URG on Kondo Model

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Contents

1 Questions

- What is the motivation behind the choice of the initial condition $\tau_{q\uparrow} = -\tau_{q\downarrow} = \frac{1}{2}$? Does that choice not violate the SU(2) symmetry of the model? Why not take a more symmetric choice like $\tau_{q\beta} = \tau_{q\bar{\beta}} = \frac{1}{2}$ for $q\beta$ below fermi level and $-1/2$ for above it?

- If we follow your notes and try to derive the equations with just the initial configuration $\tau_{q\uparrow} = -\tau_{q\downarrow} = \frac{1}{2}$, we end up with a field-like term αS_d^z in ΔH, which violates SU(2). However, if we also add the ΔH from the initial state with $\uparrow$ and $\downarrow$ flipped, then we lose the field term.

- On setting $J_z = J_t$, we do not get $\Delta J_z = \Delta J_t$. The RG equations appear to not respect the SU(2) symmetry. How do we resolve this?

- ΔJ_t obtained without summing over β is half of what you get. Why is this so?

- Is there a general prescription for choosing what part of the Hamiltonian comes down in the denominator?

2 Formulation

$$
\begin{aligned}
H &= \sum_{k\alpha} \epsilon_k \tau_{k\alpha} + J_z S_d^z \sum_{kk'} \left(c^\dagger_{k\uparrow} c_{k'\uparrow} - c^\dagger_{k\downarrow} c_{k'\downarrow} \right) \\
&\quad + J_t \sum_{kk'} \left(S_d^+ c^\dagger_{k\downarrow} c_{k'\uparrow} + S_d^- c^\dagger_{k\uparrow} c_{k'\downarrow} \right) \\
&= H^D + H^i + H^I
\end{aligned}
\tag{2.1}
$$

$$
H^D = \sum_{k\alpha} \epsilon_k \tau_{k\alpha} + J_z S_d^z \sum_{k\beta} \beta \tau_{k\beta} \tag{2.2}
$$

$$
\begin{aligned}
H^i &= J_z S_d^z \sum_{kk' \neq q} \beta \left(c^\dagger_{k\beta} c_{k'\beta} - c^\dagger_{k\bar{\beta}} c_{k'\bar{\beta}} \right) (1 - \delta_{kk'}) \\
&\quad + J_t \sum_{k' \neq q, k} \left(c^\dagger_{d\beta} c_{d\bar{\beta}} c^\dagger_{k\bar{\beta}} c_{k'\beta} + c^\dagger_{d\bar{\beta}} c_{d\beta} c^\dagger_{k'\beta} c_{k\bar{\beta}} \right)
\end{aligned}
\tag{2.3}
$$

$$
\begin{aligned}
H^I &= J_t \sum_{k \neq q} \left(c^\dagger_{d\beta} c_{d\bar{\beta}} c^\dagger_{k\bar{\beta}} c_{q\beta} + c^\dagger_{d\bar{\beta}} c_{d\beta} c^\dagger_{q\beta} c_{k\bar{\beta}} \right) \\
&\quad + J_z S_d^z \beta \sum_{k \neq q} \left(c^\dagger_{k\beta} c_{q\beta} + c^\dagger_{q\beta} c_{k\beta} \right) \\
&= c^\dagger_{q\beta} T_{q\beta} + T^\dagger_{q\beta} c_{q\beta}
\end{aligned}
\tag{2.4}
$$

where

$$
T_{q\beta} = J_z S_d^z \beta \sum_{k \neq q} c_{k\beta} + J_t c^\dagger_{d\bar{\beta}} c_{d\beta} \sum_{k \neq q} c_{k\bar{\beta}} \tag{2.5}
$$

The transformed hamiltonian is

$$
U H U^\dagger = H^D + H^i + \underbrace{c^\dagger_{q\beta} T_{q\beta} \eta}_{\text{Particle}} + \underbrace{\eta_0 c^\dagger_{q\beta} T_{q\beta}}_{\text{Hole}} \tag{2.6}
$$

where $\eta_0 = -\eta$

3 Particle, hole sectors (Left GFs)

For simpler calculations, take H^D in the green's functions of η, η_0 as

$$
H^D = \epsilon_q \tau_{q\beta} + \beta J_z S_d^z \left(\tau_{q\beta} - \tau_{q\bar{\beta}} \right) \tag{3.1}
$$

3.1 Particle Sector

$$
\begin{aligned}
c^\dagger_{q\beta}T_{q\beta}\eta &= \frac{1}{\omega - H^D}c^\dagger_{q\beta}T_{q\beta}T^\dagger_{q\beta}c_{q\beta} \\
&= \frac{1}{\omega - H^D}\sum_{kk'\neq q} c^\dagger_{q\beta}\left(J_z S^z_d \beta c_{k\beta} + J_t c^\dagger_{d\bar\beta}c_{d\beta}c_{k\bar\beta}\right)\left(J_z S^z_d \beta c^\dagger_{k'\beta} + J_t c^\dagger_{d\beta}c_{d\bar\beta}c^\dagger_{k'\bar\beta}\right)c_{q\beta}
\end{aligned} \tag{3.2}
$$

$$
\begin{aligned}
\left(c^\dagger_{q\beta}T_{q\beta}\eta\right)_1 &= J_z^2\frac{1}{\omega - H^D}\sum_{kk'\neq q}\beta S^z_d c^\dagger_{q\beta}c_{k\beta}\beta S^z_d c^\dagger_{k'\beta}c_{q\beta} \\
&= \frac{1}{4}J_z^2\frac{1}{\omega - \frac{1}{2}\epsilon_q - \beta J_z S^z_d\left(\tau_{q\beta}-\tau_{q\bar\beta}\right)}\sum_{kk'\neq q}c_{k\beta}c^\dagger_{k'\beta}n_{q\beta} \qquad (3.3a) \\
\left(c^\dagger_{q\beta}T_{q\beta}\eta\right)_2 &= J_z J_t\frac{1}{\omega - H^D}\sum_{kk'\neq q}\beta S^z_d c^\dagger_{q\beta}c_{k\beta}c^\dagger_{d\beta}c_{d\bar\beta}c^\dagger_{k'\bar\beta}c_{q\beta} \\
&= \frac{1}{2}J_z J_t\frac{1}{\omega - \frac{1}{2}\epsilon_q - \frac{1}{2}J_z\left(\tau_{q\beta}-\tau_{q\bar\beta}\right)}c^\dagger_{d\beta}c_{d\bar\beta}\sum_{kk'\neq q}c_{k\beta}c^\dagger_{k'\bar\beta}n_{q\beta} \qquad (3.3b) \\
\left(c^\dagger_{q\beta}T_{q\beta}\eta\right)_3 &= J_z J_t\frac{1}{\omega - H^D}\sum_{kk'\neq q}c^\dagger_{d\bar\beta}c_{d\beta}c^\dagger_{q\beta}c_{k\bar\beta}\beta S^z_d c^\dagger_{k'\beta}c_{q\beta} \\
&= \frac{1}{2}J_z J_t\frac{1}{\omega - \frac{1}{2}\epsilon_q + \frac{1}{2}J_z\left(\tau_{q\beta}-\tau_{q\bar\beta}\right)}c^\dagger_{d\bar\beta}c_{d\beta}\sum_{kk'\neq q}c_{k\bar\beta}c^\dagger_{k'\beta}n_{q\beta} \qquad (3.3c) \\
\left(c^\dagger_{q\beta}T_{q\beta}\eta\right)_4 &= J_t^2\frac{1}{\omega - H^D}\sum_{kk'\neq q}c^\dagger_{d\bar\beta}c_{d\beta}c^\dagger_{q\beta}c_{k\bar\beta}c^\dagger_{d\beta}c_{d\bar\beta}c^\dagger_{k'\bar\beta}c_{q\beta} \\
&= J_t^2\frac{1}{\omega - \frac{1}{2}\epsilon_q + \frac{1}{2}J_z\left(\tau_{q\beta}-\tau_{q\bar\beta}\right)}\left(\frac{1}{2} + \bar\beta S^z_d\right)\sum_{kk'\neq q}c_{k\bar\beta}c^\dagger_{k'\bar\beta}n_{q\beta} \qquad (3.3d)
\end{aligned}
$$

3.2 Hole Sector

$$
\begin{aligned}
\eta_0 c^\dagger_{q\beta}T_{q\beta} &= \frac{1}{\omega' - H^D}T^\dagger_{q\beta}c_{q\beta}c^\dagger_{q\beta}T_{q\beta} \\
&= \frac{1}{\omega' - H^D}\sum_{kk'\neq q}\left(J_z S^z_d\beta c^\dagger_{k'\beta} + J_t c^\dagger_{d\beta}c_{d\bar\beta}c^\dagger_{k'\bar\beta}\right)c_{q\beta}c^\dagger_{q\beta}\left(J_z S^z_d\beta c_{k\beta} + J_t c^\dagger_{d\bar\beta}c_{d\beta}c_{k\bar\beta}\right)
\end{aligned} \tag{3.4}
$$

$$\left(\eta_0 c^\dagger_{q\beta} T_{q\beta}\right)_1 = J_z^2 \frac{1}{\omega' - H^D} \sum_{kk'\neq q} \beta S_d^z c^\dagger_{k'\beta} c_{q\beta} \beta S_d^z c^\dagger_{q\beta} c_{k\beta}$$

$$= \frac{1}{4} J_z^2 \frac{1}{\omega' + \frac{1}{2}\epsilon_q - \beta J_z S_d^z \left(\tau_{q\beta} - \tau_{q\bar{\beta}}\right)} \sum_{kk'\neq q} c^\dagger_{k'\beta} c_{k\beta} \left(1 - n_{q\beta}\right) \tag{3.5a}$$

$$\left(\eta_0 c^\dagger_{q\beta} T_{q\beta}\right)_2 = J_z J_t \frac{1}{\omega' - H^D} \sum_{kk'\neq q} \beta S_d^z c^\dagger_{k'\beta} c_{q\beta} c^\dagger_{d\bar{\beta}} c_{d\beta} c^\dagger_{q\beta} c_{k\bar{\beta}}$$

$$= -\frac{1}{2} J_z J_t \frac{1}{\omega' + \frac{1}{2}\epsilon_q + \frac{1}{2} J_z \left(\tau_{q\beta} - \tau_{q\bar{\beta}}\right)} c^\dagger_{d\bar{\beta}} c_{d\beta} \sum_{kk'\neq q} c^\dagger_{k'\beta} c_{k\bar{\beta}} \left(1 - n_{q\beta}\right) \tag{3.5b}$$

$$\left(\eta_0 c^\dagger_{q\beta} T_{q\beta}\right)_3 = J_z J_t \frac{1}{\omega' - H^D} \sum_{kk'\neq q} c^\dagger_{d\beta} c_{d\bar{\beta}} c^\dagger_{k'\bar{\beta}} c_{q\beta} \beta S_d^z c^\dagger_{q\beta} c_{k\beta}$$

$$= -\frac{1}{2} J_z J_t \frac{1}{\omega' + \frac{1}{2}\epsilon_q - \frac{1}{2} J_z \left(\tau_{q\beta} - \tau_{q\bar{\beta}}\right)} c^\dagger_{d\beta} c_{d\bar{\beta}} \sum_{kk'\neq q} c^\dagger_{k'\bar{\beta}} c_{k\beta} \left(1 - n_{q\beta}\right) \tag{3.5c}$$

$$\left(\eta_0 c^\dagger_{q\beta} T_{q\beta}\right)_4 = J_t^2 \frac{1}{\omega' - H^D} \sum_{kk'\neq q} c^\dagger_{d\beta} c_{d\bar{\beta}} c^\dagger_{k'\bar{\beta}} c_{q\beta} c^\dagger_{d\bar{\beta}} c_{d\beta} c^\dagger_{q\beta} c_{k\bar{\beta}}$$

$$= J_t^2 \frac{1}{\omega' + \frac{1}{2}\epsilon_q - \frac{1}{2} J_z \left(\tau_{q\beta} - \tau_{q\bar{\beta}}\right)} \left(\frac{1}{2} - \bar{\beta} S_d^z\right) \sum_{kk'\neq q} c^\dagger_{k'\bar{\beta}} c_{k\bar{\beta}} \left(1 - n_{q\beta}\right) \tag{3.5d}$$

4 Decoupling $q\beta$, $q\bar{\beta}$

We consider the decoupling for the initial condition $\tau_{q\beta} = -\tau_{q\bar{\beta}} = \frac{1}{2}$ with the ansatz $\omega = -\omega'$

What is the motivation behind the initial condition $\tau_{q\beta} = -\tau_{q\bar{\beta}} = \frac{1}{2}$? Can't we take $\tau_{q\beta} = \tau_{q\bar{\beta}} = \frac{1}{2}$ for $q\beta$ below fermi level and $-1/2$ for above it?

$$c^\dagger_{q\beta} T_{q\beta} \eta = \frac{1}{4} J_z^2 \frac{1}{\omega - \frac{1}{2}\epsilon_q - \beta J_z S_d^z} \sum_{kk'\neq q} c_{k\beta} c^\dagger_{k'\beta}$$

$$+ \frac{1}{2} J_z J_t \frac{1}{\omega - \frac{1}{2}\epsilon_q - \frac{1}{2} J_z} c^\dagger_{d\beta} c_{d\bar{\beta}} \sum_{kk'\neq q} c_{k\beta} c^\dagger_{k'\bar{\beta}}$$

$$+\frac{1}{2}J_zJ_t\frac{1}{\omega-\frac{1}{2}\epsilon_q+\frac{1}{2}J_z}c^\dagger_{d\bar\beta}c_{d\beta}\sum_{kk'\neq q}c_{k\bar\beta}c^\dagger_{k'\beta}$$

$$+J_t^2\frac{1}{\omega-\frac{1}{2}\epsilon_q+\frac{1}{2}J_z}\left(\frac{1}{2}+\bar\beta S_d^z\right)\sum_{kk'\neq q}c_{k\bar\beta}c^\dagger_{k'\bar\beta} \tag{4.1}$$

$$\eta_0 c^\dagger_{q\bar\beta}T_{q\bar\beta}=-\frac{1}{4}J_z^2\frac{1}{\omega-\frac{1}{2}\epsilon_q-\bar\beta J_zS_d^z}\sum_{kk'\neq q}c^\dagger_{k'\bar\beta}c_{k\bar\beta}$$

$$+\frac{1}{2}J_zJ_t\frac{1}{\omega-\frac{1}{2}\epsilon_q+\frac{1}{2}J_z}c^\dagger_{d\beta}c_{d\bar\beta}\sum_{kk'\neq q}c^\dagger_{k'\bar\beta}c_{k\beta}$$

$$+\frac{1}{2}J_zJ_t\frac{1}{\omega-\frac{1}{2}\epsilon_q-\frac{1}{2}J_z}c^\dagger_{d\bar\beta}c_{d\beta}\sum_{kk'\neq q}c^\dagger_{k'\beta}c_{k\bar\beta}$$

$$-J_t^2\frac{1}{\omega-\frac{1}{2}\epsilon_q-\frac{1}{2}J_z}\left(\frac{1}{2}-\beta S_d^z\right)\sum_{kk'\neq q}c^\dagger_{k'\beta}c_{k\beta} \tag{4.2}$$

$$\left(c^\dagger_{q\beta}T_{q\beta}\eta\right)_2+\left(\eta_0c^\dagger_{q\bar\beta}T_{q\bar\beta}\right)_2=J_zJ_t\frac{\left(\omega-\frac{1}{2}\epsilon_q\right)}{\left(\omega-\frac{1}{2}\epsilon_q\right)^2-\left(\frac{1}{2}J_z\right)^2}c^\dagger_{d\beta}c_{d\bar\beta}\sum_{kk'\neq q}c^\dagger_{k'\bar\beta}c_{k\beta} \tag{4.3a}$$

$$\left(c^\dagger_{q\beta}T_{q\beta}\eta\right)_3+\left(\eta_0c^\dagger_{q\bar\beta}T_{q\bar\beta}\right)_3=J_zJ_t\frac{\left(\omega-\frac{1}{2}\epsilon_q\right)}{\left(\omega-\frac{1}{2}\epsilon_q\right)^2-\left(\frac{1}{2}J_z\right)^2}c^\dagger_{d\bar\beta}c_{d\beta}\sum_{kk'\neq q}c^\dagger_{k'\beta}c_{k\bar\beta} \tag{4.3b}$$

$$\left(c^\dagger_{q\beta}T_{q\beta}\eta\right)_1+\left(\eta_0c^\dagger_{q\bar\beta}T_{q\bar\beta}\right)_4=\frac{J_t^2\left(\omega-\frac{1}{2}\epsilon_q\right)+\frac{1}{2}J_z\left(J_t^2-\frac{1}{2}J_z^2\right)}{\left(\omega-\frac{1}{2}\epsilon_q\right)^2-\left(\frac{1}{2}J_z\right)^2}\beta S_d^z\sum_{kk'\neq q}c^\dagger_{k'\beta}c_{k\beta}$$

$$-\frac{1}{2}\frac{\left(\omega-\frac{1}{2}\epsilon_q\right)\left(J_t^2+\frac{1}{2}J_z^2\right)+\frac{1}{2}J_zJ_t^2}{\left(\omega-\frac{1}{2}\epsilon_q\right)^2-\left(\frac{1}{2}J_z\right)^2}\sum_{kk'\neq q}c^\dagger_{k'\beta}c_{k\beta}$$

$$+\frac{1}{4}J_z^2\frac{\left(\omega+\frac{1}{2}\epsilon_q+\beta\mathbf{J_zS_d^z}\right)}{\left(\omega-\frac{1}{2}\epsilon_q\right)^2-\left(\frac{1}{2}J_z\right)^2}\sum_{k\neq q}1 \tag{4.3c}$$

$$\left(c^\dagger_{q\beta}T_{q\beta}\eta\right)_4+\left(\eta_0c^\dagger_{q\bar\beta}T_{q\bar\beta}\right)_1=-\frac{J_t^2\left(\omega-\frac{1}{2}\epsilon_q\right)-\frac{1}{2}J_z\left(J_t^2-\frac{1}{2}J_z^2\right)}{\left(\omega-\frac{1}{2}\epsilon_q\right)^2-\left(\frac{1}{2}J_z\right)^2}\bar\beta S_d^z\sum_{kk'\neq q}c^\dagger_{k'\bar\beta}c_{k\bar\beta}$$

$$-\frac{1}{2}\frac{\left(\omega-\frac{1}{2}\epsilon_q\right)\left(J_t^2+\frac{1}{2}J_z^2\right)-\frac{1}{2}J_zJ_t^2}{\left(\omega-\frac{1}{2}\epsilon_q\right)^2-\left(\frac{1}{2}J_z\right)^2}\sum_{kk'\neq q}c^\dagger_{k'\bar\beta}c_{k\bar\beta}$$

$$+J_t^2\frac{\left(\omega+\frac{1}{2}\epsilon_q-\frac{1}{2}J_z\right)}{\left(\omega-\frac{1}{2}\epsilon_q\right)^2-\left(\frac{1}{2}J_z\right)^2}\sum_{k\neq q}1 \tag{4.3d}$$

Field terms arise in $UHU^\dagger$ if we don't sum Eqs. (4.3) over β

4.1 Scaling Equations

Without summing over β

$$\begin{aligned}
\Delta J_{z\uparrow} &= \frac{J_t^2\left(\omega - \frac{1}{2}\epsilon_q\right) + \frac{1}{2}J_z\left(J_t^2 - \frac{1}{2}J_z^2\right)}{\left(\omega - \frac{1}{2}\epsilon_q\right)^2 - \left(\frac{1}{2}J_z\right)^2} \\
\Delta J_{z\downarrow} &= -\frac{J_t^2\left(\omega - \frac{1}{2}\epsilon_q\right) - \frac{1}{2}J_z\left(J_t^2 - \frac{1}{2}J_z^2\right)}{\left(\omega - \frac{1}{2}\epsilon_q\right)^2 - \left(\frac{1}{2}J_z\right)^2} \\
\Delta J_t &= J_z J_t \frac{\left(\omega - \frac{1}{2}\epsilon_q\right)}{\left(\omega - \frac{1}{2}\epsilon_q\right)^2 - \left(\frac{1}{2}J_z\right)^2}
\end{aligned} \tag{4.4}$$

Field term:

$$\frac{1}{4}J_z^2 \frac{\beta J_z S_d^z}{\left(\omega - \frac{1}{2}\epsilon_q\right)^2 - \left(\frac{1}{2}J_z\right)^2} \sum_{k\neq q} 1 \tag{4.5}$$

Putting $J_z = J_t = \frac{J}{2}$ in ΔJ_t, we get

$$\Delta J = \frac{1}{2}J^2 \frac{\left(\omega - \frac{1}{2}\epsilon_q\right)}{\left(\omega - \frac{1}{2}\epsilon_q\right)^2 - \left(\frac{1}{4}J\right)^2} \tag{4.6}$$

On summing over β

$$\Delta J_t = 2J_z J_t \frac{\left(\omega - \frac{1}{2}\epsilon_q\right)}{\left(\omega - \frac{1}{2}\epsilon_q\right)^2 - \left(\frac{1}{2}J_z\right)^2} \tag{4.7}$$

$$\Delta J_z = \frac{J_z\left(J_t^2 - \frac{1}{2}J_z^2\right)}{\left(\omega - \frac{1}{2}\epsilon_q\right)^2 - \left(\frac{1}{2}J_z\right)^2} \tag{4.8}$$

ΔJ_z doesn't have the same form as ΔJ_t.