Hierarchical structure and topological content of entanglement of free fermions

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Introduction

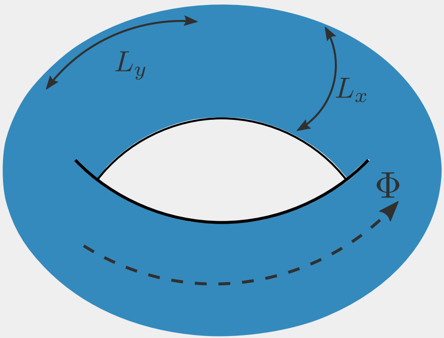
The system

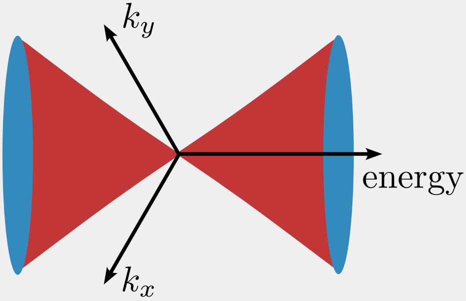
Massless Dirac fermions on a 2-torus

$$L = i\overline{\psi}\gamma_\mu \partial_\mu \psi$$

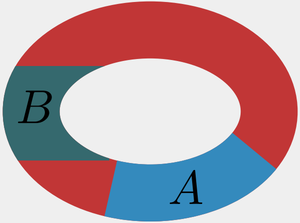
In presence of an Aharonov-Bohm flux

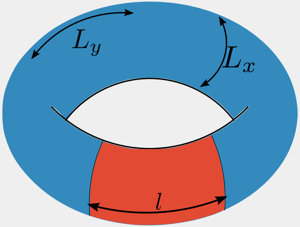
$$L = \overline{\psi}\left(i\gamma_\mu + eA_\mu\right)\partial_\mu \psi$$





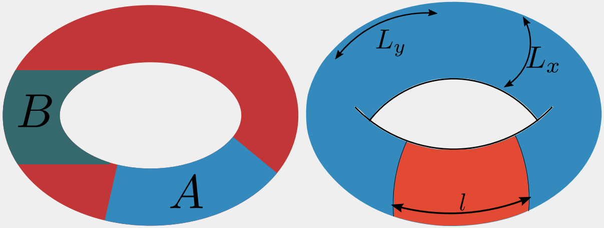
$\rho = |\Psi\rangle\langle\Psi| \longrightarrow$ **density matrix**





$\rho = |\Psi\rangle\langle\Psi| \longrightarrow$ **density matrix**

ρ_A = partial trace over system A $\longrightarrow$ **reduced DM**



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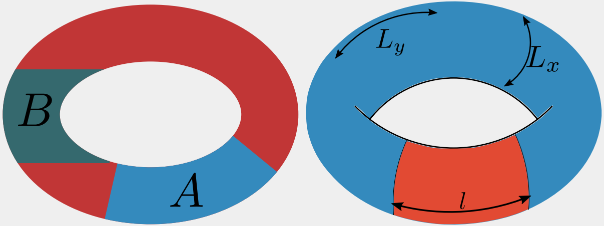

$S(A) = -\mathrm{Tr}\left[\rho_A \ln \rho_A\right] \longrightarrow$ **entanglement entropy** of A

$\longrightarrow$ quantifies information shared between A and rest

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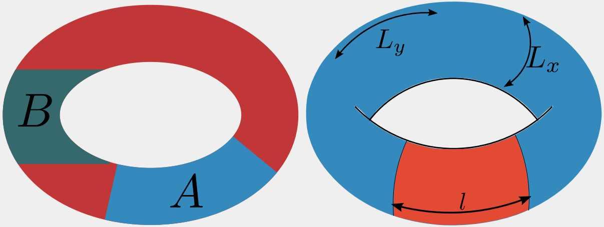

$I(A:B) = S(A) + S(B) - S(A \cup B) \longrightarrow$ **mutual information** between A and B

$\longrightarrow$ quantifies information shared between A and B

Entanglement of free fermions

Diagonal in k-space $\longrightarrow$ **Vanishing** entanglement in momentum space

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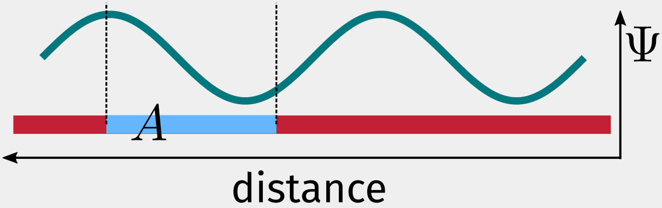
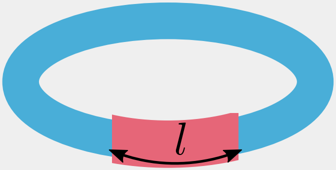
Off-diagonal in r-space $\longrightarrow$ **Fluctuations** exist in real space
$\longrightarrow$ leads to entanglement in real space

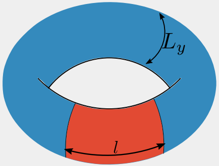


1D-ring of massless fermions: $\frac{2}{3}\ln\left(\frac{L}{\pi a}\sin\frac{\pi l}{L}\right)$



1D-line of relativistic fermions: $-\frac{1}{3}\ln(ma)$

2D-torus of massless fermions: $\alpha\frac{L_y}{\epsilon}$



Calabrese et al. 2004.

What are we going after?

What are we going after?

- Effect of a magnetic flux on the entanglement
- Distribution of the entanglement among subsystems of various sizes
- Emergent space generated by the transformations between these subsystems
- Curvature and related quantities of this space

Reduction to $(1+1)$–D systems

Reduction to (1 + 1)–D systems

In presence of flux: $L = \int dxdy \quad \overline{\Psi}(x)\left(i\gamma_\mu + eA_\mu\right)\partial_\mu\Psi(x)$

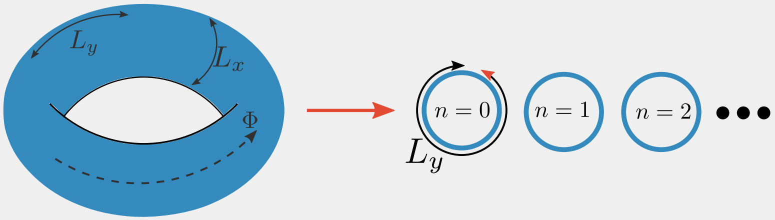
Periodic boundary conditions along $\vec{x}$: $k_x^n = \frac{2\pi n}{L_x}, \quad n \in \mathbb{Z}$

Introduce Fourier modes: $\Psi(x) = \sum_{n=-\infty}^{\infty} e^{ixk_x^n}\Psi(k_x^n)$

Reduction to (1 + 1)–D systems

Decouples into massive 1D modes: $L = \sum_n \int dy\, \overline{\Psi}(k_x, y)\left(i\gamma_\mu\partial_\mu - M\right)\Psi(k_x, y)$

Mass of each mode: $M(n, \phi) = \frac{2\pi}{L_x}|n + \phi|$



Reduction to (1 + 1)–D systems

$2D$ system is described by sum over $1D$ modes.

$\downarrow$

Modes do not couple - no inter-mode entanglement.

$\downarrow$

Total entanglement is sum of each part: $S = \sum_n S_n$

$$S_n(\phi) = \underbrace{c \log\left(\alpha L_x\right)}_{\text{modified area law}} - \underbrace{c \log|n+\phi|}_{\text{mass correction}}$$

Entanglement hierarchy in mixed momentum and real space

Creating subsystems

$$k_x^n = \frac{2\pi}{L_x} n, \quad n \in \mathbb{Z}; \quad \text{define } \textbf{distance} = \Delta n = 1$$

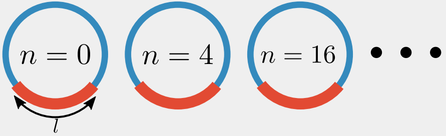
Simplest choice: the entire set

$$\text{distance} = 1 \longrightarrow n \in \{-N, -(N-1), -(N-2), \ldots, -1, 0, 1, \ldots, N-2, N-1, N\}$$

Coarser choices: increase distance

$$\text{distance} = 2 \longrightarrow n \in \{-N, -(N-2), -(N-4), \ldots, -2, 0, 2, \ldots, N-4, N-2, N\}$$

$$\text{distance} = 4 \longrightarrow n \in \{-N, -(N-4), -(N-8), \ldots, -4, 0, 4, \ldots, N-8, N-4, N\}$$



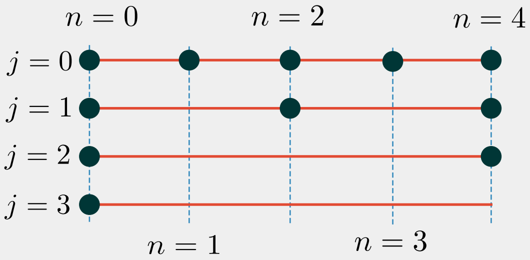
Define **sequence** of subsystems

$$A_z(j): \quad t_z(j) = 2^{j^z}$$

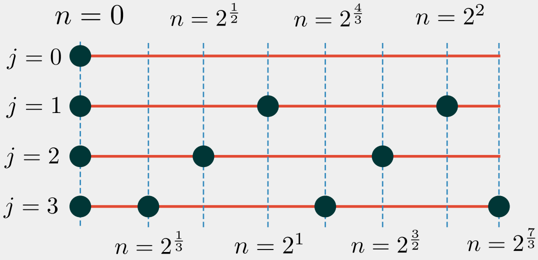
sequence index: $j = 0, 1, 2, \ldots$

strength of coarse/fine-graining: $z = \pm 1, \pm 2, \pm 3, \ldots$

$z = 1$



$z = -1$



Sequence of Hamiltonians $\longleftrightarrow$ **renormalisation** group flow

RG $\longrightarrow$ transformation of Hamiltonian via change of scale

Superset of all members: $A_z^{(0)} = \bigcup_j A_z(j)$

"Super-Hamiltonian": $H^{(0)} = \sum_{k_x \in A_z^{(0)}} H\left(k_x\right)$

RG equation: $H_z(j) = \underbrace{P_z(j)}_{\text{projector}} \; H^{(0)} \, P_z(j)$

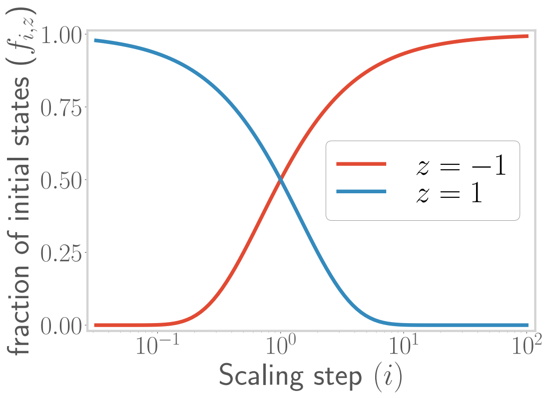
What, exactly, is getting renormalised?

Several ways to look at this

- renormalisation in **entanglement**: $\Delta \log S_z(j) \sim \Delta f_z(j)$
- renormalisation in 1-particle **spectral gap**: $M(n, \phi) \sim |n + \phi|$
- renormalisation in real space **quantum fluctuation**

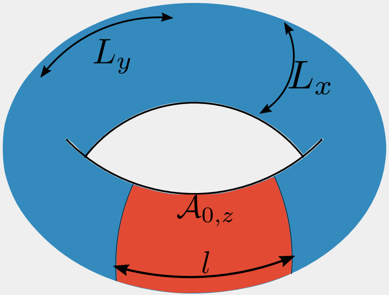
Fraction of maximum states

$f_z(j)$ = fraction of maximum states = $1/t_z(j)$



Sequence of subsystems

$j = 0$: $A_z(0)$: annulus



$\Delta n \sim \Delta k_x \sim 1/L_x$

$z > 0$: decreasing system size

$z < 0$: increasing system size

Modes are decoupled $\longrightarrow$ entanglement is additive

$$S_{A_z(j)} = \sum_{n \in A_z(j)} S_n = f_z(j) c \alpha L_x - c \log \left| 2 \sin\left(\pi f_z(j) \phi\right) \right|$$

$$i < j, \quad S_{i \cup j} = \begin{cases} S_i, & z > 0 \\ S_j, & z < 0 \end{cases}$$

Entanglement hierarchy

$$i < j, \quad S_{i \cup j} = \begin{cases} S_i, & z > 0 \\ S_j, & z < 0 \end{cases}$$

presents a **hierarchy** of entanglement $\longrightarrow$ EE distributed across levels

RG transformation $\longrightarrow$ reveals entanglement

distribution of entanglement also present in **multipartite** entanglement

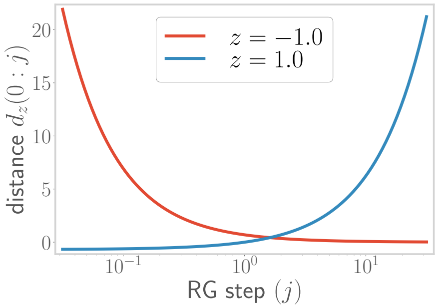
Holographic nature of the RG flow

Mutual information = distance

Mutual information: $I^2(A : B) \equiv S(A) + S(B) - S(A \cup B)$

information gained about B upon measuring A

define distance along the RG: $d_z(j) \equiv \log I^2_{\max} - \log I^2_z(0 : j) = \log t_z(j)$

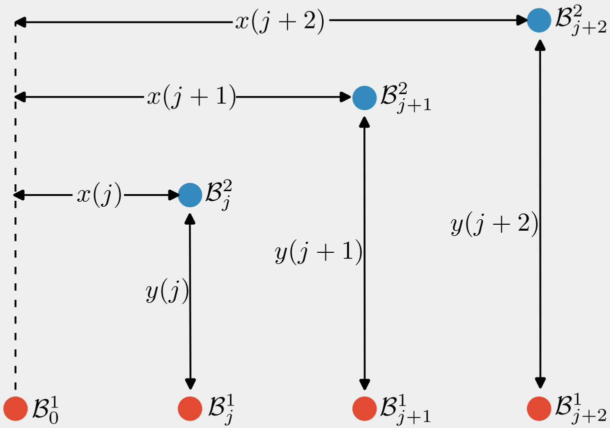


For $z > 0$:

- mut. info. is maximum for small j
- decreases for large j
- corresponds to **increasing distance**

RG EVOLUTION = EMERGENT DISTANCE

Define 2-dimensional $x - y$ structure



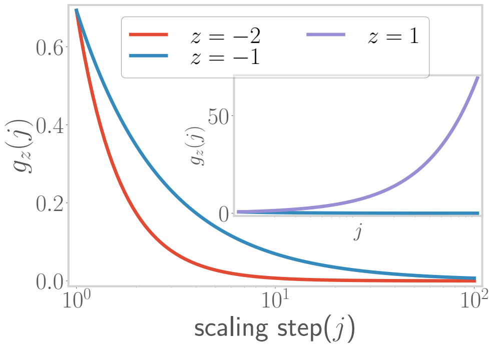
$$x_z(j) = d_z(j) = \log t_z(j)$$

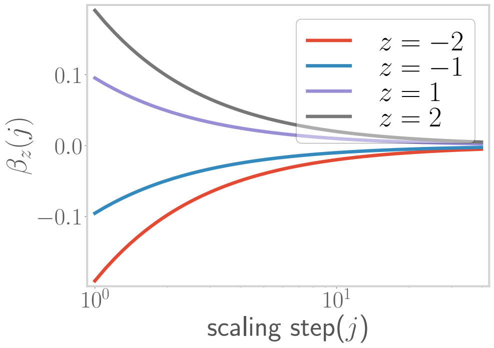
$$y_z(j) = \log I^2_{max} - \log I^2_z(B^1_j : B^2_j) = \log t_z(j \pm 1)$$

RG evolution = emergent distance

Define coupling that measures spectral gap: $g_z(j) = \log \frac{M_{n+1}(\phi) - M_n(\phi)}{2\pi / L_x} = \log t_z(j)$

RG beta function for its evolution: $\beta_z(j) = \Delta \log g_z(j) = z \log\left(1 + j^{-1}\right)$





RG evolution = emergent distance

RG beta function can be related to the x, y–distances

$$x_z = \left(e^{\frac{\beta_z}{z}} - 1\right)^{-z} \ln 2$$

$$y_z = \begin{cases} x_z e^{\beta}, & z > 0 \\ x_z \left(2 - e^{\frac{\beta}{z}}\right)^{z}, & z < 0 \end{cases}$$

explicit relation between the RG flow and the emergent **geometry**

Define first and second derivatives in emergent space

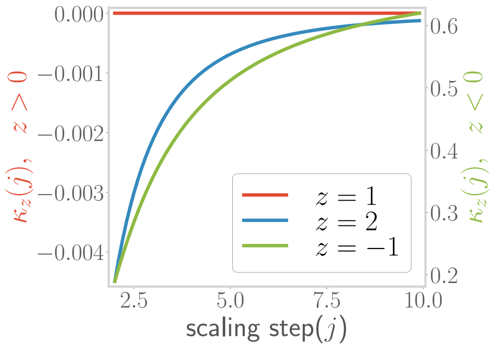
$$v_z(j) \equiv \frac{\Delta y_z(j)}{\Delta x_z(j)} = \begin{cases} \frac{(j+2)^z-(j+1)^z}{(j+1)^z-j^z}, & z > 0 \\ \frac{(j)^z-(j-1)^z}{(j+1)^z-j^z}, & z < 0 \end{cases}$$

$$v_z'(j) \equiv \frac{v_z(j+1) - v_z(j)}{x_z(j+1) - x_z(j)}$$

Define curvature using them: $\kappa_z(j) = \frac{v_z'(j)}{\left[1+v_z(j)^2\right]^{\frac{3}{2}}}$

$\longrightarrow$ can be expressed in terms of $\beta_z(j)$

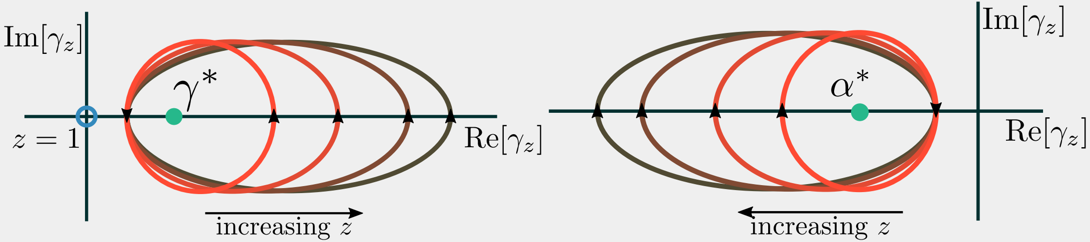
CURVATURE OF THE EMERGENT SPACE



- positive curvature for $z < 0$
- zero curvature for $z = 1$
- negative curvature for $z > 1$
- **asymptotically flat** for large j, at all z

Curvature can be written as the product of **winding numbers**:

$$\mathrm{sign}\left[\kappa_z\right] = W_z(\gamma^*) \times \left[2W_z'(\alpha^*) - 1\right]$$



winding numbers count singularities, robust against deformations

The sign of the curvature is topological

Where exactly is the topology changing?

- z acts as the **anomalous dimension** of the effective field theory
- change in z can be interpreted as a change in the underlying **interacting theory**
- change in sign of z is therefore a **topological phase transition** in the microscopic theory

References I

- Calabrese, Pasquale and John Cardy (2004). "Entanglement entropy and quantum field theory". In: *Journal of Statistical Mechanics: Theory and Experiment* 2004.06, P06002.