Decomposition of Δu_S into holon, doublon and spin contributions
The total renormalization relevant to $(\hat{n}_{d\uparrow} - \hat{n}_{d\downarrow}) \sum_{kk'} \left(c^\dagger_{k\uparrow} c_{k'\uparrow} - c^\dagger_{k\downarrow} c_{k'\downarrow} \right)$ is

$$\begin{aligned}
&\sum_{q\beta kk'} c^\dagger_{k\beta} c_{k'\beta} \left(u_A^2 + 2u_S^2 \right) \hat{n}_{d\uparrow} \hat{n}_{d\downarrow} \left(\frac{1}{\omega + \epsilon_q^- - \epsilon_k} - \frac{1}{\omega - \epsilon_q^+ + \epsilon_{k'}} \right) \\
&- \sum_{q\beta kk'} c^\dagger_{k\beta} c_{k'\beta} u_S^2 \hat{n}_d \left(\frac{1}{\omega + \epsilon_q^- - \epsilon_k} - \frac{1}{\omega - \epsilon_q^+ + \epsilon_{k'}} \right) \\
&- \sum_{q\beta kk'} c^\dagger_{k\beta} c_{k'\beta} u_A^2 \left(\frac{\hat{n}_{d\beta}}{\omega + \epsilon_q^- - \epsilon_k} - \frac{\hat{n}_{d\bar{\beta}}}{\omega - \epsilon_q^+ + \epsilon_{k'}} \right)
\end{aligned} \tag{0.124}$$

- Holon contribution is zero, because setting $\hat{n}_{d\uparrow} = \hat{n}_{d\downarrow} = 0$ gives 0
- Doublon contribution is obtained by setting $\hat{n}_{d\uparrow} = \hat{n}_{d\downarrow} = 1$. That also gives 0, because the first term is canceled by the sum of the second and third terms.
- Spin contribution is obtained by setting $\hat{n}_{d\uparrow} + \hat{n}_{d\downarrow} = 1 \implies S^z_d = \frac{1}{2}(\hat{n}_{d\uparrow} - \hat{n}_{d\downarrow}) = \hat{n}_{d\uparrow} - \frac{1}{2} \implies \hat{n}_{d\uparrow} = \frac{1}{2} + S^z_d, \hat{n}_{d\downarrow} = \frac{1}{2} - S^z_d$. The total thing becomes

$$\begin{aligned}
&\sum_{q\beta kk'} c^\dagger_{k\beta} c_{k'\beta} \left(u_A^2 + u_S^2 \right) \left(\frac{1}{\omega + \epsilon_q^- - \epsilon_k} - \frac{1}{\omega - \epsilon_q^+ + \epsilon_{k'}} \right) \\
&- \sum_{qkk'} u_A^2 \left(\frac{\hat{n}_{d\uparrow}}{\omega + \epsilon_q^- - \epsilon_k} - \frac{\hat{n}_{d\downarrow}}{\omega - \epsilon_q^+ + \epsilon_{k'}} \right) \left(c^\dagger_{k\uparrow} c_{k'\uparrow} - c^\dagger_{k\downarrow} c_{k'\downarrow} \right) \\
&= \text{terms free of impurity operator} \\
&- \sum_{qkk'} S^z_d u_A^2 \left(\frac{1}{\omega + \epsilon_q^- - \epsilon_k} + \frac{1}{\omega - \epsilon_q^+ + \epsilon_{k'}} \right) s^z_{kk'}
\end{aligned} \tag{0.125}$$