Contents

1 Anderson Model URG

$$\mathcal{H} = \sum_{k\sigma} \epsilon_k \hat{n}_{k\sigma} + \sum_{k\sigma} \left(V_k c^\dagger_{k\sigma} c_{d\sigma} + h.c. \right) + \epsilon_d \sum_\sigma \hat{n}_{d\sigma} + U \hat{n}_{d\uparrow} \hat{n}_{d\downarrow} \tag{1.1}$$

With four-Fermion interaction

$$\mathcal{H} = \sum_{k\sigma} \epsilon_k \hat{n}_{k\sigma} + \sum_{k\sigma} \left(V_k c^\dagger_{k\sigma} c_{d\sigma} + h.c. \right) + \epsilon_d \sum_\sigma \hat{n}_{d\sigma} + U \hat{n}_{d\uparrow} \hat{n}_{d\downarrow} + \sum_{\substack{k,k',q \\ \sigma\sigma'}} v_q c^\dagger_{k+q,\sigma} c^\dagger_{k'-q,\sigma'} c_{k,\sigma} c_{k',\sigma'} \tag{1.2}$$

Considering one electron on the shell,

$$\begin{aligned}\mathcal{H} = &\sum_{\substack{k<\Lambda_N \\ \sigma}} \left[\epsilon_k \hat{n}_{k\sigma} + V_k c^\dagger_{k\sigma} c_{d\sigma} + \text{h.c.} \right] + \epsilon_d \sum_\sigma \hat{n}_{d\sigma} + U \hat{n}_{d\uparrow} \hat{n}_{d\downarrow} + \sum_{\substack{k,k',q \\ \sigma\sigma'}} v_q c^\dagger_{k+q,\sigma} c^\dagger_{k'-q,\sigma'} c_{k,\sigma} c_{k',\sigma'} \\ &+ \epsilon_q \hat{n}_{q\beta} + V_q c^\dagger_{q\beta} c_{d\beta} + \text{h.c.} + \sum_{\substack{k,p \\ (k,k+p,q-p<\Lambda_N)}} \left[v_p c^\dagger_{k+p,\sigma} c^\dagger_{q-p,\beta} c_{k\sigma} c_{q\beta} + \text{h.c.} \right] + \sum_{k<\Lambda_N,\sigma} v_{k-q} \hat{n}_{k\sigma} \hat{n}_{q\beta}\end{aligned} \tag{1.3}$$

Define

$$H_0 \equiv \sum_{\substack{k<\Lambda_N \\ \sigma}} \left[\epsilon_k \hat{n}_{k\sigma} + V_k c^\dagger_{k\sigma} c_{d\sigma} + \text{h.c.} \right] + \sum_{\substack{k,k',q \\ \sigma\sigma'}} v_q c^\dagger_{k+q,\sigma} c^\dagger_{k'-q,\sigma'} c_{k,\sigma} c_{k',\sigma'} \tag{1.4}$$

$$H_{\text{imp}} = \epsilon_d \sum_\sigma \hat{n}_{d\sigma} + U \hat{n}_{d\uparrow} \hat{n}_{d\downarrow} \tag{1.5}$$

Then,

$$\mathcal{H}_1 = \frac{1}{2} \text{Tr}_{\text{all}} \left(\mathcal{H} \right) + \sum_{q\beta} \tau_{q\beta} \left\{ c^\dagger_{q\beta} \text{Tr}_{q\beta} \left(\mathcal{H} c_{q\beta} \right), \eta_{q\beta} \right\} \tag{1.6}$$

The first term is

$$\frac{1}{2} \text{Tr}_{\text{all}} \left(\mathcal{H} \right) = H_0 + H_{\text{imp}} + \frac{1}{2} \sum_{q\beta} \left[\epsilon_q + \sum_{k<\Lambda_N,\sigma} v_{k-q} \hat{n}_{k\sigma} \right] \tag{1.7}$$

The second term is:

$$\begin{aligned}c^\dagger_{q\beta} \text{Tr}_{q\beta} \left(\mathcal{H} c_{q\beta} \right) &= V_q c^\dagger_{q\beta} c_{d\beta} + \sum_{k,p} v^*_p c^\dagger_{q\beta} c^\dagger_{k\sigma} c_{k+p,\sigma} c_{q-p,\beta} \\ &= c^\dagger_{q\beta} \left(V_q c_{d\beta} + \sum_{k,p} v^*_p c^\dagger_{k\sigma} c_{k+p,\sigma} c_{q-p,\beta} \right)\end{aligned} \tag{1.8}$$

$$
\begin{aligned}
\eta_{q\beta} &= \left[V_q^* c^\dagger_{d\beta} c_{q\beta} + \sum_{k,p} v_p c^\dagger_{k+p,\sigma} c^\dagger_{q-p,\beta} c_{k\sigma} c_{q\beta}\right] \frac{1}{\hat\omega - \left[H_0^D + H_{\text{imp}} + \epsilon_q + \sum_{k<\Lambda_N,\sigma} v_{k-q}\hat n_{k\sigma}\right]\hat n_{q\beta}} \\
&= \left[\left(V_q^* c^\dagger_{d\beta} + \sum_{k,p} v_p c^\dagger_{q-p,\beta} c^\dagger_{k+p,\sigma} c_{k\sigma}\right) c_{q\beta}\right] \frac{2}{\omega - \left[\epsilon_d \hat n_{d\overline\beta} + \epsilon_q\right]}
\end{aligned} \tag{1.9}
$$

$$
\begin{aligned}
\sum_{q\beta} \tau_{q\beta}\left\{c^\dagger_{q\beta} \text{Tr}_{q\beta}\left(\mathcal{H} c_{q\beta}\right), \eta_{q\beta}\right\} = \sum_{q\beta} \frac{2\tau_{q\beta}}{\omega - \left[\epsilon_d \hat n_{d\overline\beta} + \epsilon_q\right]} &\left\{ c^\dagger_{q\beta}\left(V_q c_{d\beta} + \sum_{k,p} v_p^* c^\dagger_{k\sigma} c_{k+p,\sigma} c_{q-p,\beta}\right),\right. \\
&\left.\left(V_q^* c^\dagger_{d\beta} + \sum_{k,p} v_p c^\dagger_{q-p,\beta} c^\dagger_{k+p,\sigma} c_{k\sigma}\right) c_{q\beta}\right\}
\end{aligned} \tag{1.10}
$$

Because $2\tau_{q\beta} c_{q\beta} = -c_{q\beta}$ and $2\tau_{q\beta} c^\dagger_{q\beta} = c^\dagger_{q\beta}$, we get

$$
\begin{aligned}
\sum_{q\beta} \frac{1}{\omega - \left[\epsilon_d \hat n_{d\overline\beta} + \epsilon_q\right]} &\left[c^\dagger_{q\beta}\left(V_q c_{d\beta} + \sum_{k,p} v_p^* c^\dagger_{k\sigma} c_{k+p,\sigma} c_{q-p,\beta}\right),\right. \\
&\left.\left(V_q^* c^\dagger_{d\beta} + \sum_{k,p} v_p c^\dagger_{q-p,\beta} c^\dagger_{k+p,\sigma} c_{k\sigma}\right) c_{q\beta}\right]
\end{aligned} \tag{1.11}
$$

The first commutator renormalizes the impurity site energy:

$$
\left[V_q c^\dagger_{q\beta} c_{d\beta}, V_q^* c^\dagger_{d\beta} c_{q\beta}\right] = |V_q|^2 \left(\hat n_{q\beta} - \hat n_{d\beta}\right) \tag{1.12}
$$

The second term gives:

$$
\begin{aligned}
\left[V_q c^\dagger_{q\beta} c_{d\beta}, \sum_{k,p,\sigma} v_p c^\dagger_{q-p,\beta} c^\dagger_{k+p,\sigma} c_{k\sigma} c_{q\beta}\right] &= \sum_{k,p,\sigma} V_q v_p \left[c^\dagger_{q\beta} c_{d\beta}, c^\dagger_{q-p,\beta} c^\dagger_{k+p,\sigma} c_{k,\sigma} c_{q,\beta}\right] \\
&= -\sum_{k,p,\sigma} V_q v_p c_{d\beta} c^\dagger_{k+p,\sigma} c^\dagger_{q-p,\beta} c_{k,\sigma} \\
&= -\sum_{k,p,\sigma} V_q v_p c_{d\beta} c^\dagger_{k+p,\sigma}\left(\delta_{\substack{q=k+p\\ \beta=\sigma}} - c_{k,\sigma} c^\dagger_{q-p,\beta}\right) \\
&= \sum_{k,p,\sigma} V_q v_p c_{d\beta} c^\dagger_{k+p,\sigma} c_{k,\sigma} c^\dagger_{q-p,\beta}
\end{aligned} \tag{1.13}
$$

In the second step, I chose the $\hat n_{q\beta} = 1$ sector. In the last step I used the fact that since the only electron on the shell is the one with q, we must have $k + p \neq q$, hence the $\delta_{q=k+p}$

gives zero. The third term is just the Hermitian conjugate:

$$\left[\sum_{k,p,\sigma} v_p^* c_{k\sigma}^\dagger c_{q\beta}^\dagger c_{k+p,\sigma} c_{q-p,\beta}, V_q^* c_{d\beta}^\dagger c_{q\beta}\right] = \sum_{k,p,\sigma} V_q^* v_p^* c_{q-p,\beta} c_{k,\sigma}^\dagger c_{k+p,\sigma} c_{d,\beta}^\dagger \tag{1.14}$$

The fourth term gives (using $\hat{n}_{q\beta} = 1$),

$$\left[\sum_{k,p,\sigma} v_p^* c_{k\sigma}^\dagger c_{q\beta}^\dagger c_{k+p,\sigma} c_{q-p,\beta}, \sum_{k',p',\sigma'} v_p' c_{q-p',\beta}^\dagger c_{k'+p',\sigma'}^\dagger c_{k'\sigma'} c_{q\beta}\right]$$
$$= \sum v_p^* v_{p'} c_{k\sigma}^\dagger c_{k+p,\sigma} c_{q-p,\beta} c_{q-p',\beta}^\dagger c_{k'+p',\sigma'}^\dagger c_{k'\sigma'} \tag{1.15}$$

The total renormalization is thus

$$\Delta H = \sum_{q\beta} \frac{1}{\omega - \left[\epsilon_d \hat{n}_{d\bar{\beta}} + \epsilon_q\right]} \left[|V_q|^2 \left(\hat{n}_{q\beta} - \hat{n}_{d\beta}\right) + \sum_{k,p,\sigma} V_q v_p c_{d\beta} c_{k+p,\sigma}^\dagger c_{k,\sigma} c_{q-p,\beta}^\dagger + \text{h.c.} \right.$$
$$\left. + \sum v_p^* v_{p'} c_{k\sigma}^\dagger c_{k+p,\sigma} c_{q-p,\beta} c_{q-p',\beta}^\dagger c_{k'+p',\sigma'}^\dagger c_{k'\sigma'} \right] \tag{1.16}$$