Contents

1 Anderson Model URG

$$\mathcal{H} = \sum_{k\sigma} \epsilon_k \hat{n}_{k\sigma} + \sum_{k\sigma} \left(V_k c^\dagger_{k\sigma} c_{d\sigma} + h.c. \right) + \epsilon_d \sum_\sigma \hat{n}_{d\sigma} + U \hat{n}_{d\uparrow} \hat{n}_{d\downarrow} \tag{1.1}$$

One electron on shell

At first order, the rotated Hamiltonian is

$$\mathcal{H}_{j-1} = 2^{-n_j} \mathrm{Tr}_{1,2,\ldots,n_j} \mathcal{H}_j + \sum_{q\beta} \tau_{q\beta} \left\{ c^\dagger_{q\beta} \mathrm{Tr}_{q\beta} \left(\mathcal{H} c_{q\beta} \right), \eta_{q\beta} \right\} \tag{1.2}$$

n_j is the number of states on the shell Λ_j. We take the full Hamiltonian as our $\mathcal{H}_j$. Since this is the first step of the RG, the shell being decoupled is the highest one, which we call Λ_N.

Calculation of first term The first term, the initial trace, is a sequential trace over all the states on the shell being disentangled. At each trace, we consider only electrons on the current degree of freedom and on shells below the current shell:

$$\begin{aligned} \frac{1}{2} \mathrm{Tr}_{q\uparrow} \mathcal{H}_j &= \sum_{k<\Lambda_N, \sigma} \epsilon_k \hat{n}_{k\sigma} + \epsilon_d \sum_\sigma \hat{n}_{d\sigma} + U \hat{n}_{d\uparrow} \hat{n}_{d\downarrow} + \frac{1}{2} \mathrm{Tr}_{q\uparrow} \left\{ \epsilon_k \hat{n}_{q\uparrow} \right\} \\ &= \sum_{k<\Lambda_N, \sigma} \epsilon_k \hat{n}_{k\sigma} + \epsilon_d \sum_\sigma \hat{n}_{d\sigma} + U \hat{n}_{d\uparrow} \hat{n}_{d\downarrow} + \frac{1}{2} \epsilon_q \\ \frac{1}{2} \mathrm{Tr}_{q\downarrow} \frac{1}{2} \mathrm{Tr}_{q\uparrow} \mathcal{H}_j &= \sum_{k<\Lambda_N, \sigma} \epsilon_k \hat{n}_{k\sigma} + \epsilon_d \sum_\sigma \hat{n}_{d\sigma} + U \hat{n}_{d\uparrow} \hat{n}_{d\downarrow} + \epsilon_q \end{aligned} \tag{1.3}$$

$$\implies 2^{-n_j} \mathrm{Tr}_{1,2,\ldots,n_j} \mathcal{H}_j = \sum_{k<\Lambda_N, \sigma} \epsilon_k \hat{n}_{k\sigma} + \epsilon_d \sum_\sigma \hat{n}_{d\sigma} + U \hat{n}_{d\uparrow} \hat{n}_{d\downarrow} + \sum_{|q|=\Lambda_N} \epsilon_q \tag{1.4}$$

Calculation of second term The second term involves some other traces:

$$\begin{aligned} \mathrm{Tr}_{q\beta} \left(\mathcal{H} c_{q\beta} \right) &= \sum_{k\sigma} V_k \mathrm{Tr}_{q\beta} \left(c^\dagger_{k\sigma} c_{d\sigma} c_{q\beta} \right) \\ &= \sum_{k\sigma} V_k c_{d\sigma} \delta_{\sigma\beta} \delta_{kq} \\ &= V_q c_{d\beta} \\ \mathrm{Tr}_{q\beta} \left(c^\dagger_{q\beta} \mathcal{H} \right) &= V^*_q c^\dagger_{d\beta} \end{aligned} \tag{1.5}$$

$$
\begin{aligned}
\mathcal{H}^D &= \sum_{k\sigma} \epsilon_k \hat{n}_{k\sigma} + \epsilon_d \sum_\sigma \hat{n}_{d\sigma} + U \hat{n}_{d\uparrow}\hat{n}_{d\downarrow} \\
\mathrm{Tr}_{q\beta}\left(\mathcal{H}^D \hat{n}_{q\beta}\right) &= \sum_{k<\Lambda_N,\sigma} \epsilon_k \hat{n}_{k\sigma} + \epsilon_d \sum_\sigma \hat{n}_{d\sigma} + U \hat{n}_{d\uparrow}\hat{n}_{d\downarrow} + \epsilon_q \\
\eta_{q\beta} &= \mathrm{Tr}_{q\beta}\left(c^\dagger_{q\beta}\mathcal{H}\right) c_{q\beta} \frac{1}{\hat{\omega} - \mathrm{Tr}_{q\beta}\left(\mathcal{H}^D \hat{n}_{q\beta}\right)\hat{n}_{q\beta}} \\
&= V_q^* c^\dagger_{d\beta} c_{q\beta} \frac{1}{\hat{\omega} - \left(\sum_{k<\Lambda_N,\sigma} \epsilon_k \hat{n}_{k\sigma} + \epsilon_d \sum_\sigma \hat{n}_{d\sigma} + U \hat{n}_{d\uparrow}\hat{n}_{d\downarrow} - \epsilon_q\right)\hat{n}_{q\beta}} \\
&= V_q^* c^\dagger_{d\beta} c_{q\beta} \frac{1}{\omega\tau_{q\beta} - \left(\epsilon_d \sum_\sigma \hat{n}_{d\sigma} + U \hat{n}_{d\uparrow}\hat{n}_{d\downarrow} + \epsilon_q\right)\tau_{q\beta}}
\end{aligned} \tag{1.6}
$$

At the last step, **I replaced $\hat{\omega} - \sum_{k<\Lambda_N,\sigma} \epsilon_k \hat{n}_{k\sigma}\hat{n}_{q\beta} - \frac{1}{2}\left(\epsilon_d \sum_\sigma \hat{n}_{d\sigma} + U \hat{n}_{d\uparrow}\hat{n}_{d\downarrow} + \epsilon_q\right)$ with $\omega\tau_{q\beta}$**. Note that since this term has a $c^\dagger_{d\beta}$, it will not vanish only when acting on a state with $\hat{n}_{d\beta} = 0$. Hence we can drop the terms $\hat{n}_{d\uparrow}\hat{n}_{d\downarrow}$ and $\epsilon_{d\beta}\hat{n}_{d\beta}$ in the denominator. Also, since it has a $c_{q\beta}$, we can set the $\tau_{q\beta}$ in the denominator to $\frac{1}{2}$. Putting together the individual pieces, we can now write the second term:

$$
\begin{aligned}
\sum_{q\beta} \tau_{q\beta} \left\{ c^\dagger_{q\beta} \mathrm{Tr}_{q\beta}\left(\mathcal{H} c_{q\beta}\right), \eta_{q\beta} \right\} &= \sum_{q\beta} \tau_{q\beta} \left\{ V_q c^\dagger_{q\beta} c_{d\beta}, V_q^* c^\dagger_{d\beta} c_{q\beta} \frac{1}{\frac{1}{2}\left(\omega - \epsilon_q - \epsilon_d \hat{n}_{d\overline{\beta}}\right)} \right\} \\
&= \sum_{q\beta} 2\tau_{q\beta} \left\{ V_q c^\dagger_{q\beta} c_{d\beta}, V_q^* c^\dagger_{d\beta} c_{q\beta} \frac{1}{\omega - \epsilon_q - \epsilon_d \hat{n}_{d\overline{\beta}}} \right\}
\end{aligned} \tag{1.7}
$$

We now note that the factor with ω can be written as follows:

$$
\begin{aligned}
\frac{1}{\omega - \epsilon_q - \epsilon_d \hat{n}_{d\overline{\beta}}} &= \frac{\hat{n}_{d\overline{\beta}}}{\omega - \epsilon_q - \epsilon_d} + \frac{1 - \hat{n}_{d\overline{\beta}}}{\omega - \epsilon_q} \\
&= \hat{n}_{d\overline{\beta}} \frac{\epsilon_d}{\left(\omega - \epsilon_q - \epsilon_d\right)\left(\omega - \epsilon_q\right)} + \frac{1}{\omega - \epsilon_q}
\end{aligned} \tag{1.8}
$$

Since these terms commute with the other terms, they can be taken out of the anticommutator; what's left is

$$
\left\{ V_q c^\dagger_{q\beta} c_{d\beta}, V_q^* c^\dagger_{d\beta} c_{q\beta} \right\} = \left|V_q\right|^2 \left[\hat{n}_{q\beta}\left(1 - \hat{n}_{d\beta}\right) + \hat{n}_{d\beta}\left(1 - \hat{n}_{q\beta}\right)\right] \tag{1.9}
$$

The τ and the $\hat{n}$ can be multiplied:

$$
2\tau_{q\beta}\left(1 - \hat{n}_{q\beta}\right) = \left(\hat{n}_{q\beta} - 1\right) \tag{1.10}
$$

$$
2\tau_{q\beta}\hat{n}_{q\beta} = \hat{n}_{q\beta} \tag{1.11}
$$

The total thing becomes

$$
\begin{aligned}
\sum_{q\beta}\left|V_q\right|^2\left[\hat{n}_{d\beta}\left(\hat{n}_{q\beta}-1\right)+\hat{n}_{q\beta}\left(1-\hat{n}_{d\beta}\right)\right]&\left[\hat{n}_{d\bar{\beta}}\frac{\epsilon_d}{\left(\omega-\epsilon_q-\epsilon_d\right)\left(\omega-\epsilon_q\right)}+\frac{1}{\omega-\epsilon_q}\right]\\
=\sum_{q\beta}\left|V_q\right|^2\left[\hat{n}_{q\beta}-\hat{n}_{d\beta}\right]&\left[\hat{n}_{d\bar{\beta}}\frac{\epsilon_d}{\left(\omega-\epsilon_q-\epsilon_d\right)\left(\omega-\epsilon_q\right)}+\frac{1}{\omega-\epsilon_q}\right]
\end{aligned}
\tag{1.12}
$$

Putting $\hat{n}_{q\beta}=1$, and dropping the non-operator terms, we get

$$
\sum_\beta \hat{n}_{d\beta}\sum_q\left|V_q\right|^2\frac{\epsilon_q-\omega+2\epsilon_d}{\left(\omega-\epsilon_q\right)\left(\omega-\epsilon_q-\epsilon_d\right)}-\hat{n}_{d\uparrow}\hat{n}_{d\downarrow}\sum_{q\beta}\left|V_q\right|^2\frac{\epsilon_d}{\left(\omega-\epsilon_q\right)\left(\omega-\epsilon_q-\epsilon_d\right)}
\tag{1.13}
$$

The first term is the renormalization in on-site energy, $\sum_\beta \hat{n}_{d\beta}\Delta\epsilon_{d\beta}$, and the second term is the renormalization in the onsite repulsion, $\hat{n}_{d\uparrow}\hat{n}_{d\downarrow}\Delta U$.

Renormalized Hamiltonian Combining eqs. 1.4 and 1.13, we get

$$
\mathcal{H}_{N-1}=\sum_{k<\Lambda_N,\sigma}\epsilon_k\hat{n}_{k\sigma}+\sum_{|q|=\Lambda_N}\epsilon_q+\sum_\sigma\left(\epsilon_{d\sigma}+\Delta\epsilon_{d\sigma}\right)\hat{n}_{d\sigma}+\left(U+\Delta U\right)\hat{n}_{d\uparrow}\hat{n}_{d\downarrow}
\tag{1.14}
$$

The second term is the renormalization in the kinetic energy of the disentangled electrons, the third term is the renormalized impurity site energy and the fourth term is the renormalized onsite repulsion.

$$
\Delta\epsilon_d^N\equiv\epsilon_d\big|_{N-1}-\epsilon_d\big|_N=\sum_q\left|V_q\right|^2\frac{\epsilon_q-\omega+2\epsilon_d}{\left(\omega-\epsilon_q\right)\left(\omega-\epsilon_q-\epsilon_d\right)}
\tag{1.15}
$$

According to Hewson eq. 3.62 (page 68),

$$
\frac{\mathrm{d}\epsilon_d}{\mathrm{d}\ln D}=-\frac{\Delta}{\pi}+O(V^3)=-\rho_0|V|^2+O(V^3)
\tag{1.16}
$$

in the limit of $U+\epsilon_d\gg D$ and $|\epsilon_d|\ll D$, under the assumptions that V_k is independent of k and the conduction band is flat ($\rho(\epsilon)=\rho_0$ for $\epsilon\in[-D,D]$).

Assuming that we integrate out a ring at energy D and of thickness $-|\delta D|$, such that $\epsilon_q=D$ everywhere on the ring, the number of available states is

$$
\delta n=\frac{\mathrm{d}n}{\mathrm{d}E}\times\delta E=\rho(D)\times|\delta D|
\tag{1.17}
$$

We can then replace the summation in eq. 1.15 by δn:

$$
\delta\epsilon_d(D)=|V|^2\rho(D)|\delta D|\frac{D-\omega+2\epsilon_d}{\left(\omega-D\right)\left(\omega-D-\epsilon_d\right)}
\tag{1.18}
$$

where $\rho(D)$ is the number of single-spin states on the shell D. This can be compared to eq. 1.16. In two dimensions, the energy density of states is independent of energy. **Setting $\omega = 0$**, we get

$$\begin{aligned}\delta\epsilon_d(D) &= |V|^2\rho(D)|\delta D|\frac{D+2\epsilon_d}{D\,(D+\epsilon_d)} \\ &= |V|^2\rho(D)\frac{|\delta D|}{D}\frac{D+2\epsilon_d}{D+\epsilon_d}\end{aligned} \tag{1.19}$$

I used $\delta D = -|\delta D|$. Changing to continuum equation,

$$\frac{\mathrm{d}\epsilon_d}{\mathrm{d}\ln D} = -\frac{\Delta}{\pi}\frac{D+2\epsilon_d}{D+\epsilon_d} \tag{1.20}$$

In the regime where the single-occupied impurity level is comfortably inside the conduction band ($D \gg |\epsilon_d|$), we can approximate both the numerator and denominator as simply D. Then,

$$\frac{\mathrm{d}\epsilon_d}{\mathrm{d}\ln D} = -\frac{\Delta}{\pi} \tag{1.21}$$

$$\implies \epsilon_d + \frac{\Delta}{\pi}\log D = \text{constant} \tag{1.22}$$

Turning to the general equation 1.15, under the assumption of momentum-independent scattering, the continuum equation is

$$\begin{aligned}\frac{\mathrm{d}\epsilon_d}{\mathrm{d}\ln D} &= |V|^2 n(D)\frac{\omega-D-2\epsilon_d}{(\omega-D)\,(\omega-D-\epsilon_d)} \\ &= |V|^2 n(D)\left(\frac{2}{\omega-D}-\frac{1}{\omega-D-\epsilon_d}\right)\end{aligned} \tag{1.23}$$

$n(D)$ is not the density of states, but the total number of states on the shell at energy D. Similarly, the renormalization in U is

$$\begin{aligned}\delta U &= -\sum_{q\beta}|V_q|^2\frac{\epsilon_d}{(\omega-\epsilon_q)\,(\omega-\epsilon_q-\epsilon_d)} \\ &= -|V|^2 n(D)\sum_{\beta}\frac{\epsilon_d}{(\omega-D)\,(\omega-D-\epsilon_d)} \\ &= -2|V|^2 n(D)\frac{\epsilon_d}{(\omega-D)\,(\omega-D-\epsilon_d)} \\ \implies \frac{\mathrm{d}U}{\mathrm{d}\ln D} &= 2|V|^2 n(D)\frac{\epsilon_d}{(\omega-D)\,(\omega-D-\epsilon_d)} \\ &= 2|V|^2 n(D)\left(\frac{1}{\omega-D-\epsilon_d}-\frac{1}{\omega-D}\right)\end{aligned} \tag{1.24}$$

In the penultimate step, I used the fact that since the onsite energy for either spin is same, the summation just returns a factor of 2.

Putting $\omega = 0$,

$$\begin{aligned}\frac{\mathrm{d}\epsilon_d}{\mathrm{d}\ln D} &= |V|^2 n(D)\left(\frac{1}{D+\epsilon_d} - \frac{2}{D}\right)\\ \frac{\mathrm{d}U}{\mathrm{d}\ln D} &= 2|V|^2 n(D)\left(\frac{1}{D} - \frac{1}{D+\epsilon_d}\right)\end{aligned} \tag{1.25}$$

With higher order scattering

$$\mathcal{H} = \sum_{k\sigma}\epsilon_k \hat{n}_{k\sigma} + \sum_{k\sigma}\left(V_k c^\dagger_{k\sigma}c_{d\sigma} + h.c.\right) + \epsilon_d\sum_\sigma \hat{n}_{d\sigma} + U\hat{n}_{d\uparrow}\hat{n}_{d\downarrow} + \sum_{\substack{kk'\\ \sigma\sigma'}} V_2 c^\dagger_{d\sigma'}c^\dagger_{k\sigma}c_{d\sigma}c_{k'\sigma'} \tag{1.26}$$

Such an interaction allows both spin-flip ($d\sigma \to d\overline{\sigma}$) as well as spin-preserving ($d\sigma \to d\sigma$) scattering.

One electron on shell:

$$\begin{aligned}\mathcal{H}_N = H_0 + H_{\text{imp}} + \epsilon_q\hat{n}_{q\beta} + V_q c^\dagger_{q\beta}c_{d\beta} + V_q^* c^\dagger_{d\beta}c_{q\beta} - \sum_{k\sigma}V_2\left(c^\dagger_{q\beta}c_{k\sigma}c^\dagger_{d\sigma}c_{d\beta} + c^\dagger_{d\beta}c_{d\sigma}c^\dagger_{k\sigma}c_{q\beta}\right)\\ +V_2\hat{n}_{q\beta}\hat{n}_{d\beta}\end{aligned} \tag{1.27}$$

For $\hat{n}_{q\beta} = 1$:

$$\begin{aligned}\Delta\mathcal{H}_N &= \sum_{q\beta}\tau_{q\beta}c^\dagger_{q\beta}\left[V_q - \sum_{k\sigma}V_2 c_{k\sigma}c^\dagger_{d\sigma}\right]c_{d\beta}\times c^\dagger_{d\beta}\left[V_q^* - \sum_{k'\sigma'}V_2 c_{d\sigma'}c^\dagger_{k'\sigma'}\right]c_{q\beta}\\ &\quad\times\frac{1}{\hat{\omega} - \left(H_{\text{imp}} + V_2\hat{n}_{d\beta} + \epsilon_q\right)\hat{n}_{q\beta}}\\ &= \sum_{q\beta}\left[V_q - \sum_{k\sigma}V_2 c_{k\sigma}c^\dagger_{d\sigma}\right]\left(1-\hat{n}_{d\beta}\right)\left[V_q^* - \sum_{k'\sigma'}V_2 c_{d\sigma'}c^\dagger_{k'\sigma'}\right]\frac{1}{\omega - \left(H_{\text{imp}} + V_2\hat{n}_{d\beta} + \epsilon_q\right)}\\ &= \sum_{q\beta}\left[|V_q|^2\left(1-\hat{n}_{d\beta}\right) - \sum_{k'\sigma'}V_qV_2\left(1-\hat{n}_{d\beta}\right)c_{d\sigma'}c^\dagger_{k'\sigma'} - \sum_{k\sigma}V_q^*V_2 c_{k\sigma}c^\dagger_{d\sigma}\left(1-\hat{n}_{d\beta}\right)\right.\\ &\quad\left. + \sum_{kk'\sigma\sigma'}V_2^2 c_{k\sigma}c^\dagger_{d\sigma}\left(1-\hat{n}_{d\beta}\right)c_{d\sigma'}c^\dagger_{k'\sigma'}\right]\frac{1}{\omega - \left(H_{\text{imp}} + V_2\hat{n}_{d\beta} + \epsilon_q\right)}\end{aligned} \tag{1.28}$$

The first term in $\Delta\mathcal{H}_N$ is (calculated in the previous section)

$$\sum_{q\beta} |V_q|^2 \left(1-\hat{n}_{d\beta}\right) \frac{1}{\omega - \left(H_{\text{imp}} + V_2 \hat{n}_{d\beta} + \epsilon_q\right)} = \sum_{q\beta} |V_q|^2 \frac{\hat{n}_{d\beta}\left(\epsilon_q - \omega + 2\epsilon_d\right) - \hat{n}_{d\uparrow}\hat{n}_{d\downarrow}\epsilon_d}{\left(\omega-\epsilon_q\right)\left(\omega-\epsilon_q-\epsilon_d\right)} \tag{1.29}$$

Just as in the previous section, they renormalize the onsite energy ϵ_d and double-occupation penalty U. The third term in $\Delta\mathcal{H}_N$ gives

$$\begin{aligned}
&-\sum_{k\sigma q\beta} c_{k\sigma} c^\dagger_{d\sigma} V_q^* V_2 \left(1-\hat{n}_{d\beta}\right) \frac{1}{\omega - \left(H_{\text{imp}} + V_2 \hat{n}_{d\beta} + \epsilon_q\right)} \\
&= -\sum_{k\sigma q} c_{k\sigma} c^\dagger_{d\sigma} V_q^* V_2 \left[\frac{2}{\omega-\epsilon_q} + \frac{\sum_\beta \hat{n}_{d\beta}\left(\epsilon_q-\omega+2\epsilon_d\right)}{\left(\omega-\epsilon_q\right)\left(\omega-\epsilon_q-\epsilon_d\right)} - \frac{2\hat{n}_{d\uparrow}\hat{n}_{d\downarrow}\epsilon_d}{\left(\omega-\epsilon_q\right)\left(\omega-\epsilon_q-\epsilon_d\right)}\right] \\
&= -\sum_{k\sigma q} c_{k\sigma} c^\dagger_{d\sigma} V_q^* V_2 \left[\frac{2}{\omega-\epsilon_q} + \frac{\hat{n}_{d\overline{\sigma}}\left(\epsilon_q-\omega+2\epsilon_d\right)}{\left(\omega-\epsilon_q\right)\left(\omega-\epsilon_q-\epsilon_d\right)}\right] \\
&= \sum_{k\sigma q} c^\dagger_{d\sigma} c_{k\sigma} V_q^* V_2 \left[\frac{2}{\omega-\epsilon_q} + \frac{\hat{n}_{d\overline{\sigma}}\left(\epsilon_q-\omega+2\epsilon_d\right)}{\left(\omega-\epsilon_q\right)\left(\omega-\epsilon_q-\epsilon_d\right)}\right]
\end{aligned} \tag{1.30}$$

In the penultimate step, I used

$$c^\dagger_{d\sigma} \times \sum_\beta \hat{n}_{d\beta} = c^\dagger_{d\sigma} \times \left(\hat{n}_{d\sigma} + \hat{n}_{d\overline{\sigma}}\right) = c^\dagger_{d\sigma}\hat{n}_{d\overline{\sigma}} \tag{1.31}$$

and

$$c^\dagger_{d\sigma} \times \hat{n}_{d\uparrow}\hat{n}_{d\downarrow} = c^\dagger_{d\sigma} \times \hat{n}_{d\sigma}\hat{n}_{d\overline{\sigma}} = 0 \tag{1.32}$$

The first of these terms renormalizes the coupling $V^*_{k'}$. The second term in $\Delta\mathcal{H}_N$ gives

$$\begin{aligned}
&-\sum_{q\beta k\sigma} V_q V_2 \left(1-\hat{n}_{d\beta}\right) c_{d\sigma} c^\dagger_{k\sigma} \frac{1}{\omega - \left(H_{\text{imp}} + V_2 \hat{n}_{d\beta} + \epsilon_q\right)} \\
&= -\sum_{qk\sigma} V_q V_2 c_{d\sigma} c^\dagger_{k\sigma} \left[\frac{1}{\omega - \left(H_{\text{imp}} + V_2\hat{n}_{d\sigma} + \epsilon_q\right)} + \left(1-\hat{n}_{d\overline{\sigma}}\right)\frac{1}{\omega - \left(H_{\text{imp}} + \epsilon_q\right)}\right] \\
&= -\sum_{qk\sigma} V_q V_2 c_{d\sigma} c^\dagger_{k\sigma} \left[\frac{2\left(\omega-\epsilon_q-\epsilon_d\right) - V_2}{\left(\omega-\epsilon_d-\epsilon_q\right)\left(\omega-\epsilon_d-\epsilon_q-V_2\right)}\right. \\
&\quad + \quad \left.\hat{n}_{d\overline{\sigma}}\left\{\frac{\epsilon_d+U}{\left(\omega-2\epsilon_d-U-\epsilon_q-V_2\right)\left(\omega-\epsilon_d-\epsilon_q-V_2\right)} - \frac{1}{\omega-\epsilon_q-\epsilon_d}\right\}\right] \\
&= \sum_{qk\sigma} V_q V_2 c^\dagger_{k\sigma} c_{d\sigma} \left[\frac{2\left(\omega-\epsilon_q-\epsilon_d\right) - V_2}{\left(\omega-\epsilon_d-\epsilon_q\right)\left(\omega-\epsilon_d-\epsilon_q-V_2\right)}\right. \\
&\quad + \quad \left.\hat{n}_{d\overline{\sigma}}\left\{\frac{\epsilon_d+U}{\left(\omega-2\epsilon_d-U-\epsilon_q-V_2\right)\left(\omega-\epsilon_d-\epsilon_q-V_2\right)} - \frac{1}{\omega-\epsilon_q-\epsilon_d}\right\}\right]
\end{aligned} \tag{1.33}$$

The first of these terms renormalizes the coupling $V_{k'}$. The fourth term gives

$$
\begin{aligned}
&\sum_{kk'q\beta\sigma\sigma'} V_2^2 c_{k\sigma} c_{d\sigma}^{\dagger} \left(1-\hat{n}_{d\beta}\right) c_{d\sigma'} c_{k'\sigma'}^{\dagger} \frac{1}{\omega-\left(H_{\mathrm{imp}}+\epsilon_q\right)} \\
&= \sum_{kk'q\sigma\sigma'} V_2^2 c_{k\sigma} c_{d\sigma}^{\dagger} c_{d\sigma'} c_{k'\sigma'}^{\dagger} \left[\frac{2\left(\omega-\epsilon_q-\epsilon_d\right)-V_2}{\left(\omega-\epsilon_d-\epsilon_q\right)\left(\omega-\epsilon_d-\epsilon_q-V_2\right)}\right. \\
&\quad + \left.\hat{n}_{d\bar{\sigma}}\left\{\frac{\epsilon_d+U}{\left(\omega-2\epsilon_d-U-\epsilon_q-V_2\right)\left(\omega-\epsilon_d-\epsilon_q-V_2\right)}-\frac{1}{\omega-\epsilon_q-\epsilon_d}\right\}\right] \\
&= \sum_{kk'q\sigma\sigma'} V_2^2 \hat{n}_{d\sigma} \left[\frac{2\left(\omega-\epsilon_q-\epsilon_d\right)-V_2}{\left(\omega-\epsilon_d-\epsilon_q\right)\left(\omega-\epsilon_d-\epsilon_q-V_2\right)}\right. \\
&\quad + \left.\hat{n}_{d\bar{\sigma}}\left\{\frac{\epsilon_d+U}{\left(\omega-2\epsilon_d-U-\epsilon_q-V_2\right)\left(\omega-\epsilon_d-\epsilon_q-V_2\right)}-\frac{1}{\omega-\epsilon_q-\epsilon_d}\right\}\right] \\
&+ \sum_{kk'q\sigma\sigma'} V_2^2 c_{d\sigma}^{\dagger} c_{k'\sigma'}^{\dagger} c_{d\sigma'} c_{k\sigma} \left[\frac{2\left(\omega-\epsilon_q-\epsilon_d\right)-V_2}{\left(\omega-\epsilon_d-\epsilon_q\right)\left(\omega-\epsilon_d-\epsilon_q-V_2\right)}\right. \\
&\quad + \left.\hat{n}_{d\bar{\sigma}}\left\{\frac{\epsilon_d+U}{\left(\omega-2\epsilon_d-U-\epsilon_q-V_2\right)\left(\omega-\epsilon_d-\epsilon_q-V_2\right)}-\frac{1}{\omega-\epsilon_q-\epsilon_d}\right\}\right]
\end{aligned}
\tag{1.34}
$$

The first line in the final equation describes the renormalization of ϵ_d and U at order V_2^2. The first term in the second line of the last equation describes the renormalization of the two-particle interaction coupling, V_2.

The changes in the couplings are

$$
\begin{aligned}
\Delta\epsilon_d &= \sum_q \left[|V_q|^2 \frac{\epsilon_q-\omega+2\epsilon_d}{\left(\omega-\epsilon_q\right)\left(\omega-\epsilon_q-\epsilon_d\right)} + V_2^2 \frac{2\left(\omega-\epsilon_q-\epsilon_d\right)-V_2}{\left(\omega-\epsilon_d-\epsilon_q\right)\left(\omega-\epsilon_d-\epsilon_q-V_2\right)}\right] \\
\Delta U &= \sum_q \left[-\frac{2|V_q|^2\epsilon_d}{\left(\omega-\epsilon_q\right)\left(\omega-\epsilon_q-\epsilon_d\right)} + \frac{V_2^2\left(\epsilon_d+U\right)}{\left(\omega-2\epsilon_d-U-\epsilon_q-V_2\right)\left(\omega-\epsilon_d-\epsilon_q-V_2\right)}\right. \\
&\quad \left. - \frac{V_2^2}{\omega-\epsilon_q-\epsilon_d}\right] \\
\Delta V_k &= 2\sum_q V_q V_2 \frac{\left(\omega-\epsilon_q-\epsilon_d\right)-\frac{V_2}{2}}{\left(\omega-\epsilon_d-\epsilon_q\right)\left(\omega-\epsilon_d-\epsilon_q-V_2\right)} \\
\Delta V_k^* &= 2\sum_q \frac{V_q^* V_2}{\omega-\epsilon_q} \\
\Delta V_2 &= 2\sum_q V_2^2 \frac{\left(\omega-\epsilon_q-\epsilon_d\right)-\frac{V_2}{2}}{\left(\omega-\epsilon_d-\epsilon_q\right)\left(\omega-\epsilon_d-\epsilon_q-V_2\right)}
\end{aligned}
\tag{1.35}
$$

The renormalized Hamiltonian is

$$\begin{aligned}\mathcal{H}_{N-1} &= \sum_{k\sigma}\left[\epsilon_k^{N-1}\hat{n}_{k\sigma} + V_k^{N-1}c^\dagger_{k\sigma}c_{d\sigma} + h.c.\right] + \epsilon_d^{N-1}\sum_\sigma \hat{n}_{d\sigma} + U^{N-1}\hat{n}_{d\uparrow}\hat{n}_{d\downarrow}\\
&+ \sum_{\substack{kk'\\ \sigma\sigma'}} V_2^{N-1}c^\dagger_{d\sigma'}c^\dagger_{k\sigma}c_{d\sigma}c_{k'\sigma'} + \frac{1}{2}\sum_{q\beta}\left(\epsilon_q\tau_{q\beta} - V_2\hat{n}_{d\beta}\tau_{q\beta}\right)\\
&+ \sum_{k\sigma q} c^\dagger_{d\sigma}c_{k\sigma}\hat{n}_{d\overline{\sigma}}V_q^* V_2 C_1 - \sum_{qk\sigma} V_q V_2 c^\dagger_{k\sigma}c_{d\sigma}\hat{n}_{d\overline{\sigma}}C_2 - \sum_{kk'q\sigma\sigma'} V_2^2 c^\dagger_{k'\sigma'}c^\dagger_{d\sigma}c_{d\sigma'}c_{k\sigma}\hat{n}_{d\overline{\sigma}'}C_3\end{aligned} \tag{1.36}$$

Sanity Checks ($\omega = 0$)

1. of ϵ_d

$$\begin{aligned}\delta\epsilon_d &= \sum_q |V_q|^2\frac{\epsilon_q - \omega + 2\epsilon_d}{\left(\omega - \epsilon_q\right)\left(\omega - \epsilon_q - \epsilon_d\right)}\\
&= |V|^2 n(D)\frac{D}{D^2} \qquad \text{[D very large]}\\
&= -|V|^2\rho\delta D\frac{1}{D}\\
&= -\frac{\Delta}{\pi}\delta\ln D\\
\implies \frac{\mathrm{d}\epsilon_d}{\mathrm{d}\ln D} &= -\frac{\Delta}{\pi} \qquad \text{[matches with Hewson]}\end{aligned} \tag{1.37}$$

2. of U

$$\begin{aligned}\delta U &= -\sum_q |V_q|^2\frac{2\epsilon_d}{\left(\omega - \epsilon_q\right)\left(\omega - \epsilon_q - \epsilon_d\right)}\\
&= |V|^2\rho\delta D\frac{2\epsilon_d}{D^2} \qquad \text{[very small, matches with Hewson]}\end{aligned} \tag{1.38}$$

3. of V_1

$$\begin{aligned}
\delta V_1 &= \sum_q V_q V_2 \frac{2\left(\omega-\epsilon_q-\epsilon_d\right)-V_2}{\left(\omega-\epsilon_d-\epsilon_q\right)\left(\omega-\epsilon_d-\epsilon_q-V_2\right)} \\
&= V_1 V_2 \rho \delta D \frac{2\left(D+\epsilon_d\right)+V_2}{\left(\epsilon_d+D\right)\left(\epsilon_d+D+V_2\right)} \\
&= V_1 V_2 \frac{\delta D}{2 D_0} \frac{1}{D} \\
\implies \frac{\mathrm{d} V_1}{\mathrm{d} D} &= \frac{V_1 V_2}{D_0 D} \\
&\quad\text{[matches with Jefferson up to a D, should come from the definition of } V_2\text{]}
\end{aligned} \tag{1.39}$$

For $\hat{n}_{q\beta}=0$:

$$\begin{aligned}
&\Delta\mathcal{H}_N \\
&= \sum_{q\beta} \tau_{q\beta} \frac{1}{\hat{\omega}-H_{imp}\left(1-\hat{n}_{q\beta}\right)} c_{d\beta}^\dagger \left[V_q^* - \sum_{k'\sigma'} V_2 c_{d\sigma'} c_{k'\sigma'}^\dagger\right] c_{q\beta} c_{q\beta}^\dagger \left[V_q - \sum_{k\sigma} V_2 c_{k\sigma} c_{d\sigma}^\dagger\right] c_{d\beta} \\
&= \sum_{q\beta} \frac{1}{\omega+H_{imp}+\epsilon_q+V_2\hat{n}_{d\beta}} c_{d\beta}^\dagger \left[V_q^* - \sum_{k'\sigma'} V_2 c_{d\sigma'} c_{k'\sigma'}^\dagger\right] \left[V_q - \sum_{k\sigma} V_2 c_{k\sigma} c_{d\sigma}^\dagger\right] c_{d\beta}
\end{aligned} \tag{1.40}$$

Here I replaced $\hat{\omega}-\frac{\epsilon_q}{2}-\frac{H_{imp}+V_2\hat{n}_{d\beta}}{2}$ with $\hat{\omega}$, just as in the previous section, and then set $\hat{n}_{q\beta}=0, \tau_{q\beta}=-\frac{1}{2}$. Now,

$$\begin{aligned}
\frac{1}{\omega+H_{imp}+V_2\hat{n}_{d\beta}+\epsilon_q} c_{d\beta}^\dagger &= \frac{1}{\omega+\epsilon_d+V_2+\left(\epsilon_d+U\right)\hat{n}_{d\overline{\beta}}+\epsilon_q} c_{d\beta}^\dagger \\
&= \left[\frac{1}{\omega+\epsilon_d+V_2+\epsilon_q} - \frac{\hat{n}_{d\overline{\beta}}\left(\epsilon_d+U\right)}{\left(\omega+2\epsilon_d+U+V_2+\epsilon_q\right)\left(\omega+\epsilon_d+V_2+\epsilon_q\right)}\right] c_{d\beta}^\dagger
\end{aligned} \tag{1.41}$$

The first term in $\Delta\mathcal{H}_N$ gives

$$\begin{aligned}
&\sum_{q\beta} \frac{1}{\omega+H_{imp}+V_2+\epsilon_q} |V_q|^2 \hat{n}_{d\beta} \\
&\quad= \sum_{q\beta} \left[\frac{|V_q|^2}{\omega+\epsilon_d+V_2+\epsilon_q} - \frac{|V_q|^2 \hat{n}_{d\overline{\beta}}\left(\epsilon_d+U\right)}{\left(\omega+2\epsilon_d+U+V_2+\epsilon_q\right)\left(\omega+\epsilon_d+V_2+\epsilon_q\right)}\right] \hat{n}_{d\beta} \\
&\quad= \sum_{q\beta} \frac{\hat{n}_{d\beta}|V_q|^2}{\omega+\epsilon_d+V_2+\epsilon_q} - \hat{n}_{d\uparrow}\hat{n}_{d\downarrow} \sum_q \frac{2|V_q|^2\left(\epsilon_d+U\right)}{\left(\omega+2\epsilon_d+U+V_2+\epsilon_q\right)\left(\omega+\epsilon_d+V_2+\epsilon_q\right)}
\end{aligned} \tag{1.42}$$

The second term gives

$$\sum_{k\sigma q\beta}\frac{-V_q^*V_2}{\omega+H_{imp}+V_2+\epsilon_q}c^\dagger_{d\beta}c_{k\sigma}c^\dagger_{d\sigma}c_{d\beta}=-\sum_{k\sigma q}\frac{V_q^*V_2}{\omega+H_{imp}+V_2+\epsilon_q}\left(c^\dagger_{d\sigma}c_{k\sigma}c^\dagger_{d\sigma}c_{d\sigma}+c^\dagger_{d\bar\sigma}c_{k\sigma}c^\dagger_{d\sigma}c_{d\bar\sigma}\right) \tag{1.43}$$

The first term on the RHS is zero, because it has two consecutive $c^\dagger_{d\sigma}$.

$$\begin{aligned}
-\sum_{k\sigma q\beta}\frac{V_q^*V_2}{\omega+H_{imp}+V_2+\epsilon_q}c^\dagger_{d\beta}c_{k\sigma}c^\dagger_{d\sigma}c_{d\beta}&=-\sum_{k\sigma q}\frac{V_q^*V_2}{\omega+H_{imp}+V_2+\epsilon_q}c^\dagger_{d\bar\sigma}c_{k\sigma}c^\dagger_{d\sigma}c_{d\bar\sigma}\\
&=-\sum_{k\sigma q}\frac{V_q^*V_2}{\omega+H_{imp}+V_2+\epsilon_q}\hat{n}_{d\bar\sigma}c_{k\sigma}c^\dagger_{d\sigma}\\
&=-\sum_{k\sigma q}\frac{V_q^*V_2}{\omega+\epsilon_d+(\epsilon_d+U)\,\hat{n}_{d\sigma}+V_2+\epsilon_q}\hat{n}_{d\bar\sigma}c_{k\sigma}c^\dagger_{d\sigma}\\
&=-\sum_{k\sigma q}\frac{V_q^*V_2}{\omega+2\epsilon_d+U+V_2+\epsilon_q}\hat{n}_{d\bar\sigma}c_{k\sigma}c^\dagger_{d\sigma}
\end{aligned} \tag{1.44}$$

The third term gives

$$\begin{aligned}
-\sum_{k\sigma q\beta}\frac{V_qV_2}{\omega+H_{imp}+V_2+\epsilon_q}c^\dagger_{d\beta}c_{d\sigma}c^\dagger_{k\sigma}c_{d\beta}&=-\sum_{k\sigma q}\frac{V_qV_2}{\omega+H_{imp}+V_2+\epsilon_q}c^\dagger_{d\bar\sigma}c_{d\sigma}c^\dagger_{k\sigma}c_{d\bar\sigma}\\
&=-\sum_{k\sigma q}\frac{V_qV_2}{\omega+\epsilon_d+(\epsilon_d+U)\,\hat{n}_{d\sigma}+V_2+\epsilon_q}\hat{n}_{d\bar\sigma}c_{d\sigma}c^\dagger_{k\sigma}\\
&=-\sum_{k\sigma q}\frac{V_qV_2}{\omega+\epsilon_d+V_2+\epsilon_q}\hat{n}_{d\bar\sigma}c_{d\sigma}c^\dagger_{k\sigma}
\end{aligned} \tag{1.45}$$

The fourth term gives

$$
\begin{aligned}
&\sum_{\substack{kk'q\\ \sigma\sigma'\beta}}\frac{V_2^2}{\omega+H_{imp}+V_2+\epsilon_q}c^\dagger_{d\beta}c_{d\sigma'}c^\dagger_{k'\sigma'}c_{k\sigma}c^\dagger_{d\sigma}c_{d\beta}\\
&\quad=\sum_{\substack{kk'q\\ \sigma\sigma'\beta}}\frac{V_2^2}{\omega+H_{imp}+V_2+\epsilon_q}c^\dagger_{d\beta}c_{d\sigma'}c^\dagger_{k'\sigma'}c_{k\sigma}\left(\delta_{\beta\sigma}-c^\dagger_{d\beta}c_{d\sigma}\right)\\
&\quad=-\sum_{qkk'\sigma\sigma'}\frac{V_2^2}{\omega+\epsilon_d+V_2+\epsilon_q}c^\dagger_{d\sigma}c^\dagger_{k'\sigma'}c_{d\sigma'}c_{k\sigma}\\
&\qquad+\sum_{qkk'\sigma\sigma'}\frac{V_2^2\hat{n}_{d\overline{\sigma}}\left(\epsilon_d+U\right)}{\left(\omega+2\epsilon_d+U+V_2+\epsilon_q\right)\left(\omega+\epsilon_d+V_2+\epsilon_q\right)}c^\dagger_{d\sigma}c^\dagger_{k'\sigma'}c_{d\sigma'}c_{k\sigma}\\
&\qquad+\sum_{\substack{qkk'\sigma\\ q\beta}}\left[\frac{V_2^2\hat{n}_{d\beta}}{\omega+\epsilon_d+V_2+\epsilon_q}-\frac{V_2^2\left(\epsilon_d+U\right)\hat{n}_{d\uparrow}\hat{n}_{d\downarrow}}{\left(\omega+2\epsilon_d+U+V_2+\epsilon_q\right)\left(\omega+\epsilon_d+V_2+\epsilon_q\right)}\right]c^\dagger_{k'\sigma'}c_{k\sigma}\\
&\qquad+\sum_{\substack{qkk'\sigma\\ \sigma'q\beta}}\left[\frac{V_2^2\hat{n}_{d\beta}}{\omega+\epsilon_d+\epsilon_q}-\frac{V_2^2\left(\epsilon_d+U\right)\hat{n}_{d\uparrow}\hat{n}_{d\downarrow}}{\left(\omega+2\epsilon_d+U+\epsilon_q\right)\left(\omega+\epsilon_d+\epsilon_q\right)}\right]c^\dagger_{d\sigma}c^\dagger_{k'\sigma'}c_{d\sigma'}c_{k\sigma}
\end{aligned}
\tag{1.46}
$$

The changes in the couplings are

$$
\begin{aligned}
\Delta\epsilon_d&=\sum_q\frac{|V_q|^2}{\omega+\epsilon_d+V_2+\epsilon_q}\\
\Delta U&=\sum_q\frac{2|V_q|^2\left(\epsilon_d+U\right)}{\left(\omega+2\epsilon_d+U+V_2+\epsilon_q\right)\left(\omega+\epsilon_d+V_2+\epsilon_q\right)}\\
\Delta V_k&=\Delta V_k^*=0\\
\Delta V_2&=-\sum_q\frac{V_2^2}{\omega+\epsilon_d+V_2+\epsilon_q}
\end{aligned}
\tag{1.47}
$$

With higher order scattering

$$
\mathcal{H}=\sum_{k\sigma}\epsilon_k\hat{n}_{k\sigma}+\sum_{k\sigma}\Big(V_kc^\dagger_{k\sigma}c_{d\sigma}+h.c.\Big)+\epsilon_d\sum_\sigma\hat{n}_{d\sigma}+U\hat{n}_{d\uparrow}\hat{n}_{d\downarrow}+\sum_{\substack{kk'\\ \sigma\sigma'}}V_2c^\dagger_{d\sigma'}c^\dagger_{k\sigma}c_{d\sigma}c_{k'\sigma'}
\tag{1.48}
$$

Such an interaction allows both spin-flip ($d\sigma\to d\overline{\sigma}$) as well as spin-preserving ($d\sigma\to d\sigma$) scattering.

One electron on shell:

$$\mathcal{H}_N = H_0 + H_{\text{imp}} + \epsilon_q \hat{n}_{q\beta} + V_q c^\dagger_{q\beta} c_{d\beta} + V_q^* c^\dagger_{d\beta} c_{q\beta} + \sum_{k\sigma} V_2 \left(c^\dagger_{q\beta} c^\dagger_{d\sigma} c_{k\sigma} c_{d\beta} - c^\dagger_{d\beta} c_{d\sigma} c^\dagger_{k\sigma} c_{q\beta} \right) + V_2 \hat{n}_{q\beta} \hat{n}_{d\beta} \tag{1.49}$$

For $\hat{n}_{q\beta} = 1$:

$$\begin{aligned} \Delta\mathcal{H}_N = &\sum_{q\beta} \tau_{q\beta} c^\dagger_{q\beta} \left[V_q + \sum_{k\sigma} V_2 c^\dagger_{d\sigma} c_{k\sigma} \right] c_{d\beta} \times c^\dagger_{d\beta} \left[V_q^* - \sum_{k'\sigma'} V_2 c_{d\sigma'} c^\dagger_{k'\sigma'} \right] c_{q\beta} \\ &\times \frac{1}{\hat{\omega} - \left(H_{\text{imp}} + V_2 \hat{n}_{d\beta} + \epsilon_q \right) \hat{n}_{q\beta}} \end{aligned} \tag{1.50}$$

The only operators acting on the shell electron can be pushed to the end, and we will get $\tau_{q\beta} \hat{n}_{q\beta} = \frac{1}{2} \hat{n}_{q\beta}$. We can also simplify the last factor as

$$\begin{aligned} \frac{1}{2} \hat{n}_{q\beta} \frac{1}{\hat{\omega} - \left(H_{\text{imp}} + V_2 \hat{n}_{d\beta} + \epsilon_q \right) \hat{n}_{q\beta}} &= \frac{1}{2} \hat{n}_{q\beta} \frac{1}{2\omega\tau_{q\beta} - \left(H_{\text{imp}} + V_2 \hat{n}_{d\beta} + \epsilon_q \right) \hat{n}_{q\beta}} \\ &= \frac{1/2}{\omega - \left(H_{\text{imp}} + V_2 \hat{n}_{d\beta} + \epsilon_q \right)} \qquad [\hat{n}_{q\beta} = 1] \end{aligned} \tag{1.51}$$

$$\begin{aligned} &= \sum_{q\beta} \left[V_q - \sum_{k\sigma} V_2 c_{k\sigma} c^\dagger_{d\sigma} \right] \left(1 - \hat{n}_{d\beta} \right) \left[V_q^* - \sum_{k'\sigma'} V_2 c_{d\sigma'} c^\dagger_{k'\sigma'} \right] \frac{1/2}{\omega - \left(H_{\text{imp}} + V_2 \hat{n}_{d\beta} + \epsilon_q \right)} \\ &= \sum_{q\beta} \left[|V_q|^2 \left(1 - \hat{n}_{d\beta} \right) - \sum_{k'\sigma'} V_q V_2 \left(1 - \hat{n}_{d\beta} \right) c_{d\sigma'} c^\dagger_{k'\sigma'} - \sum_{k\sigma} V_q^* V_2 c_{k\sigma} c^\dagger_{d\sigma} \left(1 - \hat{n}_{d\beta} \right) \right. \\ &\quad \left. + \sum_{kk'\sigma\sigma'} V_2^2 c_{k\sigma} c^\dagger_{d\sigma} \left(1 - \hat{n}_{d\beta} \right) c_{d\sigma'} c^\dagger_{k'\sigma'} \right] \frac{1/2}{\omega - \left(H_{\text{imp}} + V_2 \hat{n}_{d\beta} + \epsilon_q \right)} \end{aligned} \tag{1.52}$$

The first term in $\Delta\mathcal{H}_N$ is (calculated in the previous section)

$$\sum_{q\beta} |V_q|^2 \left(1 - \hat{n}_{d\beta} \right) \frac{1/2}{\omega - \left(H_{\text{imp}} + V_2 \hat{n}_{d\beta} + \epsilon_q \right)} = \frac{1}{2} \sum_{q\beta} |V_q|^2 \frac{\hat{n}_{d\beta} \left(\epsilon_q - \omega + 2\epsilon_d \right) - \hat{n}_{d\uparrow} \hat{n}_{d\downarrow} \epsilon_d}{\left(\omega - \epsilon_q \right) \left(\omega - \epsilon_q - \epsilon_d \right)} \tag{1.53}$$

Just as in the previous section, they renormalize the onsite energy ϵ_d and double-occupation penalty U. The third term in $\Delta\mathcal{H}_N$ gives

$$
\begin{aligned}
&-\sum_{k\sigma q\beta} c_{k\sigma}c^\dagger_{d\sigma}V_q^*V_2\left(1-\hat{n}_{d\beta}\right)\frac{1/2}{\omega-\left(H_{\text{imp}}+V_2\hat{n}_{d\beta}+\epsilon_q\right)}\\
&=-\frac{1}{2}\sum_{k\sigma q} c_{k\sigma}c^\dagger_{d\sigma}V_q^*V_2\left[\frac{2}{\omega-\epsilon_q}+\frac{\sum_\beta\hat{n}_{d\beta}\left(\epsilon_q-\omega+2\epsilon_d\right)}{\left(\omega-\epsilon_q\right)\left(\omega-\epsilon_q-\epsilon_d\right)}-\frac{2\hat{n}_{d\uparrow}\hat{n}_{d\downarrow}\epsilon_d}{\left(\omega-\epsilon_q\right)\left(\omega-\epsilon_q-\epsilon_d\right)}\right]\\
&=-\frac{1}{2}\sum_{k\sigma q} c_{k\sigma}c^\dagger_{d\sigma}V_q^*V_2\left[\frac{2}{\omega-\epsilon_q}+\frac{\hat{n}_{d\bar{\sigma}}\left(\epsilon_q-\omega+2\epsilon_d\right)}{\left(\omega-\epsilon_q\right)\left(\omega-\epsilon_q-\epsilon_d\right)}\right]\\
&=\sum_{k\sigma q} c^\dagger_{d\sigma}c_{k\sigma}V_q^*V_2\left[\frac{1}{\omega-\epsilon_q}+\frac{1}{2}\frac{\hat{n}_{d\bar{\sigma}}\left(\epsilon_q-\omega+2\epsilon_d\right)}{\left(\omega-\epsilon_q\right)\left(\omega-\epsilon_q-\epsilon_d\right)}\right]
\end{aligned}
\tag{1.54}
$$

In the penultimate step, I used

$$
c^\dagger_{d\sigma}\times\sum_\beta\hat{n}_{d\beta}=c^\dagger_{d\sigma}\times\left(\hat{n}_{d\sigma}+\hat{n}_{d\bar{\sigma}}\right)=c^\dagger_{d\sigma}\hat{n}_{d\bar{\sigma}}
\tag{1.55}
$$

and

$$
c^\dagger_{d\sigma}\times\hat{n}_{d\uparrow}\hat{n}_{d\downarrow}=c^\dagger_{d\sigma}\times\hat{n}_{d\sigma}\hat{n}_{d\bar{\sigma}}=0
\tag{1.56}
$$

The first of these terms renormalizes the coupling $V^*_{k'}$. The second term in $\Delta\mathcal{H}_N$ gives

$$
\begin{aligned}
&-\sum_{q\beta k\sigma} V_qV_2\left(1-\hat{n}_{d\beta}\right)c_{d\sigma}c^\dagger_{k\sigma}\frac{1/2}{\omega-\left(H_{\text{imp}}+V_2\hat{n}_{d\beta}+\epsilon_q\right)}\\
&=-\frac{1}{2}\sum_{qk\sigma} V_qV_2c_{d\sigma}c^\dagger_{k\sigma}\left[\frac{1}{\omega-\left(H_{\text{imp}}+V_2\hat{n}_{d\sigma}+\epsilon_q\right)}+\left(1-\hat{n}_{d\bar{\sigma}}\right)\frac{1}{\omega-\left(H_{\text{imp}}+\epsilon_q\right)}\right]\\
&=-\frac{1}{2}\sum_{qk\sigma} V_qV_2c_{d\sigma}c^\dagger_{k\sigma}\left[\frac{2\left(\omega-\epsilon_q-\epsilon_d\right)-V_2}{\left(\omega-\epsilon_d-\epsilon_q\right)\left(\omega-\epsilon_d-\epsilon_q-V_2\right)}\right.\\
&\quad+\quad\left.\hat{n}_{d\bar{\sigma}}\left\{\frac{\epsilon_d+U}{\left(\omega-2\epsilon_d-U-\epsilon_q-V_2\right)\left(\omega-\epsilon_d-\epsilon_q-V_2\right)}-\frac{1}{\omega-\epsilon_q-\epsilon_d}\right\}\right]\\
&=\sum_{qk\sigma} V_qV_2c^\dagger_{k\sigma}c_{d\sigma}\left[\frac{\left(\omega-\epsilon_q-\epsilon_d\right)-V_2/2}{\left(\omega-\epsilon_d-\epsilon_q\right)\left(\omega-\epsilon_d-\epsilon_q-V_2\right)}\right.\\
&\quad+\quad\left.\frac{1}{2}\hat{n}_{d\bar{\sigma}}\left\{\frac{\epsilon_d+U}{\left(\omega-2\epsilon_d-U-\epsilon_q-V_2\right)\left(\omega-\epsilon_d-\epsilon_q-V_2\right)}-\frac{1}{\omega-\epsilon_q-\epsilon_d}\right\}\right]
\end{aligned}
\tag{1.57}
$$

The first of these terms renormalizes the coupling $V_{k'}$. The fourth term gives

$$
\begin{aligned}
&\sum_{kk'q\beta\sigma\sigma'} V_2^2 c_{k\sigma} c_{d\sigma}^\dagger \left(1-\hat{n}_{d\beta}\right) c_{d\sigma'} c_{k'\sigma'}^\dagger \frac{1/2}{\omega-\left(H_{\text{imp}}+\epsilon_q\right)} \\
&= \frac{1}{2}\sum_{kk'q\sigma\sigma'} V_2^2 c_{k\sigma} c_{d\sigma}^\dagger c_{d\sigma'} c_{k'\sigma'}^\dagger \left[\frac{2\left(\omega-\epsilon_q-\epsilon_d\right)-V_2}{\left(\omega-\epsilon_d-\epsilon_q\right)\left(\omega-\epsilon_d-\epsilon_q-V_2\right)}\right. \\
&\quad + \quad \left.\hat{n}_{d\bar{\sigma}}\left\{\frac{\epsilon_d+U}{\left(\omega-2\epsilon_d-U-\epsilon_q-V_2\right)\left(\omega-\epsilon_d-\epsilon_q-V_2\right)}-\frac{1}{\omega-\epsilon_q-\epsilon_d}\right\}\right] \\
&= \sum_{kk'q\sigma\sigma'} V_2^2 \hat{n}_{d\sigma}\left[\frac{\left(\omega-\epsilon_q-\epsilon_d\right)-V_2/2}{\left(\omega-\epsilon_d-\epsilon_q\right)\left(\omega-\epsilon_d-\epsilon_q-V_2\right)}\right. \\
&\quad + \quad \left.\frac{1}{2}\hat{n}_{d\bar{\sigma}}\left\{\frac{\epsilon_d+U}{\left(\omega-2\epsilon_d-U-\epsilon_q-V_2\right)\left(\omega-\epsilon_d-\epsilon_q-V_2\right)}-\frac{1}{\omega-\epsilon_q-\epsilon_d}\right\}\right] \\
&+ \sum_{kk'q\sigma\sigma'} V_2^2 c_{d\sigma}^\dagger c_{k'\sigma'}^\dagger c_{d\sigma'} c_{k\sigma}\left[\frac{\left(\omega-\epsilon_q-\epsilon_d\right)-V_2/2}{\left(\omega-\epsilon_d-\epsilon_q\right)\left(\omega-\epsilon_d-\epsilon_q-V_2\right)}\right. \\
&\quad + \quad \left.\frac{1}{2}\hat{n}_{d\bar{\sigma}}\left\{\frac{\epsilon_d+U}{\left(\omega-2\epsilon_d-U-\epsilon_q-V_2\right)\left(\omega-\epsilon_d-\epsilon_q-V_2\right)}-\frac{1}{\omega-\epsilon_q-\epsilon_d}\right\}\right]
\end{aligned} \tag{1.58}
$$

The first line in the final equation describes the renormalization of ϵ_d and U at order V_2^2. The first term in the second line of the last equation describes the renormalization of the two-particle interaction coupling, V_2.

The changes in the couplings are

$$
\begin{aligned}
\Delta\epsilon_d &= \sum_q \left[|V_q|^2 \frac{\epsilon_q-\omega+2\epsilon_d}{2\left(\omega-\epsilon_q\right)\left(\omega-\epsilon_q-\epsilon_d\right)} + V_2^2 \frac{\left(\omega-\epsilon_q-\epsilon_d\right)-V_2/2}{\left(\omega-\epsilon_d-\epsilon_q\right)\left(\omega-\epsilon_d-\epsilon_q-V_2\right)}\right] \\
\Delta U &= \sum_q \left[-\frac{|V_q|^2\epsilon_d}{\left(\omega-\epsilon_q\right)\left(\omega-\epsilon_q-\epsilon_d\right)} + \frac{1}{2}\frac{V_2^2\left(\epsilon_d+U\right)}{\left(\omega-2\epsilon_d-U-\epsilon_q-V_2\right)\left(\omega-\epsilon_d-\epsilon_q-V_2\right)}\right. \\
&\qquad \left. -\frac{1}{2}\frac{V_2^2}{\omega-\epsilon_q-\epsilon_d}\right] \\
\Delta V_k &= \sum_q V_q V_2 \frac{\left(\omega-\epsilon_q-\epsilon_d\right)-\frac{V_2}{2}}{\left(\omega-\epsilon_d-\epsilon_q\right)\left(\omega-\epsilon_d-\epsilon_q-V_2\right)} \\
\Delta V_k^* &= \sum_q \frac{V_q^* V_2}{\omega-\epsilon_q} \\
\Delta V_2 &= \sum_q V_2^2 \frac{\left(\omega-\epsilon_q-\epsilon_d\right)-\frac{V_2}{2}}{\left(\omega-\epsilon_d-\epsilon_q\right)\left(\omega-\epsilon_d-\epsilon_q-V_2\right)}
\end{aligned} \tag{1.59}
$$

The renormalized Hamiltonian is

$$
\begin{aligned}
\mathcal{H}_{N-1} = & \sum_{k\sigma}\left[\epsilon_k^{N-1}\hat{n}_{k\sigma} + V_k^{N-1}c^\dagger_{k\sigma}c_{d\sigma} + h.c.\right] + \epsilon_d^{N-1}\sum_{\sigma}\hat{n}_{d\sigma} + U^{N-1}\hat{n}_{d\uparrow}\hat{n}_{d\downarrow} \\
& + \sum_{\substack{kk' \\ \sigma\sigma'}} V_2^{N-1}c^\dagger_{d\sigma'}c^\dagger_{k\sigma}c_{d\sigma}c_{k'\sigma'} + \frac{1}{2}\sum_{q\beta}\left(\epsilon_q\tau_{q\beta} - V_2\hat{n}_{d\beta}\tau_{q\beta}\right) \\
& + \sum_{k\sigma q} c^\dagger_{d\sigma}c_{k\sigma}\hat{n}_{d\overline{\sigma}}V_q^* V_2 C_1 - \sum_{qk\sigma} V_q V_2 c^\dagger_{k\sigma}c_{d\sigma}\hat{n}_{d\overline{\sigma}}C_2 - \sum_{kk'q\sigma\sigma'} V_2^2 c^\dagger_{k'\sigma'}c^\dagger_{d\sigma}c_{d\sigma'}c_{k\sigma}\hat{n}_{d\overline{\sigma}'}C_3
\end{aligned}
\tag{1.60}
$$

For $\hat{n}_{q\beta} = 0$: To get the hole kinetic energy, we write the kinetic energy part as

$$
\epsilon_q\hat{n}_{q\beta} = -\epsilon_q\left(1 - \hat{n}_{q\beta}\right) + \epsilon_q \tag{1.61}
$$

and drop the extra constant term.

$$
\begin{aligned}
\Delta\mathcal{H}_N &= \sum_{q\beta}\tau_{q\beta}\frac{1}{\hat{\omega} - \mathrm{Tr}_{q\beta}\left(\mathcal{H}_N^D\left(1-\hat{n}_{q\beta}\right)\right)\left(1-\hat{n}_{q\beta}\right)}\mathrm{Tr}_{q\beta}\left(c^\dagger_{q\beta}\mathcal{H}_N\right)c_{q\beta}c^\dagger_{q\beta}\mathrm{Tr}_{q\beta}\left(\mathcal{H}_N c_{q\beta}\right) \\
&= \sum_{q\beta}\tau_{q\beta}\frac{1}{2\omega\tau_{q\beta} - \left(H_{imp} - \epsilon_q\right)\left(1-\hat{n}_{q\beta}\right)}c^\dagger_{d\beta}\left[V_q^* - \sum_{k'\sigma'}V_2 c_{d\sigma'}c^\dagger_{k'\sigma'}\right]c_{q\beta}c^\dagger_{q\beta}\left[V_q - \sum_{k\sigma}V_2 c_{k\sigma}c^\dagger_{d\sigma}\right]c_{d\beta} \\
&= \sum_{q\beta}\frac{-1/2}{-\omega - H_{imp} + \epsilon_q}c^\dagger_{d\beta}\left[V_q^* - \sum_{k'\sigma'}V_2 c_{d\sigma'}c^\dagger_{k'\sigma'}\right]\left[V_q - \sum_{k\sigma}V_2 c_{k\sigma}c^\dagger_{d\sigma}\right]c_{d\beta} \\
&= \sum_{q\beta}\frac{1/2}{\omega + H_{imp} - \epsilon_q}c^\dagger_{d\beta}\left[V_q^* - \sum_{k'\sigma'}V_2 c_{d\sigma'}c^\dagger_{k'\sigma'}\right]\left[V_q - \sum_{k\sigma}V_2 c_{k\sigma}c^\dagger_{d\sigma}\right]c_{d\beta}
\end{aligned}
\tag{1.62}
$$

Here I set $\hat{n}_{q\beta} = 0, \tau_{q\beta} = -\frac{1}{2}$. Now,

$$
\begin{aligned}
\frac{1/2}{\omega + H_{imp} - \epsilon_q}c^\dagger_{d\beta} &= \frac{1/2}{\omega + \epsilon_d + \left(\epsilon_d + U\right)\hat{n}_{d\overline{\beta}} - \epsilon_q}c^\dagger_{d\beta} \\
&= \frac{1}{2}\left[\frac{1}{\omega + \epsilon_d - \epsilon_q} - \frac{\hat{n}_{d\overline{\beta}}\left(\epsilon_d + U\right)}{\left(\omega + \left(2\epsilon_d + U\right) - \epsilon_q\right)\left(\omega + \epsilon_d - \epsilon_q\right)}\right]c^\dagger_{d\beta}
\end{aligned}
\tag{1.63}
$$

The first term in $\Delta\mathcal{H}_N$ gives

$$
\begin{aligned}
&\frac{1}{2}\sum_{q\beta}\frac{1}{\omega + H_{imp} - \epsilon_q}|V_q|^2\hat{n}_{d\beta} \\
&\quad = \frac{1}{2}\sum_{q\beta}\left[\frac{|V_q|^2}{\omega + \epsilon_d - \epsilon_q} + \frac{|V_q|^2\hat{n}_{d\overline{\beta}}\left(\epsilon_d + U\right)}{\left(\omega + \left(2\epsilon_d + U\right) - \epsilon_q\right)\left(\omega + \epsilon_d - \epsilon_q\right)}\right]\hat{n}_{d\beta} \\
&\quad = \frac{1}{2}\sum_{q\beta}\frac{\hat{n}_{d\beta}|V_q|^2}{\omega + \epsilon_d - \epsilon_q} + \hat{n}_{d\uparrow}\hat{n}_{d\downarrow}\sum_{q}\frac{|V_q|^2\left(\epsilon_d + U\right)}{\left(\omega + \left(2\epsilon_d + U\right) - \epsilon_q\right)\left(\omega + \epsilon_d - \epsilon_q\right)}
\end{aligned}
\tag{1.64}
$$

The second term gives

$$\sum_{k\sigma q\beta} \frac{-V_q^* V_2}{\omega + H_{imp} + \epsilon_q} c_{d\beta}^\dagger c_{k\sigma} c_{d\sigma}^\dagger c_{d\beta} = -\sum_{k\sigma q} \frac{V_q^* V_2}{\omega + H_{imp} + \epsilon_q} \left(c_{d\sigma}^\dagger c_{k\sigma} c_{d\sigma}^\dagger c_{d\sigma} + c_{d\bar{\sigma}}^\dagger c_{k\sigma} c_{d\sigma}^\dagger c_{d\bar{\sigma}} \right) \tag{1.65}$$

The first term on the RHS is zero, because it has two consecutive $c_{d\sigma}^\dagger$.

$$\begin{aligned} -\sum_{k\sigma q\beta} \frac{V_q^* V_2}{\omega + H_{imp} + \epsilon_q} c_{d\beta}^\dagger c_{k\sigma} c_{d\sigma}^\dagger c_{d\beta} &= -\sum_{k\sigma q} \frac{V_q^* V_2}{\omega + H_{imp} + \epsilon_q} c_{d\bar{\sigma}}^\dagger c_{k\sigma} c_{d\sigma}^\dagger c_{d\bar{\sigma}} \\ &= -\sum_{k\sigma q} \frac{V_q^* V_2}{\omega + H_{imp} + \epsilon_q} \hat{n}_{d\bar{\sigma}} c_{k\sigma} c_{d\sigma}^\dagger \\ &= -\sum_{k\sigma q} \frac{V_q^* V_2}{\omega + \epsilon_d + (\epsilon_d + U)\, \hat{n}_{d\sigma} + \epsilon_q} \hat{n}_{d\bar{\sigma}} c_{k\sigma} c_{d\sigma}^\dagger \\ &= -\sum_{k\sigma q} \frac{V_q^* V_2}{\omega + 2\epsilon_d + U + \epsilon_q} \hat{n}_{d\bar{\sigma}} c_{k\sigma} c_{d\sigma}^\dagger \end{aligned} \tag{1.66}$$

The third term gives

$$\begin{aligned} -\sum_{k\sigma q\beta} \frac{V_q V_2}{\omega + H_{imp} + V_2 + \epsilon_q} c_{d\beta}^\dagger c_{d\sigma} c_{k\sigma}^\dagger c_{d\beta} &= -\sum_{k\sigma q} \frac{V_q V_2}{\omega + H_{imp} + V_2 + \epsilon_q} c_{d\bar{\sigma}}^\dagger c_{d\sigma} c_{k\sigma}^\dagger c_{d\bar{\sigma}} \\ &= -\sum_{k\sigma q} \frac{V_q V_2}{\omega + \epsilon_d + (\epsilon_d + U)\, \hat{n}_{d\sigma} + V_2 + \epsilon_q} \hat{n}_{d\bar{\sigma}} c_{d\sigma} c_{k\sigma}^\dagger \\ &= -\sum_{k\sigma q} \frac{V_q V_2}{\omega + \epsilon_d + V_2 + \epsilon_q} \hat{n}_{d\bar{\sigma}} c_{d\sigma} c_{k\sigma}^\dagger \end{aligned} \tag{1.67}$$

The fourth term gives

$$
\begin{aligned}
&\sum_{\substack{kk'q\\ \sigma\sigma'\beta}} \frac{V_2^2}{\omega + H_{imp} + V_2 + \epsilon_q} c^\dagger_{d\beta} c_{d\sigma'} c^\dagger_{k'\sigma'} c_{k\sigma} c^\dagger_{d\sigma} c_{d\beta} \\
&\quad = \sum_{\substack{kk'q\\ \sigma\sigma'\beta}} \frac{V_2^2}{\omega + H_{imp} + V_2 + \epsilon_q} c^\dagger_{d\beta} c_{d\sigma'} c^\dagger_{k'\sigma'} c_{k\sigma} \left(\delta_{\beta\sigma} - c^\dagger_{d\beta} c_{d\sigma}\right) \\
&\quad = -\sum_{qkk'\sigma\sigma'} \frac{V_2^2}{\omega + \epsilon_d + V_2 + \epsilon_q} c^\dagger_{d\sigma} c^\dagger_{k'\sigma'} c_{d\sigma'} c_{k\sigma} \\
&\qquad + \sum_{qkk'\sigma\sigma'} \frac{V_2^2 \hat{n}_{d\overline{\sigma}} \left(\epsilon_d + U\right)}{\left(\omega + 2\epsilon_d + U + V_2 + \epsilon_q\right)\left(\omega + \epsilon_d + V_2 + \epsilon_q\right)} c^\dagger_{d\sigma} c^\dagger_{k'\sigma'} c_{d\sigma'} c_{k\sigma} \\
&\qquad + \sum_{\substack{qkk'\sigma\\ q\beta}} \left[\frac{V_2^2 \hat{n}_{d\beta}}{\omega + \epsilon_d + V_2 + \epsilon_q} - \frac{V_2^2 \left(\epsilon_d + U\right) \hat{n}_{d\uparrow} \hat{n}_{d\downarrow}}{\left(\omega + 2\epsilon_d + U + V_2 + \epsilon_q\right)\left(\omega + \epsilon_d + V_2 + \epsilon_q\right)}\right] c^\dagger_{k'\sigma'} c_{k\sigma} \\
&\qquad + \sum_{\substack{qkk'\sigma\\ \sigma' q\beta}} \left[\frac{V_2^2 \hat{n}_{d\beta}}{\omega + \epsilon_d + \epsilon_q} - \frac{V_2^2 \left(\epsilon_d + U\right) \hat{n}_{d\uparrow} \hat{n}_{d\downarrow}}{\left(\omega + 2\epsilon_d + U + \epsilon_q\right)\left(\omega + \epsilon_d + \epsilon_q\right)}\right] c^\dagger_{d\sigma} c^\dagger_{k'\sigma'} c_{d\sigma'} c_{k\sigma}
\end{aligned}
\tag{1.68}
$$

The changes in the couplings are

$$
\begin{aligned}
\Delta\epsilon_d &= \frac{1}{2} \sum_q \frac{|V_q|^2}{\omega + \epsilon_d - \epsilon_q} \\
\Delta U &= \sum_q \frac{2|V_q|^2 \left(\epsilon_d + U\right)}{\left(\omega + 2\epsilon_d + U + V_2 + \epsilon_q\right)\left(\omega + \epsilon_d + V_2 + \epsilon_q\right)} \\
\Delta V_k &= \Delta V_k^* = 0 \\
\Delta V_2 &= -\sum_q \frac{V_2^2}{\omega + \epsilon_d + V_2 + \epsilon_q}
\end{aligned}
\tag{1.69}
$$

Sanity Checks ($\omega = 0$)

1. of ϵ_d

$$
\begin{aligned}
\delta\epsilon_d &= \sum_q |V_q|^2 \frac{\epsilon_q - \omega + 2\epsilon_d}{(\omega - \epsilon_q)(\omega - \epsilon_q - \epsilon_d)} \\
&= |V|^2 n(D) \frac{D}{D^2} && \text{[D very large]} \\
&= -|V|^2 \rho \delta D \frac{1}{D} \\
&= -\frac{\Delta}{\pi} \delta \ln D \\
\implies \frac{\mathrm{d}\epsilon_d}{\mathrm{d}\ln D} &= -\frac{\Delta}{\pi} && \text{[matches with Hewson]}
\end{aligned}
\tag{1.70}
$$

2. of U

$$
\begin{aligned}
\delta U &= -\sum_q |V_q|^2 \frac{2\epsilon_d}{(\omega - \epsilon_q)(\omega - \epsilon_q - \epsilon_d)} \\
&= |V|^2 \rho \delta D \frac{2\epsilon_d}{D^2} && \text{[very small, matches with Hewson]}
\end{aligned}
\tag{1.71}
$$

3. of V_1

$$
\begin{aligned}
\delta V_1 &= \sum_q V_q V_2 \frac{2(\omega - \epsilon_q - \epsilon_d) - V_2}{(\omega - \epsilon_d - \epsilon_q)(\omega - \epsilon_d - \epsilon_q - V_2)} \\
&= V_1 V_2 \rho \delta D \frac{2(D + \epsilon_d) + V_2}{(\epsilon_d + D)(\epsilon_d + D + V_2)} \\
&= V_1 V_2 \frac{\delta D}{2D_0} \frac{1}{D} \\
\implies \frac{\mathrm{d}V_1}{\mathrm{d}D} &= \frac{V_1 V_2}{D_0 D} \\
&\text{[matches with Jefferson up to a D, should come from the definition of } V_2\text{]}
\end{aligned}
\tag{1.72}
$$

With higher order scattering

$$
\mathcal{H} = \sum_{k\sigma} \epsilon_k \hat{n}_{k\sigma} + \sum_{k\sigma} \left(V_k c^\dagger_{k\sigma} c_{d\sigma} + h.c. \right) + \epsilon_d \sum_\sigma \hat{n}_{d\sigma} + U \hat{n}_{d\uparrow} \hat{n}_{d\downarrow} + \sum_{\substack{kk' \\ \sigma\sigma'}} V_2 c^\dagger_{d\sigma'} c^\dagger_{k\sigma} c_{d\sigma} c_{k'\sigma'}
\tag{1.73}
$$

Such an interaction allows both spin-flip ($d\sigma \to d\overline{\sigma}$) as well as spin-preserving ($d\sigma \to d\sigma$) scattering.

One electron on shell:

$$\mathcal{H}_N = H_0 + H_{\text{imp}} + \epsilon_q \hat{n}_{q\beta} + V_q c^\dagger_{q\beta} c_{d\beta} + V_q^* c^\dagger_{d\beta} c_{q\beta} + \sum_{k\sigma} V_2 \left(c^\dagger_{q\beta} c^\dagger_{d\sigma} c_{k\sigma} c_{d\beta} + c^\dagger_{d\beta} c^\dagger_{k\sigma} c_{d\sigma} c_{q\beta} \right) \\ + V_2 \hat{n}_{q\beta} \hat{n}_{d\beta} \tag{1.74}$$

Particle sector: The intermediate state consists of particle states, obtained by exciting electrons to the upper bandwidth edge ($+D$).

$$\begin{aligned} \Delta\mathcal{H}_N = \sum_{q\beta} \tau_{q\beta} c^\dagger_{d\beta} \left[V_q^* + \sum_{k'\sigma'} V_2 c^\dagger_{k'\sigma'} c_{d\sigma'} \right] c_{q\beta} \times \frac{1}{\hat{\omega} - \left(H_{\text{imp}} + V_2 \hat{n}_{d\beta} + \epsilon_q \right) \hat{n}_{q\beta}} \\ \times c^\dagger_{q\beta} \left[V_q + \sum_{k\sigma} V_2 c^\dagger_{d\sigma} c_{k\sigma} \right] c_{d\beta} \end{aligned} \tag{1.75}$$

Since the Greens function is preceded by $c_{q\beta}$, we can substitute $\hat{n}_{q\beta} = 1$ in the denominator.

$$\begin{aligned} c_{q\beta} \times \frac{1}{\hat{\omega} - \left(H_{\text{imp}} + V_2 \hat{n}_{d\beta} + \epsilon_q \right) \hat{n}_{q\beta}} &= c_{q\beta} \times \frac{1}{\hat{\omega} - \left(H_{\text{imp}} + V_2 \hat{n}_{d\beta} + \epsilon_q \right)} \\ &= c_{q\beta} \times \frac{1}{\hat{\omega} - \frac{\epsilon_q}{2} - \left(H_{\text{imp}} + V_2 \hat{n}_{d\beta} + \frac{\epsilon_q}{2} \right)} \end{aligned} \tag{1.76}$$

Set $\hat{\omega} - \frac{\epsilon_q}{2} = 2\omega\tau_{q\beta} = \omega$.

$$c_{q\beta} \times \frac{1}{\omega - \left(H_{\text{imp}} + V_2 \hat{n}_{d\beta} + \frac{\epsilon_q}{2} \right)} \tag{1.77}$$

Since $\tau_{q\beta} c_{q\beta} = -\frac{1}{2} c_{q\beta}$, we get

$$\Delta\mathcal{H}_N = \sum_{q\beta} c_{q\beta} c^\dagger_{d\beta} \left[V_q^* + \sum_{k'\sigma'} V_2 c^\dagger_{k'\sigma'} c_{d\sigma'} \right] \frac{-1/2}{\omega - H_{\text{imp}} - V_2 \hat{n}_{d\beta} - \frac{\epsilon_q}{2}} \left[V_q + \sum_{k\sigma} V_2 c^\dagger_{d\sigma} c_{k\sigma} \right] c_{d\beta} c^\dagger_{q\beta} \tag{1.78}$$

There are four scattering processes.

1.

$$
\begin{aligned}
&\frac{-1}{2}\sum_{q\beta}|V_q|^2 c^\dagger_{d\beta}\frac{1}{\omega-\left(H_{\text{imp}}+V_2\hat{n}_{d\beta}+\frac{\epsilon_q}{2}\right)}c_{d\beta}\\
&\quad=\frac{-1}{2}\sum_{q\beta}|V_q|^2 c^\dagger_{d\beta}c_{d\beta}\frac{1}{\omega-\frac{1}{2}\epsilon_q-\epsilon_d\hat{n}_{d\overline{\beta}}} && [\hat{n}_{d\beta}=0]\\
&\quad=\frac{-1}{2}\sum_{q\beta}|V_q|^2\hat{n}_{d\beta}\left[\frac{\hat{n}_{d\overline{\beta}}}{\omega-\frac{1}{2}\epsilon_q-\epsilon_d}+\frac{1-\hat{n}_{d\overline{\beta}}}{\omega-\frac{1}{2}\epsilon_q}\right]\\
&\quad=\frac{-1}{2}\sum_{q\beta}|V_q|^2\hat{n}_{d\beta}\left[\frac{\hat{n}_{d\overline{\beta}}\epsilon_d}{\left(\omega-\frac{1}{2}\epsilon_q-\epsilon_d\right)\left(\omega-\frac{1}{2}\epsilon_q\right)}+\frac{1}{\omega-\frac{1}{2}\epsilon_q}\right]\\
&\quad=-\frac{1}{2}\sum_{\beta}\hat{n}_{d\beta}\sum_q\frac{|V_q|^2}{\omega-\frac{1}{2}\epsilon_q}-\hat{n}_{d\uparrow}\hat{n}_{d\downarrow}\sum_q\frac{|V_q|^2\epsilon_d}{\left(\omega-\frac{1}{2}\epsilon_q-\epsilon_d\right)\left(\omega-\frac{1}{2}\epsilon_q\right)}
\end{aligned}
\tag{1.79}
$$

2.

$$
\begin{aligned}
&-\frac{1}{2}\sum_{q\beta k\sigma}V_q^*V_2c_{q\beta}c^\dagger_{d\beta}\frac{1}{\omega-H_{\text{imp}}-V_2\hat{n}_{d\beta}-\frac{\epsilon_q}{2}}c^\dagger_{d\sigma}c_{k\sigma}c_{d\beta}c^\dagger_{q\beta}\\
&=-\frac{1}{2}\sum_{q\beta k}V_q^*V_2c_{q\beta}c^\dagger_{d\beta}\frac{1}{\omega-H_{\text{imp}}-V_2\hat{n}_{d\beta}-\frac{\epsilon_q}{2}}c^\dagger_{d\overline{\beta}}c_{k\overline{\beta}}c_{d\beta}c^\dagger_{q\beta} && \left[c^\dagger_\beta c^\dagger_\sigma=\delta_{\beta\sigma}c^\dagger_\beta\right]\\
&=-\frac{1}{2}\sum_{q\beta k}V_q^*V_2c_{q\beta}c^\dagger_{d\beta}\left[\frac{\hat{n}_{d\overline{\beta}}\epsilon_d}{\left(\omega-\frac{1}{2}\epsilon_q-\epsilon_d\right)\left(\omega-\frac{1}{2}\epsilon_q\right)}+\frac{1}{\omega-\frac{1}{2}\epsilon_q}\right]c^\dagger_{d\overline{\beta}}c_{k\overline{\beta}}c_{d\beta}c^\dagger_{q\beta}\\
&=-\frac{1}{2}\sum_{q\beta k}V_q^*V_2c_{q\beta}c^\dagger_{d\beta}\left[\frac{\epsilon_d}{\left(\omega-\frac{1}{2}\epsilon_q-\epsilon_d\right)\left(\omega-\frac{1}{2}\epsilon_q\right)}+\frac{1}{\omega-\frac{1}{2}\epsilon_q}\right]c^\dagger_{d\overline{\beta}}c_{k\overline{\beta}}c_{d\beta}c^\dagger_{q\beta} && \left[\hat{n}c^\dagger=c^\dagger\right]\\
&=-\frac{1}{2}\sum_{q\beta k}\frac{V_q^*V_2}{\omega-\frac{1}{2}\epsilon_q-\epsilon_d}\hat{n}_{d\beta}c^\dagger_{d\overline{\beta}}c_{k\overline{\beta}}\\
&=-\frac{1}{2}\sum_{q\beta k}\frac{V_q^*V_2}{\omega-\frac{1}{2}\epsilon_q-\epsilon_d}\hat{n}_{d\overline{\beta}}c^\dagger_{d\beta}c_{k\beta}
\end{aligned}
\tag{1.80}
$$

3.

$$
\begin{aligned}
-\frac{1}{2}\sum_{q\beta k\sigma}V_qV_2c_{q\beta}c^\dagger_{d\beta}c^\dagger_{k\sigma}c_{d\sigma}\frac{1}{\omega-H_{\text{imp}}-V_2\hat{n}_{d\beta}-\frac{\epsilon_q}{2}}c_{d\beta}c^\dagger_{q\beta}&=\text{h.c. of 2}\\
&=-\frac{1}{2}\sum_{\beta kq}\frac{V_qV_2}{\omega-\frac{1}{2}\epsilon_q-\epsilon_d}\hat{n}_{d\overline{\beta}}c^\dagger_{k\beta}c_{d\beta}
\end{aligned}
\tag{1.81}
$$

4. $O(V_2^2)$, ignore for now.

Hole sector The intermediate state consists of hole states, obtained by exciting electrons from the lower bandwidth edge $(-D)$.

$$\begin{aligned}\Delta\mathcal{H}_N &= \sum_{q\beta}\tau_{q\beta}c^\dagger_{q\beta}\left[V_q + \sum_{k\sigma}V_2 c^\dagger_{d\sigma}c_{k\sigma}\right]c_{d\beta} \times \frac{1}{\hat{\omega} - H_{\text{imp}}\left(1-\hat{n}_{q\beta}\right)} \\ &\quad \times c^\dagger_{d\beta}\left[V_q^* + \sum_{k'\sigma'}V_2 c^\dagger_{k'\sigma'}c_{d\sigma'}\right]c_{q\beta} \\ &= \frac{1}{2}\sum_{q\beta}c^\dagger_{q\beta}\left[V_q + \sum_{k\sigma}V_2 c^\dagger_{d\sigma}c_{k\sigma}\right]c_{d\beta}\frac{1}{\hat{\omega}-H_{\text{imp}}}c^\dagger_{d\beta}\left[V_q^* + \sum_{k'\sigma'}V_2 c^\dagger_{k'\sigma'}c_{d\sigma'}\right]c_{q\beta}\end{aligned} \tag{1.82}$$

The Green's function simplifies as

$$\begin{aligned}c_{d\beta}\frac{1}{\hat{\omega}-H_{\text{imp}}\left(1-\hat{n}_{q\beta}\right)}c^\dagger_{d\beta} &= c_{d\beta}\frac{1}{\hat{\omega}-H_{\text{imp}}}c^\dagger_{d\beta} \qquad\qquad [\hat{n}_{q\beta}=0] \\ &= c_{d\beta}\frac{1}{\hat{\omega}-\frac{1}{2}\epsilon_q+\frac{1}{2}\epsilon_q-H_{\text{imp}}}c^\dagger_{d\beta}\end{aligned} \tag{1.83}$$

Again set $\hat{\omega}-\frac{1}{2}\epsilon_q = 2\omega\tau_{q\beta} = -\omega$.

$$\begin{aligned}c_{d\beta}\frac{1}{-\omega+\frac{1}{2}\epsilon_q-H_{\text{imp}}}c^\dagger_{d\beta} &= -c_{d\beta}\frac{1}{\omega-\frac{1}{2}\epsilon_q+H_{\text{imp}}}c^\dagger_{d\beta} \\ &= -c_{d\beta}\frac{1}{\omega-\frac{1}{2}\epsilon_q+\epsilon_d+\left(\epsilon_d+U\right)\hat{n}_{d\overline{\beta}}}c^\dagger_{d\beta} \\ &= -\left(1-\hat{n}_{d\beta}\right)\left[\frac{\hat{n}_{d\overline{\beta}}}{\omega-\frac{1}{2}\epsilon_q+2\epsilon_d+U}+\frac{1-\hat{n}_{d\overline{\beta}}}{\omega-\frac{1}{2}\epsilon_q+\epsilon_d}\right] \\ &= \left(\hat{n}_{d\beta}-1\right)\left[\frac{-\left(\epsilon_d+U\right)\hat{n}_{d\overline{\beta}}}{\left(\omega-\frac{1}{2}\epsilon_q+2\epsilon_d+U\right)\left(\omega-\frac{1}{2}\epsilon_q+\epsilon_d\right)}+\frac{1}{\omega-\frac{1}{2}\epsilon_q+\epsilon_d}\right] \\ &= -\frac{1}{\omega-\frac{1}{2}\epsilon_q+\epsilon_d}+\frac{\hat{n}_{d\beta}}{\omega-\frac{1}{2}\epsilon_q+\epsilon_d} \\ &\quad+\frac{\left(\epsilon_d+U\right)\hat{n}_{d\overline{\beta}}}{\left(\omega-\frac{1}{2}\epsilon_q+2\epsilon_d+U\right)\left(\omega-\frac{1}{2}\epsilon_q+\epsilon_d\right)} \\ &\quad-\frac{\left(\epsilon_d+U\right)\hat{n}_{d\beta}\hat{n}_{d\overline{\beta}}}{\left(\omega-\frac{1}{2}\epsilon_q+2\epsilon_d+U\right)\left(\omega-\frac{1}{2}\epsilon_q+\epsilon_d\right)}\end{aligned} \tag{1.84}$$

There are again four possible scattering processes.

1.

$$
\begin{aligned}
&\frac{1}{2}\sum_{q\beta}|V_q|^2 c^\dagger_{q\beta}c_{d\beta}\frac{1}{\hat{\omega}-H_{\rm imp}}c^\dagger_{d\beta}c_{q\beta}\\
&=\frac{1}{2}\sum_{q\beta}|V_q|^2\left[\frac{\hat{n}_{d\beta}}{\omega-\frac{1}{2}\epsilon_q+\epsilon_d}+\frac{(\epsilon_d+U)\,\hat{n}_{d\overline{\beta}}}{\left(\omega-\frac{1}{2}\epsilon_q+2\epsilon_d+U\right)\left(\omega-\frac{1}{2}\epsilon_q+\epsilon_d\right)}\right.\\
&\left.-\frac{(\epsilon_d+U)\,\hat{n}_{d\beta}\hat{n}_{d\overline{\beta}}}{\left(\omega-\frac{1}{2}\epsilon_q+2\epsilon_d+U\right)\left(\omega-\frac{1}{2}\epsilon_q+\epsilon_d\right)}\right]\\
&=\frac{1}{2}\sum_{q\beta}\frac{|V_q|^2\hat{n}_{d\beta}\left(\omega-\frac{1}{2}\epsilon_q+3\epsilon_d+2U\right)}{\left(\omega-\frac{1}{2}\epsilon_q+2\epsilon_d+U\right)\left(\omega-\frac{1}{2}\epsilon_q+\epsilon_d\right)}-\sum_q\frac{(\epsilon_d+U)\,\hat{n}_{d\uparrow}\hat{n}_{d\downarrow}}{\left(\omega-\frac{1}{2}\epsilon_q+2\epsilon_d+U\right)\left(\omega-\frac{1}{2}\epsilon_q+\epsilon_d\right)}
\end{aligned}
\tag{1.85}
$$

2.

$$
\begin{aligned}
&\frac{1}{2}\sum_{k\sigma q\beta}V_2V_q c^\dagger_{q\beta}c_{d\beta}\frac{1}{\hat{\omega}-H_{\rm imp}}c^\dagger_{d\beta}c^\dagger_{k\sigma}c_{d\sigma}c_{q\beta}\\
&=\frac{1}{2}\sum_{kq\beta}V_2V_q c^\dagger_{q\beta}c_{d\beta}\frac{1}{\hat{\omega}-H_{\rm imp}}c^\dagger_{d\beta}\left(c^\dagger_{k\beta}c_{d\beta}+c^\dagger_{k\overline{\beta}}c_{d\overline{\beta}}\right)c_{q\beta}\\
&=\frac{1}{2}\sum_{kq\beta}V_2V_q\left[\frac{\hat{n}_{d\beta}-1}{\omega-\frac{1}{2}\epsilon_q+\epsilon_d}c^\dagger_{k\overline{\beta}}c_{d\overline{\beta}}+\frac{(\epsilon_d+U)\,\hat{n}_{d\overline{\beta}}-\left(\omega-\frac{\epsilon_q}{2}+2\epsilon_d+U\right)}{\left(\omega-\frac{1}{2}\epsilon_q+2\epsilon_d+U\right)\left(\omega-\frac{1}{2}\epsilon_q+\epsilon_d\right)}c^\dagger_{k\beta}c_{d\beta}\right]\\
&=\frac{1}{2}\sum_{kq\beta}V_2V_q\left[\frac{\hat{n}_{d\overline{\beta}}\left(\omega-\frac{1}{2}\epsilon_q+3\epsilon_d+2U\right)}{\left(\omega-\frac{1}{2}\epsilon_q+2\epsilon_d+U\right)\left(\omega-\frac{1}{2}\epsilon_q+\epsilon_d\right)}-\frac{2}{\omega-\frac{1}{2}\epsilon_q+\epsilon_d}\right]c^\dagger_{k\beta}c_{d\beta}\\
&=-\frac{1}{2}\sum_{kq\beta}V_2V_q\left[\frac{\hat{n}_{d\overline{\beta}}}{\omega-\frac{1}{2}\epsilon_q+2\epsilon_d+U}+\frac{2\left(1-\hat{n}_{d\overline{\beta}}\right)}{\omega-\frac{1}{2}\epsilon_q+\epsilon_d}\right]c^\dagger_{k\beta}c_{d\beta}
\end{aligned}
\tag{1.86}
$$

3.

$$
\begin{aligned}
&\frac{1}{2}\sum_{q\beta k\sigma}V_2V_q^* c^\dagger_{q\beta}c^\dagger_{d\sigma}c_{k\sigma}c_{d\beta}\frac{1}{\hat{\omega}-H_{\rm imp}}c^\dagger_{d\beta}c_{d\sigma'}c_{q\beta}\\
&=\text{h.c. of 2}\\
&=-\frac{1}{2}\sum_{kq\beta}V_2V_q\left[\frac{\hat{n}_{d\overline{\beta}}}{\omega-\frac{1}{2}\epsilon_q+2\epsilon_d+U}+\frac{2\left(1-\hat{n}_{d\overline{\beta}}\right)}{\omega-\frac{1}{2}\epsilon_q+\epsilon_d}\right]c^\dagger_{d\beta}c_{k\beta}
\end{aligned}
\tag{1.87}
$$

4. $O(V_2^2)$

Scaling equations

$$\Delta\epsilon_d = -\frac{1}{2}\sum_q \frac{|V_q|^2}{\omega - \frac{1}{2}\epsilon_q} + \frac{1}{2}\sum_q \frac{|V_q|^2\left(\omega - \frac{1}{2}\epsilon_q + 3\epsilon_d + 2U\right)}{\left(\omega - \frac{1}{2}\epsilon_q + 2\epsilon_d + U\right)\left(\omega - \frac{1}{2}\epsilon_q + \epsilon_d\right)} \tag{1.88}$$

$$\Delta U = -\sum_q \frac{|V_q|^2\epsilon_d}{\left(\omega - \frac{1}{2}\epsilon_q - \epsilon_d\right)\left(\omega - \frac{1}{2}\epsilon_q\right)} - \sum_q \frac{|V_q|^2\left(\epsilon_d + U\right)}{\left(\omega - \frac{1}{2}\epsilon_q + 2\epsilon_d + U\right)\left(\omega - \frac{1}{2}\epsilon_q + \epsilon_d\right)} \tag{1.89}$$

$$\Delta V_k^+ = -\frac{1}{2}\sum_q \frac{V_q V_2\left(2\omega - \epsilon_q + \epsilon_d + U\right)}{\left(\omega - \frac{1}{2}\epsilon_q + 2\epsilon_d + U\right)\left(\omega - \frac{1}{2}\epsilon_q - \epsilon_d\right)} \tag{1.90}$$

$$\Delta V_k^- = -\sum_q V_2 V_q \frac{1}{\omega - \frac{1}{2}\epsilon_q + \epsilon_d} \tag{1.91}$$

They can be made more compact by defining $\mathcal{W} = \omega - \frac{1}{2}\epsilon_q$:

$$\begin{aligned}
\Delta\epsilon_d &= -\frac{1}{2}\sum_q \frac{|V_q|^2}{\mathcal{W}} + \frac{1}{2}\sum_q \frac{|V_q|^2\left(\mathcal{W} + 3\epsilon_d + 2U\right)}{\left(\mathcal{W} + 2\epsilon_d + U\right)\left(\mathcal{W} + \epsilon_d\right)} \\
&= \frac{1}{2}\sum_q \frac{|V_q|^2\left(U\mathcal{W} - U\epsilon_d - 2\epsilon_d^2\right)}{\mathcal{W}\left(\mathcal{W} + 2\epsilon_d + U\right)\left(\mathcal{W} + \epsilon_d\right)} \\
\Delta U &= -\sum_q \left[\frac{|V_q|^2\epsilon_d}{\left(\mathcal{W} - \epsilon_d\right)\mathcal{W}} + \frac{|V_q|^2\left(\epsilon_d + U\right)}{\left(\mathcal{W} + 2\epsilon_d + U\right)\left(\mathcal{W} + \epsilon_d\right)}\right] \\
&= -\sum_q |V_q|^2 \frac{\left(2\epsilon_d + U\right)\left(\mathcal{W}^2 + \epsilon_d^2\right) + 2\mathcal{W}\epsilon_d^2}{\mathcal{W}\left(\mathcal{W} + 2\epsilon_d + U\right)\left(\mathcal{W}^2 - \epsilon_d^2\right)} \\
\Delta V_k^+ &= -\frac{1}{2}\sum_q \frac{V_q V_2\left(2\mathcal{W} + \epsilon_d + U\right)}{\left(\mathcal{W} + 2\epsilon_d + U\right)\left(\mathcal{W} - \epsilon_d\right)} \\
\Delta V_k^- &= -\sum_q \frac{V_2 V_q}{\mathcal{W} + \epsilon_d}
\end{aligned} \tag{1.92}$$

Sanity Checks: $\omega = D$ $(\mathcal{W} = \frac{1}{2}D)$

1. $\epsilon_d, U + \epsilon_d \ll D$

$$\begin{aligned}
\Delta\epsilon_d &= -\frac{1}{2}n(D)|V_1|^2\frac{1}{\mathcal{W}} + \frac{1}{2}n(D)|V_1|^2\left[\frac{2}{\mathcal{W} + \epsilon_d} - \frac{1}{\mathcal{W} + 2\epsilon_d + U}\right] \\
&\approx -\frac{1}{2}n(D)|V_1|^2\frac{1}{\mathcal{W}} + \frac{1}{2}n(D)|V_1|^2\left[\frac{2\left(\mathcal{W} - \epsilon_d\right)}{\mathcal{W}^2} - \frac{\mathcal{W} - 2\epsilon_d - U}{\mathcal{W}^2}\right] \\
&= \frac{1}{2}n(D)|V_1|^2\frac{U}{\mathcal{W}^2}
\end{aligned} \tag{1.93}$$

$$\begin{aligned}\Delta U &= -n(D)|V_1|^2\left[\frac{1}{\mathcal{W}-\epsilon_d}-\frac{1}{\mathcal{W}}+\frac{1}{\mathcal{W}+\epsilon_d}-\frac{1}{\mathcal{W}+2\epsilon_d+U}\right]\\ &\approx -n(D)|V_1|^2\left[\frac{\mathcal{W}+\epsilon_d}{\mathcal{W}^2}-\frac{1}{\mathcal{W}}+\frac{\mathcal{W}-\epsilon_d}{\mathcal{W}^2}-\frac{\mathcal{W}-2\epsilon_d-U}{\mathcal{W}^2}\right]\\ &= -n(D)|V_1|^2\frac{2\epsilon_d+U}{\mathcal{W}^2}\end{aligned} \tag{1.94}$$

2. $U+\epsilon_d \gg D \gg \epsilon_d$

$$\begin{aligned}\Delta\epsilon_d &= -\frac{1}{2}n(D)|V_1|^2\frac{1}{\mathcal{W}}+\frac{1}{2}n(D)|V_1|^2\left[\frac{2}{\mathcal{W}+\epsilon_d}-\frac{1}{\mathcal{W}+2\epsilon_d+U}\right]\\ &\approx -\frac{1}{2}n(D)|V_1|^2\frac{1}{\mathcal{W}}+\frac{1}{2}n(D)|V_1|^2\left[\frac{2\left(\mathcal{W}-\epsilon_d\right)}{\mathcal{W}^2}-\frac{2\epsilon_d+U-\mathcal{W}}{\left(2\epsilon_d+U\right)^2}\right]\\ &= \frac{1}{2}n(D)|V_1|^2\frac{1}{\mathcal{W}}-\frac{1}{2}n(D)|V_1|^2\left[\frac{2\epsilon_d}{\mathcal{W}^2}+\frac{1}{2\epsilon_d+U}-\frac{\mathcal{W}}{\left(2\epsilon_d+U\right)^2}\right]\end{aligned} \tag{1.95}$$

Following Hewson, the last three terms are very small and can be dropped.

$$\begin{aligned}\Delta\epsilon_d &\approx \frac{1}{2}n(D)|V_1|^2\frac{1}{\mathcal{W}} = -\rho(D)\Delta D|V_1|^2\frac{1}{D}\\ \implies \frac{\mathrm{d}\epsilon_d}{\mathrm{d}\ln D} &= -\frac{\Delta}{\pi} \qquad\qquad \left[\Delta \equiv \pi\rho|V_1|^2\right]\end{aligned} \tag{1.96}$$

$$\begin{aligned}\Delta U &= -n(D)|V_1|^2\left[\frac{1}{\mathcal{W}-\epsilon_d}-\frac{1}{\mathcal{W}}+\frac{1}{\mathcal{W}+\epsilon_d}-\frac{1}{\mathcal{W}+2\epsilon_d+U}\right]\\ &\approx -n(D)|V_1|^2\left[\frac{\mathcal{W}+\epsilon_d}{\mathcal{W}^2}-\frac{1}{\mathcal{W}}+\frac{\mathcal{W}-\epsilon_d}{\mathcal{W}^2}-\frac{2\epsilon_d+U-\mathcal{W}}{\left(2\epsilon_d+U\right)^2}\right]\\ &= -n(D)|V_1|^2\left[\frac{1}{\mathcal{W}}-\frac{1}{2\epsilon_d+U}+\frac{\mathcal{W}}{\left(2\epsilon_d+U\right)^2}\right]\end{aligned} \tag{1.97}$$

$$\begin{aligned}\Delta V_k^+ &= -\frac{1}{2}n(D)V_1V_2\left[\frac{1}{\mathcal{W}-\epsilon_d}+\frac{1}{\mathcal{W}+2\epsilon_d+U}\right]\\ &= -\frac{1}{2}n(D)V_1V_2\left[\frac{\mathcal{W}+\epsilon_d}{\mathcal{W}^2}+\frac{2\epsilon_d+U-\mathcal{W}}{\left(2\epsilon_d+U\right)^2}\right]\end{aligned} \tag{1.98}$$

Following Jefferson, $\mathcal{W}+\epsilon_d \approx \mathcal{W}$ and we can drop the second term:

$$\begin{aligned}\Delta V_k^+ &\approx -\frac{1}{2}n(D)V_1V_2\frac{1}{\mathcal{W}} = \frac{V_1V_2}{2D_0D}\Delta D \qquad\qquad \left[\rho = \frac{1}{2D_0}\right]\\ \Delta V_k^- &= -n(D)V_1V_2\frac{1}{\mathcal{W}+\epsilon_d} \approx -n(D)V_1V_2\frac{1}{\mathcal{W}} = \frac{V_1V_2}{D_0D}\Delta D\end{aligned} \tag{1.99}$$