

Lax Pairs and Double Bracket Flows

1 Definition of a Lax Pair

Two operator A and B are said to form a lax pair if they satisfy the equation

$$\frac{dA(t)}{dt} = [B(t), A(t)] \quad (1)$$

2 Unitary Nature of the Flow

It can be shown that this defines a unitary time evolution on $A(t)$, in the following manner. Let $U(t, t_0)$ be the unitary operator that carries this evolution through. We then need to construct a $U(t, t_0)$.

$$A(t) = U(t, t_0)A(t_0)U^\dagger(t, t_0) \quad (2)$$

where $A(t_0)$ is the operator A at a particular time t_0 . The time change of A can then be written as

$$\begin{aligned} \frac{dA(t)}{dt} &= \frac{dU(t, t_0)}{dt}A(t_0)U^\dagger(t, t_0) + U(t, t_0)A(t_0)\frac{dU^\dagger(t, t_0)}{dt} \\ &= \frac{dU(t, t_0)}{dt}U^\dagger(t, t_0)A(t) + A(t)U(t, t_0)\frac{dU^\dagger(t, t_0)}{dt} \quad [A(t) = UAU^\dagger] \\ &= \frac{dU(t, t_0)}{dt}U^\dagger(t, t_0)A(t) - A(t)\frac{dU(t, t_0)}{dt}U^\dagger(t, t_0) \quad [UU^\dagger = 1] \\ &= \left[\frac{dU(t, t_0)}{dt}U^\dagger(t, t_0), A(t) \right] \end{aligned} \quad (3)$$

Looking at the definition of a lax pair, we can now make the connection

$$B(t) = \frac{dU(t, t_0)}{dt}U^\dagger(t, t_0) \quad (4)$$

The equation of motion characterised by the lax pair eq. 1 can thus be said to generate a family of unitarily connected operators $A(t)$, related by the unitaries defined by eq. 4. A direct corollary is that the spectrum of $A(t)$ is preserved during this evolution.

3 Double Bracket Flow

The double bracket flows correspond to a special choice of the operator $B(t)$: $B(t) \equiv [A(t), C]$. A consequence of this choice is that the lax pair evolution then serves to minimize the commutator $[A(t), C]$. To see how, we first write down a function

$$\chi \equiv \text{Tr} \left([A(t) - C]^2 \right) = \text{Tr} [A(t)^2 + C^2 - A(t)C - CA(t)] \quad (5)$$

Since $A^2(t) = U A^2 U^\dagger$, we get $\text{Tr}(A^2(t)) = \text{Tr}(A)$. Also, from the cyclic nature of trace, we can write $\text{Tr}(A(t)C) = \text{Tr}(CA(t))$. These considerations (and the fact that C does not depend on t) allows us to write

$$\frac{d\chi}{dt} = -2\text{Tr} \left(\frac{dA(t)}{dt} C \right) = -2\text{Tr} ([B(t), A(t)] C) \quad (6)$$

Using the cyclic property of trace, this becomes

$$\begin{aligned} \text{Tr} ([B(t), A(t)] C) &= \text{Tr} (B(t)A(t)C - A(t)B(t)C) \\ &= \text{Tr} (B(t)A(t)C - B(t)A(t)C) \\ &= \text{Tr} (B(t) [A(t), C]) \end{aligned} \quad (7)$$

If we now substitute the choice of $B(t)$ we made above, we get

$$\frac{d\chi}{dt} = -2\text{Tr} \left([A(t), C]^2 \right) = -2\text{Tr} (B(t)^2) \leq 0 \quad (8)$$

Since χ , the way it is defined, must necessarily be positive semi-definite for all t , it has a global minimum at $\chi = 0$. Since the derivative $\frac{d\chi}{dt}$ is always negative, it will take χ towards its minimum. At the minimum, the time derivative must vanish, otherwise $\chi(t)$ will become negative. This gives the result

$$\lim_{t \rightarrow \infty} \frac{d\chi}{dt} = -2 \lim_{t \rightarrow \infty} \text{Tr} \left([A(t), C]^2 \right) = 0 \implies \lim_{t \rightarrow \infty} [A(t), C] = 0 \quad (9)$$

In other words, **the lax pair evolution of $A(t)$ against $[A(t), C]$ leads to the diagonalization of $A(t)$ with respect to C .** This can be used as an iterative algorithm to diagonalize a general matrix with respect to another matrix:

- Define matrices A and B, A being the one we want to diagonalize w.r.t B
- Iteratively run the next two steps until a desired accuracy is reached
- Compute a new matrix $C = A*B - B*A$
- Change A as follows: $A = A + C*A - A*C$

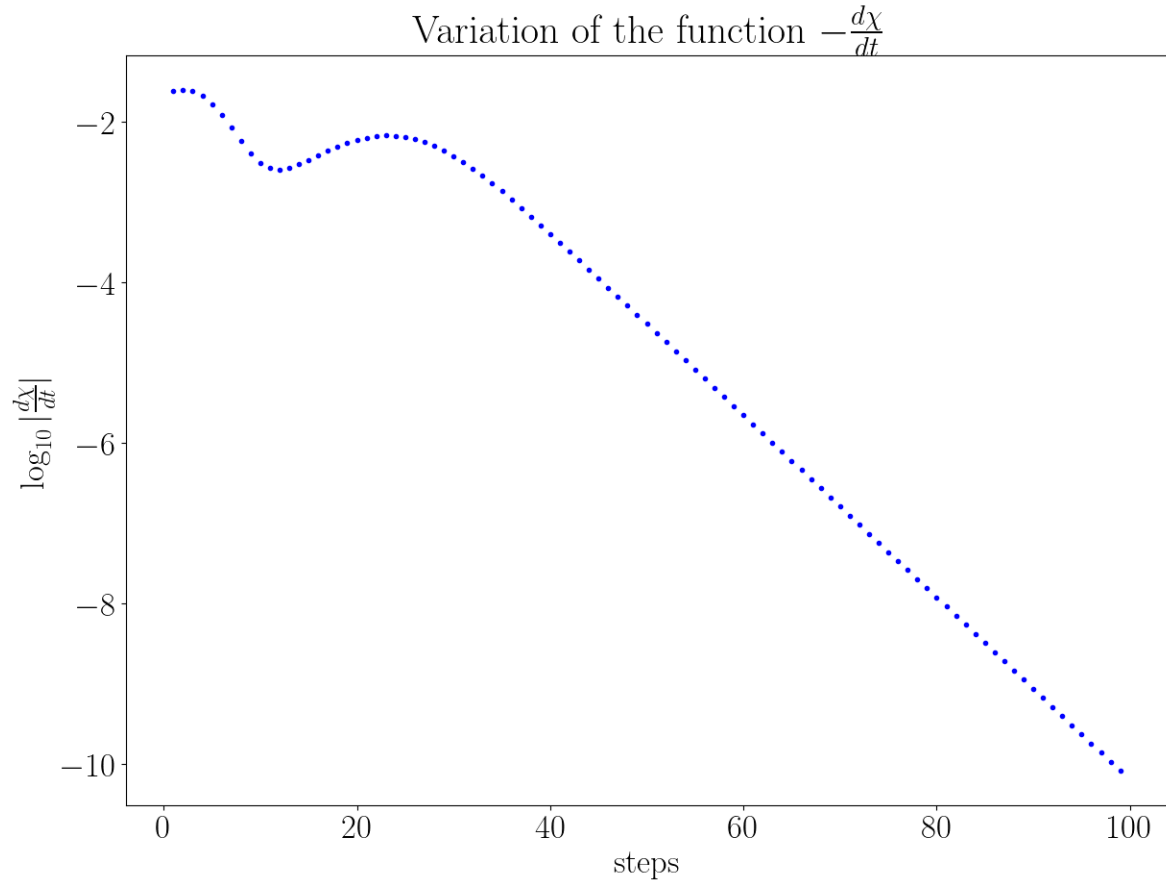


Figure 1: Variation of the function $\frac{d\chi}{dt}$ for an arbitrary choice of A and C . The convergence of A towards a diagonal form is clear.

The double bracket flow

$$\frac{dA}{dt} = [[A, C], A] \quad (10)$$

is thus a flow towards the minima of the function $\chi = \text{tr} \left((A - C)^2 \right)$.

4 URG as a double-bracket flow

The difference RG equation for URG can be written in the form

$$\Delta\mathcal{H}(\omega, D) = \frac{1}{\omega_1 - \omega_0} [G [\mathcal{H}^d, \mathcal{H}^I], \mathcal{H}] \quad (11)$$

This is not in the standard double-bracket form, primarily because it takes into account the off-diagonal terms in the Hamiltonian inside the generator of the unitary transformation. It can be given a double-bracket form by taking some approximations, as was shown in eq. ??.

Just like the standard double-bracket flow equation, the URG equation acts as an optimizer - it minimizes the function

$$\chi_j = \text{Tr} \left[(\mathcal{H}_j^I)^2 \right] \quad (12)$$

The definition of this function first requires a scheme to be defined. We can order the energy of the electrons as $\epsilon_1 < \epsilon_2 < \dots < \epsilon_j < \dots < \epsilon_N$. The URG consists of sequentially decoupling the states ϵ_N , then ϵ_{N-1} , and so on. At the j^{th} step, the Hamiltonian can be partitioned in the subspace of the electron being decoupled; the partitioning looks like

$$\mathcal{H}_j^0 + c_j^\dagger T_j + T_j^\dagger c_j \quad (13)$$

\mathcal{H}_j^0 is the part that *does not* scatter between $|\hat{n}_j\rangle = 0, 1$, while $\mathcal{H}_j^I = c_j^\dagger T_j + T_j^\dagger c_j$ is the part that *does* scatter between states with a definite value of \hat{n}_j .

The first observation that we make is that χ_j is semi-positive definite. This is because it can be expressed as the norm-squared of a state vector.

$$\chi_j = \sum_{i=1}^N \langle \psi_i | (\mathcal{H}_j^I)^2 | \psi_i \rangle = \sum_{i=1}^N \langle \phi_i | \phi_i \rangle \geq 0, \text{ [where } |\phi_i\rangle = \mathcal{H}_j^I |\psi_i\rangle] \quad (14)$$

The difference equation for χ_j is

$$\Delta\chi_j = 2\text{Tr} [\mathcal{H}_j^I \Delta\mathcal{H}_j^I] = 2\text{Tr} [\mathcal{H}_j^I (\mathcal{H}_{j-1}^I - \mathcal{H}_j^I)] \quad (15)$$

The first part of the trace is zero. To see why, note that from the nature of URG, once j has been decoupled, it is diagonal in all the subsequent Hamiltonians. Hence, \mathcal{H}_{j-1}^I will be diagonal in j , while \mathcal{H}_j^I is, by definition,

off-diagonal in j . The product $\mathcal{H}_j^I \mathcal{H}_{j-1}^I$ will hence be off-diagonal and will change \hat{n}_j . Hence, it will vanish when taken inside a trace. What remains is

$$\Delta\chi_j = -2\text{Tr} \left[(\mathcal{H}_j^I)^2 \right] = -2\chi_j \leq 0 \quad (16)$$

At the fixed point j^* of URG, the off-diagonal part of the Hamiltonian vanishes, so we can write $\mathcal{H}_{j^*}^I = 0 \implies \Delta\chi^* = 0$. Combining the three points:

$$\chi_j \geq 0, \quad \Delta\chi_j \leq 0, \quad \Delta\chi_{j^*} = 0 \quad (17)$$

we can say that URG starts from a non-minimal value of χ and flows to its minimum $\chi^* = 0$ at the fixed point.

This minimization has been demonstrated numerically in fig. 2, where URG has been performed on a very simple model of potential scatter: $\mathcal{H} = \sum_k \epsilon_k \hat{n}_k + J \sum_{k \neq k'} c_k^\dagger c_{k'}$.

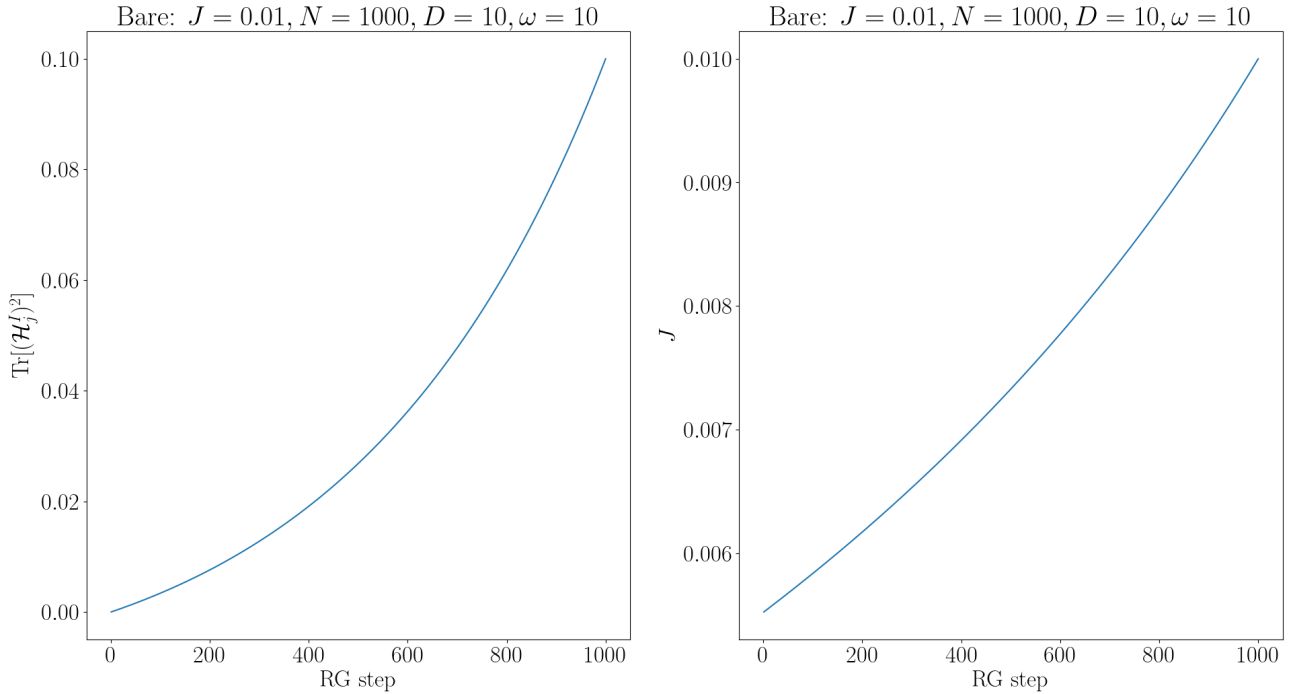


Figure 2: Variation of J and χ for the potential scattering problem.