

For a simple 2d matrix \hat{A} , we have two eigenvalue equations corresponding to the two eigenvalues λ_i :

$$\begin{aligned}\hat{A} |\psi_1\rangle &= \lambda_1 |\psi_1\rangle, \quad \text{such that } \hat{P} |\psi_1\rangle = |1\rangle \\ \hat{A} |\psi_0\rangle &= \lambda_0 |\psi_0\rangle, \quad \text{such that } \hat{P} |\psi_0\rangle = |0\rangle\end{aligned}\tag{1}$$

$|\psi_i\rangle$ are the eigenvectors of \hat{A} and \hat{P} is the similarity transformation that diagonalizes \hat{A} . $|i\rangle$ are the number diagonal states.

In URG, we consider similar equations in the subspace of the node we are decoupling at present.

$$\begin{aligned}\mathcal{H} |\psi_1\rangle &= \tilde{\mathcal{H}}_1 |\psi_1\rangle, \quad \text{such that } \hat{P} |\psi_1\rangle = |1\rangle \\ \mathcal{H} |\psi_0\rangle &= \tilde{\mathcal{H}}_0 |\psi_0\rangle, \quad \text{such that } \hat{P} |\psi_0\rangle = |0\rangle\end{aligned}\tag{2}$$

The difference here is that the kets are many-body and the eigenvalue is not really a number but a smaller matrix. The structure is nevertheless identical. If we say work with the first equation, we can finally obtain

$$|\psi_1\rangle = (1 + \eta_1) |1\rangle = \left(1 + \eta_1^\dagger\right) |0\rangle\tag{3}$$

The important fact here is that from eq. 1, we can see that there isn't any connection between $|\psi_1\rangle$ and $|0\rangle$, so *we can ignore the second equality*, and construct our transformation purely from the first equality. That gives a similarity and consequently a unitary transformation.

$$\hat{P}_1^{-1} = 1 + \eta_1, \quad U_1 = \frac{1}{\sqrt{2}} \left(1 + \eta_1^\dagger - \eta_1\right)\tag{4}$$

Crucially important is the subscript 1 on η , \hat{P} and U^\dagger . That subscript signifies that those operators are functions of a particular eigenvalue of the quantum fluctuation operator - ω_1 .

*My previous contention was that applying U_1 does not give back the final equality in eq. 3, but that is not a problem since that equality does not have any significance to begin with; $|0\rangle$ must relate to $|\psi_0\rangle$ and **not** $|\psi_1\rangle$.*

If we had instead started with the second equation in 2, we would obtain

$$|\psi_0\rangle = \left(1 + \eta_0^\dagger\right) |0\rangle = (1 + \eta_0) |1\rangle\tag{5}$$

The subscript 0 means these operators are parametrised by a different eigenvalue ω_0 . In this case we must construct the similarity as well as the unitary from the equality relating $|0\rangle$ with $|\psi_0\rangle$. That gives

$$\hat{P}_0^{-1} = 1 + \eta_0^\dagger, \quad U_0 = \frac{1}{\sqrt{2}} \left(1 - \eta_0^\dagger + \eta_0\right)\tag{6}$$

If we want the two unitaries to give the same effective Hamiltonian, we must demand $U_0^\dagger = U_1^\dagger$ which translates to demanding $\eta_1 = -\eta_0$. From the expressions of η , this becomes

$$\frac{1}{\omega_1 - \mathcal{H}^d} T^\dagger c = -\frac{1}{\omega_0 - \mathcal{H}^d} T^\dagger c \implies \omega_1 + \omega_0 = 2\mathcal{H}^d\tag{7}$$

Note the **minus sign** on the RHS. This minus sign takes care of the apparent difference between URG and PMS. The final relation is also in line with the fact that at the fixed point, the ω eigenvalues match up with the Hamiltonian eigenvalues ($\omega_1 = \omega_0 = \mathcal{H}^d$); away from the fixed point, they change such that the total thing remains equal to twice the diagonal component. It is a manifestation of the trace-preserving nature of the transformation:

$$2\mathcal{H}^d = \omega_1 + \omega_0 = \tilde{H}_1 + \tilde{H}_0 - \mathcal{H}_0^i - \mathcal{H}_1^i \implies \tilde{H}_1 + \tilde{H}_0 - \mathcal{H}_0^i - \mathcal{H}_1^i = 2\mathcal{H}^d\tag{8}$$

Summary: Both the unitaries are perfectly correct but they relate to different values of $\hat{\omega}$.