## CM Cheat Sheet Xiaoye Wu, Page 1 of 2

# 1 Coordinate systems

#### 1.1 Cylindrical Basis vectors

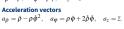
$$\mathbf{e}_{\rho} = \left| \frac{\partial \mathbf{r}}{\partial \rho} \right|^{-1} \frac{\partial \mathbf{r}}{\partial \rho} = (\cos \phi, \sin \phi, 0),$$

$$\begin{split} e_{\phi} &= \left| \frac{\partial r}{\partial \phi} \right|^{-1} \frac{\partial r}{\partial \phi} = (-\sin \phi, -\cos \phi, 0), \\ e_{z} &= \left| \frac{\partial r}{\partial z} \right|^{-1} \frac{\partial r}{\partial z} = (0, 0, 1). \end{split}$$

### Velocity vectors

$$\begin{split} \dot{\mathbf{e}}_{\rho} &= \dot{\phi} \mathbf{e}_{\phi}, \quad \dot{\mathbf{e}}_{\phi} = -\dot{\phi} \mathbf{e}_{\rho}, \quad \dot{\mathbf{e}}_{z} = 0. \\ v_{\rho} &= \dot{\rho}, \quad v_{\phi} = \rho \dot{\phi}, \quad v_{z} = \dot{z}. \end{split}$$

Acceleration vectors
$$a_0 = \ddot{\rho} - \rho \dot{\phi}^2, \quad a_0 = \rho \ddot{\phi} + 2\dot{\rho} \dot{\phi}, \quad a_2 = 0$$



#### 1.2 Spherical **Basis vectors**

$$\mathbf{e}_r = \left| \frac{\partial \mathbf{r}}{\partial r} \right|^{-1} \frac{\partial \mathbf{r}}{\partial r} \\ = (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta) = \frac{\mathbf{r}}{r},$$

$$= (\sin \theta \cos \varphi, \sin \theta \sin \varphi, \cos \theta) = \frac{1}{2}$$

$$\mathbf{e}_{\theta} = \left| \frac{\partial \mathbf{r}}{\partial \theta} \right|^{-1} \frac{\partial \mathbf{r}}{\partial \theta}$$

= 
$$(\cos \theta \cos \varphi, \cos \theta \sin \varphi, -\sin \theta),$$
  
 $\mathbf{e}_{\varphi} = \left| \frac{\partial \mathbf{r}}{\partial \varphi} \right|^{-1} \frac{\partial \mathbf{r}}{\partial \varphi} = (\sin \varphi, \cos \varphi, 0).$ 

$$\begin{split} \dot{\mathbf{e}}_{r} &= \frac{\partial \mathbf{e}_{r}}{\partial \theta} \dot{\boldsymbol{\theta}} + \frac{\partial \mathbf{e}_{r}}{\partial \phi} \dot{\boldsymbol{\phi}} = \dot{\boldsymbol{\theta}} \mathbf{e}_{\theta} + \sin \theta \dot{\boldsymbol{\phi}} \mathbf{e}_{\phi}, \\ \dot{\mathbf{e}}_{\theta} &= \frac{\partial \mathbf{e}_{\theta}}{\partial \theta} \dot{\boldsymbol{\theta}} + \frac{\partial \mathbf{e}_{\theta}}{\partial \theta} \dot{\boldsymbol{\phi}} = -\dot{\boldsymbol{\theta}} \mathbf{e}_{r} + \cos \theta \dot{\boldsymbol{\phi}} \mathbf{e}_{\phi}, \end{split}$$

$$\dot{\mathbf{e}}_{\varphi} = \frac{\partial \mathbf{e}_{\varphi}}{\partial \varphi} \dot{\varphi} = -(\cos \varphi, \sin \varphi, 0) \dot{\varphi} = -\dot{\varphi}(\sin \theta \mathbf{e}_r + \cos \theta \mathbf{e}_{\theta}).$$

$$v_r = \dot{r}, \quad v_\theta = r\dot{\theta}, \quad v_\phi = r\sin\theta\dot{\phi}.$$
Acceleration vectors

$$a_r = \ddot{r} - r\dot{\theta}^2 - r\sin^2\theta \dot{\phi}^2, \quad a_{\theta} = r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\sin\theta\cos\theta \dot{\phi}^2,$$

$$a_{\phi} = r\sin\theta \dot{\phi} + 2\sin\theta \dot{r}\dot{\phi} + 2r\cos\theta \dot{\theta}\dot{\phi}.$$

## Newtonian Mechanics

# 2.1 Newton's Laws 1. In the absence of external forces, the momentum of a particle remains con-

stant.

2. If an external force F acts on a particle, the rate of variation of its momentum is give by  $\mathbf{p} = \mathbf{F} \iff \mathbf{F} = m\ddot{\mathbf{r}} (\Leftarrow \text{eqn. of motion of a particle}).$ 

# 2.2 Conservation laws

 $\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\mathbf{r} \times \dot{\mathbf{r}}, \quad \dot{\mathbf{L}} = \mathbf{N} = \mathbf{r} \times \mathbf{F} = 0 \iff \mathbf{F} \parallel \mathbf{r}.$ A force is central  $\iff$   $\mathbf{F} = g(t, \mathbf{r}, \dot{\mathbf{r}})\mathbf{r}$ . Let  $T = \frac{1}{2}m\dot{\mathbf{r}}^2$ , then  $\frac{dT}{dt} = m\dot{\mathbf{r}}\dot{\mathbf{r}} = \mathbf{F}\dot{\mathbf{r}}$ Kinetic energy is conserved  $\iff \mathbf{F} \perp \dot{\mathbf{r}}, \forall t$ .

A force is conservative 
$$\iff$$
  $\mathbf{F} \perp \mathbf{r}, \forall t$ .

A force is conservative  $\iff$   $\mathbf{F}(\mathbf{r}) = -\nabla V(\mathbf{r}) \equiv -\sum_{i=1}^{3} \frac{\partial V(\mathbf{r})}{\partial x_i} \mathbf{e}_i \equiv -\frac{\partial V(\mathbf{r})}{\partial \mathbf{r}}$ 

If **F** conservative, then E = T + V conserved.

A force is irrotational  $\iff \exists V(t, \mathbf{r}) \text{ s.t. } \mathbf{F}(t, \mathbf{r}) = -\frac{\partial V(t, \mathbf{r})}{\partial u} \Rightarrow \frac{\mathrm{d}E}{\partial t} = \frac{\partial V}{\partial t}$ 

# Gravitational and Electrostatic forces $\mathbf{F} = f(r)\mathbf{e}_r$

 $f(r) = -\frac{GMm}{2}$ ,  $V(r) = -\frac{GMm}{r} \Rightarrow \mathbf{a} = \frac{\mathbf{F}}{m} = -\frac{GM}{3}\mathbf{r}$  independent of m.  $f(r) = k \frac{qQ}{2}, V(r) = k \frac{qQ}{r}.$ 

In general, for forces between a particle and a continuous distribution of

mass/charge: 
$$\begin{split} \mathbf{F}(\mathbf{r}) &= -Gm \int \rho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \, \mathrm{d}^3\mathbf{r} = -m \frac{\partial \Phi(\mathbf{r})}{\partial \mathbf{r}}, \ \Phi(\mathbf{r}) = -G \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \, \mathrm{d}^3\mathbf{r} \\ \mathbf{F}(\mathbf{r}) &= -kq \int \rho(\mathbf{r}') \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \, \mathrm{d}^3\mathbf{r} = -q \frac{\partial \Phi(\mathbf{r})}{\partial \mathbf{r}}, \ \Phi(\mathbf{r}) = -k \int \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3} \, \mathrm{d}^3\mathbf{r} \end{split}$$

 $\mathbf{F}(t, \mathbf{r}, \dot{\mathbf{r}}) = q(\mathbf{E}(t, \mathbf{r}) + \dot{\mathbf{r}} \times \mathbf{B}(t, \mathbf{r})), T + q\Phi(\mathbf{r}) \equiv$  "electromechanical energy"

# 2.3 Motion of a particle in a one-dimensional potential

Law of conservation of energy:  $\frac{1}{2}m\dot{x}^2 + V(x) = E \Rightarrow \dot{x}(m\ddot{x} - F(x)) = 0$ 

(if  $\dot{x} \neq 0 \Rightarrow m\ddot{x} - F(x) = 0$ ). Said ODE is an **autonomous system** (i.e. it's of the form  $\ddot{x} = f(x, \dot{x})$ ), and it's thus invariant under transformations  $t \rightarrow t + c$  and  $t \rightarrow -t$ 

Equilibrium, and trajectory

**Equilibria** are the points  $x_0 \in \mathbb{R}$  s.t.  $x(t) = x_0$  is a solution of the EOM  $\Rightarrow \ddot{x}(t) = 0 \Rightarrow F(x(t)) = F(x_0) = -V(x_0) = 0.$ The accessible region for a given E is the set  $\{x \in \mathbb{R} \mid V(x) \leq E\}$ , which, in

general, are disjoint union of closed intervals. The endpoints of the intervals, if not equilibriums, are the turning points of the trajectories:  $x_i$  turning point  $\iff V(x_i) = E, V'(x_i) \neq 0$ .

In general, the relation between time and displacement is given as:

$$t - t_0 = \pm \sqrt{\frac{m}{2}} \int \frac{\mathrm{d}x}{\sqrt{E - V(x)}} \Rightarrow t = \sqrt{\frac{m}{2}} \int_{x_0}^x \frac{\mathrm{d}s}{\sqrt{E - V(s)}} \equiv \theta(x)$$

## Interval with two turning points:

The time for the particle to go from  $x_0$  to  $x_1$  is  $\frac{\tau}{2}$ , being

$$\tau = 2\theta(x_1) = \sqrt{2m} \int_{x_0}^{x_1} \frac{ds}{\sqrt{E - V(s)}}.$$

The function  $\theta(x)$  is mon. increasing for  $x \in (x_0, x_1)$ , therefore  $\theta$  is invertible and we can write x in terms of t:

$$x(t) = \theta^{-1}(t), 0 \le t \le \tfrac{\tau}{2}, \qquad x(t) = \theta^{-1}(\tau - t), \tfrac{\tau}{2} \le t \le \tau.$$

#### Semi-infinite interval with turning point at one side $[x_0, \infty)$ :

For a particle with initial positon  $x(0) = x_0$  and moving towards infinity, the

$$t_{\infty}=\theta(\infty)=\sqrt{\frac{m}{2}}\int_{x_0}^{\infty}\frac{\mathrm{d}s}{\sqrt{E-V(s)}}$$
, it can be convergent or divergent.  $x$  can be expressed as:

$$x(t) = \theta^{-1}(t), 0 \le t \le t_{\infty}, \qquad x(t) = \theta^{-1}(-t), -t_{\infty} \le t \le 0.$$

Accessible region is  $\mathbb{R}$ :

The time for the particle to reach infinity from an initial positon 
$$x_0$$
 is:
$$t_{+++} = \theta(+\infty) = \sqrt{\frac{\pi}{2}} \int_{-\infty}^{+\infty} \frac{ds}{s}$$

$$t_{\pm\infty} = \theta(\pm\infty) = \sqrt{\frac{m}{2}} \int_{x_0}^{\pm\infty} \frac{ds}{\sqrt{E-V(s)}}$$

$$x(t) = \theta^{-1}(t), t_{-\infty} \le t \le t_{\infty}, \quad x(t) = \theta^{-1}(-t), -t_{\infty} \le t \le t_{-\infty}.$$

#### Stability of equilibria, oscillation approximation

An equilibrium is stable  $\iff$  it is a relative minimum of the potential. For an open ball centered at a stable eq.  $(x_{eq} - \varepsilon, x_{eq} + \varepsilon)$ , the Taylor expansion of V is of the form:

$$\begin{split} V(x) &= V(x_{\rm eq}) + \tfrac{1}{2} V''(x_{\rm eq})(x - x_{\rm eq})^2 + O(x^3) \\ \text{Then the motion of a particle within said open ball can be approximated to } \ddot{x} &= \frac{F(x)}{m} = -\frac{V'(x)}{m} \times \frac{V''(x_{\rm eq})}{m} (x - x_{\rm eq}) \quad \Rightarrow \quad \ddot{\xi} + \omega^2 \xi = 0 \\ \text{with } \xi \equiv x - x_{\rm eq}, \; \omega \equiv \sqrt{\frac{V''(x_{\rm eq})}{m}}. \end{split}$$

The solution of this ODE is:  $\xi = A\cos(\omega t + \alpha)$ ,  $A \in (0, |x - x_{eq}|)$ , and the period of the oscillation is  $\tau \simeq \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{m}{V''(x_{eq})}}$ .

# 2.4 Dynamics of a system of particles

The law of motion of N elements with masses  $m_i$ , positions  $\mathbf{r}_i$  is:

$$m_i\ddot{\mathbf{r}}_i = \sum_{j=1}^{N} \mathbf{F}_{ij} + \mathbf{F}_i^{(e)}, \quad i = 1,...,N,$$

where  $\mathbf{F}_{ij}$  are the forces exerted by j on i, and  $\mathbf{F}_{i}^{(c)}$  are the total external force exerted on i. The solution of this set of N equations is unique if given the initial

$$\mathbf{r}_i(t_0) = \mathbf{r}_{i_0}, \quad \dot{\mathbf{r}}_i(t_0) = \mathbf{v}_{i_0}, \quad i \in \{1,...,N\}.$$

$$\mathbf{F}^{(\mathbf{c})} = \sum_{i=1}^{N} \mathbf{F}_{i}^{(\mathbf{c})}, \quad \sum_{i=1}^{N} m_{i} \ddot{\mathbf{r}}_{i} = \mathbf{F}^{(\mathbf{c})}, \quad \mathbf{R} \equiv \mathbf{M}^{-1} \sum_{i=1}^{N} m_{i} \mathbf{r}_{i}, \quad \mathbf{M} \ddot{\mathbf{R}} = \mathbf{F}^{(\mathbf{c})}.$$

$$\mathbf{P} \equiv \sum\limits_{i=1}^{N} m_i \dot{\mathbf{r}}_i = M \dot{\mathbf{R}} \Rightarrow \dot{\mathbf{P}} = \mathbf{F}^{(\mathbf{c})} \Rightarrow \mathbf{P} \text{ conserved } \Longleftrightarrow \mathbf{F}^{(\mathbf{c})} = 0.$$
Ansular momentum:

$$\begin{split} \mathbf{L} &= M\mathbf{R} \times \dot{\mathbf{R}} + \sum_{i=1}^{N} m_i \mathbf{r}_i' \times \dot{\mathbf{r}}_i' \equiv \mathbf{L}_{CM} + \sum_{i=1}^{N} m_i \mathbf{r}_i' \times \dot{\mathbf{r}}_i', & \mathbf{r}_i = \mathbf{R} + \mathbf{r}_i', \\ \dot{\mathbf{L}} &= \sum_{i=1}^{N} \mathbf{r}_i \times \mathbf{F}_i^{(c)} \equiv \mathbf{N}^{(c)} \quad \Rightarrow & \mathbf{L} \text{ conserved} \iff \mathbf{N}^{(c)} = \mathbf{0} \end{split}$$

Kinetic energy, potential, and total mechanical energy:

$$T = \frac{1}{2} \sum_{i=1}^{N} m_i \hat{\mathbf{r}}_i^2 = \frac{1}{2} M \hat{\mathbf{R}}^2 + \frac{1}{2} \sum_{i=1}^{N} m_i (\mathbf{r}_i')^2.$$
 If the forces acting on the system

is conservative, in other words, if  $\mathbf{F}_i \equiv \mathbf{F}_i^{(e)} + \sum_{i=1}^{N} \mathbf{F}_{ij} = -\frac{\partial V}{\partial \mathbf{r}_i}$ , i = 1,...,N; then the total mechanical energy  $E = T + V(\mathbf{r}_1, ..., \mathbf{r}_N)$  of the system is con-

The potential of the system has the form:  $V = \sum_{i=1}^{N} V_i(\mathbf{r}_i) + \sum_{1 \le i < j < N} V_{ij}(\mathbf{r}_i, \mathbf{r}_j)$ .

Defining  $V_{ij}(\mathbf{r}_i, \mathbf{r}_j) = U_{ij}(\mathbf{r}_i - \mathbf{r}_j) \Rightarrow V = \sum\limits_{i=1}^N V_i(\mathbf{r}_i) + \sum\limits_{1 \leq i < j < N} U_{ij}(\mathbf{r}_i - \mathbf{r}_j).$ 

# 3 Central forces

### 3.1 Basic definitions

#### Conditions for a central force

 $\mathbf{F}_{12}=f(|\mathbf{r}_1-\mathbf{r}_2|)\,rac{\mathbf{r}_1-\mathbf{r}_2}{|\mathbf{r}_1-\mathbf{r}_2|}\Rightarrow \mathbf{F}(\mathbf{r})=\mu\ddot{\mathbf{r}}=f(r)\,rac{\mathbf{r}}{r}$  , in such case the following equations hold:  $\ddot{r} - r\dot{\theta}^2 = \frac{f(r)}{r}$ ,  $r\ddot{\theta} + 2\dot{r}\dot{\theta} = 0$ 

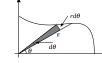
# 3.2 Conserved quantities

 $\mathbf{L} = \mu \mathbf{r} \times \dot{\mathbf{r}} = \mu r^2 \dot{\boldsymbol{\theta}} \mathbf{e}_z, \; |\mathbf{L}| = L_z = \dot{\boldsymbol{\theta}} r^2 \mu > 0$  $E = \frac{1}{2}\mu r^2 + U(r) = \frac{L^2}{2\mu}(u'^2 + u^2) + V(\frac{1}{\mu})$ 

Law of a reas 
$$A(\theta) = \frac{1}{2} \int_{\theta_0}^{\theta} r^2(\alpha) \mathrm{d}\alpha,$$
 
$$\dot{A} = \frac{\mathrm{d}A}{\mathrm{d}\theta} \dot{\theta} = \frac{1}{2} r^2 \dot{\theta} = \frac{L}{2\mu} \in \mathbb{R}$$

# 3.3 Important equations

$$f(r) = -\frac{L^2}{\mu r^2} (u'' + u)$$



U(r) = V(r) + 
$$\frac{L^2}{2\mu r^2}$$
,  $-\frac{\partial}{\partial r} \left(\frac{L^2}{2\mu r^2}\right) = \frac{\mu v_\theta^2}{r^2} e_r$   
 $t = \pm \sqrt{\frac{\mu}{2}} \int_{r_1}^{r_2} \frac{dr}{\sqrt{E - U(r)}} = \frac{\mu}{L} \int_{r_2}^{r_2} r^2(\theta) d\theta$ ,

$$\theta = \frac{L}{\mu} \int_{t_1}^{t_2} \frac{dt}{r^2(t)} = \pm \frac{L}{\sqrt{2\mu}} \int_{-\frac{1}{\mu}}^{\frac{1}{\mu}} \frac{du}{\sqrt{E - U(\frac{1}{\mu})}},$$

$$\begin{split} \tau_r &= \sqrt{2\mu} \int_{r_1}^{r_2} = \frac{\mathrm{d}r}{\sqrt{E-U(r)}}, \ \Delta\theta = \sqrt{\frac{2\mu^2}{\mu}} \int_{-\frac{1}{r_2}}^{\frac{1}{r_1}} = \frac{\mathrm{d}u}{\sqrt{E-U(\frac{1}{\mu})}}, \\ \text{An orbit is periodic} &\iff \Delta\theta = q\pi, \ q \in \mathbb{Q}. \end{split}$$

$$\begin{split} A_T &= \pi r^2 = \pi ab = \dot{A} \cdot \tau = \frac{L}{2\mu} \, \tau, \\ v &= \frac{L}{u} \, \sqrt{u'^2 + u^2} \end{split}$$

### 3.4 Kepler's potential

$$\mathbf{F}(\mathbf{r}) = -\frac{k}{r^2}\,\mathbf{e}_r,\,V(r) = -\frac{k}{r}\,,\,\ddot{\mathbf{r}} = -GM\,\frac{\mathbf{r}}{r^3}\,,u'' + u = \frac{\mu k}{L^2}$$
 Equation of trajectory and its clasifications

$$u = \frac{\mu k}{72} (1 + e \cos(\theta - \theta_0)), e, \theta_0 \in \mathbb{R}.$$

$$\begin{split} r &= \frac{\alpha}{1 + ccos(\theta)}, \ \alpha = \frac{L^2}{\mu^2}, \ e = \sqrt{1 + \frac{2EL^2}{\mu^2}} \ge 0 \\ e &= 1 &\iff E > 0 &\implies \text{psyncholo} \\ e &= 1 &\iff E = 0 &\implies \text{psyncholo} \\ e &= (0, 1) &\iff E \in (-\frac{\mu L^2}{2L^2}, 0) \implies \text{dilipse} \\ e &= 0 &\iff E = -\frac{\mu L^2}{\mu^2} \implies \text{circle} \end{split}$$

#### 3.5 Planetary motion Study of ellipses

C = {
$$(1-e^2)(x+\frac{\alpha e}{1-e^2})^2+y^2=\frac{\alpha^2}{1-e^2}$$
}  
 $a=\frac{\alpha}{1-e^2}$ ,  $b=\frac{\alpha}{\sqrt{1-e^2}}$ ,  $c=\varepsilon a=ea$ ,  $e=\varepsilon$ ,



$$\begin{split} E &= -\frac{k}{2a}, \ \langle r \rangle = \left(1 + \frac{r^2}{2}\right)a, \\ \tau &= 2\pi\sqrt{\frac{\mu}{k}}\,a^{3/2} = \pi k\sqrt{\frac{\mu}{2}}\,|E|^{-3/2} = \frac{2\pi a^{3/2}}{\sqrt{GM}} \simeq \frac{2\pi a^{3/2}}{\sqrt{GM_{\odot}}} \\ p &= \frac{\alpha}{1+\sigma} = a(1-e), \ p' = \frac{\alpha}{1-\sigma} = a(1+e), \\ v^2 &= \frac{k^2}{L^2}\left(1 + e^2 + 2e\cos\theta\right) \Rightarrow \left\{ \begin{array}{c} v_p &= \frac{k}{L}(1+e) = \sqrt{\frac{k}{\mu p'}(1+e)} \\ v_{p'} &= \frac{k}{L}(1-e) = \sqrt{\frac{k}{\mu p'}(1-e)} \end{array} \right. \end{split}$$

# 4 Lagrangian & Hamiltonian

## 4.1 Basic Definitions

 $F[y] = \int_{0}^{\infty} dx$ , where  $F \equiv$  functional, and  $f \equiv$  density.

#### 4.2 Important equations and theorems **Euler-Lagrange equations:**

 $\frac{d}{dx}\frac{\partial f}{\partial y^l} - \frac{\partial f}{\partial y} = 0$ ;  $\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q}$ . If  $\frac{\partial f}{\partial x} = 0 \Rightarrow y'\frac{\partial f}{\partial y^l} - f$  is constant. If  $\frac{\partial f}{\partial v} = 0 \Rightarrow \frac{\partial f}{\partial v^2}$  is constant. Two densities differing by a total derivative has

#### Hamilton's principle for N particles:

$$L(t, \mathbf{r}_1, ..., \mathbf{r}_N, \dot{\mathbf{r}}_1, ..., \dot{\mathbf{r}}_N) = \frac{1}{2} \sum_{i}^{N} m_i \dot{\mathbf{r}}_i^2 - V(t, \mathbf{r}_1, ..., \mathbf{r}_N).$$
If  $\frac{\partial L}{\partial t} = 0 \Rightarrow \sum \dot{\mathbf{r}}_i \frac{\partial L}{\partial \dot{\mathbf{r}}_i} - L = T + V$  conserved.

If  $\frac{\partial L}{\partial x} = 0 \Rightarrow \frac{\partial L}{\partial \xi}$  is conserved.

 $p_i = \frac{\partial L}{\partial \hat{q}_i} \Rightarrow \dot{p}_i = \frac{\partial L}{\partial q_i}$ . If  $p_i = 0$  then the coordinate  $q_i$  is cyclic or ignorable.

# $H = \sum_{i=1}^{n} \dot{q}_{i} \frac{\partial L}{\partial \dot{q}_{i}} - L = \sum_{i=1}^{n} p_{i} \dot{q}_{i} - L = T + V (\Leftarrow \text{ If natural}).$

# Hamilton's canonical eqns.:

 $\dot{\mathbf{q}} = \frac{\partial H}{\partial \mathbf{p}}(t, \mathbf{q}, \mathbf{p}); \, \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{q}}(t, \mathbf{q}, \mathbf{p})$ 

Jacobi-Poisson Theorem
If f and g are two first integrals of H's canonical equations, so is  $\{f, g\}$ 

#### 4.3 Procedures

# Find the EOM of N particles with I holonomic constraints:

1. Introduce 3N - l = n generalized coordinates  $(q_1, ..., q_n) = \mathbf{q}$  to the constraint manifold  $\{\Phi_1, ..., \Phi_l(t, \mathbf{q}) = 0\}$ .

- 2. Express  $T = \frac{1}{2} \sum_{i=1}^{N} m_i \dot{\mathbf{r}}_i^2$  and V in terms of  $(t, \mathbf{q}, \dot{\mathbf{q}}) \Rightarrow$
- $T \equiv T(t, \mathbf{q}, \dot{\mathbf{q}}); V \equiv V(t, \mathbf{q}).$ 3. E-L equations of L = T - V are the *n* equations of motion in coordinates  $q_i$  $\Rightarrow \frac{d}{dt} \frac{\partial L}{\partial a_i} - \frac{\partial L}{\partial a_i}, i = 1,...,n.$
- 4. Constraint forces on i-th particle are expressed as:  $\mathbf{F}_{i}^{(c)} = m_{i}\ddot{\mathbf{r}}_{i} + \frac{\partial V}{\partial \mathbf{r}_{i}}, i = 1,...,N.$

## Write Hamilton's canonical equations:

- 1. Calculate the canonical momenta  $p_i = \frac{\partial L}{\partial \hat{x}}(t, \mathbf{q}, \dot{\mathbf{q}}), i = 1, ..., n.$
- 2. Express the generalized velocities as  $\dot{q}_i \equiv \dot{q}_i(t, \mathbf{q}, \mathbf{p}), i = 1, ..., n$ .
- 3. Compute the Hamiltonian  $H(t, \mathbf{q}, \mathbf{p}) = \mathbf{p} \cdot \mathbf{q}(t, \mathbf{q}, \mathbf{p}) L(t, \mathbf{q}, \mathbf{p})$ . 4. The partial derivatives of H w.r.t the variables  $q_1, p_1$  are the canonical equations. The first n equations are the ones in step 2, the rest are  $\dot{p}_i = -\frac{\partial H}{\partial x}$ , i =1, ..., n. There are in total 2n equations.

### 4.4 Hamiltonian conservations

$$\partial H \dot{q}_i = 0 \Rightarrow p_i \text{ const. } \frac{\partial H}{\partial p_i} = 0 \Rightarrow q_i \text{ const.}$$

 $\frac{dH}{dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial a}\dot{\mathbf{q}} + \frac{\partial H}{\partial a}\dot{\mathbf{p}} = \frac{\partial H}{\partial t} \Rightarrow \text{If } \frac{\partial H}{\partial t} = 0 \Rightarrow H \text{ const. Moreover, since}$  $\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$ , H is conserved  $\iff$  L indep. of t.

#### 4.5 Poisson brackets

 $(\mathbf{q}, \mathbf{p}) \equiv$  "phase space", for every smooth function  $f(t, \mathbf{q}, \mathbf{p})$  the following holds:  $\dot{f} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \mathbf{q}} \dot{\mathbf{q}} + \frac{\partial f}{\partial \mathbf{p}} \dot{\mathbf{p}} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \mathbf{q}} \frac{\partial H}{\partial \mathbf{p}} + \frac{\partial f}{\partial \mathbf{p}} \frac{\partial H}{\partial \mathbf{q}} = \frac{\partial f}{\partial t} + \{f, H\}$ 

Being  $\{f,g\} \equiv \frac{\partial f}{\partial \mathbf{q}} \frac{\partial g}{\partial \mathbf{p}} - \frac{\partial f}{\partial \mathbf{p}} \frac{\partial g}{\partial \mathbf{q}} = \sum_{i=1}^{n} \left( \frac{\partial f}{\partial q_{i}} \frac{\partial g}{\partial p_{i}} - \frac{\partial f}{\partial p_{i}} \frac{\partial g}{\partial q_{i}} \right), f, g \text{ any func-}$ 

#### tion of $(t, \mathbf{q}, \mathbf{p})$ .

Poisson brackets properties

$$\begin{split} \{f,g\} &= -\{g,f\}, \text{ in particular } \{f,f\} = 0. \\ \{\lambda f + \mu g,h\} &= \lambda \{f,h\} + \mu \{g,h\}. \end{split}$$
(Antisymmetry) (Bilinearity)  ${fg,h} = f{g,h} + g{f,h}.$ (Leibniz's rule)  $\{\{f,g\},h\}+\{\{g,h\},f\}+\{\{h,f\},g\}=0.$ (Jacobi identity)  $\frac{\partial}{\partial t} \{f, g\} = \{\frac{\partial f}{\partial t}, g\} + \{f, \frac{\partial g}{\partial t}\} \Rightarrow \frac{d}{dt} \{f, g\} = \{f, g\} + \{f, g\}$ (J-P id.)  $\dot{q}_i = \{q_i, H\}, \ \dot{p}_i = \{p_i, H\}; \ \{q_i, q_i\} = \{p_i, p_i\} = 0, \ \{q_i, p_i\} = \delta_{ij}$ Canonical change of variable

Consists in changing  $(\mathbf{q}, \mathbf{p}) \mapsto (\tilde{\mathbf{q}}, \tilde{\mathbf{p}})$ , s.t.  $\exists \tilde{H}$  Hamiltonian where  $\dot{\tilde{\mathbf{q}}} = \frac{\partial \tilde{H}}{\partial \tilde{\mathbf{q}}}$  and

The changes 
$$\left\{ \begin{array}{l} (\mathbf{q},\mathbf{p}) \mapsto (\mathbf{p},\mathbf{q}), \ \tilde{H}(t,\tilde{\mathbf{q}},\tilde{\mathbf{p}}) = -H(t,\mathbf{q},\mathbf{p}) \\ (\mathbf{q},\mathbf{p}) \mapsto (\mathbf{p},-\mathbf{q}), \ \tilde{H}(t,\tilde{\mathbf{q}},\tilde{\mathbf{p}}) = H(t,\mathbf{q},\mathbf{p}) \end{array} \right. \text{ are canonical.}$$

In general, a transformation is canonical  $\iff \left\{ \begin{array}{l} \{\tilde{q}_i, \tilde{q}_j\} = \{\tilde{p}_i, \tilde{p}_j\} = 0 \\ \{\tilde{q}_i, \tilde{p}_j\} = \lambda \delta_{ij}, \ \lambda \neq 0. \end{array} \right.$ If  $\lambda = 1$  then  $(\tilde{\mathbf{q}}, \tilde{\mathbf{p}})$  are canonically conjugate.

If  $\lambda \neq 1$ , the change  $(\mathbf{q}, \mathbf{p}) \mapsto (\tilde{\mathbf{q}}, \tilde{\mathbf{p}}/\lambda)$  is canonically conjugate

#### Hamilton-Jacobi theory It's always possible to find a can. transf. that has $\tilde{H} = 0 \Rightarrow \dot{\tilde{\mathbf{q}}} = 0 = \dot{\tilde{\mathbf{p}}}$

 $\Rightarrow \tilde{\mathbf{p}} = \tilde{\mathbf{p}}_0$ ;  $\tilde{\mathbf{q}} = \tilde{\mathbf{q}}_0$  constant solutions, and reversing the transformation we will get the original  $q_0$ ,  $p_0$  constant solutions of the original H.

# Charged particle mass in an EM field $\Phi$ (not natural):

 $L(t, \mathbf{r}, \dot{\mathbf{r}}) = \frac{1}{2} m \dot{\mathbf{r}}^2 - e \Phi(t, \mathbf{r}) + e \dot{\mathbf{r}} \cdot \mathbf{A}(t, \mathbf{r})$  $H(t,\mathbf{r},\mathbf{p}) = \frac{1}{2m} (\mathbf{p} - e\mathbf{A}(t,\mathbf{r}))^2 + e\Phi(t,\mathbf{r})$   $p_i = m\dot{x}_i + eA_i(t,\mathbf{r}), i = 1,2,3.$ 

# 5 Non-inertial frame

# 5.1 Basic definitions

 $S' \equiv \text{inertial reference frame}, S \equiv \text{non inertial reference frame}.$  $\left(\frac{d}{dt}\right)_{\mathcal{E}}t \equiv \text{time derivative w.r.t } \mathcal{E}', \left(\frac{d}{dt}\right)_{\mathcal{E}}t \equiv \text{time derivative w.r.t } \mathcal{E},$ 

the vectors between the frames are related by  $\mathbf{e}'_i(t) = O(t)\mathbf{e}_i$ , being O(t) the

orthogonal linear application.   
**Rotation matrices and relations between ref. frames** 
$$SO(3) = \{M \in \mathbb{R}^{3 \times 3} | MM^T = M^TM = z, \det M = 1\}$$

$$\begin{split} \forall M \in SO(3), \; \exists n \in \mathbb{R}^3 \; \text{and reference frame} \; \mathcal{S}' = \{e_1, e_2, n\} \\ \text{s.t.} \; \max_{\mathcal{S}'}(M) \equiv R_3(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{split}$$

$$\frac{d}{d\theta} (R_{\mathbf{n}}(\theta)) = \frac{d}{d\epsilon} \Big|_{\epsilon=0} R_{\mathbf{n}}(\theta + \epsilon) = \frac{d}{d\epsilon} \Big|_{\epsilon=0} R_{\mathbf{n}}(\epsilon) R_{\mathbf{n}}(\theta)$$

$$= \Omega R_{\mathbf{n}}(\theta) = \mathbf{n} \times R_{\mathbf{n}}(\theta)$$

$$O(t) \in SO(3), \forall t, \text{ and } \dot{O}(t) = \Omega(t)O(t),$$

where 
$$\Omega(t) = \begin{bmatrix} 0 & -\omega_1(t) & \omega_2(t) \\ \omega_1(t) & 0 & -\widetilde{\omega}_1(t) \end{bmatrix}$$
 antisymmetric matrix.  
 $\dot{O}(t)c = \omega(t) \times O(t)c, \quad \omega = (\omega_1, \omega_2, \omega_3). \quad \forall t \in \mathbb{R} \text{ and } \forall c \in \mathbb{R}^3$   
" $\omega(t)x = \Omega(t) = \dot{O}(t)O(t)^{-1} = \dot{O}(t)O(t)^{-1} \iff$ 

$$\mathbf{\omega}(t_0) \times = \frac{d}{dt} \Big|_{t=t_0} O(t) O^{-1}(t_0).$$
  

$$\dot{\mathbf{e}}_i(t) = \mathbf{\omega}(t) \times O(t) \mathbf{e}'_i = \mathbf{\omega}(t) \times \mathbf{e}_i(t).$$

 $\mathbf{\omega}(t) = \dot{\boldsymbol{\theta}}(t)\mathbf{n}(t)$ , in words: the direction of the vector  $\mathbf{\omega}(t)$  is equal to the axis of rotation  $\mathbf{n}(t)$ , and the magnitude of  $\boldsymbol{\omega}(t)$  is equal to the angular veloc-

#### ity of the moving axes w.r.t the fixed ones. 5.2 Time derivative in different reference frames

5.2 Time derivative in different reference fram 
$$\left(\frac{\mathrm{d} A(t)}{dt}\right)_f = \left(\frac{\mathrm{d} A(t)}{dt}\right)_m + \omega(t) \times A(t),$$
 in particular  $\left(\frac{\mathrm{d} \omega(t)}{dt}\right)_f = \left(\frac{\mathrm{d} \omega(t)}{dt}\right)_m = \dot{\omega}(t)$  If the origin of  $\mathcal{S}$  is displacing from that of  $\mathcal{S}'$  with  $R(t)$ , then:

# $\left(\frac{\mathrm{d}\mathbf{A}(t)}{\mathrm{d}t}\right)_{\mathrm{f}} = \left(\frac{\mathrm{d}\mathbf{A}(t)}{\mathrm{d}t}\right)_{\mathrm{m}} + \boldsymbol{\omega}\left(t\right) \times \mathbf{A}(t) + \mathbf{V}(t),$ 5.3 Dynamics in non-inertial ref. frame

In the general case, the origin of S is displaced from that of S' by a timedependent vector  $\mathbf{R}(t)$ . If  $\mathbf{r}$  is the position of a particle w.r.t  $\mathcal{S}$ , then the position

w.r.t 
$$\mathcal{S}'$$
 is given by  $\mathbf{r}' = \mathbf{r} + \mathbf{R}$ . We'll define the following vectors: 
$$V = \left(\frac{d\mathbf{r}}{dt}\right)_f \cdot \mathbf{A} = \left(\frac{d\mathbf{r}}{dt}\right)_f = \left(\frac{d^2\mathbf{R}}{dt^2}\right)_f \cdot \mathbf{v}_f = \left(\frac{d\mathbf{r}'}{dr}\right)_f \cdot \mathbf{v}_m = \left(\frac{d\mathbf{r}'}{dr}\right)_m \\ \mathbf{a}_f = \left(\frac{d\mathbf{r}_f}{dt^2}\right)_f = \left(\frac{d^2\mathbf{r}}{dt^2}\right)_f \cdot \mathbf{a}_m = \left(\frac{d\mathbf{r}_m}{dr}\right)_m = \left(\frac{d^2\mathbf{r}}{dt^2}\right)_m.$$

Equations of motion

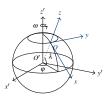
 $v_f = v_m + \omega \times r + V$ ,  $\mathbf{a}_{f} = \mathbf{a}_{m} + \mathbf{A} + 2\boldsymbol{\omega} \times \mathbf{v}_{m} + \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) + \dot{\boldsymbol{\omega}} \times \mathbf{r},$  $m\mathbf{a}_{m} = \mathbf{F} - m\mathbf{A} - 2m\boldsymbol{\omega} \times \mathbf{v}_{m} - m\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}) - m\boldsymbol{\dot{\omega}} \times \mathbf{r} \equiv \mathbf{F} + \mathbf{F}_{i}$ 

where  $\mathbf{F}_i$  is the fictitious force:

 $\mathbf{F_i} = -m\mathbf{A} - 2m\mathbf{\omega} \times \mathbf{v_m} - m\mathbf{\omega} \times (\mathbf{\omega} \times \mathbf{r}) - m\dot{\mathbf{\omega}} \times \mathbf{r}$ Centrifugal force  $\equiv -m \omega \times (\omega \times \mathbf{r}),$ Coriolis force  $\equiv -2m \omega \times v_m$ , depends on the velocity of the particle.

For the equation  $\dot{\mathbf{A}}(t) = \boldsymbol{\omega} \times \mathbf{A}(t) + \mathbf{c}$ , where  $\boldsymbol{\omega} \perp \mathbf{c}$  are both constant vectors, the general solution of the ODE is:  $\mathbf{A}(t) = R_{\mathbf{n}}(\omega t)\mathbf{u} + \frac{1}{\omega}\mathbf{n} \times \mathbf{c}$ , with  $\mathbf{u} \in \mathbb{R}^3$ ,  $\mathbf{n} = \boldsymbol{\omega}^{-1} \boldsymbol{\omega}$ ,.

# 5.4 Motion of a particle relative to the rotating Earth



 $e_1 = \sin\lambda\cos\phi e_1' + \sin\lambda\sin\phi e_2' - \cos\lambda e_2',$  $\mathbf{e}_2 = -\sin \varphi \mathbf{e}_1' + \cos \varphi \mathbf{e}_2'$  $e_3 = \cos \lambda \cos \varphi e'_1 + \cos \lambda \sin \varphi e'_2 + \sin \lambda e'_3$ .

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 $\omega = \omega(-\cos\lambda \mathbf{e}_1 + \sin\lambda \mathbf{e}_3)$   $\mathbf{g}_0 = -\frac{GM}{c^3}\mathbf{R} = -\frac{GM}{c^2}\mathbf{e}_3 \equiv -g_0\mathbf{e}_3, g_0 \simeq 9.8 \text{ m} \cdot \text{s}^{-2}$ 

 $\ddot{\mathbf{r}} = \mathbf{g} - 2\boldsymbol{\omega} \times \dot{\mathbf{r}} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r}), \text{ where } \mathbf{g} = \mathbf{g}_0 - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{R})$ With additional force:  $\ddot{\mathbf{r}} = \frac{\mathbf{F}}{m} + \mathbf{g} - 2\boldsymbol{\omega} \times \dot{\mathbf{r}} - \boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{r})$ .

The angle between  $-\mathbf{e}_3$  and  $\mathbf{g}$  is  $\delta(\lambda)$ , where  $\tan \delta(\lambda) = \frac{\omega^2 R \sin \lambda \cos \lambda}{g_0 - \omega^2 R \cos^2 \lambda}$ 

Approximately, the equation of motion of a particle moving close to the Earth's surface, is given by  $\mathbf{r} \simeq \mathbf{r}_0 + \mathbf{v}_0 t + \mathbf{g}_0 \frac{t^2}{2} - t^2 \omega \times \mathbf{v}_0 - \frac{t^3}{3} \omega \times \mathbf{g}_0$ . Sometimes, to approximate, we often ignore some terms inside the above equation because of the following ratio:  $\frac{g_0}{\omega} \simeq 1.35 \cdot 10^5 \, \text{m s}^{-1} >> v_0$ 

# 5.5 Foucault's pendulum

Consists of a pendulum with length that is far greater than the coordinates x, y, z of its bob. The equations of motion of the coordinates x, y are:

 $\ddot{x} + \alpha^2 x = 2\omega y \sin \lambda$ , where  $\alpha \equiv \sqrt{\frac{g_0}{l}}$ . The motion of the coordinate z can be ignored. Letting u = x + iy, the general solution of u is:  $u = e^{-i\Omega t} \left( c_1 e^{i\alpha t} + c_2 e^{-i\alpha t} \right)$ , where  $c_1, c_2 \in \mathbb{C}$  and  $\Omega \equiv \omega \sin \lambda$ .

A solution with initial conditions  $\begin{cases} x(0) = x_0 > 0, y(0) = 0 \\ \dot{y} = \dot{y} = 0 \end{cases}$ , would be:  $u = x_0 e^{-i\Omega t} \cos(\alpha t) \Rightarrow x = x_0 \cos(\Omega t) \cos(\alpha t), \ y = -x_0 \sin(\Omega t) \cos(\alpha t).$ The projection of the vector  $\mathbf{r}$  onto the X-Y plane is given by  $x_0 \cos(\alpha t) [\cos(\Omega t) e_1 - \sin(\Omega t) e_2]$ . In the northern hemisphere the pendulum's plane rotates clockwise, i.e., in the east-south direction (since  $\dot{\theta} = -\Omega =$ 

 $-\omega \sin \lambda < 0$ ), with angular velocity  $\Omega = \omega \sin \lambda$ . In the southern hemisphere the rotation of the pendulum's plane is counterclockwise (since  $\sin \lambda < 0$ ), and in the equator ( $\lambda = 0$ ) no such rotation occurs. In each period  $2\pi/\alpha$  of the pendulum the angle between the pendulum's plane and, the X-Y plane increases by  $-2\pi\Omega/\alpha$ , which is a very small quantity. The period of the rotation of the pendulum's plane is given by

# $\tau = \frac{2\pi}{\Omega} = \frac{2\pi}{\omega} \csc \lambda$ sidereal days. 6 Rigid body motion

Note: some of the notations in this part is mixed with the ones used by Amores in Una base para el álgebra lineal, in which column vectors are not represented in hold

**6.1 Basic definitions** A **rigid body** is a system of particles of mass  $m_{\alpha}$  ( $\alpha = 1,...,N$ ) in which the distance  $|\mathbf{r}_{\alpha} - \mathbf{r}_{\beta}|$  between any two particles is constant. In other words, a rigid body is a mechanical system of N particles subject to the N(N-1)/2time-independent holonomic constraints (that may or may not be mutually in-

In a generic rigid body it's always possible to construct a set of moving axes S(body axes), with respect to which all it's particles are fixed. A generic rigid body has 6 d. of freedom, 3 for the coordinate of the origin of

the body axes, and 3 for the matrix  $O(t) \in SO(3)$  that relates S' and S. Centre of mass of a r.b.:  $\mathbf{R} = \frac{1}{M} \int \rho(\mathbf{r}) \mathbf{r} d^3 \mathbf{r}$ , where  $M = \int \rho(\mathbf{r}) d^3 \mathbf{r}$ .

# 6.2 Angular momentum and T of a rigid body

Letting the centre of mass be the origin of S:  $\mathbf{r}'_{\alpha} = \mathbf{R} + \mathbf{r}_{\alpha}, \ \mathbf{v}'_{\alpha} = \mathbf{V} + \boldsymbol{\omega} \times \mathbf{r}_{\alpha}, \ \mathbf{P} = M\mathbf{V}.$  $\mathbf{L} = M\mathbf{R} \times \mathbf{V} + \mathbf{L}_{CM}$ , where  $\mathbf{L}_{CM} = \sum m_{\alpha} \mathbf{r}_{\alpha} \times (\boldsymbol{\omega} \times \mathbf{r}_{\alpha})$  $T = \frac{1}{2}MV^2 + T_{\text{rot}}$ , where  $T_{\text{rot}} = \frac{1}{2}\sum m_{\alpha}(\omega \times \mathbf{r}_{\alpha})^2$ 

# $\Rightarrow \mathbf{L}_{\mathrm{CM}} = \sum m_{\alpha} [r_{\alpha}^{2} \boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \mathbf{r}_{\alpha}) \mathbf{r}_{\alpha}], \quad T_{\mathrm{rot}} = \frac{1}{2} \boldsymbol{\omega} \cdot \mathbf{L}_{\mathrm{CM}}$

**6.3 Inertia tensor**Note: the inertia tensor *I*, without any subscript, is calculated w.r.t. the body axes (with CM as it's origin). But it can also be calculated w.r.t other points

(Steiner's theorem)

The inertial tensor 
$$(I \in \mathbb{R}^{3 \times 3})$$
 is a symmetric matrix with elements:  

$$I_{ij} = \sum_{\alpha} m_{\alpha} (\delta_{ij} r_{\alpha}^2 - x_{\alpha i} x_{\alpha j}) \Rightarrow \begin{cases} I_{ii} = m_{\alpha} (x_{\alpha j}^2 + x_{\alpha k}^2) \\ I_{ij} = -\sum_{\alpha} m_{\alpha} x_{\alpha i} x_{\alpha j} \end{cases}$$
If the r.b. is continuous:  

$$\begin{cases} I_{ii} = \int_{\Omega} \rho(\mathbf{r})(x_j^2 + x_k^2) d^3\mathbf{r}, \ i = 1, 2, 3 \\ I_{ij} = -\int_{\Omega} \rho(\mathbf{r}) x_i x_j d^3\mathbf{r}, \ 1 \le i \ne j \le 3 \end{cases}$$

 $T_{\text{rot}} = \frac{1}{2} \omega \cdot (I \omega) = \frac{1}{2} \omega^{\mathsf{T}} I \omega$  ( $\Leftarrow$  here  $\omega$  is a column (lin. algebraic) vector.  $\Rightarrow T_{\text{rot}} = \frac{1}{2}I_{\text{n}}\omega^2$ , where  $I_{\text{n}}$  is the moment of inertia w.r.t. instantaneous axis of rotation  $\mathbf{n} = \frac{\omega}{\omega}$ 

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The inertia moment computed w.r.t a point P fixed w.r.t body axes is given as:  $(I_P)_{ij} = \sum m_{\alpha} (\delta_{ij} \tilde{\mathbf{r}}_{\alpha}^2 - \tilde{\mathbf{x}}_{\alpha i} \tilde{\mathbf{x}}_{\alpha j})$ . Taking into account  $\tilde{\mathbf{r}}_{\alpha} = \mathbf{r}_{\alpha} - \mathbf{a}$ ,  $\mathbf{a} = \overrightarrow{CP}$ 

 $\Rightarrow$   $(I_P)_{ij} = I_{ij} + M(\mathbf{a}^2 \delta_{ij} - a_i a_j).$ 

If P is also fixed w.r.t S', then, placing O' = P, we can reduce the expressions:  $\mathbf{V} = \boldsymbol{\omega} \times \mathbf{R}, \quad \mathbf{v}'_{\alpha} = \boldsymbol{\omega} \times \mathbf{r}'_{\alpha}, \quad \mathbf{L} = l' \boldsymbol{\omega}, \quad (l' = l_P), \quad T = \frac{1}{2} \boldsymbol{\omega} \cdot l' \boldsymbol{\omega}.$ I' can be computed from  $I: I'_{ij} = I_{ij} + M(\mathbb{R}^2 \delta_{ij} - X_i X_j)$ , where  $X_h$  are components of the vector R

### Principal axes of inertia

Let  $A \in SO(3)$  be a constant change of basis matrix, and  $\tilde{\mathbf{e}}_i = \sum a_{ii} \mathbf{e}_i$ .

$$\Rightarrow x = A\tilde{x}, \quad \tilde{I} = A^{-1}IA = A^{T}IA.$$

Since I is a symmetric matrix, there is always a matrix  $A \in SO(3)$ , s.t.  $\tilde{I} = A^T I A$ is a diagonal matrix, in other words, the moment of inertia w.r.t the alternate body axes e. (which are eigenvectors) is expressed as:

$$\tilde{I} = \begin{bmatrix} I_1 & & \\ & I_2 & \\ & & I_3 \end{bmatrix}$$
, with  $I_i$  the eigenvalues corresponding to  $\tilde{e}_i$ .  
In such case,  $\tilde{e}_i$  are principal axes of inertia, and  $I_i$  are principal axes.

In such case,  $\tilde{\mathbf{e}}_i$  are principal axes of inertia, and  $I_i$  are principal moments of inertia, which are the roots of the caracteristic polinomial of I

If 
$$e_i$$
 are p.a.i. then:  $L_{CM} = \sum_i I_i \omega_i e_i$ ,  $T_{rot} = \frac{1}{2} \sum_i I_i \omega_i^2$ 

If the origin of a set of p.a.i. are also fixed in S', then L, T can be calculated with the expressions above, replacing I by I' w.r.t. the point O'. Clasification of r.b. in terms of the multiplicity of the eigenvalues  $I_i$ 

Type of symmetry	Definition	
Asymmetric tops: Axially symmetric tops: Spherically symmetric tops:	$I_j \neq I_i, \forall i \neq j,$ $I_i = I_j \neq I_k,$ $I_i = I_j, \forall i, j$	

Deduce  $I_i$  of a rigid body by its invariance under transformations Note: both  $\Omega$  and  $\rho$  have to be invariant under the transformations.

# Inv. under Behaviour of $I_i$

$x_i \mapsto -x_i$	$I_{ij} = 0, \forall j \neq i$
	$I_{ii} = I_{jj}$ , $I_{ik} = I_{jk}$ , $k \neq i, j$

In particular, for a solid of revolution with homogeneous density, taking  $x_3$  as the axis of rotation and arbitrary perpendicular axes  $x_1, x_2$ , it satisfy the following invariances:  $x_1 \mapsto -x_1$ ,  $x_2 \mapsto -x_2$ ,  $x_1 \mapsto x_2$ , and thus:

$$\begin{split} I_{11} &\equiv I_1 = I_2 \equiv I_{22}, \ I_{ij} = 0 \ (i \neq j), \ \text{with:} \\ I_1 &= \pi \rho \int_{z_2}^{z_2} z^2 \ f^2(z) \mathrm{d}z + \frac{\pi \rho}{z_1} \int_{z_1}^{z_2} f^4(z) \mathrm{d}z, \qquad I_3 \equiv I_{33} = \frac{\pi \rho}{2} \int_{z_1}^{z_2} f^4(z) \mathrm{d}z. \qquad \textit{O is ver}_1, \text{ i.s.} \end{split}$$

# 6.4 EOM of a rigid body

## EOM in an inertial frame

If  $\mathbf{F} = 0$  then N computed at any point in S' is the same

A r.b. is in equilibrium in an  $S \iff F = N = 0 \iff F = N_{CM} = 0$ Forces due to a constant field f  $\lambda_{\alpha}$  ="charge" of a particle  $\Rightarrow$   $F_{\alpha} = \lambda_{\alpha} f \Rightarrow$   $F = \sum = f \sum \lambda_{\alpha} = \Lambda f$ ,

$$\mathbf{N} = \left(\sum_{\alpha} \lambda_{\alpha} \mathbf{r}'_{\alpha}\right) \times \mathbf{f}. \tag{$\downarrow$ Note that $\mathbf{X}$ is fixed in the body)}$$

If  $\Lambda \neq 0$ , defining  $\mathbf{X} = \frac{1}{\Lambda} \sum_{\alpha} \lambda_{\alpha} \mathbf{r}'_{\alpha} = \frac{1}{\Lambda} \sum_{\alpha} \lambda_{\alpha} (\mathbf{R} + \mathbf{r}_{\alpha}) = \mathbf{R} + \frac{1}{\Lambda} \sum_{\alpha} \lambda_{\alpha} \mathbf{r}_{\alpha}$ we have that  $N = X \times F \Rightarrow \text{If } \Lambda \neq 0$ , then the total torque of the external forces coincides with the torque of the total external forces applied at the centre of charge. (For the grav. field g,  $\lambda = m$  and C. charge is equal to CM).

Torque of the external forces w.r.t. CM is  $N_{CM} = (X - R) \times F$  (for g it is zero). When  $\Lambda = 0$  we use the former equation for N and the torque of the external forces is independent of the reference point, since  $\mathbf{F} = \Lambda \mathbf{f} = 0$ .

 $-(I_2 - I_3)\omega_2\omega_3 = N_1$  $I_2 \dot{\omega}_2 - (I_3 - I_1) \omega_1 \omega_3 = N_2$   $I_3 \dot{\omega}_3 - (I_1 - I_2) \omega_1 \omega_2 = N_3$ Euler's equations is the system { and I are computed w.r.t. CM or a point fixed in the body and S'.  $\omega$  and N

are in a frame of p.a.i. (in general not inertial). If N = 0 and the origin O' of S' is a point in the body,  $L_{O'} \equiv L$  and T are conserved. Similarly, if  $N_{CM} = 0$  then  $L_{CM}$  and  $T_{rot}$  are conserved.

## 6.5 Inertial motion of a symmetric top

The angular velocity (1) of a set of axes w.r.t. to another is additive.

For an axially (e<sub>3</sub>) symmetric r.b., by E's eqns, the following conditions holds: 
$$\omega_3$$
,  $\Omega = \frac{I_3 - I_1}{I_1} \omega_3$ ,  $\omega_0 = \sqrt{\omega_1^2 + \omega_2^2}$ ,  $\alpha = \arctan(\omega_0/\omega_3)$  constants.

In frame of body axes, the vector  $\omega$  rotates around  $e_3$  with  $\Omega$  (traces out the body cone fixed w.r.t. body axes). The angular velocity  $\Omega > 0$  for  $I_3 > I_1$ , and

Relative to body axes, L rotates with  $\Omega$  around  $e_3$ . L fixed relative to S'Relative to S',  $\omega$  traces out the space cone around L (fixed in S') with  $\Omega_p$ 

The angle between  $e_3$  and  $\omega$  is  $\alpha$ , between L and  $e_3$  is  $\theta = \arctan \frac{I_1}{I_2} \tan \alpha$ .

$$\Omega_p = \frac{L}{l_1} = \omega \sqrt{1 + \frac{l_2^2 - l_1^2}{l_1^2} \cos^2 \alpha}, \ \omega = \Omega_p \frac{L}{L} - \Omega e_3.$$

$$L \qquad \omega \qquad e_3$$

$$Space cone$$

$$Body cone$$

$$Body cone$$

#### 6.6 Miscellaneous

In general, for a solid of revolution rolling without sliding down a inclined ( $\alpha$ ) plane around the principal axis e<sub>3</sub>, the horizontal acceleration is expressed as:

Body	Definition	Inertia tensor
Solid Sphere	$x^2 + y^2 + z^2 \le r^2$	$\frac{2}{3} mr^2 \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}$
Hollow Sphere	$x^2 + y^2 + z^2 = r^2$	$\frac{2}{3}mr^2\begin{bmatrix}1&&\\&1&\\&&1\end{bmatrix}$
Cuboid	$\begin{cases} -\frac{d}{2} \le x \le \frac{d}{2} \\ -\frac{w}{2} \le y \le \frac{w}{2} \\ -\frac{h}{2} \le y \le \frac{h}{2} \end{cases}$	$\begin{bmatrix} \frac{m}{12} & h^2 + d^2 \\ & w^2 + d^2 \\ & h^2 + w \end{bmatrix}$
Cylinder	$\begin{cases} x^2 + y^2 \le r^2 \\ -\frac{h}{2} \le z \le \frac{h}{2} \end{cases}$	$\frac{m}{12} \begin{bmatrix} 3r^2 + h^2 \\ 3r^2 + h^2 \\ 6r^2 \end{bmatrix}$
Rod about end	$\begin{cases} x^2 + y^2 \le r^2 \\ 0 \le z \le h \\ h >> r \\ \omega \parallel x \end{cases}$	$\frac{1}{3}mt^2\begin{bmatrix}1&0&\\&1\end{bmatrix}$
Rod about center	$\begin{cases} x^2 + y^2 \le r^2 \\ -\frac{h}{2} \le z \le \frac{h}{2} \\ h >> r \end{cases}$ $\omega \parallel x$	$\frac{1}{12}ml^2\begin{bmatrix}1\\0\\1\end{bmatrix}$

# 7 Special relativity

### 7.1 Principle of special relativity

In Galilean relativity, the acceleration of a particle is the same in all inertial frames => all inertial frames are equivalent from the point of view of Newtonian mechanics

## Postulates of special relativity:

- 1. The laws of physics are the same in all inertial frames (relativity principle).
- 2. The speed of electromagnetic waves in vacuum is universal:  $c = \frac{1}{\sqrt{\epsilon_0 \mu_0}}$ (Implication of 1. & 2.) speed of electromagnetic waves in vacuum is c in

#### 7.2 Lorentz transformations Basic equations of transformation

We will use the notation  $x_0 \equiv ct$ ,  $x_{0'} \equiv ct'$  as the "time coordinate" of spacetime. The Lorentz factor for a speed of v is defined as:

$$\gamma(v) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}, \text{ dimensionless.}$$

The L. transformation relating the frames  $x_{\mu}$  and  $x'_{\mu}$ , (the velocity of O' w.r.t.

$$t' = \gamma(v)\left(t - \frac{vx_1}{c^2}\right), \quad x'_1 = \gamma(v)(x_1 - vt), \quad x'_k = x_k, \quad k = 2, 3$$

### Consequences of L.T.

1. The relative speed between two inertial frames is strictly less than c in vac-

 The speed of all mass is less than c. 3. The propagation speed any information cannot exceed c.

The general L.T. with v the velocity of O' w.r.t. S is given by:

$$t' = \gamma(v)\left(t - \frac{\mathbf{v} \cdot \mathbf{x}}{2}\right), \quad \mathbf{x}' = \mathbf{x} + (\gamma(v) - 1)\frac{\mathbf{v} \cdot \mathbf{x}}{2} - \gamma(v)\mathbf{v}t$$

(Relativistic law for the addition of velocities.) For a particle with constant velocity  $\mathbf{u} = u\mathbf{e}_1$  w.r.t. S, its velocity w.r.t. S' (whose origin moves at constant velocity ve1) is also constant, with components:

is also constant, with components:  

$$u_1 = \frac{u'_1 + v}{1 + \frac{u'_1 v}{2}}, \quad u_k = \frac{u'_k}{\gamma(v) \left(1 + \frac{u'_1 v}{2}\right)}, \quad (k = 2, 3)$$

the reversed relations can be obtained by changing the signs and switching primes. Intervals

The quadratic form  $c^2t^2 - \mathbf{x}^2 \equiv x_0^2 - \mathbf{x}^2$  is invariant under L.T..

The square of the **interval** between two events with s-t coordinates  $x_{tt}$  and

$$\Delta s^2 = c^2 \Delta t^2 - \sum_{i=1}^{3} \Delta x_i^2 = \Delta x_0^2 - \sum_{i=1}^{3} \Delta x_i^2 \equiv \Delta x_0^2 - \Delta x^2;$$

said interval is invariant under L.T.  $\Rightarrow \Delta s^2 = \Delta x_0^2 - \Delta x^2 = \Delta (x_0')^2 - \Delta (x')^2$ . The interval between two events is time-like if  $\Delta s^2 > 0$ , light-like if  $\Delta s^2 = 0$ , and space-like if  $\Delta s^2 < 0$ .

If the interval between two events is time-like:

Let the original S s.t.  $\Delta x_2$ ,  $\Delta x_3 = 0$ , and a second S' with velocity  $\frac{\Delta x_1}{\Delta t} e_1$ . Then, in  $\mathcal{S}'$  the two events occur at the same point in space, with time interval (proper time lapse)  $\Delta \tau = \frac{\Delta s}{c} = \Delta t \sqrt{1 - \frac{\Delta x^2}{\Delta r^2}}$ 

If the interval between two events is snace-like:

Let the original S' s.t.  $\Delta x_2$ ,  $\Delta x_3 = 0$ , and a second S' with velocity  $\frac{c^2 \Delta t}{\Delta x_1} \mathbf{e}_1 = \frac{c \Delta x_0}{\Delta x_1} \mathbf{e}_1$ . Then in  $\mathcal{S}'$  the two events occur simultaneously, with dis-

tance (**proper distance**)  $\sqrt{-\Delta s^2} = |\Delta x_1'|$ . The proper distance is less or equal to the spatial distance  $|\Delta x|$  in any other inertial frame.

## Minkowski product

For two s-t coordinates in S,  $x = (x_0, \mathbf{x})$  and  $y = (y_0, \mathbf{y}) \in \mathbb{R}^4$ , their Minkowski

$$x \odot y \equiv x_0 y_0 - \mathbf{x} \cdot \mathbf{y} \Rightarrow x^2 \equiv x \odot x = x_0^2 - \mathbf{x}^2, \quad \Delta x \odot \Delta x = \Delta s^2$$
 is also invariant under L.T.. The vector space  $(\mathbb{R}^4, \odot)$  is the *Minkowski space*.

L(v) is the "basic" L.T.

The procedure of relating the s-t coordinates of an event in two reference 
The wave vector is defined as: frames S and S' (S' moves at v) is the following:

$$S \to S'' \to S''' \to S''' \to S'$$
  
1. Consider  $S''$  at rest w.r.t.  $S$ , whose  $x_i''$  is in the direction of  $\mathbf{v} \Rightarrow x'' = R_1 x$ , where  $R_1 \in \mathbb{R}^{4 \times 4}$  but only rotates the spatial coordinates

where  $R_1 \in \mathbb{R}^{4 \times 4}$  but only rotates the spatial coordinates. 2. S''' is the frame that moves with  $\mathbf{v} = v\mathbf{e}_1''$  away from  $S'' \Rightarrow x''' = L(v)x''$ .

3. The frames S''' and S' are related by a rotation  $\Rightarrow x' = R_2 x'''$ Combining everything we get

 $x' = R_2 L(v) R_1 x \equiv \Lambda x$ 

the transformation  $\Lambda \in \mathbb{R}^{4 \times 4}$  is the general Lorentz transformation. It is the most genearl form of L.T. that relates the space-time coord. of an event in two reference frames whose space-time origin coincide  $(t = x_i = 0 \iff t' = x_i' =$ 0). If the space-time origin of the frames differs by a vector  $a \in \mathbb{R}^4$ , then the transformation  $x' = \Lambda x + a$  is called the *Poincaré tr.*. The basic I. T can be expressed as a matrix:

$$L(v) = \begin{bmatrix} \gamma(v) & -\beta(v)\gamma(v) & 0 & 0 \\ -\beta(v)\gamma(v) & \gamma(v) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \beta(v) \equiv 0$$

$$G = \begin{bmatrix} 1 & & & \\ & -1 & \\ & & -1 \end{bmatrix}, \quad \Rightarrow \quad x \odot y = x^{\mathsf{T}} G y, \quad \Rightarrow \quad \Lambda^{\mathsf{T}} G \Lambda = G$$

The Minkowski product (and the square of intervals), are invariant under general Lorentz transformations. The set of matrices  $\Lambda$  that satisfy the equation above form the Lorentz group

Performing the change of variable: 
$$\beta(\nu) = \tanh \phi \Rightarrow \gamma(\nu) = \cosh \phi \Rightarrow L(\nu) = \begin{bmatrix} \cosh \phi & -\sinh \phi & 0 & 0 \\ -\sinh \phi & \cosh \phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

And the result of successive Lorentz boosts with speeds  $v_1 = c \tanh \phi_1$  and  $v_2 = c \tanh \phi_2$  is another L. boost with rapidity  $\phi_1 + \phi_2$ :

 $v = c \tanh(\phi_1 + \phi_2) \equiv \frac{v_1 + v_2}{1 + \frac{v_1 + v_2}{2}}$  similar to the eqn. of velocity of a prtcl.

7.3 Dilations and contractions The proper time measured in  ${\cal S}$  is larger than the coordinate time measured in S'. In other words, from an observer at rest with S, the time recorded by a watch at rest with S', t', is slower than the proper time t that he has measured. We say that t' dilated, the dilation effect is symmetrical in **inertial frames**:

$$t \equiv \frac{t'}{\sqrt{1 - \frac{v^2}{\sigma^2}}} = \frac{t'}{\gamma(v)} > t'$$

For a particle following a trajectory  $\mathbf{r} \equiv \mathbf{r}(t)$ , the proper time increment (time according to an observer at rest with said particle) is expressed as:

$$d\tau = \sqrt{1 - \frac{v^2(t)}{c^2}} \Rightarrow \Delta\tau(C) = \int_{t_1}^{t_2} \sqrt{1 - \frac{v^2(t)}{c^2}} dt = \int_{t_1}^{t_2} \sqrt{1 - \frac{\dot{\tau}^2}{c^2}} dt$$

being C the path taken by the particle.  $\Delta \tau \leq \Delta t$  and is equal only if  $\mathbf{v} = 0, \forall t$ . Lorentz-Fitzgerald contraction has to do with contraction of an object that's at rest with S' but in motion wrt to S in the direction of its velocity. In mathematical terms, the length component in the direction of its movement,  $x_1$ , is

$$\Delta x_1' \equiv l_0 = \gamma(v)(\Delta x_1 - v\Delta t) \equiv \gamma(v)l \Rightarrow l = \frac{l_0}{\gamma(v)} = l\sqrt{1 - \frac{v^2}{c^2}} < l_0,$$

where  $l_0$  is the **rest length** or the length directly measured by an observer at rest with the object

### 7.4 Four-velocity and four-momentum

The four-velocity is defined as:

$$u=rac{\mathrm{d}x}{\mathrm{d} au}\in\mathbb{R}^4,\quad x\equiv$$
 "s-t coordinates",  $\mathrm{d} au=\sqrt{1-rac{v^2}{c^2}}\,\mathrm{d}t\equivrac{\mathrm{d}t}{\gamma(v)}$  and it has the following properties:

$$u=\gamma(v)(c,\mathbf{v})\in\mathbb{R}^4,\quad u'=\Lambda u,\quad u^2=c^2.$$
 The four-momentum is: 
$$p=mu\Rightarrow p^2=m^2c^2$$

Here we will use the vector  $\mathbf{p} = (p_1, p_2, p_3) \equiv m\gamma(v)\mathbf{v} \in \mathbb{R}^3$  which coincides with the non-r. momentum only when  $v \to 0$ . Then:

$$p_0 = \sqrt{\mathbf{p}^2 + m^2 c^2}, \Rightarrow \mathbf{v} = \frac{c\mathbf{p}}{p_0}$$

and the relativistic kinetic energy is given as:

$$T = cp_0 - mc^2 \equiv mc^2(\gamma(v) - 1) \quad \Rightarrow \quad p_0 = \frac{1}{c} (mc^2 + T)$$

# Conservation of four-momentum

 $p = \text{const.} \iff cp_0 = mc^2 + T = \text{const.} \iff p_i = m\gamma(v)v_i = \text{const.}$ The total four-momentum of a system of N particles is:

$$P = \sum_{n=1}^{N} p_n \equiv (P_0, P), \quad P_0 = \sum_{n=1}^{N} p_{n,0} \equiv c \sum_{n=1}^{N} m_n \gamma(v_n),$$

$$P = \sum_{n=1}^{N} p_n = \sum_{n=1}^{N} m_n \gamma(v_n),$$

 $\mathbf{P} = \sum_{n=1}^N \mathbf{p}_n \equiv \sum_{n=1}^N m_n \gamma(v_n) \mathbf{v}_n.$  The conservation of P during the collision of said N particles is verified even when the collision is not elastic:

$$P_i = P_f \iff (Mc^2 + T)_i = (Mc^2 + T)_f$$
  
During the process of collision, The mass and the kinetic energy can vary together with the relation:

 $\Delta T = -\Delta (Mc^2)$ The total relativistic energy can be defined as:

e total relativistic energy can be defined as:  

$$E = cP_0 = mc^2 + T = mc^2 \gamma(v) = c\sqrt{\mathbf{p}^2 + m^2c^2} = mc^2 \sqrt{1 + \frac{\mathbf{p}^2}{m^2c^2}}$$

#### $\Rightarrow P = (E/c, \mathbf{P}) \Rightarrow \mathbf{v} = \frac{c^2 \mathbf{p}}{F}$ 7.5 Massless particles

For particle with mass  $m \to 0$ , its energy is  $E = c|\mathbf{p}|$  and its four-momentum:  $p = (|\mathbf{p}|, \mathbf{p}) \Rightarrow \mathbf{v} = c \frac{\mathbf{p}}{|\mathbf{p}|}$ 

#### Motion of a photon

Photon is the only massless particle known. The relation between its energy and the frequency ω of its associated EM wave is given as:

$$E = \hbar \omega \equiv hv = \frac{hc}{2}$$
,  $\lambda \equiv$  "wavelength"

$$\omega = c|\mathbf{k}|, \quad \mathbf{v} = c \frac{\mathbf{k}}{|\mathbf{k}|} \Rightarrow \mathbf{p} = \hbar \mathbf{k}$$

We can define the wave four-vector:  $k = (k_0, \mathbf{k}) = (|\mathbf{k}|, \mathbf{k}) = \frac{p}{\hbar} \in \mathbb{R}^4$ . The wave four-vector is proportional to the four-momentum, therefore, if S' is another inertial system and  $x' = \Lambda x$  we have:  $k' = \Lambda k$ 

### 7.6 Longitudinal Doppler effect

If an EM wave propagates in the direction of the relative motion between the observer S and the source S'

### 8 Definitions

### 8.1 Lagrange-Hamilton

A system is natural when the Hamiltonian of the system is T + V = E (the total

A constraint is *ideal* when the constraint force for each particle is perpendicular to the constraint manifold for that particle.

A constraint is holonomic when it's independent of the velocities of the parti-

#### 8.2 Rigid body motion

A rigid body is generic if it contains three non-collinear particles.

# 9 Miscellaneous

# 9.1 Vectorial calculus

 $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}, \quad (\mathbf{a} \times \mathbf{b})^2 = \mathbf{a}^2\mathbf{b}^2 - (\mathbf{a} \cdot \mathbf{b})^2$ 

# 9.2 Chain rule (Partial derivatives)

Given a function  $f \equiv f(t, x_1, ..., x_n), x_i \equiv x_i(t), \forall i$ , then

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \frac{\partial f}{\partial t} + \sum_{i} \frac{\partial f}{\partial x_{i}} \frac{\mathrm{d}x_{i}}{\mathrm{d}t} \equiv \frac{\partial f}{\partial t} + \sum_{i} \frac{\partial f}{\partial x_{i}} \dot{x_{i}}.$$

1. 
$$m_1\ddot{\mathbf{r}}_1 = \mathbf{F}_{12}(t, \mathbf{r}_1, \mathbf{r}_2, \dot{\mathbf{r}}_1, \dot{\mathbf{r}}_2), m_1\ddot{\mathbf{r}}_1 = \mathbf{F}_{21} = -\mathbf{F}_{12}$$
  
2.  $\mathbf{R} = \frac{1}{M}(m_1\mathbf{r}_1 + m_2\mathbf{r}_2)\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2 \Rightarrow \mu\ddot{\mathbf{r}} = \mathbf{F}_{12} = \mathbf{F}(t, \mathbf{r}, \dot{\mathbf{r}})$   
3.  $\mu = \frac{m_1m_2}{m_1+m_2}$ 

$$3 \quad \mu = \frac{m_1 m_2}{m_1 m_2}$$

4.  $\mathbf{r}_1 = \mathbf{R} + \frac{\tilde{m}_2}{M} \mathbf{r}$ ,  $\mathbf{r}_2 = \mathbf{R} - \frac{m_1}{M} \mathbf{r}$ . Since  $\ddot{\mathbf{R}} = 0$  we can place the origin at  $\mathbf{R}(t) = 0$  and the ref. frame will still be inertial.

5. 
$$\Rightarrow \mathbf{r}_1 = \frac{m_2}{M} \mathbf{r}, \mathbf{r}_2 = -\frac{m_1}{M} \mathbf{r}$$

5. 
$$\Rightarrow$$
  $\mathbf{r}_1 = \frac{1}{M}\mathbf{r}$ ,  $\mathbf{r}_2 = -\frac{1}{M}\mathbf{r}$   
6. When  $m_2$  is much larger than  $m_1 \Rightarrow \begin{cases} \frac{m_1}{M} \simeq 0 \Rightarrow \mathbf{r}_2 \simeq 0 \\ \frac{m_2}{M} \simeq 1 \Rightarrow \mathbf{r}_1 \simeq \mathbf{r} \end{cases}$ 

# 9.4 Integrals

Integrals
• 
$$x = \sqrt{\frac{1}{k}} \cosh z \Rightarrow \int \frac{dx}{\sqrt{k-2} - 1} = \int \sqrt{\frac{1}{k}} dz = \sqrt{\frac{1}{k}} \cdot z$$

#### 9.5 ODEs

DDEs  

$$u'' + Cu = 0, \quad C \in \mathbb{R}$$
If  $C < 0 \Rightarrow u = ae^{\gamma\theta} + be^{-\gamma\theta}, \quad \gamma \equiv \sqrt{|C|} > 0 \Rightarrow r = u^{-1}$ 

$$\Rightarrow r = Ascch(\gamma(\theta - \theta_0)) \quad (a, b > 0)$$

$$\Rightarrow r = Ae^{\pm\gamma(\theta - \theta_0)} \quad (a = 0 \text{ or } b = 0)$$

$$\Rightarrow r = Acsch(\gamma(\theta - \theta_0)) \quad (ab < 0)$$
If  $C = 0 \Rightarrow u = a + b\theta \iff r = \frac{1}{a + b\theta}$   
If  $C < 0 \Rightarrow u = \frac{1}{4} \cos(\gamma(\theta - \theta_0))) \iff r = A \sec(\gamma(\theta - \theta_0))$ 

# 9.6 Trigonometric functions

 $\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$ ,  $\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$  $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta, \cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$  $\cos(2x) = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$  $\sin(2x) = 2\sin x \cos x$ ,  $\tan(2x) = \frac{2\tan x}{1+x^2}$  $\sin^2 x - 1 = -\cos(2x), \cos^2 x - 1 = -\sin^2 x$  $\sin(\frac{x}{2}) = \sqrt{\frac{1-\cos x}{2}}$ ,  $\cos\frac{x}{2} = \sqrt{\frac{1+\cos x}{2}}$ ,  $\tan\frac{x}{2} = \sqrt{\frac{1-\cos x}{1+\cos x}}$  $\int \sin^2(x) dx = \frac{x}{2} - \frac{\sin(2x)}{4} + C, \quad \int \cos^2(x) dx = \frac{x}{2} + \frac{\sin(2x)}{4} + C$ 

# $\int \sin^{3}(x)dx = \frac{\cos^{3}x}{3} - \cos x + C, \int \cos^{3}(x)dx = \sin x - \frac{\sin^{3}x}{3} + C$ 9.7 Hyperbolic functions

$$\begin{aligned} &\sinh x = \frac{e^x - e^{-x}}{2}, &\cosh x = \frac{e^x + e^{-x}}{2} \\ &\cosh^2 x - \sinh^2 x = 1 \\ &\sinh^2 x + 9 & \sinh x \cos y + \cosh x \sinh y, \\ &\cosh x + y = \cosh x \cosh y + \sinh x \sinh y, \\ &\cosh x + y = \cosh x \cosh y + \sinh x \sinh y, \\ &\sinh x - y = \cosh x \cosh y - \sinh x \sinh y, \\ &\sinh 2x = 2 \sinh x \cosh y - \sinh x \sinh y, \\ &\sinh 2x = 2 \sinh x \cosh y - \sinh x \sinh y, \\ &\sinh 2x = 2 \sinh x \cosh x \cosh x \sinh x \sin^2 x = \frac{\cosh^2 x}{2} + \frac{1 + \cosh^2 x}{1 + \cosh^2 x}, &\sinh^2 x = \frac{\cosh^2 x}{2} + \frac{1 + \cosh^2 x}{2} \\ &\cosh^{\frac{2}{3}} = \sqrt{\frac{1 + \cosh x}{dx}}, &\sinh \frac{x}{2} = \frac{1}{\sqrt{\cosh x - 1}}, \\ &\frac{d}{dx} \sinh x = \cosh x, &\frac{d}{dx} \cosh x = \sinh x, &\frac{d}{dx} \tanh x = \operatorname{sech}^2 x \\ &\tanh (\phi_1 + \phi_2) = \frac{\tanh \phi_1 + \tanh \phi_2}{1 + \tanh \phi_1 \tanh \phi_2} \\ &\frac{1}{\sqrt{1 - \tanh^2 \phi}} = \cosh \phi. \end{aligned}$$

# 9.8 Constants and units

Earth radius  $\equiv 6.3674 \cdot 10^6$ Centrifugal acceleration on Earth  $\equiv 5.93031 \cdot 10^{-3} \text{m s}^{-2}$  $h = 2\pi\hbar = \text{Planck's constant} = 6.62607015 \times 10^{-34} \text{ J} \cdot \text{s}$ 

# 9.9 Dynamic equations

Hooke's Law:  $\mathbf{F}_s = -k(l_0 - x)\mathbf{e}_x$ ,  $V_s = -k/2(x - l_0)^2$ Electromagnetic (Static fields) force:  $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ 

#### 9.10 Vectorial operators

Gradients & diverge

Cylindrical:

 $\nabla U = \frac{\partial U}{\partial a} \mathbf{u}_{\rho} + \frac{1}{a} \frac{\partial U}{\partial w} \mathbf{u}_{\phi} + \frac{\partial U}{\partial x} \mathbf{k}$  $\nabla \cdot \mathbf{F} = \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho F_{\rho}) + \frac{1}{\rho} \frac{\partial F_{\phi}}{\partial \phi} + \frac{\partial F_{z}}{\partial z}$ 

$$\begin{pmatrix} \frac{1}{\rho} \frac{\partial F_c}{\partial \phi} - \frac{\partial F_o}{\partial z} \end{pmatrix} \mathbf{u}_\rho + \begin{pmatrix} \frac{\partial F_D}{\partial z} - \frac{\partial F_c}{\partial \rho} \end{pmatrix} \mathbf{u}_\phi + \begin{pmatrix} \frac{1}{\rho} \frac{\partial}{\partial \rho} (\rho F_\phi) - \frac{\partial F_D}{\partial \phi} \end{pmatrix} \mathbf{k}$$
Spherical:
$$\nabla U = \frac{\partial U}{\partial r} \mathbf{u}_r + \frac{1}{r} \frac{\partial U}{\partial \theta} \mathbf{u}_\theta + \frac{1}{r \sin \theta} \frac{\partial U}{\partial \phi} \mathbf{u}_\phi$$

$$\nabla \cdot \mathbf{F} = \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 F_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta F_\theta) + \frac{1}{r \sin \theta} \frac{\partial F_\phi}{\partial \phi}$$

$$\begin{split} \nabla U &= \frac{\partial U}{\partial r} \mathbf{u}_r + \frac{1}{r} \frac{\partial U}{\partial \theta} \mathbf{u}_{\theta} + \frac{1}{r \sin \theta} \frac{\partial U}{\partial \phi} \mathbf{u}_{\phi} \\ \nabla \cdot \mathbf{F} &= \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 F_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta F_{\theta}) + \frac{1}{r \sin \theta} \frac{\partial F_{\phi}}{\partial \phi} \\ &\left[ \frac{1}{r \sin \theta} \left( \frac{\partial}{\partial \theta} (\sin \theta F_{\phi}) - \frac{\partial F_{\theta}}{\partial \phi} \right) \right] \mathbf{u}_r + \left[ \frac{1}{r} \left( \frac{1}{\sin \theta} \frac{\partial F_r}{\partial \phi} - \frac{\partial}{\partial r} (r F_{\phi}) \right) \right] \mathbf{u}_{\theta} + \right. \\ &\left. + \left[ \frac{1}{r} \left( \frac{\partial}{\partial r} (r F_{\theta}) - \frac{\partial F_r}{\partial \theta} \right) \right] \mathbf{u}_{\phi} \end{split}$$