For 2 momentum Hubbard model, we have :-

$$H = \sum_{k\sigma}(\epsilon^0_{k\sigma} - \mu_{eff})\hat{n}_{k\sigma} + U_0(\tilde{\mathbf{C}}^2 - \tilde{\mathbf{S}}^2)$$

Here, $\tilde{\mathbf{C}} = \sum_{k\sigma}\vec{C}_{k\sigma}$, and $\tilde{\mathbf{S}} = \sum_{k\sigma}\vec{S}_{k\sigma}$. $\vec{S}_{k\sigma} = \frac{1}{2}f^{s\dagger}_k\vec{\sigma}f^s_k$ and $\vec{C}_{k\sigma} = \frac{1}{2}f^{c\dagger}_k\vec{\sigma}f^c_k$. The f^s and f^c are defined as :-

$$f^s_k = \begin{bmatrix} c_{k\uparrow} \\ c_{-k\downarrow} \end{bmatrix}$$

and

$$f^c_k = \begin{bmatrix} c_{k\uparrow} \\ c^\dagger_{-k\downarrow} \end{bmatrix}$$

Using this notation, we ultimately get the Hamiltonian as (for $\mu_{eff} = 0$) :-

$$H = \begin{bmatrix} H_{\Lambda\downarrow,e}n_{\Lambda\downarrow} & c^\dagger_{\Lambda\downarrow}T_{\Lambda\downarrow} \\ T^\dagger_{\Lambda\downarrow}c_{\Lambda\downarrow} & H_{\Lambda\downarrow,h}(1-n_{\Lambda\downarrow}) \end{bmatrix}$$

where

$$H_{\Lambda\downarrow,e} = \epsilon_0 + \frac{1}{2} + \frac{1}{4}[n_{\Lambda\downarrow} - n_{\Lambda\uparrow} + n_{-\Lambda\downarrow} - n_{\Lambda\uparrow}]$$
$$H_{\Lambda\downarrow,e} = \frac{1}{2} + \frac{1}{4}[(1-n_{\Lambda\downarrow}) - n_{\Lambda\uparrow} + (1-n_{-\Lambda\downarrow}) - n_{\Lambda\uparrow}]$$
$$T_{\Lambda\downarrow} = (c^\dagger_{-\Lambda\uparrow}c_{\Lambda\uparrow} - c^\dagger_{\Lambda\uparrow}c_{-\Lambda\uparrow})c_{-\Lambda\downarrow}$$

Now, we have

$$\hat{\eta} = \frac{1}{(E_{\Lambda\downarrow,h} - H_{\Lambda\downarrow,h}(1-n_{\Lambda\downarrow})}T^\dagger_{\Lambda\downarrow}c_{\Lambda\downarrow} = G_hE_{\Lambda\downarrow}T^\dagger_{\Lambda\downarrow}c_{\Lambda\downarrow}$$

and

$$U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & \hat{\eta}^\dagger \\ -\hat{\eta} & 1 \end{bmatrix}$$

We want to rotate away the $\Lambda\downarrow$ node, i.e. block diagonalize the H-matrix.
After applying the unitary transformation, we have :-

$$\begin{gathered} H^{(1)} \\ = UHU^\dagger \\ = \begin{bmatrix} H_{\Lambda\downarrow,e}n_{\Lambda\downarrow} + \hat{\eta}^\dagger T^\dagger_{\Lambda\downarrow}c_{\Lambda\downarrow} + c^\dagger_{\Lambda\downarrow}T_{\Lambda\downarrow}\hat{\eta} + \hat{\eta}^\dagger H_{\Lambda\downarrow,h}(1-n_{\Lambda\downarrow})\hat{\eta} & c^\dagger_{\Lambda\downarrow}T_{\Lambda\downarrow} - H_{\Lambda\downarrow,e}n_{\Lambda\downarrow}\hat{\eta}^\dagger - \hat{\eta}^\dagger T^\dagger_{\Lambda\downarrow}c_{\Lambda\downarrow}\hat{\eta} + \hat{\eta}^\dagger H_{\Lambda\downarrow,h}(1-n_{\Lambda\downarrow}) \\ T^\dagger_{\Lambda\downarrow}c_{\Lambda\downarrow} - \hat{\eta}H_{\Lambda\downarrow,e}n_{\Lambda\downarrow} - \hat{\eta}^\dagger c^\dagger_{\Lambda\downarrow}T_{\Lambda\downarrow}\hat{\eta} + H_{\Lambda\downarrow,h}(1-n_{\Lambda\downarrow})\hat{\eta} & H_{\Lambda\downarrow,h}(1-n_{\Lambda\downarrow}) + \hat{\eta}H_{\Lambda\downarrow,e}n_{\Lambda\downarrow}\hat{\eta}^\dagger - T^\dagger_{\Lambda\downarrow}c_{\Lambda\downarrow}\hat{\eta}^\dagger - \hat{\eta}c^\dagger_{\Lambda\downarrow}T_{\Lambda\downarrow} \end{bmatrix} \end{gathered}$$

Calculating the off-diagonal element term by term, we get :-

$$\hat{\eta}^\dagger T^\dagger_{\Lambda\downarrow}c_{\Lambda\downarrow}\hat{\eta} = -c^\dagger_{\Lambda\downarrow}T_{\Lambda\downarrow}G_hE_{\Lambda\downarrow}T_{\Lambda\downarrow}c_{\Lambda\downarrow}c^\dagger_{\Lambda\downarrow}T_{\Lambda\downarrow}G_hE_{\Lambda\downarrow} = -c^\dagger_{\Lambda\downarrow}T_{\Lambda\downarrow}[\frac{1}{(E_{\Lambda\downarrow} - H_{\Lambda\downarrow,h})^2(1-n_{\Lambda\downarrow})}\{n_{\Lambda\uparrow}(1-n_{-\Lambda\uparrow}) + n_{-\Lambda\uparrow}(1-n_{\Lambda\uparrow})\}n_{-\Lambda\downarrow}]$$
$$H_{\Lambda\downarrow,e}n_{\Lambda\downarrow}\hat{\eta}^\dagger = c^\dagger_{\Lambda\downarrow}T_{\Lambda\downarrow}\frac{\epsilon_0 - H_{\Lambda\downarrow,h}}{(\epsilon_0 - E_{\Lambda\downarrow}) - H_{\Lambda\downarrow,h}}$$
$$\hat{\eta}^\dagger H_{\Lambda\downarrow,h}(1-n_{\Lambda\downarrow}) = c^\dagger_{\Lambda\downarrow}T_{\Lambda\downarrow}\frac{H_{\Lambda\downarrow,h}}{(E_{\Lambda\downarrow}) - H_{\Lambda\downarrow,h}}$$

This gives an equation for $E_{\Lambda\downarrow}$, whose solution is the required operator-valued eigenvalue for the block diagonalization.