Holographic Entanglement in Free Fermionic Quantum Matter

Aspects of Hierarchy and Topology in Entanglement

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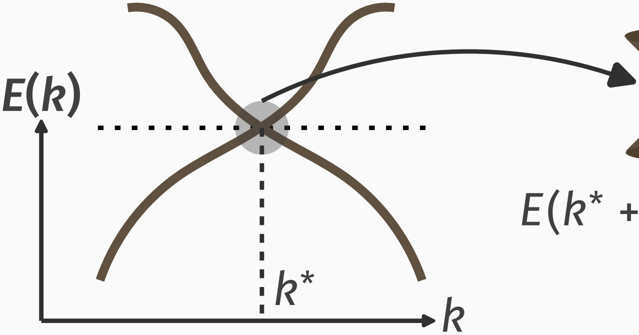
Introduction

Some Prerequisites

- The system: 2D Dirac electrons
- Entanglement of free fermions
- Reduction of a 2D system to sum of 1D systems
- Entanglement in topologically ordered phases
- The holographic principle

The System: 2D Dirac Electrons

Dispersion is linear in momentum space



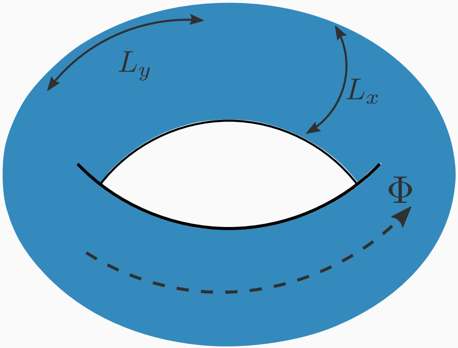
$$E(k^* + q) = \cancelto{0}{E(k^*)} + \frac{\partial E}{\partial k^*} q + \dots$$

- Describe the low-energy theory near gap-closing points
- Emerge at boundaries of topological insulators

The System: 2D Dirac Electrons

- Place on a torus (periodic boundary conditions)
- Insert a vector potential (flux-tuning)

$$H = v_F \vec{\alpha}\cdot\left(\vec{p} - \underbrace{e\vec{A}}_{\text{vector potential}}\right) + \underbrace{\beta m}_{\text{mass term}}$$



Measures of Entanglement

$$\rho = |\Psi\rangle\langle\Psi| \longrightarrow \text{density matrix}$$

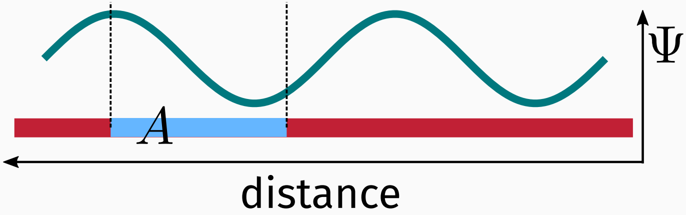
$$\rho_A = \text{partial trace over system A} \longrightarrow \text{reduced DM}$$

- $S(A) = -\mathrm{Tr}\left[\rho_A \log \rho_A\right] \longrightarrow$ entanglement entropy of A
- $I(A:B) = S(A) + S(B) - S(A \cup B) \longrightarrow$ mutual information between A and B
- quantifies amount of information shared between subsystems

Entanglement of Free Fermions

Diagonal in k−space : $H = i\overline{\psi}\left(\gamma_\mu \partial_\mu + m\right)\psi$

- Vanishing entanglement in momentum space
- Off-diagonal in r−space $\longrightarrow$ Fluctuations exist in real space
- Leads to entanglement in real space



Entanglement of Free Fermions

Some existing results on fermionic entanglement:

- massless fermions in d-dimensions: $L^{d-1} \log L$

- massive fermions in 1-dimension: $\frac{1}{3} \log (L/\epsilon) - \frac{1}{6} (mL \log mL)^2$

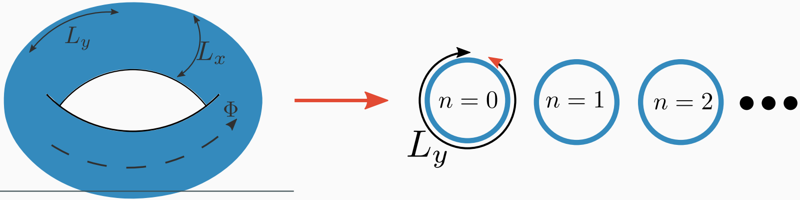
(ϵ = short-distance cutoff, m = mass gap in the spectrum)

Calabrese and Cardy 2004a; Casini, Fosco, and Huerta 2005; Gioev and Klich 2006; Wolf 2006; Li et al. 2006; Casini and Huerta 2009.

Reduction of 2D System into Sum of 1D Systems

In presence of flux: $L = \int dxdy\, \overline{\Psi}(x)\left(i\gamma_\mu + eA_\mu\right)\partial_\mu \Psi(x)$

- PBC along $\vec{x}$: $\Psi(x) = \sum_{n=-\infty}^{\infty} e^{ixk_x^n}\Psi(k_x^n)$, $\quad k_x^n = \frac{2\pi n}{L_x}$, $\quad n \in \mathbb{Z}$
- Lagrangian decouples: $L = \sum_n \int dy\, \overline{\Psi}_n(y)\left(i\gamma_\mu\partial_\mu - M_n\right)\Psi_n(y)$
- Mass of each 1D mode: $M_n = \frac{2\pi}{L_x}|n + \phi|$

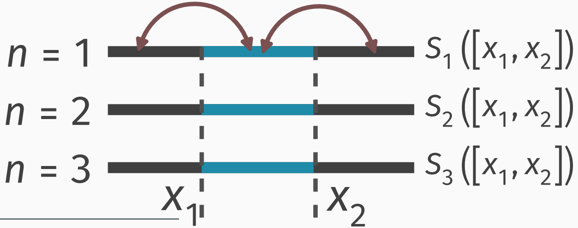


Chung and Peschel 2000; Arias, Blanco, and Casini 2015; Chen et al. 2017; Murciano, Ruggiero, and Calabrese 2020.

Reduction of 2D System into Sum of 1D Systems

- $H = \sum_n H_n \implies \rho = \exp(-\beta H) = \otimes_n \rho_n \implies$ no entanglement in k_x-space
- Entanglement reduces to sum over 1D modes: $S([x_1, x_2]) = \sum_n S_n([x_1, x_2])$

$$S_n(\phi) = \underbrace{c \log(\alpha L_x)}_{\text{modified area law}} - \underbrace{c \log|n + \phi|}_{\text{mass correction}}, \quad \alpha \longrightarrow \text{cutoff dependent constant}$$



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Entanglement in Topologically Ordered Phases

Gapped quantum liquids arising from strong inter-electron correlations

- FQHE, Toric Code, Kitaev's honeycomb model, QSLs
- robust ground-state degeneracy on closed manifolds (for eg., torus),
- long-ranged entanglement: $S(L) = \alpha L - \gamma + O(1/L)$.

N-partite information measure depends on γ and the Euler characteristic χ of the manifold: $|I_N| = \gamma\chi$.

Wen 1989; Kitaev and Preskill 2006; Patra, Basu, and Lal 2022.

The AdS-CFT Correspondence: A Holographic Duality Relation

What is a duality?

Different Hamiltonian/action describing the same system

Example: Quantum-Classical Mapping

Ising model in 1D		Partition function / Path integral		Quantum spin in 0D
$H = -J \sum_{<i,j>} S_i^z S_j^z$	$\rightarrow$	$Z = \sum \langle x \lvert e^{-\beta H} \rvert x_1 \rangle \ldots \langle x_N \lvert e^{-\beta H} \rvert x \rangle$	$\leftarrow$	$H = \frac{\Delta}{2} \sigma^x$

Another example: Maxwell's equations

$$\nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{B} = 0, \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t}$$

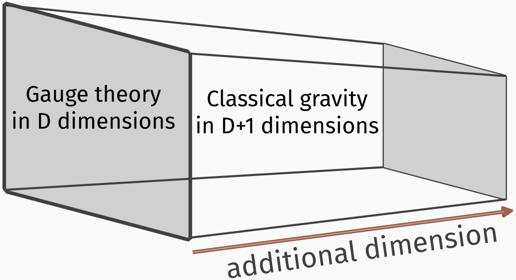
under the transformation $\quad \mathbf{E} \rightarrow -\mathbf{B}$

The AdS-CFT Correspondence: A Holographic Duality Relation

What is AdS-CFT?

Duality between a gravity theory and a conformal field theory

$$\underbrace{Z_Q}_{\text{D dims}} \sim \underbrace{\exp\left(-S_{\text{cl}}\right)}_{\text{D+1 dims}}$$



CFT: Remains invariant under conformal transformations

$$g_{\mu\nu}(x) \rightarrow \Lambda(x) g_{\mu\nu}(x)$$

Green 2013; Sachdev 2012.

The AdS-CFT Correspondence: A Holographic Duality Relation

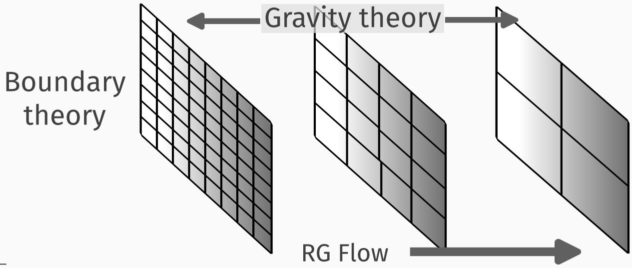
What is Holography?

Amount of information within a region is bounded by the surface area!

Entropy of a black hole: $S_{\mathrm{BH}} = \frac{k_B}{4l_P^2} A_H$

Physical Interpretation of the Additional Dimension

renormalisation group flow of boundary CFT



Bekenstein 1973; Akhmedov 1998; Álvarez and Gómez 1999.

What are we going after?

What Are We Going After?

- Distribution of entanglement across subsystems and scales (RG flow of entanglement)
- Topological aspects of entanglement (link to Fermi volume)
- Emergent space generated by this entanglement (holography)
- Curvature and related quantities of this emergent space (curvature transition)
- Effect of boundary phase transition on the emergent space (phase transition = wormhole geomtry)

Entanglement Hierarchy in Mixed Momentum and Real Space

Creating Subsystems

$$k_x^n = \frac{2\pi}{L_x} n, \quad n \in \mathbb{Z}; \quad \text{define distance} = \Delta n = 1$$

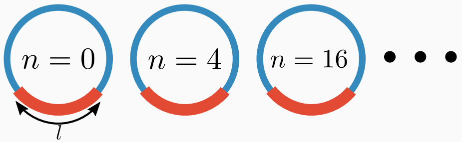
Simplest choice: the entire set

$$\text{distance} = 1 \longrightarrow n \in \{0, 1, \ldots, N-2, N-1, N\}$$

Coarser choices: increase distance

$$\text{distance} = 2 \longrightarrow n \in \{0, 2, \ldots, N-4, N-2, N\}$$

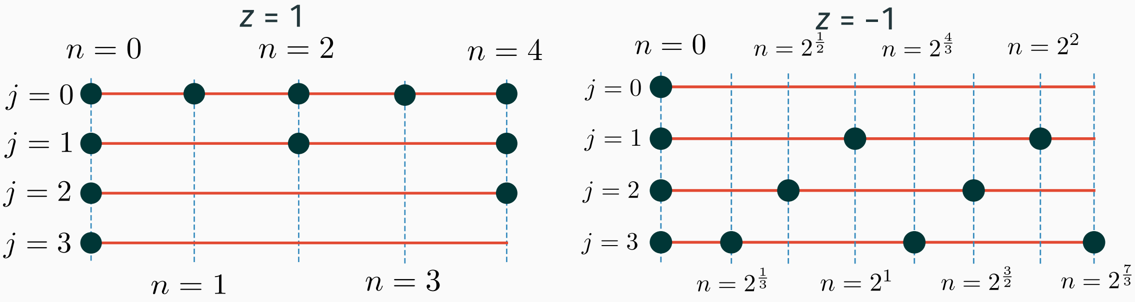
$$\text{distance} = 4 \longrightarrow n \in \{0, 4, \ldots, N-8, N-4, N\}$$



Define Sequence of Subsystems

$$k_x^j = \frac{2\pi}{L_x} t_z(j), \quad t_z(j) = 2^{j^z}; \quad \text{sequence index: } j = 0, 1, 2, \ldots$$

strength of coarse/fine-graining: $z = \pm 1, \pm 2, \pm 3, \ldots$



Interpreting the Set of Transformations as an RG Flow

Sequence of Hamiltonians ↔ renormalisation group flow

RG → transformation of Hamiltonian via change of scale

Superset of all members: $A_z^{(0)} = \bigcup_j A_z(j)$

"Super-Hamiltonian": $H^{(0)} = \sum_{k_x \in A_z^{(0)}} H\left(k_x\right)$

RG equation: $H_z(j) = \underbrace{P_z(j)}_{\text{projector}} \; H^{(0)} P_z(j)$

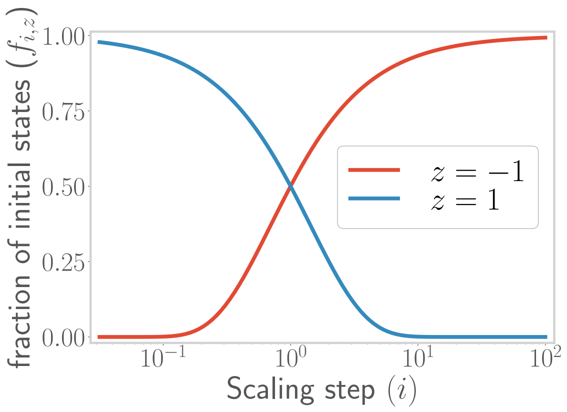
So What, Exactly, is Getting Renormalised?

Several ways to look at this

- renormalisation in entanglement: $\Delta \log S_z(j) \sim \Delta f_z(j)$
- renormalisation in 1-particle spectral gap: $M(n, \phi) \sim |n + \phi|$
- renormalisation in real space quantum fluctuation

Fraction of Maximum States

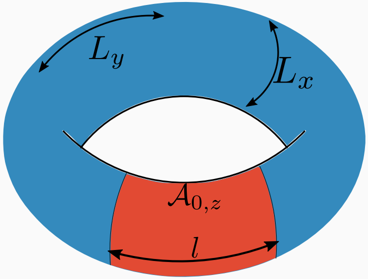
$f_z(j)$ = fraction of maximum states = $1/t_z(j)$



Simplest Limit

Simplest case: $j = 0$

- no coarse-graining or fine-graining
- $A_z(0) \longrightarrow$ cylindrical section



In general:

$$\Delta n \sim \Delta k_x \sim 1/L_x \quad \longrightarrow \quad \begin{array}{l} z > 0 \,:\, \text{decreasing system size} \\ z < 0 \,:\, \text{increasing system size} \end{array}$$

Subsystem Entanglement Entropy

Modes are decoupled $\longrightarrow$ entanglement is additive

$$S_n(\phi) = c \log\left(\alpha L_x\right) - c \log|n + \phi|$$

$$S_{A_z(j)} = \sum_{n \in A_z(j)} S_n = f_z(j) c \alpha L_x - c \log\left|2 \sin\left(\pi f_z(j) \phi\right)\right|$$

$$i < j, \; S_{i \cup j} = \begin{cases} S_i, & z > 0 \\ S_j, & z < 0 \end{cases}$$

Calabrese and Cardy 2004b; Casini, Fosco, and Huerta 2005; Arias, Blanco, and Casini 2015; Chen et al. 2017; Murciano, Ruggiero, and Calabrese 2020.

Entanglement Hierarchy

$$i < j, \;\; S_{i \cup j} = \begin{cases} S_i, & z > 0 \\ S_j, & z < 0 \end{cases}$$


- presents a hierarchy of entanglement $\longrightarrow$ EE distributed across RG steps:

  RG transformation $\longrightarrow$ reveals entanglement

- distribution of entanglement also present in multipartite entanglement:

  mutual information and higher order measures, within one RG step or spread across the flow

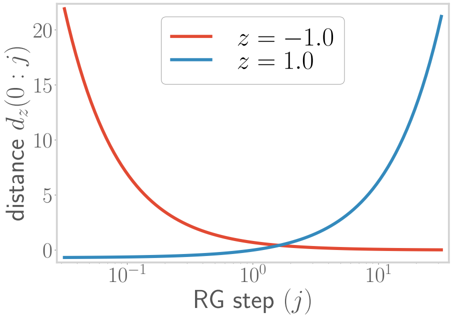
Holographic Nature of the RG Flow

Mutual Information = Distance

Mutual information: $I^2(A : B) \equiv S(A) + S(B) - S(A \cup B)$ (non-negative)

information gained about B upon measuring A

define distance along the RG: $d_z(j) \equiv \log I^2_{max} - \log I^2_z(0 : j) = \log t_z(j)$



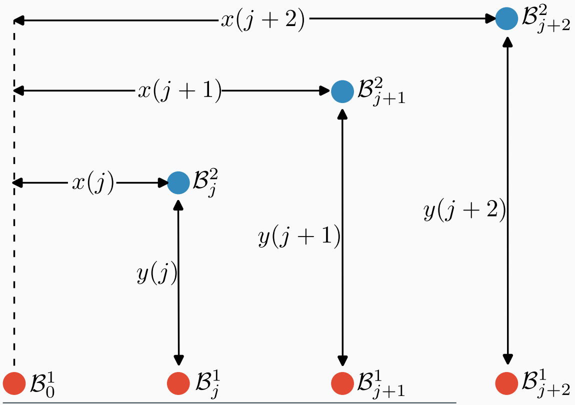
For $z > 0$:

- mut. info. is maximum for small j
- decreases for large j
- corresponds to increasing distance

Van Raamsdonk 2010; Lee and Qi 2016; Mukherjee and Lal 2022.

RG evolution = Emergent Distance

Define 2-dimensional x – y structure



Red Circle: RG steps

Blue Circle: subsystems within an RG step

$$x_z(j) = d_z(j) = \log t_z(j)$$

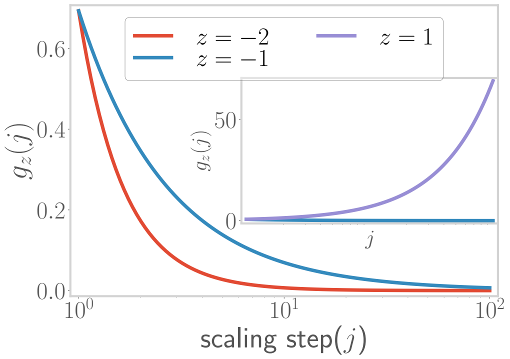
$$y_z(j) = \log I^2_{max} - \log I^2_z(B^1_j : B^2_j)$$
$$= \log t_z(j \pm 1)$$

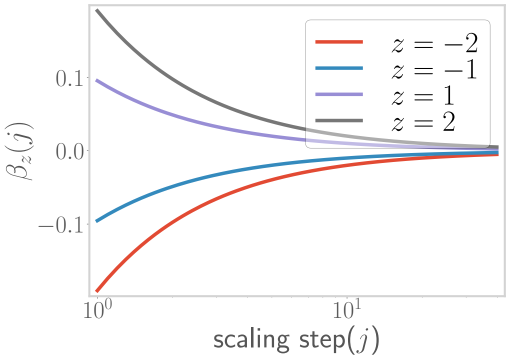
Lee 2010; Mukherjee and Lal 2020; Ryu and Takayanagi 2006; Nozaki, Ryu, and Takayanagi 2012.

RG evolution = Emergent Distance

Define coupling that measures spectral gap: $g_z(j) = \log \frac{M_{n+1}(\phi) - M_n(\phi)}{2\pi / L_x} = \log t_z(j)$

RG beta function for its evolution: $\beta_z(j) = \Delta \log g_z(j) = z \log\left(1 + j^{-1}\right)$





Lee 2010; Mukherjee and Lal 2020; Ryu and Takayanagi 2006; Nozaki, Ryu, and Takayanagi 2012.

RG evolution = Emergent Distance

RG beta function can be related to the x, y-distances

$$x_z = \left(e^{\frac{\beta_z}{z}} - 1\right)^{-z} \log 2$$

$$y_z = \begin{cases} x_z e^{\beta}, & z > 0 \\ x_z \left(2 - e^{\frac{\beta}{z}}\right)^{z}, & z < 0 \end{cases}$$

explicit relation between the RG flow and the emergent geometry

Lee 2010; Mukherjee and Lal 2020; Ryu and Takayanagi 2006; Nozaki, Ryu, and Takayanagi 2012.

Curvature of Emergent Space

Define first and second derivatives in emergent space

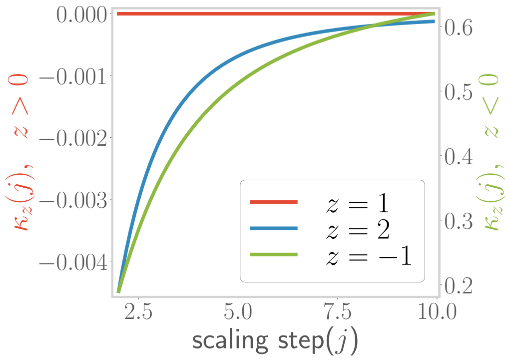
$$v_z(j) \equiv \frac{\Delta y_z(j)}{\Delta x_z(j)} = \begin{cases} \frac{(j+2)^z-(j+1)^z}{(j+1)^z-j^z}, & z > 0 \\ \frac{(j)^z-(j-1)^z}{(j+1)^z-j^z}, & z < 0 \end{cases}$$

$$v_z'(j) \equiv \frac{v_z(j+1) - v_z(j)}{x_z(j+1) - x_z(j)}$$

Define curvature using them: $\kappa_z(j) = \frac{v_z'(j)}{\left[1+v_z(j)^2\right]^{\frac{3}{2}}}$

$\longrightarrow$ can be expressed in terms of $\beta_z(j)$

Curvature of Emergent Space



- positive curvature for $z < 0$
- zero curvature for $z = 1$
- negative curvature for $z > 1$
- asymptotically flat for large j, at all z

The Sign of Curvature is Topological!

$$\gamma_z(j) \equiv 1 - v_z(j+1)/v_z(j), \quad \alpha_z(j) = y_z(j+1) - y_z(j)$$

$$\kappa_z(j) = -\frac{\alpha_z(j)\,\gamma_z(j)}{\left(\Delta x_z(j)\right)^2\left[1 + v_z(j)^2\right]^{\frac{3}{2}}} \implies \mathrm{sign}\left[\kappa_z(j)\right] = -\mathrm{sign}\left[\alpha_z(j)\right]\mathrm{sign}\left[\gamma_z(j)\right]$$

$$\mathrm{sign}\left[\kappa_z\right] = \begin{cases} -1, & z \geq 1 \\ 1, & z \leq -1 \end{cases} = \begin{cases} -\mathrm{sign}\left[\gamma_z(j)\right], & z \geq 1 \\ -\mathrm{sign}\left[\alpha_z(j)\right], & z \leq -1 \end{cases}$$

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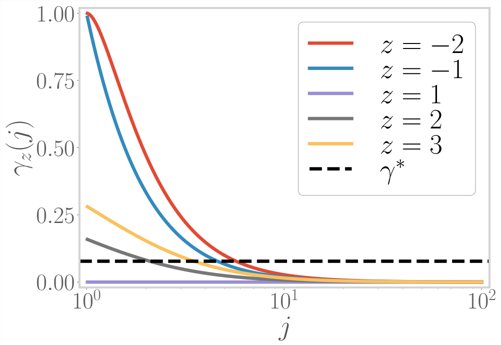
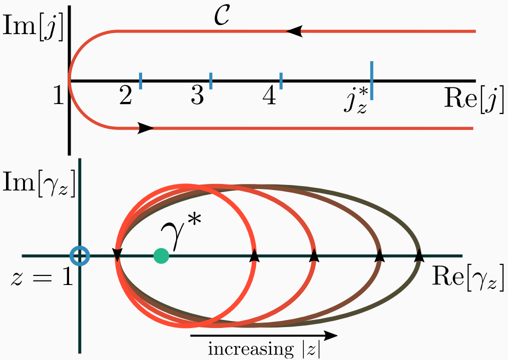
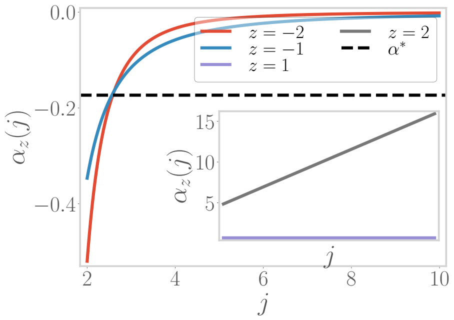
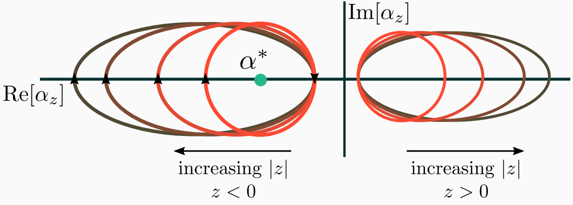




- $\ln(\gamma - \gamma^*)$ has branch point at γ^*, can be avoided for $z = 1$, contour is trivial
- cannot be avoided for $z \neq 1 \longrightarrow$ presence of singularity $\longrightarrow$ encoded through winding number

The Sign of Curvature is Topological!





very similar thing holds for α_z

- singularity exists only for $z < 0$
- otherwise contour can be trivialised

Curvature can be written as the product of winding numbers:

$$\mathrm{sign}\left[\kappa_z\right] = W_z(\gamma^*) \times \left[2W_z'(\alpha^*) - 1\right]$$

Winding numbers count singularities, robust against deformations

The Sign of Curvature is Topological!

Significance of change in topology

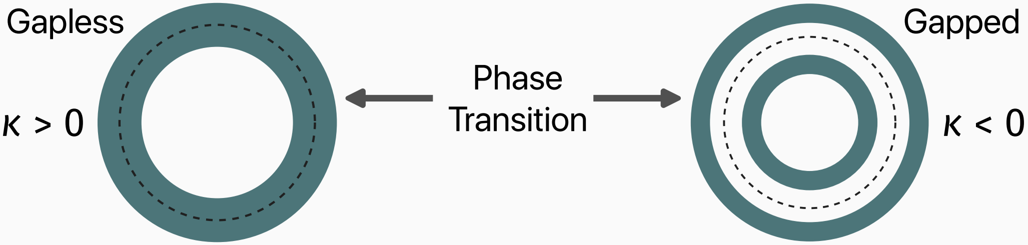
- sign of z reflects the RG relevance/irrelevance of g_z in the microscopic fermionic theory
- change in sign of z is hence a phase transition in the microscopic theory that changes the topology of the Fermi surface

Entanglement Holography and Fermionic Criticality

Critical Fermi Surface = Wormhole Geometry

Between $z < 0$ and $z > 0$, two topological transitions occur:

- Curvature changes sign
- Fermi surface becomes gapped → change in Luttinger's volume
- Reflects a phase transition in the underlying interacting fermionic theory



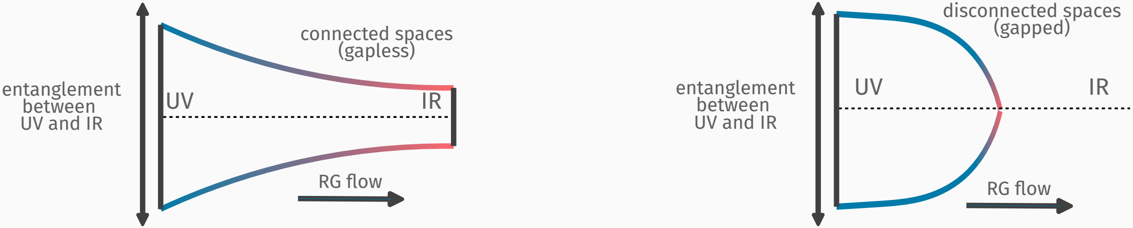
Mukherjee and Lal 2020; Heath and Bedell 2020.

Critical Fermi Surface = Wormhole Geometry

Also involves transition in nature of UV-IR entanglement

- Finite entanglement between UV and IR for $z < 0$ (connected spaces)
- Vanishing entanglement between UV and IR for $z > 0$ (disconnected spaces)

At transition, minimal entanglement between two almost disconnected spaces → wormhole geometry!



Van Raamsdonk 2010; Cao, Carroll, and Michalakis 2017.

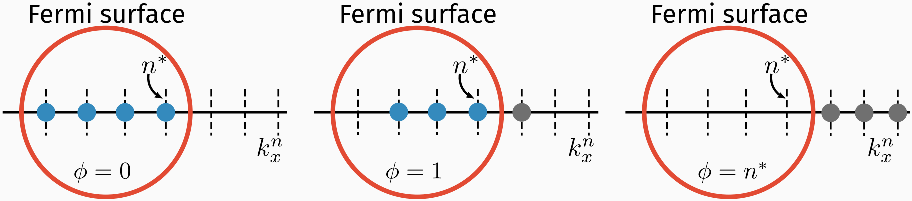
Topological Content of Entanglement

Luttinger Volume and Flux-Dependent Entanglement

Spectral flow: $k_n = 2\pi n / L_x, \quad n \to n + \phi(\text{flux})$

- Tuning flux by one unit removes one k–state from Fermi volume
- Fermi momentum is therefore linked to the maximum flux ϕ^*

No. of states within Fermi volume = number of integers between 0^+ and ϕ^{*+}.



Oshikawa 2000.

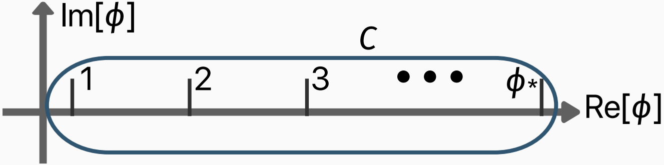
Luttinger Volume and Flux-Dependent Entanglement

Consider function $Q(\phi) = f\left[\frac{1}{\sqrt{2}} - |\sin \pi\phi f|\right]$

Goes through zero twice when ϕf changes by one unit: $\phi f = 1/4, 3/4$

Fermi volume = no. of poles of Q^{-1}
(residue ∝ no. of poles):

$$\sim \frac{1}{2}\oint_C \frac{d\phi}{Q(\phi)} \sim \oint_{Y(C)} \frac{dY}{Y}$$

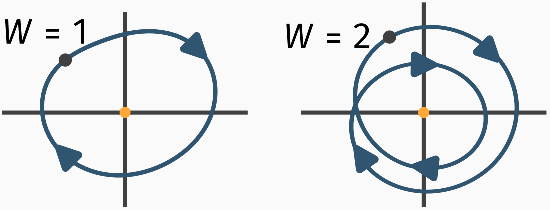


Integral is quantised!

$$Y = re^{i\theta} \implies W = \oint_{Y(C)} \frac{dY}{Y} \sim \oint d\theta = 0, 2\pi, 4\pi, \ldots$$

Luttinger Volume and Flux-Dependent Entanglement

- Fermi volume is the winding number of $Y(C)$ around $Y = 0$
- Topological in nature: Invariant under small deformations of the contour C



Key points

- Topological content of entanglement is the link to LV via spectral flow
- Yet another route to visualising LV as a topological invariant
- Boundary conditions are important: No entanglement flow in localised states

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Oshikawa 2000; Seki and Yunoki 2017.

Concluding Remarks

Summary of Results

- Entanglement renormalisation = emergent distance scale.
- Nature of emergent space depends on anomalous dimension of RG flow.
- Change in curvature corresponds to fermionic phase transition and a wormhole geometry.
- Topological structure of entanglement spectrum determines LV.

Some Results I Didn't Have The Patience to Build Slides For

(But then I felt bad so I added this slide.)

- We can construct a 'discrete metric' for our emergent dimension, by calculating the minimum distance between two points (geodesics).

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- We can define an expansion parameter (change of area of RG trajectories) that relates to the curvature, leading to equations similar to Raychaudhuri equation.
- Exploring the entanglement for a gapped system with a magnetic field (QHE) allows us to relate the topology of the entanglement to the Chern number.

Thank you!

References

Akhmedov, E.T. (1998). "A remark on the AdS/CFT correspondence and the renormalization group flow". In: *Physics Letters B* 442.1, pp. 152–158.
Arias, Raúl E, David D Blanco, and Horacio Casini (2015). "Entanglement entropy as a witness of the Aharonov–Bohm effect in QFT". In: *Journal of Physics A: Mathematical and Theoretical* 48.14, p. 145401.
Bekenstein, Jacob D. (1973). "Black Holes and Entropy". In: *Phys. Rev. D* 7 (8), pp. 2333–2346.
Calabrese, Pasquale and John Cardy (2004a). "Entanglement entropy and quantum field theory". In: *Journal of Statistical Mechanics: Theory and Experiment* 2004.06, P06002.
— (2004b). "Entanglement entropy and quantum field theory". In: *Journal of Statistical Mechanics: Theory and Experiment* 2004.06, P06002.
Cao, ChunJun, Sean M. Carroll, and Spyridon Michalakis (2017). "Space from Hilbert space: Recovering geometry from bulk entanglement". In: *Phys. Rev. D* 95 (2), p. 024031.
Casini, H, C D Fosco, and M Huerta (2005). "Entanglement and alpha entropies for a massive Dirac field in two dimensions". In: *Journal of Statistical Mechanics: Theory and Experiment* 2005.07, P07007–P07007.
Casini, H and M Huerta (2009). "Entanglement entropy in free quantum field theory". In: *Journal of Physics A: Mathematical and Theoretical* 42.50, p. 504007.
Chen, Xiao et al. (2017). "Two-cylinder entanglement entropy under a twist". In: *Journal of Statistical Mechanics: Theory and Experiment* 2017.4, p. 043104.
Chung, Ming-Chiang and Ingo Peschel (2000). "Density-matrix spectra for two-dimensional quantum systems". In: *Phys. Rev. B* 62 (7), pp. 4191–4193.
Gioev, Dimitri and Israel Klich (2006). "Entanglement entropy of fermions in any dimension and the Widom conjecture". In: *Physical review letters* 96.10, p. 100503.
Green, A.G. (2013). "An introduction to gauge-gravity duality and its application in condensed matter". In: *Contemporary Physics* 54.1, pp. 33–48.
Heath, Joshuah T and Kevin S Bedell (2020). "Necessary and sufficient conditions for the validity of Luttinger's theorem". In: *New Journal of Physics* 22.6, p. 063011.

Kitaev, Alexei and John Preskill (2006). "Topological entanglement entropy". In: *Physical review letters* 96.11, p. 110404.
Lee, Ching Hua and Xiao-Liang Qi (2016). "Exact holographic mapping in free fermion systems". In: *Physical Review B* 93.3, p. 035112.
Lee, Sung-Sik (2010). "Holographic description of quantum field theory". In: *Nuclear Physics B* 832.3, pp. 567–585.
Li, Weifei et al. (2006). "Scaling behavior of entanglement in two- and three-dimensional free-fermion systems". In: *Phys. Rev. B* 74 (7), p. 073103.
Mukherjee, Anirban and Siddhartha Lal (2020). "Unitary renormalisation group for correlated electrons-I: a tensor network approach". In: *Nuclear Physics B* 960, p. 115170.
— (2022). "Superconductivity from repulsion in the doped 2D electronic Hubbard model: an entanglement perspective". In: *Journal of Physics: Condensed Matter* 34.27, p. 275601.
Murciano, Sara, Paola Ruggiero, and Pasquale Calabrese (2020). "Symmetry resolved entanglement in two-dimensional systems via dimensional reduction". In: *Journal of Statistical Mechanics: Theory and Experiment* 2020.8, p. 083102.
Nozaki, Masahiro, Shinsei Ryu, and Tadashi Takayanagi (2012). "Holographic geometry of entanglement renormalization in quantum field theories". In: *Journal of High Energy Physics* 2012.10, p. 193.
Oshikawa, Masaki (2000). "Topological approach to Luttinger's theorem and the Fermi surface of a Kondo lattice". In: *Physical Review Letters* 84.15, p. 3370.
Patra, Siddhartha, Somnath Basu, and Siddhartha Lal (2022). "Unveiling topological order through multipartite entanglement". In: *Phys. Rev. A* 105 (5), p. 052428.
Ryu, Shinsei and Tadashi Takayanagi (2006). "Holographic derivation of entanglement entropy from the anti–de sitter space/conformal field theory correspondence". In: *Physical review letters* 96.18, p. 181602.
Sachdev, Subir (2012). "What Can Gauge-Gravity Duality Teach Us About Condensed Matter Physics?" In: *Annual Review of Condensed Matter Physics* 3.Volume 3, 2012, pp. 9–33.
Seki, Kazuhiro and Seiji Yunoki (2017). "Topological interpretation of the Luttinger theorem". In: *Physical Review B* 96.8, p. 085124.

Van Raamsdonk, Mark (2010). "Building up spacetime with quantum entanglement". In: *General Relativity and Gravitation* 42.10, pp. 2323–2329.

Wen, X. G. (1989). "Vacuum degeneracy of chiral spin states in compactified space". In: *Phys. Rev. B* 40 (10), pp. 7387–7390.

Wolf, Michael M. (2006). "Violation of the Entropic Area Law for Fermions". In: *Phys. Rev. Lett.* 96 (1), p. 010404.

Álvarez, Enrique and César Gómez (1999). "Geometric holography, the renormalization group and the c-theorem". In: *Nuclear Physics B* 541.1, pp. 441–460. ISSN: 0550-3213.