# 1 Coordinate systems

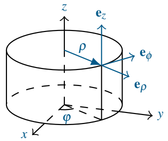
## 1.1 Cylindrical

**Basis vectors**

$\mathbf{e}_\rho = \left|\frac{\partial\mathbf{r}}{\partial\rho}\right|^{-1}\frac{\partial\mathbf{r}}{\partial\rho} = (\cos\varphi,\sin\varphi,0)$

$\mathbf{e}_\varphi = \left|\frac{\partial\mathbf{r}}{\partial\varphi}\right|^{-1}\frac{\partial\mathbf{r}}{\partial\varphi} = (\sin\varphi,-\cos\varphi,0)$

$\mathbf{e}_z = \left|\frac{\partial\mathbf{r}}{\partial z}\right|^{-1}\frac{\partial\mathbf{r}}{\partial z} = (0,0,1)$




**Velocity vectors**

$\dot{\mathbf{e}}_\rho = \dot\varphi\mathbf{e}_\varphi, \quad \dot{\mathbf{e}}_\varphi = -\dot\varphi\mathbf{e}_\rho, \quad \dot{\mathbf{e}}_z = 0,$

$v_\rho = \dot\rho, \quad v_\varphi = \rho\dot\varphi, \quad v_z = \dot z.$

**Acceleration vectors**

$a_\rho = \ddot\rho - \rho\dot\varphi^2, \quad a_\varphi = \rho\ddot\varphi + 2\dot\rho\dot\varphi, \quad a_z = \ddot z.$

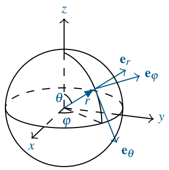
## 1.2 Spherical

**Basis vectors**

$\mathbf{e}_r = \left|\frac{\partial\mathbf{r}}{\partial r}\right|^{-1}\frac{\partial\mathbf{r}}{\partial r}$
$= (\sin\theta\cos\varphi,\sin\theta\sin\varphi,\cos\theta) = \frac{\mathbf{r}}{r},$

$\mathbf{e}_\theta = \left|\frac{\partial\mathbf{r}}{\partial\theta}\right|^{-1}\frac{\partial\mathbf{r}}{\partial\theta}$
$= (\cos\theta\cos\varphi,\cos\theta\sin\varphi,-\sin\theta)$

$\mathbf{e}_\varphi = \left|\frac{\partial\mathbf{r}}{\partial\varphi}\right|^{-1}\frac{\partial\mathbf{r}}{\partial\varphi} = (\sin\varphi,\cos\varphi,0).$




**Velocity vectors**

$\dot{\mathbf{e}}_r = \frac{\partial\mathbf{e}_r}{\partial\theta}\dot\theta + \frac{\partial\mathbf{e}_r}{\partial\varphi}\dot\varphi = \dot\theta\mathbf{e}_\theta + \sin\theta\dot\theta\mathbf{e}_\varphi,$

$\dot{\mathbf{e}}_\theta = \frac{\partial\mathbf{e}_\theta}{\partial\theta}\dot\theta + \frac{\partial\mathbf{e}_\theta}{\partial\varphi}\dot\varphi = -\dot\theta\mathbf{e}_r + \cos\theta\dot\varphi\mathbf{e}_\varphi,$

$\dot{\mathbf{e}}_\varphi = \frac{\partial\mathbf{e}_\varphi}{\partial\varphi}\dot\varphi = -(\cos\varphi,\sin\varphi,0)\dot\varphi = -\dot\varphi(\sin\theta\mathbf{e}_r + \cos\theta\mathbf{e}_\theta).$

$v_r = \dot r, \quad v_\theta = r\dot\theta, \quad v_\varphi = r\sin\theta\dot\varphi.$

**Acceleration vectors**

$a_r = \ddot r - r\dot\theta^2 - r\sin^2\theta\dot\varphi^2, \quad a_\theta = r\ddot\theta + 2\dot r\dot\theta - r\sin\theta\cos\theta\dot\varphi^2,$

$a_\varphi = r\sin\theta\ddot\varphi + 2\sin\theta\dot r\dot\varphi + 2r\cos\theta\dot\theta\dot\varphi.$

# 2 Newtonian Mechanics

## 2.1 Newton's Laws

**I.** In the absence of external forces, the momentum of a particle remains constant.
**II.** If an external force $\mathbf{F}$ acts on a particle, the rate of variation of its momentum is given by $\dot{\mathbf{p}} = \mathbf{F} \iff \mathbf{F} = m\ddot{\mathbf{r}}$ ($\Leftarrow$ eqn. of motion of a particle).

## 2.2 Conservation laws

**Basic definitions**

$\mathbf{L} = \mathbf{r}\times\mathbf{p} = m\mathbf{r}\times\dot{\mathbf{r}}, \quad \dot{\mathbf{L}} = \mathbf{N} = \mathbf{r}\times\mathbf{F} = 0 \iff \mathbf{F}\parallel\mathbf{r}.$ (Angular momentum)

$T = \frac{1}{2}m\dot r^2, \ \frac{dT}{dt} = m\dot{\mathbf{r}}\ddot{\mathbf{r}} = \mathbf{F}\dot{\mathbf{r}}$, $T$ is conserved $\iff \mathbf{F}\perp\dot{\mathbf{r}},\ \forall t.$ (Kinetic energy)

If $\exists g(t,\mathbf{r},\dot{\mathbf{r}})$ s.t. $\mathbf{F} = g(t,\mathbf{r},\dot{\mathbf{r}})\mathbf{r}$, then $\mathbf{F}$ is central. (Central force)

If $\exists V(\mathbf{r})$ s.t. $\mathbf{F}(\mathbf{r}) = -\nabla V(\mathbf{r})$, then $\mathbf{F}$ is conservative. (Conservative force)

If $\mathbf{F}$ conservative, then $E = T + V$ conserved. (Conservation of energy)

If $\exists V(t,\mathbf{r})$ s.t. $\mathbf{F}(t,\mathbf{r}) = -\frac{\partial V(t,\mathbf{r})}{\partial\mathbf{r}}$, then $\mathbf{F}$ is irrotational. (Irrotational force)

If $\mathbf{F}$ irrotational, then $\frac{dE}{dt} = \frac{\partial V}{\partial t}$.

**Gravitational and Electrostatic forces**

$\mathbf{F} = f(r)\mathbf{e}_r$

$f(r) = -\frac{GMm}{r^2}, \quad V(r) = -\frac{GMm}{r}.$ (Gravitational force)

$\Rightarrow \mathbf{a} = \frac{\mathbf{F}}{m} = -\frac{GM}{r^3}\mathbf{r}$ is independent of $m$

$f(r) = k\frac{qQ}{r^2}, \quad V(r) = k\frac{qQ}{r}.$ (Electrostatic force)

In general, for forces between a particle and a continuous distribution of mass/charge:

$\mathbf{F}(\mathbf{r}) = -Gm\int\rho(\mathbf{r}')\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\,d^3\mathbf{r} = -m\frac{\partial\Phi(\mathbf{r})}{\partial\mathbf{r}}, \ \Phi(\mathbf{r}) = -G\int\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d^3\mathbf{r}$

$\mathbf{F}(\mathbf{r}) = -kq\int\rho(\mathbf{r}')\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\,d^3\mathbf{r} = -q\frac{\partial\Phi(\mathbf{r})}{\partial\mathbf{r}}, \ \Phi(\mathbf{r}) = -k\int\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,d^3\mathbf{r}$

**Electromagnetic force**

$\mathbf{F}(t,\mathbf{r},\dot{\mathbf{r}}) = q(\mathbf{E}(t,\mathbf{r}) + \dot{\mathbf{r}}\times\mathbf{B}(t,\mathbf{r})),\ T + q\Phi(\mathbf{r}) \equiv$"electromechanical energy".

## 2.3 Motion of a particle in a one-dimensional potential

$\frac{1}{2}m\dot x^2 + V(x) = E, \dot x(m\ddot x - F(x)) = 0$, (if $\dot x \neq 0 \Rightarrow m\ddot x - F(x) = 0$).

Said ODE is an **autonomous system** (i.e. it's of the form $\ddot x = f(x,\dot x)$), and it's thus invariant under transformations $t\to t+c$ and $t\to -t$.

**Equilibrium, and trajectory**

**Equilibria** are the points $x_0\in\mathbb{R}$ s.t. $x(t) = x_0$ is a solution of the EOM $\Rightarrow \ddot x(t) = 0 \Rightarrow F(x(t)) = F(x_0) = -V(x_0) = 0$.

The **accessible region** for a given $E$ is the set $\{x\in\mathbb{R}\mid V(x)\leq E\}$, which, in general, are disjoint union of closed intervals.

The endpoints of the intervals, if not equilibriums, are the **turning points** of the trajectories: $x_i$ turning point $\iff V(x_i) = E,\ V'(x_i)\neq 0$.

In general, the relation between time and displacement is given as:

$$t - t_0 = \pm\sqrt{\frac{m}{2}}\int\frac{dx}{\sqrt{E-V(x)}} \Rightarrow t = \sqrt{\frac{m}{2}}\int_{x_0}^{x}\frac{ds}{\sqrt{E-V(s)}} \equiv \theta(x)$$

**Interval with two turning points**:

The time for the particle to go from $x_0$ to $x_1$ is $\frac{\tau}{2}$, being:

$$\tau = 2\theta(x_1) = \sqrt{2m}\int_{x_0}^{x_1}\frac{ds}{\sqrt{E-V(s)}}.$$

The function $\theta(x)$ is mon. increasing for $x\in(x_0,x_1)$, therefore $\theta$ is invertible and we can write $x$ in terms of t:

$$x(t) = \theta^{-1}(t), 0\leq t\leq\frac{\tau}{2}, \qquad x(t) = \theta^{-1}(\tau - t), \frac{\tau}{2}\leq t\leq\tau.$$

**Semi-infinite interval with turning point at one side** $[x_0,\infty)$:

For a particle with initial positon $x(0) = x_0$ and moving towards infinity, the time is:

$$t_\infty = \theta(\infty) = \sqrt{\frac{m}{2}}\int_{x_0}^{\infty}\frac{ds}{\sqrt{E-V(s)}}, \text{ it can be convergent or divergent.}$$

$x$ can be expressed as:

$$x(t) = \theta^{-1}(t), 0\leq t\leq t_\infty, \qquad x(t) = \theta^{-1}(-t), -t_\infty\leq t\leq 0.$$

**Accessible region is** $\mathbb{R}$:

The time for the particle to reach infinity from an initial positon $x_0$ is:

$$t_{\pm\infty} = \theta(\pm\infty) = \sqrt{\frac{m}{2}}\int_{x_0}^{\pm\infty}\frac{ds}{\sqrt{E-V(s)}}$$

$$x(t) = \theta^{-1}(t), t_{-\infty}\leq t\leq t_\infty, \qquad x(t) = \theta^{-1}(-t), -t_\infty\leq t\leq t_{-\infty}.$$

**Stability of equilibria, oscillation approximation**

An equilibrium is stable $\iff$ it is a *relative minimum* of the potential.

For an open ball centered at a stable eq. $(x_{eq}-\varepsilon, x_{eq}+\varepsilon)$, the Taylor expansion of $V$ is of the form:

$$V(x) = V(x_{eq}) + \frac{1}{2}V''(x_{eq})(x - x_{eq})^2 + O(x^3)$$

Then the motion of a particle within said open ball can be approximated to:

$$\ddot x = \frac{F(x)}{m} = -\frac{V'(x)}{m} \simeq -\frac{V''(x_{eq})}{m}(x - x_{eq}) \quad\Rightarrow\quad \ddot\xi + \omega^2\xi = 0$$

with $\xi \equiv x - x_{eq},\ \omega \equiv \sqrt{\frac{V''(x_{eq})}{m}}$.

The solution of this ODE is: $\xi = A\cos(\omega t + \alpha),\ A\in(0,|x - x_{eq}|)$, and the period of the oscillation is $\tau \simeq \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{V''(x_{eq})}}$.

## 2.4 Dynamics of a system of particles

The law of motion of $N$ elements with masses $m_i$, positions $\mathbf{r}_i$ is:

$$m_i\ddot{\mathbf{r}}_i = \sum_{j=1}^{N}\mathbf{F}_{ij} + \mathbf{F}_i^{(e)}, \quad i = 1,\ldots,N,$$

where $\mathbf{F}_{ij}$ are the forces exerted by $j$ on $i$, and $\mathbf{F}_i^{(e)}$ are the total external force exerted on $i$. The solution of this set of $N$ equations is unique if given the initial conditions

$$\mathbf{r}_i(t_0) = \mathbf{r}_{i_0}, \quad \dot{\mathbf{r}}_i(t_0) = \mathbf{v}_{i_0}, \quad i\in\{1,\ldots,N\}.$$

**Center of mass**

$$\mathbf{F}^{(e)} = \sum_{i=1}^{N}\mathbf{F}_i^{(e)}, \quad \sum_{i=1}^{N}m_i\ddot{\mathbf{r}}_i = \mathbf{F}^{(e)}, \quad \mathbf{R}\equiv M^{-1}\sum_{i=1}^{N}m_i\mathbf{r}_i, \quad M\ddot{\mathbf{R}} = \mathbf{F}^{(e)}.$$

**Conservation**

*Linear momentum:*

$\mathbf{P}\equiv\sum_{i=1}^{N}m_i\dot{\mathbf{r}}_i = M\dot{\mathbf{R}} \Rightarrow \dot{\mathbf{P}} = \mathbf{F}^{(e)} \quad\Rightarrow\quad \mathbf{P}$ conserved $\iff \mathbf{F}^{(e)} = 0.$

*Angular momentum:*

$\mathbf{L} = M\mathbf{R}\times\dot{\mathbf{R}} + \sum_{i=1}^{N}m_i\mathbf{r}_i'\times\dot{\mathbf{r}}_i' \equiv \mathbf{L}_{CM} + \sum_{i=1}^{N}m_i\mathbf{r}_i'\times\dot{\mathbf{r}}_i', \qquad \mathbf{r}_i = \mathbf{R} + \mathbf{r}_i',$

$\dot{\mathbf{L}} = \sum_{i=1}^{N}\mathbf{r}_i\times\mathbf{F}_i^{(e)} \equiv \mathbf{N}^{(e)} \quad\Rightarrow\quad \mathbf{L}$ conserved $\iff \mathbf{N}^{(e)} = 0$

*Kinetic energy, potential, and total mechanical energy:*

$T = \frac{1}{2}\sum_{i=1}^{N}m_i\dot{\mathbf{r}}_i^2 = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\sum_{i=1}^{N}m_i(\mathbf{r}_i')^2$. If the forces acting on the system is *conservative*, in other words, if $\mathbf{F}_i \equiv \mathbf{F}_i^{(e)} + \sum_{j=1}^{N}\mathbf{F}_{ij} = -\frac{\partial V}{\partial\mathbf{r}_i}, \quad i = 1,\ldots,N$; then the total mechanical energy $E = T + V(\mathbf{r}_1,\ldots,\mathbf{r}_N)$ of the system is conserved.

The potential of the system has the form: $V = \sum_{i=1}^{N}V_i(\mathbf{r}_i) + \sum_{1\leq i<j\leq N}V_{ij}(\mathbf{r}_i,\mathbf{r}_j)$.

Defining $V_{ij}(\mathbf{r}_i,\mathbf{r}_j) = U_{ij}(\mathbf{r}_i - \mathbf{r}_j) \Rightarrow \quad V = \sum_{i=1}^{N}V_i(\mathbf{r}_i) + \sum_{1\leq i<j\leq N}U_{ij}(\mathbf{r}_i - \mathbf{r}_j).$

# 3 Central forces

## 3.1 Basic definitions

**Conditions for a central force**

$\mathbf{F}_{12} = f(|\mathbf{r}_1 - \mathbf{r}_2|)\frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|} \Rightarrow \mathbf{F}(\mathbf{r}) = \mu\ddot{\mathbf{r}} = f(r)\frac{\mathbf{r}}{r}$, in such case the following equations hold: $\ddot r - r\dot\theta^2 = \frac{f(r)}{\mu},\ r\ddot\theta + 2\dot r\dot\theta = 0$

## 3.2 Conserved quantities

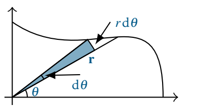
$\mathbf{L} = \mu\mathbf{r}\times\dot{\mathbf{r}} = \mu r^2\dot\theta\mathbf{e}_z,\ |\mathbf{L}| = L_z = \dot\theta r^2\mu > 0$

$E = \frac{1}{2}\mu\dot r^2 + U(r) = \frac{L^2}{2\mu}(u'^2 + u^2) + V(\frac{1}{u})$

**Law of areas**

$A(\theta) = \frac{1}{2}\int_{\theta_0}^{\theta}r^2(\alpha)\,d\alpha,$

$\dot A = \frac{dA}{d\theta}\dot\theta = \frac{1}{2}r^2\dot\theta = \frac{L}{2\mu}\in\mathbb{R}$




## 3.3 Important equations

**Binet's eqn.**

$f(r) = -\frac{L^2}{\mu r^2}(u'' + u)$

**Other equations**

$U(r) = V(r) + \frac{L^2}{2\mu r^2},\ -\frac{\partial}{\partial r}\left(\frac{L^2}{2\mu r^2}\right) = \frac{\mu v_\theta^2}{r}\mathbf{e}_r$

$t = \pm\sqrt{\frac{\mu}{2}}\int_{r_1}^{r_2}\frac{dr}{\sqrt{E - U(r)}} = \frac{\mu}{L}\int_{r_1}^{r_2}r^2(\theta)\,d\theta,$

$\theta = \frac{L}{\mu}\int_{t_1}^{t_2}\frac{dt}{r^2(t)} = \pm\frac{L}{\sqrt{2\mu}}\int^{\frac{1}{r}}\frac{du}{\sqrt{E - U(\frac{1}{u})}},$

$\tau_r = \sqrt{2\mu}\int_{r_1}^{r_2}\frac{dr}{\sqrt{E - U(r)}},\ \Delta\theta = \sqrt{\frac{2L^2}{\mu}}\int_{\frac{1}{r_2}}^{\frac{1}{r_1}} = \frac{du}{\sqrt{E - U(\frac{1}{u})}},$

An orbit is periodic $\iff \Delta\theta = q\pi,\ q\in\mathbb{Q}.$

$A_T = \pi r^2 = \pi ab = \dot A\cdot\tau = \frac{L}{2\mu}\tau,$

$v = \frac{L}{\mu}\sqrt{u'^2 + u^2}$

## 3.4 Kepler's potential

**Force field and Binet's eqn.**

$\mathbf{F}(\mathbf{r}) = -\frac{k}{r^2}\mathbf{e}_r,\ V(r) = -\frac{k}{r},\ \ddot{\mathbf{r}} = -GM\frac{\mathbf{r}}{r^3},\ u'' + u = \frac{\mu k}{L^2}$

**Equation of trajectory and its clasifications**

$u = \frac{\mu k}{L^2}(1 + e\cos(\theta - \theta_0)),\ e,\theta_0\in\mathbb{R}.$

$r = \frac{\alpha}{1 + e\cos(\theta)},\ \alpha\equiv\frac{L^2}{\mu k},\ e = \sqrt{1 + \frac{2EL^2}{\mu k^2}}\geq 0$

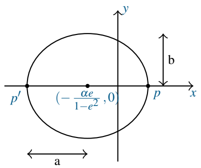
$$\begin{cases} e > 1 & \iff & E > 0 & \Rightarrow & \text{hyperbola} \\ e = 1 & \iff & E = 0 & \Rightarrow & \text{parabola} \\ e\in(0,1) & \iff & E\in(-\frac{\mu k^2}{2L^2},0) & \Rightarrow & \text{ellipse} \\ e = 0 & \iff & E = -\frac{\mu k^2}{2L^2} & \Rightarrow & \text{circle} \end{cases}$$

## 3.5 Planetary motion

**Study of ellipses**

$\mathbf{C}\equiv\{(1 - e^2)(x + \frac{\alpha e}{1 - e^2})^2 + y^2 = \frac{\alpha^2}{1 - e^2}\}$

$a = \frac{\alpha}{1 - e^2},\ b = \frac{\alpha}{\sqrt{1 - e^2}},\ c = \varepsilon a = ea,\ e = \varepsilon,$




$E = -\frac{k}{2a},\ \langle r\rangle = \left(1 + \frac{e^2}{2}\right)a,$

$\tau = 2\pi\sqrt{\frac{\mu}{k}}a^{3/2} = \pi k\sqrt{\frac{\mu}{2}}|E|^{-3/2} = \frac{2\pi a^{3/2}}{\sqrt{GM}} \simeq \frac{2\pi a^{3/2}}{\sqrt{GM_\odot}}$

$p = \frac{\alpha}{1 + e} = a(1 - e),\ p' = \frac{\alpha}{1 - e} = a(1 + e),$

$$v^2 = \frac{k^2}{L^2}(1 + e^2 + 2e\cos\theta) \Rightarrow \begin{cases} v_p = \frac{k}{L}(1 + e) = \sqrt{\frac{k}{\mu p}}(1 + e) \\ v_{p'} = \frac{k}{L}(1 - e) = \sqrt{\frac{k}{\mu p'}}(1 - e) \end{cases}$$

# 4 Lagrangian & Hamiltonian

## 4.1 Basic Definitions

$F[y] = \int_{x_1}^{x_2}f\,dx$, where $F\equiv$functional, and $f\equiv$ density.

## 4.2 Important equations and theorems

**Euler-Lagrange equations:**

$\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = 0;\ \frac{d}{dt}\frac{\partial L}{\partial\dot{\mathbf{q}}} - \frac{\partial L}{\partial\mathbf{q}}$. If $\frac{\partial f}{\partial x} = 0 \Rightarrow y'\frac{\partial f}{\partial y'} - f$ is constant. If $\frac{\partial f}{\partial y} = 0 \Rightarrow \frac{\partial f}{\partial y'}$ is constant. Two densities differing by a total derivative has the same E-L eqn.

**Hamilton's principle for $N$ particles:**

$L(t,\mathbf{r}_1,\ldots,\mathbf{r}_N,\dot{\mathbf{r}}_1,\ldots,\dot{\mathbf{r}}_N) = \frac{1}{2}\sum_i^N m_i\dot{\mathbf{r}}_i^2 - V(t,\mathbf{r}_1,\ldots,\mathbf{r}_N).$

If $\frac{\partial L}{\partial t} = 0 \Rightarrow \sum_i\dot{\mathbf{r}}_i\frac{\partial L}{\partial\dot{\mathbf{r}}_i} - L = T + V$ conserved.

If $\frac{\partial L}{\partial x_i} = 0 \Rightarrow \frac{\partial L}{\partial\dot x_i}$ is conserved.

**Canonical momenta:**

$p_i = \frac{\partial L}{\partial\dot q_i} \Rightarrow \dot p_i = \frac{\partial L}{\partial q_i}$. If $p_i = 0$ then the coordinate $q_i$ is *cyclic* or *ignorable*.

**Hamiltonian:**

$H = \sum_i^n\dot q_i\frac{\partial L}{\partial\dot q_i} - L = \sum_i^n p_i\dot q_i - L = T + V$ ($\Leftarrow$ If natural).

**Hamilton's canonical eqns.:**

$\dot{\mathbf{q}} = \frac{\partial H}{\partial\mathbf{p}};\ \dot{\mathbf{p}} = -\frac{\partial H}{\partial\mathbf{q}}$

**Jacobi-Poisson Theorem**

If $f$ and $g$ are two first integrals of H's canonical equations, so is $\{f,g\}$.

## 4.3 Procedures

**Find the EOM of $N$ particles with $l$ holonomic constraints:**

1. Introduce $3N - l = n$ generalized coordinates $(q_1,\ldots,q_n) = \mathbf{q}$ to the constraint manifold $\{\Phi_1,\ldots,\Phi_l(t,\mathbf{q}) = 0\}$.

2. Express $T = \frac{1}{2}\sum_i^N m_i\dot{\mathbf{r}}_i^2$ and $V$ in terms of $(t,\mathbf{q},\dot{\mathbf{q}}) \Rightarrow$
$$T = T(t,\mathbf{q},\dot{\mathbf{q}});\ V \equiv V(t,\mathbf{q}).$$

3. E-L equations of $L = T - V$ are the $n$ equations of motion in coordinates $q_i$
$$\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial\dot q_i} - \frac{\partial L}{\partial q_i},\ i = 1,\ldots,n.$$

4. Constraint forces on $i$-th particle are expressed as:
$$\mathbf{F}_i^{(c)} = m_i\ddot{\mathbf{r}}_i + \frac{\partial V}{\partial\mathbf{r}_i},\ i = 1,\ldots,N.$$

**Write Hamilton's canonical equations:**

1. Calculate the canonical momenta $p_i = \frac{\partial L}{\partial\dot q_i},\ i = 1,\ldots,n.$
2. Express the generalized velocities as $\dot q_i \equiv \dot q_i(t,\mathbf{q},\mathbf{p}),\ i = 1,\ldots,n.$
3. Compute the Hamiltonian $H(t,\mathbf{q},\mathbf{p}) = \mathbf{p}\cdot\dot{\mathbf{q}}(t,\mathbf{q},\mathbf{p}) - L(t,\mathbf{q},\mathbf{p}).$
4. The partial derivatives of $H$ w.r.t the variables $q_i, p_i$ are the canonical equations. The first $n$ equations are the ones in step 2, the rest are $\dot p_i = -\frac{\partial H}{\partial q_i},\ i = 1,\ldots,n$. There are in total $2n$ equations.

## 4.4 Hamiltonian conservations

$\frac{\partial H}{\partial q_i} = 0 \Rightarrow p_i$ const. $\frac{\partial H}{\partial p_i} = 0 \Rightarrow q_i$ const.

$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial\mathbf{q}}\dot{\mathbf{q}} + \frac{\partial H}{\partial\mathbf{p}}\dot{\mathbf{p}} = \frac{\partial H}{\partial t} \Rightarrow$ If $\frac{\partial H}{\partial t} = 0 \Rightarrow H$ const. Moreover, since $\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$, $H$ is conserved $\iff L$ indep. of $t$.

## 4.5 Poisson brackets

$(\mathbf{q},\mathbf{p})\equiv$"phase space", for every smooth function $f(t,\mathbf{q},\mathbf{p})$ the following holds:

$\dot f = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial\mathbf{q}}\dot{\mathbf{q}} + \frac{\partial f}{\partial\mathbf{p}}\dot{\mathbf{p}} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial\mathbf{q}}\frac{\partial H}{\partial\mathbf{p}} + \frac{\partial f}{\partial\mathbf{p}}\frac{\partial H}{\partial\mathbf{q}} = \frac{\partial f}{\partial t} + \{f,H\}$

Being $\{f,g\}\equiv\frac{\partial f}{\partial\mathbf{q}}\frac{\partial g}{\partial\mathbf{p}} - \frac{\partial f}{\partial\mathbf{p}}\frac{\partial g}{\partial\mathbf{q}} = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right)$, $f,g$ any function of $(t,\mathbf{q},\mathbf{p})$.

**Poisson brackets properties**

$\{f,g\} = -\{g,f\}$, in particular $\{f,f\} = 0.$ (Antisymmetry)
$\{\lambda f + \mu g,h\} = \lambda\{f,h\} + \mu\{g,h\}.$ (Bilinearity)
$\{fg,h\} = f\{g,h\} + g\{f,h\}.$ (Leibniz's rule)
$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0.$ (Jacobi identity)
$\frac{\partial}{\partial t}\{f,g\} = \left\{\frac{\partial f}{\partial t},g\right\} + \left\{f,\frac{\partial g}{\partial t}\right\} \Rightarrow \frac{d}{dt}\{f,g\} = \left\{\dot f,g\right\} + \{f,\dot g\}$ (J-P id.)
$\dot q_i = \{q_i,H\},\ \dot p_i = \{p_i,H\};\ \{q_i,q_j\} = \{p_i,p_j\} = 0,\ \{q_i,p_j\} = \delta_{ij}$

**Canonical change of variable**

Consists in changing $(\mathbf{q},\mathbf{p})\mapsto(\tilde{\mathbf{q}},\tilde{\mathbf{p}})$, s.t. $\exists\tilde H$ Hamiltonian where $\dot{\tilde{\mathbf{q}}} = \frac{\partial\tilde H}{\partial\tilde{\mathbf{p}}}$ and $\dot{\tilde{\mathbf{p}}} = -\frac{\partial\tilde H}{\partial\tilde{\mathbf{q}}}$.

The changes $\begin{cases}(\mathbf{q},\mathbf{p})\mapsto(\mathbf{p},\mathbf{q}),\ \tilde H(t,\tilde{\mathbf{q}},\tilde{\mathbf{p}}) = -H(t,\mathbf{q},\mathbf{p}) \\ (\mathbf{q},\mathbf{p})\mapsto(\mathbf{p},-\mathbf{q}),\ \tilde H(t,\tilde{\mathbf{q}},\tilde{\mathbf{p}}) = H(t,\mathbf{q},\mathbf{p})\end{cases}$ are canonical.

In general, a transformation is canonical $\iff \begin{cases}\{\tilde q_i,\tilde q_j\} = \{\tilde p_i,\tilde p_j\} = 0 \\ \{\tilde q_i,\tilde p_j\} = \lambda\delta_{ij},\ \lambda\neq 0.\end{cases}$

If $\lambda = 1$ then $(\tilde{\mathbf{q}},\tilde{\mathbf{p}})$ are *canonically conjugate*.
If $\lambda\neq 1$, the change $(\mathbf{q},\mathbf{p})\mapsto(\tilde{\mathbf{q}},\tilde{\mathbf{p}}/\lambda)$ is canonically conjugate.

**Hamilton-Jacobi theory**

It's always possible to find a can. transf. that has $\tilde H = 0 \Rightarrow \dot{\tilde{\mathbf{q}}} = 0 = \dot{\tilde{\mathbf{p}}}$ $\Rightarrow \tilde{\mathbf{p}} = \tilde{\mathbf{p}}_0;\ \tilde{\mathbf{q}} = \tilde{\mathbf{q}}_0$ constant solutions, and reversing the transformation we will get the original $\mathbf{q}_0,\mathbf{p}_0$ constant solutions of the original $H$.

## 4.6 Misc.

**Charged particle mass in an EM field $\Phi$ (*not* natural):**

$$\begin{cases} L(t,\mathbf{r},\dot{\mathbf{r}}) = \frac{1}{2}m\dot{\mathbf{r}}^2 - e\Phi(t,\mathbf{r}) + e\dot{\mathbf{r}}\cdot\mathbf{A}(t,\mathbf{r}) \\ H(t,\mathbf{r},\mathbf{p}) = \frac{1}{2m}(\mathbf{p} - e\mathbf{A}(t,\mathbf{r}))^2 + e\Phi(t,\mathbf{r}) \\ \qquad p_i = m\dot x_i + eA_i(t,\mathbf{r}),\ i = 1,2,3. \end{cases}$$

# 5 Non-inertial frame

## 5.1 Basic definitions

**Reference frames**

$\mathcal{S}'\equiv$ inertial reference frame, $\mathcal{S}\equiv$ non inertial reference frame.

$\left(\frac{d}{dt}\right)_{f}\equiv$ time derivative w.r.t $\mathcal{S}'$, $\left(\frac{d}{dt}\right)_{f}\equiv$ time derivative w.r.t $\mathcal{S}$,

the vectors between the frames are related by $\mathbf{e}_i'(t) = O(t)\mathbf{e}_i$, being $O(t)$ the orthogonal linear application.

**Rotation matrices and relations between ref. frames**

$SO(3) = \{M\in\mathbb{R}^{3\times 3}|MM^\intercal = M^\intercal M = \mathbb{1},\ \det\{M\} = 1\}$

$\forall M\in SO(3),\ \exists\backslash\in\mathbb{R}^3$ and reference frame $\mathcal{S}' = \{\mathbf{e}_1,\mathbf{e}_2,\mathbf{n}\}$

s.t. $\text{mat}_{\mathcal{S}'}^{\mathcal{S}'}(M) = R_3(\theta) = \begin{bmatrix}\cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1\end{bmatrix}.$

$\frac{d}{d\theta}(R_\mathbf{n}(\theta)) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}R_\mathbf{n}(\theta + \varepsilon) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}R_\mathbf{n}(\varepsilon)R_\mathbf{n}(\theta)$
$= \Omega R_\mathbf{n}(\theta) = \mathbf{n}\times R_\mathbf{n}(\theta)$

$O(t)\in SO(3),\ \forall t$, and $\dot O(t) = \Omega(t)O(t),$

where $\Omega(t) = \begin{bmatrix}0 & -\omega_3(t) & \omega_2(t) \\ \omega_3(t) & 0 & -\omega_1(t) \\ -\omega_2(t) & \omega_1(t) & 0\end{bmatrix}$ antisymmetric matrix.

$\dot O(t)\mathbf{c} = \boldsymbol{\omega}(t)\times O(t)\mathbf{c},\ \boldsymbol{\omega} = (\omega_1,\omega_2,\omega_3). \qquad \forall t\in\mathbb{R}$ and $\forall\mathbf{c}\in\mathbb{R}^3$

"$\boldsymbol{\omega}(t)\times$"$= \Omega(t) = \dot O(t)O(t)^\intercal = \dot O(t)O(t)^{-1} \iff$

$\boldsymbol{\omega}(t_0)\times = \frac{d}{dt}\Big|_{t=t_0}O(t)O^{-1}(t_0).$

$\dot{\mathbf{e}}_i(t) = \boldsymbol{\omega}(t)\times O(t)\mathbf{e}_i' = \boldsymbol{\omega}(t)\times\mathbf{e}_i(t).$

$\boldsymbol{\omega}(t) = \dot\theta(t)\mathbf{n}(t)$, in words: the direction of the vector $\boldsymbol{\omega}(t)$ is equal to the axis of rotation $\mathbf{n}(t)$, and the magnitude of $\boldsymbol{\omega}(t)$ is equal to the angular velocity of the moving axes w.r.t the fixed ones.

## 5.2 Time derivative in different reference frames

$\left(\frac{d\mathbf{A}(t)}{dt}\right)_{f} = \left(\frac{d\mathbf{A}(t)}{dt}\right)_{m} + \boldsymbol{\omega}(t)\times\mathbf{A}(t),$

in particular $\left(\frac{d\boldsymbol{\omega}(t)}{dt}\right)_{f} = \left(\frac{d\boldsymbol{\omega}(t)}{dt}\right)_{m} = \dot{\boldsymbol{\omega}}(t)$

If the origin of $\mathcal{S}$ is displacing from that of $\mathcal{S}'$ with $\mathbf{R}(t)$, then:

$\left(\frac{d\mathbf{A}(t)}{dt}\right)_{f} = \left(\frac{d\mathbf{A}(t)}{dt}\right)_{m} + \boldsymbol{\omega}(t)\times\mathbf{A}(t) + \mathbf{V}(t),$

## 5.3 Dynamics in non-inertial ref. frame

**Preamble**

In the general case, the origin of $\mathcal{S}$ is displaced from that of $\mathcal{S}'$ by a time-dependent vector $\mathbf{R}(t)$. If $\mathbf{r}$ is the position of a particle w.r.t $\mathcal{S}$, then the position w.r.t $\mathcal{S}'$ is given by $\mathbf{r}' = \mathbf{r} + \mathbf{R}$. We'll define the following vectors:

$\mathrm{V}\equiv\left(\frac{d\mathbf{R}}{dt}\right)_{f},\ \mathbf{A}\equiv\left(\frac{d\mathbf{V}}{dt}\right)_{f} = \left(\frac{d^2\mathbf{R}}{dt^2}\right)_{f},\ \mathrm{v}_f = \left(\frac{d\mathbf{r}'}{dt}\right)_{f},\ \mathrm{v}_m = \left(\frac{d\mathbf{r}}{dt}\right)_{m}$

$\mathbf{a}_f\equiv\left(\frac{d\mathbf{v}_f}{dt}\right)_{f} = \left(\frac{d^2\mathbf{r}'}{dt^2}\right)_{f},\ \mathbf{a}_m\equiv\left(\frac{d\mathbf{v}_m}{dt}\right)_{m} = \left(\frac{d^2\mathbf{r}}{dt^2}\right)_{m}.$

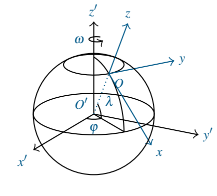
**Equations of motion**

$\mathbf{v}_f = \mathbf{v}_m + \boldsymbol{\omega}\times\mathbf{r} + \mathbf{V},$
$\mathbf{a}_f = \mathbf{a}_m + \mathbf{A} + 2\boldsymbol{\omega}\times\mathbf{v}_m + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) + \dot{\boldsymbol{\omega}}\times\mathbf{r},$
$m\mathbf{a}_m = \mathbf{F} - m\mathbf{A} - 2m\boldsymbol{\omega}\times\mathbf{v}_m - m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) - m\dot{\boldsymbol{\omega}}\times\mathbf{r}\equiv\mathbf{F} + \mathbf{F}_i,$
where $\mathbf{F}_i$ is the fictitious force:
$\mathbf{F}_i = -m\mathbf{A} - 2m\boldsymbol{\omega}\times\mathbf{v}_m - m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) - m\dot{\boldsymbol{\omega}}\times\mathbf{r}$
Centrifugal force $= -m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}),$
Coriolis force $= -2m\boldsymbol{\omega}\times\mathbf{v}_m$, depends on the velocity of the particle.

For the equation $\dot{\mathbf{A}}(t) = \boldsymbol{\omega}\times\mathbf{A}(t) + \mathbf{c}$, where $\boldsymbol{\omega}\perp\mathbf{c}$ are both constant vectors, the general solution of the ODE is: $\mathbf{A}(t) = R_\mathbf{n}(\omega t)\mathbf{u} + \frac{1}{\omega}\mathbf{n}\times\mathbf{c}$, with $\mathbf{u}\in\mathbb{R}^3,\ \mathbf{n} = \omega^{-1}\boldsymbol{\omega},$.

## 5.4 Motion of a particle relative to the rotating Earth

**Basis vectors**




$\mathbf{e}_1 = \sin\lambda\cos\varphi\mathbf{e}_1' + \sin\lambda\sin\varphi\mathbf{e}_2' - \cos\lambda\mathbf{e}_3',$
$\mathbf{e}_2 = -\sin\varphi\mathbf{e}_1' + \cos\varphi\mathbf{e}_2',$
$\mathbf{e}_3 = \cos\lambda\cos\varphi\mathbf{e}_1' + \cos\lambda\sin\varphi\mathbf{e}_2' + \sin\lambda\mathbf{e}_3'.$

**Kinetics & Dynamics**

$\boldsymbol{\omega} = \omega(-\cos\lambda\mathbf{e}_1 + \sin\lambda\mathbf{e}_3)$
$\mathbf{g}_0 = -\frac{GM}{R^3}\mathbf{R} = -\frac{GM}{R^2}\mathbf{e}_3\equiv -g_0\mathbf{e}_3,\ g_0\simeq 9.8\ \mathrm{m\cdot s^{-2}}$
$\ddot{\mathbf{r}} = \mathbf{g} - 2\boldsymbol{\omega}\times\dot{\mathbf{r}} - \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$, where $\mathbf{g} = \mathbf{g}_0 - \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{R}).$
With additional force: $\ddot{\mathbf{r}} = \frac{\mathbf{F}}{m} + \mathbf{g} - 2\boldsymbol{\omega}\times\dot{\mathbf{r}} - \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}).$

The angle between $-\mathbf{e}_3$ and $\mathbf{g}$ is $\delta(\lambda)$, where $\tan\delta(\lambda) = \frac{\omega^2R\sin\lambda\cos\lambda}{g_0 - \omega^2R\cos^2\lambda}$.

Approximately, the equation of motion of a particle moving close to the Earth's surface, is given by $\mathbf{r}\simeq\mathbf{r}_0 + \mathbf{v}_0t + \mathbf{g}_0\frac{t^2}{2} - t^2\boldsymbol{\omega}\times\mathbf{v}_0 - \frac{t^3}{3}\boldsymbol{\omega}\times\mathbf{g}_0$.

Sometimes, to approximate, we often ignore some terms inside the above equation because of the following ratio: $\frac{g_0}{\omega}\simeq 1.35\cdot 10^5\,\mathrm{m\,s^{-1}} >> v_0$

## 5.5 Foucault's pendulum

Consists of a pendulum with length that is far greater than the coordinates $x,y,z$ of its bob. The equations of motion of the coordinates $x,y$ are:

$\begin{cases}\ddot x + \alpha^2x = 2\omega\dot y\sin\lambda \\ \ddot y + \alpha^2y = -2\omega\dot x\sin\lambda\end{cases}$, where $\alpha\equiv\sqrt{\frac{g_0}{l}}$. The motion of the coordinate $z$ can be ignored. Letting $u = x + iy$, the general solution of $u$ is:

$u = e^{-i\Omega t}\left(c_1e^{i\alpha t} + c_2e^{-i\alpha t}\right)$, where $c_1,c_2\in\mathbb{C}$ and $\Omega\equiv\omega\sin\lambda$.

A solution with initial conditions $\begin{cases}x(0) = x_0 > 0,\ y(0) = 0 \\ \dot x = \dot y = 0\end{cases}$, would be:

$u = x_0e^{-i\Omega t}\cos(\alpha t) \Rightarrow x = x_0\cos(\Omega t)\cos(\alpha t),\ y = -x_0\sin(\Omega t)\cos(\alpha t).$

The projection of the vector $\mathbf{r}$ onto the $X - Y$ plane is given by $x_0\cos(\alpha t)[\cos(\Omega t)\mathbf{e}_1 - \sin(\Omega t)\mathbf{e}_2]$. In the northern hemisphere the pendulum's plane rotates clockwise, i.e., in the east-south direction (since $\dot\theta = -\Omega = -\omega\sin\lambda < 0$), with angular velocity $\Omega = \omega\sin\lambda$. In the southern hemisphere the rotation of the pendulum's plane is counterclockwise (since $\sin\lambda < 0$), and in the equator ($\lambda = 0$) no such rotation occurs.

In each period $2\pi/\alpha$ of the pendulum the angle between the pendulum's plane and the $X - Y$ plane increases by $-2\pi\Omega/\alpha$, which is a very small quantity. The period of the rotation of the pendulum's plane is given by $\tau = \frac{2\pi}{\Omega} = \frac{2\pi}{\omega}\csc\lambda$ sidereal days.

# 6 Rigid body motion

**Note:** some of the notations in this part is mixed with the ones used by *Amores* in Una base para el álgebra lineal, in which column vectors are *not* represented in bold.

## 6.1 Basic definitions

A **rigid body** is a system of particles of mass $m_\alpha$ ($\alpha = 1,\ldots,N$) in which the *distance* $|\mathbf{r}_\alpha - \mathbf{r}_\beta|$ between any two particles is *constant*. In other words, a rigid body is a mechanical system of $N$ particles subject to the $N(N-1)/2$ time-independent holonomic constraints (that may or may not be mutually independent).

In a generic rigid body it's always possible to construct a set of moving axes $\mathcal{S}$ (**body axes**), with respect to which all it's particles are *fixed*.

A generic rigid body has 6 d. of freedom, 3 for the coordinate of the origin of the body axes, and 3 for the matrix $O(t)\in SO(3)$ that relates $\mathcal{S}'$ and $\mathcal{S}$.

Centre of mass of a r.b.: $\mathbf{R} = \frac{1}{M}\int\rho(\mathbf{r})\mathbf{r}\,d^3\mathbf{r}$, where $M = \int\rho(\mathbf{r})\,d^3\mathbf{r}$.

## 6.2 Angular momentum and $T$ of a rigid body

Letting the centre of mass be the origin of $\mathcal{S}$:
$\mathbf{r}_\alpha' = \mathbf{R} + \mathbf{r}_\alpha,\ \mathbf{v}_\alpha' = \mathbf{V} + \boldsymbol{\omega}\times\mathbf{r}_\alpha,\ \mathbf{P} = M\mathbf{V}.$
$\mathbf{L} = M\mathbf{R}\times\mathbf{V} + \mathbf{L}_{CM}$, where $\mathbf{L}_{CM} = \sum_\alpha m_\alpha\mathbf{r}_\alpha\times(\boldsymbol{\omega}\times\mathbf{r}_\alpha)$

$T = \frac{1}{2}M\mathbf{V}^2 + T_{rot}$, where $T_{rot} = \frac{1}{2}\sum_\alpha m_\alpha(\boldsymbol{\omega}\times\mathbf{r}_\alpha)^2$

$\Rightarrow\mathbf{L}_{CM} = \sum_\alpha m_\alpha[r_\alpha^2\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\mathbf{r}_\alpha)\mathbf{r}_\alpha], \quad T_{rot} = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L}_{CM}$

## 6.3 Inertia tensor

**Note:** the inertia tensor $I$, without any subscript, is calculated w.r.t. the body axes (with CM as it's origin). But it can also be calculated w.r.t other points (Steiner's theorem).

**Definition**

The inertial tensor ($I\in\mathbb{R}^{3\times 3}$) is a symmetric matrix with elements:

$I_{ij} = \sum_\alpha m_\alpha(\delta_{ij}r_\alpha^2 - x_{\alpha i}x_{\alpha j}) \Rightarrow \begin{cases}I_{ii} = m_\alpha(x_{\alpha j}^2 + x_{\alpha k}^2) \\ I_{ij} = -\sum_\alpha m_\alpha x_{\alpha i}x_{\alpha j}\end{cases}$

If the r.b. is continuous: $\begin{cases}I_{ii} = \int_\Omega\rho(\mathbf{r})(x_j^2 + x_k^2)\,d^3\mathbf{r},\ i = 1,2,3 \\ I_{ij} = -\int_\Omega\rho(\mathbf{r})x_ix_j\,d^3\mathbf{r},\ 1\leq i\neq j\leq 3\end{cases}$

### Application

$T_{\mathrm{rot}} = \frac{1}{2}\,\boldsymbol{\omega}\cdot(I\,\boldsymbol{\omega}) = \frac{1}{2}\,\boldsymbol{\omega}^{\mathsf T} I\boldsymbol{\omega}$ ($\Leftarrow$ here $\omega$ is a column (lin. algebraic) vector.
$\Rightarrow T_{\mathrm{rot}} = \frac{1}{2} I_{\mathbf n}\omega^2$, where $I_{\mathbf n}$ is the moment of inertia w.r.t. instantaneous axis of rotation $\mathbf{n} = \frac{\boldsymbol{\omega}}{\omega}$.

### Steiner's theorem

The inertia moment computed w.r.t a point $P$ fixed w.r.t body axes is given as: $(I_P)_{ij} = \sum_\alpha m_\alpha\left(\delta_{ij}\tilde{\mathbf r}_\alpha^2 - \tilde x_{\alpha i}\tilde x_{\alpha j}\right)$. Taking into account $\tilde{\mathbf r}_\alpha = \mathbf r_\alpha - \mathbf a$, $\mathbf a = \overrightarrow{CP}$
$\Rightarrow (I_P)_{ij} = I_{ij} + M(\mathbf a^2\delta_{ij} - a_i a_j)$.
If $P$ is also fixed w.r.t $\mathcal S'$, then, placing $O' = P$, we can reduce the expressions: $\mathbf V = \boldsymbol\omega\times\mathbf R$, $\mathbf v'_\alpha = \boldsymbol\omega\times\mathbf r'_\alpha$, $\mathbf L = I'\,\boldsymbol\omega$, $(I' = I_P)$, $T = \frac12\,\boldsymbol\omega\cdot I'\boldsymbol\omega$.
$I'$ can be computed from $I$: $I'_{ij} = I_{ij} + M(\mathbf R^2\delta_{ij} - X_iX_j)$, where $X_h$ are components of the vector $\mathbf R$.

### Principal axes of inertia

Let $A\in SO(3)$ be a constant change of basis matrix, and $\tilde{\mathbf e}_i = \sum_j a_{ji}\mathbf e_j$.
$\Rightarrow x = A\tilde x$, $\quad \tilde I = A^{-1}IA = A^{\mathsf T}IA$.
Since $I$ is a symmetric matrix, there is always a matrix $A\in SO(3)$, s.t. $\tilde I = A^{\mathsf T}IA$ is a diagonal matrix, in other words, the moment of inertia w.r.t the alternate body axes $\tilde{\mathbf e}_i$ (which are eigenvectors) is expressed as:

$$\tilde I = \begin{bmatrix} I_1 & & \\ & I_2 & \\ & & I_3\end{bmatrix}$$, with $I_i$ the eigenvalues corresponding to $\tilde{\mathbf e}_i$.

In such case, $\tilde{\mathbf e}_i$ are **principal axes of inertia**, and $I_i$ are **principal moments of inertia**, which are the roots of the caracteristic polinomial of $I$.
If $\mathbf e_i$ are p.a.i. then: $\mathbf L_{\mathrm{CM}} = \sum_i I_i\omega_i\mathbf e_i$, $\quad T_{\mathrm{rot}} = \frac12\sum_i I_i\omega_i^2$.
If the origin of a set of p.a.i. are also fixed in $\mathcal S'$, then $\mathbf L, T$ can be calculated with the expressions above, replacing $I$ by $I'$ w.r.t. the point $O'$.

### Clasification of r.b. in terms of the multiplicity of the eigenvalues $I_i$

| Type of symmetry | Definition |
|---|---|
| Asymmetric tops: | $I_i\neq I_j, \forall i\neq j$, |
| Axially symmetric tops: | $I_i = I_j\neq I_k$, |
| Spherically symmetric tops: | $I_i = I_j,\ \forall i,j$ |

### Deduce $I_i$ of a rigid body by its invariance under transformations

**Note:** both $\Omega$ and $\rho$ have to be invariant under the transfomations.

| Inv. under | Behaviour of $I_i$ |
|---|---|
| $x_i\mapsto -x_i$ | $I_{ij} = 0,\ \forall j\neq i$ |
| $x_i\mapsto x_j$ | $I_{ii} = I_{jj}$, $\quad I_{ik} = I_{jk},\ k\neq i,j$ |

In particular, for a *solid of revolution* with homogeneous density, taking $x_3$ as the axis of rotation and arbitrary perpendicular axes $x_1, x_2$, it satisfy the following invariances: $x_1\mapsto -x_1$, $x_2\mapsto -x_2$, $x_1\mapsto x_2$, and thus:
$I_{11}\equiv I_1 = I_2\equiv I_{22}$, $I_{ij} = 0$ $(i\neq j)$, with:

$$I_1 = \pi\rho\int_{z_1}^{z_2} z^2 f^2(z)\,\mathrm dz + \frac{\pi\rho}{4}\int_{z_1}^{z_2} f^4(z)\,\mathrm dz,\qquad I_3\equiv I_{33} = \frac{\pi\rho}{2}\int_{z_1}^{z_2} f^4(z)\,\mathrm dz.$$

## 6.4 EOM of a rigid body

### EOM in an inertial frame

If $\mathbf F = 0$ then $\mathbf N$ computed at any point in $\mathcal S'$ is the same.
A r.b. is in equilibrium in an $\mathcal S \iff \mathbf F = \mathbf N = 0 \iff \mathbf F = \mathbf N_{\mathrm{CM}} = 0$

### Forces due to a constant field f

$\lambda_\alpha\equiv$"charge" of a particle $\Rightarrow \mathbf F_\alpha = \lambda_\alpha\mathbf f \Rightarrow \mathbf F = \sum_\alpha = \sum_\alpha \mathbf f\lambda_\alpha = \Lambda\mathbf f$,

$\mathbf N = \left(\sum_\alpha \lambda_\alpha\mathbf r'_\alpha\right)\times\mathbf f$. $\qquad$ ($\downarrow$ Note that $\mathbf X$ is fixed in the body)

If $\Lambda\neq 0$, defining $\mathbf X = \frac1\Lambda\sum_\alpha\lambda_\alpha\mathbf r'_\alpha = \frac1\Lambda\sum_\alpha\lambda_\alpha(\mathbf R + \mathbf r_\alpha) = \mathbf R + \frac1\Lambda\sum_\alpha\lambda_\alpha\mathbf r_\alpha$

we have that $\mathbf N = \mathbf X\times\mathbf F \Rightarrow$ If $\Lambda\neq 0$, then the total torque of the external forces coincides with the torque of the total external forces applied at the *centre of charge*. (For the grav. field $\mathbf g$, $\lambda = m$ and C. charge is equal to CM).
Torque of the external forces w.r.t. CM is $\mathbf N_{\mathrm{CM}} = (\mathbf X - \mathbf R)\times\mathbf F$ (for $\mathbf g$ it is zero).
When $\Lambda = 0$ we use the former equation for $\mathbf N$ and the torque of the external forces is independent of the reference point, since $\mathbf F = \Lambda\mathbf f = 0$.

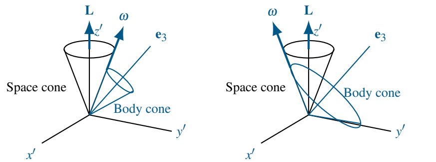
### Euler's equations

Euler's equations is the system $\begin{cases} I_1\dot\omega_1 - (I_2 - I_3)\omega_2\omega_3 = N_1,\\ I_2\dot\omega_2 - (I_3 - I_1)\omega_1\omega_3 = N_2,\\ I_3\dot\omega_3 - (I_1 - I_2)\omega_1\omega_2 = N_3.\end{cases}$ where $\mathbf N$ and $I$ are computed w.r.t. CM or a point fixed in the body and $\mathcal S'$. $\boldsymbol\omega$ and $\mathbf N$ are in a frame of p.a.i. (in general not inertial).
If $\mathbf N = 0$ and the origin $O'$ of $\mathcal S'$ is a point in the body, $\mathbf L_{O'}\equiv\mathbf L$ and $T$ are conserved. Similarly, if $\mathbf N_{\mathrm{CM}} = 0$ then $\mathbf L_{\mathrm{CM}}$ and $T_{\mathrm{rot}}$ are conserved.

## 6.5 Inertial motion of a symmetric top

The angular velocity $\boldsymbol\omega$ of a set of axes w.r.t. to another is *additive*.
For an axially ($\mathbf e_3$) symmetric r.b., by E's eqns, the following conditions holds:
$\omega_3,\ \Omega = \frac{I_3 - I_1}{I_1}\,\omega_3,\ \omega_0 = \sqrt{\omega_1^2 + \omega_2^2},\ \alpha = \arctan(\omega_0/\omega_3)$ constants.
In frame of body axes, the vector $\boldsymbol\omega$ rotates around $\mathbf e_3$ with $\Omega$ (traces out the **body cone** fixed w.r.t. body axes). The angular velocity $\Omega > 0$ for $I_3 > I_1$, and negative otherwise.
Relative to body axes, $\mathbf L$ rotates with $\Omega$ around $\mathbf e_3$. $\mathbf L$ fixed relative to $\mathcal S'$.
Relative to $\mathcal S'$, $\boldsymbol\omega$ traces out the **space cone** around $\mathbf L$ (fixed in $\mathcal S'$) with $\Omega_p$.
The angle between $\mathbf e_3$ and $\boldsymbol\omega$ is $\alpha$, between $\mathbf L$ and $\mathbf e_3$ is $\theta = \arctan\frac{I_1}{I_3}\tan\alpha$.

$\Omega_p = \frac{L}{I_1} = \omega\sqrt{1 + \frac{I_3^2 - I_1^2}{I_1^2}\cos^2\alpha}$, $\ \boldsymbol\omega = \Omega_p\frac{\mathbf L}{L} - \Omega\mathbf e_3$.




## 6.6 Miscellaneous

In general, for a solid of revolution rolling without sliding down a inclined ($\alpha$) plane around the principal axis $\mathbf e_3$, the horizontal acceleration is expressed as:
$\ddot x = \frac{g\sin\alpha}{1 + \frac{I_3}{Ma^2}} \Rightarrow$.

### Inertia tensor of some regular bodies

| Body | Definition | Inertia tensor |
|---|---|---|
| Solid Sphere | $x^2+y^2+z^2\le r^2$ | $\frac25 mr^2\begin{bmatrix}1&&\\&1&\\&&1\end{bmatrix}$ |
| Hollow Sphere | $x^2+y^2+z^2 = r^2$ | $\frac23 mr^2\begin{bmatrix}1&&\\&1&\\&&1\end{bmatrix}$ |
| Cuboid | $\begin{cases}-\frac d2\le x\le\frac d2\\ -\frac w2\le y\le\frac w2\\ -\frac h2\le z\le\frac h2\end{cases}$ | $\frac{m}{12}\begin{bmatrix}h^2+d^2&&\\&w^2+d^2&\\&&h^2+w^2\end{bmatrix}$ |
| Cylinder | $\begin{cases}x^2+y^2\le r^2\\ -\frac h2\le z\le\frac h2\end{cases}$ | $\frac{m}{12}\begin{bmatrix}3r^2+h^2&&\\&3r^2+h^2&\\&&6r^2\end{bmatrix}$ |
| Rod about end | $\begin{cases}x^2+y^2\le r^2\\ 0\le z\le h\\ h>>r\\ \boldsymbol\omega\parallel\mathbf x\end{cases}$ | $\frac13 ml^2\begin{bmatrix}1&&\\&0&\\&&1\end{bmatrix}$ |
| Rod about center | $\begin{cases}x^2+y^2\le r^2\\ -\frac h2\le z\le\frac h2\\ h>>r\\ \boldsymbol\omega\parallel\mathbf x\end{cases}$ | $\frac1{12} ml^2\begin{bmatrix}1&&\\&0&\\&&1\end{bmatrix}$ |

# 7 Special relativity

## 7.1 Principle of special relativity

In Galilean relativity, the acceleration of a particle is the same in all inertial frames $\Rightarrow$ all inertial frames are *equivalent* from the point of view of Newtonian mechanics.

**Postulates of special relativity:**
1. The laws of physics are the same in all inertial frames (**relativity principle**).
2. The speed of electromagnetic waves in vacuum is universal: $c = \frac{1}{\sqrt{\varepsilon_0\mu_0}}$.
3. (Implication of 1. & 2.) speed of electromagnetic waves in vacuum is $c$ in all inertial frames.

## 7.2 Lorentz transformations

### Basic equations of transformation

We will use the notation $x_0\equiv ct,\ x_{0'}\equiv ct'$ as the "time coordinate" of space-time. The Lorentz factor for a speed of $v$ is defined as:

$$\gamma(v) = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}},\ \text{dimensionless.}$$

The L. transformation relating the frames $x_\mu$ and $x'_\mu$, (the velocity of $O'$ w.r.t. $O$ is $v\mathbf e_1$), is:

$$t' = \gamma(v)\left(t - \frac{vx_1}{c^2}\right),\quad x'_1 = \gamma(v)(x_1 - vt),\quad x'_k = x_k,\quad k = 2,3$$

### Consequences of L.T.

1. The relative speed between two inertial frames is strictly less than $c$ in vacuum.
2. The speed of all mass is less than $c$.
3. The propagation speed any information cannot exceed $c$.

### General transformations

The general L.T. with $\mathbf v$ the velocity of $O'$ w.r.t. $\mathcal S$ is given by:

$$t' = \gamma(v)\left(t - \frac{\mathbf v\cdot\mathbf x}{c^2}\right),\quad \mathbf x' = \mathbf x + (\gamma(v) - 1)\frac{\mathbf v\cdot\mathbf x}{v^2} - \gamma(v)\mathbf vt$$

(*Relativistic law for the addition of velocities.*) For a particle with constant velocity $\mathbf u = u\mathbf e_1$ w.r.t. $\mathcal S$, its velocity w.r.t. $\mathcal S'$ (whose origin moves at constant velocity $v\mathbf e_1$) is also constant, with components:

$$u_1 = \frac{u'_1 + v}{1 + \frac{u'_1 v}{c^2}},\quad u_k = \frac{u'_k}{\gamma(v)\left(1 + \frac{u'_1 v}{c^2}\right)},\quad (k = 2,3),$$

the reversed relations can be obtained by changing the signs and switching primes.

### Intervals

The quadratic form $c^2t^2 - \mathbf x^2\equiv x_0^2 - \mathbf x^2$ is *invariant* under L.T..
The square of the **interval** between two events with s-t coordinates $x_\mu$ and $x_\mu + \Delta x_\mu$ is defined as:

$$\Delta s^2 = c^2\Delta t^2 - \sum_{i=1}^3\Delta x_i^2 = \Delta x_0^2 - \sum_{i=1}^3\Delta x_i^2\equiv\Delta x_0^2 - \Delta\mathbf x^2;$$

said interval is invariant under L.T.$\Rightarrow\Delta s^2 = \Delta x_0^2 - \Delta\mathbf x^2 = \Delta(x'_0)^2 - \Delta(\mathbf x')^2$.
The interval between two events is *time-like* if $\Delta s^2 > 0$, *light-like* if $\Delta s^2 = 0$, and *space-like* if $\Delta s^2 < 0$.

If the interval between two events is *time-like*:
Let the original $\mathcal S$ s.t. $\Delta x_2,\ \Delta x_3 = 0$, and a second $\mathcal S'$ with velocity $\frac{\Delta x_1}{\Delta t}\mathbf e_1$.
Then, in $\mathcal S'$ the two events occur at the same point in space, with time interval (**proper time lapse**) $\Delta\tau = \frac{\Delta s}{c} = \Delta t\sqrt{1 - \frac{\Delta\mathbf x^2}{\Delta x_0^2}}$.

If the interval between two events is *space-like*:
Let the original $\mathcal S'$ s.t. $\Delta x_2,\ \Delta x_3 = 0$, and a second $\mathcal S'$ with velocity $\frac{c^2\Delta t}{\Delta x_1}\mathbf e_1 = \frac{c\Delta x_0}{\Delta x_1}\mathbf e_1$. Then in $\mathcal S'$ the two events occur simultaneously, with distance (**proper distance**) $\sqrt{-\Delta s^2} = |\Delta x'_1|$. The proper distance is less or equal to the spatial distance $|\Delta\mathbf x|$ in any other inertial frame.

### Minkowski product

For two s-t coordinates in $\mathcal S$, $x = (x_0,\mathbf x)$ and $y = (y_0,\mathbf y)\in\mathbb R^4$, their *Minkowski product*:

$$x\odot y\equiv x_0y_0 - \mathbf x\cdot\mathbf y\quad\Rightarrow\quad x^2\equiv x\odot x = x_0^2 - \mathbf x^2,\quad \Delta x\odot\Delta x = \Delta s^2$$

is also invariant under L.T.. The vector space $(\mathbb R^4,\odot)$ is the *Minkowski space*.

### Lorentz group

The procedure of relating the s-t coordinates of an event in two reference frames $\mathcal S$ and $\mathcal S'$ ($\mathcal S'$ moves at $\mathbf v$) is the following:

$$\mathcal S\to\mathcal S''\to\mathcal S'''\to\mathcal S'$$

1. Consider $\mathcal S''$ at rest w.r.t. $\mathcal S$, whose $x''_i$ is in the direction of $\mathbf v\Rightarrow x'' = R_1x$, where $R_1\in\mathbb R^{4\times4}$ but only rotates the spatial coordinates.
2. $\mathcal S'''$ is the frame that moves with $\mathbf v = v\mathbf e''_1$ away from $\mathcal S''\Rightarrow x''' = L(v)x''$. $L(v)$ is the "basic" L.T.
3. The frames $\mathcal S'''$ and $\mathcal S'$ are related by a rotation $\Rightarrow x' = R_2x'''$.
Combining everything we get

$$x' = R_2L(v)R_1x\equiv\Lambda x$$

the transformation $\Lambda\in\mathbb R^{4\times4}$ is the *general Lorentz tranformation*. It is the most genearl form of L.T. that relates the space-time coord. of an event in two reference frames whose space-time origin coincide ($t = x_i = 0\iff t' = x'_i = 0$).
If the space-time origin of the frames differs by a vector $a\in\mathbb R^4$, then the transformation $x' = \Lambda x + a$ is called the *Poincaré tr.*.
The basic L.T. can be expressed as a matrix:

$$L(v) = \begin{bmatrix}\gamma(v) & -\beta(v)\gamma(v) & 0 & 0\\ -\beta(v)\gamma(v) & \gamma(v) & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix},\quad\beta(v)\equiv\frac vc$$

The bilinear form associated with the M. product is:

$$G = \begin{bmatrix}1 &&&\\ & -1 &&\\ && -1 &\\ &&& -1\end{bmatrix},\quad\Rightarrow\quad x\odot y = x^{\mathsf T}G\,y,\quad\Rightarrow\quad\Lambda^{\mathsf T}G\Lambda = G$$

The Minkowski product (and the square of intervals), are invariant under general Lorentz transformations. The set of matrices $\Lambda$ that satisfy the equation above form the *Lorentz group*.
Performing the change of variable:

$$\beta(v) = \tanh\phi\Rightarrow\gamma(v) = \cosh\phi\Rightarrow L(v) = \begin{bmatrix}\cosh\phi & -\sinh\phi & 0 & 0\\ -\sinh\phi & \cosh\phi & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix},$$

And the result of successive Lorentz boosts with speeds $v_1 = c\tanh\phi_1$ and $v_2 = c\tanh\phi_2$ is another L. boost with rapidity $\phi_1 + \phi_2$:
$v = c\tanh(\phi_1 + \phi_2)\equiv\frac{v_1 + v_2}{1 + \frac{v_1v_2}{c^2}}$ similar to the eqn. of velocity of a prtcl.

## 7.3 Dilations and contractions

The **proper time** measured in $\mathcal S$ is larger than the **coordinate time** measured in $\mathcal S'$. In other words, from an observer at rest with $\mathcal S$, the time recorded by a watch at rest with $\mathcal S'$, $t'$, is slower than the proper time $t$ that he has measured. We say that $t'$ dilated, the dilation effect is symmetrical in **inertial frames**:

$$t\equiv\frac{t'}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{t'}{\gamma(v)} > t'$$

For a particle following a trajectory $\mathbf r\equiv\mathbf r(t)$, the proper time increment (time according to an observer at rest with said particle) is expressed as:

$$\mathbf d\tau = \sqrt{1 - \frac{\mathrm v^2(t)}{c^2}}\Rightarrow\Delta\tau(C) = \int_{t_1}^{t_2}\sqrt{1 - \frac{\mathrm v^2(t)}{c^2}}\,\mathbf dt = \int_{t_1}^{t_2}\sqrt{1 - \frac{\dot{\mathbf r}^2}{c^2}}\,\mathrm dt$$

being $C$ the path taken by the particle. $\Delta\tau\le\Delta t$ and is equal only if $\mathbf v = 0,\ \forall t$.

Lorentz-Fitzgerald contraction has to do with contraction of an object that's at rest with $\mathcal S'$ but in motion w.r.t. to $\mathcal S$ in the direction of its velocity. In mathematical terms, the length component in the direction of its movement, $x_1$, is contracted:

$$\Delta x'_1\equiv l_0 = \gamma(v)(\Delta x_1 - v\Delta t)\equiv\gamma(v)l\Rightarrow l = \frac{l_0}{\gamma(v)} = l\sqrt{1 - \frac{\mathbf v^2}{c^2}} < l_0,$$

where $l_0$ is the **rest length** or the length directly measured by an observer at rest with the object.

## 7.4 Four-velocity and four-momentum

The *four-velocity* is defined as:

$$u = \frac{\mathrm dx}{\mathrm d\tau}\in\mathbb R^4,\quad x\equiv\text{“s-t coordinates”},\ \mathrm d\tau = \sqrt{1 - \frac{\mathbf v^2}{c^2}}\,\mathrm dt\equiv\frac{\mathrm dt}{\gamma(v)}$$

and it has the following properties:

$$u = \gamma(v)(c,\mathbf v)\in\mathbb R^4,\quad u' = \Lambda u,\quad u^2 = c^2.$$

The four-momentum is: $\qquad p = mu\Rightarrow p^2 = m^2c^2$
Here we will use the vector $\mathbf p = (p_1,p_2,p_3)\equiv m\gamma(v)\mathbf v\in\mathbb R^3$ which coincides with the non-r. momentum only when $v\to 0$. Then:

$$p_0 = \sqrt{\mathbf p^2 + m^2c^2},\Rightarrow\mathbf v = \frac{c\mathbf p}{p_0}$$

and the relativistic kinetic energy is given as:

$$T = cp_0 - mc^2\equiv mc^2(\gamma(v) - 1)\quad\Rightarrow\quad p_0 = \frac1c(mc^2 + T)$$

### Conservation of four-momentum

$p = \text{const.}\iff cp_0 = mc^2 + T = \text{const.}\iff p_i = m\gamma(v)v_i = \text{const.}$
The total four-momentum of a system of $N$ particles is:

$$P = \sum_{n=1}^N p_n\equiv(P_0,\mathbf P),\quad P_0 = \sum_{n=1}^N p_{n,0}\equiv c\sum_{n=1}^N m_n\gamma(v_n),$$

$$\mathbf P = \sum_{n=1}^N\mathbf p_n\equiv\sum_{n=1}^N m_n\gamma(v_n)\mathbf v_n.$$

The conservation of $P$ during the collision of said $N$ particles is verified even when the collision is not elastic:

$$P_i = P_f\iff(Mc^2 + T)_i = (Mc^2 + T)_f$$

During the process of collision, The mass and the kinetic energy can vary together with the relation:

$$\Delta T = -\Delta(Mc^2)$$

The total relativistic energy can be defined as:

$$E = cP_0 = mc^2 + T = mc^2\gamma(v) = c\sqrt{\mathbf p^2 + m^2c^2} = mc^2\sqrt{1 + \frac{\mathbf p^2}{m^2c^2}}$$

$$\Rightarrow P = (E/c,\mathbf P)\Rightarrow\mathbf v = \frac{c^2\mathbf p}{E}$$

## 7.5 Massless particles

For particle with mass $m\to 0$, its energy is $E = c|\mathbf p|$ and its four-momentum:

$$p = (|\mathbf p|,\mathbf p)\Rightarrow\mathbf v = c\frac{\mathbf p}{|\mathbf p|}.$$

### Motion of a photon

**Photon** is the only massless particle known. The relation between its energy and the frequency $\omega$ of its associated EM wave is given as:

$$E = \hbar\omega\equiv h\nu = \frac{hc}{\lambda},\quad\lambda\equiv\text{“wavelength”}$$

The wave vector is defined as:

$$\omega = c|\mathbf k|,\quad\mathbf v = c\frac{\mathbf k}{|\mathbf k|}\quad\Rightarrow\quad\mathbf p = \hbar\mathbf k$$

We can define the **wave four-vector**: $\quad k = (k_0,\mathbf k) = (|\mathbf k|,\mathbf k) = \frac{p}{\hbar}\in\mathbb R^4$. The wave four-vector is proportional to the four-momentum, therefore, if $\mathcal S'$ is another inertial system and $x' = \Lambda x$ we have: $\quad k' = \Lambda k$

## 7.6 Longitudinal Doppler effect

If an EM wave propagates in the direction of the relative motion between the observer $\mathcal S$ and the source $\mathcal S'$

# 8 Definitions

## 8.1 Lagrange-Hamilton

A system is *natural* when the Hamiltonian of the system is $T + V = E$ (the total energy).
A constraint is *ideal* when the constraint force for each particle is perpendicular to the constraint manifold for that particle.
A constraint is *holonomic* when it's independent of the velocities of the particles.

## 8.2 Rigid body motion

A rigid body is *generic* if it contains three non-collinear particles.

# 9 Miscellaneous

## 9.1 Vectorial calculus

### Cross/dot product

$\mathbf a\times(\mathbf b\times\mathbf c) = (\mathbf a\cdot\mathbf c)\mathbf b - (\mathbf a\cdot\mathbf b)\mathbf c,\quad(\mathbf a\times\mathbf b)^2 = \mathbf a^2\mathbf b^2 - (\mathbf a\cdot\mathbf b)^2$

## 9.2 Chain rule (Partial derivatives)

Given a function $f\equiv f(t,x_1,\dots,x_n)$, $x_i\equiv x_i(t),\forall i$, then:
$\frac{\mathrm df}{\mathrm dt} = \frac{\partial f}{\partial t} + \sum_i\frac{\partial f}{\partial x_i}\frac{\mathrm dx_i}{\mathrm dt}\equiv\frac{\partial f}{\partial t} + \sum_i\frac{\partial f}{\partial x_i}\dot x_i.$

## 9.3 Reduction of two body syst. into one under central field

1. $m_1\ddot{\mathbf r}_1 = \mathbf F_{12}(t,\mathbf r_1,\mathbf r_2,\dot{\mathbf r}_1,\dot{\mathbf r}_2)$, $m_1\ddot{\mathbf r}_1 = \mathbf F_{21} = -\mathbf F_{12}$
2. $\mathbf R = \frac1M(m_1\mathbf r_1 + m_2\mathbf r_2)$ $\mathbf r = \mathbf r_1 - \mathbf r_2\Rightarrow\mu\ddot{\mathbf r} = \mathbf F_{12} = \mathbf F(t,\mathbf r,\dot{\mathbf r})$
3. $\mu = \frac{m_1m_2}{m_1 + m_2}$
4. $\mathbf r_1 = \mathbf R + \frac{m_2}{M}\mathbf r$, $\mathbf r_2 = \mathbf R - \frac{m_1}{M}\mathbf r$. Since $\ddot{\mathbf R} = 0$ we can place the origin at $\mathbf R(t) = 0$ and the ref. frame will still be inertial.
5. $\Rightarrow\mathbf r_1 = \frac{m_2}{M}\mathbf r$, $\mathbf r_2 = -\frac{m_1}{M}\mathbf r$
6. When $m_2$ is much larger than $m_1\Rightarrow\begin{cases}\frac{m_1}{M}\simeq= 0\Rightarrow\mathbf r_2\simeq 0\\ \frac{m_2}{M}\simeq= 1\Rightarrow\mathbf r_1\simeq\mathbf r\end{cases}$

## 9.4 Integrals

- $x = \sqrt{\frac1k}\cosh z\Rightarrow\int\frac{dx}{\sqrt{kx^2 - 1}} = \int\sqrt{\frac1k}\,dz = \sqrt{\frac1k}\cdot z$

## 9.5 ODEs

- $u'' + Cu = 0,\quad C\in\mathbb R$
  If $C < 0\Rightarrow u = ae^{\gamma\theta} + be^{-\gamma\theta},\quad\gamma\equiv\sqrt{|C|} > 0\Rightarrow r = u^{-1}$
  $\Rightarrow r = A\,\mathrm{sech}(\gamma(\theta - \theta_0))\quad(a,b > 0)$
  $\Rightarrow r = Ae^{\pm\gamma(\theta - \theta_0)}\quad(a = 0\text{ or }b = 0)$
  $\Rightarrow r = A\,\mathrm{csch}(\gamma(\theta - \theta_0))\quad(ab < 0)$
  If $C = 0\Rightarrow u = a + b\theta\iff r = \frac{1}{a + b\theta}$
  If $C > 0\Rightarrow u = \frac1A\cos(\gamma(\theta - \theta_0))\iff r = A\sec(\gamma(\theta - \theta_0))$

## 9.6 Trigonometric identities

$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$, $\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$
$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$, $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$
$\cos(2x) = \cos^2x - \sin^2x = 2\cos^2x - 1 = 1 - 2\sin^2x$
$\sin(2x) = 2\sin x\cos x$, $\tan(2x) = \frac{2\tan x}{1 - \tan^2x}$
$\sin^2x - 1 = -\cos(2x)$, $\cos^2x - 1 = -\sin^2x$
$\sin\left(\frac x2\right) = \sqrt{\frac{1 - \cos x}{2}}$, $\cos\frac x2 = \sqrt{\frac{1 + \cos x}{2}}$, $\tan\frac x2 = \sqrt{\frac{1 - \cos x}{1 + \cos x}}$
$\int\sin^2(x)\,\mathrm dx = \frac x2 - \frac{\sin(2x)}{4} + C$, $\int\cos^2(x)\,\mathrm dx = \frac x2 + \frac{\sin(2x)}{4} + C$
$\int\sin^3(x)\,\mathrm dx = \frac{\cos^3x}{3} - \cos x + C$, $\int\cos^3(x)\,\mathrm dx = \sin x - \frac{\sin^3x}{3} + C$

## 9.7 Hyperbolic identities

$\sinh x = \frac{e^x - e^{-x}}{2}$, $\cosh x = \frac{e^x + e^{-x}}{2}$
$\cosh^2x - \sinh^2x = 1$
$\sinh x + y = \sinh x\cosh y + \cosh x\sinh y$,
$\cosh x + y = \cosh x\cosh y + \sinh x\sinh y$
$\sinh x - y = \sinh x\cosh y - \cosh x\sinh y$,
$\cosh x - y = \cosh x\cosh y - \sinh x\sinh y$
$\sinh 2x = 2\sinh x\cosh x$, $\cosh(2x) = \cosh^2x + \sinh^2x$
$\cosh^2x = \frac{1 + \cosh 2x}{2}$, $\sinh^2x = \frac{\cosh 2x - 1}{2}$
$\cosh\frac x2 = \sqrt{\frac{1 + \cosh x}{2}}$, $\sinh\frac x2 = \pm\sqrt{\frac{\cosh x - 1}{2}}$
$\frac{d}{dx}\sinh x = \cosh x$, $\frac{d}{dx}\cosh x = \sinh x$, $\frac{d}{dx}\tanh x = \mathrm{sech}^2x$
$\tanh(\phi_1 + \phi_2) = \frac{\tanh\phi_1 + \tanh\phi_2}{1 + \tanh\phi_1\tanh\phi_2}$
$\frac{1}{\sqrt{1 - \tanh^2\phi}} = \cosh\phi$

## 9.8 Constants and units

Earth radius $\equiv 6.3674\cdot10^6\,\mathrm m$
Centrifugal acceleration on Earth$\equiv 5.93031\cdot10^{-3}\,\mathrm{m\,s^{-2}}$
$h = 2\pi\hbar\equiv$ Planck's constant $\equiv 6.62607015\times10^{-34}\,\mathrm{J\cdot s}$

## 9.9 Dynamic equations

Hooke's Law: $\mathbf F_s = -k(l_0 - x)\mathbf e_x$, $V_s = -k/2(x - l_0)^2$
Electromagnetic (Static fields) force: $\mathbf F = q(\mathbf E + \mathbf v\times\mathbf B)$

## 9.10 Vectorial operators

### Gradients & divergence

Cylindrical:

$$\nabla U = \frac{\partial U}{\partial\rho}\mathbf u_\rho + \frac1\rho\frac{\partial U}{\partial\varphi}\mathbf u_\varphi + \frac{\partial U}{\partial z}\mathbf k$$

$$\nabla\cdot\mathbf F = \frac1\rho\frac{\partial}{\partial\rho} + \frac1\rho\frac{\partial F_\varphi}{\partial\varphi} + \frac{\partial F_z}{\partial z}$$

$$\left(\frac1\rho\frac{\partial F_z}{\partial\varphi} - \frac{\partial F_\varphi}{\partial z}\right)\mathbf u_\rho + \left(\frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial\rho}\right)\mathbf u_\varphi + \left(\frac1\rho\frac{\partial}{\partial\rho} - \frac{\partial F_\rho}{\partial\varphi}\right)\mathbf k$$

Spherical:

$$\nabla U = \frac{\partial U}{\partial r}\mathbf u_r + \frac1r\frac{\partial U}{\partial\theta}\mathbf u_\theta + \frac{1}{r\sin\theta}\frac{\partial U}{\partial\varphi}\mathbf u_\varphi$$

$$\nabla\cdot\mathbf F = \frac{1}{r^2}\frac{\partial}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial F_\varphi}{\partial\varphi}$$

$$\left[\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta} - \frac{\partial F_\theta}{\partial\varphi}\right)\right]\mathbf u_r + \left[\frac1r\left(\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\varphi} - \frac{\partial}{\partial r}\right)\right]\mathbf u_\theta +$$

$$+\left[\frac1r\left(\frac{\partial}{\partial r} - \frac{\partial F_r}{\partial\theta}\right)\right]\mathbf u_\varphi$$