculate the canonical momenta $p_i = \frac{\partial L}{\partial \dot{q}_i}(\mathbf{r}, \mathbf{q}, \dot{\mathbf{q}}), i = 1, \dots, n$.

2. Express the generalized velocities as $\dot{q}_i \equiv \dot{q}_i(\mathbf{r}, \mathbf{q}, \mathbf{p}), i = 1, \dots, n$.

3. Compute the Hamiltonian $H(\mathbf{r}, \mathbf{q}, \mathbf{p}) = \mathbf{p} \cdot \dot{\mathbf{q}}(\mathbf{r}, \mathbf{q}, \mathbf{p}) - L(\mathbf{r}, \mathbf{q}, \mathbf{p})$.

4. The partial derivatives of H w.r.t the variables q_i, p_i are the canonical equations. The first n equations are the ones in step 2, the rest are $\dot{p}_i = -\frac{\partial H}{\partial q_i}, i = 1, \dots, n$. There are in total $2n$ equations.

4.4 Hamiltonian conservations

Steiner's theorem

The inertia moment computed w.r.t a point P fixed w.r.t body axes is given as: $(I_P)_{ij} = \sum m a_i (\delta_{ij} j_a^2 - \bar{x}_{ai} \bar{x}_{aj})$. Taking into account $\bar{\mathbf{r}}_{\mathbf{A}} = \mathbf{r}_{\mathbf{A}} - \mathbf{a}$, $\mathbf{a} = \overrightarrow{CP} \Rightarrow (I_P)_{ij} = I_{ij} + M(\mathbf{a}^2 \delta_{ij} - a_i a_j)$.

If P is also fixed w.r.t S' , then, placing $O' = P$, we can reduce the expressions: $\mathbf{V} = \boldsymbol{\omega} \times \mathbf{R}$, $\mathbf{v}'_{\mathbf{A}} = \boldsymbol{\omega} \times \mathbf{r}'_{\mathbf{A}}$, $\mathbf{L} = \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{I}$, $(I' = I_P)$, $T = \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{I}' \cdot \boldsymbol{\omega}$. I' can be computed from I : $I'_{ij} = I_{ij} + M(\mathbf{R}^2 \delta_{ij} - X_i X_j)$, where X_k are components of the vector \mathbf{R} .
Principal axes of inertia
Let $A \in \text{SO}(3)$ be a constant change of basis matrix, and $\tilde{\mathbf{e}}_i = \sum_j A_{ji} \mathbf{e}_j$.

$\Rightarrow x = A\tilde{x}$, $\tilde{I} = A^{-1} I A = A^T I A$.

Since I is a symmetric matrix, there is always a matrix $A \in \text{SO}(3)$, s.t. $\tilde{I} = A^T I A$ is a diagonal matrix, in other words, the moment of inertia w.r.t the alternate body axes $\tilde{\mathbf{e}}_i$ (which are eigenvectors) is expressed as:

$\tilde{I} = \begin{bmatrix} I_1 & & \\ & I_2 & \\ & & I_3 \end{bmatrix}$, with I_i the eigenvalues corresponding to $\tilde{\mathbf{e}}_i$.

In such case, $\tilde{\mathbf{e}}_i$ are **principal axes of inertia**, and I_i are **principal moments of inertia**, which are the roots of the characteristic polynomial of I .

If \mathbf{e}_i are p.a.i. then: $\mathbf{L}_{\text{CM}} = \sum I_i \omega_i \mathbf{e}_i$, $T_{\text{rot}} = \frac{1}{2} \sum I_i \omega_i^2$.

If the origin of a set of p.a.i. are also fixed in S' , then \mathbf{L} , T can be calculated with the expressions above, replacing I by I' w.r.t. the point O' .

Classification of r.b. in terms of the multiplicity of the eigenvalues I_i

Type of symmetry	Definition
Asymmetric tops:	$I_i \neq I_j, \forall i \neq j$.
Axially symmetric tops:	$I_i = I_j \neq I_k$.
Spherically symmetric tops:	$I_i = I_j, \forall i, j$.

Deduce I' of a rigid body by its invariance under transformations

Note: both Ω and ρ have to be invariant under the transformations.

Inv. under	Behaviour of I_i
$x_i \mapsto -x_i$	$I_{ij} = 0, \forall j \neq i$
$x_i \mapsto x_j$	$I_{ii} = I_{jj}$, $I_{ik} = I_{jk}, k \neq i, j$

In particular, for a *solid of revolution* with homogeneous density, taking x_3 as the axis of rotation and arbitrary perpendicular axes x_1, x_2 , it satisfy the following invariances: $x_1 \mapsto -x_1, x_2 \mapsto -x_2, x_1 \mapsto x_2$, and thus:

$I_{11} \equiv I_1 = I_2 \equiv I_{22}, I_{ij} = 0$ ($i \neq j$), with:

$$I_1 = \pi \rho \int_{z_1}^{z_2} z^2 f^2(z) dz + \frac{\pi \rho}{2} \int_{z_1}^{z_2} f^4(z) dz, \quad I_{33} \equiv I_{33} = \frac{\pi \rho}{2} \int_{z_1}^{z_2} f^4(z) dz.$$

6.4 EOM of a rigid body

EOM in an inertial frame

If $\Lambda \neq 0$, defining $\mathbf{X} = \frac{1}{\Lambda} \sum \lambda_a \mathbf{r}'_a = \frac{1}{\Lambda} \sum \lambda_a (\mathbf{R} + \mathbf{r}_a) = \mathbf{R} + \frac{1}{\Lambda} \sum \lambda_a \mathbf{r}_a$

A r.b. is in equilibrium in an $S \iff \mathbf{F} = \mathbf{N} = 0 \iff \mathbf{F} = \mathbf{N}_{\text{CM}} = 0$

Forces due to a constant field \mathbf{f}
 $\lambda_a \equiv$ "charge" of a particle $\Rightarrow \mathbf{F}_{\mathbf{A}} = \lambda_a \mathbf{f} \Rightarrow \mathbf{F} = \sum_a \lambda_a \mathbf{f} = \Lambda \mathbf{f}$.

$$\mathbf{N} = \left(\sum \lambda_a \mathbf{r}'_a \right) \times \mathbf{f}. \quad (\downarrow \text{ Note that } \mathbf{X} \text{ is fixed in the body})$$

If $\Lambda \neq 0$, defining $\mathbf{X} = \frac{1}{\Lambda} \sum \lambda_a \mathbf{r}'_a = \frac{1}{\Lambda} \sum \lambda_a (\mathbf{R} + \mathbf{r}_a) = \mathbf{R} + \frac{1}{\Lambda} \sum \lambda_a \mathbf{r}_a$ we have that $\mathbf{N} = \mathbf{X} \times \mathbf{F} \Rightarrow$ If $\Lambda \neq 0$, then the total torque of the external forces coincides with the torque of the total external forces applied at the *centre of charge*. (For the grav. field \mathbf{g} , $\lambda = m$ and C. charge is equal to CM). Torque of the external forces w.r.t. CM is $\mathbf{CM} = (\mathbf{X} - \mathbf{R}) \times \mathbf{F}$ (for \mathbf{g} it is zero). When $\Lambda = 0$ we use the former equation for \mathbf{N} and the torque of the external forces is independent of the reference point, since $\mathbf{F} = \Lambda \mathbf{f} = 0$.

Euler's equations

Euler's equations is the system $\begin{cases} I_1 \dot{\omega}_1 - (I_2 - I_3) \omega_2 \omega_3 = N_1, \\ I_2 \dot{\omega}_2 - (I_3 - I_1) \omega_1 \omega_3 = N_2, \\ I_3 \dot{\omega}_3 - (I_1 - I_2) \omega_1 \omega_2 = N_3, \end{cases}$ where \mathbf{N} and I are computed w.r.t. CM or a point fixed in the body and S' . $\boldsymbol{\omega}$ and \mathbf{N} are in a frame of p.a.i. (in general not inertial). If $\mathbf{N} = 0$ and the origin O' of S' is a point in the body, $\mathbf{L}_{O'} = \mathbf{L}$ and T are conserved. Similarly, if $\mathbf{N}_{\text{CM}} = 0$ then \mathbf{L}_{CM} and T_{rot} are conserved.

6.5 Inertial motion of a symmetric top

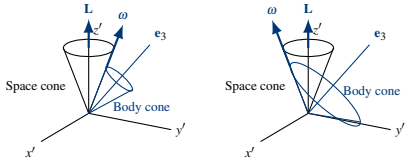
The angular velocity $\boldsymbol{\omega}$ of a set of axes w.r.t. to another is *additive*.

For an axially (\mathbf{e}_3) symmetric r.b., by E's eqns, the following conditions holds: $\omega_3, \Omega = \frac{I_3 - I_1}{I_3} \omega_3, \omega_1 = \sqrt{\omega_1^2 + \omega_2^2}, \alpha = \arctan(\omega_1/\omega_3)$ constants. In the frame of body axes, the vector $\boldsymbol{\omega}$ rotates around \mathbf{e}_3 with Ω (traces out the body cone fixed w.r.t. body axes). The angular velocity $\Omega > 0$ for $I_3 > I_1$, and negative otherwise.

Relative to body axes, \mathbf{L} rotates with Ω around \mathbf{e}_3 . \mathbf{L} fixed relative to S' . Relative to S' , $\boldsymbol{\omega}$ traces out the **space cone** around \mathbf{L} (fixed in S') with Ω_P .

The angle between \mathbf{e}_3 and $\boldsymbol{\omega}$ is α , between \mathbf{L} and \mathbf{e}_3 is $\theta = \arctan \frac{I_3}{I_1} \tan \alpha$.

$$\Omega_P = \frac{I_1}{I_1} = \frac{I_1}{I_1} \sqrt{1 + \frac{I_3^2 - I_1^2}{I_1^2} \cos^2 \alpha}, \quad \boldsymbol{\omega} = \Omega_P \frac{\mathbf{L}}{L} - \Omega \mathbf{e}_3.$$



6.6 Miscellaneous

In general, for a solid of revolution rolling without sliding down a inclined (α) plane around the principal axis \mathbf{e}_3 , the horizontal acceleration is expressed as:

$$\ddot{x} = \frac{g \sin \alpha}{1 + \frac{I_3}{M a^2}} \Rightarrow.$$

Inertia tensor of some regular bodies

Body	Definition	Inertia tensor
Solid Sphere	$x^2 + y^2 + z^2 \leq r^2$	$\frac{83}{320} m r^2 \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$
Hollow Sphere	$x^2 + y^2 + z^2 = r^2$	$\frac{83}{320} m r^2 \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$
Cuboid	$\begin{cases} -\frac{d}{2} \leq x \leq \frac{d}{2} \\ -\frac{b}{2} \leq y \leq \frac{b}{2} \\ -\frac{h}{2} \leq z \leq \frac{h}{2} \end{cases}$	$\frac{m}{12} \begin{bmatrix} b^2 + d^2 & & \\ & d^2 + h^2 & \\ & & b^2 + h^2 \end{bmatrix}$
Cylinder	$\begin{cases} x^2 + y^2 \leq r^2 \\ -\frac{h}{2} \leq z \leq \frac{h}{2} \end{cases}$	$\frac{m}{12} \begin{bmatrix} 3r^2 + h^2 & & \\ & 3r^2 + h^2 & \\ & & 6r^2 \end{bmatrix}$
Rod about end	$\begin{cases} x^2 + y^2 \leq r^2 \\ 0 \leq z \leq h \\ h \gg r \end{cases}$	$\frac{1}{12} m l^2 \begin{bmatrix} 1 & & \\ & 0 & \\ & & 1 \end{bmatrix}$
Rod about center	$\begin{cases} x^2 + y^2 \leq r^2 \\ -\frac{h}{2} \leq z \leq \frac{h}{2} \\ h \gg r \\ \omega \parallel \mathbf{x} \end{cases}$	$\frac{1}{12} m l^2 \begin{bmatrix} 1 & & \\ & 0 & \\ & & 1 \end{bmatrix}$

7 Special relativity

7.1 Principle of special relativity

In Galilean relativity, the acceleration of a particle is the same in all inertial frames \Rightarrow all inertial frames are *equivalent* from the point of view of Newtonian mechanics.

Postulates of special relativity:

- The laws of physics are the same in all inertial frames (**relativity principle**).
- The speed of electromagnetic waves in vacuum is universal: $c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$.
- (Implication of 1. & 2.) speed of electromagnetic waves in vacuum is c in all inertial frames.

7.2 Lorentz transformations

Basic equations of transformation

We will use the notation $x_0 \equiv ct$, $x_\mu \equiv ct'$ as the "time coordinate" of space-time. The Lorentz factor for a speed of v is defined as:

$$\gamma(v) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}, \text{ dimensionless.}$$

The L . transformation relating the frames x_μ and x'_μ , (the velocity of O' w.r.t. O is $\mathbf{v} \mathbf{e}_1$), is:

$$t' = \gamma(v) \left(t - \frac{v x_1}{c^2} \right), \quad x'_1 = \gamma(v) (x_1 - vt), \quad x'_k = x_k, \quad k = 2, 3$$

Consequences of L.T.

- The relative speed between two inertial frames is strictly less than c in vacuum.
- The speed of all mass is less than c .
- The propagation speed any information cannot exceed c .

General transformations

The general L.T. with \mathbf{v} the velocity of O' w.r.t. S is given by:

$$t' = \gamma(v) \left(t - \frac{\mathbf{x} \cdot \mathbf{v}}{c^2} \right), \quad \mathbf{x}' = \mathbf{x} + (\gamma(v) - 1) \frac{\mathbf{v} \mathbf{v}}{v^2} - \gamma(v) \mathbf{v} t$$

(*Relativistic law for the addition of velocities*). For a particle with constant velocity $\mathbf{u} = u \mathbf{e}_1$ w.r.t. S , its velocity w.r.t. S' (whose origin moves at constant velocity $\mathbf{v} \mathbf{e}_1$) are also constant, with components:

$$u_1 = \frac{u + v}{1 + \frac{uv}{c^2}}, \quad u_k = \frac{u_k}{\gamma(v) \left(1 + \frac{v u_1}{c^2} \right)}, \quad (k = 2, 3),$$

the reversed relations can be obtained by changing the signs and switching primes.

Intervals

The quadratic form $c^2 t^2 - \mathbf{x}^2 = x_0^2 - \mathbf{x}^2$ is *invariant* under L.T.. The square of the **interval** between two events with s-t coordinates x_μ and $x_\mu + \Delta x_\mu$ is defined as:

$$\Delta s^2 = c^2 \Delta t^2 - \sum_{i=1}^3 \Delta x_i^2 = \Delta x_0^2 - \sum_{i=1}^3 \Delta x_i^2 = \Delta x_0^2 - \Delta \mathbf{x}^2;$$

said interval is invariant under L.T. $\Rightarrow \Delta s^2 = \Delta x_0^2 - \Delta \mathbf{x}^2 = \Delta (x'_0)^2 - \Delta (\mathbf{x}')^2$. The interval between two events is *time-like* if $\Delta s^2 > 0$, *light-like* if $\Delta s^2 = 0$, and *space-like* if $\Delta s^2 < 0$.

If the interval between two events is *time-like*:

Let the original S s.t. $\Delta x_2, \Delta x_3 = 0$, and a second S' with velocity $\frac{\Delta x_1}{\Delta x_0} \mathbf{e}_1$. Then, in S' the two events occur at the same point in space, with time interval (**proper time lapse**) $\Delta \tau = \frac{\Delta s}{c} = \Delta t \sqrt{1 - \frac{\Delta x_1^2}{\Delta x_0^2}}$.

If the interval between two events is *space-like*:

Let the original S' s.t. $\Delta x_2, \Delta x_3 = 0$, and a second S' with velocity $\frac{c^2 \Delta t}{\Delta x_1} \mathbf{e}_1 = \frac{c \Delta x_0}{\Delta x_1} \mathbf{e}_1$. Then in S' the two events occur simultaneously, with distance (**proper distance**) $\sqrt{-\Delta s^2} = |\Delta \mathbf{x}'|$. The proper distance is less or equal to the spatial distance $|\Delta \mathbf{x}|$ in any other inertial frame.

Minkowski product

For two s-t coordinates in S , $\mathbf{x} = (x_0, \mathbf{x})$ and $\mathbf{y} = (y_0, \mathbf{y}) \in \mathbb{R}^4$, their *Minkowski product*:

$$x \odot y \equiv x_0 y_0 - \mathbf{x} \cdot \mathbf{y} \quad \Rightarrow \quad x^2 \equiv x \odot x = x_0^2 - \mathbf{x}^2, \quad \Delta x \odot \Delta x = \Delta s^2$$

is also invariant under L.T.. The vector space (\mathbb{R}^4, \odot) is the *Minkowski space*.

Lorentz group

The procedure of relating the s-t coordinates of an event in two reference frames S and S' (moves at \mathbf{v}) is the following:

- Consider S'' at rest w.r.t. S , whose x'_1 is in the direction of $\mathbf{v} \Rightarrow x'' = R_1 x$, where $R_1 \in \mathbb{R}^{4 \times 4}$ but only rotates the spatial coordinates.
- S'' is the frame that moves with $\mathbf{v} = v \mathbf{e}_1$ away from $S'' \Rightarrow x'' = L(v) x''$. $L(v)$ is the "basic" L.T.
- The frames S'' and S' are related by a rotation $\Rightarrow x' = R_2 x''$. Combining everything we get

$x' = R_2 L(v) R_1 x \equiv \Lambda x$
the transformation $\Lambda \in \mathbb{R}^{4 \times 4}$ is the *general Lorentz transformation*. It is the most general form of L.T. that relates the space-time coord. of an event in two reference frames whose space-time origin coincide ($t = x_1 = 0 \iff t' = x'_1 = 0$). If the space-time origin of the frames differs by a vector $\mathbf{a} \in \mathbb{R}^4$, then the transformation $x' = \Lambda x + \mathbf{a}$ is called the *Poincaré tr.*. The basic L.T. can be expressed as a matrix:

$$L(v) = \begin{bmatrix} \gamma(v) & -\beta(v) \gamma(v) & 0 & 0 \\ -\beta(v) \gamma(v) & \gamma(v) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \beta(v) \equiv \frac{v}{c}$$

The bilinear form associated with the M. product is:

$$G = \begin{bmatrix} 1 & & \\ & -1 & \\ & & -1 \end{bmatrix}, \quad \Rightarrow \quad x \odot y = x^T G y, \quad \Rightarrow \quad \Lambda^T G \Lambda = G$$

The Minkowski product (and the square of intervals), are invariant under general Lorentz transformations. The set of matrices Λ that satisfy the equation above form the *Lorentz group*. Performing the change of variable:

$$\beta(v) = \tanh \phi \Rightarrow \gamma(v) = \cosh \phi \Rightarrow L(v) = \begin{bmatrix} \cosh \phi & -\sinh \phi & 0 & 0 \\ -\sinh \phi & \cosh \phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

And the result of successive Lorentz boosts with speeds $v_1 = c \tanh \phi_1$ and $v_2 = c \tanh \phi_2$ is another L . boost with rapidity $\phi_1 + \phi_2$:
 $v = c \tanh (\phi_1 + \phi_2) \equiv \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}$ similar to the eqn. of velocity of a ptcl.

7.3 Dilations and contractions

The **proper time** measured in S is larger than the **coordinate time** measured in S' . In other words, from an observer at rest with S , the time recorded by a watch at rest with S' , t' , is slower than the proper time t that he has measured. We say that t' dilated, the dilation effect is symmetrical in **inertial frames**:

$$t \equiv \frac{t'}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{t'}{\gamma(v)} > t'$$

For a particle following a trajectory $\mathbf{r} = \mathbf{r}(t)$, the proper time increment (time according to an observer at rest with said particle) is expressed as:

$$d\tau = \sqrt{1 - \frac{v^2(t)}{c^2}} \Rightarrow \Delta \tau(C) = \int_{t_1}^{t_2} \sqrt{1 - \frac{v^2(t)}{c^2}} dt = \int_{t_1}^{t_2} \sqrt{1 - \frac{v^2}{c^2}} dt$$

being C the path taken by the particle. $\Delta t \leq \Delta \tau$ and is equal only if $\mathbf{v} = 0$, $\forall t$.

Lorentz-Fitzgerald contraction has to do with contraction of an object that's at rest with S' but in motion w.r.t. to S in the direction of its velocity. In mathematical terms, the length component in the direction of its movement, x_1 , is contracted:

$$\Delta x'_1 \equiv l_0 = \gamma(v) (\Delta x_1 - v \Delta t) \equiv \gamma(v) l \Rightarrow l = \frac{l_0}{\gamma(v)} = l \sqrt{1 - \frac{v^2}{c^2}} < l_0,$$

where l_0 is the **rest length** or the length directly measured by an observer at rest with the object.

7.4 Four-velocity and four-momentum

The *four-velocity* is defined as:

$$u = \frac{dx}{d\tau} \in \mathbb{R}^4, \quad x \equiv \text{"s-t coordinates"}, \quad d\tau = \sqrt{1 - \frac{v^2}{c^2}} dt \equiv \frac{dt}{\gamma(v)}$$

and it has the following properties:

$$u = \gamma(v) (c, \mathbf{v}) \in \mathbb{R}^4, \quad u' = \Lambda u, \quad u^2 = c^2.$$

The four-momentum is: $p = m u \Rightarrow p^2 = m^2 c^2$

Here we will use the vector $\mathbf{p} = (p_1, p_2, p_3) \equiv m \gamma(v) \mathbf{v} \in \mathbb{R}^3$ which coincides with the non-r. momentum only when $v \rightarrow 0$. Then:

$$p_0 = \sqrt{\mathbf{p}^2 + m^2 c^2} \Rightarrow \mathbf{v} = \frac{c \mathbf{p}}{p_0}$$

$$T = c p_0 - m c^2 = m c^2 (\gamma(v) - 1) \quad \Rightarrow \quad p_0 = \frac{1}{c} (m c^2 + T)$$

Conservation of four-momentum

$p = \text{const.} \iff c p_0 = m c^2 + T = \text{const.} \iff p_i = m \gamma(v) v_i = \text{const.}$
The total four-momentum of a system of N particles is:

$$P = \sum_{n=1}^N p_n \equiv (P_0, \mathbf{P}), \quad P_0 = \sum_{n=1}^N p_{n,0} \equiv c \sum_{n=1}^N m_n \gamma(v_n),$$

$$\mathbf{P} = \sum_{n=1}^N p_n \equiv \sum_{n=1}^N m_n \gamma(v_n) \mathbf{v}_n.$$

The conservation of P during the collision of said N particles is verified even when the collision is not elastic:

$$P_i = P_j \iff (M c^2 + T)_i = (M c^2 + T)_j$$

During the process of collision, The mass and the kinetic energy can vary to-gether with the relation:

$$\Delta T = -\Delta (M c^2)$$

The total relativistic energy can be defined as:

$$E = c P_0 = m c^2 + T = m c^2 \gamma(v) = c \sqrt{\mathbf{p}^2 + m^2 c^2} = m c^2 \sqrt{1 + \frac{\mathbf{p}^2}{m^2 c^2}}$$

$$\Rightarrow P = (E/c, \mathbf{P}) \Rightarrow \mathbf{v} = \frac{c^2 \mathbf{P}}{E}$$

7.5 Massless particles

For particle with mass $m \rightarrow 0$, its energy is $E = c |\mathbf{p}|$ and its four-momentum: $p = (|\mathbf{p}|, \mathbf{p}) \Rightarrow \mathbf{v} = c \frac{\mathbf{p}}{|\mathbf{p}|}$.

Motion of a photon

Photon is the only massless particle known. The relation between its energy and the frequency ω of its associated EM wave is given as:

$$E = h \omega \equiv h v = \frac{h c}{\lambda}, \quad \lambda \equiv \text{"wavelength"}$$

The wave vector is defined as:

$$\boldsymbol{\omega} = c |\mathbf{k}|, \quad \mathbf{v} = c \frac{\mathbf{k}}{|\mathbf{k}|} \quad \Rightarrow \quad \mathbf{p} = \hbar \mathbf{k}$$

We can