In the simple example, you write down the expression for $\eta^\dagger$ from the *first off-diagonal equation*, take its conjugate to find η and go on to find the correct eigenvalues (and eigenvectors). An issue apparently arises if I take, instead, the *second off-diagonal equation* to calculate η, hence calculate $\eta^\dagger$ and try to find the eigenstates and values.

Eigenvalues

From second-off diagonal equation,

$$\begin{aligned}
\eta_1 &= \frac{-t}{\omega + \mu\hat{n}_2(1-\hat{n}_1} c_2^\dagger c_1 \\
&\implies \eta_1^\dagger = c_1^\dagger c_2 \frac{-t}{\omega + \mu\hat{n}_2(1-\hat{n}_1)} \\
\eta_1\eta_1^\dagger = 1-\hat{n}_1 &\implies 1-\hat{n}_1 = \frac{1}{\omega + \mu\hat{n}_2(1-\hat{n}_1)} t^2\hat{n}_2(1-\hat{n}_1) \frac{1}{\omega + \mu\hat{n}_2(1-\hat{n}_1)} \\
&\implies \left[\omega + \mu\hat{n}_2(1-\hat{n}_1)\right]^2 (1-\hat{n}_1) = t^2\hat{n}_2(1-\hat{n}_1) \\
&\implies \boxed{\omega(1-\hat{n}_1) = t\hat{n}_2(1-\hat{n}_1) - \mu\hat{n}_2(1-\hat{n}_1)}
\end{aligned} \tag{1}$$

From what I understand, $\omega = \begin{pmatrix} \hat{E}_1 & \\ & \hat{E}_1 \end{pmatrix}$, so $\omega(1-\hat{n}_1)$ should just be the eigenblock $\hat{E}_1 = (V-2\mu)\hat{n}_2 + (t-\mu)(1-\hat{n}_2)$. In that case, their is a contradiction between the two methods.

Eigenstates

The correct eigenstate corresponding to $|10\rangle$ is $U^\dagger|10\rangle = |10\rangle - |01\rangle$, where $|10\rangle$ means $|\hat{n}_1 = 1, \hat{n}_2 = 0\rangle$. If I now attempt to calculate the eigenstates using the alternative approach (just as I did for the eigenstates), the following happens:

$$\begin{aligned}
&U^\dagger|10\rangle = (1-\eta+\eta^\dagger)|10\rangle = |10\rangle - \eta|10\rangle \qquad\qquad (\eta^\dagger|10\rangle = 0) \\
&\text{Now, } \eta|10\rangle = \frac{-t}{\omega + \mu\hat{n}_2(1-\hat{n}_1)} c_2^\dagger c_1|10\rangle = \frac{-t}{\omega + \mu\hat{n}_2(1-\hat{n}_1)}|01\rangle \\
&\text{But, } (\omega + \mu\hat{n}_2(1-\hat{n}_1))|01\rangle = \left[\begin{pmatrix} \hat{E}_1 & \\ & \hat{E}_1 \end{pmatrix} + \mu\right]\begin{pmatrix} 0 \\ |\hat{n}_2 = 1\rangle \end{pmatrix} = \begin{pmatrix} 0 \\ (\hat{E}_1 + \mu)|\hat{n}_2 = 1\rangle \end{pmatrix}
\end{aligned} \tag{2}$$

Using $\hat{E}_1|\hat{n}_2 = 1\rangle = [(V-2\mu)\hat{n}_2 + (t-\mu)(1-\hat{n}_2)]|\hat{n}_2 = 1\rangle = (V-2\mu)|\hat{n}_2 = 1\rangle$, the final expression for $(\omega + \mu\hat{n}_2(1-\hat{n}_1))|10\rangle$ becomes

$$\begin{aligned}
(\omega + \mu\hat{n}_2(1-\hat{n}_1))|10\rangle &= \begin{pmatrix} 0 \\ (V-2\mu+\mu)|\hat{n}_2 = 1\rangle \end{pmatrix} = (V-\mu)|01\rangle \\
\text{therefore, } \eta|10\rangle &= \frac{-t}{V-\mu}|01\rangle \\
\text{which means, } & \boxed{U^\dagger|10\rangle = |10\rangle + \frac{t}{V-\mu}|01\rangle}
\end{aligned} \tag{3}$$

This is in conflict with the actual eigenvalue $|10\rangle - |01\rangle$.