We define $\xi \equiv \epsilon_{k+q} - \epsilon_k + \hbar\omega_q$. The renormalization in the total Hamiltonian becomes

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}l} = \left[\eta, \mathcal{H}\right] \tag{4.3.20}$$

The flow equation for the electron-phonon coupling is

$$\frac{\mathrm{d}g_q}{\mathrm{d}l} = -\xi^2 g_q \implies g_q(l) = g_q(0)\exp\left(-\xi^2 l\right) \tag{4.3.21}$$

A new electron-electron coupling $V_{kk^q}c^\dagger_{k+q}c^\dagger_{k'-q}c_{k'}c_k$ is also generated. For the Cooper channel ($k' = -k$), the flow equation is

$$V_{k,-k,q}(\infty) = V_{k,-k,-q}(0) - \frac{g_q^2\omega_q}{\omega_q^2 + \left(\epsilon_{k+q} - \epsilon_k\right)^2} \tag{4.3.22}$$

Off-diagonal terms that connect larger energy differences ξ decay the fastest.

CUT RG from URG

We will now see that the renormalization in URG can also be put into a similar form. From eq. 3.1.47, we can write the URG renormalization in the diagonal part as

$$\Delta\mathcal{H}^0 = \frac{1}{2}\left[\eta^\dagger - \eta, \mathcal{H}\right] \tag{4.3.23}$$

where $\mathcal{H}^0 = \mathcal{H}^d + \mathcal{H}^i$. The URG generator can be recast (starting from the definitions of η, eqs. 3.1.14) as

$$\begin{aligned}
\eta^\dagger - \eta &= G_1 c^\dagger T - G_0 T^\dagger c \\
&= \frac{1}{\omega_1 - \omega_0}\left[G_1\left(\omega_1 - \omega_0\right)c^\dagger T - G_0\left(\omega_1 - \omega_0\right)T^\dagger c\right] \\
&= \frac{1}{\omega_1 - \omega_0}\left[G_1\omega_1 c^\dagger T - c^\dagger T\omega_0 G_0 - T^\dagger c\omega_1 G_1 + G_0\omega_0 T^\dagger c\right]
\end{aligned} \tag{4.3.24}$$

In the last step, we changed the second and fourth terms using the constraints $G_1 c^\dagger T = c^\dagger T G_0$ and $G_0 T^\dagger c = T^\dagger c G_1$, eq. 3.1.16. We now add and subtract $G_1 G_1^{-1} c^\dagger T = c^\dagger T$ and $G_0 G_0^{-1} T^\dagger c = T^\dagger c$ for each term.

$$\begin{aligned}
\eta^\dagger - \eta = &\frac{1}{\omega_1 - \omega_0}\left[G_1\left(\omega_1 - G_1^{-1}\right)c^\dagger T + c^\dagger T - c^\dagger T\left(\omega_0 - G_0^{-1}\right)G_0 - c^\dagger T\right. \\
&\left. - T^\dagger c\left(\omega_1 - G_1^{-1}\right)G_1 - T^\dagger c + G_0\left(\omega_0 - G_0^{-1}\right)T^\dagger c + T^\dagger c\right] \\
= &\frac{1}{\omega_1 - \omega_0}\left[G_1\left(\omega_1 - G_1^{-1}\right)c^\dagger T - c^\dagger T\left(\omega_0 - G_0^{-1}\right)G_0\right. \\
&\left. - T^\dagger c\left(\omega_1 - G_1^{-1}\right)G_1 + G_0\left(\omega_0 - G_0^{-1}\right)T^\dagger c\right]
\end{aligned} \tag{4.3.25}$$

In the last step, the extra $c^\dagger T$ and $T^\dagger c$ terms canceled out. In the second and third terms, we can bring the Greens function closer to the operators $c^\dagger T$ and $T^\dagger c$ because $\left(\omega_j - G_j^{-1}\right) G_j = G_j \left(\omega_j - G_j^{-1}\right)$:

$$\begin{aligned}\eta^\dagger - \eta &= \frac{1}{\omega_1 - \omega_0}\left[G_1\left(\omega_1 - G_1^{-1}\right)c^\dagger T - c^\dagger T G_0\left(\omega_0 - G_0^{-1}\right)\right.\\ &\qquad\left. - T^\dagger c G_1\left(\omega_1 - G_1^{-1}\right) + G_0\left(\omega_0 - G_0^{-1}\right)T^\dagger c\right]\\ &= \frac{1}{\omega_1 - \omega_0}\left[G_1\left(\omega_1 - G_1^{-1}\right)c^\dagger T - G_1 c^\dagger T\left(\omega_0 - G_0^{-1}\right)\right.\\ &\qquad\left. - G_0 T^\dagger c\left(\omega_1 - G_1^{-1}\right) + G_0\left(\omega_0 - G_0^{-1}\right)T^\dagger c\right]\end{aligned} \tag{4.3.26}$$

In the last step, we again used the constraint $G_1 c^\dagger T = c^\dagger T G_0$ and its partner. From the definition of the Green's function $G = \left(\omega - \mathcal{H}^d\right)^{-1}$, we can write $\omega_j - G_j^{-1} = \mathcal{H}_j^d$. Therefore,

$$\begin{aligned}\eta^\dagger - \eta &= \frac{1}{\omega_1 - \omega_0}\left[G_1\mathcal{H}_1^d c^\dagger T - G_1 c^\dagger T\mathcal{H}_0^d - G_0 T^\dagger c\mathcal{H}_1^d + G_0\mathcal{H}_0^d T^\dagger c\right]\\ &= \frac{1}{\omega_1 - \omega_0}\left[G\mathcal{H}^d c^\dagger T - G c^\dagger T\mathcal{H}^d - G T^\dagger c\mathcal{H}^d + G\mathcal{H}^d T^\dagger c\right]\\ &= \frac{1}{\omega_1 - \omega_0} G\left[\mathcal{H}^d, c^\dagger T + T^\dagger c\right]\end{aligned} \tag{4.3.27}$$

where $\mathcal{H}^d = \mathcal{H}_1^d \hat{n} + \mathcal{H}_0^d\left(1 - \hat{n}\right)$ and $G = G_1\hat{n} + G_0\left(1 - \hat{n}\right) = \left(\hat{\omega} - \mathcal{H}^d\right)^{-1}$. For URG, the relevant off-diagonal part of the Hamiltonian for the current node is $\mathcal{H}^I = c^\dagger T + T^\dagger c$. Therefore,

$$\eta^\dagger - \eta = \frac{1}{\omega_1 - \omega_0} G\left[\mathcal{H}^d, \mathcal{H}^I\right] = \left[\mathcal{H}^d, \frac{1}{\omega_1 - \omega_0} G\mathcal{H}^I\right] \tag{4.3.28}$$

The last equality comes about because both G and $\mathcal{H}^d$ are completely diagonal and hence commute. The renormalization in the Hamiltonian under URG, which is a function of both the quantum fluctuation scale ω and the running bandwidth D, can thus be written as

$$\Delta\mathcal{H}(\omega, D) = \left[\left[\mathcal{H}^d, \tilde{\mathcal{H}}^I\right], \mathcal{H}\right] - \mathcal{H}_X \tag{4.3.29}$$

The most obvious difference with the CUT version is the presence of the off-diagonal piece $-\mathcal{H}_X$. This is a result of the philosophical difference between URG and CUT-RG - while CUT-RG gradually suppresses the off-diagonal matrix elements, URG makes the off-diagonal components in each 2×2 block vanish completely. We can instead look at the renormalization in the diagonal part of the Hamiltonian under URG:

$$\Delta\mathcal{H}^0(\omega, D) = \left[\left[\mathcal{H}^d, \tilde{\mathcal{H}}^I\right], \mathcal{H}\right] \tag{4.3.30}$$

where $\tilde{\mathcal{H}}^I = \frac{1}{\omega_1-\omega_0} G\mathcal{H}^I$. This can be compared to the analogous equation for CUT (eq. 4.3.4),

$$\Delta\mathcal{H}(\lambda) = \Delta\lambda \left[\left[\mathcal{H}_d, \mathcal{H}_X\right], \mathcal{H}\right] \tag{4.3.31}$$

Leaving aside the obvious differences in the philosophies (the presence of ω in URG or the fact that while URG decouples as a flow in the bandwidth, CUT uses a general parameter λ or the algorithmic difference that while URG decouples one specific node, CUT tries to make the Hamiltonian band-diagonal), the major physical difference is the presence of the total Green's function in the URG equation. It must be noted that while CUT employs the entire off-diagonal part in $\mathcal{H}_X$, the $\mathcal{H}^I$ of URG contains only those parts that are off-diagonal with respect to the node being decoupled at this step.

To bring the URG form closer to CUT, we can make some approximations.

$$G = \left(\hat{\omega} - \mathcal{H}^d\right)^{-1} \approx -\left(\mathcal{H}^d\right)^{-1} \tag{4.3.32}$$

where we assumed that the quantum fluctuations are small and can be ignored w.r.t the diagonal contribution $\mathcal{H}^d$. This gives

$$\frac{\Delta\mathcal{H}^0(\omega, D)}{\left[\mathcal{H}_1^d\left(\omega_0 - \omega_1\right)\right]^{-1}} = \left[\left[\mathcal{H}^d, \mathcal{H}^I\right], \mathcal{H}\right] \tag{4.3.33}$$

We can thus make the connection,

$$\Delta\lambda \sim \left[\mathcal{H}_1^d\left(\omega_0 - \omega_1\right)\right]^{-1} \tag{4.3.34}$$

Note that in going from eq. 4.3.30 to the simplified form eq. 4.3.33, we had to drop all quantum fluctuations in the denominator and we lose the fixed point structure in the process. This results in the distinction that while URG can reach a fixed point theory in a finite number of steps, CUT cannot do so.

4.4 Comparison of the Canonical Transformations

We have considered three canonical transformations in this section: the Poor Man's scaling (PMS), the Schrieffer-Wolff transformation (SWT) and the continuous unitary transformation renormalization group (CUT-RG). PMS and SWT are more or less identical; they differ in the context in which they are used. PMS is used when there is an entire spectrum of energies in the model, ranging from a low energy limit to a high energy; it is then employed to decouple the highest energy modes in an iterative fashion. SWT is used when the Hamiltonian can be split into one high and one low energy part, and we need to decouple these two modes. It is clear when seen from this perspective that SWT is like a one-shot PMS; it decouples the UV from the IR in one step, compared to the iterative approach of PMS. However, as has been shown in the previous subsections, both

PMS and SWT can be formalized in an identical fashion, so that one can be switched for the other in both contexts.

Now that we have established that PMS and SWT are formally identical, we can relate them to URG. URG is similar in philosophy to PMS in the fact that URG also successively decouples high energy modes from the low energy ones. The difference is that URG takes care of the quantum fluctuations, at least in principle, by introducing a new set of energy scales ω. These ω then parameterise the RG flows of URG, compared to the single RG flow of PMS. Since SWT is formally the same as PMS, these is also a comparison between SWT and URG. Both PMS and SWT trivialize the quantum fluctuation scales of URG by replacing it with the bare energy values.

CUT-RG is philosophically different from the other transformations. Its goal is to gradually reduce the contributions of the off-diagonal terms by making them decay along a certain scale l. Off-diagonal terms that connect states with large energy differences decay faster. In this sense, there is no sequential dropping out of off-diagonal terms; all off-diagonal terms disappear at $l = \infty$. In this sense, it is like a continuous version of SWT. While SWT strives to remove an entire off-diagonal term in one-shot, CUT RG does this gradually by introducing a scale l. This separation of scales is absent in SWT. It does exist in URG and PMS, albeit in a different fashion. There, the separation comes in when we decouple single electron states starting from the Brillouin zone boundary Λ_N and come down to the Fermi surface Λ_0. Each Λ provides a natural energy scale for separating the high and low energy physics.

If one integrates the continuum generator η over all the scales, one should recover something analogous to the SWT generator. This generator is however necessarily perturbative in the off-diagonal term, since by definition η will only be of first order in $\mathcal{H}_X$. This is in contrast to the non-perturbative generators in URG and, at least in principle, PMS. This non-perturbative form is encapsulated in the presence of a second completely different set of energy scales ω (or E in PMS). This second energy scale is absent in CUT RG because it takes care of at most the second order term.

Another point to note is that since SWT keeps the entire off-diagonal piece in the generator, new terms will almost certainly by generated at every step, and they have to be truncated. This is not the case with URG or PMS, becasue in those methods, we decouple just a single-electron at each step, and so those electrons become integrals of motion at that step, leading to their removal from the off-diagonal piece, and very often the Hamiltonian retains the same form as the bare model. This makes the philosophy of URG and PMS easier to work with. Tied to this is the fact that CUT RG often takes a certain type of interaction in the generator part, and not the entire off-diagonal piece. Hence, at the limit of the flow parameter going to ∞, the chosen off-diagonal piece goes to zero but the remaining off-diagonal pieces still remain. As a result, the Hamiltonian is atmost block diagonal at this stage. On the other hand, URG progressively decouples single electrons, meaning all scattering terms corresponding to that electron vanish at each step.

One last thing to note is that URG, being unitarily implemented with a well-defined generator, does not accomodate for spontaneous symmetry breaking (SSB). In order to see SSB, the symmetry-breaking term has to be added to the bare model explicitly; if this term grows under the RG, then the ground state will be symmetry-broken. CUT RG, however, does allow for SSB through the idea that the generator is not uniquely defined. If the generator commutes with a particular symmetry, then the family of Hamiltonians will also have the symmetry [30]. However, if the generator is replaced by something that is normal ordered w.r.t a particular expectation value, then the CUT RG flow will usually be towards either the symmetry-preserved state or the symmetry-broken state, depending on whichever is stable [32].

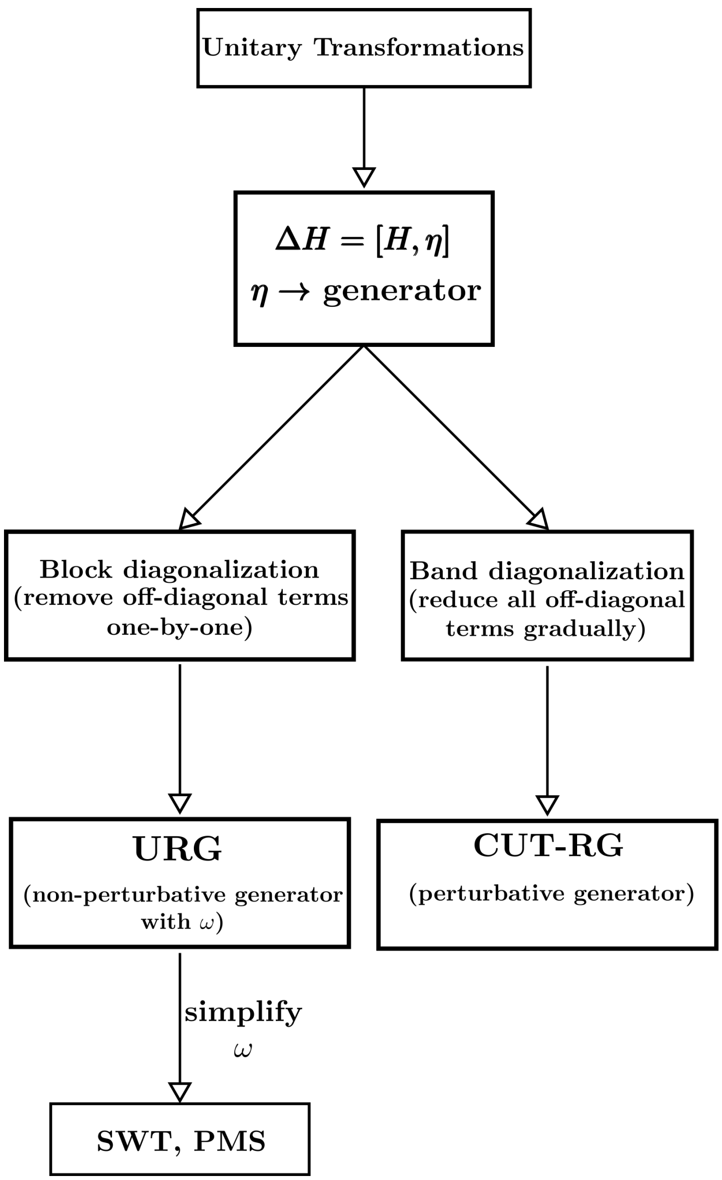

Figure 4.3: Comparison of the various unitary transformations, depicting how each of them are related to one another.

Chapter 5

URG of the SIAM and its Spin and Charge Generalizations

5.1 URG of the SIAM

5.1.1 Setting up the various terms in the URG

The model is the usual single-impurity Anderson model Hamiltonian.

$$\mathcal{H} = \sum_{k\sigma} \epsilon_k \hat{n}_{k\sigma} + \sum_{k\sigma} \left(V_k c^\dagger_{k\sigma} c_{d\sigma} + h.c. \right) + \epsilon_d \sum_\sigma \hat{n}_{d\sigma} + U \hat{n}_{d\uparrow} \hat{n}_{d\downarrow} \tag{5.1.1}$$

To allow the calculation of both particle and hole kinetic energies, we will write the kinetic energy part as $\sum_{k\sigma} \epsilon_k \tau_{k\sigma}$, where $\tau = \hat{n} - \frac{1}{2}$ and drop the extra constant part.

$$\mathcal{H} = \sum_{k\sigma} \epsilon_k \tau_{k\sigma} + \sum_{k\sigma} \left(V_k c^\dagger_{k\sigma} c_{d\sigma} + h.c. \right) + \epsilon_d \sum_\sigma \hat{n}_{d\sigma} + U \hat{n}_{d\uparrow} \hat{n}_{d\downarrow} \tag{5.1.2}$$

The renormalization is

$$c^\dagger_{q\beta} T \eta + T^\dagger c_{q\beta} \eta^\dagger_0 \tag{5.1.3}$$

We will be decoing an electron $q\beta$ at the energy shell ϵ_q. The diagonal part (that comes down in the denominator) is

$$\mathcal{H}_d = \epsilon_q \tau_{q\beta} + \epsilon_d \sum_\sigma \hat{n}_{d\sigma} + U \hat{n}_{d\uparrow} \hat{n}_{d\downarrow} \tag{5.1.4}$$

Since this is the first step of the RG, the shell being decoupled is the highest one, which we call Λ_N.

5.1.2 Calculation of renormalization

The η is

$$\begin{aligned} \eta = \frac{1}{\hat{\omega} - \mathcal{H}_d} T^\dagger c_{q\beta} &= \frac{1}{\hat{\omega} - \epsilon_q \tau_{q\beta} - \epsilon_d \sum_\sigma \hat{n}_{d\sigma} - U \hat{n}_{d\uparrow} \hat{n}_{d\downarrow}} V^*_q c^\dagger_{d\beta} c_{q\beta} \\ &= \frac{1}{\hat{\omega} + \frac{1}{2}\epsilon_q - \epsilon_d - \left(\epsilon_d + U \right) \hat{n}_{d\overline{\beta}}} V^*_q c^\dagger_{d\beta} c_{q\beta} \end{aligned} \tag{5.1.5}$$

We substituted $\tau_{q\beta} = -\frac{1}{2}$ and $\hat{n}_{d\beta} = 1$ because of the off-diagonal terms in the numerator. The renormalization in the particle sector ($\hat{n}_{q\beta} = \hat{n}_{q\overline{\beta}} = 1$), at an energy $-\epsilon_q$ is

$$\Delta\mathcal{H} = c^\dagger_{q\beta} T\eta = \frac{|V_q|^2}{\hat{\omega} - \frac{1}{2}\epsilon_q - \epsilon_d - \left(\epsilon_d + U\right)\hat{n}_{d\overline{\beta}}} \left(1 - \hat{n}_{d\beta}\right)\hat{n}_{q\beta} \tag{5.1.6}$$

Similarly, the renormalization in the hole sector ($\hat{n}_{q\beta} = \hat{n}_{q\overline{\beta}} = 0$), at an energy ϵ_q is:

$$\Delta\mathcal{H} = T^\dagger c\eta^\dagger_0 = \frac{|V_q|^2}{\hat{\omega} - \frac{1}{2}\epsilon_q - \epsilon_d\hat{n}_{d\overline{\beta}}} \left(1 - \hat{n}_{q\beta}\right)\hat{n}_{d\beta} \tag{5.1.7}$$

Therefore, the total renormalization obtained from decoupling a particle state ($\hat{n}_{q\beta} = 1$) at $-\epsilon_q$ and a hole state ($\hat{n}_{q\beta} = 0$) at ϵ_q is

$$\begin{aligned}\Delta\mathcal{H} &= \frac{|V_q|^2\left(1 - \hat{n}_{d\beta}\right)}{\hat{\omega} - \frac{1}{2}\epsilon_q - \epsilon_d - \left(\epsilon_d + U\right)\hat{n}_{d\overline{\beta}}} + \frac{|V_q|^2\hat{n}_{d\beta}}{\hat{\omega} - \frac{1}{2}\epsilon_q - \epsilon_d\hat{n}_{d\overline{\beta}}} \\ &= |V_q|^2\left[\frac{\left(1 - \hat{n}_{d\beta}\right)\hat{n}_{d\overline{\beta}}}{\omega_1 - \frac{1}{2}\epsilon_q - 2\epsilon_d - U} + \frac{\left(1 - \hat{n}_{d\beta}\right)\left(1 - \hat{n}_{d\overline{\beta}}\right)}{\omega_2 - \frac{1}{2}\epsilon_q - \epsilon_d} + \frac{\hat{n}_{d\beta}\hat{n}_{d\overline{\beta}}}{\omega_3 - \frac{1}{2}\epsilon_q - \epsilon_d} + \frac{\hat{n}_{d\beta}\left(1 - \hat{n}_{d\overline{\beta}}\right)}{\omega_4 - \frac{1}{2}\epsilon_q}\right]\end{aligned} \tag{5.1.8}$$

To relate the ω_i, we look at the their values obtained by substituting the energies of the initial states:

$$\begin{aligned}\omega_1 = \omega_4 &= -\frac{1}{2}\epsilon_q + \epsilon_d \\ \omega_2 &= -\frac{1}{2}\epsilon_q \\ \omega_3 &= -\frac{1}{2}\epsilon_q + 2\epsilon_d + U\end{aligned} \tag{5.1.9}$$

Defining $\omega \equiv \omega_2$ (because ω_2 has the simplest form), we can write

$$\begin{aligned}\omega_1 = \omega_4 &= \omega + \epsilon_d \\ \omega_2 &= \omega \\ \omega_3 &= \omega + 2\epsilon_d + U\end{aligned} \tag{5.1.10}$$

The renormalization then becomes

$$\Delta\mathcal{H} = |V_q|^2\left[\frac{\left(1 - \hat{n}_{d\beta}\right)\hat{n}_{d\overline{\beta}}}{\omega - \frac{1}{2}\epsilon_q - \epsilon_d - U} + \frac{\left(1 - \hat{n}_{d\beta}\right)\left(1 - \hat{n}_{d\overline{\beta}}\right)}{\omega - \frac{1}{2}\epsilon_q - \epsilon_d} + \frac{\hat{n}_{d\beta}\hat{n}_{d\overline{\beta}}}{\omega - \frac{1}{2}\epsilon_q + \epsilon_d + U} + \frac{\hat{n}_{d\beta}\left(1 - \hat{n}_{d\overline{\beta}}\right)}{\omega - \frac{1}{2}\epsilon_q + \epsilon_d}\right] \tag{5.1.11}$$

5.1.3 Scaling equations

Once we have the renormalization for decoupling one electronic or hole state, we can just sum over the spins and momenta to get the total renormalization upon decoupling the entire shells $\pm\epsilon_q$. From the structure of $\Delta\mathcal{H}$ in eq. 5.1.11, we can see that there are renormalizations to all three configuration energies of the impurity: the doublon energy E_2 corresponding to the state $\hat{n}_{d\beta}\hat{n}_{d\overline{\beta}}$, the single energy E_1 corresponding to $(\hat{n}_{d\beta}(1-\hat{n}_{d\overline{\beta}})+\hat{n}_{d\overline{\beta}}(1-\hat{n}_{d\beta})$, and the holon energy E_0 corresponding to $(1-\hat{n}_{d\beta})(1-\hat{n}_{d\overline{\beta}})+\hat{n}_{d\overline{\beta}}(1-\hat{n}_{d\beta})$.

$$\begin{aligned}
\Delta E_2 &= +2\sum_q \frac{|V_q|^2}{\omega-\frac{1}{2}\epsilon_q+\epsilon_d+U}\\
\Delta E_1 &= +\sum_q \frac{|V_q|^2}{\omega-\frac{1}{2}\epsilon_q+\epsilon_d}+\sum_q \frac{|V_q|^2}{\omega-\frac{1}{2}\epsilon_q-\epsilon_d-U}\\
\Delta E_0 &= +2\sum_q \frac{|V_q|^2}{\omega-\frac{1}{2}\epsilon_q-\epsilon_d}
\end{aligned} \tag{5.1.12}$$

Using the relations $\epsilon_d = E_1 - E_0$ and $U = E_2 + E_0 - 2E_1$, we can write

$$\begin{aligned}
\Delta\epsilon_d &= +\sum_q \frac{|V_q|^2}{\omega-\frac{1}{2}\epsilon_q+\epsilon_d}+\sum_q \frac{|V_q|^2}{\omega-\frac{1}{2}\epsilon_q-\epsilon_d-U}-2\sum_q \frac{|V_q|^2}{\omega-\frac{1}{2}\epsilon_q-\epsilon_d}\\
\Delta U &= \sum_q \frac{2|V_q|^2}{\omega-\frac{1}{2}\epsilon_q+\epsilon_d+U}+\sum_q \frac{2|V_q|^2}{\omega-\frac{1}{2}\epsilon_q-\epsilon_d}-\sum_q \frac{2|V_q|^2}{\omega-\frac{1}{2}\epsilon_q+\epsilon_d}-\sum_q \frac{2|V_q|^2}{\omega-\frac{1}{2}\epsilon_q-\epsilon_d-U}
\end{aligned} \tag{5.1.13}$$

5.1.4 Connection to Poor Man's scaling

To obtain the results of Poor Man's scaling [26][33], we can look at various regimes. First we look at the case when both ϵ_d and U are small such that both the singly-occupied and doubly-occupied subspaces of the impurity are comfortably inside the bandwidth, $U, \epsilon_d \ll \epsilon_q$. We can then ignore the ϵ_d and U in the denominator compared to the ϵ_q.

$$\begin{aligned}
\Delta\epsilon_d &= \sum_q \frac{|V_q|^2}{\omega-\frac{1}{2}\epsilon_q}+\sum_q \frac{|V_q|^2}{\omega-\frac{1}{2}\epsilon_q}-2\sum_q \frac{|V_q|^2}{\omega-\frac{1}{2}\epsilon_q}\\
\Delta U &= 2\sum_q \frac{|V_q|^2}{\omega-\frac{1}{2}\epsilon_q}+2\sum_q \frac{|V_q|^2}{\omega-\frac{1}{2}\epsilon_q}-2\sum_q \frac{|V_q|^2}{\omega-\frac{1}{2}\epsilon_q}-2\sum_q \frac{|V_q|^2}{\omega-\frac{1}{2}\epsilon_q}
\end{aligned} \tag{5.1.14}$$

Assuming the upper and lower band edges are symmetrical such that $\sum_{-D} = \sum_D$, we get $\Delta\epsilon_d = \Delta U = 0$.

In the regime $U \gg \epsilon_q \gg \epsilon_d$, the doubly-occupied state is far above the bandwidth. We can now ignore the terms that have U in the denominator. We get

$$\begin{aligned}
\Delta\epsilon_d &= \sum_q \frac{|V_q|^2}{\omega-\frac{1}{2}\epsilon_q}-2\sum_q \frac{|V_q|^2}{\omega-\frac{1}{2}\epsilon_q}\\
\Delta U &= 2\sum_q \frac{|V_q|^2}{\omega-\frac{1}{2}\epsilon_q}-2\sum_q \frac{|V_q|^2}{\omega-\frac{1}{2}\epsilon_q}
\end{aligned} \tag{5.1.15}$$

Again assuming symmetrical upper and lower edges, and isotropic dispersion $\epsilon_q = D$ and $\sum_q |V|^2 = \frac{\Delta}{\pi}|\delta D|$, we get

$$\Delta U = 0$$
$$\delta\epsilon_d = -\frac{\Delta}{\pi}\frac{1}{\omega - \frac{1}{2}D} \tag{5.1.16}$$

There we replaced the difference symbol Δ with δ to avoid confusion with the hybridisation $\Delta \sim \sum V^2$. For low quantum fluctuations, we can ignore the renormalization in the couplings and replace ω with the initial conduction electron energy: $\omega = \epsilon_q \tau_{q\beta} = -\frac{1}{2}D$.

$$\delta\epsilon_d = \frac{\Delta}{\pi}\frac{\delta D}{D} \tag{5.1.17}$$

This is the one-loop scaling equation.

5.1.5 Particle-Hole symmetry

The Anderson model Hamiltonian, eq. 5.2.7, has an impurity particle-hole symmetry for a certain condition of the couplings. To see this, we apply the particle-hole transformation $c_k \to c_k^\dagger, c_d \to -c_d^\dagger$ to the Hamiltonian. Since we are looking at the impurity symmetry, we will only look at the terms involving the impurity. The particle-hole symmetry of the conduction bath is a separate thing and that requires a specific lattice. Hence we will not consider kinetic energy term in this discussion. The rest of the terms transform as

$$\epsilon_d \sum_\sigma \hat{n}_{d\sigma} \to 2\epsilon_d - \epsilon_d \sum_\sigma \hat{n}_{d\sigma} \tag{5.1.18}$$
$$U\hat{n}_{d\uparrow}\hat{n}_{d\downarrow} \to U\hat{n}_{d\uparrow}\hat{n}_{d\downarrow} - U\sum_\sigma \hat{n}_{d\sigma} + U \tag{5.1.19}$$
$$\sum_{k\sigma} V(k)c_{k\sigma}^\dagger c_{d\sigma} + hc \to \sum_{k\sigma} -V(k)c_{k\sigma}c_{d\sigma}^\dagger + hc = \sum_{k\sigma} V^*(k)c_{k\sigma}^\dagger c_{d\sigma} + hc \tag{5.1.20}$$
$$S^z \sum_{kq} s_{kq}^z \to \left(-S^z\right)\sum_{kq}\left(-s_{kq}^z\right) = S^z \sum_{kq} s_{kq}^z \tag{5.1.21}$$
$$S^\pm \sum_{kq} s_{kq}^\mp \to \left(-S^\pm\right)\sum_{kq}\left(-s_{kq}^\mp\right) = S^\pm \sum_{kq} s_{kq}^\mp \tag{5.1.22}$$

The transformation of the spin terms, eqs. 5.1.21 and 5.1.22, can be understood from the fact that since a spin degree of freedom can be written in terms of the number operator as $\hat{S} = \hat{n} - \frac{1}{2}$, it must transform by flipping its sign: $\hat{S} = \hat{n} - \frac{1}{2} \to \frac{1}{2} - \hat{n} = -\hat{S}$. The spin terms are thus invariant under the particle-hole transformation. The impurity-bath hopping term can be made symmetric by making $V(k)$ real; then we would have, from eq. 5.1.20,

$$V(k)\left(c_{k\sigma}^\dagger c_{d\sigma} + c_{d\sigma}^\dagger c_{k\sigma}\right) \to V(k)\left(c_{d\sigma}^\dagger c_{k\sigma} + c_{k\sigma}^\dagger c_{d\sigma}\right) \tag{5.1.23}$$

The impurity diagonal terms, ϵ_d and U, require a specific condition. Combining eqs. 5.1.18 and 5.1.19,

$$\epsilon_d \hat{n}_{d\sigma} + U\hat{n}_{d\uparrow}\hat{n}_{d\downarrow} \to \left(-\epsilon_d - U\right)\hat{n}_{d\sigma} + U\hat{n}_{d\uparrow}\hat{n}_{d\downarrow} \tag{5.1.24}$$

We dropped some constant terms in the transformed Hamiltonian. For particle-hole symmetry, the left and right hand sides must be same. The required condition is thus

$$\epsilon_d = -\epsilon_d - U \implies \epsilon_d + \frac{1}{2}U = 0 \tag{5.1.25}$$

This same condition can be obtained in a more physical way. If we consider the singly-occupied state of the impurity as the reference state, the doubly-occupied state is the particle-excitation and the vacant state is the hole excitation. The energy of this particle state is $E_2 = 2\epsilon_d + U$ and that of the hole state is $E_0 = 0$. Particle-hole symmetry then requires the particle and hole levels to be degenerate, which means $E_2 = E_0$, and we recover the condition eq. 5.1.25.

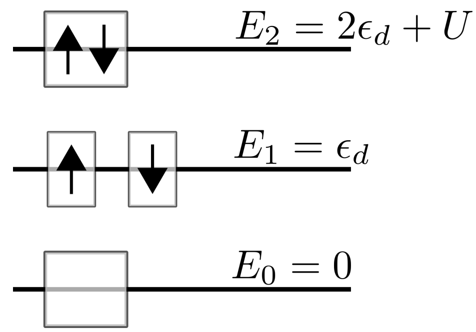

Figure 5.1: Particle and hole excitations of the impurity

Since the URG is unitary, if we start from a model that is particle-hole symmetric, the RG equations should uphold that symmetry. What this means is that if we have $\epsilon_d + \frac{1}{2}U = 0$ in the bare model, the new couplings should also satisfy $\epsilon_d' + \frac{1}{2}U' = 0$. This means we must have

$$\Delta\left(\epsilon_d + \frac{1}{2}U\right) = 0 \tag{5.1.26}$$

The quantity $\gamma = \epsilon_d + \frac{1}{2}U$ is thus an RG-invariant for the particle-hole symmetric model; it does not change under the RG flow. It is often referred to as the asymmetry parameter; it quantifies the asymmetry in the model. We need to check if our equations satisfy this. Looking at the RG equations for ϵ_d and U, we can find the RG equation for the asymmetry parameter. The slightly easier way is to just note that the renormalization in E_2 should be equal to the renormalization in E_0, in order for p-h symmetry to hold.

$$\Delta E_2 = 2\frac{\Delta}{\pi}\frac{1}{\omega - D + \epsilon_d + U}, \Delta E_0 \quad = 2\frac{\Delta}{\pi}\frac{1}{\omega - D - \epsilon_d} \tag{5.1.27}$$

If we start with a particle-hole symmetric model, we will have $-\epsilon_d = \epsilon_d + U$. Substituting that gives $\Delta E_2 = \Delta E_0$. This shows that the doublon and holon states remain equidistant from the single-particle level, thus maintaining particle-hole symmetry along the flow.

5.1.6 Numerical analysis of symmetric SIAM

We will specialize to the particle-hole symmetric case, $2\epsilon_d + U = 0$, and a symmetric energy shell $\epsilon_q = D$, and look at the scaling behavior of ϵ_d.

$$\Delta\epsilon_d = -4|V|^2 \frac{\epsilon_d}{\left(\omega - \frac{1}{2}D\right)^2 - \epsilon_d^2} \tag{5.1.28}$$

Since the equation is symmetric under $\epsilon_d \to -\epsilon_d$, we might as well work with the magnitude of the onsite energy:

$$\Delta|\epsilon_d| = -4|V|^2 \frac{|\epsilon_d|}{\left(\omega - \frac{1}{2}D\right)^2 - \epsilon_d^2} \tag{5.1.29}$$

Depending on the signature of the denominator, the flows will be either relevant or irrelevant. For the flow to the local moment fixed point, the fixed point value of $|\epsilon_d|$ grows

Figure 5.2: Left: Irrelevant flow towards $|\epsilon_d| = 0$, at low ω. Right: Relevant flow towards large $|\epsilon_d|$, at large ω. The former can be thought of as the projection of the strong-coupling flow on to the $\epsilon_d - D$ plane. The latter is the flow towards the local moment fixed point, if we start from a negative ϵ_d.

as we increase the bandwidth. This implies that for a thermodynamically large system, the local moment fixed point will be at $-\epsilon_d \to \infty$. This behavior is shown in fig. 5.3.

5.2 Anderson-Kondo (spin) model URG

In order to obtain a renormalization in V, we will introduce a spin-spin interaction between the impurity and the mobile electrons. Such terms are generated when one does a Schrieffer-Wolff transformation on the SIAM, but we will find it prudent to keep these terms in the bare model itself.

5.2.1 Spin-spin interaction

We first consider a general four-Fermion interaction of the form

$$\mathcal{H}_I = \sum_{k,k',\sigma_i} u c^\dagger_{d\sigma_2} c_{d\sigma_4} c_{k'\sigma_3} c^\dagger_{k\sigma_1} \delta_{(\sigma_1+\sigma_2=\sigma_3+\sigma_4)} \tag{5.2.1}$$

Figure 5.3: Change in fixed point value of $|\epsilon_d|$ with system size.

The u in general depends on the spin and the momenta. Expanding the summation by using the delta gives

$$\mathcal{H}_I = \underbrace{\sum_{k,k',\sigma,\sigma'} u_1 \hat{n}_{d\sigma'} c^\dagger_{k\sigma} c_{k'\sigma}}_{\text{spin-preserving scattering}} + \overbrace{\sum_{k,k',\sigma} u_2 c^\dagger_{d\overline{\sigma}} c_{d\sigma} c^\dagger_{k\sigma} c_{k'\overline{\sigma}}}^{\text{spin-flip scattering}} \tag{5.2.2}$$

At this point, we drop the dependence of u on the momenta and assume it depends only on the spin transfer. The first term (attached with u_1) involves no spin-flip between the scattering momenta or the scattering impurity electrons ($k\sigma \to k'\sigma, d\sigma' \to d\sigma'$). We label this coupling as u_P. The other coupling involves a spin-flip scattering, so we label that as u_A.

$$\mathcal{H}_{I,N} = \sum_{k,k',\sigma,\sigma'} u_P \hat{n}_{d\sigma'} c^\dagger_{k\sigma} c_{k'\sigma} + \sum_{k,k',\sigma} u_A c^\dagger_{d\overline{\sigma}} c_{d\sigma} c^\dagger_{k\sigma} c_{k'\overline{\sigma}} \tag{5.2.3}$$

where the N in the denominator means the sum is over all momenta up to $|k| = \Lambda_N$. The parallel scattering has two components, when expanded, is of the form

$$u_{\uparrow\uparrow} \hat{n}_{d\uparrow} c^\dagger_{k\uparrow} c_{k'\uparrow} + u_{\downarrow\downarrow} \hat{n}_{d\downarrow} c^\dagger_{k\downarrow} c_{k'\uparrow} + u_{\uparrow\downarrow} \hat{n}_{d\uparrow} c^\dagger_{k\downarrow} c_{k'\downarrow} + u_{\downarrow\uparrow} \hat{n}_{d\downarrow} c^\dagger_{k\uparrow} c_{k'\uparrow} \tag{5.2.4}$$

We define J_z and J_t such that this term can be written as

$$\begin{aligned} \mathcal{H}_I &= J_z \frac{\hat{n}_{d\uparrow} - \hat{n}_{d\downarrow}}{2} \sum_{kk'} \left(c^\dagger_{k\uparrow} c_{k'\uparrow} - c^\dagger_{k\downarrow} c_{k'\downarrow} \right) + J_t \sum_{kk'} \left[c^\dagger_{d\uparrow} c_{d\downarrow} c^\dagger_{k\downarrow} c_{k'\uparrow} + c^\dagger_{d\downarrow} c_{d\uparrow} c^\dagger_{k\uparrow} c_{k'\downarrow} \right] \\ &= 2 J_z S^z_d s^z + J_t \left(S^+_d s^- + S^-_d s^+ \right) \end{aligned} \tag{5.2.5}$$

The spin-like operators are defined as

$$\begin{aligned} S_d^z &\equiv \frac{1}{2}\left(\hat{n}_{d\uparrow} - \hat{n}_{d\downarrow}\right) & S_d^+ &\equiv c^\dagger_{d\uparrow}c_{d\downarrow} & S_d^- &\equiv c^\dagger_{d\downarrow}c_{d\uparrow} \\ s_{kk'}^z &\equiv \frac{1}{2}\left(c^\dagger_{k\uparrow}c_{k'\uparrow} - c^\dagger_{k\downarrow}c_{k'\downarrow}\right) & s_{kk'}^+ &\equiv c^\dagger_{k\uparrow}c_{k'\downarrow} & s_{kk'}^- &\equiv c^\dagger_{k\downarrow}c_{k'\uparrow} \\ && s^a \equiv \sum_{kk'} s^a_{kk'} \end{aligned} \tag{5.2.6}$$

This is the same interaction that constitutes the Kondo model and gives rise to the quenching of the local moment at low energies. The total Hamiltonian for this *Anderson-Kondo model* is thus

$$\mathcal{H} = \sum_{k\sigma}\left(\epsilon_k \hat{n}_{k\sigma} + V_k c^\dagger_{k\sigma}c_{d\sigma} + h.c.\right) + \epsilon_d \sum_\sigma \hat{n}_{d\sigma} + U\hat{n}_{d\uparrow}\hat{n}_{d\downarrow} + 2J_z S_d^z s^z + J_t\left(S_d^+ s^- + S_d^- s^+\right) \tag{5.2.7}$$

For the special case of $2J_z = 2J_t = J$, we get the SU(2) symmetric Heisenberg-like interaction

$$\mathcal{H}_I = J\left[S_d^z s^z + \frac{1}{2}\left(S_d^+ s^- + S_d^- s^+\right)\right] = J\mathbf{S_d}\cdot\mathbf{s} \tag{5.2.8}$$

For the URG, we take two electrons on the shell Λ_N, $q\beta$ and $q\overline{\beta}$, then decouple the electron $q\beta$. The reason for taking two electrons is to allow the symmetries to be preserved. For simplicity, we will only consider those diagonal terms in the denominator that either have both $q\beta$ and $q\overline{\beta}$ or both $q\beta$ and d or both $q\overline{\beta}$ and d. Terms that have purely $q\overline{\beta}$ will not be considered. Also, the scattering between just d and $q\overline{\beta}$ can be ignored since it is diagonal in $q\beta$. The Hamiltonian for such a system is

$$\begin{aligned} \mathcal{H}_N &= H_{N-1} + H_{\text{imp}} + \epsilon_q \hat{n}_{q\beta} + 2J_z S_d^z s_q^z + V_q c^\dagger_{q\beta}c_{d\beta} + \text{h.c.} + \\ &\sum_{k<\Lambda_N}\left[J_z S_d^z \beta\left(c^\dagger_{k\beta}c_{q\beta} + c^\dagger_{q\beta}c_{k\beta}\right) + J_t\left(c^\dagger_{d\beta}c_{d\overline{\beta}}c^\dagger_{k\overline{\beta}}c_{q\beta} + c^\dagger_{d\overline{\beta}}c_{d\beta}c^\dagger_{q\beta}c_{k\overline{\beta}}\right)\right] \\ &+ J_t\left(c^\dagger_{d\beta}c_{d\overline{\beta}}c^\dagger_{q\overline{\beta}}c_{q\beta} + c^\dagger_{d\overline{\beta}}c_{d\beta}c^\dagger_{q\beta}c_{q\overline{\beta}}\right) \end{aligned} \tag{5.2.9}$$

where $s_q^z = \frac{1}{2}\left(\hat{n}_{q\uparrow} - \hat{n}_{q\downarrow}\right)$ and H_{imp} is the impurity-diagonal part of the Hamiltonian ($\epsilon_d\hat{n}_d + U\hat{n}_{d\uparrow}\hat{n}_{d\downarrow}$) and

$$H_{N-1} = \sum_{k<\Lambda_N,\sigma}\left[\left(\epsilon_k + \sigma J_z S_d^z\right)\hat{n}_{k\sigma} + V_k c^\dagger_{k\sigma}c_{d\sigma} + \text{h.c.}\right] + H_{I,N-1} \tag{5.2.10}$$

The diagonal (number-preserving) part is

$$\mathcal{H}_D = H^D_{N-1} + \epsilon_q\left(\hat{n}_{q\beta} + \hat{n}_{q\overline{\beta}}\right) + 2J_z S_d^z s_q^z + H_{imp} \tag{5.2.11}$$

In line with the simplifications mentioned above, we will work with the following terms:

$$\mathcal{H}_D = \epsilon_q \hat{n}_{q\beta} + 2J_z S_d^z s_q^z + H_{imp} \tag{5.2.12}$$

To allow the calculation of hole and particle energies on an equal footing, we will make a transformation at the bare model itself:

$$\sum_{k\sigma} \epsilon_k \hat{n}_{k\sigma} = \sum_{k\sigma} \epsilon_k \hat{\tau}_{k\sigma} + C \tag{5.2.13}$$

where $\tau \equiv \hat{n} - \frac{1}{2}$ and C is non-dynamic and will hence be dropped. This transforms the diagonal part $\mathcal{H}^D$. Eq. 5.2.12 becomes

$$\mathcal{H}_D = \epsilon_q \tau_{q\beta} + 2 J_z S_d^z s_q^z + H_{imp} \tag{5.2.14}$$

The entire off-diagonal piece can be split into 6 parts:

$$\begin{aligned}\mathcal{H}_X = &\underbrace{V_1^* c^\dagger_{d\beta} c_{q\beta} \hat{n}_{d\overline{\beta}}}_{T_1^\dagger c_{q\beta}} + \overbrace{V_0^* c^\dagger_{d\beta} c_{q\beta}\left(1 - \hat{n}_{d\overline{\beta}}\right)}^{T_2^\dagger c_{q\beta}} + \underbrace{\sum_{k<\Lambda_N} J_0^z \hat{n}_{d\beta}\left(1 - \hat{n}_{d\overline{\beta}}\right) c^\dagger_{k\beta} c_{q\beta}}_{T_3^\dagger c_{q\beta}} \\ &- \overbrace{\sum_{k<\Lambda_N} J_1^z \hat{n}_{d\overline{\beta}}\left(1 - \hat{n}_{d\beta}\right) c^\dagger_{k\beta} c_{q\beta}}^{T_4^\dagger c_{q\beta}} + \underbrace{\sum_{k<\Lambda_N} J^t c^\dagger_{d\beta} c_{d\overline{\beta}} c^\dagger_{k\overline{\beta}} c_{q\beta}}_{T_5^\dagger c_{q\beta}} + \overbrace{J^t c^\dagger_{d\beta} c_{d\overline{\beta}} c^\dagger_{q\overline{\beta}} c_{q\beta}}^{T_6^\dagger c_{q\beta}} + \text{h.c.}\end{aligned} \tag{5.2.15}$$

The various parts of the off-diagonal piece are

$$\begin{aligned} T_1 &= V_1 c_{d\beta} \hat{n}_{d\overline{\beta}} \\ T_2 &= V_0 c_{d\beta}\left(1 - \hat{n}_{d\overline{\beta}}\right) \\ T_3 &= \sum_{k<\Lambda_N} J_0^z \hat{n}_{d\beta}\left(1 - \hat{n}_{d\overline{\beta}}\right) c_{k\beta} \\ T_4 &= -\sum_{k<\Lambda_N} J_1^z \hat{n}_{d\overline{\beta}}\left(1 - \hat{n}_{d\beta}\right) c_{k\beta} \\ T_5 &= \sum_{k<\Lambda_N} J^t c^\dagger_{d\overline{\beta}} c_{d\beta} c_{k\overline{\beta}} \end{aligned} \tag{5.2.16}$$

5.2.2 Calculation of renormalization in particle sector

We will first look at the renormalization of ϵ_d, U and the interaction couplings for the lower shell electrons, so we can ignore T_6 for the time-being. The renormalization in the particle sector ($\hat{n}_{q\beta} = 1$) is of the form

$$c^\dagger_{q\beta} T \eta \tag{5.2.17}$$

Since we have $\hat{n}_{q\beta} = 1$ in the initial state, this will be at energy $-\epsilon_q$. The generator η will have five parts:

$$\eta = \sum_{i=1}^{5} \frac{1}{\omega_i - E_i^0} T_i^\dagger c_{q\beta} \tag{5.2.18}$$

We need to compute a quantity of the form

$$\begin{aligned}
&\sum_{ij} \left(\frac{1}{\omega_i - E_i^0} + \frac{1}{\omega - E_j^0} \right) T_i T_j^\dagger \\
&= \frac{1}{\omega_1 - E_1^0} T_1 T_1^\dagger + \frac{1}{\omega_2 - E_2^0} T_2 T_2^\dagger + \frac{1}{\omega_3 - E_3^0} T_3 T_3^\dagger + \frac{1}{\omega_4 - E_4^0} T_4 T_4^\dagger + \frac{1}{\omega_5 - E_5^0} T_5 T_5^\dagger \\
&+ \left(\frac{1}{\omega_2 - E_2^0} + \frac{1}{\omega_5 - E_5^0} \right) \left(T_2 T_5^\dagger + T_5 T_2^\dagger \right) + \left(\frac{1}{\omega_2 - E_2^0} + \frac{1}{\omega_3 - E_3^0} \right) \left(T_2 T_3^\dagger + T_3 T_2^\dagger \right) \\
&+ \left(\frac{1}{\omega_3 - E_3^0} + \frac{1}{\omega_5 - E_5^0} \right) \left(T_3 T_5^\dagger + T_5 T_3^\dagger \right)
\end{aligned}$$

The diagonal parts E_i^0 are

$$\begin{aligned}
E_1^0 &= \frac{\epsilon_q}{2} + 2\epsilon_d + U \\
E_2^0 = E_5^0 = E_4^0 &= \frac{\epsilon_q}{2} + \epsilon_d - \frac{J_z}{2} \\
E_3^0 &= \frac{\epsilon_q}{2} + \epsilon_d + \frac{J_z}{2}
\end{aligned} \tag{5.2.19}$$

Note that in writing these diagonal parts, we have considered the effect of $c_k^\dagger$ on the J_z part. For example, if there is a $\hat{n}_{d\beta} \left(1 - \hat{n}_{d\overline{\beta}}\right) c_{k\beta}^\dagger$ in front of the propagator, that means the intermediate state has $\hat{n}_{d\beta} - \hat{n}_{d\overline{\beta}} = 1 = \hat{n}_{k\beta}$ and that will contribute a term $\frac{J_z}{2} \left(\hat{n}_{d\beta} - \hat{n}_{d\overline{\beta}}\right) \hat{n}_{k\beta} = \frac{J_z}{2}$ to E^0. Also, while calculating E_5, we have ignored the presence of $\hat{n}_{q\overline{\beta}}$, because it violates the spin reversal symmetry for this term.

Defining $\xi_i \equiv \omega_i - E_i^0$ and evaluating the terms $T_i T_j^\dagger$ gives

$$\begin{aligned}
&\sum_{ij}\left(\frac{1}{\omega_i - E_i^0} + \frac{1}{\omega - E_j^0}\right) T_i T_j^\dagger \\
&= \frac{|V_1|^2}{\xi_1}\left(1-\hat{n}_{d\beta}\right)\hat{n}_{d\overline{\beta}} + \frac{|V_0|^2}{\xi_2}\left(1-\hat{n}_{d\beta}\right)\left(1-\hat{n}_{d\overline{\beta}}\right) \\
&+ \frac{1}{4}\left[{J_0^z}^2 \frac{\hat{n}_{d\beta}\left(1-\hat{n}_{d\overline{\beta}}\right)}{\xi_3} + {J_1^z}^2 \frac{\hat{n}_{d\overline{\beta}}\left(1-\hat{n}_{d\beta}\right)}{\xi_4}\right] c_{k'\beta} c^\dagger_{k\beta} + \frac{1}{\xi_5} J_t^2 \left(1-\hat{n}_{d\beta}\right)\hat{n}_{d\overline{\beta}} c_{k'\overline{\beta}} c^\dagger_{k\overline{\beta}} \\
&- \frac{1}{2}\left(\frac{1}{\xi_2} + \frac{1}{\xi_5}\right) J_t \left(1-\hat{n}_{d\beta}\right)\left(V_0 c^\dagger_{k\overline{\beta}} c_{d\overline{\beta}} + \text{h.c.}\right) - \frac{1}{2}\left(\frac{1}{\xi_2} + \frac{1}{\xi_3}\right)\frac{J_0^z}{2}\left(1-\hat{n}_{d\overline{\beta}}\right)\left(V_0 c^\dagger_{k\beta} c_{d\beta} + \text{h.c.}\right) \\
&- \frac{1}{2}\left(\frac{1}{\xi_3} + \frac{1}{\xi_5}\right)\frac{J_t J_0^z}{2}\left(c^\dagger_{d\beta} c_{d\overline{\beta}} c^\dagger_{k\overline{\beta}} c_{k'\beta} + \text{h.c.}\right)
\end{aligned} \tag{5.2.20}$$

The indices k and k' are being summed over, wherever they appear.

5.2.3 Calculation of renormalization in hole sector

The renormalization in the particle sector ($\hat{n}_{q\beta} = 0$) is of the form

$$T^\dagger_{q\beta} c \eta_0^\dagger \tag{5.2.21}$$

where $\eta_0^\dagger$ is of the form

$$\eta_0^\dagger = \sum_{i=1}^{5} \frac{1}{\omega_i' - E_i^1} c^\dagger_{q\beta} T_i \tag{5.2.22}$$

This will be at an energy $+\epsilon_q$, because the state is occupied. The scattering terms T_i are already written down in the previous subsection. The diagonal parts are

$$\begin{aligned}
E_1^1 = E_4^1 = E_5^1 &= \frac{\epsilon_q}{2} + \epsilon_d - \frac{J_z}{2} \\
E_2^1 &= \frac{\epsilon_q}{2} \\
E_3^1 &= \frac{\epsilon_q}{2} + \epsilon_d + \frac{J_z}{2}
\end{aligned} \tag{5.2.23}$$

The renormalization can be computed by calculating $\sum_{ij} T_i^\dagger T_j$.

$$\begin{aligned}
&\sum_{ij}\left(\frac{1}{\omega_i' - E_i^1} + \frac{1}{\omega' - E_j^1}\right) T_i^\dagger T_j \\
&= \frac{|V_1|^2}{\xi_1'}\hat{n}_{d\overline{\beta}}\hat{n}_{d\beta} + \frac{|V_0|^2}{\xi_2'}\hat{n}_{d\beta}\left(1-\hat{n}_{d\overline{\beta}}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{4}\left[{J_0^z}^2\frac{\hat{n}_{d\beta}\left(1-\hat{n}_{d\overline{\beta}}\right)}{\xi_3'}+{J_1^z}^2\frac{\hat{n}_{d\overline{\beta}}\left(1-\hat{n}_{d\beta}\right)}{\xi_4'}\right]c^\dagger_{k\beta}c_{k'\beta}+\frac{1}{\xi_5'}J_t^2\left(1-\hat{n}_{d\overline{\beta}}\right)\hat{n}_{d\beta}c^\dagger_{k\overline{\beta}}c_{k'\overline{\beta}}\\
&-\frac{1}{2}\left(\frac{1}{\xi_4'}+\frac{1}{\xi_1'}\right)\frac{J_1^z}{2}\hat{n}_{d\overline{\beta}}\left(V_1c^\dagger_{k\beta}c_{d\beta}+\text{h.c.}\right)-\frac{1}{2}\left(\frac{1}{\xi_1'}+\frac{1}{\xi_5'}\right)J_t\hat{n}_{d\beta}\left(V_1c^\dagger_{k\overline{\beta}}c_{d\overline{\beta}}+\text{h.c.}\right)\\
&-\frac{1}{2}\left(\frac{1}{\xi_4'}+\frac{1}{\xi_5'}\right)\frac{J_tJ_1^z}{2}\left(c^\dagger_{d\beta}c_{d\overline{\beta}}c^\dagger_{k\overline{\beta}}c_{k'\beta}+\text{h.c.}\right)
\end{aligned}
$$

where ξ_i' is defined similar to the particle sector: $\xi_i' = \omega_i' - E_i^1$. The indices k and k' are being summed over, wherever they appear.

5.2.4 Relating the ω_i and ω_i'

To relate the ω_i and their primed counterparts, we will look at their bare non-interacting values:

$$
\begin{aligned}
\omega_1 &= -\frac{1}{2}\epsilon_q+\epsilon_d\\
\omega_2 &= -\frac{1}{2}\epsilon_q\\
\omega_3 &= -\frac{1}{2}\epsilon_q+\epsilon_d+\frac{J_z}{2}\\
\omega_5 = \omega_4 &= -\frac{1}{2}\epsilon_q+\epsilon_d-\frac{J_z}{2}
\end{aligned}
\tag{5.2.24}
$$

We will *assume* that the relations between these values of ω_i will hold for the URG ω_i along the flow as well. We want to write all the ω_i in terms of a single $\omega = -\frac{\epsilon_q}{2} - \frac{1}{2}J_z$.

$$
\begin{aligned}
\omega_1 &= \omega+\frac{J_z}{2}+\epsilon_d\\
\omega_2 &= \omega+\frac{J_z}{2}\\
\omega_3 &= \omega+J_z+\epsilon_d\\
\omega_5 = \omega_4 &= \omega+\epsilon_d
\end{aligned}
\tag{5.2.25}
$$

The denominators can now be written in terms of the single ω:

$$
\begin{aligned}
\xi_1 &= \omega-\frac{\epsilon_q}{2}-\epsilon_d-U+\frac{J_z}{2}\\
\xi_2 &= \omega-\frac{\epsilon_q}{2}-\epsilon_d+J_z\\
\xi_3 = \xi_4 = \xi_5 &= \omega-\frac{\epsilon_q}{2}+\frac{J_z}{2}
\end{aligned}
\tag{5.2.26}
$$

Similarly, for the ω_i' of the hole sector, we have

$$\begin{aligned} \omega_1' &= -\frac{1}{2}\epsilon_q + 2\epsilon_d + U \\ \omega_2' &= -\frac{1}{2}\epsilon_q + \epsilon_d \\ \omega_3' &= -\frac{1}{2}\epsilon_q + \epsilon_d + \frac{J_z}{2} \\ \omega_4' = \omega_5' &= -\frac{1}{2}\epsilon_q + \epsilon_d - \frac{J_z}{2} \end{aligned} \tag{5.2.27}$$

Writing these in terms of $\omega = -\frac{\epsilon_q}{2} - \frac{1}{2}J_z$ gives

$$\begin{aligned} \omega_1' &= \omega + U + \frac{J_z}{2} \\ \omega_2' &= \omega + \epsilon_d + \frac{J_z}{2} \\ \omega_3' &= \omega + \epsilon_d + J_z \\ \omega_4' &= \omega_5' = \omega + \epsilon_d \end{aligned} \tag{5.2.28}$$

The denominators are therefore

$$\begin{aligned} \xi_1' &= \omega - \frac{\epsilon_q}{2} + \epsilon_d + U + J_z \\ \xi_2' &= \omega - \frac{\epsilon_q}{2} + \epsilon_d + \frac{J_z}{2} \\ \xi_3' = \xi_4' = \xi_5' &= \omega - \frac{\epsilon_q}{2} + \frac{J_z}{2} = \xi_3 = \xi_4 = \xi_5 \end{aligned} \tag{5.2.29}$$

5.2.5 Making sense of the various terms

We will now look at each of the renormalizations separately. Note that the indices k and k' are being summed over, wherever they appear. The first two terms in each sector form a part of the renormalization in ϵ_d and U.

$$\frac{|V_1|^2}{\xi_1}\left(1-\hat{n}_{d\beta}\right)\hat{n}_{d\overline{\beta}} + \frac{|V_0|^2}{\xi_2}\left(1-\hat{n}_{d\beta}\right)\left(1-\hat{n}_{d\overline{\beta}}\right) + \frac{|V_1|^2}{\xi_1'}\hat{n}_{d\overline{\beta}}\hat{n}_{d\beta} + \frac{|V_0|^2}{\xi_2'}\hat{n}_{d\beta}\left(1-\hat{n}_{d\overline{\beta}}\right) \tag{5.2.30}$$

We can read off the renormalizations in the doublon, spin and holon states. Note that the spin ($\hat{n}_d = 1$) are only renormalizedby half, because the other half of the renormalization comes when you decouple the other spin $\overline{\beta}$.

$$\Delta E_2 = \frac{|V_1|^2}{\xi_1'}, \quad \Delta E_1 = \frac{|V_1|^2}{2\xi_1} + \frac{|V_0|^2}{2\xi_2'}, \quad \Delta E_0 = \frac{|V_0|^2}{\xi_2} \tag{5.2.31}$$

Using $\epsilon_d = E_1 - E_0$ and $U = E_2 + E_0 - 2E_1$, we can write

$$\begin{aligned} \Delta\epsilon_d &= \frac{|V_1|^2}{2\xi_1} + \frac{|V_0|^2}{2\xi_2'} - \frac{|V_0|^2}{\xi_2} \\ \Delta U &= \frac{|V_1|^2}{\xi_1'} + \frac{|V_0|^2}{\xi_2} - \frac{|V_1|^2}{\xi_1} - \frac{|V_0|^2}{\xi_2'} \end{aligned} \tag{5.2.32}$$

The J_z^2 terms, together, give

$$\frac{J_z^2}{4}\left[\frac{\hat{n}_{d\beta}\left(1-\hat{n}_{d\overline{\beta}}\right)}{\xi_3}+\frac{\hat{n}_{d\overline{\beta}}\left(1-\hat{n}_{d\beta}\right)}{\xi_4}\right]c_{k'\beta}c^\dagger_{k\beta}+\frac{1}{4}\left[\frac{\hat{n}_{d\beta}\left(1-\hat{n}_{d\overline{\beta}}\right)}{\xi'_3}+\frac{\hat{n}_{d\overline{\beta}}\left(1-\hat{n}_{d\beta}\right)}{\xi'_4}\right]c^\dagger_{k\beta}c_{k'\beta} \quad (5.2.33)$$

There we used $J_0^z = J_1^z = J_z$. From the expressions of ξ_i and ξ'_i, we know that $\xi_3 = \xi_4 = \xi'_3 = \xi'_4$. Therefore, the terms in the box brackets are identical, and we can simplify this to

$$\frac{J_z^2}{4\xi_3}\left[\hat{n}_{d\beta}\left(1-\hat{n}_{d\overline{\beta}}\right)+\hat{n}_{d\overline{\beta}}\left(1-\hat{n}_{d\beta}\right)\right]\left(c_{k'\beta}c^\dagger_{k\beta}+c^\dagger_{k\beta}c_{k'\beta}\right)=\frac{J_z^2}{4\xi_3}\left[\hat{n}_d-2\hat{n}_{d\beta}\hat{n}_{d\overline{\beta}}\right]\delta_{kk'} \quad (5.2.34)$$

This will further renormalize $\epsilon_d\hat{n}_d$ and $U\hat{n}_{d\beta}\hat{n}_{d\overline{\beta}}$, now at J^2 order:

$$\begin{aligned}\Delta\epsilon_d &= \frac{J_z^2}{4\xi_3}\delta_{kk'}\\ \Delta U &= -2\frac{J_z^2}{4\xi_3}\delta_{kk'}\end{aligned} \quad (5.2.35)$$

We next look at the J_zV terms:

$$-\frac{J_z}{4}\left[\left(\frac{1}{\xi_2}+\frac{1}{\xi_3}\right)\left(1-\hat{n}_{d\overline{\beta}}\right)V_0c^\dagger_{k\beta}c_{d\beta}+\left(\frac{1}{\xi'_4}+\frac{1}{\xi'_1}\right)\hat{n}_{d\overline{\beta}}V_1c^\dagger_{k\beta}c_{d\beta}\right]+\text{h.c.} \quad (5.2.36)$$

The first term in the box bracket renormalizes $V_0c^\dagger_{k\beta}c_{d\beta}\left(1-\hat{n}_{d\overline{\beta}}\right)$, while the second term renormalizes $V_1c^\dagger_{k\beta}c_{d\beta}\hat{n}_{d\overline{\beta}}$. Because this renormalizes only one spin component (β), the other spin component will get renormalized when we decouple $\overline{\beta}$, and so we attribute only half of this to the total renormalization.

$$\begin{aligned}\Delta V_0 &= -\frac{1}{2}\frac{J_z}{2}V_0\left(\frac{1}{\xi_2}+\frac{1}{\xi_3}\right)\\ \Delta V_1 &= -\frac{1}{2}\frac{J_z}{2}V_1\left(\frac{1}{\xi'_1}+\frac{1}{\xi_3}\right)\end{aligned} \quad (5.2.37)$$

where we used $\xi'_4 = \xi_3$. The terms with J_tV also renormalize the same terms. Combining this with the previous renormalization gives the total renormalization of V_0 and V_1:

$$\begin{aligned}\Delta V_0 &= -\left(\frac{J_z}{4}V_0+J_tV_0\right)\left(\frac{1}{\xi_2}+\frac{1}{\xi_3}\right)\\ \Delta V_1 &= -\left(\frac{J_z}{4}V_1+J_tV_1\right)\left(\frac{1}{\xi'_1}+\frac{1}{\xi_3}\right)\end{aligned} \quad (5.2.38)$$

There we used $\xi_5' = \xi_5 = \xi_3$.

The remaining terms are all of order J^2. First we look at the J_t^2 terms:

$$
\begin{aligned}
&\frac{1}{\xi_5}J_t^2\left(1-\hat{n}_{d\beta}\right)\hat{n}_{d\overline{\beta}}c_{k'\overline{\beta}}c^\dagger_{k\overline{\beta}} + \frac{1}{\xi_5'}J_t^2\left(1-\hat{n}_{d\overline{\beta}}\right)\hat{n}_{d\beta}c^\dagger_{k\overline{\beta}}c_{k'\overline{\beta}} \\
&= \frac{1}{\xi_3}J_t^2c^\dagger_{k\overline{\beta}}c_{k'\overline{\beta}}\left[\left(1-\hat{n}_{d\overline{\beta}}\right)\hat{n}_{d\beta} - \left(1-\hat{n}_{d\beta}\right)\hat{n}_{d\overline{\beta}}\right] + \delta_{kk'}\frac{1}{\xi_3}J_t^2\left(1-\hat{n}_{d\beta}\right)\hat{n}_{d\overline{\beta}} \\
&= -\frac{1}{\xi_3}2J_t^2c^\dagger_{k\overline{\beta}}c_{k'\overline{\beta}}\frac{\hat{n}_{d\overline{\beta}}-\hat{n}_{d\beta}}{2} + \delta_{kk'}\frac{1}{\xi_3}J_t^2\left(1-\hat{n}_{d\beta}\right)\hat{n}_{d\overline{\beta}}
\end{aligned}
\tag{5.2.39}
$$

In the second step, we used $\xi_5' = \xi_5 = \xi_3$ and $c_kc^\dagger_{k'} = \delta_{kk'} - c^\dagger_{k'}c_k$. The first term in the final expression renormalizes half of the Ising Kondo coupling $J_zS_d^zs^z$, the other half will be renormalized when we decouple $q\overline{\beta}$.

$$
\Delta J_z = -\frac{1}{2\xi_3}2J_t^2 \tag{5.2.40}
$$

The other term in the final expression renormalizes U and half of ϵ_d (only $\overline{\beta}$).

$$
\begin{aligned}
\Delta\epsilon_d &= \frac{1}{2\xi_3}J_t^2\delta_{kk'} \\
\Delta U &= -\frac{1}{\xi_3}J_t^2\delta_{kk'}
\end{aligned}
\tag{5.2.41}
$$

The remaining terms are those with J_zJ_t:

$$
\left[-\frac{1}{2}\left(\frac{1}{\xi_3}+\frac{1}{\xi_5}\right)\frac{J_tJ_0^z}{2}c^\dagger_{d\beta}c_{d\overline{\beta}}c^\dagger_{k\overline{\beta}}c_{k'\beta} - \frac{1}{2}\left(\frac{1}{\xi_4'}+\frac{1}{\xi_5'}\right)\frac{J_tJ_1^z}{2}c^\dagger_{d\beta}c_{d\overline{\beta}}c^\dagger_{k\overline{\beta}}c_{k'\beta}\right] + \text{h.c.} \tag{5.2.42}
$$

They renormalize the transverse Kondo coupling. Noting that $\xi_5 = \xi_4' = \xi_5' = \xi_3$, we get

$$
\Delta J_t = -\frac{1}{\xi_3}J_tJ_0^z \tag{5.2.43}
$$

5.2.6 Scaling equations

Looking at eqs. 5.2.32, 5.2.35, 5.2.41, 5.2.38, 5.2.40 and 5.2.43, and summing over β, we can write down the scaling equations for the Anderson-Kondo model.

$$
\begin{aligned}
\Delta\epsilon_d &= \frac{|V_1|^2}{\xi_1} + \frac{|V_0|^2}{\xi_2'} - \frac{2|V_0|^2}{\xi_2} + \sum_k \frac{1}{\xi_3}\left(J_t^2 + \frac{1}{2}J_z^2\right) \\
\Delta U &= \frac{2|V_1|^2}{\xi_1'} + \frac{2|V_0|^2}{\xi_2} - \frac{2|V_1|^2}{\xi_1} - \frac{2|V_0|^2}{\xi_2'} - \sum_k \frac{1}{\xi_3}\left(2J_t^2 + J_z^2\right) \\
\Delta V_0 &= -V_0\left(\frac{J_z}{2} + J_t\right)\left(\frac{1}{\xi_2} + \frac{1}{\xi_3}\right) \\
\Delta V_1 &= -V_1\left(\frac{J_z}{2} + J_t\right)\left(\frac{1}{\xi_1'} + \frac{1}{\xi_3}\right) \\
\Delta J_z &= -\frac{2}{\xi_3}J_t^2 \\
\Delta J_t &= -\frac{2}{\xi_3}J_tJ_0^z
\end{aligned}
\tag{5.2.44}
$$

$$
\begin{aligned}
\xi_1 &= \omega - \frac{\epsilon_q}{2} - \epsilon_d - U + \frac{J_z}{2} \\
\xi_2 &= \omega - \frac{\epsilon_q}{2} - \epsilon_d + J_z \\
\xi_1' &= \omega - \frac{\epsilon_q}{2} + \epsilon_d + U + J_z \\
\xi_2' &= \omega - \frac{\epsilon_q}{2} + \epsilon_d + \frac{J_z}{2} \\
\xi_3' &= \xi_4' = \xi_5' \\
&= \omega - \frac{\epsilon_q}{2} + \frac{J_z}{2} \\
&= \xi_3 = \xi_4 = \xi_5
\end{aligned}
$$

5.2.7 Particle-Hole symmetry

As discussed in the previous section, the particle-hole symmetry condition for the basic SIAM ($J = 0$) is $\epsilon_d + U = -\epsilon_d$. With the inclusion of J, we will need to see what the new condition is. We will first write the impurity part of the Hamiltonian and see how it transforms under a particle-hole transformation.

$$
\begin{aligned}
&\epsilon_d\hat{n}_d + U\hat{n}_{d\uparrow}\hat{n}_{d\downarrow} + J_z\left(\hat{n}_{d\uparrow} - \hat{n}_{d\downarrow}\right)\left(\hat{n}_{q\uparrow} - \hat{n}_{q\downarrow}\right) \\
&\to 2\epsilon_d + U - \left(\epsilon_d + U\right)\hat{n}_d + U\hat{n}_{d\uparrow}\hat{n}_{d\downarrow} + J_z\left(\hat{n}_{d\downarrow} - \hat{n}_{d\uparrow}\right)\left(\hat{n}_{q\downarrow} - \hat{n}_{q\uparrow}\right)
\end{aligned}
\tag{5.2.45}
$$

This gives us the same condition as in the $J = 0$ case. The p-h symmetry condition implies that $2\epsilon_d + U$ must be an RG-invariant. The RG equation for $2\epsilon_d + U$ is

$$
\Delta\left(2\epsilon_d + U\right) = \frac{|V_1|^2}{\xi_1'} - \frac{|V_1|^2}{\xi_2}
\tag{5.2.46}
$$

For $2\epsilon_d + U = 0$ in the bare model, we can write $\xi_2 = \xi_1'$, which means $\Delta\left(2\epsilon_d + U\right) = 0$.

5.2.8 "Poor Man's" one-loop form for asymmetric Anderson model

In the limit of $\epsilon_d, J \ll D \ll U$, we can ignore the J^2 terms in ϵ_d and U, and the remaining terms simplify:

$$
\begin{aligned}
\frac{1}{\xi_1} &= \frac{1}{\xi_1'} \approx 0 \\
\xi_2 &= \xi_2' \approx \omega - \frac{\epsilon_q}{2}
\end{aligned}
\tag{5.2.47}
$$

These give

$$\Delta U \approx 0$$
$$\Delta\epsilon_d \approx -\frac{|V_0|^2}{\xi_2} \approx -\frac{|V_0|^2}{\omega - \frac{\epsilon_q}{2}} \tag{5.2.48}$$

This is the same form that we had in the pure SIAM, and we can again repeat what we did in subsection 5.1.4.

5.2.9 SU(2) invariance and Kondo model one-loop form

Setting $J_z = J_t = \frac{1}{2}J$ makes the interaction $SU(2)$ symmetric; the last two RG equations can then be written in the common form:

$$\Delta J = -\frac{1}{\xi_3}J^2 = -J^2\frac{1}{\omega - \frac{\epsilon_q}{2} + \frac{J_z}{2}} \tag{5.2.49}$$

For low quantum fluctuations we can ignore the renormalization and replace ω with the bare initial energy value $-\frac{1}{2}\epsilon_q$.

$$\Delta J = -J^2\frac{1}{-\epsilon_q + \frac{1}{4}J} \tag{5.2.50}$$

We can now expand the denominator in powers of J and keep only the lowest order, we get

$$\Delta J = J^2\frac{1}{\epsilon_q} \tag{5.2.51}$$

This is the Kondo model one-loop form.

5.3 Anderson-Kondo (charge) model URG

Performing the Schrieffer-Wolff transformation on the SIAM generates four types of terms. The simplest terms are ones that renormalize the impurity scales ϵ_d and U. The next are the potential scattering terms that describe interactions between mobile electrons with the impurity simply acting as a stationary potential. The third term is the familiar Kondo model interaction terms that involve a Heisenberg-like interaction between the impurity spin $S_d^z = \frac{1}{2}\left(\hat{n}_{d\uparrow} - \hat{n}_{d\downarrow}\right)$ and the global spin of the mobile electrons $\frac{1}{2}\sum_{kk'\alpha\beta} c^\dagger_{k\alpha}\vec{\sigma}_{\alpha\beta}c_{k'\beta}$. The fourth term is the interactions that modify the charge of either entity by 2, $c^\dagger_{k\alpha}c^\dagger_{k'\overline{\alpha}}c_{d\alpha}c_{d\overline{\alpha}}$. We will be considering the last kind of terms in this section. For that, we define the Nambu spinor [34, 35].

$$\psi^k = \begin{pmatrix} c_{k\uparrow} \\ c^\dagger_{k\downarrow} \end{pmatrix} \tag{5.3.1}$$

and the charge isospin [36] for the mobile conduction electrons

$$\vec{C} = \sum_{kk'} \psi^{k\dagger} \vec{S} \psi^{k'} = \frac{1}{2} \sum_{kk'\alpha\beta} \psi^{k\dagger}_{\alpha} \vec{\sigma}_{\alpha\beta} \psi^{k'}_{\beta} \tag{5.3.2}$$

The various components of the isospin are

$$\begin{aligned}
C^z &= \sum_{kk'\sigma} \frac{1}{2} \psi^{k\dagger}_{\sigma} \sigma^z_{\sigma\sigma} \psi^{k'}_{\sigma} = \frac{1}{2} \sum_{kk'} \left(c^{\dagger}_{k\uparrow} c_{k'\uparrow} - c_{k'\downarrow} c^{\dagger}_{k\downarrow} \right) = \frac{1}{2} \sum_{kk'} \left(c^{\dagger}_{k\uparrow} c_{k'\uparrow} + c^{\dagger}_{k\downarrow} c_{k'\downarrow} - \delta_{kk'} \right) \\
&= \frac{1}{2} \sum_{kk'\sigma} \left(c^{\dagger}_{k\sigma} c_{k'\sigma} - \frac{1}{2} \delta_{kk'} \right) \\
C^x &= \sum_{kk'\sigma} \frac{1}{2} \psi^{k\dagger}_{\sigma} \sigma^x_{\sigma\overline{\sigma}} \psi^{k'}_{\overline{\sigma}} = \frac{1}{2} \sum_{kk'} \left(c^{\dagger}_{k\uparrow} c^{\dagger}_{k'\downarrow} + c_{k\downarrow} c_{k'\uparrow} \right) = \sum_{kk'\sigma} \frac{\sigma}{4} \left(c^{\dagger}_{k\sigma} c^{\dagger}_{k'\overline{\sigma}} + \text{h.c.} \right) \\
C^y &= \sum_{kk'\sigma} \frac{1}{2} \psi^{k\dagger}_{\sigma} \sigma^y_{\sigma\overline{\sigma}} \psi^{k'}_{\overline{\sigma}} = -\frac{i}{2} \sum_{kk'} \left(c^{\dagger}_{k\uparrow} c^{\dagger}_{k'\downarrow} - c_{k\downarrow} c_{k'\uparrow} \right) = \sum_{kk'\sigma} -\frac{i\sigma}{4} \left(c^{\dagger}_{k\sigma} c^{\dagger}_{k'\overline{\sigma}} - \text{h.c.} \right)
\end{aligned} \tag{5.3.3}$$

It is easy to verify that these operators satisfy the SU(2) commutation algebra. For example, if we write $C^x = A + A^{\dagger}$ and $C^y = B + B^{\dagger}$, then $\left[C^x, C^y\right] = \left[A, B^{\dagger}\right] - \text{h.c.}$, where

$$\left[A, B^{\dagger}\right] = \frac{1}{4} \sum_{kk',qq'} \left[c^{\dagger}_{k\uparrow} c^{\dagger}_{k'\downarrow}, i c_{q'\downarrow} c_{q\uparrow} \right] = \frac{i}{4} \sum_{kq} \left(c^{\dagger}_{k\uparrow} c_{q\uparrow} - c_{k\downarrow} c^{\dagger}_{q\downarrow} \right) \tag{5.3.4}$$

and therefore

$$\implies \left[C^x, C^y\right] = \frac{i}{2} \sum_{kq} \left(c^{\dagger}_{k\uparrow} c_{q\uparrow} - c_{k\downarrow} c^{\dagger}_{q\downarrow} \right) = i C^z \tag{5.3.5}$$

There are similar operators for the impurity electron:

$$\begin{aligned}
\psi_d &= \begin{pmatrix} c_{d\uparrow} \\ c^{\dagger}_{d\downarrow} \end{pmatrix} \\
C^z_d &= \frac{1}{2} \left(c^{\dagger}_{d\uparrow} c_{d\uparrow} + c^{\dagger}_{d\downarrow} c_{d\downarrow} - 1 \right) = \frac{1}{2} \left(\hat{n}_d - 1 \right) \\
C^x_d &= \frac{1}{2} \left(c^{\dagger}_{d\uparrow} c^{\dagger}_{d\downarrow} + c_{d\downarrow} c_{d\uparrow} \right) = \sum_{\sigma} \frac{\sigma}{4} \left(c^{\dagger}_{d\sigma} c^{\dagger}_{d\overline{\sigma}} + \text{h.c.} \right) \\
C^y_d &= -i \frac{1}{2} \left(c^{\dagger}_{d\uparrow} c^{\dagger}_{d\downarrow} - c_{d\downarrow} c_{d\uparrow} \right) = -i \sum_{\sigma} \frac{\sigma}{4} \left(c^{\dagger}_{d\sigma} c^{\dagger}_{d\overline{\sigma}} - \text{h.c.} \right)
\end{aligned} \tag{5.3.6}$$

The full charge-Kondo interaction can now be written down in terms of these isospins:

$$4 K_z C^z_d C^z + K_t \left(C^+_d C^- + C^-_d C^+ \right) \tag{5.3.7}$$

where $C^{\pm} \equiv C^x \pm iC^y$.

$$C^+ = \sum_{kk'} c^\dagger_{k\uparrow} c^\dagger_{k'\downarrow}, \quad C^- = \sum_{kk'} c_{k'\downarrow} c_{k\uparrow} \tag{5.3.8}$$

For $4K_z = 2K_t = K$, we get an $SU(2)$-charge symmetric model:

$$KC^z_d C^z + \frac{1}{2}K\left(C^+_d C^- + C^-_d C^+\right) = K\vec{C}_d \cdot \vec{C} \tag{5.3.9}$$

To proceed with the URG, we start with the outermost shell Λ_N and consider an electron $q\beta$ on that shell. The URG then involves decoupling this electron. In the subspace of $q\beta$, the diagonal and off-diagonal parts are

$$\begin{aligned} \mathcal{H}_d &= \epsilon_q \tau_{q\beta} + H_{imp} + K_z\left(\hat{n}_d - 1\right)\tau_{q\beta} \\ \mathcal{H}_X &= V_q c^\dagger_{q\beta} c_{d\beta} + \text{h.c.} + K_z\left(\hat{n}_d - 1\right)\sum_k \left(c^\dagger_{q\beta} c_{k\beta} + \text{h.c.}\right) + K_t \sum_k \left(c^\dagger_{d\beta} c^\dagger_{d\overline{\beta}} c_{k\overline{\beta}} c_{q\beta} + \text{h.c.}\right) \end{aligned} \tag{5.3.10}$$

As usual, we have considered only one mobile electron on the shell we are decoupling, and we keep only the energy of that electron and the impurity in the diagonal part which comes down in the denominator. Note that the factors of half in C^z are cancelled by the factor of 4 in $4K_z$. The last term in $\mathcal{H}_d$ is obtained by setting $k = k' = q$ and $\sigma = \beta$ in eq. 5.3.3, and then recognizing that $\hat{n}_{q\beta} - \frac{1}{2} = \tau_{q\beta}$. The K_z part of $\mathcal{H}_X$ is obtained by noting:

$$\begin{aligned} C^z_{q\neq k} &= \frac{1}{2}\left(c^\dagger_{q\uparrow} c_{k\uparrow} + c^\dagger_{k\uparrow} c_{q\uparrow} - c_{k\downarrow} c^\dagger_{q\downarrow} - c_{q\downarrow} c^\dagger_{k\downarrow}\right) \\ &= \frac{1}{2}\left(c^\dagger_{q\uparrow} c_{k\uparrow} + c^\dagger_{k\uparrow} c_{q\uparrow} + c^\dagger_{q\downarrow} c_{k\downarrow} + c^\dagger_{k\downarrow} c_{q\downarrow}\right) \\ &= \frac{1}{2}\sum_\sigma \left(c^\dagger_{q\sigma} c_{k\sigma} + \text{h.c.}\right) \end{aligned} \tag{5.3.11}$$

The calculation of renormalization will proceed similar to the spin-Kondo Anderson URG. We will again separate the off-diagonal piece $\mathcal{H}^X_d$ into separate parts T_i, calculate the renormalization in particle and hole sectors, and then finally relate the ω_i using their bare values. The off-diagonal parts for this problem are

$$\begin{aligned} T_1 &= V_1 c_{d\beta} \hat{n}_{d\overline{\beta}} \\ T_2 &= V_0 c_{d\beta}\left(1 - \hat{n}_{d\overline{\beta}}\right) \\ T_3 &= K^1_z \hat{n}_{d\beta} \hat{n}_{d\overline{\beta}} c_{k\beta} \\ T_4 &= -K^0_z\left(1 - \hat{n}_{d\beta}\right)\left(1 - \hat{n}_{d\overline{\beta}}\right) c_{k\beta} \\ T_5 &= K_t c^\dagger_{k\overline{\beta}} c_{d\overline{\beta}} c_{d\beta} \end{aligned} \tag{5.3.12}$$

5.3.1 Calculation of renormalization in particle sector

The renormalization in this sector is

$$c^\dagger_{q\beta} T \eta \tag{5.3.13}$$

This is at energy $-\epsilon_q$. Using the expressions of the T_i, this becomes

$$\begin{aligned}
&\left(1-\hat{n}_{d\beta}\right)\left[\hat{n}_{d\overline{\beta}}\frac{|V_1|^2}{\xi_1}+\left(1-\hat{n}_{d\overline{\beta}}\right)\frac{|V_0|^2}{\xi_2}\right]+\left[\frac{{K_z^1}^2}{\xi_3}\hat{n}_{d\beta}\hat{n}_{d\overline{\beta}}+\frac{{K_z^0}^2}{\xi_4}\left(1-\hat{n}_{d\beta}\right)\left(1-\hat{n}_{d\overline{\beta}}\right)\right]c_{k'\beta}c^\dagger_{k\beta}\\
&+\frac{K_t^2}{\xi_5}\left(1-\hat{n}_{d\beta}\right)\left(1-\hat{n}_{d\overline{\beta}}\right)c^\dagger_{k\overline{\beta}}c_{k'\overline{\beta}}-\frac{1}{2}\left(\frac{1}{\xi_1}+\frac{1}{\xi_3}\right)V_1K_z^1\hat{n}_{d\overline{\beta}}\left(c^\dagger_{k\beta}c_{d\beta}+\text{h.c.}\right)\\
&+\frac{1}{2}\left(\frac{1}{\xi_1}+\frac{1}{\xi_5}\right)V_1K_t\left(1-\hat{n}_{d\beta}\right)\left(c^\dagger_{k\overline{\beta}}c_{d\overline{\beta}}+\text{h.c.}\right)-\frac{1}{2}\left(\frac{1}{\xi_3}+\frac{1}{\xi_5}\right)K_z^1K_t\left(c^\dagger_{d\beta}c^\dagger_{d\overline{\beta}}c_{k'\overline{\beta}}c_{k\beta}+\text{h.c.}\right)
\end{aligned}\tag{5.3.14}$$

The indices k, k' are summed over. ξ_i is defined exactly as before, $\omega_i - E_i^0$.

The intermediate energies, E_i^0, are

$$\begin{aligned}
E_1^0 = E_3^0 &= \frac{\epsilon_q}{2}+2\epsilon_d+U\\
E_2^0 &= \frac{\epsilon_q}{2}+\epsilon_d\\
E_4^0 &= \frac{\epsilon_q}{2}\\
E_5^0 &= \frac{\epsilon_q}{2}+2\epsilon_d+U-K_z
\end{aligned}\tag{5.3.15}$$

The contribution of $k\overline{\beta}$ in the denominators of $E^0_{3,4,5}$ has been considered.

5.3.2 Calculation of renormalization in hole sector

The renormalization in the hole sector is given by

$$T^\dagger c_{q\beta}\eta_0^\dagger \tag{5.3.16}$$

at energy ϵ_q. That comes out to be

$$\begin{aligned}
&\frac{|V_1|^2}{\xi_1'}\hat{n}_{d\beta}\hat{n}_{d\overline{\beta}}+\frac{|V_0|^2}{\xi_2'}\hat{n}_{d\beta}\left(1-\hat{n}_{d\overline{\beta}}\right)+\left[\frac{{K_z^1}^2}{\xi_3'}\hat{n}_{d\beta}\hat{n}_{d\overline{\beta}}+\frac{{K_z^0}^2}{\xi_4'}\left(1-\hat{n}_{d\beta}\right)\left(1-\hat{n}_{d\overline{\beta}}\right)\right]c^\dagger_{k\beta}c_{k'\beta}\\
&+\frac{K_t^2}{\xi_5'}\hat{n}_{d\beta}\hat{n}_{d\overline{\beta}}c_{k'\overline{\beta}}c^\dagger_{k\overline{\beta}}-\frac{1}{2}\left(\frac{1}{\xi_2'}+\frac{1}{\xi_4'}\right)V_0K_z^0\left(1-\hat{n}_{d\overline{\beta}}\right)\left(c^\dagger_{k\beta}c_{d\beta}+\text{h.c.}\right)\\
&+\frac{1}{2}\left(\frac{1}{\xi_2'}+\frac{1}{\xi_5'}\right)V_0K_t\hat{n}_{d\beta}\left(c^\dagger_{k\overline{\beta}}c_{d\overline{\beta}}+\text{h.c.}\right)-\frac{1}{2}\left(\frac{1}{\xi_4'}+\frac{1}{\xi_5'}\right)K_z^0K_t\left(c^\dagger_{d\beta}c^\dagger_{d\overline{\beta}}c_{k\overline{\beta}}c_{k'\beta}+\text{h.c.}\right)
\end{aligned}\tag{5.3.17}$$

where $\xi' = \omega' = E_i^1$. The intermediate energies in this sector are

$$\begin{aligned} E_1^1 &= \frac{\epsilon_q}{2} + \epsilon_d \\ E_2^1 = E_4^1 &= \frac{\epsilon_q}{2} \\ E_3^1 &= \frac{\epsilon_q}{2} + 2\epsilon_d + U \\ E_5^1 &= \frac{\epsilon_q}{2} - K_z \end{aligned} \tag{5.3.18}$$

5.3.3 Relating the ω

Just as in the previous subsection, we relate the ω_i using their diagonal values.

First the particle sector ω_i:

$$\begin{aligned} \omega_1 &= -\frac{\epsilon_q}{2} + \epsilon_d \\ \omega_2 &= \omega_5 = -\frac{\epsilon_q}{2} - K_z \\ \omega_3 &= -\frac{\epsilon_q}{2} + 2\epsilon_d + U \\ \omega_4 &= -\frac{\epsilon_q}{2} \end{aligned} \tag{5.3.19}$$

Rewriting everything in terms of $\omega = -\frac{\epsilon_q}{2} - K_z$ gives

$$\begin{aligned} \omega_1 &= \omega + \epsilon_d + K_z \\ \omega_2 &= \omega_5 = \omega \\ \omega_3 &= \omega + 2\epsilon_d + U + K_z \\ \omega_4 &= \omega + K_z \end{aligned} \tag{5.3.20}$$

For the hole sector, we get

$$\begin{aligned} \omega_1' &= \omega_5' = -\frac{\epsilon_q}{2} + 2\epsilon_d + U - K_z \\ \omega_2' &= -\frac{\epsilon_q}{2} + \epsilon_d \\ \omega_3' &= -\frac{\epsilon_q}{2} + 2\epsilon_d + U \\ \omega_4' &= -\frac{\epsilon_q}{2} \end{aligned} \tag{5.3.21}$$

and, in terms of ω,

$$\begin{aligned} \omega_1' &= \omega_5' = \omega + 2\epsilon_d + U \\ \omega_2' &= \omega + \epsilon_d + K_z \\ \omega_3' &= \omega + 2\epsilon_d + U + K_z \\ \omega_4' &= \omega + K_z \end{aligned} \tag{5.3.22}$$