1 URG as a double-bracket flow

The difference RG equation for URG can be written in the form

$$\Delta\mathcal{H}(\omega, D) = \frac{1}{\omega_1 - \omega_0}\left[G\left[\mathcal{H}^d, \mathcal{H}^I\right], \mathcal{H}\right] \tag{1}$$

This is not in the standard double-bracket form, primarily because it takes into account the off-diagonal terms in the Hamiltonian inside the generator of the unitary transformation. It can be given a double-bracket form by taking some approximations, as was shown in eq. **??**.

Just like the standard double-bracket flow equation, the URG equation acts as an optimizer - it minimizes the function

$$\chi_j = \mathrm{Tr}\left[\left(\mathcal{H}^I_j\right)^2\right] \tag{2}$$

The definition of this function first requires a scheme to be defined. We can order the energy of the electrons as $\epsilon_1 < \epsilon_2 < ... < \epsilon_j < ... < \epsilon_N$. The URG consists of sequentially decoupling the states ϵ_N, then ϵ_{N-1}, and so on. At the j^{th} step, the Hamiltonian can be partitioned in the subspace of the electron being decoupled; the partitioning looks like

$$\mathcal{H}^0_j + c^\dagger_j T_j + T^\dagger_j c_j \tag{3}$$

$\mathcal{H}^0_j$ is the part that *does not* scatter between $|\hat{n}_j\rangle = 0, 1$, while $\mathcal{H}^I_j = c^\dagger_j T_j + T^\dagger_j c_j$ is the part that *does* scatter between states with a definite value of $\hat{n}_j$.

The first observation that we make is that χ_j is semi-positive definite. This is because it can be expressed as the norm-squared of a state vector.

$$\chi_j = \sum_{i=1}^{N} \langle\psi_i|\left(\mathcal{H}^I_j\right)^2|\psi_i\rangle = \sum_{i=1}^{N}\langle\phi_i|\phi_i\rangle \geq 0, \left[\text{where } |\phi_i\rangle = \mathcal{H}^I_j|\psi_i\rangle\right] \tag{4}$$

The difference equation for χ_j is

$$\Delta\chi_j = 2\mathrm{Tr}\left[\mathcal{H}^I_j \Delta\mathcal{H}^I_j\right] = 2\mathrm{Tr}\left[\mathcal{H}^I_j\left(\mathcal{H}^I_{j-1} - \mathcal{H}^I_j\right)\right] \tag{5}$$

The first part of the trace is zero. To see why, note that from the nature of URG, once j has been decoupled, it is diagonal in all the subsequent Hamiltonians. Hence, $\mathcal{H}^I_{j-1}$ will be diagonal in j, while $\mathcal{H}^I_j$ is, by definition, off-diagonal in j. The product $\mathcal{H}^I_j\mathcal{H}^I_{j-1}$ will hence be off-diagonal and will change $\hat{n}_j$. Hence, it will vanish when taken inside a trace. What remains is

$$\Delta\chi_j = -2\mathrm{Tr}\left[\left(\mathcal{H}^I_j\right)^2\right] = -2\chi_j \leq 0 \tag{6}$$

At the fixed point j^* of URG, the off-diagonal part of the Hamiltonian vanishes, so we can write $\mathcal{H}^I_{j^*} = 0 \implies \Delta\chi^* = 0$. Combining the three points:

$$\chi_j \geq 0, \quad \Delta\chi_j \leq 0, \quad \Delta\chi_{j^*} = 0 \tag{7}$$

we can say that URG starts from a non-minimal value of χ and flows to its minimum $\chi^* = 0$ at the fixed point.