URG on Kondo Model

April 6, 2021

Contents

1 Questions

Most of the questions are hyperlinks, so you can click them and go to relevant portions (the targets are colored green).

- What is the motivation behind the choice of the initial condition $\tau_{q\uparrow} = -\tau_{q\downarrow} = \frac{1}{2}$? Does that choice not violate the SU(2) symmetry of the model? Why not take a more symmetric choice like $\tau_{q\beta} = \tau_{q\bar{\beta}} = \frac{1}{2}$ for $q\beta$ below fermi level and $-1/2$ for above it?

- If we follow your notes and try to derive the equations with just the initial configuration $\tau_{q\uparrow} = -\tau_{q\downarrow} = \frac{1}{2}$, we end up with a field-like term αS_d^z in ΔH, which violates SU(2). However, if we consider anothe initial configuration with $\uparrow$ and $\downarrow$ flipped, and add the ΔH arising from such a configuration, we can get rid of the field term.

- On setting $J_z = J_t$, we do not get $\Delta J_z = \Delta J_t$. The RG equations appear to not respect the SU(2) symmetry. How do we resolve this?

- The ΔJ_t that we obtained without summing over β is half of what you get. Why is this so?

- Is there a general prescription for choosing what part of the Hamiltonian comes down in the denominator?

2 Calculations

These subsections contain the calculations which back up the questions mentioned above.

2.1 Formulation

$$\begin{aligned} H &= \sum_{k\alpha} \epsilon_k \tau_{k\alpha} + J_z S_d^z \sum_{kk'} \left(c_{k\uparrow}^\dagger c_{k'\uparrow} - c_{k\downarrow}^\dagger c_{k'\downarrow} \right) \\ &\quad + J_t \sum_{kk'} \left(S_d^+ c_{k\downarrow}^\dagger c_{k'\uparrow} + S_d^- c_{k\uparrow}^\dagger c_{k'\downarrow} \right) \\ &= H^D + H^i + H^I \end{aligned} \tag{2.1}$$

$$H^D = \sum_{k\alpha} \epsilon_k \tau_{k\alpha} + J_z S_d^z \sum_{k\beta} \beta \tau_{k\beta} \tag{2.2}$$

$$\begin{aligned} H^i &= J_z S_d^z \sum_{kk' \neq q} \beta \left(c_{k\beta}^\dagger c_{k'\beta} - c_{k\bar{\beta}}^\dagger c_{k'\bar{\beta}} \right) (1 - \delta_{kk'}) \\ &\quad + J_t \sum_{k' \neq q, k} \left(c_{d\beta}^\dagger c_{d\bar{\beta}} c_{k\bar{\beta}}^\dagger c_{k'\beta} + c_{d\bar{\beta}}^\dagger c_{d\beta} c_{k'\beta}^\dagger c_{k\bar{\beta}} \right) \end{aligned} \tag{2.3}$$

$$\begin{aligned} H^I &= J_t \sum_{k \neq q} \left(c_{d\beta}^\dagger c_{d\bar{\beta}} c_{k\bar{\beta}}^\dagger c_{q\beta} + c_{d\bar{\beta}}^\dagger c_{d\beta} c_{q\beta}^\dagger c_{k\bar{\beta}} \right) \\ &\quad + J_z S_d^z \beta \sum_{k \neq q} \left(c_{k\beta}^\dagger c_{q\beta} + c_{q\beta}^\dagger c_{k\beta} \right) \\ &= c_{q\beta}^\dagger T_{q\beta} + T_{q\beta}^\dagger c_{q\beta} \end{aligned} \tag{2.4}$$

where

$$T_{q\beta} = J_z S_d^z \beta \sum_{k \neq q} c_{k\beta} + J_t c_{d\bar{\beta}}^\dagger c_{d\beta} \sum_{k \neq q} c_{k\bar{\beta}} \tag{2.5}$$

The transformed hamiltonian is

$$U H U^\dagger = H^D + H^i + \underbrace{c_{q\beta}^\dagger T_{q\beta} \eta}_{\text{Particle}} + \underbrace{\eta_0 c_{q\beta}^\dagger T_{q\beta}}_{\text{Hole}} \tag{2.6}$$

where $\eta_0 = -\eta$

For simpler calculations, take H^D in the green's functions of η, η_0 as

$$H^D = \epsilon_q \tau_{q\beta} + \beta J_z S_d^z \left(\tau_{q\beta} - \tau_{q\bar{\beta}}\right) \tag{2.7}$$

2.2 Particle Sector

$$\begin{aligned}
c^\dagger_{q\beta} T_{q\beta} \eta &= \frac{1}{\omega - H^D} c^\dagger_{q\beta} T_{q\beta} T^\dagger_{q\beta} c_{q\beta} \\
&= \frac{1}{\omega - H^D} \sum_{kk' \neq q} c^\dagger_{q\beta} \left(J_z S_d^z \beta c_{k\beta} + J_t c^\dagger_{d\bar{\beta}} c_{d\beta} c_{k\bar{\beta}}\right) \left(J_z S_d^z \beta c^\dagger_{k'\beta} + J_t c^\dagger_{d\beta} c_{d\bar{\beta}} c^\dagger_{k'\bar{\beta}}\right) c_{q\beta}
\end{aligned} \tag{2.8}$$

$$\begin{aligned}
\left(c^\dagger_{q\beta} T_{q\beta} \eta\right)_1 &= J_z^2 \frac{1}{\omega - H^D} \sum_{kk' \neq q} \beta S_d^z c^\dagger_{q\beta} c_{k\beta} \beta S_d^z c^\dagger_{k'\beta} c_{q\beta} \\
&= \frac{1}{4} J_z^2 \frac{1}{\omega - \frac{1}{2}\epsilon_q - \beta J_z S_d^z \left(\tau_{q\beta} - \tau_{q\bar{\beta}}\right)} \sum_{kk' \neq q} c_{k\beta} c^\dagger_{k'\beta} n_{q\beta} && \text{(2.9a)} \\
\left(c^\dagger_{q\beta} T_{q\beta} \eta\right)_2 &= J_z J_t \frac{1}{\omega - H^D} \sum_{kk' \neq q} \beta S_d^z c^\dagger_{q\beta} c_{k\beta} c^\dagger_{d\beta} c_{d\bar{\beta}} c^\dagger_{k'\bar{\beta}} c_{q\beta} \\
&= \frac{1}{2} J_z J_t \frac{1}{\omega - \frac{1}{2}\epsilon_q - \frac{1}{2} J_z \left(\tau_{q\beta} - \tau_{q\bar{\beta}}\right)} c^\dagger_{d\beta} c_{d\bar{\beta}} \sum_{kk' \neq q} c_{k\beta} c^\dagger_{k'\bar{\beta}} n_{q\beta} && \text{(2.9b)} \\
\left(c^\dagger_{q\beta} T_{q\beta} \eta\right)_3 &= J_z J_t \frac{1}{\omega - H^D} \sum_{kk' \neq q} c^\dagger_{d\bar{\beta}} c_{d\beta} c^\dagger_{q\beta} c_{k\bar{\beta}} \beta S_d^z c^\dagger_{k'\beta} c_{q\beta} \\
&= \frac{1}{2} J_z J_t \frac{1}{\omega - \frac{1}{2}\epsilon_q + \frac{1}{2} J_z \left(\tau_{q\beta} - \tau_{q\bar{\beta}}\right)} c^\dagger_{d\bar{\beta}} c_{d\beta} \sum_{kk' \neq q} c_{k\bar{\beta}} c^\dagger_{k'\beta} n_{q\beta} && \text{(2.9c)} \\
\left(c^\dagger_{q\beta} T_{q\beta} \eta\right)_4 &= J_t^2 \frac{1}{\omega - H^D} \sum_{kk' \neq q} c^\dagger_{d\bar{\beta}} c_{d\beta} c^\dagger_{q\beta} c_{k\bar{\beta}} c^\dagger_{d\beta} c_{d\bar{\beta}} c^\dagger_{k'\bar{\beta}} c_{q\beta} \\
&= J_t^2 \frac{1}{\omega - \frac{1}{2}\epsilon_q + \frac{1}{2} J_z \left(\tau_{q\beta} - \tau_{q\bar{\beta}}\right)} \left(\frac{1}{2} + \bar{\beta} S_d^z\right) \sum_{kk' \neq q} c_{k\bar{\beta}} c^\dagger_{k'\bar{\beta}} n_{q\beta} && \text{(2.9d)}
\end{aligned}$$

2.3 Hole Sector

$$\eta_0 c^\dagger_{q\beta} T_{q\beta} = \frac{1}{\omega' - H^D} T^\dagger_{q\beta} c_{q\beta} c^\dagger_{q\beta} T_{q\beta}$$

$$= \frac{1}{\omega' - H^D} \sum_{kk' \neq q} \left(J_z S_d^z \beta c^\dagger_{k'\beta} + J_t c^\dagger_{d\beta} c_{d\bar{\beta}} c^\dagger_{k'\bar{\beta}} \right) c_{q\beta} c^\dagger_{q\beta} \left(J_z S_d^z \beta c_{k\beta} + J_t c^\dagger_{d\bar{\beta}} c_{d\beta} c_{k\bar{\beta}} \right) \tag{2.10}$$

$$\left(\eta_0 c^\dagger_{q\beta} T_{q\beta}\right)_1 = J_z^2 \frac{1}{\omega' - H^D} \sum_{kk' \neq q} \beta S_d^z c^\dagger_{k'\beta} c_{q\beta} \beta S_d^z c^\dagger_{q\beta} c_{k\beta}$$

$$= \frac{1}{4} J_z^2 \frac{1}{\omega' + \frac{1}{2}\epsilon_q - \beta J_z S_d^z \left(\tau_{q\beta} - \tau_{q\bar{\beta}}\right)} \sum_{kk' \neq q} c^\dagger_{k'\beta} c_{k\beta} \left(1 - n_{q\beta}\right) \tag{2.11a}$$

$$\left(\eta_0 c^\dagger_{q\beta} T_{q\beta}\right)_2 = J_z J_t \frac{1}{\omega' - H^D} \sum_{kk' \neq q} \beta S_d^z c^\dagger_{k'\beta} c_{q\beta} c^\dagger_{d\bar{\beta}} c_{d\beta} c^\dagger_{q\beta} c_{k\bar{\beta}}$$

$$= -\frac{1}{2} J_z J_t \frac{1}{\omega' + \frac{1}{2}\epsilon_q + \frac{1}{2} J_z \left(\tau_{q\beta} - \tau_{q\bar{\beta}}\right)} c^\dagger_{d\bar{\beta}} c_{d\beta} \sum_{kk' \neq q} c^\dagger_{k'\beta} c_{k\bar{\beta}} \left(1 - n_{q\beta}\right) \tag{2.11b}$$

$$\left(\eta_0 c^\dagger_{q\beta} T_{q\beta}\right)_3 = J_z J_t \frac{1}{\omega' - H^D} \sum_{kk' \neq q} c^\dagger_{d\beta} c_{d\bar{\beta}} c^\dagger_{k'\bar{\beta}} c_{q\beta} \beta S_d^z c^\dagger_{q\beta} c_{k\beta}$$

$$= -\frac{1}{2} J_z J_t \frac{1}{\omega' + \frac{1}{2}\epsilon_q - \frac{1}{2} J_z \left(\tau_{q\beta} - \tau_{q\bar{\beta}}\right)} c^\dagger_{d\beta} c_{d\bar{\beta}} \sum_{kk' \neq q} c^\dagger_{k'\bar{\beta}} c_{k\beta} \left(1 - n_{q\beta}\right) \tag{2.11c}$$

$$\left(\eta_0 c^\dagger_{q\beta} T_{q\beta}\right)_4 = J_t^2 \frac{1}{\omega' - H^D} \sum_{kk' \neq q} c^\dagger_{d\beta} c_{d\bar{\beta}} c^\dagger_{k'\bar{\beta}} c_{q\beta} c^\dagger_{d\bar{\beta}} c_{d\beta} c^\dagger_{q\beta} c_{k\bar{\beta}}$$

$$= J_t^2 \frac{1}{\omega' + \frac{1}{2}\epsilon_q - \frac{1}{2} J_z \left(\tau_{q\beta} - \tau_{q\bar{\beta}}\right)} \left(\frac{1}{2} - \bar{\beta} S_d^z\right) \sum_{kk' \neq q} c^\dagger_{k'\bar{\beta}} c_{k\bar{\beta}} \left(1 - n_{q\beta}\right) \tag{2.11d}$$

2.4 Decoupling $q\beta$, $q\bar{\beta}$

We consider the decoupling for the initial condition $\tau_{q\beta} = -\tau_{q\bar{\beta}} = \frac{1}{2}$ with the ansatz $\omega = -\omega'$

Question 1 : **What is the motivation behind the initial condition $\tau_{q\beta} = -\tau_{q\bar{\beta}} = \frac{1}{2}$? Can't we take $\tau_{q\beta} = \tau_{q\bar{\beta}} = \frac{1}{2}$ for $q\beta$ below fermi level and $-1/2$ for above it?**

$$
\begin{aligned}
c^{\dagger}_{q\beta}T_{q\beta}\eta = \frac{1}{4}J_z^2\frac{1}{\omega-\frac{1}{2}\epsilon_q-\beta J_z S_d^z}\sum_{kk'\neq q}c_{k\beta}c^{\dagger}_{k'\beta} \\
+\frac{1}{2}J_zJ_t\frac{1}{\omega-\frac{1}{2}\epsilon_q-\frac{1}{2}J_z}c^{\dagger}_{d\beta}c_{d\bar{\beta}}\sum_{kk'\neq q}c_{k\beta}c^{\dagger}_{k'\bar{\beta}} \\
+\frac{1}{2}J_zJ_t\frac{1}{\omega-\frac{1}{2}\epsilon_q+\frac{1}{2}J_z}c^{\dagger}_{d\bar{\beta}}c_{d\beta}\sum_{kk'\neq q}c_{k\bar{\beta}}c^{\dagger}_{k'\beta} \\
+J_t^2\frac{1}{\omega-\frac{1}{2}\epsilon_q+\frac{1}{2}J_z}\left(\frac{1}{2}+\bar{\beta}S_d^z\right)\sum_{kk'\neq q}c_{k\bar{\beta}}c^{\dagger}_{k'\bar{\beta}}
\end{aligned}
\tag{2.12}
$$

$$
\begin{aligned}
\eta_0 c^{\dagger}_{q\bar{\beta}}T_{q\bar{\beta}} = -\frac{1}{4}J_z^2\frac{1}{\omega-\frac{1}{2}\epsilon_q-\bar{\beta} J_z S_d^z}\sum_{kk'\neq q}c^{\dagger}_{k'\bar{\beta}}c_{k\bar{\beta}} \\
+\frac{1}{2}J_zJ_t\frac{1}{\omega-\frac{1}{2}\epsilon_q+\frac{1}{2}J_z}c^{\dagger}_{d\beta}c_{d\bar{\beta}}\sum_{kk'\neq q}c^{\dagger}_{k'\bar{\beta}}c_{k\beta} \\
+\frac{1}{2}J_zJ_t\frac{1}{\omega-\frac{1}{2}\epsilon_q-\frac{1}{2}J_z}c^{\dagger}_{d\bar{\beta}}c_{d\beta}\sum_{kk'\neq q}c^{\dagger}_{k'\beta}c_{k\bar{\beta}} \\
-J_t^2\frac{1}{\omega-\frac{1}{2}\epsilon_q-\frac{1}{2}J_z}\left(\frac{1}{2}-\beta S_d^z\right)\sum_{kk'\neq q}c^{\dagger}_{k'\beta}c_{k\beta}
\end{aligned}
\tag{2.13}
$$

$$
\left(c^{\dagger}_{q\beta}T_{q\beta}\eta\right)_2+\left(\eta_0c^{\dagger}_{q\bar{\beta}}T_{q\bar{\beta}}\right)_2 = J_zJ_t\frac{\left(\omega-\frac{1}{2}\epsilon_q\right)}{\left(\omega-\frac{1}{2}\epsilon_q\right)^2-\left(\frac{1}{2}J_z\right)^2}c^{\dagger}_{d\beta}c_{d\bar{\beta}}\sum_{kk'\neq q}c^{\dagger}_{k'\bar{\beta}}c_{k\beta} \tag{2.14a}
$$

$$
\left(c^{\dagger}_{q\beta}T_{q\beta}\eta\right)_3+\left(\eta_0c^{\dagger}_{q\bar{\beta}}T_{q\bar{\beta}}\right)_3 = J_zJ_t\frac{\left(\omega-\frac{1}{2}\epsilon_q\right)}{\left(\omega-\frac{1}{2}\epsilon_q\right)^2-\left(\frac{1}{2}J_z\right)^2}c^{\dagger}_{d\bar{\beta}}c_{d\beta}\sum_{kk'\neq q}c^{\dagger}_{k'\beta}c_{k\bar{\beta}} \tag{2.14b}
$$

$$
\begin{aligned}
\left(c^{\dagger}_{q\beta}T_{q\beta}\eta\right)_1+\left(\eta_0c^{\dagger}_{q\bar{\beta}}T_{q\bar{\beta}}\right)_4 = \frac{J_t^2\left(\omega-\frac{1}{2}\epsilon_q\right)+\frac{1}{2}J_z\left(J_t^2-\frac{1}{2}J_z^2\right)}{\left(\omega-\frac{1}{2}\epsilon_q\right)^2-\left(\frac{1}{2}J_z\right)^2}\beta S_d^z\sum_{kk'\neq q}c^{\dagger}_{k'\beta}c_{k\beta} \\
-\frac{1}{2}\frac{\left(\omega-\frac{1}{2}\epsilon_q\right)\left(J_t^2+\frac{1}{2}J_z^2\right)+\frac{1}{2}J_zJ_t^2}{\left(\omega-\frac{1}{2}\epsilon_q\right)^2-\left(\frac{1}{2}J_z\right)^2}\sum_{kk'\neq q}c^{\dagger}_{k'\beta}c_{k\beta} \\
+\frac{1}{4}J_z^2\frac{\left(\omega+\frac{1}{2}\epsilon_q+\beta\mathbf{J_z S_d^z}\right)}{\left(\omega-\frac{1}{2}\epsilon_q\right)^2-\left(\frac{1}{2}J_z\right)^2}\sum_{k\neq q}1
\end{aligned}
\tag{2.14c}
$$

$$
\left(c^{\dagger}_{q\beta}T_{q\beta}\eta\right)_4+\left(\eta_0c^{\dagger}_{q\bar{\beta}}T_{q\bar{\beta}}\right)_1 = -\frac{J_t^2\left(\omega-\frac{1}{2}\epsilon_q\right)-\frac{1}{2}J_z\left(J_t^2-\frac{1}{2}J_z^2\right)}{\left(\omega-\frac{1}{2}\epsilon_q\right)^2-\left(\frac{1}{2}J_z\right)^2}\bar{\beta} S_d^z\sum_{kk'\neq q}c^{\dagger}_{k'\bar{\beta}}c_{k\bar{\beta}}
$$

$$- \frac{1}{2} \frac{\left(\omega - \frac{1}{2}\epsilon_q\right)\left(J_t^2 + \frac{1}{2}J_z^2\right) - \frac{1}{2}J_z J_t^2}{\left(\omega - \frac{1}{2}\epsilon_q\right)^2 - \left(\frac{1}{2}J_z\right)^2} \sum_{kk' \neq q} c^\dagger_{k'\bar{\beta}} c_{k\bar{\beta}}$$
$$+ J_t^2 \frac{\left(\omega + \frac{1}{2}\epsilon_q - \frac{1}{2}J_z\right)}{\left(\omega - \frac{1}{2}\epsilon_q\right)^2 - \left(\frac{1}{2}J_z\right)^2} \sum_{k \neq q} 1 \tag{2.14d}$$

Question 2: A magnetic field-like term (eq. (2.14c)) arises in ΔH if we don't sum Eqs. (2.14) over β.

2.5 Scaling Equations

Without summing over β

$$\Delta J_{z\uparrow} = \frac{J_t^2\left(\omega - \frac{1}{2}\epsilon_q\right) + \frac{1}{2}J_z\left(J_t^2 - \frac{1}{2}J_z^2\right)}{\left(\omega - \frac{1}{2}\epsilon_q\right)^2 - \left(\frac{1}{2}J_z\right)^2}$$
$$\Delta J_{z\downarrow} = -\frac{J_t^2\left(\omega - \frac{1}{2}\epsilon_q\right) - \frac{1}{2}J_z\left(J_t^2 - \frac{1}{2}J_z^2\right)}{\left(\omega - \frac{1}{2}\epsilon_q\right)^2 - \left(\frac{1}{2}J_z\right)^2} \tag{2.15}$$
$$\Delta J_t = J_z J_t \frac{\left(\omega - \frac{1}{2}\epsilon_q\right)}{\left(\omega - \frac{1}{2}\epsilon_q\right)^2 - \left(\frac{1}{2}J_z\right)^2}$$

The field-like term mentioned in question 2:

$$\frac{1}{4}J_z^2 \frac{\beta J_z S_d^z}{\left(\omega - \frac{1}{2}\epsilon_q\right)^2 - \left(\frac{1}{2}J_z\right)^2} \sum_{k \neq q} 1 \tag{2.16}$$

Putting $J_z = J_t = \frac{J}{2}$ in ΔJ_t, we get

$$\Delta J = \frac{1}{2}J^2 \frac{\left(\omega - \frac{1}{2}\epsilon_q\right)}{\left(\omega - \frac{1}{2}\epsilon_q\right)^2 - \left(\frac{1}{4}J\right)^2} \tag{2.17}$$

Question 4: This is double of what you get.

On summing over β

$$\Delta J_t = 2J_z J_t \frac{\left(\omega - \frac{1}{2}\epsilon_q\right)}{\left(\omega - \frac{1}{2}\epsilon_q\right)^2 - \left(\frac{1}{2}J_z\right)^2} \tag{2.18}$$
$$\Delta J_z = \frac{J_z\left(J_t^2 - \frac{1}{2}J_z^2\right)}{\left(\omega - \frac{1}{2}\epsilon_q\right)^2 - \left(\frac{1}{2}J_z\right)^2} \tag{2.19}$$

Question 3: **ΔJ_z doesn't have the same form as ΔJ_t.**