Hierarchical structure and topological content of entanglement of free fermions

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Introduction

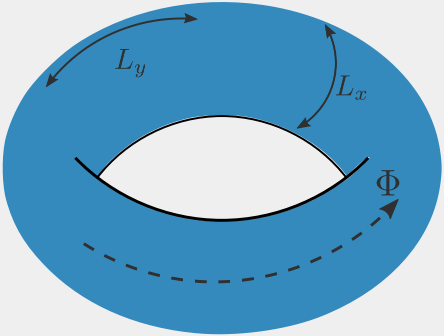
The system

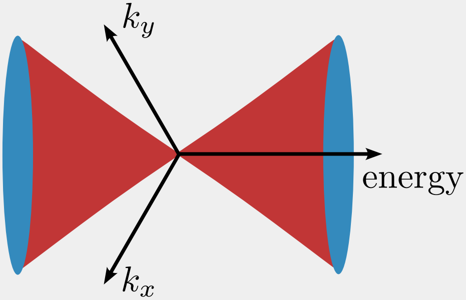
Massless Dirac fermions on a 2-torus

$$L = i\overline{\psi}\gamma_\mu \partial_\mu \psi$$

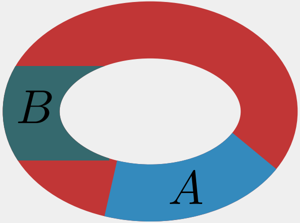
In presence of an Aharonov-Bohm flux

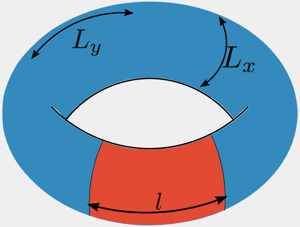
$$L = \overline{\psi}\left(i\gamma_\mu + eA_\mu\right)\partial_\mu \psi$$





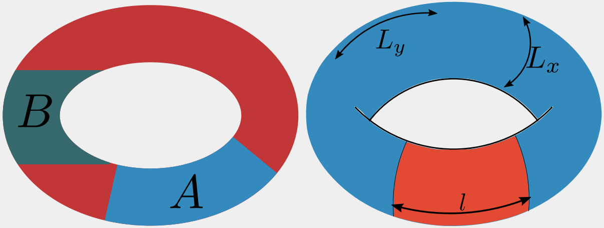
$\rho = |\Psi\rangle\langle\Psi| \longrightarrow$ **density matrix**



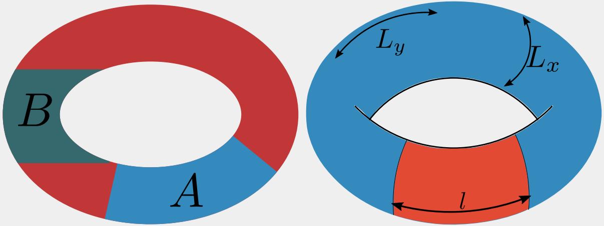


$\rho = |\Psi\rangle\langle\Psi| \longrightarrow$ **density matrix**

ρ_A = partial trace over system A
$\longrightarrow$ **reduced DM**



$\rho = |\Psi\rangle\langle\Psi| \longrightarrow$ **density matrix**



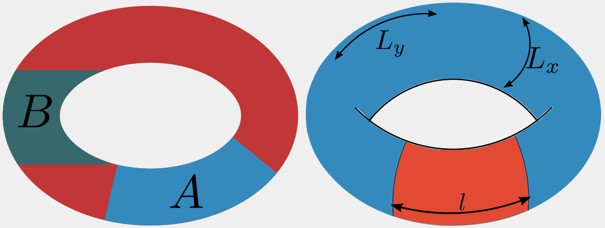
ρ_A = partial trace over system A
$\longrightarrow$ **reduced DM**

$S(A) = -\mathrm{Tr}\left[\rho_A \log \rho_A\right] \longrightarrow$ **entanglement entropy** of A

$\longrightarrow$ quantifies information shared between A and rest

$\rho = |\Psi\rangle\langle\Psi| \longrightarrow$ **density matrix**

ρ_A = partial trace over system A
$\longrightarrow$ **reduced DM**



$I(A:B) = S(A) + S(B) - S(A \cup B) \longrightarrow$ **mutual information** between A and B

$\longrightarrow$ quantifies information shared between A and B

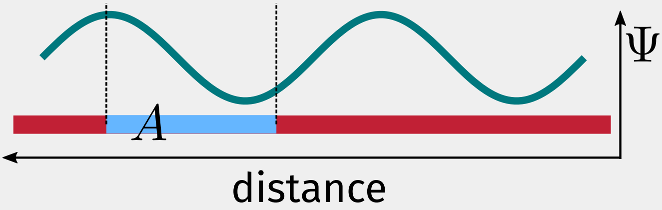
$$L = i\overline{\psi}\gamma_\mu\partial_\mu\psi$$

Diagonal in k-space $\longrightarrow$ **Vanishing** entanglement in momentum space

$$L = i\bar{\psi}\gamma_{\mu}\partial_{\mu}\psi$$

Diagonal in k–space $\longrightarrow$ **Vanishing** entanglement in momentum space

Off-diagonal in r–space $\longrightarrow$ **Fluctuations** exist in real space
$\longrightarrow$ leads to entanglement in real space



Entanglement of free fermions

Some existing results on fermionic entanglement:

- massless fermion on 1-d line: $\frac{1}{3}\log(l/\epsilon)$
- massive fermions on 1-d line: $\frac{1}{3}\log(l/\epsilon) - \frac{1}{6}(ml\log ml)^2$
- massless fermions in higher dims.: $l^{d-1}\log l$

Calabrese et al. 2004a; Casini et al. 2005; Gioev et al. 2006; Wolf 2006; Li et al. 2006; Casini et al. 2009.

Reduction of 2–D system to $(1 + 1)$–D systems

In presence of flux: $L = \int dxdy \;\; \overline{\Psi}(x)\left(i\gamma_\mu + eA_\mu\right)\partial_\mu \Psi(x)$

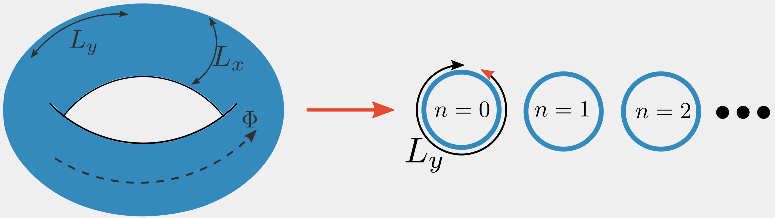
Periodic boundary conditions along $\vec{x}$: $k_x^n = \frac{2\pi n}{L_x}, \quad n \in \mathbb{Z}$

Introduce Fourier modes: $\Psi(x) = \sum_{n=-\infty}^{\infty} e^{ixk_x^n} \Psi(k_x^n)$

Chung et al. 2000; Arias et al. 2015; Chen et al. 2017; Murciano et al. 2020.

Decouples into massive 1*D* modes: $L = \sum_n \int dy\, \overline{\Psi}(k_x, y)\left(i\gamma_\mu \partial_\mu - M\right)\Psi(k_x, y)$

Mass of each mode: $M(n, \phi) = \frac{2\pi}{L_x}|n + \phi|$



Chung et al. 2000; Arias et al. 2015; Chen et al. 2017; Murciano et al. 2020.

Reduction to (1 + 1)–D systems

$2D$ system is described by sum over $1D$ modes.

↓

Modes do not couple - no inter-mode entanglement in k–space

↓

Total position-space entanglement is sum of each part: $S = \sum_n S_n$

$$S_n(\phi) = \underbrace{c \log(\alpha L_x)}_{\text{modified area law}} - \underbrace{c \log|n+\phi|}_{\text{mass correction}}$$

$\alpha \longrightarrow$ non-universal cutoff dependent constant

Chung et al. 2000; Arias et al. 2015; Chen et al. 2017; Murciano et al. 2020.

What are we going after?

What are we going after?

- Distribution of entanglement across subsystems and scales
- Emergent space generated by this entanglement (**holography**)
- Curvature and related quantities of this emergent space

Entanglement hierarchy in mixed momentum and real space

Creating subsystems

$$k_x^n = \frac{2\pi}{L_x} n, \quad n \in \mathbb{Z}; \quad \text{define } \textbf{distance} = \Delta n = 1$$

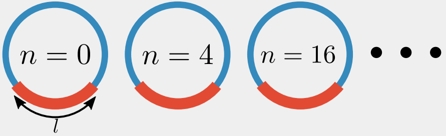
Simplest choice: the entire set

$$\text{distance} = 1 \longrightarrow n \in \{-N, -(N-1), -(N-2), \dots, -1, 0, 1, \dots, N-2, N-1, N\}$$

Coarser choices: increase distance

$$\text{distance} = 2 \longrightarrow n \in \{-N, -(N-2), -(N-4), \dots, -2, 0, 2, \dots, N-4, N-2, N\}$$

$$\text{distance} = 4 \longrightarrow n \in \{-N, -(N-4), -(N-8), \dots, -4, 0, 4, \dots, N-8, N-4, N\}$$



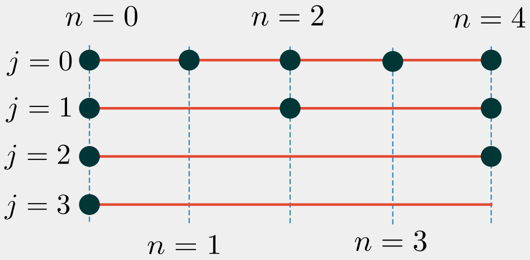
Define **sequence** of subsystems

$$A_z(j): \quad t_z(j) = 2^{j^z}$$

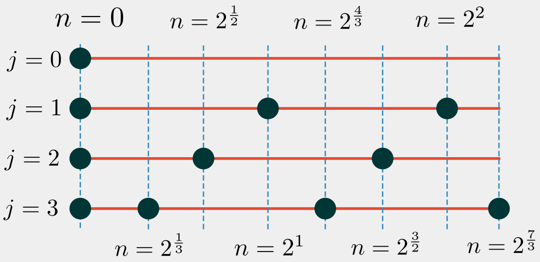
sequence index: $j = 0, 1, 2, \ldots$

strength of coarse/fine-graining: $z = \pm 1, \pm 2, \pm 3, \ldots$

$z = 1$



$z = -1$



The sequence as a renormalisation group flow

Sequence of Hamiltonians $\longleftrightarrow$ **renormalisation** group flow

RG $\longrightarrow$ transformation of Hamiltonian via change of scale

Superset of all members: $A_z^{(0)} = \bigcup_j A_z(j)$

"Super-Hamiltonian": $H^{(0)} = \sum_{k_x \in A_z^{(0)}} H\left(k_x\right)$

RG equation: $H_z(j) = \underbrace{P_z(j)}_{\text{projector}} \; H^{(0)} \, P_z(j)$

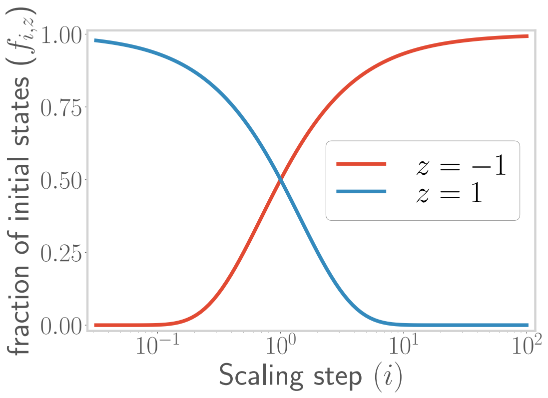
What, exactly, is getting renormalised?

Several ways to look at this

- renormalisation in **entanglement**: $\Delta \log S_z(j) \sim \Delta f_z(j)$
- renormalisation in 1-particle **spectral gap**: $M(n, \phi) \sim |n + \phi|$
- renormalisation in real space **quantum fluctuation**

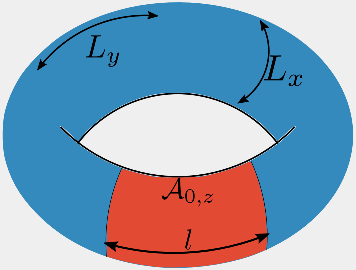
Fraction of maximum states

$f_z(j)$ = fraction of maximum states = $1/t_z(j)$



Simplest case: $j = 0$

- no coarse-graining or fine-graining
- $A_z(0) \longrightarrow$ **short cylinder**



In general:

$\Delta n \sim \Delta k_x \sim 1/L_x \longrightarrow$

$z > 0$: decreasing system size

$z < 0$: increasing system size

Modes are decoupled $\longrightarrow$ entanglement is additive

$$S_{A_z(j)} = \sum_{n \in A_z(j)} S_n = f_z(j) c \alpha L_x - c \log \left| 2 \sin \left(\pi f_z(j) \phi \right) \right|$$

$$i < j, \quad S_{i \cup j} = \begin{cases} S_i, & z > 0 \\ S_j, & z < 0 \end{cases}$$

Calabrese et al. 2004b; Casini et al. 2005; Arias et al. 2015; Chen et al. 2017; Murciano et al. 2020.

$$i < j, \quad S_{i \cup j} = \begin{cases} S_i, & z > 0 \\ S_j, & z < 0 \end{cases}$$

- presents a **hierarchy** of entanglement $\longrightarrow$ EE distributed across RG steps:

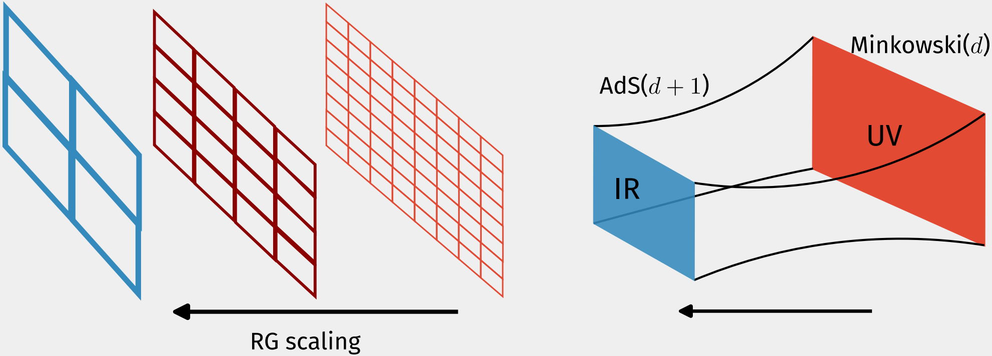
  RG transformation $\longrightarrow$ reveals entanglement

- distribution of entanglement also present in **multipartite** entanglement:

  mutual information and higher order measures, within one RG step or spread across the flow

Holographic nature of the RG flow

Conformal FT in d–dimensions $\longleftrightarrow$ Anti-de-Sitter space-time in $d+1$-dimensions



extra dimension in bulk corresponds to **RG flow**

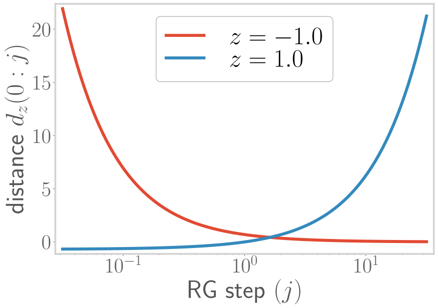
J. McGreevy (McGreevy,2010)

Mutual information = distance

Mutual information: $I^2(A:B) \equiv S(A) + S(B) - S(A \cup B)$ (non-negative)

information gained about B upon measuring A

define distance along the RG: $d_z(j) \equiv \log I^2_{\max} - \log I^2_z(0:j) = \log t_z(j)$

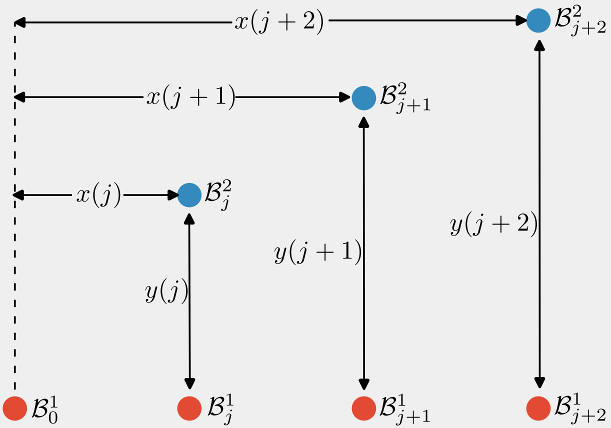


Van Raamsdonk 2010; Lee et al. 2016; Mukherjee et al. 2022.

For $z > 0$:

- mut. info. is maximum for small j
- decreases for large j
- corresponds to **increasing distance**

Define 2-dimensional $x - y$ structure



: RG steps

: subsystems within an RG step

$$x_z(j) = d_z(j) = \log t_z(j)$$

$$y_z(j) = \log I^2_{max} - \log I^2_z(B^1_j : B^2_j)$$

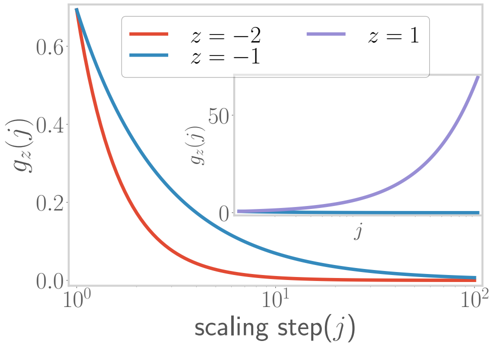
$$= \log t_z(j \pm 1)$$

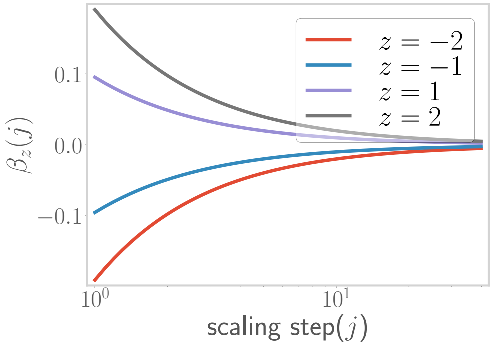
Lee 2010; Lee 2014; Qi 2013; Lee et al. 2016; Mukherjee et al. 2020a; Mukherjee et al. 2020b; Ryu et al. 2006b; Ryu et al. 2006a; Nozaki et al. 2012.

RG EVOLUTION = EMERGENT DISTANCE

Define coupling that measures spectral gap: $g_z(j) = \log \frac{M_{n+1}(\phi) - M_n(\phi)}{2\pi/L_x} = \log t_z(j)$

RG beta function for its evolution: $\beta_z(j) = \Delta \log g_z(j) = z \log\left(1 + j^{-1}\right)$





Lee 2010; Lee 2014; Qi 2013; Lee et al. 2016; Mukherjee et al. 2020a; Mukherjee et al. 2020b; Ryu et al. 2006b; Ryu et al. 2006a; Nozaki et al. 2012.

RG beta function can be related to the x, y–distances

$$x_z = \left(e^{\frac{\beta_z}{z}} - 1\right)^{-z} \log 2$$

$$y_z = \begin{cases} x_z e^{\beta}, & z > 0 \\ x_z \left(2 - e^{\frac{\beta}{z}}\right)^{z}, & z < 0 \end{cases}$$

explicit relation between the RG flow and the emergent **geometry**

Lee 2010; Lee 2014; Qi 2013; Lee et al. 2016; Mukherjee et al. 2020a; Mukherjee et al. 2020b; Ryu et al. 2006b; Ryu et al. 2006a; Nozaki et al. 2012.

Define first and second derivatives in emergent space

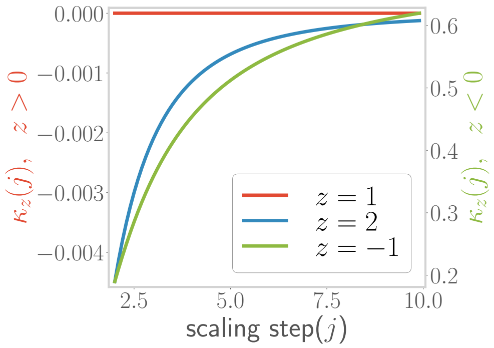
$$v_z(j) \equiv \frac{\Delta y_z(j)}{\Delta x_z(j)} = \begin{cases} \frac{(j+2)^z-(j+1)^z}{(j+1)^z-j^z}, & z > 0 \\ \frac{(j)^z-(j-1)^z}{(j+1)^z-j^z}, & z < 0 \end{cases}$$

$$v_z'(j) \equiv \frac{v_z(j+1) - v_z(j)}{x_z(j+1) - x_z(j)}$$

Define curvature using them: $\kappa_z(j) = \frac{v_z'(j)}{\left[1+v_z(j)^2\right]^{\frac{3}{2}}}$

$\longrightarrow$ can be expressed in terms of $\beta_z(j)$

Curvature of the emergent space



- positive curvature for $z < 0$
- zero curvature for $z = 1$
- negative curvature for $z > 1$
- **asymptotically flat** for large j, at all z

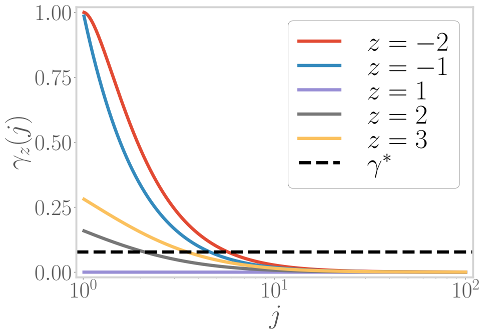
Is there a name for such spaces?

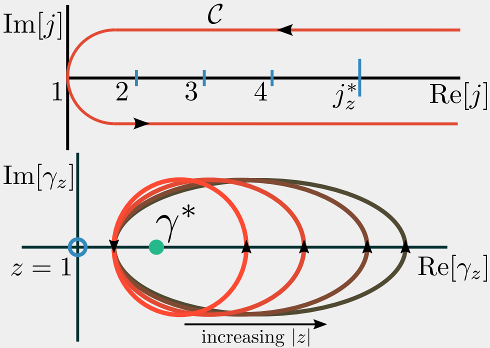
The sign of the curvature is topological

$$\gamma_z(j) \equiv 1 - v_z(j+1)/v_z(j), \quad \alpha_z(j) = y_z(j+1) - y_z(j)$$

$$\kappa_z(j) = -\frac{\alpha_z(j)\,\gamma_z(j)}{\left(\Delta x_z(j)\right)^2\left[1 + v_z(j)^2\right]^{\frac{3}{2}}} \implies \operatorname{sign}\left[\kappa_z(j)\right] = -\operatorname{sign}\left[\alpha_z(j)\right]\operatorname{sign}\left[\gamma_z(j)\right]$$

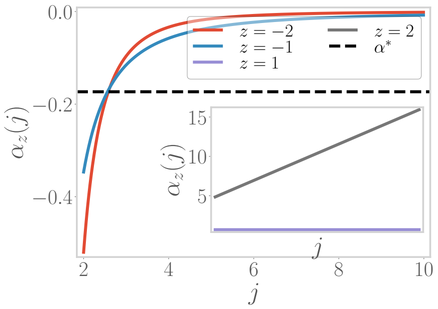
$$\operatorname{sign}\left[\kappa_z\right] = \begin{cases} -1, & z \geq 1 \\ 1, & z \leq -1 \end{cases} = \begin{cases} -\operatorname{sign}\left[\gamma_z(j)\right], & z \geq 1 \\ -\operatorname{sign}\left[\alpha_z(j)\right], & z \leq -1 \end{cases}$$





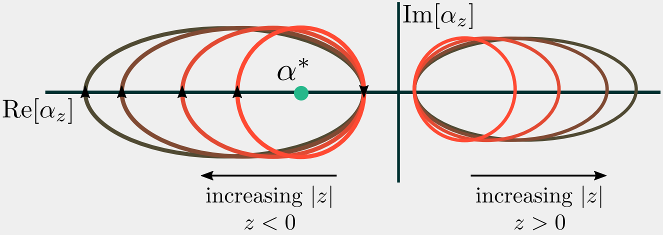
- $\ln(\gamma - \gamma^*)$ has branch point at γ^*, can be avoided for $z = 1$, **contour is trivial**
- cannot be avoided for $z \neq 1 \longrightarrow$ presence of **singularity** $\longrightarrow$ encoded through **winding number**

THE SIGN OF THE CURVATURE IS TOPOLOGICAL



very similar thing holds for α_z

- singularity exists only for $z < 0$
- otherwise contour can be trivialised



The sign of the curvature is topological

Curvature can be written as the product of **winding numbers**:

$$\mathrm{sign}\left[\kappa_z\right] = W_z(\gamma^*) \times \left[2W'_z(\alpha^*) - 1\right]$$

- winding numbers count singularities
- robust against deformations

The sign of the curvature is topological

What does this change in topology really mean?

- z is the **anomalous dimension** of the spectral gap g_z in the effective field theory
- sign of z reflects the RG relevance/irrelevance of g_z in the microscopic fermionic theory
- change in z can be interpreted as a change in the underlying **interacting theory**
- change in sign of z is hence a **phase transition** in the microscopic theory that changes the topology of the Fermi surface

Evolution of expansion parameter

- Define an expansion parameter
- can be related to RG flow through β_z
- related to change in area of flows of g_z

$$\theta_z(j) = \frac{1}{\sqrt{1 + v_z^{-2}}}$$

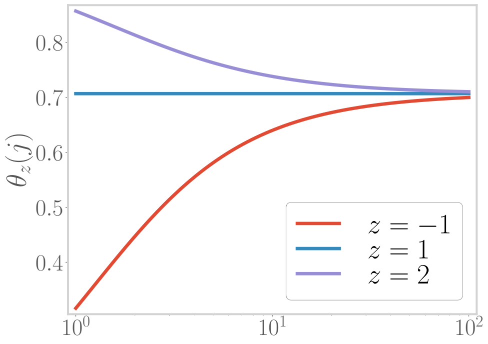
$$\theta_z \sim \frac{1}{\sqrt{\Delta x^2 + \Delta y^2}} \Delta g_z(j + 1)$$

Kar 2001; Kar et al. 2007; Balasubramanian et al. 1999.

- Expansion parameter satisfies "Raychaudhuri-like" equation

$$\frac{d\theta_z}{dx_z} = \kappa$$

- No attractive θ^2 term: fixed points reached only at $j \to \infty$



Kar 2001; Kar et al. 2007; Balasubramanian et al. 1999.

Conclusions

- hierarchy of entanglement, across scales as well as number of parties

$$S_{A\cup B} = S_{\text{larger}}$$

Conclusions

- hierarchy of entanglement, across scales as well as number of parties

- RG beta function gives rise to emergent distances

$$x_z(\beta), y_z(\beta)$$

Conclusions

- hierarchy of entanglement, across scales as well as number of parties
- RG beta function gives rise to emergent distances
- anomalous dimension z determines sign of curvature

$$\kappa \begin{cases} > 0 \text{ if } z < 0 \\ = 0 \text{ if } z = 1 \\ < 0 \text{ if } z > 1 \end{cases}$$

Conclusions

- hierarchy of entanglement, across scales as well as number of parties
- RG beta function gives rise to emergent distances
- anomalous dimension z determines sign of curvature
- sign of curvature is topological

$$\mathrm{sign}\left[\kappa_z\right] = W_z(\gamma^*) \times \left[2W_z'(\alpha^*) - 1\right]$$

Conclusions

- hierarchy of entanglement, across scales as well as number of parties
- RG beta function gives rise to emergent distances
- anomalous dimension z determines sign of curvature
- sign of curvature is topological
- θ satisfies ”Raychaudhuri-like” equation

$$\frac{d\theta_z}{dx_z} = \kappa$$

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Other Stuff

- Transformation to a different space

$$\tilde{\theta} = \frac{1}{1 - \sqrt{2}\theta}, \qquad \frac{d\tilde{\theta}}{dx_z} = \sqrt{2}\tilde{\theta}^2\kappa$$

- Does generate θ^2 term
- Effective curvature is zero