4 Connection between Unitary Renormalization Group and Poor Man's Scaling

We first motivate the formalism of PMS method. The problem is defined as

$$\mathcal{H}\left|\Psi\right\rangle = E\left|\Psi\right\rangle \tag{4.1}$$

$\mathcal{H}$ is the total Hamiltonian and $|\Psi\rangle$ and E are the exact eigenstate and eigenvalue of $\mathcal{H}$. The problems we deal with typically have a bath of mobile electrons, with energies spanning from $-D$ to D. We are interested in finding effective Hamiltonians after "removing" the highest shell in the conduction bath. This will give us a Hamiltonian to which we can again apply the same procedure.

We want to decouple one electron at momentum q. We can split the exact wavefunction as

$$\left|\Psi\right\rangle = \left|\Psi_0\right\rangle + \left|\Psi_1\right\rangle \tag{4.2}$$

where $|\Psi_0\rangle = (1-\hat{n}_q)\,|\Psi^N\rangle$ is that part of the wavefunction where the state q is occupied. $|\Psi_1^N\rangle = \hat{n}_q\,|\Psi\rangle$ is that part of the wavefunction where the state q is occupied. We can also split the Hamiltonian as

$$\mathcal{H} = \mathcal{H}^d + V_0 + V_+ + V_- \tag{4.3}$$

$\mathcal{H}^d$ is the diagonal part; it has the purely energy terms as well as self-energies that may arise from the diagonal parts of interactions; V_0 is the purely off-diagonal term that does not change $\hat{n}_q$; it is the scattering *inside* the low energy subspace. V_+ and V_- are the purely off-diagonal terms that *do* change $\hat{n}_q$; V_+ takes you from $\hat{n}_q = 0$ to $\hat{n}_q = 1$ and V_- does the opposite.

Substituting eqs. 4.3 and 4.2 in eq. 4.1 gives

$$\left(\mathcal{H}^d + V_0 + V_+ + V_-\right)\left(\left|\Psi_0\right\rangle + \left|\Psi_1\right\rangle\right) = E\left(\left|\Psi_0\right\rangle + \left|\Psi_1\right\rangle\right) \tag{4.4}$$

Gathering the kets with $\hat{n}_q = 0, 1$ gives

$$\begin{aligned}\left(\mathcal{H}_0^d + V_0\right)\left|\Psi_0\right\rangle + V_-\left|\Psi_1\right\rangle &= E\left|\Psi_0\right\rangle \\ \left(\mathcal{H}_1^d + V_0\right)\left|\Psi_1\right\rangle + V_+\left|\Psi_0\right\rangle &= E\left|\Psi_1\right\rangle\end{aligned} \tag{4.5}$$

Eliminating $|\Psi_1\rangle$ from the two equations gives

$$\left(\mathcal{H}_0^d + V_0 + V_-\frac{1}{E - \mathcal{H}_1^d - V_0}V_+\right)\left|\Psi_0\right\rangle = E\left|\Psi_0\right\rangle \tag{4.6}$$

This new Hamiltonian,

$$\tilde{\mathcal{H}}_0 = \mathcal{H}_0^d + V_0 + V_-\frac{1}{E - \mathcal{H}_1^d - V_0}V_+ \tag{4.7}$$

has the high energy mode removed; the scattering terms start from the low energy subspace and end at the low energy subspace as well. The renormalization in the low energy subspace scatterings is

$$\Delta V_0\big|_{PMS} = V_- \frac{1}{E - \mathcal{H}_1^d - V_0} V_+ \tag{4.8}$$

We can write down the renormalized Schrodinger equation in the low energy subspace, from eq. 4.6,

$$\tilde{\mathcal{H}}_0 \left|\Psi_0\right\rangle = E \left|\Psi_0\right\rangle \tag{4.9}$$

and again repeat the entire process. $\tilde{\mathcal{H}}_0$ now takes the place of $\mathcal{H}$ and $|\Psi_0\rangle$ takes the place of $|\Psi\rangle$ in eq. 4.1.

The expression for URG is obtained in an almost identical way. The only difference is that instead of starting with the exact eigenpair $(E, |\Psi\rangle)$, we start with a more general pair $(\tilde{\mathcal{H}}, |\Phi\rangle)$ where $|\Phi\rangle$ is not necessarily an exact eigenstate of $\mathcal{H}$. It is defined by $\mathcal{H}'$, which is in turn defined as $\hat{n}_q \mathcal{H}' (1 - \hat{n}_q) = 0$. $|\Phi\rangle$ is then defined by

$$\mathcal{H} \left|\Phi\right\rangle = \mathcal{H}' \left|\Phi\right\rangle \tag{4.10}$$

This definition of $\mathcal{H}'$ is the very minimum that we must have in order to fulfill our goal (decouple q). The final expression for the renormalization can be written directly from the PMS expression simply by replacing E with $\mathcal{H}'$.

$$\Delta V_0\big|_{URG} = V_- \frac{1}{\mathcal{H}_1' - \mathcal{H}_1^d - V_0} V_+ \tag{4.11}$$

This allows us to write down a unitary operator that decouples the entangled state:

$$U \left|\Phi\right\rangle = \left|\Phi_0\right\rangle \text{ and } \left[\hat{n}_q, \tilde{\mathcal{H}}\right] = 0 \tag{4.12}$$

where $\tilde{\mathcal{H}} = U^\dagger \mathcal{H} U$. We can now write down a new problem in this decoupled space with the rotated items and attempt to decouple another electron q'. We will again choose some general eigenpair $(\mathcal{H}', |\Phi\rangle)$ such that $\tilde{\mathcal{H}} |\Phi\rangle = \mathcal{H}' |\Phi\rangle$ and $[\mathcal{H}', \hat{n}_{q'}] = 0$.

Summarizing, the general Hamiltonian is not diagonal in the Fock space basis. URG, in order to proceed, selects one non-Fock basis of states $|\Phi\rangle$ such that q is decoupled in that Hamiltonian. Since there can be lots of such basis, there is a freedom in this choice. With this basis in mind, URG then finds a unitary operator which when operated on the Hamiltonian takes me to the form in which it is diagonal in the Fock space basis. Note that this form is a function of the chosen $|\Psi\rangle$. We then select the second degree of freedom and repeat the process. What PMS does is, it exploits the freedom of choice and selects the exact eigenstate $|\Phi\rangle$ of the Hamiltonian as the non-Fock basis $|\Phi\rangle$. Doing that returns a rotated Hamiltonian which is diagonal in q, and is a function of the chosen state, same as URG. The conclusion is that depending on which state we choose as our diagonal non-Fock

basis, URG and PMS will cause flows along different lines in general.

In the context of URG, we define a quantum fluctuation energy scale $\hat{\omega}$:

$$\hat{\omega} = \mathcal{H}_1' - V_0 \tag{4.13}$$

This allows us to write

$$\Delta V_0\big|_{URG} = V_- \frac{1}{\hat{\omega} - \mathcal{H}_1^d} V_+ \tag{4.14}$$

As the couplings flow, V_0 will also flow, leading to a flow of $\hat{\omega}$. Just at the fixed point, the denominator of URG vanishes, giving the equation

$$\left(\hat{\omega} - \mathcal{H}_1^d\right) V_+ \left|\Psi_0\right\rangle \text{ or } \left(\hat{\omega} - \mathcal{H}_1^d\right) V_- \left|\Psi_1\right\rangle \tag{4.15}$$

This means that one of the eigenvalues of $\hat{\omega}$ matches with the eigenvalue of the diagonal part $\mathcal{H}^d$, either in the occupied sector ($\mathcal{H}_1^d$) or unoccupied sector ($\mathcal{H}_1^d$). Since the eigenvalues are unchanged during the unitary renormalization, this implies that ω takes up one of the eigenvalues of the whole Hamiltonian $\mathcal{H}$. This will correspond to the fixed point obtained from PMS if we had started PMS with that eigenvalue.

In short, while the PMS flow is parametrised by one of the exact energy eigenvalues E, the URG flow is parametrised by a non-trivial operator $\hat{\omega}$ which incorporates both a diagonal part and an off-diagonal part and itself flows under the URG. At the fixed point, the off-diagonal part cancels out and the $\hat{\omega}$ finally flows to one of the energy eigenvalues and the URG fixed point matches with one of the PMS fixed points.

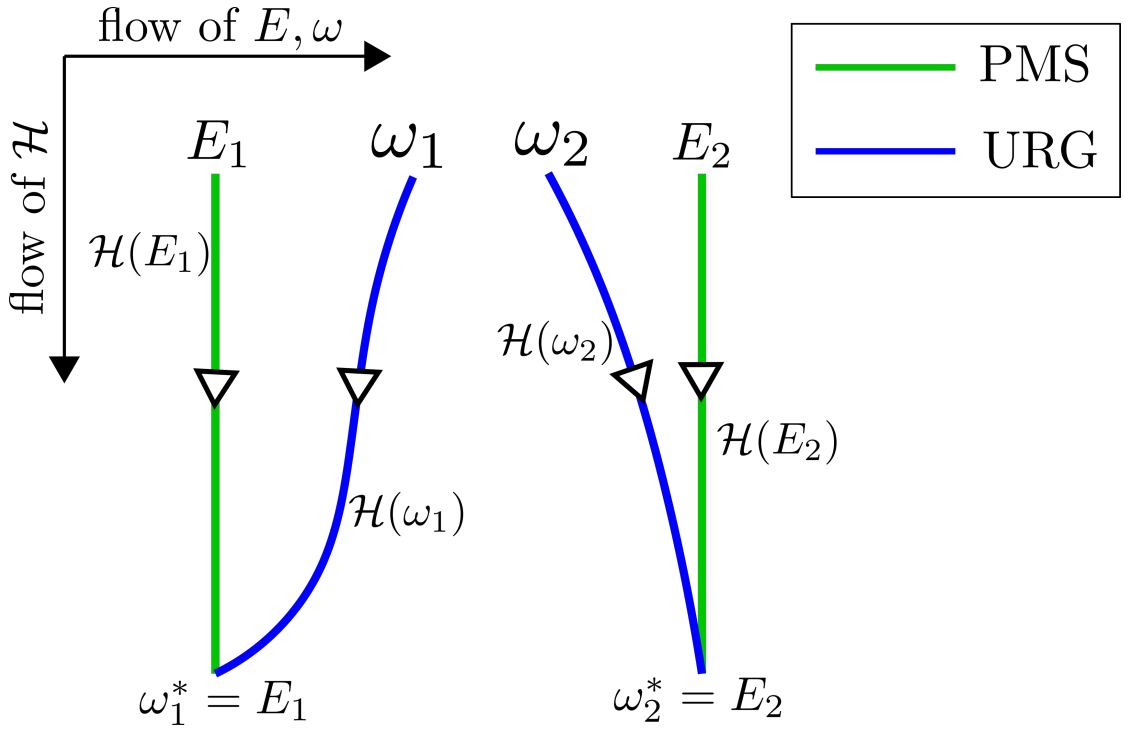

Figure 5: Flows of PMS(green) and URG(blue)