Lax Pairs and Double Bracket Flows

1 Definition of a Lax Pair

Two operator A and B are said to form a lax pair if they satisfy the equation

$$\frac{\mathrm{d}A(t)}{\mathrm{d}t} = [B(t), A(t)] \tag{1}$$

2 Unitary Nature of the Flow

It can be shown that this defines a unitary time evolution on $A(t)$, in the following manner. Let $U(t, t_0)$ be the unitary operator that carries this evolution through. We then need to construct a $U(t, t_0)$.

$$A(t) = U(t, t_0)A(t_0)U^\dagger(t, t_0) \tag{2}$$

where $A(t_0)$ is the operator A at a particular time t_0. The time change of A can then be written as

$$\begin{aligned}
\frac{\mathrm{d}A(t)}{\mathrm{d}t} &= \frac{\mathrm{d}U(t,t_0)}{\mathrm{d}t}A(t_0)U^\dagger(t,t_0) + U(t,t_0)A(t_0)\frac{\mathrm{d}U^\dagger(t,t_0)}{\mathrm{d}t} \\
&= \frac{\mathrm{d}U(t,t_0)}{\mathrm{d}t}U^\dagger(t,t_0)A(t) + A(t)U(t,t_0)\frac{\mathrm{d}U^\dagger(t,t_0)}{\mathrm{d}t} \qquad \left[A(t) = UAU^\dagger\right] \\
&= \frac{\mathrm{d}U(t,t_0)}{\mathrm{d}t}U^\dagger(t,t_0)A(t) - A(t)\frac{\mathrm{d}U(t,t_0)}{\mathrm{d}t}U^\dagger(t,t_0) \qquad \left[UU^\dagger = 1\right] \\
&= \left[\frac{\mathrm{d}U(t,t_0)}{\mathrm{d}t}U^\dagger(t,t_0), A(t)\right]
\end{aligned} \tag{3}$$

Looking at the definition of a lax pair, we can now make the connection

$$B(t) = \frac{\mathrm{d}U(t,t_0)}{\mathrm{d}t}U^\dagger(t,t_0) \tag{4}$$

The equation of motion characterised by the lax pair eq. 1 can thus be said to generate a family of unitarily connected operators $A(t)$, related by the unitaries defined by eq. 4. A direct corrolary is that the spectrum of $A(t)$ is preserved during this evolution.

3 Double Bracket Flow

The double bracket flows correspond to a special choice of the operator $B(t)$: $B(t) \equiv [A(t), C]$. A consequence of this choice is that the lax pair evolution then serves to minimize the commutator $[A(t), C]$. To see how, we first write down a function

$$\chi \equiv \text{Tr}\left([A(t) - C]^2\right) = \text{Tr}\left[A(t)^2 + C^2 - A(t)C - CA(t)\right] \tag{5}$$

Since $A^2(t) = UA^2U^\dagger$, we get $\text{Tr}(A^2(t)) = \text{Tr}(A)$. Also, from the cyclic nature of trace, we can write $\text{Tr}(A(t)C) = \text{Tr}(CA(t))$. These considerations (and the fact that C does not depend on t) allows us to write

$$\frac{\mathrm{d}\chi}{\mathrm{d}t} = -2\text{Tr}\left(\frac{\mathrm{d}A(t)}{\mathrm{d}t}C\right) = -2\text{Tr}\left([B(t), A(t)]\,C\right) \tag{6}$$

Using the cyclic property of trace, this becomes

$$\begin{aligned}\text{Tr}\left([B(t), A(t)]\,C\right) &= \text{Tr}\left(B(t)A(t)C - A(t)B(t)C\right)\\ &= \text{Tr}\left(B(t)A(t)C - B(t)A(t)C\right)\\ &= \text{Tr}\left(B(t)\,[A(t), C]\right)\end{aligned} \tag{7}$$

If we now substitute the choice of $B(t)$ we made above, we get

$$\frac{\mathrm{d}\chi}{\mathrm{d}t} = -2\text{Tr}\left([A(t), C]^2\right) \leq 0 \tag{8}$$

Since χ, the way it is defined, must necessarily be positive semi-definite for all t, the derivative $\frac{\mathrm{d}\chi}{\mathrm{d}t}$ must vanish in the limit $t \to \infty$, otherwise $\chi(t)$ will become negative. This gives the result

$$\lim_{t\to\infty}\frac{\mathrm{d}\chi}{\mathrm{d}t} = -2\lim_{t\to\infty}\text{Tr}\left([A(t), C]^2\right) = 0 \implies \lim_{t\to\infty}[A(t), C] = 0 \tag{9}$$

In other words, **the lax pair evolution of $A(t)$ against $[A(t), C]$ leads to the diagonalization of $A(t)$ with respect to C.** This can be used as an iterative algorithm to diagonalize a general matrix with respect to another matrix:

- Define matrices A and B, A being the one we want to diagonalize w.r.t B
- Iteratively run the next two steps until a desired accuracy is reached
- Compute a new matrix C = A*B - B*A
- Change A as follows: A = A + C*A - A*c