Lax Pairs and Double Bracket Flows

1 Definition of a Lax Pair

Two operator A and B are said to form a lax pair if they satisfy the equation

$$\frac{\mathrm{d}A(t)}{\mathrm{d}t} = [B(t), A(t)] \tag{1}$$

2 Unitary Nature of the Flow

It can be shown that this defines a unitary time evolution on $A(t)$, in the following manner. Let $U(t, t_0)$ be the unitary operator that carries this evolution through. We then need to construct a $U(t, t_0)$.

$$A(t) = U(t, t_0)A(t_0)U^\dagger(t, t_0) \tag{2}$$

where $A(t_0)$ is the operator A at a particular time t_0. The time change of A can then be written as

$$\begin{aligned}
\frac{\mathrm{d}A(t)}{\mathrm{d}t} &= \frac{\mathrm{d}U(t, t_0)}{\mathrm{d}t}A(t_0)U^\dagger(t, t_0) + U(t, t_0)A(t_0)\frac{\mathrm{d}U^\dagger(t, t_0)}{\mathrm{d}t} \\
&= \frac{\mathrm{d}U(t, t_0)}{\mathrm{d}t}U^\dagger(t, t_0)A(t) + A(t)U(t, t_0)\frac{\mathrm{d}U^\dagger(t, t_0)}{\mathrm{d}t} \qquad \left[A(t) = UAU^\dagger\right] \\
&= \frac{\mathrm{d}U(t, t_0)}{\mathrm{d}t}U^\dagger(t, t_0)A(t) - A(t)\frac{\mathrm{d}U(t, t_0)}{\mathrm{d}t}U^\dagger(t, t_0) \qquad \left[UU^\dagger = 1\right] \\
&= \left[\frac{\mathrm{d}U(t, t_0)}{\mathrm{d}t}U^\dagger(t, t_0), A(t)\right]
\end{aligned} \tag{3}$$

Looking at the definition of a lax pair, we can now make the connection

$$B(t) = \frac{\mathrm{d}U(t, t_0)}{\mathrm{d}t}U^\dagger(t, t_0) \tag{4}$$

The equation of motion characterised by the lax pair eq. 1 can thus be said to generate a family of unitarily connected operators $A(t)$, related by the unitaries defined by eq. 4. A direct corrolary is that the spectrum of $A(t)$ is preserved during this evolution.

3 Double Bracket Flow

The double bracket flows correspond to a special choice of the operator $B(t)$: $B(t) \equiv [A(t), C]$. A consequence of this choice is that the lax pair evolution then serves to minimize the commutator $[A(t), C]$. To see how, we first write down a function

$$\chi \equiv \text{Tr}\left([A(t) - C]^2\right) = \text{Tr}\left[A(t)^2 + C^2 - A(t)C - CA(t)\right] \tag{5}$$

Since $A^2(t) = UA^2U^\dagger$, we get $\text{Tr}(A^2(t)) = \text{Tr}(A)$. Also, from the cyclic nature of trace, we can write $\text{Tr}(A(t)C) = \text{Tr}(CA(t))$. These considerations (and the fact that C does not depend on t) allows us to write

$$\frac{\mathrm{d}\chi}{\mathrm{d}t} = -2\text{Tr}\left(\frac{\mathrm{d}A(t)}{\mathrm{d}t}\,C\right) = -2\text{Tr}\left(\left[B(t), A(t)\right]C\right) \tag{6}$$

Using the cyclic property of trace, this becomes

$$\begin{aligned} \text{Tr}\left(\left[B(t), A(t)\right]C\right) &= \text{Tr}\left(B(t)A(t)C - A(t)B(t)C\right) \\ &= \text{Tr}\left(B(t)A(t)C - B(t)A(t)C\right) \\ &= \text{Tr}\left(B(t)\left[A(t), C\right]\right) \end{aligned} \tag{7}$$

If we now substitute the choice of $B(t)$ we made above, we get

$$\frac{\mathrm{d}\chi}{\mathrm{d}t} = -2\text{Tr}\left([A(t), C]^2\right) = -2\text{Tr}\left(B(t)^2\right) \leq 0 \tag{8}$$

Since χ, the way it is defined, must necessarily be positive semi-definite for all t, it has a global minimum at $\chi = 0$. Since the derivative $\frac{\mathrm{d}\chi}{\mathrm{d}t}$ is always negative, it will take χ towards its minimum. At the minimum, the time derivative must vanish, otherwise $\chi(t)$ will become negative. This gives the result

$$\lim_{t\to\infty}\frac{\mathrm{d}\chi}{\mathrm{d}t} = -2\lim_{t\to\infty}\text{Tr}\left([A(t), C]^2\right) = 0 \implies \lim_{t\to\infty}[A(t), C] = 0 \tag{9}$$

In other words, **the lax pair evolution of $A(t)$ against $[A(t), C]$ leads to the diagonalization of $A(t)$ with respect to C.** This can be used as an iterative algorithm to diagonalize a general matrix with respect to another matrix:

- Define matrices A and B, A being the one we want to diagonalize w.r.t B
- Iteratively run the next two steps until a desired accuracy is reached
- Compute a new matrix C = A*B - B*A
- Change A as follows: A = A + C*A - A*c

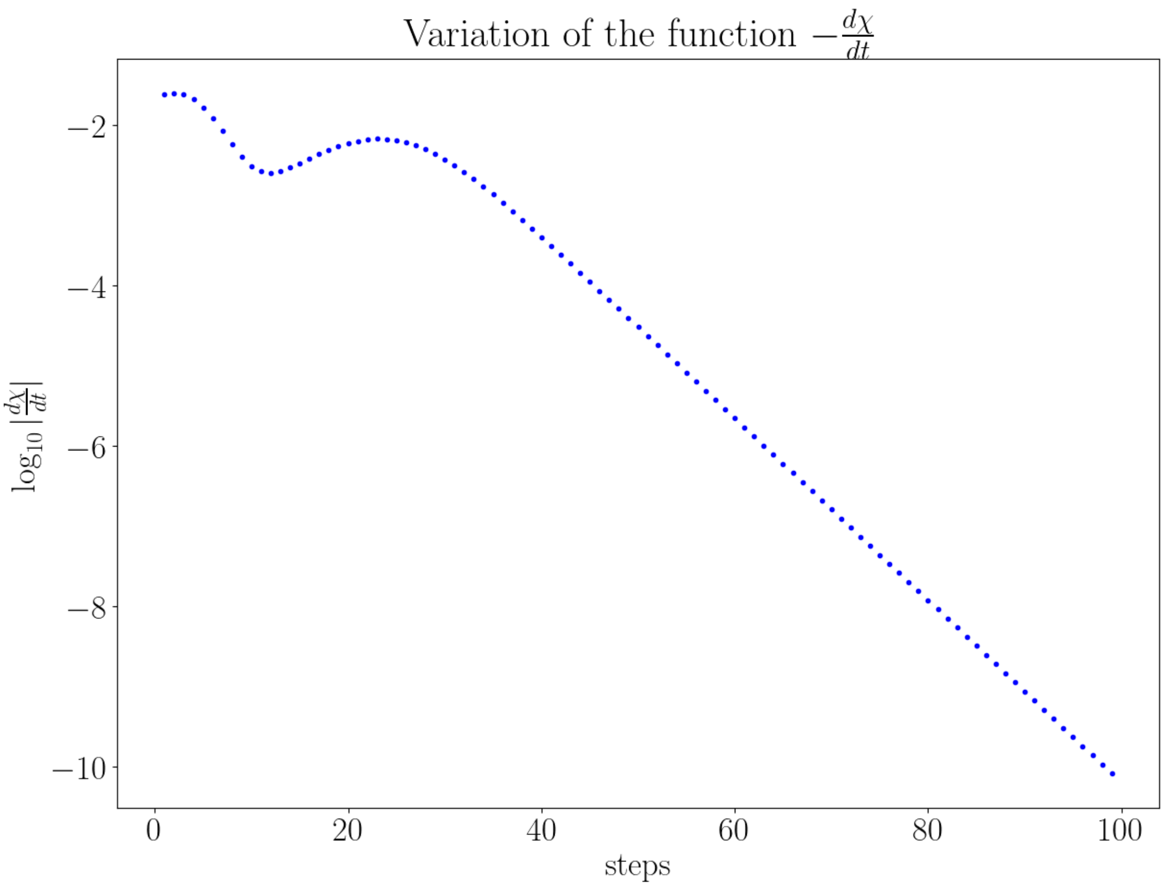

Figure 1: Variation of the function $\frac{\mathrm{d}\chi}{\mathrm{d}t}$ for an arbitrary choice of A and C. The convergence of A towards a diagonal form is clear.

The double bracket flow

$$\frac{\mathrm{d}A}{\mathrm{d}t} = [[A, C], A] \tag{10}$$

is thus a flow towards the minima of the function $\chi = \mathrm{tr}\left((A-C)^2\right)$.

4 URG as a double-bracket flow

The difference RG equation for URG can be written in the form

$$\Delta\mathcal{H}(\omega, D) = \frac{1}{\omega_1 - \omega_0}\left[G\left[\mathcal{H}^d, \mathcal{H}^I\right], \mathcal{H}\right] \tag{11}$$

This is not in the standard double-bracket form, primarily because it takes into account the off-diagonal terms in the Hamiltonian inside the generator of the unitary transformation. It can be given a double-bracket form by taking some approximations, as was shown in eq. **??**.

Just like the standard double-bracket flow equation, the URG equation acts as an optimizer - it minimizes the function

$$\chi_j = \mathrm{Tr}\left[\left(\mathcal{H}^I_j\right)^2\right] \tag{12}$$

The definition of this function first requires a scheme to be defined. We can order the energy of the electrons as $\epsilon_1 < \epsilon_2 < ... < \epsilon_j < ... < \epsilon_N$. The URG consists of sequentially decoupling the states ϵ_N, then ϵ_{N-1}, and so on. At the j^{th} step, the Hamiltonian can be partitioned in the subspace of the electron being decoupled; the partitioning looks like

$$\mathcal{H}^0_j + c^\dagger_j T_j + T^\dagger_j c_j \tag{13}$$

$\mathcal{H}^0_j$ is the part that *does not* scatter between $|\hat{n}_j\rangle = 0, 1$, while $\mathcal{H}^I_j = c^\dagger_j T_j + T^\dagger_j c_j$ is the part that *does* scatter between states with a definite value of $\hat{n}_j$.

The first observation that we make is that χ_j is semi-positive definite. This is because it can be expressed as the norm-squared of a state vector.

$$\chi_j = \sum_{i=1}^{N} \langle\psi_i| \left(\mathcal{H}^I_j\right)^2 |\psi_i\rangle = \sum_{i=1}^{N}\langle\phi_i|\phi_i\rangle \geq 0, \left[\text{where } |\phi_i\rangle = \mathcal{H}^I_j |\psi_i\rangle\right] \tag{14}$$

The difference equation for χ_j is

$$\Delta\chi_j = 2\mathrm{Tr}\left[\mathcal{H}^I_j \Delta\mathcal{H}^I_j\right] = 2\mathrm{Tr}\left[\mathcal{H}^I_j\left(\mathcal{H}^I_{j-1} - \mathcal{H}^I_j\right)\right] \tag{15}$$

The first part of the trace is zero. To see why, note that from the nature of URG, once j has been decoupled, it is diagonal in all the subsequent Hamiltonians. Hence, $\mathcal{H}^I_{j-1}$ will be diagonal in j, while $\mathcal{H}^I_j$ is, by definition, off-diagonal in j. The product $\mathcal{H}_j^I\mathcal{H}_{j-1}^I$ will hence be off-diagonal and will change $\hat{n}_j$. Hence, it will vanish when taken inside a trace. What remains is

$$\Delta\chi_j = -2\mathrm{Tr}\left[\left(\mathcal{H}_j^I\right)^2\right] = -2\chi_j \leq 0 \tag{16}$$

At the fixed point j^* of URG, the off-diagonal part of the Hamiltonian vanishes, so we can write $\mathcal{H}_{j^*}^I = 0 \implies \Delta\chi^* = 0$. Combining the three points:

$$\chi_j \geq 0, \quad \Delta\chi_j \leq 0, \quad \Delta\chi_{j^*} = 0 \tag{17}$$

we can say that URG starts from a non-minimal value of χ and flows to its minimum $\chi^* = 0$ at the fixed point.

This minimization has been demonstrated numerically in fig. 2, where URG has been performed on a very simple model of potential scatter:$\mathcal{H} = \sum_k \epsilon_k\hat{n}_k + J\sum_{k\neq k'} c_k^\dagger c_{k'}$.

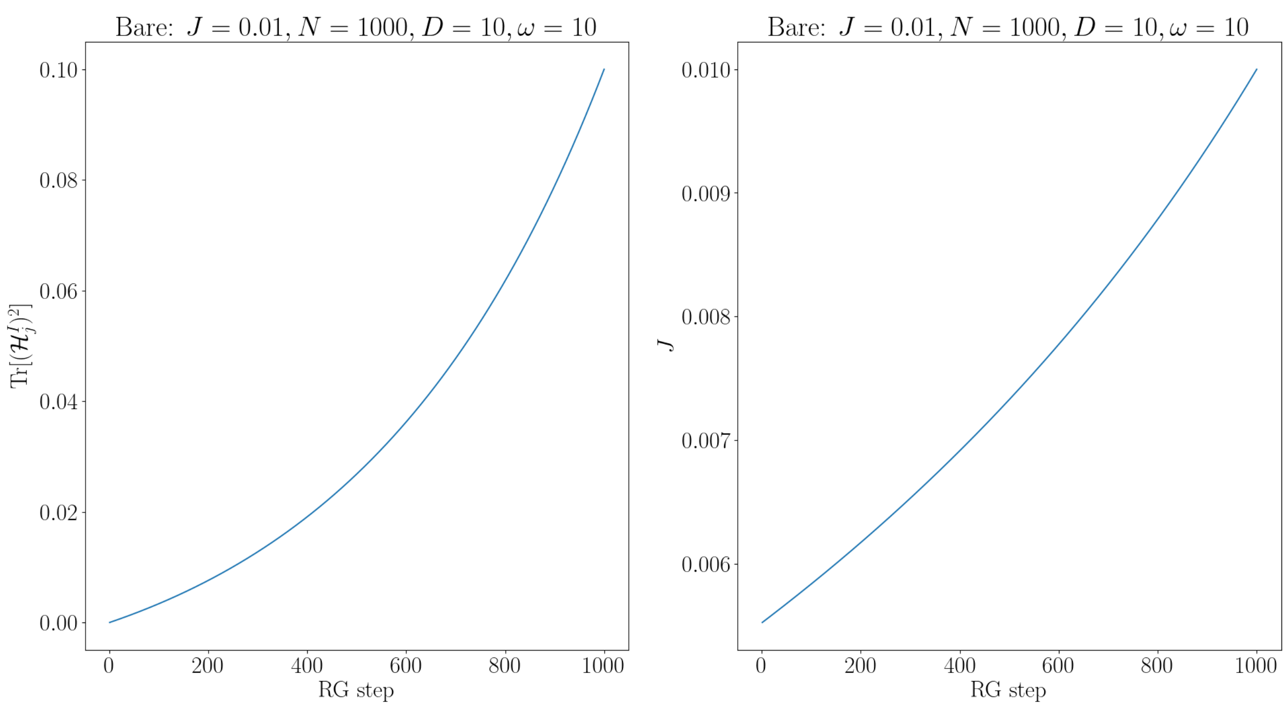

Figure 2: Variation of J and χ for the potential scattering problem.