$$\rho_1 = \frac{1}{2(1+\alpha^{-2})} \left[\underbrace{|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|}_{\text{singlet part}} + \overbrace{\alpha^{-2}\left(|\uparrow\downarrow\rangle\langle\uparrow\downarrow| + |0\rangle\langle0|\right)}^{\text{doublon-holon part}} \right] \tag{1}$$

Projecting out the doublon-holon part,

$$\rho_{\text{sing}} = \frac{1}{2(1+\alpha^{-2})}\left(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|\right) \tag{2}$$

To make trace 1, drop the α^{-2} in denominator.

$$\rho_{\text{sing}} = \frac{1}{2}\left(|\uparrow\rangle\langle\uparrow| + |\downarrow\rangle\langle\downarrow|\right) \tag{3}$$

Is this the procedure you meant? Is it all right to throw the α part in order to renormalise the ρ_{sing}?