

The density of states at the Fermi surface is given by

$$\begin{aligned}
 \rho_d(0) &= \frac{1}{\pi} \text{Im } G_d(\omega) \Big|_{\omega=0} \\
 &= \frac{1}{\pi} \frac{Z\Delta^*}{\left(\omega - \epsilon_d^*\right)^2 + \Delta^{*2}} \Big|_{\omega=0} \\
 &= \frac{1}{\pi} \frac{Z\Delta^*}{\epsilon_d^{*2} + \Delta^{*2}}
 \end{aligned} \tag{2.6.25}$$

The total Green's function for the conduction electrons can be expressed in powers of the scattering potential V :

$$\begin{aligned}
 G &= G^{(0)} + G^{(0)}VG_d^{(0)}VG^{(0)} + G^{(0)}VG_d^{(0)}VG^{(0)}VG_d^{(0)}VG^{(0)} + \dots \\
 &= G^{(0)} + G^{(0)}V \left[G_d^{(0)} + G_d^{(0)}VG^{(0)}VG_d^{(0)} \right] VG^{(0)} \\
 &= G^{(0)} + G^{(0)}V^2G_dG^{(0)}
 \end{aligned} \tag{2.6.26}$$

Here, $G^{(0)}$ are the bare Green functions of the conduction and impurity electron and G_d is the interaction impurity Green's function. Comparing with

$$G = G_0 + G_0TG_0 \tag{2.6.27}$$

we can write

$$T = V^2G_d \tag{2.6.28}$$

where T is the T -matrix for scattering of conduction electrons off the impurity. From the optical theorem, we know that the S -matrix ($S(\omega) \equiv e^{2i\delta(\omega)}$) is related to the T -matrix as

$$\begin{aligned}
 e^{2i\delta(\omega)} &= 1 - 2\pi i \rho T(\omega) \\
 \implies T &= V^2G_d = \frac{1}{2\pi i \rho} \left(1 - e^{2i\delta(\omega)} \right) = \frac{e^{i\delta(\omega)}}{2\pi i \rho} (-2i \sin \delta) \\
 \implies G_d &= -\frac{e^{i\delta(\omega)}}{V^2 \pi \rho} \sin \delta
 \end{aligned} \tag{2.6.29}$$

Since $-\frac{1}{V^2 \pi \rho} \sin \delta$ is real, we can write

$$G_d = |G_d| e^{i\delta(\omega)} \tag{2.6.30}$$

From the expression for G_d in eq. 2.6.24, we can find the phase of G_d :

$$\begin{aligned}
 \delta(\omega) &= \tan^{-1} \frac{\Delta^*}{\omega - \epsilon_d^*} \\
 \implies \epsilon_d^* &= -\Delta^* \cot \delta(0)
 \end{aligned} \tag{2.6.31}$$

Substituting this in the density of states expression gives

$$\rho_d(0) = \frac{Z \sin^2 \delta(0)}{\pi \Delta^*} \tag{2.6.32}$$

Substituting this expression for the density of states in the expression for the Wilson ratio gives

$$R = 1 + \frac{UZ \sin^2 \delta(0)}{\pi \Delta^*} \quad (2.6.33)$$

From the definition $\Delta^* \equiv Z\Delta$, we get

$$R = 1 + \frac{U}{\pi \Delta} \sin^2 \delta(0) \quad (2.6.34)$$

The case of $\langle n_d \rangle = 1$

Exactly at the strong-coupling fixed point, for particle-hole symmetry, we expect the occupancy of the impurity to be $\langle n_d \rangle = 1$, because the singly-occupied state is below the Fermi level while the doubly occupied state is above. If we now lower the Fermi level by $\Delta\mu$ while keeping the particle-hole symmetry intact (by suitably shifting the impurity levels), the resonance in the spectral function at the Fermi surface will persist, because the electrons at the Fermi surface will always form a singlet with the impurity and go into a bound state.

Since the energies are measured relative to the Fermi level, all quasiparticle energies will increase by $\Delta\epsilon_{k\sigma} = \Delta\mu$. However, some of the quasiparticles closer to the Fermi surface will now come below it, so that the number of quasiparticles will decrease by $\Delta n = -\Delta\mu \rho_d(0)$. The net change in n_\uparrow is thus

$$\begin{aligned} \Delta n_\uparrow &= \delta n_\uparrow(\epsilon_{k\uparrow} + \mu) - \delta n_\uparrow(\epsilon_{k\uparrow}) \\ &= \rho_d(0) (\Delta\mu + U \Delta n_\downarrow) \\ &= \rho_d(0) (\Delta\mu - U \rho_d(0) \Delta\mu) \\ &= \rho_d(0) \Delta\mu (1 - U \rho_d(0)) \end{aligned} \quad (2.6.35)$$

At the Kondo limit, the impurity occupation is fixed at 1 because the resonance in the spectral function of the conduction electrons is pinned at the Fermi energy. This means that even if we shift the Fermi energy, the resonance moves with it, and there should be no change Δn_\uparrow . Hence,

$$1 - U \rho_d(0) = 0 \implies U \rho_d(0) = 1 \quad (2.6.36)$$

Substituting $\langle n_{d\sigma} \rangle = \frac{1}{2}$ and $\epsilon_d = -\frac{U}{2}$ in the density of states eq. 2.6.7 gives $\rho_d(0) = \frac{1}{\pi \Delta} = \frac{1}{U}$. This can be substituted in the Wilson ratio to give

$$R = 1 + \sin^2 \delta(0) \quad (2.6.37)$$

2.7 Topological interpretation of Wilson ratio

From the Friedel sum rule[21], we can relate the phase shift $\delta(0)$ due to scattering (at the Fermi surface) off a local impurity to the number of electrons bound in the potential

well produced by that impurity:

$$\tilde{N} = \frac{1}{2\pi i} \text{Tr} \ln S(0) = \int_{\Gamma} dz \partial_z \frac{1}{2\pi i} \text{Tr} \ln S(0) \quad (2.7.1)$$

From the optical theorem, we can write

$$S = 1 + T G_0 = \frac{G}{G_0} \quad [\text{eq. 2.6.27}] \quad (2.7.2)$$

This allows us to write [22]

$$\tilde{N} = \int_{\Gamma} dz \partial_z \frac{1}{2\pi i} \text{Tr} \ln \frac{G}{G_0} \quad (2.7.3)$$

Since $\text{Tr} \ln \hat{O} = \sum_{\lambda} \ln O_{\lambda} = \ln \prod_{\lambda} O_{\lambda} = \ln \text{Det} \hat{O}$, we get

$$\begin{aligned} \tilde{N} &= \int_{\Gamma} dz \partial_z \frac{1}{2\pi i} \ln \text{Det} \frac{G}{G_0} \\ &= - \int_{\Gamma} dz \partial_z \frac{1}{2\pi i} \ln \frac{\text{Det} G_0}{\text{Det} G} \\ &\equiv - \int_{\Gamma} dz \partial_z \frac{1}{2\pi i} \ln D \\ &= - \int_{\Gamma(D)} \frac{dD}{D} \end{aligned} \quad (2.7.4)$$

From the work of Seki and Yunoki [23], we know that this quantity is essentially the winding number of the curve $\Gamma(D)$ in the complex plane spanned by the real and imaginary parts of D , and is equal to the change in Luttinger's volume V_L at $T = 0$.

$$\tilde{N} = - \int_{\Gamma(D)} \frac{dD}{D} = -\Delta V_L \quad (2.7.5)$$

The incoming electrons can have $\sigma = \uparrow, \downarrow$. Since the impurity singlet ground state is rotationally invariant, we have $\delta_{\uparrow} = \delta_{\downarrow} = \delta(0)$.

$$\begin{aligned} \tilde{N} &= \frac{1}{\pi} \sum_{\sigma} \delta_{\sigma}(0) \\ \implies \delta(0) &= \frac{\pi}{2} \tilde{N} = -\frac{\pi}{2} \Delta V_L \end{aligned} \quad (2.7.6)$$

$$\begin{aligned} R &= 1 + \sin^2 \left(\frac{\pi}{2} \tilde{N} \right) \\ &= 1 + \sin^2 \left(\frac{\pi}{2} \Delta V_L \right) \end{aligned} \quad (2.7.7)$$

We note that this connection between R and ΔV_L has not been obtained in the existing literature thus far. In the unitary limit, $\delta(0) = \frac{\pi}{2}$, giving $\Delta V_L = -1 = -\tilde{N}$ [24] (i.e., one electronic state from the impurity has been absorbed into the Luttinger volume of the conduction bath), such that $R = 2$ in this limit. In this way, we see that a change in the topological quantum number \tilde{N} causes the well known renormalisation of the Wilson ratio R from its non-interacting value (1) to the value (2) obtained for the local Fermi liquid [8].

2.8 Renormalized perturbation theory

This is a perturbative expansion of the Hamiltonian in terms of the renormalized interaction \tilde{U} , and the second order results obtained from this approach coincide with the phenomenological results at $T, \hbar \rightarrow 0$. This approach is obviously more general as all terms in the original Hamiltonian are retained. This is an alternative to the full microscopic approach. In the microscopic approach, we take the exact microscopic Hamiltonian and calculate observables from it. In the renormalized perturbation, we separate the Hamiltonian into a non-interacting quasiparticle Hamiltonian which is like the low-energy free Hamiltonian, and an interacting part, and also a counter-term to prevent divergences. The original parameters of the model get replaced by renormalized parameters, and we can analyze the model perturbatively in powers of the renormalized interaction.

To do a perturbative expansion of the Hamiltonian in terms of the interaction U , it is useful to introduce the self energy $\Sigma(E) = \Sigma(0) + E\Sigma' + \Sigma^{\text{rem}}(E)$. In the absence of interaction, the impurity Green's function is

$$G_d^0 = \frac{1}{E - \epsilon_d + i\Delta} \quad (2.8.1)$$

Including the self energy gives

$$G_d = \frac{1}{E - \epsilon_d + i\Delta - \Sigma(E)} \quad (2.8.2)$$

As shown previously in section 2.4.4, the impurity Green's function can be shown to take the form

$$G_d = \frac{Z}{E - \tilde{\epsilon}_d + i\tilde{\Delta} - \tilde{\Sigma}(E)} \quad (2.8.3)$$

where the $\tilde{}$ represents the renormalised quantities

$$\begin{aligned} \tilde{\epsilon}_d &= Z(\epsilon_d + \Sigma(0)) \\ \tilde{\Delta} &= Z\Delta \\ \tilde{\Sigma} &= Z\Sigma^{\text{rem}}(E) \\ Z^{-1} &= 1 - \Sigma(0)' \\ \tilde{\Gamma}_{\sigma\sigma'}(E, E') &= Z^2\Gamma_{\sigma\sigma'}(E, E') \\ \tilde{U} &= Z^2\Gamma_{\uparrow\downarrow}(0, 0) \end{aligned} \quad (2.8.4)$$

The perturbative expansion is about the bare Hamiltonian, that is, the one with $\tilde{\Sigma} = 0$. The corresponding Greens function (non-interacting quasiparticle Green's function) is

$$\tilde{G}_d = \frac{1}{E - \tilde{\epsilon}_d + i\tilde{\Delta}} \quad (2.8.5)$$

The Anderson hamiltonian

$$H = \epsilon_d n_d + U n_{d\uparrow} n_{d\downarrow} + \sum_k \epsilon_k n_k + \sum_k \left(V_k c_{d\sigma}^\dagger c_{k\sigma} + V_k^* c_{k\sigma}^\dagger c_{d\sigma} \right) \quad (2.8.6)$$

can be written in the form

$$H = \tilde{H}_{qp} - \tilde{H}_c \quad (2.8.7)$$

$\tilde{H}_{qp} = \tilde{H}_{qp}^0 + \tilde{H}_{qp}^I$ is the total quasiparticle Hamiltonian, consisting of a non-interacting part \tilde{H}_{qp}^0 and an interaction \tilde{H}_{qp}^I .

$$\tilde{H}_{qp}^0 = \tilde{\epsilon}_d \tilde{n}_d + \sum_k \epsilon_k n_k + \sum_k \left(\tilde{V}_k \tilde{c}_{d\sigma}^\dagger c_{k\sigma} + \tilde{V}_k^* c_{k\sigma}^\dagger \tilde{c}_{d\sigma} \right) \quad (2.8.8)$$

$$\tilde{H}_{qp}^I = \tilde{U} \tilde{n}_{d\uparrow} \tilde{n}_{d\downarrow} \quad (2.8.9)$$

The renormalised parameters are defined in eq. 2.8.4. The renormalised operators are

$$\tilde{c}_d^\dagger = \sqrt{z} c_d^\dagger \quad (2.8.10)$$

$$\tilde{c}_d = \sqrt{z} c_d \quad (2.8.11)$$

The \tilde{H}_c that satisfies eq. 2.8.7 is

$$\tilde{H}_c = \lambda_1 \tilde{n}_d + \lambda_2 \tilde{n}_{d\uparrow} \tilde{n}_{d\downarrow} \quad (2.8.12)$$

where

$$\lambda_1 = z \Sigma(0, 0) \quad (2.8.13)$$

$$\lambda_2 = z^2 \left[\Gamma_{\uparrow\downarrow}(0, 0) - U \right] \quad (2.8.14)$$

\tilde{H}_{qp} is the effective Hamiltonian close to the strong-coupling fixed point. \tilde{H}_c is the counter-term. It is introduced to cancel divergences. Close to the Fermi surface, we want the renormalised self-energy $\tilde{\Sigma}(E)$ to vary as E^2 . That gives two constraints

$$\begin{aligned} \tilde{\Sigma}(0) &= 0 \\ \tilde{\Sigma}'(0) &= 0 \end{aligned} \quad (2.8.15)$$

Close to the Fermi surface, we also have

$$\tilde{\Gamma}_{\uparrow\downarrow}(0) = \tilde{U} \quad (2.8.16)$$

$$\tilde{\Gamma}_{\sigma\sigma}(0) = 0 \quad (2.8.17)$$

$$\implies \Gamma_{\sigma\sigma'}(0) = \tilde{U}(1 - \delta_{\sigma\sigma'}) \quad (2.8.18)$$

This is the third constraint. The perturbation expansion is in powers of the renormalised interaction \tilde{U} . The parameters that are determined by the expansion are λ_1, λ_2, z . Hence, they should be expanded in powers of \tilde{U} .

$$\lambda_i = \sum_n \lambda_i^{(n)} \tilde{U}^n \quad (2.8.19)$$

$$z = \sum_n z^{(n)} \tilde{U}^n \quad (2.8.20)$$

$$(2.8.21)$$

The expansion is about the non-interacting quasiparticle Hamiltonian. The corresponding Green's function is

$$G^0 = \frac{1}{E - \tilde{\epsilon}_d + i\tilde{\Delta}} \quad (2.8.22)$$

From the Friedel sum rule in the next section, we get

$$\langle n_{d\sigma} \rangle = \frac{1}{2} - \frac{1}{\pi} \tan^{-1} \frac{\epsilon_d + \Sigma(0, h)}{\Delta} \quad (2.8.23)$$

Multiplying the numerator and denominator by z , we get the same occupancy in terms of the renormalised parameters.

$$\langle n_{d\sigma} \rangle = \frac{1}{2} - \frac{1}{\pi} \tan^{-1} \frac{\tilde{\epsilon}_d + \tilde{\Sigma}(0, h)}{\tilde{\Delta}} \quad (2.8.24)$$

For $T, h \rightarrow 0$, the counter-term cancels appropriate terms from the quasiparticle Hamiltonian leading to the vanishing of the effects of the self-energy, eq. 2.8.15. In that case, $\langle n_{d\sigma} \rangle = \langle n_{d\sigma}^0 \rangle$, that is, the quasiparticle distribution becomes the same as the free fermionic distribution.

The first order Feynman diagram for the self-energy is of the Hartree type. They give a contribution

$$\tilde{\Sigma}(\omega, H, T) = \tilde{U} \left(n_{d\sigma}^{(0)}(0, H, T) - n_{d\sigma}^{(0)}(0, 0, 0) \right) \quad (2.8.25)$$

This satisfies the constraint eq. 2.8.15. That is, $\Sigma^{(1)}(0, 0) = 0$. With the expression for self-energy, we can write down the impurity magnetic susceptibility, $\chi_d = \frac{\partial m}{\partial B}$, where

$$m = \frac{g\mu_B}{2} \langle n_{d\uparrow} - n_{d\downarrow} \rangle \quad (2.8.26)$$

We can substitute the expression for the self-energy into eq. 2.8.24. That gives

$$\chi_d = \frac{1}{2} (g\mu_B)^2 \frac{\partial \langle n_{d\uparrow} - n_{d\downarrow} \rangle}{\partial h} = \frac{1}{2\pi} (g\mu_B)^2 \frac{\partial}{\partial h} \left(\tan^{-1} \frac{\tilde{\epsilon}_{d\downarrow}}{\tilde{\Delta}} - \tan^{-1} \frac{\tilde{\epsilon}_{d\uparrow}}{\tilde{\Delta}} \right) \quad (2.8.27)$$

where $h = g\mu_B B$ and $\tilde{\epsilon}_{d\sigma} = \tilde{\epsilon}_d + \tilde{U}n_{d\sigma}^{(0)}$. Performing the derivative and taking the limits of $T \rightarrow 0$ and $B \rightarrow 0$ gives

$$\chi_d = \frac{1}{2\pi} (g\mu_B)^2 \frac{1}{1 + \left(\frac{\tilde{\epsilon}_d}{\tilde{\Delta}} \right)^2} \frac{1}{\tilde{\Delta}} \frac{\partial}{\partial h} \left[\tilde{\epsilon}_{d\downarrow} - \tilde{\epsilon}_{d\uparrow} \right] \quad (2.8.28)$$

We can recognize that

$$\frac{1}{1 + \left(\frac{\tilde{\epsilon}_d}{\tilde{\Delta}} \right)^2} \frac{1}{\pi \tilde{\Delta}} = \frac{1}{\pi} \frac{\tilde{\Delta}}{\tilde{\Delta}^2 + \tilde{\epsilon}_d^2} = \rho_d(0) \quad (2.8.29)$$

Therefore,

$$\chi_d = \frac{1}{2} (g\mu_B)^2 \rho_d(0) \frac{\partial}{\partial h} [\tilde{\epsilon}_{d\downarrow} - \tilde{\epsilon}_{d\uparrow}] \quad (2.8.30)$$

Up to first order, we can write

$$\tilde{\epsilon}_{d\downarrow} - \tilde{\epsilon}_{d\uparrow} = \epsilon_{d\downarrow} - \epsilon_{d\uparrow} + \tilde{U} (n_{d\downarrow}^{(0)} - n_{d\uparrow}^{(0)}) = 2\epsilon_d + h + \tilde{U} (n_{d\downarrow}^{(0)} - n_{d\uparrow}^{(0)}) \quad (2.8.31)$$

where I used $\epsilon_{d\sigma}(h) = \epsilon_d - \frac{h}{2}\sigma$. Substituting this in the expression for χ_d gives

$$\frac{\partial}{\partial h} [\tilde{\epsilon}_{d\downarrow} - \tilde{\epsilon}_{d\uparrow}] = 1 + \tilde{U} \left(\frac{\partial n_{d\downarrow}^{(0)}}{\partial \epsilon_{d\downarrow}} \frac{\partial \epsilon_{d\downarrow}}{\partial h} - \frac{\partial n_{d\uparrow}^{(0)}}{\partial \epsilon_{d\uparrow}} \frac{\partial \epsilon_{d\uparrow}}{\partial h} \right) \quad (2.8.32)$$

Up to first order, we can approximate $\frac{\partial \epsilon_{d\sigma}}{\partial h} = \frac{\sigma}{2}$, therefore,

$$\frac{\partial}{\partial h} [\tilde{\epsilon}_{d\downarrow} - \tilde{\epsilon}_{d\uparrow}] = 1 + \tilde{U} \rho_d(0) \quad (2.8.33)$$

Substituting in to the parent equation, we get

$$\chi_d = \frac{1}{2} (g\mu_B)^2 \rho_d(0) (1 + \tilde{U} \rho_d(0)) \quad (2.8.34)$$

which is same as the one obtained from mean-field.

It is possible to take higher order contributions into account, but there are identities which show that these results are exact.

$$\left(\frac{\partial}{\partial E} + \frac{\partial}{\partial \mu} \right) \Sigma(E) \Big|_{E=0} = -\rho_{d\sigma}(0) \Gamma_{\uparrow\downarrow}(0, 0) \quad (2.8.35)$$

$$\left(\frac{\partial}{\partial h} - \frac{\partial}{\partial E} \right) \Sigma(E) \Big|_{E=0} = -\rho_{d\sigma}(0) \Gamma_{\uparrow\downarrow}(0, 0) \quad (2.8.36)$$

Multiplying both equations throughout by Z , we get

$$\left(\frac{\partial}{\partial E} + \frac{\partial}{\partial \mu} \right) \tilde{\Sigma}(E) \Big|_{E=0} = -Z \rho_{d\sigma}(0) \Gamma_{\uparrow\downarrow}(0, 0) = -\frac{1}{Z} \rho_{d\sigma}(0) \tilde{U} \quad (2.8.37)$$

$$\left(\frac{\partial}{\partial h} - \frac{\partial}{\partial E} \right) \tilde{\Sigma}(E) \Big|_{E=0} = -Z \rho_{d\sigma}(0) \Gamma_{\uparrow\downarrow}(0, 0) = -\frac{1}{Z} \rho_{d\sigma}(0) \tilde{U} \quad (2.8.38)$$

where I used $Z^2 \Gamma = \tilde{U}$. We also have the relation $\tilde{\rho}_d = \frac{1}{Z} \rho_d$, because

$$\tilde{\rho}(0) \sim \frac{\tilde{\Delta}}{\tilde{\epsilon}_d^2 + \tilde{\Delta}^2} = \frac{1}{Z} \frac{\Delta}{\epsilon_d^2 + \Delta^2} = \frac{1}{Z} \rho_d(0) \quad (2.8.39)$$

Noting that the derivative of the renormalised self energy goes to zero at the Fermi surface, we get

$$\left. \frac{\partial \tilde{\Sigma}(E)}{\partial \mu} \right|_{E=0} = \left. \frac{\partial \tilde{\Sigma}(E)}{\partial h} \right|_{E=0} = -\tilde{\rho}_{d\sigma}(0)\tilde{U} \quad (2.8.40)$$

These can be used to prove the mean-field results regarding specific heat and the susceptibilities. Similar to the mean field treatment, close to $T = 0$, the effects of the self energy vanish, and the specific heat linear term, γ , involves only the non-interacting density of states.

$$\tilde{\gamma} \sim \tilde{\rho}_d(0) \quad (2.8.41)$$

The susceptibilities are given by

$$\chi_m = - \left(\frac{g\mu_B}{2} \right)^2 \tilde{\rho}_d \sum_{\sigma} \frac{\partial (\epsilon_k - h\sigma + \tilde{\Sigma})}{\partial h} = \frac{g^2 \mu_B^2}{2} \tilde{\rho}_d \left(1 - \frac{\partial \tilde{\Sigma}}{\partial h} \right) \quad (2.8.42)$$

Substituting from eq. 2.8.40,

$$\chi_m = \frac{g^2 \mu_B^2}{2} \tilde{\rho}_d \left(1 + \tilde{U} \tilde{\rho}_d(0) \right) \quad (2.8.43)$$

Similarly,

$$\chi_c = \frac{dn}{d\mu} = \tilde{\rho}_d(0) \sum_{\sigma} \frac{d(\epsilon_F + \tilde{\Sigma})}{d\mu} = 2\tilde{\rho}_d(0) \left(1 + \frac{\partial \tilde{\Sigma}}{\partial \mu} \right) = 2\tilde{\rho}_d(0) \left(1 - \tilde{U} \tilde{\rho}_d(0) \right) \quad (2.8.44)$$

Friedel sum rule

Looking at eq. 2.8.2, we can write down the density of states and hence the average occupancy of the impurity site

$$\rho_{d\sigma} = \frac{-1}{\pi} \text{Im} [G_d] \quad (2.8.45)$$

$$\langle n_{d\sigma} \rangle = \int_{-\infty}^0 dE \rho_{d\sigma} = -\frac{1}{\pi} \text{Im} \int_{-\infty}^0 dE G_d(E) \quad (2.8.46)$$

Luttinger proved that

$$\int_{-\infty}^0 \Sigma' G dE = 0 \quad (2.8.47)$$

In order to use this, note that

$$\begin{aligned} \ln G_d^{-1} &= \ln (E - \epsilon_d + i\Delta - \Sigma) \\ \implies \frac{d \ln G_d^{-1}}{dE} &= \frac{1}{E - \epsilon_d + i\Delta - \Sigma} (1 - \Sigma') = G_d - \Sigma' G_d \\ \implies G_d &= \Sigma' G_d - \frac{d \ln G_d}{dE} \end{aligned} \quad (2.8.48)$$

Substituting this expression for G_d in eq. 2.8.46,

$$\langle n_{d\sigma} \rangle = -\frac{1}{\pi} \text{Im} \int_{-\infty}^0 \Sigma' G dE + \frac{1}{\pi} \text{Im} \int_{-\infty}^0 \frac{d \ln G_d}{dE} dE \quad (2.8.49)$$

The first integral is zero, courtesy Luttinger. We get

$$\begin{aligned} \langle n_{d\sigma} \rangle &= \frac{1}{\pi} \text{Im} \left[\ln \frac{1}{E - \epsilon_d + i\Delta - \Sigma} \right]_{-\infty}^0 \\ &= -\frac{1}{\pi} \text{Im} \left[\ln (E - \epsilon_d + i\Delta - \Sigma) \right]_{-\infty}^0 \\ &= -\frac{1}{\pi} \text{Im} \left[\ln e^{i\theta} \right]_{-\infty}^0 \quad \left[\tan \theta = \frac{\Delta}{E - \epsilon_d - \Sigma} \right] \\ &= -\frac{1}{\pi} [\theta(0) - \theta(-\infty)] \\ &= \frac{1}{\pi} \tan^{-1} \frac{\Delta}{\epsilon_d + \Sigma(0, h)} \\ &= \frac{1}{2} - \frac{1}{\pi} \tan^{-1} \frac{\epsilon_d + \Sigma(0, h)}{\Delta} \end{aligned} \quad (2.8.50)$$

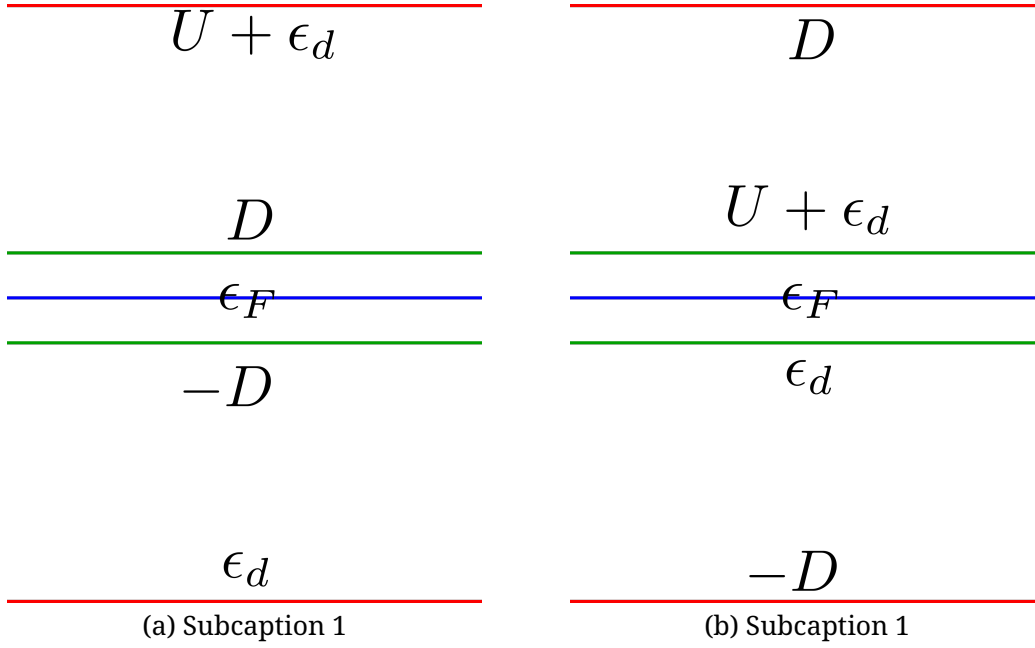
We can now relate the average occupancy with the density of states. From eq. 2.8.45,

$$\begin{aligned} \rho_{d\sigma} &= \frac{1}{\pi} \frac{\Delta}{(\epsilon_d + \Sigma(0))^2 + \Delta^2} \\ &= \frac{1}{\pi\Delta} \left[1 + \left(\frac{\epsilon_d + \Sigma(0)}{\Delta} \right)^2 \right]^{-1} \\ &= \frac{1}{\pi\Delta} \left[1 + \cot^2 \pi \langle n_{d\sigma} \rangle \right]^{-1} \\ &= \frac{\sin^2 \pi \langle n_{d\sigma} \rangle}{\pi\Delta} \end{aligned} \quad (2.8.51)$$

2.9 Poor Man's scaling of the SIAM

First consider the case in which $\epsilon_d \ll -D, U + \epsilon_d \gg D$. The situation is such that both the impurity levels are far outside the bandwidth, as shown in fig. 2.2a. The maximum energy scale at which scattering with conduction electrons can take place is of the order of the bandwidth D . Since the impurity energies are much higher than the bandwidth, no charge fluctuation can take place. The impurity state will be fixed at ϵ_d . The only remaining degree of freedom will be the spin fluctuations, and we can then do the S-W transformation.

On the other hand, if we take the situation in fig. 2.2b where both the impurity levels are far inside the bandwidth, then both the impurity levels will be on energy scales completely different from the bandwidth. So there won't be any renormalization of the impurity levels. From another perspective, it can be said that there won't be any renormalization because both the impurity levels $|0\rangle, |1\rangle$ will be able to hybridise with two states each: $|0\rangle$ can hybridize with $|k \uparrow\rangle, |k \downarrow\rangle$ and $|1\rangle$ can hybridize with $|0\rangle, |2\rangle$. The



limit where there will be some renormalization is the following. We are working with the asymmetric Anderson model, that is,
 $U + \epsilon_d \gg D \gg |\epsilon_d|, \Delta$. The total Hamiltonian is

$$H = \sum_{k\sigma} \epsilon_{k\sigma} n_{k\sigma} + \epsilon_d \sum_{\sigma} n_{d\sigma} + U n_{d\uparrow} n_{d\downarrow} + \sum_{k\sigma} \left(V_{kd} c_{k\sigma}^{\dagger} c_{d\sigma} + V_{kd}^* c_{d\sigma}^{\dagger} c_{k\sigma} \right) \quad (2.9.1)$$

This means that the doubly-occupied state is decoupled from the conduction band; it cannot hybridize through the V_{kd} because the virtual transition will involve a huge amount of energy and so it is practically impossible.

At the first iteration, we will reduce the cut-off from D to $D - \delta D$. The zeroth approximation to this Hamiltonian is

$$H^{(0)} = \sum_{k < D - \delta D, \sigma} \epsilon_{k\sigma} n_{k\sigma} + \epsilon_d \sum_{\sigma} n_{d\sigma} + \sum_{k < D - \delta D, \sigma} \left(V_{kd} c_{k\sigma}^{\dagger} c_{d\sigma} + V_{kd}^* c_{d\sigma}^{\dagger} c_{k\sigma} \right) \quad (2.9.2)$$

As is apparent, the zeroth approximation involves completely ignoring the region to be integrated out. All kinetic energies and actual scatterings are strictly within the smaller region $[-D + \delta D, D - \delta D]$. The higher approximations allow these states to make virtual

transitions to the band edge states and then come back. The Hamiltonian term for the virtual excitation in to the upper band edge (with a particle in the intermediate state) is

$$H_{\sigma}^{(1,p)} = \sum_{k \in k^+} \alpha_{k\sigma} c_{d\sigma}^{\dagger} c_{k\sigma} c_{k\sigma}^{\dagger} c_{d\sigma} \quad (2.9.3)$$

There are two things to note here. Firstly, $\alpha_{k\sigma}$ is the probability of such a virtual transition and is found from perturbation theory. Secondly, the summation k^+ is over the states in $[D - \delta D, D]$. To calculate $\alpha_{k\sigma}$, note that such a virtual excitation can take place only from the state $1_{d\sigma}$. Therefore, we look at the first order correction to this state under the perturbation V_{kd} .

$$\alpha_{k\sigma} = \frac{\langle 1_{d\sigma} | V_{kd}^* c_{d\sigma}^{\dagger} c_{k\sigma} | k\sigma \rangle \langle k\sigma | V_{kd} c_{k\sigma}^{\dagger} c_{d\sigma} | 1_{d\sigma} \rangle}{E_{1_{d\sigma}} - E_{k\sigma}} = \frac{|V_{kd}|^2}{\epsilon_d - \epsilon_k} \quad (2.9.4)$$

The analogous term in the same order for the virtual transition to the lower edge consists of a hole in the intermediate state, because the lower edge states are already filled. This term is of the form

$$H^{(1,h)} = \sum_{k \in k^-, \sigma} \beta_{k\sigma} c_{k\sigma}^{\dagger} c_{d\sigma} c_{k\sigma} c_{d\sigma}^{\dagger} \quad (2.9.5)$$

$\beta_{k\sigma}$ is calculated similarly, using perturbation theory.

$$\beta_{k\sigma} = \frac{\langle 0 | V_{kd}^* c_{d\sigma} c_{k\sigma}^{\dagger} | k\sigma \rangle \langle k\sigma | V_{kd} c_{k\sigma} c_{d\sigma}^{\dagger} | 0 \rangle}{E_0 - E_{k\sigma}} = \frac{|V_{kd}|^2}{\epsilon_k - \epsilon_d} \quad (2.9.6)$$

The total first order correction to the Hamiltonian is of the form

$$H^{(1)} = \sum_{k^+, \sigma} \alpha_{k\sigma} T_{k\sigma}^+ + \sum_{k^-, \sigma} \beta_{k\sigma} T_{k\sigma}^- \quad (2.9.7)$$

$T^{+,-}$ represent virtual transitions to the upper and lower edges. Since these terms do not cause any real fluctuations in the impurity sites, they renormalize only the impurity energy ϵ_d , and not the hybridisation coupling V_{kd} . To find the renormalization in the site energies ϵ_0 and ϵ_1 (and hence in $\epsilon_d \equiv \epsilon_1 - \epsilon_0$), note that the term T^+ virtually excites the state $n_{d\sigma} = 1$, and hence the change in ϵ_1 is

$$\delta\epsilon_1 = \alpha_{k\sigma} = \sum_{k^+} \frac{|V_{kd}|^2}{\epsilon_d - \epsilon_k} \quad (2.9.8)$$

We can write this summation in terms of $\Delta(E) = \pi N(E) V^2(E)$, under the assumption $\Delta(E) \approx \Delta$ for $E \in \{-D, D\}$.

$$\delta\epsilon_1 = \sum_{k^+} \frac{|V_{kd}|^2}{\epsilon_d - \epsilon_k} = \int_{D-\delta D}^D dE N(E) \frac{|V(E)|^2}{\epsilon_d - E} \approx \frac{\Delta}{\pi} \frac{|\delta D|}{\epsilon_d - D} \quad (2.9.9)$$

The change in ϵ_0 is

$$\delta\epsilon_0 = \sum_{\sigma} \beta_{k\sigma} \approx -2 \frac{\Delta}{\pi} \frac{|\delta D|}{\epsilon_d + D} \quad (2.9.10)$$

The change in the denominator occurs because in the lower edge, $\epsilon_k = -D$. The change in ϵ_d is

$$\delta\epsilon_d = \delta\epsilon_1 - \delta\epsilon_0 = \frac{\Delta|\delta D|}{\pi} \left[\frac{1}{\epsilon_d - D} + \frac{2}{\epsilon_d + D} \right] = \frac{\Delta|\delta D|}{\pi} \frac{1}{D} = -\frac{\Delta}{\pi} \delta \ln D \quad (2.9.11)$$

We assumed $D \gg \epsilon_d$. In the limit of infinitesimal change, we get the equation

$$\frac{d\epsilon_d}{d \ln D} = -\frac{\Delta}{\pi} \quad (2.9.12)$$

If we had allowed the $|1_{d\sigma}\rangle$ to hybridize with the state $|2_d\rangle$ (that is, if we had assumed both U and ϵ_d to be $\ll D$), then $\alpha_{k\sigma}$ would have had another term added to it:

$$\frac{|V_{kd}|^2}{\epsilon_k - U - \epsilon_d} \approx \frac{|V|^2}{-D - U - \epsilon_d} \quad (2.9.13)$$

$-(U + \epsilon_d)$ is the change in energy from $|1_d\rangle$ to $|2_d\rangle$ and $-D$ is the energy of the hole created in the process. The renormalization in ϵ_d would then have been

$$\delta\epsilon_d = \frac{\Delta|\delta D|}{\pi} \left(\frac{1}{\epsilon_d - D} - \frac{1}{D + U + \epsilon_d} + \frac{2}{\epsilon_d + D} \right) \quad (2.9.14)$$

which is zero in the limit of $U, |\epsilon_d| \ll D$. This is the equal renormalization in ϵ_0 and ϵ_1 discussed earlier.

We do not yet know whether Δ is a function of the cutoff D . To find the renormalization of Δ , we need to find the renormalization of V_{kd} . Note that the lowest order virtual transitions do not cause any actual charge fluctuation, and hence they do not renormalize V_{kd} . To see the renormalization of V_{kd} , we need to consider one order higher. These higher order terms involve transitions within the lower subspace along with virtual transitions into the higher subspaces.

$$H^{(2)} = \sum_{k^+, q, \sigma} \alpha_{k\sigma} T_{k\sigma}^+ \gamma_{q, k, \sigma} c_{d\sigma}^\dagger c_{q\sigma} + \sum_{k^-, q, \sigma} \beta_{k\sigma} T_{k\sigma}^- \gamma_{q, k, \sigma} c_{d\sigma} c_{q\sigma}^\dagger \quad (2.9.15)$$

The $\gamma_{k\sigma}$ can be calculated as

$$\begin{aligned} \alpha_{k\sigma} \gamma_{q, k, \sigma} &= \frac{\langle 1_{d\sigma} | V_{kd}^* c_{d\sigma}^\dagger c_{k\sigma} | k\sigma \rangle \langle k\sigma | V_{kd} c_{k\sigma}^\dagger c_{d\sigma} | 1_{d\sigma} \rangle \langle 1_{d\sigma} | V_{kd} c_{q\sigma} c_{d\sigma}^\dagger | q\sigma \rangle}{(E_{1_{d\sigma}} - E_{k\sigma})(E_q - E_k)} \\ &= \alpha_{k\sigma} \frac{V_{kd}}{\epsilon_q - \epsilon_k} \end{aligned} \quad (2.9.16)$$

The renormalization in V_{kd} is therefore

$$\delta V_{kd} = \frac{\Delta}{\pi} \frac{|\delta D|}{\epsilon_d - D} \frac{V_{kd}}{\epsilon_q - \epsilon_k} \quad (2.9.17)$$

Close to the band edge, we get

$$\delta V = \frac{\Delta}{\pi} \frac{|\delta D|}{\epsilon_d - D} \frac{V}{\epsilon_q - D} \approx \frac{\Delta}{\pi} \frac{|\delta D|}{D^2} V \quad (2.9.18)$$

Therefore,

$$\delta\Delta \sim V\delta V = \frac{\Delta V^2}{\pi D^2} |\delta D| \implies \frac{d\Delta}{dD} \sim \left(\frac{\Delta}{D}\right)^2 \quad (2.9.19)$$

For $D \gg \Delta$, this will vanish very quickly. Hence, in this regime, there is no renormalization of Δ , and we can take it to be a constant in the renormalization flow. Integrating eq. 2.9.12 gives

$$\epsilon_d = -\frac{\Delta}{\pi} \ln D + \text{constant} \quad (2.9.20)$$

Defining the constant as

$$\text{constant} = \epsilon_d^* + \frac{\Delta}{\pi} \ln \Delta \quad (2.9.21)$$

we get

$$\epsilon_d = -\frac{\Delta}{\pi} \ln D + \epsilon_d^* + \frac{\Delta}{\pi} \ln \Delta \quad (2.9.22)$$

$$\implies \epsilon_d = \epsilon_d^* - \frac{\Delta}{\pi} \ln \frac{D}{\Delta} \quad (2.9.23)$$

This result is in the regime $U + \epsilon_d \gg D \gg |\epsilon_d|$. Even if $U \ll D$ initially, scaling will begin once $D \sim U$. Until then, as mentioned previously, both ϵ_1 and ϵ_0 will change equally and there won't be any scaling in ϵ_d . If we start with $U \ll D$, under scaling, as D will decrease, there won't be any renormalization until we reach the point $D \sim U$.

Say, as a result of scaling, the bandwidth decreases and ϵ_d increases (which it will, as is apparent from the eq. 2.9.23). At some point, $-D \lesssim \epsilon_d$. At this point, perturbation theory breaks down and we resort to SWT. We denote this point of the scaling by $D = -a\tilde{\epsilon}_d$, $a > 1$. We can then express the SWT coupling constant \tilde{J} by replacing ϵ_d with $\tilde{\epsilon}_d$ in eq. 2.4.26. For simplicity set $U = \infty$. Then,

$$\tilde{J} = -\frac{|V|^2}{\tilde{\epsilon}_d} = \frac{a|V|^2}{D} \quad (2.9.24)$$

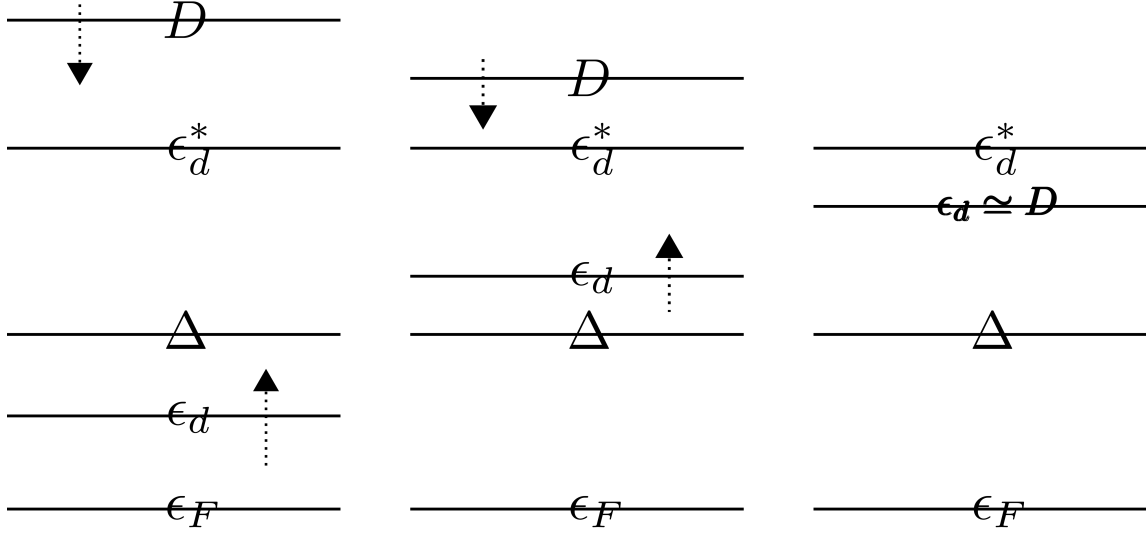
We can then do the poor man's scaling with this coupling. From eq. 2.4.170,

$$\begin{aligned} T_K &\sim D \sqrt{\tilde{J}N(0)^2} \exp\left(-\frac{1}{2\tilde{J}N(0)^2}\right) = \sqrt{\Delta D} \exp\left(-\frac{D}{2\Delta}\right) \\ &\sim D \sqrt{\frac{\Delta}{D}} \exp\left(\frac{\epsilon_d}{2\Delta}\right) \end{aligned} \quad (2.9.25)$$

A different result is obtained if one is in the regime of $\epsilon_d < -D$. This is the situation mentioned at the very beginning of the discussion, fig. 2.2a. Assuming $U \rightarrow \infty$ and ϵ_d outside the conduction band, we can do a SWT and the T_K obtained is q. 2.4.170,

$$J = -\frac{V^2}{\epsilon_d} \quad (2.9.26)$$

$$g = J\rho = -\frac{\Delta}{\epsilon_d} \quad (2.9.27)$$


 Figure 2.2: Renormalization in the energy levels when $\epsilon_d^* \gg \Delta$

$$\Rightarrow T_K = D \sqrt{\frac{\Delta}{\epsilon_d}} \exp\left(\frac{\epsilon_d}{2\Delta}\right) \quad (2.9.28)$$

The two forms of the Kondo temperature show that the prefactor is not a universal function; it depends on the starting conditions (the microscopic Hamiltonian from which we start the scaling). But the universal fact is that in the local moment regime ($U \rightarrow \infty$), all physical quantities will involve only one energy scale, T_K . This T_K itself might be different based on the starting Hamiltonian.

For $\epsilon_d^* \gg \Delta$, the renormalization will stop at $D \sim \epsilon_d$. Note that we had assumed $D \gg \epsilon_d$. That was the starting condition, that is, ϵ_d deep inside the Fermi surface. During the renormalization, D will keep on decreasing and ϵ_d will continuously increase. At some value of D , they will become equal and the impurity level will go outside the Fermi surface. At this point, none of the impurity levels can renormalize any more, because the relevant energy scales are greater than the cutoff. Hence the renormalization stops at this point. This point is given by $\bar{D} = a\epsilon_d(\bar{D}) \equiv \bar{\epsilon}_d$ where a is a constant of order unity. It satisfies the equation

$$\bar{\epsilon}_d = \epsilon_d^* - \frac{\Delta}{\pi} \ln \frac{a\bar{\epsilon}_d}{\Delta} \quad (2.9.29)$$

which is just eq. 2.9.23 with the substitution $D = a\bar{\epsilon}_d$. In this regime, because $\epsilon_d \gg \Delta$, we can do a perturbative expansion of the bare Hamiltonian in terms of $\frac{\Delta}{\epsilon_d}$. The susceptibility is

$$\chi_d = \frac{\Delta}{2\pi} \left(\frac{g\mu_B}{\epsilon_d} \right)^2 \left[1 + \frac{2\Delta}{\pi\epsilon_d} \ln \frac{\epsilon_d}{D} + \dots \right] \quad (2.9.30)$$

From the scaling, we know that D can be decreased to \bar{D} . We can hence substitute $D = a\bar{\epsilon}_d$, $\epsilon_d = \bar{\epsilon}_d$. With this in mind, the susceptibility becomes

$$\begin{aligned}\chi_d &= \frac{\Delta}{2\pi} \left(\frac{g\mu_B}{\bar{\epsilon}_d} \right)^2 \left[1 + \frac{2\Delta}{\pi\bar{\epsilon}_d} \ln a + \dots \right] \\ &= \frac{\Delta}{2\pi} \left(\frac{g\mu_B}{\bar{\epsilon}_d} \right)^2 \left[1 + O\left(\frac{2\Delta}{\pi\bar{\epsilon}_d}\right) \right]\end{aligned}\tag{2.9.31}$$

where I used the fact that $\ln a$ will be of order 1. As we go on decreasing the cutoff, the impurity level will go on moving farther away from the Fermi level, and impurity site will become null occupied: $\langle n_d \rangle \approx 0$. The critical cutoff \bar{D} can be associated with a temperature scale $k_b\bar{T} = \bar{D}$. At temperatures sufficiently below this temperature ($T \ll \bar{T}$), the susceptibility becomes (again from perturbation theory)

$$\chi_d(T) = \frac{\Delta}{2\pi} \left(\frac{g\mu_B}{\bar{\epsilon}_d} \right)^2 + \frac{1}{4T} \left[1 + \frac{1}{2} \exp\left(\frac{T^*}{T}\right) \right]^{-1}\tag{2.9.32}$$

For temperatures sufficiently low, which we demarcate by a temperature T_{FL} , the denominator in the second term will be sufficiently large so that we can ignore that term with respect to the first term:

$$T \gg T_{FL} \implies e^{\frac{T^*}{T}} \gg 1 \implies \left[1 + \frac{1}{2} \exp\left(\frac{T^*}{T}\right) \right]^{-1} \approx 0\tag{2.9.33}$$

The susceptibility in this low temperature range can thus be written as

$$\chi_d = \frac{\Delta}{2\pi} \left(\frac{g\mu_B}{\bar{\epsilon}_d} \right)^2\tag{2.9.34}$$

This is analogous to the result obtained in eq. 2.6.20, from the mean field version of the Fermi liquid theory, and also obtained from a renormalized perturbation theory of Anderson model. To see how, note that since we are in the limit $\langle n_d \rangle = 0$, the onsite repulsion term U can be dropped because there is no probability of double occupation. Eq. 2.6.20 then becomes

$$\chi_d = \frac{g^2\mu_B^2}{2} \rho_d(0) = \frac{g^2\mu_B^2}{2} \frac{\Delta}{\pi \bar{\epsilon}_d^2 + \Delta^2}\tag{2.9.35}$$

Next note that we had assumed at the beginning that $\epsilon_d^* \gg \Delta$. We need to find the relative order difference between $\bar{\epsilon}_d$ and Δ . From eq. 2.9.29, we can drop the π and a because they are of order 1.

$$\bar{\epsilon}_d = \epsilon_d^* - \Delta \ln \frac{\bar{\epsilon}_d}{\Delta}\tag{2.9.36}$$

Dividing through by Δ and defining $x_1 = \frac{\bar{\epsilon}_d}{\Delta}$, $x_2 = \frac{\epsilon_d^*}{\Delta}$, we get

$$x_1 + \ln x_1 = x_2\tag{2.9.37}$$

Since $O(\ln x_1) \leq O(x_1)$, we can write

$$O(x_1) = O(x_2) \quad (2.9.38)$$

$$\implies O\left(\frac{\bar{\epsilon}_d}{\Delta}\right) = O\left(\frac{\epsilon_d^*}{\Delta}\right) \quad (2.9.39)$$

$$\implies O(\bar{\epsilon}_d) = O(\epsilon_d^*) \quad (2.9.40)$$

$$(2.9.41)$$

Since $\bar{\epsilon}_d$ and ϵ_d^* are of the same order, we can say:

$$\epsilon_d^* \gg \Delta \implies \bar{\epsilon}_d \gg \Delta \quad (2.9.42)$$

Applying this to eq. 2.9.35 means

$$\chi_d \approx \frac{g^2 \mu_B^2 \Delta}{2 \pi \bar{\epsilon}_d^2} \quad (2.9.43)$$

which is the same as eq. 2.9.34. This tells us that scaling all the way down to very low temperatures in regime $\epsilon_d^* \gg \Delta$ brings us into a Fermi liquid state, characterized by a temperature-independent susceptibility (as is standard in a Fermi liquid). The crossovers can be seen by looking at the variation of the Curie constant χT .

Since the susceptibility is proportional to the magnetic moment, presence of degeneracy will reduce this moment because the probability of occupying the states will decrease. As a result, the Curie constant is also a measure of the effective degeneracy of the impurity orbital. At very high temperatures $T \gg U, \epsilon_d$, all the impurity levels $0, \epsilon_d$ and $2\epsilon_d + U$ will become degenerate on energy scales of the order of $k_B T$. As a result, the Curie constant is approximately $\frac{1}{8}$ in this range. The impurity occupancy is $n_d = 1$, because there are 4 degenerate states and the average number of electrons on them is 1. At lower temperatures $U \gg T \gg T^*$, the degeneracy gets lowered; now, only the vacant and single-occupied states are degenerate. Here the Curie constant is $\frac{1}{6}$. In this case, the average occupancy is $n_d = \frac{0+1+1}{3} = \frac{2}{3}$. At still lower temperatures, we saw that the impurity becomes vacant and $n_d = 0$. The Curie constant becomes linear in temperature, going down to 0. More formally,

$$m = \frac{1}{\beta} \frac{\partial \ln Z}{\partial B} \implies \chi = \lim_{B \rightarrow 0} \frac{\partial m}{\partial B} = \lim_{B \rightarrow 0} \frac{1}{Z^2 \beta} \left[Z \frac{\partial^2 Z}{\partial B^2} - \left(\frac{\partial Z}{\partial B} \right)^2 \right] \quad (2.9.44)$$

For the case of four-fold degeneracy, all the states can be assumed to be at zero energy. Then, under a magnetic field B ($h = \frac{g\mu_B}{2} B$), the partition function is

$$Z = 1 + \exp(\beta h) + \exp(-\beta h) + 1 = 2(1 + \cosh \beta h) \quad (2.9.45)$$

$$\implies \frac{\partial Z}{\partial B} = g\mu_B \beta \sinh \beta h \quad (2.9.46)$$

$$\implies \frac{\partial^2 Z}{\partial B^2} = \frac{1}{2} (g\mu_B)^2 \beta^2 \cosh \beta h \quad (2.9.47)$$

Since $\lim_{h \rightarrow 0} \sinh \beta h = 0$ and $\lim_{h \rightarrow 0} \cosh \beta h = 1$, we get

$$\chi = \frac{\beta g^2 \mu_B^2}{2Z(h=0)} \quad (2.9.48)$$

Setting $g\mu_B = k_B = 1$, we get

$$\chi T = \frac{1}{2\mathcal{D}} \quad (2.9.49)$$

where $Z(h=0) = 2 + 2 = 4 = \mathcal{D}$ is the degeneracy.

Similarly, for the triplet case (ϵ_d and 0 are degenerate while $U \gg T$), the doubly occupied case is essentially cut off from the available states, so $Z = 1 + 2 \cosh \beta h$. The proof again goes through similarly. But this time, we have $Z(h=0) = 1 + 2 = 3 = \mathcal{D}$.

For $\epsilon_d = k_B T^* > k_B T$ such that $k_B T^* \gg \Delta$, we can find the magnetic moment in a perturbative fashion. At the zeroth order, we can neglect the hybridisation Δ . Then,

$$m^{(0)} = \frac{1}{\beta} \frac{\partial \ln Z(h)}{\partial B} \quad (2.9.50)$$

where

$$Z(h) = 1 + e^{-\beta(k_B T^* - h)} + e^{-\beta(k_B T^* + h)} = 1 + e^{-\frac{\beta}{\beta^*}} 2 \cosh \beta h \quad (2.9.51)$$

Therefore,

$$\chi^{(0)} = \lim_{h \rightarrow 0} \frac{1}{\beta Z} \frac{\partial^2 Z}{\partial B^2} = \lim_{h \rightarrow 0} \frac{g^2 \mu_B^2}{4\beta Z} \frac{\partial^2 Z}{\partial h^2} = \frac{g^2 \mu_B^2}{4} \beta \frac{2e^{-\frac{\beta}{\beta^*}}}{1 + 2e^{-\frac{\beta}{\beta^*}}} \quad (2.9.52)$$

Again setting $g\mu_B = k_B = 1$, we get,

$$\chi^{(0)} = \frac{1}{4T} \frac{2e^{-\frac{\beta}{\beta^*}}}{1 + 2e^{-\frac{\beta}{\beta^*}}} = \frac{1}{4T} \frac{2}{e^{\frac{\beta}{\beta^*}} + 2} \quad (2.9.53)$$

As a first approximation, we can include the hybridisation by using the expression for the average number of spin up or spin down impurity as obtained from the non-interacting treatment, eq. 2.3.17

$$m^{(1)} = \frac{g\mu_B}{2} (n_{\uparrow} - n_{\downarrow}) = \frac{g\mu_B}{2\pi} \left[\tan^{-1} \frac{\Delta}{k_B T^* - h} - \tan^{-1} \frac{\Delta}{k_B T^* + h} \right] \quad (2.9.54)$$

Since $\Delta \ll T^*$, we can expand the arctan in a Taylor series. Up to first order, we get

$$m^{(1)} = \frac{g\mu_B}{2\pi} \left[\frac{\Delta}{k_B T^* - h} - \frac{\Delta}{k_B T^* + h} \right] = \frac{g\mu_B \Delta}{\pi} \frac{h}{k_B (T^*)^2 - h^2} \quad (2.9.55)$$

Differentiating with B gives

$$\chi^{(1)} = \lim_{h \rightarrow 0} \frac{\partial m^{(1)}}{\partial B} = \frac{g^2 \mu_B^2 \Delta}{2} \frac{1}{\pi k_B^2 T^{*2}} = \frac{\Delta}{2\pi T^{*2}} \quad (2.9.56)$$

Combining the zeroth and first order terms, the susceptibility in the regime $T \lesssim T^*$ is

$$\chi = \frac{1}{4T} \frac{2}{e^{\frac{\beta}{\beta^*}} + 2} + \frac{\Delta}{2\pi T^{*2}} \quad (2.9.57)$$

Below some temperature $T_{FL} \ll T^*$, the susceptibility reduces to

$$\chi \approx \frac{1}{4T} \frac{2}{e^{\frac{\beta}{\beta^*}} + 2} + \frac{\Delta}{2\pi T^{*2}} \approx \frac{\Delta}{2\pi T^{*2}} \quad (2.9.58)$$

$$\implies \chi T \propto T \quad (2.9.59)$$

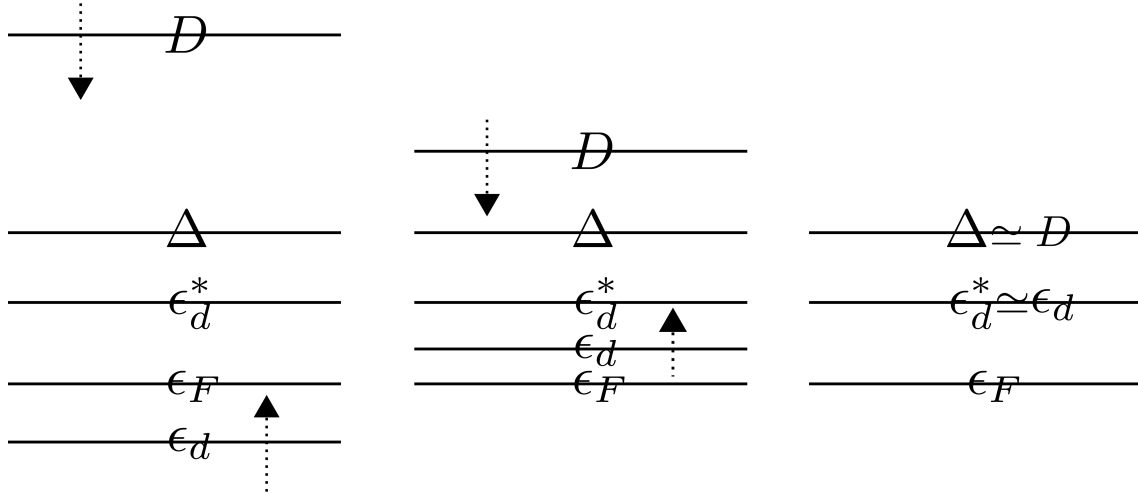
We can now visualize the various phases as the temperature is changed. For $T \gg U, \epsilon_d$, all the four states $|0\rangle, |\uparrow\rangle, |\downarrow\rangle, |2\rangle$ are degenerate ($\mathcal{D} = 4$), the average occupancy is $\langle n_d \rangle = \frac{0+1+1+2}{4} = 1$ and the effective Curie constant is $\frac{1}{2\mathcal{D}} = \frac{1}{8}$. At lower temperatures $U \gg T \gg T^*$, the level $|2\rangle$ is disconnected from the conduction band and the three remaining states are now degenerate ($\mathcal{D} = 3$). The average occupancy becomes $\frac{0+1+1}{3} = \frac{2}{3}$ and the effective Curie constant is now $\frac{1}{2 \times 3} = \frac{1}{6}$. At still lower temperatures $T^* \gg T$, the singly-occupied levels become disconnected and the impurity occupancy becomes 0. The effective Curie constant in this regime is linear in T .

$$\begin{array}{ccccc} n_d = 1 & & n_d = \frac{2}{3} & & n_d = 0 \\ \chi T \sim \frac{1}{8} & \implies & \chi T \sim \frac{1}{6} & \implies & \chi T \sim T \\ T \gg U & & T \gg T^* & & T \ll T^* \end{array}$$

Next we consider the mixed valence regime, described by $|\epsilon_d^*| < \Delta$. It is clear that since the impurity level is within an interval of the hybridisation from the Fermi surface, the charge fluctuations can cause transitions between the various states of the impurity. This means that the occupation number of the impurity site is not a good quantum number in this regime, and the average number of impurity electrons will be fractional. This definition is a bit arbitrary because any observed sample will display an eigenstate in which the impurity states have contributions from both $\langle n_d \rangle = 0$ and $\langle n_d \rangle = 1$, so any sample will be mixed in that sense. However, if we are not in the mixed valence regime ($|\epsilon_d| \gg \Delta$), then the contribution from any one state will far outweigh the other. If $\epsilon_d > 0$, then the impurity level is far above the Fermi level and it will most probably not be occupied and the majority of the contribution will come from $\langle n_d \rangle = 0$. Similarly, if $\epsilon_d < 0$, then the impurity level is far below the Fermi level and the average occupation will be close to 1. The regime of mixed valence is one in which these two contributions are comparable.

Since we have $|\epsilon_d^*| \lesssim \Delta$, as we renormalize, the decreasing cutoff will first match Δ or $k_B T$, whichever is greater. From eq. 2.9.19, we know that if D comes close to Δ , our analysis will break down because we can no longer ignore that term. Since that term represents the broadening of the impurity level, this same broadening can also be brought about by the thermal fluctuations which are of the scale $k_B T$. This means that real valence fluctuations will now renormalize the potential V_{kd} . Hence, our analysis will stop at $D = \max\{\Delta, k_B T\}$. For the simpler situation in which $T = 0$, the renormalization will stop at $D = \Delta$. From eq. 2.9.23, putting $D = \Delta$, we get

$$(\epsilon_d)_{MV} = \epsilon_d^* \quad (2.9.60)$$


 Figure 2.3: Renormalization in energy levels when $|\epsilon_d^*| \lesssim \Delta$

This is the renormalized impurity level in the mixed valence regime. A characteristic feature of this regime is that the charge fluctuations can be thermally excited. This can be seen as follows. The probability of a transition from, say, $|n_d = 0\rangle$ to $|n_d = 1\rangle$ is

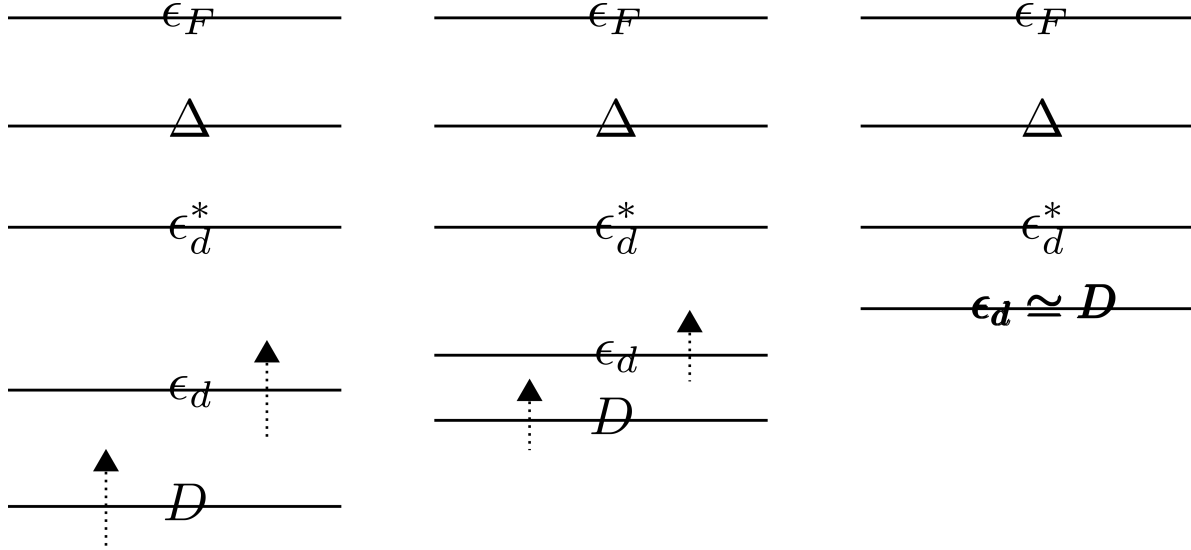
$$\sim \frac{k_B T}{\epsilon_d} \quad (2.9.61)$$

Assuming the thermal fluctuations are more or less of the order Δ , for $\epsilon_d \gg \Delta$, this transition will not be possible. However, in the mixed valence regime, because $\epsilon_d \sim \Delta$, these excitations do occur. These fluctuations, as well as the ones from the hybridisation with the conduction band, are responsible for the mixing of the singly-occupied and null-occupied states.

The crossovers in the mixed valence regime are as follows. Similar to the previous case, at high and intermediate temperatures, we have $n_d = 1$ and $n_d = \frac{2}{3}$ respectively. However, while the triplet degeneracy lasted upto $T \sim T^*$ in the previous case, here it continues up to $T \sim \Delta$ because that is where the scaling breaks down. That is, $T = \Delta$ is the point where we can no longer ignore the renormalization in V and it begins to increase with scaling. Beyond this point, the impurity occupation remains fractional and not much else can be said.

$$\begin{array}{ccccc} n_d = 1 & & n_d = \frac{2}{3} & & n_d = \text{fractional} \\ \chi T = \frac{1}{8} & \implies & \chi T = \frac{1}{6} & \implies & \chi T \propto T \\ T \gg U & & T \gg \Delta & & T \ll \Delta \end{array}$$

For $\epsilon_d^* \ll -\Delta$, the scaling will stop when the impurity level again goes out of the Fermi surface. But this time, it goes out from below. This again decouples the singly-occupied state from the conduction band and the scaling stops. This happens at say $\bar{D} = -\tilde{\epsilon}_d = \tilde{T}$. Since the singly-occupied impurity level is now well below $-D$, we have $\langle n_d \rangle = 1$ and we are comfortably in the Kondo limit and the SWT and a consequent poor man's scaling can be performed, which will give eqs. 2.9.24 through 2.9.28. The result of the Schrieffer-Wolff


 Figure 2.4: Renormalization in energy levels when $\epsilon_d^* \ll -\Delta$

transformation is a Hamiltonian that couples the impurity to the conduction electrons only through their spins; there is no charge fluctuation. At high temperatures $T \gg T_K$, the impurity is essentially decoupled and we get a susceptibility of the form eq. 2.9.49, but with a degeneracy of 2. To go to lower temperatures, we can do a Poor Man's scaling which suggests that the Hamiltonian at $T \ll T_K$ is one with a large coupling between the impurity and the conduction electrons.

$$\begin{array}{ccccc} n_d = 1 & & n_d = \frac{2}{3} & & n_d = 1 \\ \chi T = \frac{1}{8} & \Rightarrow & \chi T = \frac{1}{6} & \Rightarrow & \chi T \propto T \\ T \gg U & & T \gg \tilde{T} & & T \ll \tilde{T}_K \end{array}$$

Jefferson's calculation

Jefferson did a slightly more rigorous calculation to obtain the scaling equation. He divided the Hamiltonian into two parts

$$H = \sum_{k\sigma} \epsilon_{k\sigma} n_{k\sigma} + \epsilon_d n_d + \sum_{k\sigma} \left(V_{kd}^- c_{k\sigma}^\dagger c_{d\sigma} + V_{kd}^+ c_{d\sigma}^\dagger c_{k\sigma} \right) = H_0 + V \quad (2.9.62)$$

Before scaling, $V^+ = V^- = V$. The Schrödinger equation we want to solve is

$$H\psi = E\psi \quad (2.9.63)$$

We know the eigenstates ψ_0 of H_0 . They are the states $\{|n_{k_i\sigma}, n_{d\sigma'}\rangle\}$. These states of course span the entire Hilbert space. A subset of these states form the model subspace. We call these states ϕ . For our case, that is the subspace with all conduction electrons inside $D - \delta D$. The projection operator for this subspace is

$$P = \sum |\phi\rangle \langle \phi| = \sum_{|k| < D - \delta D, \sigma = \pm 1, n_{d\sigma} = 0, 1} |n_{k\sigma}, n_{d\sigma'}\rangle \quad (2.9.64)$$

Its orthogonal subspace has a projection operator

$$Q = 1 - P = \sum_{D-\delta D < |k| < D, \sigma=\pm 1, n_{d\sigma}=0,1} |n_{k\sigma}, n_{d\sigma'}\rangle \quad (2.9.65)$$

If the dimension of model subspace is d , we can say that P takes d eigenstates ψ of the total Hamiltonian to d eigenstates in the model subspace:

$$P\{\psi\}_d = \{\phi\} \quad (2.9.66)$$

This is of course true in the non-interacting limit. There, the ψ_0 are the exact eigenstates, and the action of P is basically

$$P\psi_0 \Big|_{|k| < D-\delta D} = \psi_0 \Big|_{|k| < D-\delta D} \quad (2.9.67)$$

Now, as we turn on the interactions adiabatically, it is safe to assume that these d non-interacting eigenstates flow into d interacting eigenstates. This means that we can define an inverse for the P operator which takes a non-interacting eigenstate from the model subspace into the interacting eigenstate:

$$\Omega\{\phi\} = \{\psi\} \quad (2.9.68)$$

Since Ω can only act on states in the model subspace, we define

$$\Omega\{\phi\}^\perp = 0 \quad (2.9.69)$$

This allows us to write

$$\Omega P\phi = \Omega\phi \quad (2.9.70)$$

$$\Omega P\phi^\perp = \Omega \times 0 = 0 = \Omega\phi^\perp \quad (2.9.71)$$

In the first equation, I used $P\phi = \phi$ because the projection of ϕ into the model subspace is ϕ itself. Together these two identities give

$$\Omega P = \Omega \quad (2.9.72)$$

With these definitions, we now change the problem a bit. We want to solve the Schrödinger equation only in the model subspace. To this end we write the Schrödinger equation as

$$H\Omega\phi = E\Omega\phi \quad (2.9.73)$$

Since we want to write down an equation only in the model subspace, the equation should operate only on the ϕ . To remove the Ω on the right side, operate on this equation with P from the left. This gives

$$PH\Omega\phi = EP\Omega\phi = E\phi \quad (2.9.74)$$

This is the effective Schrödinger equation in the model subspace. The effective Hamiltonian for the model subspace is

$$H_{\text{eff}} = PH\Omega = PH_0P + PV\Omega = PH_0P + PV\Omega \quad (2.9.75)$$

To solve for the Ω , apply eq. 2.9.72 on the Schrödinger equation $(E - H_0)\psi = V\psi$:

$$\Omega V\psi = (E\Omega P - \Omega PH_0)\psi \quad (2.9.76)$$

Now, since P is made up of the eigenstates of H_0 , those two will commute: $[H_0, P] = 0$. The equation then becomes

$$\Omega V\psi = (E - \Omega H_0 P)\psi \quad (2.9.77)$$

Subtracting the Schrödinger equation from the last equation gives

$$\begin{aligned} (\Omega - 1) V\psi &= (H_0 - \Omega H_0 P) \psi \\ \implies (\Omega - 1) V\Omega\phi &= (H_0 - \Omega H_0 P) \Omega\phi \\ \implies (\Omega - 1) V\Omega\phi &= (H_0\Omega - \Omega H_0) \phi \\ \implies (\Omega - 1) V\Omega &= [H_0, \Omega] \end{aligned} \quad (2.9.78)$$

This is the main equation. To make progress, we expand the operator Ω in powers of the interaction V :

$$\Omega = \sum_n c_n V^n = \sum_n \Lambda_n \quad (2.9.79)$$

The zeroth term in the main equation becomes

$$[H_0, \Lambda_0] = 0 \implies \Lambda_0 = P \quad (2.9.80)$$

The first order equation is

$$[H_0, \Lambda_1] = (\Lambda_0 - 1) V\Lambda_0 = (P - 1) VP = -QVP \quad (2.9.81)$$

The second order equation is

$$[H_0, \Lambda_2] = -V\Lambda_1 + \Lambda_0 V\Lambda_1 + \Lambda_1 V\Lambda_0 = -QV\Lambda_1 + \Lambda_1 VP \quad (2.9.82)$$

These equations are of the form $[H_0, \Lambda_n] = A_n$, where A_n is an operator in terms of Λ_{n-1} and lower orders.

$$A_1 = -QVP \quad (2.9.83)$$

$$A_2 = -QV\Lambda_1 + \Lambda_1 VP \quad (2.9.84)$$

Let $|l\rangle$ and $|h\rangle$ belong to the model subspace and its orthogonal subspace respectively. Then, taking matrix element between $\langle h|$ and $|l\rangle$ of the general form equation gives

$$\langle h| A_n |l\rangle = (E_h - E_l) \langle h| \Lambda_n |l\rangle \implies \langle h| \Lambda_n |l\rangle = \frac{\langle h| A_n |l\rangle}{E_h - E_l} \quad (2.9.85)$$

If we define an operator S by its action on a general operator A as

$$\langle h|SA|l\rangle = \frac{\langle h|A|l\rangle}{E_l - E_h} \quad (2.9.86)$$

we can write the solution

$$\Lambda_n = -S(A_n) \quad (2.9.87)$$

The expression of SA can be written as

$$\begin{aligned} SA &= \sum_{h,l} |h\rangle \langle l| \frac{\langle h|A|l\rangle}{E_l - E_h} \\ &= \sum_{h,l} \frac{1}{E_l - E_h} |h\rangle \langle h|A|l\rangle \langle l| \\ &= \sum_l \frac{1}{E_l - H_0} \left(\sum_h |h\rangle \langle h| \right) A |l\rangle \langle l| \\ &= \sum_l G_l A P_l \end{aligned} \quad (2.9.88)$$

where $P_l = |l\rangle \langle l|$ and $G_l = \frac{1}{E_l - H_0} Q$.

S has the property

$$\begin{aligned} \langle h|SQA|l\rangle &= \frac{\langle h|QA|l\rangle}{E_l - E_h} = \frac{\langle h|A|l\rangle}{E_l - E_h} = \langle h|SA|l\rangle \\ \implies S(QA) &= S(A) \end{aligned} \quad (2.9.89)$$

The lowest order solutions are thus

$$\Lambda_1 = S(QVP) = S(VP) \quad (2.9.90)$$

$$\Lambda_2 = S(QV\Lambda_1) - S(\Lambda_1VP) = S(VS(VP)) - S(S(VP)VP) \quad (2.9.91)$$

We can now expand the effective Hamiltonian in powers of V . From eq. 2.9.75, the interacting part of the effective Hamiltonian becomes

$$\begin{aligned} H_{\text{eff}} - PH_0P &= PV\Omega \\ &\approx PV(\Lambda_0 + \Lambda_1 + \Lambda_2) \\ &= PV[P + S(VP) + S(VS(VP)) - S(S(VP)VP)] \\ &= PVP + PVS(VP) + PVS(VS(VP)) - PVS(S(VP)VP) \end{aligned} \quad (2.9.92)$$

Therefore,

$$H_{\text{eff}} = PHP + PVS(VP) + PVS(VS(VP)) - PVS(S(VP)VP) \quad (2.9.93)$$

The first term is the obvious lowest approximation; you just project the entire Hamiltonian into the model subspace. The second term is

$$PVSVP = PV \sum_l G_l V P P_l = PV \sum_l G_l V P_l \quad (2.9.94)$$

where I used $PP_l = \sum_{l'} |l'\rangle \langle l'| |l\rangle \langle l| = \sum_{l'} |l'\rangle \langle l'| \delta_{ll'} = P_l$. The third term becomes

$$\begin{aligned} PVS SVSP &= PVS V \sum_l G_l V P_l = PV \sum_l S V G_l V P_l \\ &= PV \sum_{l,l'} G_{l'} V G_l V P_l P_{l'} = PV \sum_l G_l V G_l V P_l \end{aligned} \quad (2.9.95)$$

The fourth term is

$$\begin{aligned} PVS(S(VP)VP) &= PVS\left(\sum_l G_l V P P_l V P\right) = PV \sum_{l,l'} G_{l'} G_l V P_l V P P_{l'} \\ &= PV \sum_{l'} G_{l'} \left(\sum_l G_l V P_l\right) V P_{l'} \end{aligned} \quad (2.9.96)$$

The effective Hamiltonian up to third order in V is

$$\begin{aligned} H_{\text{eff}} &= P H_0 P + PV \sum_l G_l V P_l + PV \sum_l G_l V G_l V P_l \\ &\quad - PV \sum_{l,l'} G_{l'} G_l V P_l V P_{l'} \end{aligned} \quad (2.9.97)$$

These results have been more or less general. We now need to write these in terms of the creation and annihilation operators of our Hamiltonian. The model subspace for our problem is the part of the conduction band up to $D - \delta D$. Here on, \sum represent sum over the model subspace momenta and \sum' represent sum over the remaining momenta. To facilitate writing the effective Hamiltonian in terms of the creation and annihilation operators, we change the projection operators from the bra-ket representation to operator representation:

$$|k_1\rangle \langle k_2| = c_{k_1}^\dagger c_{k_2} \quad (2.9.98)$$

$$P_k = |k, n_{d\sigma}\rangle \langle k, n_{d\sigma}| = c_k^\dagger c_k c_{d\sigma}^\dagger c_{d\sigma} = n_{k\sigma} n_{d\sigma} \quad (2.9.99)$$

The first term becomes

$$P H_0 P = \sum_{k\sigma} \epsilon_{k\sigma} n_{k\sigma} + \epsilon_d n_d + \sum_{k\sigma} \left(V_{kd} c_{k\sigma}^\dagger c_{d\sigma} + \text{h.c.} \right) \quad (2.9.100)$$

The second term involves two potential terms that scatter from the model subspace to the high energy subspace and then back to the model subspace. Hence this term is

$$\begin{aligned} PV \sum_l G_l V P_l &= V \sum_{q\sigma} \left(\frac{V_q}{\epsilon_d - \epsilon_q} c_{q\sigma}^\dagger c_{d\sigma} + \frac{V_q^*}{\epsilon_q - \epsilon_d} c_{d\sigma}^\dagger c_{q\sigma} \right) \\ &= \sum_{q\sigma}^+ \frac{|V_q|^2 c_{d\sigma}^\dagger c_{q\sigma} c_{q\sigma}^\dagger c_{d\sigma}}{\epsilon_d - \epsilon_q} + \sum_{q\sigma}^- \frac{|V_q|^2 c_{q\sigma}^\dagger c_{d\sigma} c_{d\sigma}^\dagger c_{q\sigma}}{\epsilon_q - \epsilon_d} \\ &= \sum_{q\sigma}^+ \frac{|V_q|^2 n_{d\sigma} (1 - n_{q\sigma})}{\epsilon_d - \epsilon_q} + \sum_{q\sigma}^- \frac{|V_q|^2 n_{q\sigma} (1 - n_{d\sigma})}{\epsilon_q - \epsilon_d} \end{aligned} \quad (2.9.101)$$

In the high energy subspaces, $n_q^+ = 1 - n_q^- = 0$. Therefore,

$$\begin{aligned}
 PV \sum_l G_l V P_l &= \sum_q^+ \frac{|V_q|^2 n_{d\sigma}}{\epsilon_d - \epsilon_q} + \sum_q^- \frac{|V_q|^2 (1 - n_{d\sigma})}{\epsilon_q - \epsilon_d} \\
 &= n_d \left(\sum_q^+ \frac{|V_q|^2}{\epsilon_d - \epsilon_q} + 2 \sum_q^- \frac{|V_q|^2}{\epsilon_d - \epsilon_q} \right) \\
 &= n_d \delta \epsilon_d
 \end{aligned} \tag{2.9.102}$$

The third term is zero in our case. The part $G_l V G_l V$ will do the following.

$$|k, n_{d\sigma}\rangle \rightarrow \begin{cases} |q_e, n_d = 0\rangle \rightarrow \begin{cases} |q_e, n_d = 1\rangle \\ |q_e, q'_h, n_d = 1\rangle \end{cases} \\ |q_h, n_d = 1\rangle \rightarrow \begin{cases} |q_h, q'_e, n_d = 0\rangle \\ |q_h, n_d = 0\rangle \end{cases} \end{cases} \tag{2.9.103}$$

None of the four final states belong to the model subspace, so this term is zero.

The fourth term involves a first scattering between two model states, followed by a scattering to a high energy subspace and then a scattering back to the model subspace. One way for going through such a process is

$$|k, n_d = 0\rangle \xrightarrow{\Delta E = \epsilon_k - \epsilon_q} |n_d = 1\rangle \xrightarrow{\Delta E = \epsilon_q - \epsilon_d} |q_e, n_d = 0\rangle \xrightarrow{\Delta E = \epsilon_q - \epsilon_d} |k', n_d = 1\rangle \tag{2.9.104}$$

Another way is to start with c_d instead of c_d^\dagger

$$|n_{d\sigma} = 1\rangle \xrightarrow{\Delta E = \epsilon_k - \epsilon_q} |k\sigma, n_d = 0\rangle \rightarrow \begin{cases} |q_h \uparrow, n_{d\uparrow} = 1\rangle \\ |q_h \downarrow, n_{d\downarrow} = 1\rangle \end{cases} \xrightarrow{\Delta E = \epsilon_q - \epsilon_d} |n_d = 0\rangle \tag{2.9.105}$$

Combining the two processes gives

$$\begin{aligned}
 \sum_q^+ \sum_{k\sigma} \frac{|V_q|^2 c_{d\sigma}^\dagger c_{q\sigma} c_{q\sigma}^\dagger c_{d\sigma} c_{d\sigma}^\dagger c_{k\sigma}}{(\epsilon_q - \epsilon_d)(\epsilon_k - \epsilon_q)} + \sum_{q\sigma'}^- \sum_{k\sigma} \frac{|V_q|^2 c_{q\sigma'}^\dagger c_{d\sigma'} c_{d\sigma'}^\dagger c_{q\sigma'} c_{k\sigma}^\dagger c_{d\sigma}}{(\epsilon_q - \epsilon_d)(\epsilon_k - \epsilon_q)} \\
 = \sum_{k\sigma} \left(c_{k\sigma}^\dagger c_{d\sigma} \delta V_k^- + c_{d\sigma}^\dagger c_{k\sigma} \delta V_k^- \right)
 \end{aligned} \tag{2.9.106}$$

where

$$\begin{aligned}
 \delta V^+ &= \sum_q^+ \frac{|V_q|^2}{(\epsilon_q - \epsilon_d)(\epsilon_k - \epsilon_q)} \\
 \delta V^- &= \sum_q^- 2 \frac{|V_q|^2}{(\epsilon_q - \epsilon_d)(\epsilon_k - \epsilon_q)}
 \end{aligned} \tag{2.9.107}$$

The total Hamiltonian can be written in the form

$$H_{\text{eff}} = \sum_{k\sigma} \epsilon_{k\sigma} n_{k\sigma} + (\epsilon_d + \delta\epsilon_d) n_d + \sum_{k\sigma} \left\{ (V_k^- + \delta V_k^-) c_{k\sigma}^\dagger c_{d\sigma} + (V_k^+ + \delta V_k^+) c_{d\sigma}^\dagger c_{k\sigma} \right\} \quad (2.9.108)$$

We now evaluate the changes:

$$\begin{aligned} \delta\epsilon_d &= \left(\sum_q^+ \frac{|V_q|^2}{\epsilon_d - \epsilon_q} + 2 \sum_q^- \frac{|V_q|^2}{\epsilon_d - \epsilon_q} \right) \\ &\approx |V|^2 \rho |\delta D| \left(\frac{1}{\epsilon_d - D} + \frac{2}{\epsilon_d + D} \right) \\ &= |V|^2 \rho |\delta D| \frac{D - 3\epsilon_d}{D^2 - \epsilon_d^2} \end{aligned} \quad (2.9.109)$$

I used the approximation

$$\sum_{q=D-\delta D}^D f(q) = \int_{D-\delta D}^D dE \rho(E) f(E) \approx \rho f(D) \delta D \quad (2.9.110)$$

Also,

$$\begin{aligned} \delta V_k^+ &= \sum_q^+ \frac{|V_q|^2}{(\epsilon_q - \epsilon_d)(\epsilon_k - \epsilon_q)} \\ &\approx |V|^2 \rho |\delta D| \frac{1}{(D - \epsilon_d)(\epsilon_k - D)} \\ \delta V_k^- &= 2 \sum_q^- \frac{|V_q|^2}{(\epsilon_q - \epsilon_d)(\epsilon_k - \epsilon_q)} \\ &\approx -|V|^2 \rho |\delta D| \frac{2}{(D + \epsilon_d)(\epsilon_k + D)} \end{aligned} \quad (2.9.111)$$

We now make the following assumptions:

- k is close to the Fermi level ($\epsilon_k \approx 0$)
- Because k is close to the Fermi surface, we assume the potential is independent of momenta: $V_k^+ \equiv v^+$, $V_k^- \equiv v^-$
- Since we truncated at third order, we need $D - |\epsilon_d| \gg v^\pm$. This gives us $D \gg |\epsilon_d|$.

With these assumptions, we get the scaling equations similar to the ones obtained previously.