1 Coordinate systems

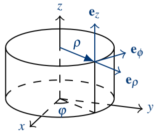
1.1 Cylindrical

Basis vectors

$\mathbf{e}_\rho = \left|\frac{\partial \mathbf{r}}{\partial \rho}\right|^{-1}\frac{\partial \mathbf{r}}{\partial \rho} = (\cos\varphi, \sin\varphi, 0)$,

$\mathbf{e}_\varphi = \left|\frac{\partial \mathbf{r}}{\partial \varphi}\right|^{-1}\frac{\partial \mathbf{r}}{\partial \varphi} = (-\sin\varphi, -cos\varphi, 0)$,

$\mathbf{e}_z = \left|\frac{\partial \mathbf{r}}{\partial z}\right|^{-1}\frac{\partial \mathbf{r}}{\partial z} = (0,0,1)$.



Velocity vectors

$\dot{\mathbf{e}}_\rho = \dot\varphi\mathbf{e}_\varphi, \quad \dot{\mathbf{e}}_\varphi = -\dot\varphi\mathbf{e}_\rho, \quad \dot{\mathbf{e}}_z = 0.$

$v_\rho = \dot\rho, \quad v_\varphi = \rho\dot\varphi, \quad v_z = \dot z.$

Acceleration vectors

$a_\rho = \ddot\rho - \rho\dot\varphi^2, \quad a_\varphi = \rho\ddot\varphi + 2\dot\rho\dot\varphi, \quad a_z = \ddot z.$

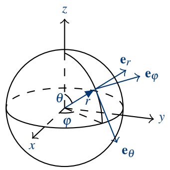
1.2 Spherical

Basis vectors

$\mathbf{e}_r = \left|\frac{\partial \mathbf{r}}{\partial r}\right|^{-1}\frac{\partial \mathbf{r}}{\partial r}$
$= (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) = \frac{\mathbf{r}}{r}$,

$\mathbf{e}_\theta = \left|\frac{\partial \mathbf{r}}{\partial \theta}\right|^{-1}\frac{\partial \mathbf{r}}{\partial \theta}$
$= (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta)$,

$\mathbf{e}_\varphi = \left|\frac{\partial \mathbf{r}}{\partial \varphi}\right|^{-1}\frac{\partial \mathbf{r}}{\partial \varphi} = (\sin\varphi, \cos\varphi, 0)$.



Velocity vectors

$\dot{\mathbf{e}}_r = \frac{\partial \mathbf{e}_r}{\partial\theta}\dot\theta + \frac{\partial \mathbf{e}_r}{\partial\varphi}\dot\varphi = \dot\theta\mathbf{e}_\theta + \sin\theta\dot\varphi\mathbf{e}_\varphi$,

$\dot{\mathbf{e}}_\theta = \frac{\partial \mathbf{e}_\theta}{\partial\theta}\dot\theta + \frac{\partial \mathbf{e}_\theta}{\partial\varphi}\dot\varphi = -\dot\theta\mathbf{e}_r + \cos\theta\dot\varphi\mathbf{e}_\varphi$,

$\dot{\mathbf{e}}_\varphi = \frac{\partial \mathbf{e}_\varphi}{\partial\varphi}\dot\varphi = -(\cos\varphi, \sin\varphi, 0)\dot\varphi = -\dot\varphi(\sin\theta\mathbf{e}_r + \cos\theta\mathbf{e}_\theta)$.

$v_r = \dot r, \quad v_\theta = r\dot\theta, \quad v_\varphi = r\sin\theta\dot\varphi.$

Acceleration vectors

$a_r = \ddot r - r\dot\theta^2 - r\sin^2\theta\dot\varphi^2, \quad a_\theta = r\ddot\theta + 2\dot r\dot\theta - r\sin\theta\cos\theta\dot\varphi^2$,

$a_\varphi = r\sin\theta\ddot\varphi + 2\sin\theta\dot r\dot\varphi + 2r\cos\theta\dot\theta\dot\varphi$.

2 Newtonian Mechanics

2.1 Newton's Laws

1. In the absence of external forces, the momentum of a particle remains constant.
2. If an external force $\mathbf{F}$ acts on a particle, the rate of variation of its momentum is give by $\dot{\mathbf{p}} = \mathbf{F} \iff \mathbf{F} = m\ddot{\mathbf{r}}$ ($\Leftarrow$ eqn. of motion of a particle).

2.2 Conservation laws

Basic definitions

$\mathbf{L} = \mathbf{r}\times\mathbf{p} = m\mathbf{r}\times\dot{\mathbf{r}}, \quad \dot{\mathbf{L}} = \mathbf{N} = \mathbf{r}\times\mathbf{F} = 0 \iff \mathbf{F}\parallel\mathbf{r}$.

A force is central $\iff \mathbf{F} = g(t,\mathbf{r},\dot{\mathbf{r}})\mathbf{r}$. Let $T = \frac{1}{2}m\dot{\mathbf{r}}^2$, then $\frac{dT}{dt} = m\dot{\mathbf{r}}\ddot{\mathbf{r}} = \mathbf{F}\dot{\mathbf{r}}$

Kinetic energy is conserved $\iff \mathbf{F}\perp\dot{\mathbf{r}}, \forall t$.

A force is conservative $\iff \mathbf{F}(\mathbf{r}) = -\nabla V(\mathbf{r}) \equiv -\sum_{i=1}^{3}\frac{\partial V(\mathbf{r})}{\partial x_i}\mathbf{e}_i \equiv -\frac{\partial V(\mathbf{r})}{\partial \mathbf{r}}$.

If $\mathbf{F}$ conservative, then $E = T + V$ conserved.

A force is irrotational $\iff \exists V(t,\mathbf{r})$ s.t. $\mathbf{F}(t,\mathbf{r}) = -\frac{\partial V(t,\mathbf{r})}{\partial \mathbf{r}} \Rightarrow \frac{dE}{dt} = \frac{\partial V}{\partial t}$.

Gravitational and Electrostatic forces

$\mathbf{F} = f(r)\mathbf{e}_r$

$f(r) = -\frac{GMm}{r^2}$, $V(r) = -\frac{GMm}{r} \Rightarrow \mathbf{a} = \frac{\mathbf{F}}{m} = -\frac{GM}{r^3}\mathbf{r}$ independent of m.

$f(r) = k\frac{qQ}{r^2}$, $V(r) = k\frac{qQ}{r}$.

In general, for forces between a particle and a continuous distribution of mass/charge:

$\mathbf{F}(\mathbf{r}) = -Gm\int\rho(\mathbf{r}')\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}d^3\mathbf{r} = -m\frac{\partial\Phi(\mathbf{r})}{\partial\mathbf{r}}$, $\Phi(\mathbf{r}) = -G\int\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}$

$\mathbf{F}(\mathbf{r}) = -kq\int\rho(\mathbf{r}')\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}d^3\mathbf{r} = -q\frac{\partial\Phi(\mathbf{r})}{\partial\mathbf{r}}$, $\Phi(\mathbf{r}) = -k\int\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}d^3\mathbf{r}$

Electromagnetic force

$\mathbf{F}(t,\mathbf{r},\dot{\mathbf{r}}) = q(\mathbf{E}(t,\mathbf{r}) + \dot{\mathbf{r}}\times\mathbf{B}(t,\mathbf{r}))$, $T + q\Phi(\mathbf{r}) \equiv$"electromechanical energy".

2.3 Motion of a particle in a one-dimensional potential

Law of conservation of energy: $\frac{1}{2}m\dot x^2 + V(x) = E \Rightarrow \dot x(m\ddot x - F(x)) = 0$
(if $\dot x \neq 0 \Rightarrow m\ddot x - F(x) = 0$).

Said ODE is an **autonomous system** (i.e. it's of the form $\ddot x = f(x,\dot x)$), and it's thus invariant under transformations $t\to t+c$ and $t\to -t$.

Equilibrium, and trajectory

Equilibria are the points $x_0\in\mathbb{R}$ s.t. $x(t) = x_0$ is a solution of the EOM $\Rightarrow \ddot x(t) = 0 \Rightarrow F(x(t)) = F(x_0) = -V(x_0) = 0$.

The **accessible region** for a given E is the set $\{x\in\mathbb{R}\mid V(x)\le E\}$, which, in general, are disjoint union of closed intervals.

The endpoints of the intervals, if not equilibriums, are the **turning points** of the trajectories: x_i turning point $\iff V(x_i) = E,\ V'(x_i)\neq 0$.

In general, the relation between time and displacement is given as:

$$t - t_0 = \pm\sqrt{\frac{m}{2}}\int\frac{dx}{\sqrt{E-V(x)}} \Rightarrow t = \sqrt{\frac{m}{2}}\int_{x_0}^{x}\frac{ds}{\sqrt{E-V(s)}} \equiv \theta(x)$$

Interval with two turning points:

The time for the particle to go from x_0 to x_1 is $\frac{\tau}{2}$, being:

$$\tau = 2\theta(x_1) = \sqrt{2m}\int_{x_0}^{x_1}\frac{ds}{\sqrt{E-V(s)}}.$$

The function $\theta(x)$ is mon. increasing for $x\in(x_0,x_1)$, therefore θ is invertible and we can write x in terms of t:

$$x(t) = \theta^{-1}(t), 0\le t\le\frac{\tau}{2}, \qquad x(t) = \theta^{-1}(\tau - t), \frac{\tau}{2}\le t\le\tau.$$

Semi-infinite interval with turning point at one side $[x_0,\infty)$:

For a particle with initial positon $x(0) = x_0$ and moving towards infinity, the time is:

$$t_\infty = \theta(\infty) = \sqrt{\frac{m}{2}}\int_{x_0}^{\infty}\frac{ds}{\sqrt{E-V(s)}}$$, it can be convergent or divergent.

x can be expressed as:

$$x(t) = \theta^{-1}(t), 0\le t\le t_\infty, \qquad x(t) = \theta^{-1}(-t), -t_\infty\le t\le 0.$$

Accessible region is $\mathbb{R}$:

The time for the particle to reach infinity from an initial positon x_0 is:

$$t_{\pm\infty} = \theta(\pm\infty) = \sqrt{\frac{m}{2}}\int_{x_0}^{\pm\infty}\frac{ds}{\sqrt{E-V(s)}}$$

$$x(t) = \theta^{-1}(t), t_{-\infty}\le t\le t_\infty, \qquad x(t) = \theta^{-1}(-t), -t_\infty\le t\le t_{-\infty}.$$

Stability of equilibria, oscillation approximation

An equilibrium is stable $\iff$ it is a *relative minimum* of the potential.

For an open ball centered at a stable eq. $(x_{eq}-\varepsilon, x_{eq}+\varepsilon)$, the Taylor expansion of V is of the form:

$$V(x) = V(x_{eq}) + \frac{1}{2}V''(x_{eq})(x-x_{eq})^2 + O(x^3)$$

Then the motion of a particle within said open ball can be approximated to:

$$\ddot x = \frac{F(x)}{m} = -\frac{V'(x)}{m} \simeq -\frac{V''(x_{eq})}{m}(x-x_{eq}) \quad\Rightarrow\quad \ddot\xi + \omega^2\xi = 0$$

with $\xi \equiv x - x_{eq}$, $\omega\equiv\sqrt{\frac{V''(x_{eq})}{m}}$.

The solution of this ODE is: $\xi = A\cos(\omega t+\alpha)$, $A\in(0,|x-x_{eq}|)$, and the period of the oscillation is $\tau\simeq\frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{V''(x_{eq})}}$.

2.4 Dynamics of a system of particles

The law of motion of N elements with masses m_i, positions $\mathbf{r}_i$ is:

$$m_i\ddot{\mathbf{r}}_i = \sum_{j=1}^{N}\mathbf{F}_{ij} + \mathbf{F}_i^{(e)}, \quad i = 1,\dots,N,$$

where $\mathbf{F}_{ij}$ are the forces exerted by j on i, and $\mathbf{F}_i^{(e)}$ are the total external force exerted on i. The solution of this set of N equations is unique if given the initial conditions

$$\mathbf{r}_i(t_0) = \mathbf{r}_{i_0}, \quad \dot{\mathbf{r}}_i(t_0) = \mathbf{v}_{i_0}, \quad i\in\{1,\dots,N\}.$$

Center of mass

$$\mathbf{F}^{(e)} = \sum_{i=1}^{N}\mathbf{F}_i^{(e)}, \quad \sum_{i=1}^{N}m_i\ddot{\mathbf{r}}_i = \mathbf{F}^{(e)}, \quad \mathbf{R}\equiv M^{-1}\sum_{i=1}^{N}m_i\mathbf{r}_i, \quad M\ddot{\mathbf{R}} = \mathbf{F}^{(e)}.$$

Conservation

Linear momentum:

$\mathbf{P}\equiv\sum_{i=1}^{N}m_i\dot{\mathbf{r}}_i = M\dot{\mathbf{R}} \Rightarrow \dot{\mathbf{P}} = \mathbf{F}^{(e)} \quad\Rightarrow\quad \mathbf{P}$ conserved $\iff \mathbf{F}^{(e)} = 0$.

Angular momentum:

$\mathbf{L} = M\mathbf{R}\times\dot{\mathbf{R}} + \sum_{i=1}^{N}m_i\mathbf{r}'_i\times\dot{\mathbf{r}}'_i \equiv \mathbf{L}_{CM} + \sum_{i=1}^{N}m_i\mathbf{r}'_i\times\dot{\mathbf{r}}'_i, \qquad \mathbf{r}_i = \mathbf{R} + \mathbf{r}'_i$,

$\dot{\mathbf{L}} = \sum_{i=1}^{N}\mathbf{r}_i\times\mathbf{F}_i^{(e)} \equiv \mathbf{N}^{(e)} \quad\Rightarrow\quad \mathbf{L}$ conserved $\iff \mathbf{N}^{(e)} = 0$

Kinetic energy, potential, and total mechanical energy:

$T = \frac{1}{2}\sum_{i=1}^{N}m_i\dot{\mathbf{r}}_i^2 = \frac{1}{2}M\dot{\mathbf{R}}^2 + \frac{1}{2}\sum_{i=1}^{N}m_i(\dot{\mathbf{r}}'_i)^2$. If the forces acting on the system is *conservative*, in other words, if $\mathbf{F}_i\equiv\mathbf{F}_i^{(e)} + \sum_{j=1}^{N}\mathbf{F}_{ij} = -\frac{\partial V}{\partial\mathbf{r}_i}$, $i = 1,\dots,N$; then the total mechanical energy $E = T + V(\mathbf{r}_1,\dots,\mathbf{r}_N)$ of the system is conserved.

The potential of the system has the form: $V = \sum_{i=1}^{N}V_i(\mathbf{r}_i) + \sum_{1\le i<j\le N}V_{ij}(\mathbf{r}_i,\mathbf{r}_j)$.

Defining $V_{ij}(\mathbf{r}_i,\mathbf{r}_j) = U_{ij}(\mathbf{r}_i-\mathbf{r}_j) \Rightarrow V = \sum_{i=1}^{N}V_i(\mathbf{r}_i) + \sum_{1\le i<j\le N}U_{ij}(\mathbf{r}_i-\mathbf{r}_j)$.

3 Central forces

3.1 Basic definitions

Conditions for a central force

$\mathbf{F}_{12} = f(|\mathbf{r}_1-\mathbf{r}_2|)\frac{\mathbf{r}_1-\mathbf{r}_2}{|\mathbf{r}_1-\mathbf{r}_2|} \Rightarrow \mathbf{F}(\mathbf{r}) = \mu\ddot{\mathbf{r}} = f(r)\frac{\mathbf{r}}{r}$, in such case the following equations hold: $\ddot r - r\dot\theta^2 = \frac{f(r)}{\mu}$, $r\ddot\theta + 2\dot r\dot\theta = 0$

3.2 Conserved quantities

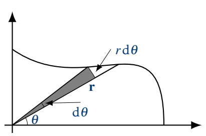
$\mathbf{L} = \mu\mathbf{r}\times\dot{\mathbf{r}} = \mu r^2\dot\theta\mathbf{e}_z$, $|\mathbf{L}| = L_z = \dot\theta r^2\mu > 0$

$E = \frac{1}{2}\mu\dot r^2 + U(r) = \frac{L^2}{2\mu}(u'^2 + u^2) + V(\frac{1}{u})$

Law of areas

$A(\theta) = \frac{1}{2}\int_{\theta_0}^{\theta}r^2(\alpha)\,d\alpha$,

$\dot A = \frac{dA}{d\theta}\dot\theta = \frac{1}{2}r^2\dot\theta = \frac{L}{2\mu}\in\mathbb{R}$



3.3 Important equations

Binet's eqn.

$f(r) = -\frac{L^2}{\mu r^2}(u'' + u)$

Other equations

$U(r) = V(r) + \frac{L^2}{2\mu r^2}$, $-\frac{\partial}{\partial r}\left(\frac{L^2}{2\mu r^2}\right) = \frac{\mu v_\theta^2}{r}\mathbf{e}_r$

$t = \pm\sqrt{\frac{\mu}{2}}\int_{r_1}^{r_2}\frac{dr}{\sqrt{E-U(r)}}$, $t = \frac{\mu}{L}\int_{r_1}^{r_2}r^2(\theta)\,d\theta$,

$\theta = \frac{L}{\mu}\int_{t_1}^{t_2}\frac{dt}{r^2(t)} = \pm\frac{L}{\sqrt{2\mu}}\int^{\frac{1}{r}}\frac{du}{\sqrt{E-U(\frac{1}{u})}}$,

$\tau_r = \sqrt{2\mu}\int_{r_1}^{r_2} = \frac{dr}{\sqrt{E-U(r)}}$, $\Delta\theta = \sqrt{\frac{2L^2}{\mu}}\int_{\frac{1}{r_2}}^{\frac{1}{r_1}} = \frac{du}{\sqrt{E-U(\frac{1}{u})}}$,

An orbit is periodic $\iff \Delta\theta = q\pi$, $q\in\mathbb{Q}$.

$A_T = \pi r^2 = \pi ab = \dot A\cdot\tau = \frac{L}{2\mu}\tau$,

$v = \frac{L}{\mu}\sqrt{u'^2 + u^2}$

3.4 Kepler's potential

Force field and Binet's eqn.

$\mathbf{F}(r) = -\frac{k}{r^2}\mathbf{e}_r$, $V(r) = -\frac{k}{r}$, $\ddot{\mathbf{r}} = -GM\frac{\mathbf{r}}{r^3}$, $u'' + u = \frac{\mu k}{L^2}$

Equation of trajectory and its clasifications

$u = \frac{\mu k}{L^2}(1 + e\cos(\theta-\theta_0))$, $e, \theta_0\in\mathbb{R}$.

$r = \frac{\alpha}{1+e\cos(\theta)}$, $\alpha\equiv\frac{L^2}{\mu k}$, $e = \sqrt{1 + \frac{2EL^2}{\mu k^2}}\ge 0$

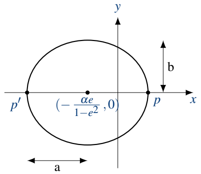
$$\begin{cases} e>1 & \iff & E>0 & \Rightarrow & \text{hyperbola}\\ e=1 & \iff & E=0 & \Rightarrow & \text{parabola}\\ e\in(0,1) & \iff & E\in(-\frac{\mu k^2}{2L^2},0) & \Rightarrow & \text{ellipse}\\ e=0 & \iff & E=-\frac{\mu k^2}{2L^2} & \Rightarrow & \text{circle}\end{cases}$$

3.5 Planetary motion

Study of ellipses

$\mathcal{C}\equiv\{(1-e^2)(x + \frac{\alpha e}{1-e^2})^2 + y^2 = \frac{\alpha^2}{1-e^2}\}$

$a = \frac{\alpha}{1-e^2}$, $b = \frac{\alpha}{\sqrt{1-e^2}}$, $c = \varepsilon a = ea$, $e = \varepsilon$,



$E = -\frac{k}{2a}$, $\langle r\rangle = \left(1 + \frac{e^2}{2}\right)a$,

$\tau = 2\pi\sqrt{\frac{\mu}{k}}a^{3/2} = \pi k\sqrt{\frac{\mu}{2}}|E|^{-3/2} = \frac{2\pi a^{3/2}}{\sqrt{GM}} \simeq \frac{2\pi a^{3/2}}{\sqrt{GM_\odot}}$

$p = \frac{\alpha}{1+e} = a(1-e)$, $p' = \frac{\alpha}{1-e} = a(1+e)$,

$$v^2 = \frac{k^2}{L^2}(1 + e^2 + 2e\cos\theta) \Rightarrow \begin{cases} v_p = \frac{k}{L}(1+e) = \sqrt{\frac{k}{\mu p}}(1+e)\\ v_{p'} = \frac{k}{L}(1-e) = \sqrt{\frac{k}{\mu p'}}(1-e)\end{cases}$$

4 Lagrangian & Hamiltonian

4.1 Basic Definitions

$F[y] = \int_{x_1}^{x_2}f\,dx$, where $F\equiv$functional, and $f\equiv$ density.

4.2 Important equations and theorems

Euler-Lagrange equations:

$\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = 0$; $\frac{d}{dt}\frac{\partial L}{\partial\dot q} - \frac{\partial L}{\partial q}$. If $\frac{\partial f}{\partial x} = 0 \Rightarrow y'\frac{\partial f}{\partial y'} - f$ is constant. If $\frac{\partial f}{\partial y} = 0 \Rightarrow \frac{\partial f}{\partial y'}$ is constant. Two densities differing by a total derivative has the same E-L eqn.

Hamilton's principle for N particles:

$L(t,\mathbf{r}_1,\dots,\mathbf{r}_N,\dot{\mathbf{r}}_1,\dots,\dot{\mathbf{r}}_N) = \frac{1}{2}\sum_i^N m_i\dot{\mathbf{r}}_i^2 - V(t,\mathbf{r}_1,\dots,\mathbf{r}_N)$.

If $\frac{\partial L}{\partial t} = 0 \Rightarrow \sum_i\dot{\mathbf{r}}_i\frac{\partial L}{\partial\dot{\mathbf{r}}_i} - L = T + V$ conserved.

If $\frac{\partial L}{\partial x_i} = 0 \Rightarrow \frac{\partial L}{\partial\dot x_i}$ is conserved.

Canonical momenta:

$p_i = \frac{\partial L}{\partial\dot q_i} \Rightarrow \dot p_i = \frac{\partial L}{\partial q_i}$. If $p_i = 0$ then the coordinate q_i is *cyclic* or *ignorable*.

Hamiltonian:

$H = \sum_i^n\dot q_i\frac{\partial L}{\partial\dot q_i} - L = \sum_i p_i\dot q_i - L = T + V(\Leftarrow$ If natural).

Hamilton's canonical eqns.:

$\dot{\mathbf{q}} = \frac{\partial H}{\partial\mathbf{p}}$; $\dot{\mathbf{p}} = -\frac{\partial H}{\partial\mathbf{q}}$

Jacobi-Poisson Theorem

If f and g are two first integrals of H's canonical equations, so is $\{f,g\}$.

4.3 Procedures

Find the EOM of N particles with l holonomic constraints:

1. Introduce $3N - l = n$ generalized coordinates $(q_1,\dots,q_n) = \mathbf{q}$ to the constraint manifold $\{\Phi_1,\dots,\Phi_l(t,\mathbf{q}) = 0\}$.
2. Express $T = \frac{1}{2}\sum_i^N m_i\dot{\mathbf{r}}_i^2$ and V in terms of $(t,\mathbf{q},\dot{\mathbf{q}}) \Rightarrow$
$$T\equiv T(t,\mathbf{q},\dot{\mathbf{q}});\ V\equiv V(t,\mathbf{q}).$$
3. E-L equations of $L = T - V$ are the n equations of motion in coordinates q_i $\Rightarrow \frac{d}{dt}\frac{\partial L}{\partial\dot q_i} - \frac{\partial L}{\partial q_i}$, $i = 1,\dots,n$.
4. Constraint forces on i-th particle are expressed as:
$$\mathbf{F}_i^{(c)} = m_i\ddot{\mathbf{r}}_i + \frac{\partial V}{\partial\mathbf{r}_i}, \ i = 1,\dots,N.$$

Write Hamilton's canonical equations:

1. Calculate the canonical momenta $p_i = \frac{\partial L}{\partial\dot q_i}$, $i = 1,\dots,n$.
2. Express the generalized velocities as $\dot q_i = \dot q_i(t,\mathbf{q},\mathbf{p})$, $i = 1,\dots,n$.
3. Compute the Hamiltonian $H(t,\mathbf{q},\mathbf{p}) = \mathbf{p}\cdot\dot{\mathbf{q}}(t,\mathbf{q},\mathbf{p}) - L(t,\mathbf{q},\mathbf{p})$.
4. The partial derivatives of H w.r.t the variables q_i, p_i are the canonical equations. The first n equations are the ones in step 2, the rest are $\dot p_i = -\frac{\partial H}{\partial q_i}$, $i = 1,\dots,n$. There are in total $2n$ equations.

4.4 Hamiltonian conservations

$\frac{\partial H}{\partial q_i} = 0 \Rightarrow p_i$ const. $\frac{\partial H}{\partial p_i} = 0 \Rightarrow q_i$ const.

$\frac{dH}{dt} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial\mathbf{q}}\dot{\mathbf{q}} + \frac{\partial H}{\partial\mathbf{p}}\dot{\mathbf{p}} = \frac{\partial H}{\partial t} \Rightarrow$ If $\frac{\partial H}{\partial t} = 0 \Rightarrow H$ const. Moreover, since $\frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$, H is conserved $\iff L$ indep. of t.

4.5 Poisson brackets

$(\mathbf{q},\mathbf{p})\equiv$"phase space", for every smooth function $f(t,\mathbf{q},\mathbf{p})$ the following holds:

$\dot f = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial\mathbf{q}}\dot{\mathbf{q}} + \frac{\partial f}{\partial\mathbf{p}}\dot{\mathbf{p}} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial\mathbf{q}}\frac{\partial H}{\partial\mathbf{p}} + \frac{\partial f}{\partial\mathbf{p}}\frac{\partial H}{\partial\mathbf{q}} = \frac{\partial f}{\partial t} + \{f,H\}$

Being $\{f,g\}\equiv\frac{\partial f}{\partial\mathbf{q}}\frac{\partial g}{\partial\mathbf{p}} - \frac{\partial f}{\partial\mathbf{p}}\frac{\partial g}{\partial\mathbf{q}} = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}\right)$, f,g any function of $(t,\mathbf{q},\mathbf{p})$.

Poisson brackets properties

$\{f,g\} = -\{g,f\}$, in particular $\{f,f\} = 0$. (Antisymmetry)
$\{\lambda f + \mu g, h\} = \lambda\{f,h\} + \mu\{g,h\}$. (Bilinearity)
$\{fg,h\} = f\{g,h\} + g\{f,h\}$. (Leibniz's rule)
$\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0$. (Jacobi identity)
$\frac{\partial}{\partial t}\{f,g\} = \left\{\frac{\partial f}{\partial t},g\right\} + \left\{f,\frac{\partial g}{\partial t}\right\} \Rightarrow \frac{d}{dt}\{f,g\} = \left\{\dot f,g\right\} + \{f,\dot g\}$ (J-P id.)
$\dot q_i = \{q_i,H\}$, $\dot p_i = \{p_i,H\}$; $\{q_i,q_j\} = \{p_i,p_j\} = 0$, $\{q_i,p_j\} = \delta_{ij}$

Canonical change of variable

Consists in changing $(\mathbf{q},\mathbf{p})\mapsto(\tilde{\mathbf{q}},\tilde{\mathbf{p}})$, s.t. $\exists\tilde H$ Hamiltonian where $\dot{\tilde{\mathbf{q}}} = \frac{\partial\tilde H}{\partial\tilde{\mathbf{p}}}$ and $\dot{\tilde{\mathbf{p}}} = -\frac{\partial\tilde H}{\partial\tilde{\mathbf{q}}}$.

The changes $\begin{cases}(\mathbf{q},\mathbf{p})\mapsto(\mathbf{p},\mathbf{q}),\ \tilde H(t,\tilde{\mathbf{q}},\tilde{\mathbf{p}}) = -H(t,\mathbf{q},\mathbf{p})\\ (\mathbf{q},\mathbf{p})\mapsto(\mathbf{p},-\mathbf{q}),\ \tilde H(t,\tilde{\mathbf{q}},\tilde{\mathbf{p}}) = H(t,\mathbf{q},\mathbf{p})\end{cases}$ are canonical.

In general, a transformation is canonical $\iff \begin{cases}\{\tilde q_i,\tilde q_j\} = \{\tilde p_i,\tilde p_j\} = 0\\ \{\tilde q_i,\tilde p_j\} = \lambda\delta_{ij},\ \lambda\neq 0.\end{cases}$

If $\lambda = 1$ then $(\tilde{\mathbf{q}},\tilde{\mathbf{p}})$ are *canonically conjugate*.
If $\lambda\neq 1$, the change $(\mathbf{q},\mathbf{p})\mapsto(\tilde{\mathbf{q}},\tilde{\mathbf{p}}/\lambda)$ is canonically conjugate.

Hamilton-Jacobi theory

It's always possible to find a can. transf. that has $\tilde H = 0 \Rightarrow \dot{\tilde{\mathbf{q}}} = 0 = \dot{\tilde{\mathbf{p}}} \Rightarrow \tilde{\mathbf{p}} = \tilde{\mathbf{p}}_0$; $\tilde{\mathbf{q}} = \tilde{\mathbf{q}}_0$ constant solutions, and reversing the transformation we will get the original $\mathbf{q}_0, \mathbf{p}_0$ constant solutions of the original H.

4.6 Misc.

Charged particle mass in an EM field Φ (*not natural*):

$$\begin{cases} L(t,\mathbf{r},\dot{\mathbf{r}}) = \frac{1}{2}m\dot{\mathbf{r}}^2 - e\Phi(t,\mathbf{r}) + e\dot{\mathbf{r}}\cdot\mathbf{A}(t,\mathbf{r})\\ H(t,\mathbf{r},\mathbf{p}) = \frac{1}{2m}(\mathbf{p} - e\mathbf{A}(t,\mathbf{r}))^2 + e\Phi(t,\mathbf{r})\\ \qquad p_i = m\dot x_i + eA_i(t,\mathbf{r}),\ i = 1,2,3.\end{cases}$$

5 Non-inertial frame

5.1 Basic definitions

Reference frames

$\mathcal{S}'\equiv$ inertial reference frame, $\mathcal{S}\equiv$ non inertial reference frame.

$\left(\frac{d}{dt}\right)_f t\equiv$ time derivative w.r.t $\mathcal{S}'$, $\left(\frac{d}{dt}\right)_f t\equiv$ time derivative w.r.t $\mathcal{S}$,

the vectors between the frames are related by $\mathbf{e}'_i(t) = O(t)\mathbf{e}_i$, being $O(t)$ the orthogonal linear application.

Rotation matrices and relations between ref. frames

$SO(3) = \{M\in\mathbb{R}^{3\times3}|MM^{\mathsf{T}} = M^{\mathsf{T}}M = \mathbb{1},\ \det\{M\} = 1\}$

$\forall M\in SO(3)$, $\exists\backslash\in\mathbb{R}^3$ and reference frame $\mathcal{S}' = \{\mathbf{e}_1,\mathbf{e}_2,\mathbf{n}\}$

s.t. $\mathrm{mat}_{\mathcal{S}'}^{\mathcal{S}'}(M)\equiv R_3(\theta) = \begin{bmatrix}\cos\theta & -\sin\theta & 0\\ \sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$.

$\frac{d}{d\theta}(R_n(\theta)) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}R_n(\theta+\varepsilon) = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0}R_n(\varepsilon)R_n(\theta)$
$= \Omega R_n(\theta) = \mathbf{n}\times R_n(\theta)$

$O(t)\in SO(3)$, $\forall t$, and $\dot O(t) = \Omega(t)O(t)$,

where $\Omega(t) = \begin{bmatrix}0 & -\omega_3(t) & \omega_2(t)\\ \omega_3(t) & 0 & -\omega_1(t)\\ -\omega_2(t) & \omega_1(t) & 0\end{bmatrix}$ antisymmetric matrix.

$\dot O(t)\mathbf{c} = \boldsymbol{\omega}(t)\times O(t)\mathbf{c}$, $\boldsymbol{\omega} = (\omega_1,\omega_2,\omega_3)$. $\qquad\forall t\in\mathbb{R}$ and $\forall\mathbf{c}\in\mathbb{R}^3$

"$\boldsymbol{\omega}(t)\times$"$= \Omega(t) = \dot O(t)O(t)^{\mathsf{T}} = \dot O(t)O(t)^{-1} \iff$

$\boldsymbol{\omega}(t_0)\times = \frac{d}{dt}\Big|_{t=t_0}O(t)O^{-1}(t_0)$.

$\dot{\mathbf{e}}_i(t) = \boldsymbol{\omega}(t)\times O(t)\mathbf{e}'_i = \boldsymbol{\omega}(t)\times\mathbf{e}_i(t)$.

$\boldsymbol{\omega}(t) = \dot\theta(t)\mathbf{n}(t)$, in words: the direction of the vector $\boldsymbol{\omega}(t)$ is equal to the axis of rotation $\mathbf{n}(t)$, and the magnitude of $\boldsymbol{\omega}(t)$ is equal to the angular velocity of the moving axes w.r.t the fixed ones.

5.2 Time derivative in different reference frames

$\left(\frac{d\mathbf{A}(t)}{dt}\right)_f = \left(\frac{d\mathbf{A}(t)}{dt}\right)_m + \boldsymbol{\omega}(t)\times\mathbf{A}(t)$,

in particular $\left(\frac{d\boldsymbol{\omega}(t)}{dt}\right)_f = \left(\frac{d\boldsymbol{\omega}(t)}{dt}\right)_m = \dot{\boldsymbol{\omega}}(t)$

If the origin of $\mathcal{S}$ is displacing from that of $\mathcal{S}'$ with $\mathbf{R}(t)$, then:

$\left(\frac{d\mathbf{A}(t)}{dt}\right)_f = \left(\frac{d\mathbf{A}(t)}{dt}\right)_m + \boldsymbol{\omega}(t)\times\mathbf{A}(t) + \mathbf{V}(t)$,

5.3 Dynamics in non-inertial ref. frame

Preamble

In the general case, the origin of $\mathcal{S}$ is displaced from that of $\mathcal{S}'$ by a time-dependent vector $\mathbf{R}(t)$. If $\mathbf{r}$ is the position of a particle w.r.t $\mathcal{S}$, then the position w.r.t $\mathcal{S}'$ is given by $\mathbf{r}' = \mathbf{r} + \mathbf{R}$. We'll define the following vectors:

$\mathbf{V}\equiv\left(\frac{d\mathbf{R}}{dt}\right)_f$, $\mathbf{A}\equiv\left(\frac{d\mathbf{V}}{dt}\right)_f = \left(\frac{d^2\mathbf{R}}{dt^2}\right)_f$, $\mathbf{v}_f = \left(\frac{d\mathbf{r}'}{dt}\right)_f$, $\mathbf{v}_m = \left(\frac{d\mathbf{r}}{dt}\right)_m$

$\mathbf{a}_f\equiv\left(\frac{d\mathbf{v}_f}{dt}\right)_f = \left(\frac{d^2\mathbf{r}'}{dt^2}\right)_f$, $\mathbf{a}_m\equiv\left(\frac{d\mathbf{v}_m}{dt}\right)_m = \left(\frac{d^2\mathbf{r}}{dt^2}\right)_m$.

Equations of motion

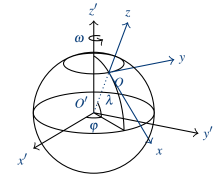
$\mathbf{v}_f = \mathbf{v}_m + \boldsymbol{\omega}\times\mathbf{r} + \mathbf{V}$,
$\mathbf{a}_f = \mathbf{a}_m + \mathbf{A} + 2\boldsymbol{\omega}\times\mathbf{v}_m + \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) + \dot{\boldsymbol{\omega}}\times\mathbf{r}$,
$m\mathbf{a}_m = \mathbf{F} - m\mathbf{A} - 2m\boldsymbol{\omega}\times\mathbf{v}_m - m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) - m\dot{\boldsymbol{\omega}}\times\mathbf{r}\equiv\mathbf{F} + \mathbf{F}_i$,
where $\mathbf{F}_i$ is the fictitious force:
$\mathbf{F}_i = -m\mathbf{A} - 2m\boldsymbol{\omega}\times\mathbf{v}_m - m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r}) - m\dot{\boldsymbol{\omega}}\times\mathbf{r}$

Centrifugal force $\equiv -m\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$,
Coriolis force $\equiv -2m\boldsymbol{\omega}\times\mathbf{v}_m$, depends on the velocity of the particle.

For the equation $\dot{\mathbf{A}}(t) = \boldsymbol{\omega}\times\mathbf{A}(t) + \mathbf{c}$, where $\boldsymbol{\omega}\perp\mathbf{c}$ are both constant vectors, the general solution of the ODE is: $\mathbf{A}(t) = R_{\mathbf{n}}(\omega t)\mathbf{u} + \frac{1}{\omega}\mathbf{n}\times\mathbf{c}$, with $\mathbf{u}\in\mathbb{R}^3$, $\mathbf{n} = \omega^{-1}\boldsymbol{\omega}$, .

5.4 Motion of a particle relative to the rotating Earth

Basis vectors



$\mathbf{e}_1 = \sin\lambda\cos\varphi\mathbf{e}'_1 + \sin\lambda\sin\varphi\mathbf{e}'_2 - \cos\lambda\mathbf{e}'_3$,
$\mathbf{e}_2 = -\sin\varphi\mathbf{e}'_1 + \cos\varphi\mathbf{e}'_2$,
$\mathbf{e}_3 = \cos\lambda\cos\varphi\mathbf{e}'_1 + \cos\lambda\sin\varphi\mathbf{e}'_2 + \sin\lambda\mathbf{e}'_3$.

Kinetics & Dynamics

$\boldsymbol{\omega} = \omega(-\cos\lambda\mathbf{e}_1 + \sin\lambda\mathbf{e}_3)$
$\mathbf{g}_0 = -\frac{GM}{R^3}\mathbf{R} = -\frac{GM}{R^2}\mathbf{e}_3\equiv -g_0\mathbf{e}_3$, $g_0\simeq 9.8\ \mathrm{m\cdot s^{-2}}$
$\ddot{\mathbf{r}} = \mathbf{g} - 2\boldsymbol{\omega}\times\dot{\mathbf{r}} - \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$, where $\mathbf{g} = \mathbf{g}_0 - \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{R})$.
With additional force: $\ddot{\mathbf{r}} = \frac{\mathbf{F}}{m} + \mathbf{g} - 2\boldsymbol{\omega}\times\dot{\mathbf{r}} - \boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{r})$.

The angle between $-\mathbf{e}_3$ and $\mathbf{g}$ is $\delta(\lambda)$, where $\tan\delta(\lambda) = \frac{\omega^2R\sin\lambda\cos\lambda}{g_0 - \omega^2R\cos^2\lambda}$.

Approximately, the equation of motion of a particle moving close to the Earth's surface, is given by $\mathbf{r}\simeq\mathbf{r}_0 + \mathbf{v}_0t + \mathbf{g}_0\frac{t^2}{2} - t^2\boldsymbol{\omega}\times\mathbf{v}_0 - \frac{t^3}{3}\boldsymbol{\omega}\times\mathbf{g}_0$.

Sometimes, to approximate, we often ignore some terms inside the above equation because of the following ratio: $\frac{g_0}{\omega}\simeq 1.35\cdot10^5\,\mathrm{m\,s^{-1}} >> v_0$

5.5 Foucault's pendulum

Consists of a pendulum with length that is far greater than the coordinates x,y,z of its bob. The equations of motion of the coordinates x,y are:

$\begin{cases}\ddot x + \alpha^2x = 2\omega\dot y\sin\lambda\\ \ddot y + \alpha^2y = -2\omega\dot x\sin\lambda\end{cases}$, where $\alpha\equiv\sqrt{\frac{g_0}{l}}$. The motion of the coordinate z can be ignored. Letting $u = x + iy$, the general solution of u is:

$u = e^{-i\Omega t}\left(c_1e^{i\alpha t} + c_2e^{-i\alpha t}\right)$, where $c_1,c_2\in\mathbb{C}$ and $\Omega\equiv\omega\sin\lambda$.

A solution with initial conditions $\begin{cases}x(0) = x_0 > 0,\ y(0) = 0\\ \dot x = \dot y = 0\end{cases}$, would be:

$u = x_0e^{-i\Omega t}\cos(\alpha t) \Rightarrow x = x_0\cos(\Omega t)\cos(\alpha t)$, $y = -x_0\sin(\Omega t)\cos(\alpha t)$.

The projection of the vector onto the $X - Y$ plane is given by $x_0\cos(\alpha t)[\cos(\Omega t)\mathbf{e}_1 - \sin(\Omega t)\mathbf{e}_2]$. In the northern hemisphere the pendulum's plane rotates clockwise, i.e., in the east-south direction (since $\dot\theta = -\Omega = -\omega\sin\lambda < 0$), with angular velocity $\Omega = \omega\sin\lambda$. In the southern hemisphere the rotation of the pendulum's plane is counterclockwise (since $\sin\lambda < 0$), and in the equator ($\lambda = 0$) no such rotation occurs.

In each period $2\pi/\alpha$ of the pendulum the angle between the pendulum's plane and. the $X - Y$ plane increases by $-2\pi\Omega/\alpha$, which is a very small quantity. The period of the rotation of the pendulum's plane is given by $\tau = \frac{2\pi}{\Omega} = \frac{2\pi}{\omega}\csc\lambda$ sidereal days.

6 Rigid body motion

Note: some of the notations in this part is mixed with the ones used by *Amores* in Una base para el álgebra lineal, in which column vectors are *not* represented in bold.

6.1 Basic definitions

A **rigid body** is a system of particles of mass m_α $(\alpha = 1,\dots,N)$ in which the *distance* $|\mathbf{r}_\alpha - \mathbf{r}_\beta|$ between any two particles is *constant*. In other words, a rigid body is a mechanical system of N particles subject to the $N(N-1)/2$ time-independent holonomic constraints (that may or may not be mutually independent).

In a generic rigid body it's always possible to construct a set of moving axes $\mathcal{S}$ (**body axes**), with respect to which all it's particles are *fixed*.

A generic rigid body has 6 d. of freedom, 3 for the coordinate of the origin of the body axes, and 3 for the matrix $O(t)\in SO(3)$ that relates $\mathcal{S}'$ and $\mathcal{S}$.

Centre of mass of a r.b.: $\mathbf{R} = \frac{1}{M}\int\rho(\mathbf{r})\mathbf{r}\,d^3\mathbf{r}$, where $M = \int\rho(\mathbf{r})\,d^3\mathbf{r}$.

6.2 Angular momentum and T of a rigid body

Letting the centre of mass be the origin of $\mathcal{S}$:

$\mathbf{r}'_\alpha = \mathbf{R} + \mathbf{r}_\alpha$, $\mathbf{v}'_\alpha = \mathbf{V} + \boldsymbol{\omega}\times\mathbf{r}_\alpha$, $\mathbf{P} = M\mathbf{V}$.
$\mathbf{L} = M\mathbf{R}\times\mathbf{V} + \mathbf{L}_{CM}$, where $\mathbf{L}_{CM} = \sum_\alpha m_\alpha\mathbf{r}_\alpha\times(\boldsymbol{\omega}\times\mathbf{r}_\alpha)$

$T = \frac{1}{2}MV^2 + T_{rot}$, where $T_{rot} = \frac{1}{2}\sum_\alpha m_\alpha(\boldsymbol{\omega}\times\mathbf{r}_\alpha)^2$

$\Rightarrow\mathbf{L}_{CM} = \sum_\alpha m_\alpha[r_\alpha^2\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\mathbf{r}_\alpha)\mathbf{r}_\alpha]$, $\quad T_{rot} = \frac{1}{2}\boldsymbol{\omega}\cdot\mathbf{L}_{CM}$

6.3 Inertia tensor

Note: the inertia tensor I, without any subscript, is calculated w.r.t. the body axes (with CM as it's origin). But it can also be calculated w.r.t other points (Steiner's theorem).

Definition

The inertial tensor $(I\in\mathbb{R}^{3\times3})$ is a symmetric matrix with elements:

$$I_{ij} = \sum_\alpha m_\alpha(\delta_{ij}r_\alpha^2 - x_{\alpha i}x_{\alpha j}) \Rightarrow \begin{cases} I_{ii} = m_\alpha(x_{\alpha j}^2 + x_{\alpha k}^2)\\ I_{ij} = -\sum_\alpha m_\alpha x_{\alpha i}x_{\alpha j}\end{cases}$$

If the r.b. is continuous: $\begin{cases} I_{ii}=\int_\Omega \rho(\mathbf r)(x_j^2+x_k^2)\,\mathrm d^3\mathbf r,\ i=1,2,3 \\ I_{ij}=-\int_\Omega \rho(\mathbf r)x_ix_j\,\mathrm d^3\mathbf r,\ 1\le i\neq j\le 3\end{cases}$

Application

$T_{\text{rot}}=\frac12\,\boldsymbol\omega\cdot(I\,\boldsymbol\omega)=\frac12\,\omega^\mathsf T I\omega$ ($\Leftarrow$ here ω is a column (lin. algebraic) vector.
$\Rightarrow T_{\text{rot}}=\frac12 I_{\mathbf n}\omega^2$, where $I_{\mathbf n}$ is the moment of inertia w.r.t. instantaneous axis of rotation $\mathbf n=\frac{\boldsymbol\omega}{\omega}$

Steiner's theorem

The inertia moment computed w.r.t a point P fixed w.r.t body axes is given as: $(I_P)_{ij}=\sum_\alpha m_\alpha(\delta_{ij}\tilde{\mathbf r}_\alpha^2-\tilde x_{\alpha i}\tilde x_{\alpha j})$. Taking into account $\tilde{\mathbf r}_\alpha=\mathbf r_\alpha-\mathbf a,\ \mathbf a=\overrightarrow{CP}$
$\Rightarrow (I_P)_{ij}=I_{ij}+M(\mathbf a^2\delta_{ij}-a_ia_j)$.
If P is also fixed w.r.t $\mathcal S'$, then, placing $O'=P$, we can reduce the expressions:
$\mathbf V=\boldsymbol\omega\times\mathbf R,\quad \mathbf v'_\alpha=\boldsymbol\omega\times\mathbf r'_\alpha,\quad \mathbf L=I'\,\boldsymbol\omega,\ (I'=I_P),\quad T=\frac12\,\boldsymbol\omega\cdot I'\,\boldsymbol\omega.$
I' can be computed from I: $I'_{ij}=I_{ij}+M(\mathbf R^2\delta_{ij}-X_iX_j)$, where X_h are components of the vector $\mathbf R$.

Principal axes of inertia

Let $A\in\mathrm{SO}(3)$ be a constant change of basis matrix, and $\tilde{\mathbf e}_i=\sum_j a_{ji}\mathbf e_j$.

$\Rightarrow x=A\tilde x,\quad \tilde I=A^{-1}IA=A^\mathsf T IA.$
Since I is a symmetric matrix, there is always a matrix $A\in\mathrm{SO}(3)$, s.t. $\tilde I=A^\mathsf T IA$ is a diagonal matrix, in other words, the moment of inertia w.r.t the alternate body axes $\tilde{\mathbf e}_i$ (which are eigenvectors) is expressed as:

$\tilde I=\begin{bmatrix}I_1&&\\&I_2&\\&&I_3\end{bmatrix}$, with I_i the eigenvalues corresponding to $\tilde{\mathbf e}_i$.

In such case, $\tilde{\mathbf e}_i$ are **principal axes of inertia**, and I_i are **principal moments of inertia**, which are the roots of the caracteristic polinomial of I.
If $\mathbf e_i$ are p.a.i. then: $\mathbf L_{\text{CM}}=\sum_i I_i\omega_i\mathbf e_i,\quad T_{\text{rot}}=\frac12\sum_i I_i\omega_i^2.$

If the origin of a set of p.a.i. are also fixed in $\mathcal S'$, then $\mathbf L, T$ can be calculated with the expressions above, replacing I by I' w.r.t. the point O'.

Clasification of r.b. in terms of the multiplicity of the eigenvalues I_i

Type of symmetry	Definition
Asymmetric tops:	$I_j\neq I_i,\forall i\neq j,$
Axially symmetric tops:	$I_i=I_j\neq I_k,$
Spherically symmetric tops:	$I_i=I_j,\ \forall i,j$

Deduce I_i of a rigid body by its invariance under transformations

Note: both Ω and ρ have to be invariant under the transfomations.

Inv. under	Behaviour of I_i
$x_i\mapsto -x_i$	$I_{ij}=0,\ \forall j\neq i$
$x_i\mapsto x_j$	$I_{ii}=I_{jj},\quad I_{ik}=I_{jk},\ k\neq i,j$

In particular, for a *solid of revolution* with homogeneous density, taking x_3 as the axis of rotation and arbitrary perpendicular axes x_1,x_2, it satisfy the following invariances: $x_1\mapsto -x_1,\ x_2\mapsto -x_2,\ x_1\mapsto x_2$, and thus:
$I_{11}\equiv I_1=I_2\equiv I_{22},\ I_{ij}=0\ (i\neq j)$, with:
$$I_1=\pi\rho\int_{z_1}^{z_2}z^2f^2(z)\,\mathrm dz+\frac{\pi\rho}{4}\int_{z_1}^{z_2}f^4(z)\,\mathrm dz,\qquad I_3\equiv I_{33}=\frac{\pi\rho}{2}\int_{z_1}^{z_2}f^4(z)\,\mathrm dz.$$

6.4 EOM of a rigid body

EOM in an inertial frame

If $\mathbf F=0$ then $\mathbf N$ computed at any point in $\mathcal S'$ is the same.
A r.b. is in equilibrium in an $\mathcal S\iff \mathbf F=\mathbf N=0\iff\mathbf F=\mathbf N_{\text{CM}}=0$

Forces due to a constant field f

$\lambda_\alpha\equiv$"charge" of a particle $\Rightarrow\mathbf F_\alpha=\lambda_\alpha\mathbf f\Rightarrow\mathbf F=\sum_\alpha=\mathbf f\sum_\alpha\lambda_\alpha=\Lambda\mathbf f,$

$\mathbf N=\left(\sum_\alpha\lambda_\alpha\mathbf r'_\alpha\right)\times\mathbf f.$ ($\downarrow$ Note that $\mathbf X$ is fixed in the body)

If $\Lambda\neq0$, defining $\mathbf X=\frac1\Lambda\sum_\alpha\lambda_\alpha\mathbf r'_\alpha=\frac1\Lambda\sum_\alpha\lambda_\alpha(\mathbf R+\mathbf r_\alpha)=\mathbf R+\frac1\Lambda\sum_\alpha\lambda_\alpha\mathbf r_\alpha$
we have that $\mathbf N=\mathbf X\times\mathbf F\Rightarrow$ If $\Lambda\neq0$, then the total torque of the external forces coincides with the torque of the total external forces applied at the *centre of charge*. (For the grav. field $\mathbf g$, $\lambda=m$ and C. charge is equal to CM).
Torque of the external forces w.r.t. CM is $\mathbf N_{\text{CM}}=(\mathbf X-\mathbf R)\times\mathbf F$ (for $\mathbf g$ it is zero).
When $\Lambda=0$ we use the former equation for $\mathbf N$ and the torque of the external forces is independent of the reference point, since $\mathbf F=\Lambda\mathbf f=0$.

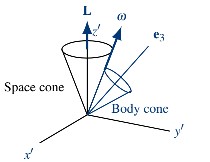
Euler's equations

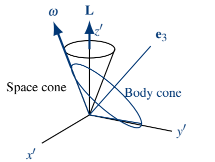
Euler's equations is the system $\begin{cases}I_1\dot\omega_1-(I_2-I_3)\omega_2\omega_3=N_1,\\ I_2\dot\omega_2-(I_3-I_1)\omega_1\omega_3=N_2,\\ I_3\dot\omega_3-(I_1-I_2)\omega_1\omega_2=N_3.\end{cases}$ where $\mathbf N$ and I are computed w.r.t. CM or a point fixed in the body and $\mathcal S'$. $\boldsymbol\omega$ and $\mathbf N$ are in a frame of p.a.i. (in general not inertial).
If $\mathbf N=0$ and the origin O' of $\mathcal S'$ is a point in the body, $\mathbf L_{O'}\equiv\mathbf L$ and T are conserved. Similarly, if $\mathbf N_{\text{CM}}=0$ then $\mathbf L_{\text{CM}}$ and T_{rot} are conserved.

6.5 Inertial motion of a symmetric top

The angular velocity $\boldsymbol\omega$ of a set of axes w.r.t. to another is *additive*.
For an axially ($\mathbf e_3$) symmetric r.b., by E's eqns, the following conditions holds:
$\omega_3,\ \Omega=\frac{I_3-I_1}{I_1}\,\omega_3,\ \omega_0=\sqrt{\omega_1^2+\omega_2^2},\ \alpha=\arctan(\omega_0/\omega_3)$ constants.
In frame of body axes, the vector $\boldsymbol\omega$ rotates around $\mathbf e_3$ with Ω (traces out the **body cone** fixed w.r.t. body axes). The angular velocity $\Omega>0$ for $I_3>I_1$, and negative otherwise.
Relative to body axes, $\mathbf L$ rotates with Ω around $\mathbf e_3$. $\mathbf L$ fixed relative to $\mathcal S'$.
Relative to $\mathcal S'$, $\boldsymbol\omega$ traces out the **space cone** around $\mathbf L$ (fixed in $\mathcal S'$) with Ω_p.
The angle between $\mathbf e_3$ and $\boldsymbol\omega$ is α, between $\mathbf L$ and $\mathbf e_3$ is $\theta=\arctan\frac{I_1}{I_3}\tan\alpha$.

$$\Omega_p=\frac{L}{I_1}=\omega\sqrt{1+\frac{I_3^2-I_1^2}{I_1^2}\cos^2\alpha},\quad \boldsymbol\omega=\Omega_p\frac{\mathbf L}{L}-\Omega\mathbf e_3.$$





6.6 Miscellaneous

In general, for a solid of revolution rolling without sliding down a inclined (α) plane around the principal axis $\mathbf e_3$, the horizontal acceleration is expressed as:
$\ddot x=\frac{g\sin\alpha}{1+\frac{I_3}{Ma^2}}\Rightarrow$.

Inertia tensor of some regular bodies

Body	Definition	Inertia tensor
Solid Sphere	$x^2+y^2+z^2\le r^2$	$\frac25 mr^2\begin{bmatrix}1&&\\&1&\\&&1\end{bmatrix}$
Hollow Sphere	$x^2+y^2+z^2=r^2$	$\frac23 mr^2\begin{bmatrix}1&&\\&1&\\&&1\end{bmatrix}$
Cuboid	$\begin{cases}-\frac d2\le x\le\frac d2\\-\frac w2\le y\le\frac w2\\-\frac h2\le z\le\frac h2\end{cases}$	$\frac{m}{12}\begin{bmatrix}h^2+d^2&&\\&w^2+d^2&\\&&h^2+w^2\end{bmatrix}$
Cylinder	$\begin{cases}x^2+y^2\le r^2\\-\frac h2\le z\le\frac h2\end{cases}$	$\frac{m}{12}\begin{bmatrix}3r^2+h^2&&\\&3r^2+h^2&\\&&6r^2\end{bmatrix}$
Rod about end	$\begin{cases}x^2+y^2\le r^2\\0\le z\le h\\h>>r\\\boldsymbol\omega\parallel\mathbf x\end{cases}$	$\frac13 ml^2\begin{bmatrix}1&&\\&0&\\&&1\end{bmatrix}$
Rod about center	$\begin{cases}x^2+y^2\le r^2\\-\frac h2\le z\le\frac h2\\h>>r\\\boldsymbol\omega\parallel\mathbf x\end{cases}$	$\frac1{12}ml^2\begin{bmatrix}1&&\\&0&\\&&1\end{bmatrix}$

7 Special relativity

7.1 Principle of special relativity

In Galilean relativity, the acceleration of a particle is the same in all inertial frames $\Rightarrow$ all inertial frames are *equivalent* from the point of view of Newtonian mechanics.

Postulates of special relativity:
1. The laws of physics are the same in all inertial frames (**relativity principle**).
2. The speed of electromagnetic waves in vacuum is universal: $c=\frac{1}{\sqrt{\epsilon_0\mu_0}}$.
3. (Implication of 1. & 2.) speed of electromagnetic waves in vacuum is c in all inertial frames.

7.2 Lorentz transformations

Basic transformation

We will use the notation $x_0\equiv ct,\ x_{0'}\equiv ct'$ as the "time coordinate" of space-time. The Lorentz factor for a speed of v is defined as:
$$\gamma(v)=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}},\ \text{dimensionless.}$$
The L. transformation relating the frames x_μ and x'_μ, (the velocity of O' w.r.t. O is $v\mathbf e_1$), is:
$$t'=\gamma(v)\left(t-\frac{vx_1}{c^2}\right),\quad x_1'=\gamma(v)(x_1-vt),\quad x_k'=x_k,\quad k=2,3$$

Consequences of L.T.

1. The relative speed between two inertial frames is strictly less than c in vacuum.
2. The speed of all mass is less than c.
3. The propagation speed any information cannot exceed c.

General transformations

The general L.T. with $\mathbf v$ the velocity of O' w.r.t. $\mathcal S$ is given by:
$$t'=\gamma(v)\left(t-\frac{\mathbf v\cdot\mathbf x}{c^2}\right),\quad \mathbf x'=\mathbf x+(\gamma(v)-1)\frac{\mathbf v\cdot\mathbf x}{v^2}-\gamma(v)\mathbf vt$$
(*Relativistic law for the addition of velocities.*) For a particle with constant velocity $\mathbf u=u\mathbf e_1$ w.r.t. $\mathcal S$, its velocity w.r.t. $\mathcal S'$ (whose origin moves at constant velocity $v\mathbf e_1$) is also constant, with components:
$$u_1=\frac{u_1'+v}{1+\frac{u_1'v}{c^2}},\quad u_k=\frac{u_k'}{\gamma(v)\left(1+\frac{u_1'v}{c^2}\right)},\quad (k=2,3),$$
the reversed relations can be obtained by changing the signs and switching primes.

Intervals

The quadratic form $c^2t^2-\mathbf x^2\equiv x_0^2-\mathbf x^2$ is *invariant* under L.T..
The square of the **interval** between two events with s-t coordinates x_μ and $x_\mu+\Delta x_\mu$ is defined as:
$$\Delta s^2=c^2\Delta t^2-\sum_{i=1}^3\Delta x_i^2=\Delta x_0^2-\sum_{i=1}^3\Delta x_i^2\equiv\Delta x_0^2-\Delta\mathbf x^2;$$
said interval is invariant under L.T.$\Rightarrow\Delta s^2=\Delta x_0^2-\Delta\mathbf x^2=\Delta(x_0')^2-\Delta(\mathbf x')^2$.
The interval between two events is *time-like* if $\Delta s^2>0$, *light-like* if $\Delta s^2=0$, and *space-like* if $\Delta s^2<0$.

If the interval between two events is *time-like*:
Let the original $\mathcal S$ s.t. $\Delta x_2,\ \Delta x_3=0$, and a second $\mathcal S'$ with velocity $\frac{\Delta x_1}{\Delta t}\mathbf e_1$.
Then, in $\mathcal S'$ the two events occur at the same point in space, with time interval (**proper time lapse**) $\Delta\tau=\frac{\Delta s}{c}=\Delta t\sqrt{1-\frac{\Delta x_1^2}{\Delta x_0^2}}$.

If the interval between two events is *space-like*:
Let the original $\mathcal S'$ s.t. $\Delta x_2,\ \Delta x_3=0$, and a second $\mathcal S'$ with velocity $\frac{c^2\Delta t}{\Delta x_1}\mathbf e_1=\frac{c\Delta x_0}{\Delta x_1}\mathbf e_1$. Then in $\mathcal S'$ the two events occur simultaneously, with distance (**proper distance**) $\sqrt{-\Delta s^2}=|\Delta x_1'|$. The proper distance is less or equal to the spatial distance $|\Delta\mathbf x|$ in any other inertial frame.

Minkowski product

For two s-t coordinates in $\mathcal S$, $x=(x_0,\mathbf x)$ and $y=(y_0,\mathbf y)\in\mathbb R^4$, their *Minkowski product*:
$$x\odot y\equiv x_0y_0-\mathbf x\cdot\mathbf y\quad\Rightarrow\quad x^2\equiv x\odot x=x_0^2-\mathbf x^2,\quad\Delta x\odot\Delta x=\Delta s^2$$
is also invariant under L.T.. The vector space $(\mathbb R^4,\odot)$ is the *Minkowski space*.

Lorentz group

The procedure of relating the s-t coordinates of an event in two reference frames $\mathcal S$ and $\mathcal S'$ ($\mathcal S'$ moves at $\mathbf v$) is the following:
$$\mathcal S\to\mathcal S''\to\mathcal S'''\to\mathcal S'$$
1. Consider $\mathcal S''$ at rest w.r.t. $\mathcal S$, whose x_1'' is in the direction of $\mathbf v\Rightarrow x''=R_1x$, where $R_1\in\mathbb R^{4\times4}$ but only rotates the spatial coordinates.
2. $\mathcal S'''$ is the frame that moves with $\mathbf v=v\mathbf e_1''$ away from $\mathcal S''\Rightarrow x'''=L(v)x''$. $L(v)$ is the "basic" L.T.
3. The frames $\mathcal S'''$ and $\mathcal S'$ are related by a rotation $\Rightarrow x'=R_2x'''$.
Combining everything we get
$$x'=R_2L(v)R_1x\equiv\Lambda x$$
the transformation $\Lambda\in\mathbb R^{4\times4}$ is the *general Lorentz tranformation*. It is the most genearl form of L.T. that relates the space-time coord. of an event in two reference frames whose space-time origin coincide ($t=x_i=0\iff t'=x_i'=0$).
If the space-time origin of the frames differs by a vector $a\in\mathbb R^4$, then the transformation $x'=\Lambda x+a$ is called the *Poincaré tr.*.
The basic L.T. can be expressed as a matrix:
$$L(v)=\begin{bmatrix}\gamma(v)&-\beta(v)\gamma(v)&0&0\\-\beta(v)\gamma(v)&\gamma(v)&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},\quad\beta(v)\equiv\frac vc$$
The bilinear form associated with the M. product is:
$$G=\begin{bmatrix}1&&&\\&-1&&\\&&-1&\\&&&-1\end{bmatrix},\quad\Rightarrow\quad x\odot y=x^\mathsf TG\,y,\quad\Rightarrow\quad\Lambda^\mathsf TG\Lambda=G$$
The Minkowski product (and the square of intervals), are invariant under general Lorentz transformations. The set of matrices Λ that satisfy the equation above form the *Lorentz group*.
Performing the change of variable:
$$\beta(v)=\tanh\phi\Rightarrow\gamma(v)=\cosh\phi\Rightarrow L(v)=\begin{bmatrix}\cosh\phi&-\sinh\phi&0&0\\-\sinh\phi&\cosh\phi&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix},$$
And the result of successive Lorentz boosts with speeds $v_1=c\tanh\phi_1$ and $v_2=c\tanh\phi_2$ is another L. boost with rapidity $\phi_1+\phi_2$:
$$v=c\tanh(\phi_1+\phi_2)=\frac{v_1+v_2}{1+\frac{v_1v_2}{c^2}}\ \text{similar to the eqn. of velocity of a prtcl.}$$

7.3 Dilations and contractions

The **proper time** measured in $\mathcal S$ is larger than the **coordinate time** measured in $\mathcal S'$. In other words, from an observer at rest with $\mathcal S$, the time recorded by a watch at rest with $\mathcal S'$, t', is slower than the proper time t that he has measured. We say that t' dilated, the dilation effect is symmetrical in **inertial frames**:
$$t\equiv\frac{t'}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{t'}{\gamma(v)}>t'$$
For a particle following a trajectory $\mathbf r\equiv\mathbf r(t)$, the proper time increment (time according to an observer at rest with said particle) is expressed as:
$$\mathrm d\tau=\sqrt{1-\frac{v^2(t)}{c^2}}\Rightarrow\Delta\tau(C)=\int_{t_1}^{t_2}\sqrt{1-\frac{v^2(t)}{c^2}}\,\mathrm dt=\int_{t_1}^{t_2}\sqrt{1-\frac{\dot{\mathbf r}^2}{c^2}}\,\mathrm dt$$
being C the path taken by the particle. $\Delta\tau\le\Delta t$ and is equal only if $\mathbf v=0,\ \forall t$.

Lorentz-Fitzgerald contraction has to do with contraction of an object that's at rest with $\mathcal S'$ but in motion w.r.t. to $\mathcal S$ in the direction of its velocity. In mathematical terms, the length component in the direction of its movement, x_1, is contracted:
$$\Delta x_1'\equiv l_0=\gamma(v)(\Delta x_1-v\Delta t)\equiv\gamma(v)l\Rightarrow l=\frac{l_0}{\gamma(v)}=l\sqrt{1-\frac{v^2}{c^2}}<l_0,$$
where l_0 is the **rest length** or the length directly measured by an observer at rest with the object.

7.4 Four-velocity and four-momentum

The *four-velocity* is defined as:
$$u=\frac{\mathrm dx}{\mathrm d\tau}\in\mathbb R^4,\quad x\equiv\text{"s-t coordinates"},\ \mathrm d\tau=\sqrt{1-\frac{v^2}{c^2}}\,\mathrm dt\equiv\frac{\mathrm dt}{\gamma(v)}$$
and it has the following properties:
$$u=\gamma(v)(c,\mathbf v)\in\mathbb R^4,\quad u'=\Lambda u,\quad u^2=c^2.$$
The four-momentum is: $\quad p=mu\Rightarrow p^2=m^2c^2$
Here we will use the vector $\mathbf p=(p_1,p_2,p_3)\equiv m\gamma(v)\mathbf v\in\mathbb R^3$ which coincides with the non-r. momentum only when $v\to0$. Then:
$$p_0=\sqrt{\mathbf p^2+m^2c^2},\Rightarrow\mathbf v=\frac{c\mathbf p}{p_0}$$
and the relativistic kinetic energy is given as:
$$T=cp_0-mc^2\equiv mc^2(\gamma(v)-1)\quad\Rightarrow\quad p_0=\frac1c(mc^2+T)$$

Conservation of four-momentum

$p=$ const. $\iff cp_0=mc^2+T=$ const. $\iff p_i=m\gamma(v)v_i=$ const.
The total four-momentum of a system of N particles is:
$$P=\sum_{n=1}^N p_n\equiv(P_0,\mathbf P),\quad P_0=\sum_{n=1}^N p_{n,0}\equiv c\sum_{n=1}^N m_n\gamma(v_n),$$
$$\mathbf P=\sum_{n=1}^N\mathbf p_n\equiv\sum_{n=1}^N m_n\gamma(v_n)\mathbf v_n.$$
The conservation of P during the collision of said N particles is verified even when the collision is not elastic:
$$P_i=P_f\iff(Mc^2+T)_i=(Mc^2+T)_f$$
During the process of collision, The mass and the kinetic energy can vary together with the relation:
$$\Delta T=-\Delta(Mc^2)$$
The total relativistic energy can be defined as:
$$E=cP_0=mc^2+T=mc^2\gamma(v)=c\sqrt{\mathbf p^2+m^2c^2}=mc^2\sqrt{1+\frac{\mathbf p^2}{m^2c^2}}$$
$$\Rightarrow P=(E/c,\mathbf P)\Rightarrow\mathbf v=\frac{c^2\mathbf p}{E}$$

7.5 Massless particles

For particle with mass $m\to0$, its energy is $E=c|\mathbf p|$ and its four-momentum:
$$p=(|\mathbf p|,\mathbf p)\Rightarrow\mathbf v=c\frac{\mathbf p}{|\mathbf p|}.$$

Motion of a photon

Photon is the only massless particle known. The relation between its energy and the frequency ω of its associated EM wave is given as:
$$E=\hbar\omega\equiv h\nu=\frac{hc}{\lambda},\quad\lambda\equiv\text{"wavelength"}$$
The wave vector is defined as:
$$\omega=c|\mathbf k|,\quad\mathbf v=c\frac{\mathbf k}{|\mathbf k|}\quad\Rightarrow\quad\mathbf p=\hbar\mathbf k$$
We can define the **wave four-vector**: $\quad k=(k_0,\mathbf k)=(|\mathbf k|,\mathbf k)=\frac{p}{\hbar}\in\mathbb R^4$. The wave four-vector is proportional to the four-momentum, therefore, if $\mathcal S'$ is another inertial system and $x'=\Lambda x$ we have: $\quad k'=\Lambda k$

7.6 Longitudinal Doppler effect

If an EM wave propagates in the direction of the relative motion between the observer $\mathcal S$ and the source $\mathcal S'$

8 Definitions

8.1 Lagrange-Hamilton

A system is *natural* when the Hamiltonian of the system is $T+V=E$ (the total energy).
A constraint is *ideal* when the constraint force for each particle is perpendicular to the constraint manifold for that particle.
A constraint is *holonomic* when it's independent of the velocities of the particles.

8.2 Rigid body motion

A rigid body is *generic* if it contains three non-collinear particles.

9 Miscellaneous

9.1 Vectorial calculus

Cross/dot product

$\mathbf a\times(\mathbf b\times\mathbf c)=(\mathbf a\cdot\mathbf c)\mathbf b-(\mathbf a\cdot\mathbf b)\mathbf c,\quad(\mathbf a\times\mathbf b)^2=\mathbf a^2\mathbf b^2-(\mathbf a\cdot\mathbf b)^2$

9.2 Chain rule (Partial derivatives)

Given a function $f\equiv f(t,x_1,...,x_n),\ x_i\equiv x_i(t),\forall i$, then:
$\frac{\mathrm df}{\mathrm dt}=\frac{\partial f}{\partial t}+\sum_i\frac{\partial f}{\partial x_i}\frac{\mathrm dx_i}{\mathrm dt}\equiv\frac{\partial f}{\partial t}+\sum_i\frac{\partial f}{\partial x_i}\dot x_i.$

9.3 Reduction of two body problem into one under central field

1. $m_1\ddot{\mathbf r}_1=\mathbf F_{12}(t,\mathbf r_1,\mathbf r_2,\dot{\mathbf r}_1,\dot{\mathbf r}_2),\ m_1\ddot{\mathbf r}_1=\mathbf F_{21}=-\mathbf F_{12}$
2. $\mathbf R=\frac1M(m_1\mathbf r_1+m_2\mathbf r_2)\ \mathbf r=\mathbf r_1-\mathbf r_2\Rightarrow\mu\ddot{\mathbf r}=\mathbf F_{12}=\mathbf F(t,\mathbf r,\dot{\mathbf r})$
3. $\mu=\frac{m_1m_2}{m_1+m_2}$
4. $\mathbf r_1=\mathbf R+\frac{m_2}{M}\mathbf r,\ \mathbf r_2=\mathbf R-\frac{m_1}{M}\mathbf r$. Since $\ddot{\mathbf R}=0$ we can place the origin at $\mathbf R(t)=0$ and the ref. frame will still be inertial.
5. $\Rightarrow\mathbf r_1=\frac{m_2}{M}\mathbf r,\ \mathbf r_2=-\frac{m_1}{M}\mathbf r$
6. When m_2 is much larger than $m_1\Rightarrow\begin{cases}\frac{m_1}{M}\simeq=0\Rightarrow\mathbf r_2\simeq0\\\frac{m_2}{M}\simeq=1\Rightarrow\mathbf r_1\simeq\mathbf r\end{cases}$

9.4 Integrals

- $x=\sqrt{\frac1k}\cosh z\Rightarrow\int\frac{dx}{\sqrt{kx^2-1}}=\int\sqrt{\frac1k}dz=\sqrt{\frac1k}\cdot z$

9.5 ODEs

- $u''+Cu=0,\quad C\in\mathbb R$
  If $C<0\Rightarrow u=ae^{\gamma\theta}+be^{-\gamma\theta},\quad\gamma\equiv\sqrt{|C|}>0\Rightarrow r=u^{-1}$
  $\Rightarrow r=A\,\mathrm{sech}(\gamma(\theta-\theta_0))\quad(a,b>0)$
  $\Rightarrow r=Ae^{\pm\gamma(\theta-\theta_0)}\quad(a=0\text{ or }b=0)$
  $\Rightarrow r=A\,\mathrm{csch}(\gamma(\theta-\theta_0))\quad(ab<0)$
  If $C=0\Rightarrow u=a+b\theta\iff r=\frac{1}{a+b\theta}$
  If $C>0\Rightarrow u=\frac1A\cos(\gamma(\theta-\theta_0))\iff r=A\sec(\gamma(\theta-\theta_0))$

9.6 Trigonometric functions

$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta,\ \cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$
$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta,\ \cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$
$\cos(2x)=\cos^2x-\sin^2x=2\cos^2x-1=1-2\sin^2x$
$\sin(2x)=2\sin x\cos x,\ \tan(2x)=\frac{2\tan x}{1-\tan^2x}$
$\sin^2x-1=-\cos(2x),\ \cos^2x-1=-\sin^2x$
$\sin\left(\frac x2\right)=\sqrt{\frac{1-\cos x}{2}},\ \cos\frac x2=\sqrt{\frac{1+\cos x}{2}},\ \tan\frac x2=\sqrt{\frac{1-\cos x}{1+\cos x}}$
$\int\sin^2(x)\,\mathrm dx=\frac x2-\frac{\sin(2x)}{4}+C,\ \int\cos^2(x)\,\mathrm dx=\frac x2+\frac{\sin(2x)}{4}+C$
$\int\sin^3(x)\,\mathrm dx=\frac{\cos^3x}{3}-\cos x+C,\ \int\cos^3(x)\,\mathrm dx=\sin x-\frac{\sin^3x}{3}+C$

9.7 Hyperbolic functions

$\sinh x=\frac{e^x-e^{-x}}{2},\ \cosh x=\frac{e^x+e^{-x}}{2}$
$\cosh^2x-\sinh^2x=1$
$\sinh x+y=\sinh x\cosh y+\cosh x\sinh y,$
$\cosh x+y=\cosh x\cosh y+\sinh x\sinh y$
$\sinh x-y=\sinh x\cosh y-\cosh x\sinh y,$
$\cosh x-y=\cosh x\cosh y-\sinh x\sinh y$
$\sinh 2x=2\sinh x\cosh x,\ \cosh(2x)=\cosh^2x+\sinh^2x$
$\cosh^2x=\frac{1+\cosh 2x}{2},\ \sinh^2x=\frac{\cosh 2x-1}{2}$
$\cosh\frac x2=\sqrt{\frac{1+\cosh x}{2}},\ \sinh\frac x2=\pm\sqrt{\frac{\cosh x-1}{2}}$
$\frac{d}{dx}\sinh x=\cosh x,\ \frac{d}{dx}\cosh x=\sinh x,\ \frac{d}{dx}\tanh x=\mathrm{sech}^2x$
$\tanh(\phi_1+\phi_2)=\frac{\tanh\phi_1+\tanh\phi_2}{1+\tanh\phi_1\tanh\phi_2}$
$\frac{1}{\sqrt{1-\tanh^2\phi}}=\cosh\phi$

9.8 Constants and units

Earth radius $\equiv6.3674\cdot10^6\,\mathrm m$
Centrifugal acceleration on Earth$\equiv5.93031\cdot10^{-3}\,\mathrm{m\,s^{-2}}$
$h=2\pi\hbar\equiv$ Planck's constant $\equiv6.62607015\times10^{-34}\,\mathrm{J\cdot s}$

9.9 Dynamic equations

Hooke's Law: $\mathbf F_s=-k(l_0-x)\mathbf e_x,\ V_s=-k/2(x-l_0)^2$
Electromagnetic (Static fields) force: $\mathbf F=q(\mathbf E+\mathbf v\times\mathbf B)$

9.10 Vectorial operators

Gradients & divergence

Cylindrical:
$$\nabla U=\frac{\partial U}{\partial\rho}\mathbf u_\rho+\frac1\rho\frac{\partial U}{\partial\varphi}\mathbf u_\varphi+\frac{\partial U}{\partial z}\mathbf k$$
$$\nabla\cdot\mathbf F=\frac1\rho\frac{\partial}{\partial\rho}+\frac1\rho\frac{\partial F_\varphi}{\partial\varphi}+\frac{\partial F_z}{\partial z}$$
$$\left(\frac1\rho\frac{\partial F_z}{\partial\varphi}-\frac{\partial F_\varphi}{\partial z}\right)\mathbf u_\rho+\left(\frac{\partial F_\rho}{\partial z}-\frac{\partial F_z}{\partial\rho}\right)\mathbf u_\varphi+\left(\frac1\rho\frac{\partial}{\partial\rho}-\frac{\partial F_\rho}{\partial\varphi}\right)\mathbf k$$
Spherical:
$$\nabla U=\frac{\partial U}{\partial r}\mathbf u_r+\frac1r\frac{\partial U}{\partial\theta}\mathbf u_\theta+\frac{1}{r\sin\theta}\frac{\partial U}{\partial\varphi}\mathbf u_\varphi$$
$$\nabla\cdot\mathbf F=\frac{1}{r^2}\frac{\partial}{\partial r}+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}+\frac{1}{r\sin\theta}\frac{\partial F_\varphi}{\partial\varphi}$$
$$\left[\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}-\frac{\partial F_\theta}{\partial\varphi}\right)\right]\mathbf u_r+\left[\frac1r\left(\frac{1}{\sin\theta}\frac{\partial F_r}{\partial\varphi}-\frac{\partial}{\partial r}\right)\right]\mathbf u_\theta+$$
$$+\left[\frac1r\left(\frac{\partial}{\partial r}-\frac{\partial F_r}{\partial\theta}\right)\right]\mathbf u_\varphi$$