Holographic entanglement in free fermionic quantum matter: hierarchy & topology

Abhirup Mukherjee, Siddhartha Patra, Siddhartha Lal

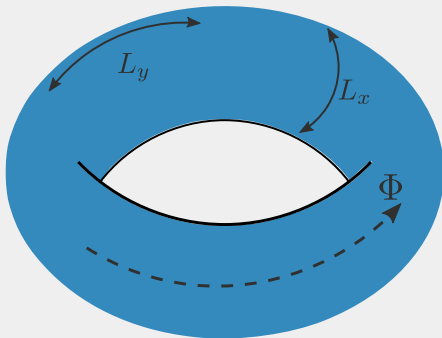Department of Physical Sciences, IISER Kolkata, Mohanpur

June 30, 2022

Introduction

Massless Dirac fermions on a 2-torus

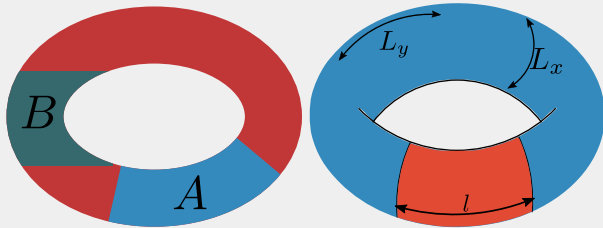$$L = i\overline{\psi}\gamma_\mu\partial_\mu\psi$$

In presence of an Aharonov-Bohm flux

$$L = \overline{\psi}\left(i\gamma_\mu + eA_\mu\right)\partial_\mu\psi$$
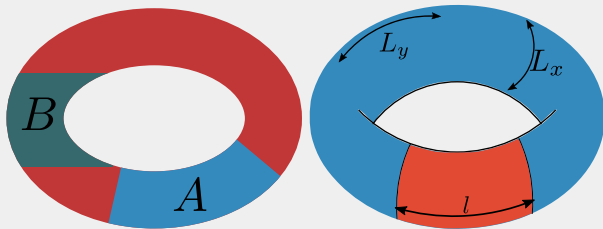
$\rho = |\Psi\rangle\langle\Psi| \longrightarrow$ **density matrix**

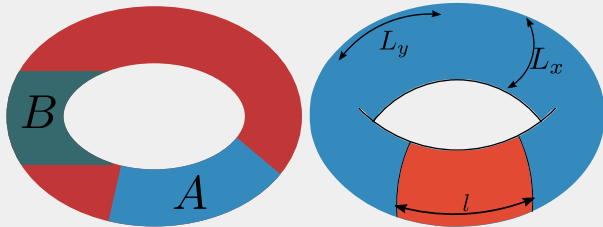$\rho = |\Psi\rangle\langle\Psi| \longrightarrow$ **density matrix**

ρ_A = partial trace over system A
$\longrightarrow$ **reduced DM**

$\rho = |\Psi\rangle\langle\Psi| \longrightarrow$ **density matrix**

ρ_A = partial trace over system A
$\longrightarrow$ **reduced DM**

$S(A) = -\text{Tr}\left[\rho_A \log \rho_A\right] \longrightarrow$ **entanglement entropy** of A

$\longrightarrow$ quantifies information shared between *A* and rest

$\rho = |\Psi\rangle\langle\Psi| \longrightarrow$ **density matrix**

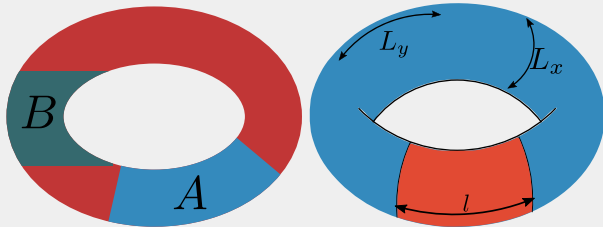ρ_A = partial trace over system A
$\longrightarrow$ **reduced DM**

$I(A : B) = S(A) + S(B) - S(A \cup B) \longrightarrow$ **mutual information** between A and B

$\longrightarrow$ quantifies information shared between A and B

$$L = i\overline{\psi}\gamma_\mu\partial_\mu\psi$$

Diagonal in k-space $\longrightarrow$ **Vanishing** entanglement in momentum space

$$L = i\overline{\psi}\gamma_\mu\partial_\mu\psi$$

Diagonal in k–space $\longrightarrow$ **Vanishing** entanglement in momentum space

Off-diagonal in r–space $\longrightarrow$ **Fluctuations** exist in real space
$\longrightarrow$ leads to entanglement in real space
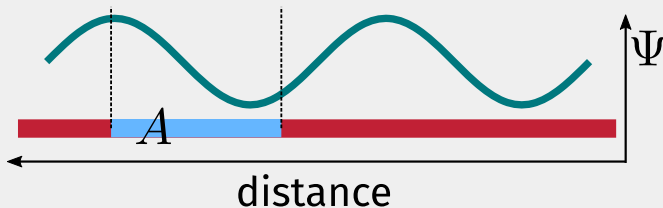
Some existing results on fermionic entanglement:

- massless fermion on 1–d line: $\frac{1}{3}\log(l/\epsilon)$

- massive fermions on 1–d line: $\frac{1}{3}\log(l/\epsilon) - \frac{1}{6}(ml\log ml)^2$

- massless fermions in higher dims.: $l^{d-1}\log l$

Calabrese et al. 2004a; Casini et al. 2005; Gioev et al. 2006; Wolf 2006; Li et al. 2006; Casini et al. 2009.

Reduction of 2–D system to (1 + 1)–D systems

In presence of flux:
$$L = \int dx\, dy\ \overline{\Psi}(x)\left(i\gamma_\mu + eA_\mu\right)\partial_\mu \Psi(x)$$
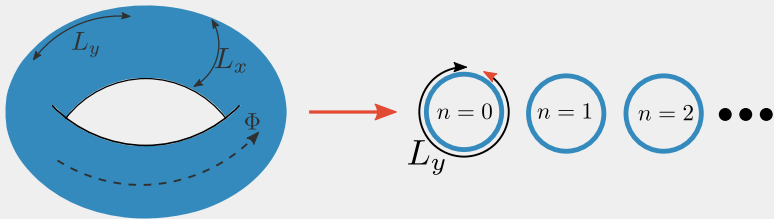
Periodic boundary conditions along $\vec{x}$:
$$k_x^n = \frac{2\pi n}{L_x}, \quad n \in \mathbb{Z}$$

Introduce Fourier modes:
$$\Psi(x) = \sum_{n=-\infty}^{\infty} e^{ix k_x^n}\Psi(k_x^n)$$

Chung et al. 2000; Arias et al. 2015; Chen et al. 2017; Murciano et al. 2020.

Decouples into massive 1*D* modes: $L = \sum_n \int dy \, \overline{\Psi}(k_x, y)\left(i\gamma_\mu \partial_\mu - M\right)\Psi(k_x, y)$

Mass of each mode: $M(n, \phi) = \frac{2\pi}{L_x}|n + \phi|$

Chung et al. 2000; Arias et al. 2015; Chen et al. 2017; Murciano et al. 2020.

2*D* system is described by sum over 1*D* modes.

↓

Modes do not couple - no inter-mode entanglement in k–space

↓

Total position-space entanglement is sum of each part: $S = \sum_n S_n$

$$S_n(\phi) = \underbrace{c \log\left(\alpha L_x\right)}_{\text{modified area law}} - \underbrace{c \log |n + \phi|}_{\text{mass correction}}$$

$\alpha \longrightarrow$ non-universal cutoff dependent constant

Chung et al. 2000; Arias et al. 2015; Chen et al. 2017; Murciano et al. 2020.

What are we going after?

- Distribution of entanglement across subsystems and scales

- Emergent space generated by this entanglement (**holography**)

- Curvature and related quantities of this emergent space

Entanglement hierarchy in mixed momentum and real space

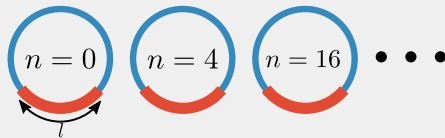$$k_x^n = \frac{2\pi}{L_x}n, \quad n \in \mathbb{Z}; \quad \text{define } \textbf{distance} = \Delta n = 1$$

Simplest choice: the entire set

distance = 1 $\longrightarrow$ $n \in \{-N, -(N-1), -(N-2), \ldots, -1, 0, 1, \ldots, N-2, N-1, N\}$

Coarser choices: increase distance

distance = 2 $\longrightarrow$ $n \in \{-N, -(N-2), -(N-4), \ldots, -2, 0, 2, \ldots, N-4, N-2, N\}$

distance = 4 $\longrightarrow$ $n \in \{-N, -(N-4), -(N-8), \ldots, -4, 0, 4, \ldots, N-8, N-4, N\}$
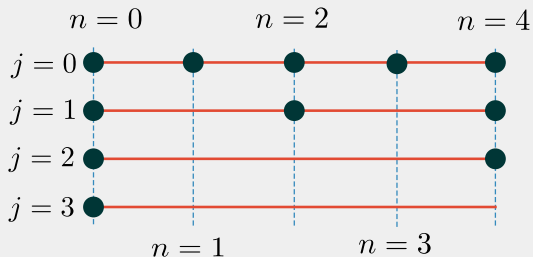
Define **sequence** of subsystems

$$A_z(j): \quad t_z(j) = 2^{j^z}$$

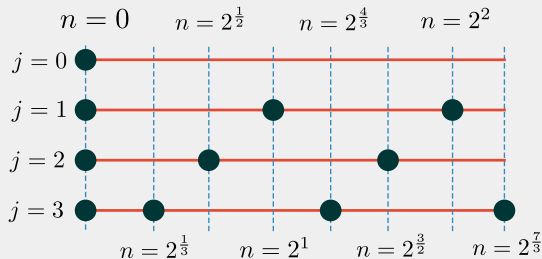sequence index: $\quad j = 0, 1, 2, \ldots$

strength of coarse/fine-graining: $\quad z = \pm1, \pm2, \pm3, \ldots$

Sequence of Hamiltonians $\longleftrightarrow$ **renormalisation** group flow

RG $\longrightarrow$ transformation of Hamiltonian via change of scale

Superset of all members: $A_z^{(0)} = \bigcup_j A_z(j)$

"Super-Hamiltonian": $H^{(0)} = \sum_{k_x \in A_z^{(0)}} H(k_x)$

RG equation: $H_z(j) = \underbrace{P_z(j)}_{\text{projector}} H^{(0)} P_z(j)$

Several ways to look at this

- renormalisation in **entanglement**: $\Delta \log S_z(j) \sim \Delta f_z(j)$
- renormalisation in 1-particle **spectral gap**: $M(n, \phi) \sim |n + \phi|$
- renormalisation in real space **quantum fluctuation**

$f_z(j)$ = fraction of maximum states = $1/t_z(j)$
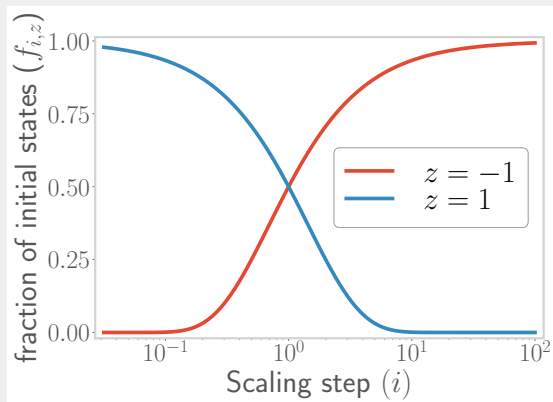
Simplest case: $j = 0$

- no coarse-graining or fine-graining

- $A_z(0) \longrightarrow$ **short cylinder**
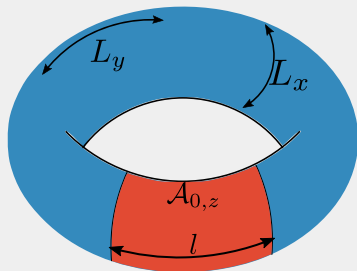
In general:

$$\Delta n \sim \Delta k_x \sim 1/L_x \quad \longrightarrow$$

$z > 0$: decreasing system size

$z < 0$: increasing system size

Modes are decoupled $\longrightarrow$ entanglement is additive

$$S_{A_z(j)} = \sum_{n \in A_z(j)} S_n = f_z(j) c \alpha L_x - c \log \left| 2 \sin \left(\pi f_z(j) \phi \right) \right|$$

$$i < j, \quad S_{i \cup j} = \begin{cases} S_i, & z > 0 \\ S_j, & z < 0 \end{cases}$$

Calabrese et al. 2004b; Casini et al. 2005; Arias et al. 2015; Chen et al. 2017; Murciano et al. 2020.

$$i < j, \ S_{i \cup j} = \begin{cases} S_i, & z > 0 \\ S_j, & z < 0 \end{cases}$$

- presents a **hierarchy** of entanglement $\longrightarrow$ EE distributed across RG steps:

  RG transformation $\longrightarrow$ reveals entanglement

- distribution of entanglement also present in **multipartite** entanglement:

  mutual information and higher order measures, within one RG step or spread across the flow

Holographic nature of the RG flow

Conformal FT in d–dimensions $\longleftrightarrow$ Anti-de-Sitter space-time in $d + 1$-dimensions

RG scaling
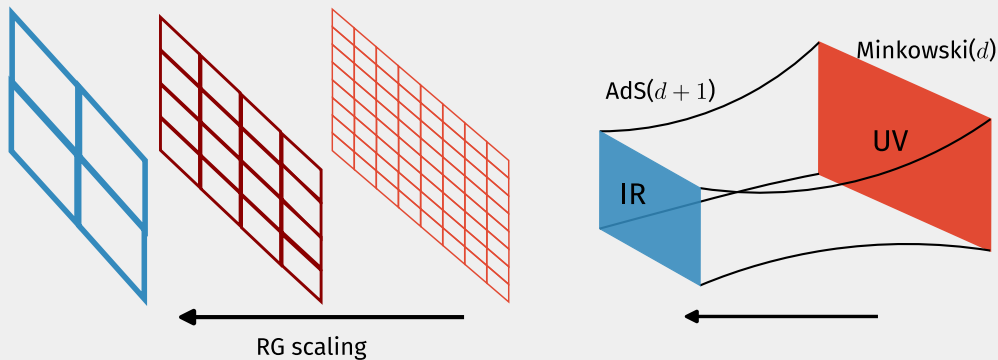
extra dimension in bulk corresponds to **RG flow**

J. McGreevy (McGreevy,2010)

Mutual information: $I^2(A : B) \equiv S(A) + S(B) - S(A \cup B)$ (non-negative)

information gained about B upon measuring A

define distance along the RG: $d_z(j) \equiv \log I^2_{\max} - \log I^2_z(0 : j) = \log t_z(j)$
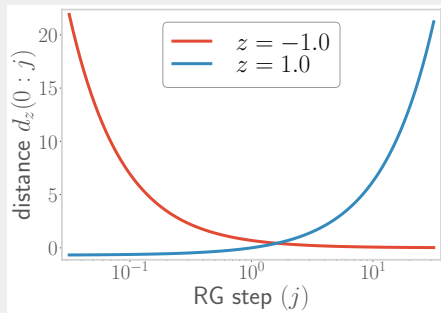
For $z > 0$:

- mut. info. is maximum for small j

- decreases for large j

- corresponds to **increasing distance**

Van Raamsdonk 2010; Lee et al. 2016; Mukherjee et al. 2022.

Define 2-dimensional x - y structure
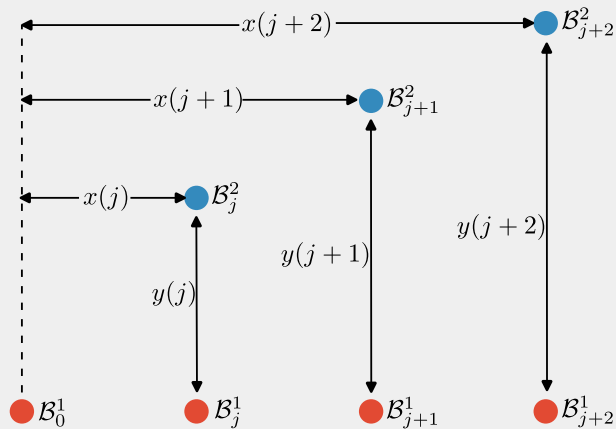
🔴 : RG steps

🔵 : subsystems within an RG step

$$x_z(j) = d_z(j) = \log t_z(j)$$

$$y_z(j) = \log l_{max}^2 - \log l_z^2(B_j^1 : B_j^2)$$

$$= \log t_z(j \pm 1)$$

Lee 2010; Lee 2014; Qi 2013; Lee et al. 2016; Mukherjee et al. 2020a; Mukherjee et al. 2020b; Ryu et al. 2006b; Ryu et al. 2006a; Nozaki et al. 2012.

Define coupling that measures spectral gap: $\quad g_z(j) = \log \frac{M_{n+1}(\phi) - M_n(\phi)}{2\pi/L_x} = \log t_z(j)$

RG beta function for its evolution: $\quad \beta_z(j) = \Delta \log g_z(j) = z \log\left(1 + j^{-1}\right)$

Lee 2010; Lee 2014; Qi 2013; Lee et al. 2016; Mukherjee et al. 2020a; Mukherjee et al. 2020b; Ryu et al. 2006b; Ryu et al. 2006a; Nozaki et al. 2012.
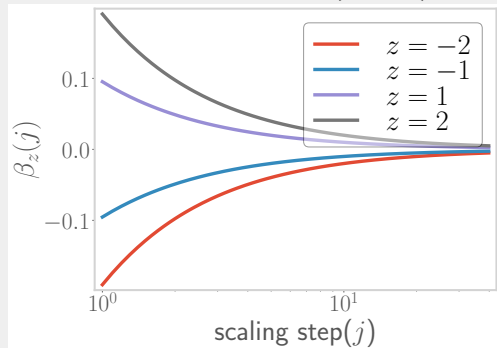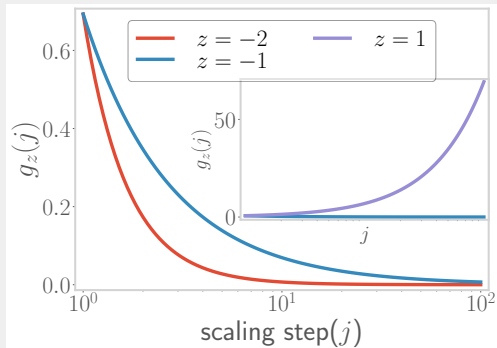
RG beta function can be related to the x, y–distances

$$x_z = \left(e^{\frac{\beta_z}{z}} - 1 \right)^{-z} \log 2$$

$$y_z = \begin{cases} x_z e^{\beta}, & z > 0 \\ x_z \left(2 - e^{\frac{\beta}{z}} \right)^z, & z < 0 \end{cases}$$

explicit relation between the RG flow and the emergent **geometry**

Lee 2010; Lee 2014; Qi 2013; Lee et al. 2016; Mukherjee et al. 2020a; Mukherjee et al. 2020b; Ryu et al. 2006b; Ryu et al. 2006a; Nozaki et al. 2012.

Define first and second derivatives in emergent space

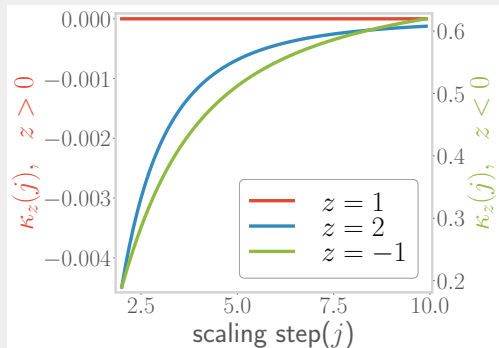$$v_z(j) \equiv \frac{\Delta y_z(j)}{\Delta x_z(j)} = \begin{cases} \frac{(j+2)^z - (j+1)^z}{(j+1)^z - j^z}, & z > 0 \\ \frac{(j)^z - (j-1)^z}{(j+1)^z - j^z}, & z < 0 \end{cases}$$

$$v_z'(j) \equiv \frac{v_z(j+1) - v_z(j)}{x_z(j+1) - x_z(j)}$$

Define curvature using them: $\kappa_z(j) = \dfrac{v_z'(j)}{\left[1 + v_z(j)^2\right]^{\frac{3}{2}}}$

$\longrightarrow$ can be expressed in terms of $\beta_z(j)$

- positive curvature for $z < 0$

- zero curvature for $z = 1$

- negative curvature for $z > 1$

- **asymptotically flat** for large j, at all z

Is there a name for such spaces?

$$\gamma_z(j) \equiv 1 - v_z(j+1)/v_z(j), \quad \alpha_z(j) = y_z(j+1) - y_z(j)$$

$$\kappa_z(j) = -\frac{\alpha_z(j)\,\gamma_z(j)}{\left(\Delta x_z(j)\right)^2 \left[1 + v_z(j)^2\right]^{\frac{3}{2}}} \implies \text{sign}\left[\kappa_z(j)\right] = -\text{sign}\left[\alpha_z(j)\right]\text{sign}\left[\gamma_z(j)\right]$$

$$\text{sign}\left[\kappa_z\right] = \begin{cases} -1, & z \geq 1 \\ 1, & z \leq -1 \end{cases} = \begin{cases} -\text{sign}\left[\gamma_z(j)\right], & z \geq 1 \\ -\text{sign}\left[\alpha_z(j)\right], & z \leq -1 \end{cases}$$
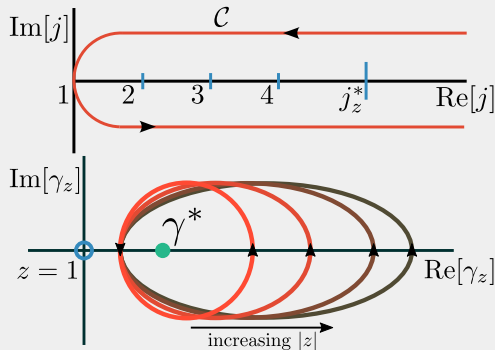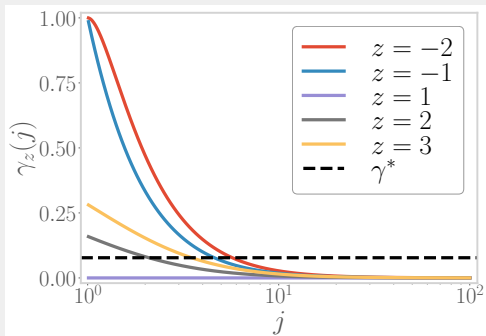
- $\ln(\gamma - \gamma^*)$ has branch point at γ^*, can be avoided for $z = 1$, **contour is trivial**

- cannot be avoided for $z \neq 1$ $\longrightarrow$ presence of **singularity** $\longrightarrow$ encoded through **winding number**
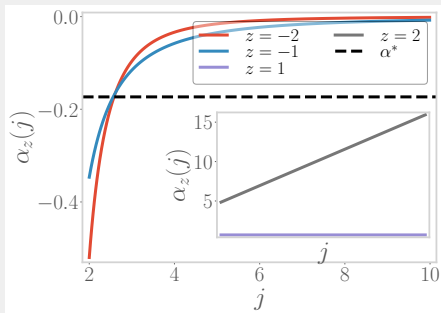
very similar thing holds for α_z

- singularity exists only for $z < 0$

- otherwise contour can be trivialised
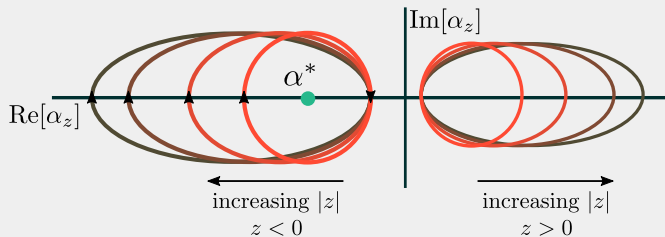
Curvature can be written as the product of **winding numbers**:

$$\text{sign}\left[\kappa_z\right] = W_z\left(\gamma^*\right) \times \left[2W_z'\left(\alpha^*\right) - 1\right]$$

- winding numbers count singularities
- robust against deformations

What does this change in topology really mean?

- z is the **anomalous dimension** of the spectral gap g_z in the effective field theory

- sign of z reflects the RG relevance/irrelevance of g_z in the microscopic fermionic theory

- change in z can be interpreted as a change in the underlying **interacting theory**

- change in sign of z is hence a **phase transition** in the microscopic theory that changes the topology of the Fermi surface

- Define an expansion parameter

$$\theta_z(j) = \frac{1}{\sqrt{1 + v_z^{-2}}}$$

- can be related to RG flow through β_z

- related to change in area of flows of g_z

$$\theta_z \sim \frac{1}{\sqrt{\Delta x^2 + \Delta y^2}} \Delta g_z(j + 1)$$

Kar 2001; Kar et al. 2007; Balasubramanian et al. 1999.

- Expansion parameter satisfies "Raychaudhuri-like" equation

$$\frac{d\theta_z}{dx_z} = \kappa$$

- No attractive θ^2 term: fixed points reached only at $j \to \infty$

Kar 2001; Kar et al. 2007; Balasubramanian et al. 1999.
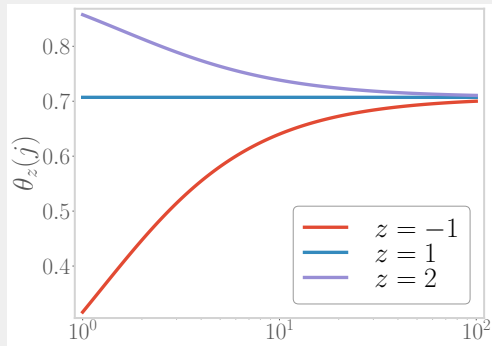
Conclusions

- hierarchy of entanglement, across scales as well as number of parties

$$S_{A \cup B} = S_{\text{larger}}$$

- hierarchy of entanglement, across scales as well as number of parties

- RG beta function gives rise to emergent distances

$$x_z(\beta), y_z(\beta)$$

- hierarchy of entanglement, across scales as well as number of parties

- RG beta function gives rise to emergent distances

- anomalous dimension z determines sign of curvature

$$\kappa \begin{cases} > 0 \text{ if } z < 0 \\ = 0 \text{ if } z = 1 \\ < 0 \text{ if } z > 1 \end{cases}$$

- hierarchy of entanglement, across scales as well as number of parties

- RG beta function gives rise to emergent distances

- anomalous dimension *z* determines sign of curvature

- sign of curvature is topological

$$\text{sign}\left[\kappa_z\right] = W_z\left(\gamma^*\right) \times \left[2W_z'\left(\alpha^*\right) - 1\right]$$

- hierarchy of entanglement, across scales as well as number of parties

- RG beta function gives rise to emergent distances

- anomalous dimension z determines sign of curvature

- sign of curvature is topological

- θ satisfies "Raychaudhuri-like" equation

$$\frac{d\theta_z}{dx_z} = \kappa$$

► Arias, Raúl E, David D Blanco, and Horacio Casini (2015). "Entanglement entropy as a witness of the Aharonov–Bohm effect in QFT". In: *Journal of Physics A: Mathematical and Theoretical* 48.14, p. 145401.

► Balasubramanian, Vijay and Per Kraus (1999). "Spacetime and the holographic renormalization group". In: *Physical Review Letters* 83.18, p. 3605.

► Calabrese, Pasquale and John Cardy (2004a). "Entanglement entropy and quantum field theory". In: *Journal of Statistical Mechanics: Theory and Experiment* 2004.06, P06002.

► — (2004b). "Entanglement entropy and quantum field theory". In: *Journal of Statistical Mechanics: Theory and Experiment* 2004.06, P06002.

► Casini, H, C D Fosco, and M Huerta (2005). "Entanglement and alpha entropies for a massive Dirac field in two dimensions". In: *Journal of Statistical Mechanics: Theory and Experiment* 2005.07, P07007–P07007.

► Casini, H and M Huerta (2009). "Entanglement entropy in free quantum field theory". In: *Journal of Physics A: Mathematical and Theoretical* 42.50, p. 504007.

► Chen, Xiao et al. (2017). "Two-cylinder entanglement entropy under a twist". In: *Journal of Statistical Mechanics: Theory and Experiment* 2017.4, p. 043104.

► Chung, Ming-Chiang and Ingo Peschel (2000). "Density-matrix spectra for two-dimensional quantum systems". In: *Phys. Rev. B* 62 (7), pp. 4191–4193.

► Gioev, Dimitri and Israel Klich (2006). "Entanglement entropy of fermions in any dimension and the Widom conjecture". In: *Physical review letters* 96.10, p. 100503.

► Kar, Sayan (2001). "Geometry of renormalization group flows in theory space". In: *Phys. Rev. D* 64 (10), p. 105017.

- Kar, Sayan and Soumitra Sengupta (2007). "The Raychaudhuri equations: A brief review". In: *Pramana* 69.1, pp. 49–76.
- Lee, Ching Hua and Xiao-Liang Qi (2016). "Exact holographic mapping in free fermion systems". In: *Physical Review B* 93.3, p. 035112.
- Lee, Sung-Sik (2010). "Holographic description of quantum field theory". In: *Nuclear Physics B* 832.3, pp. 567–585.
- — (2014). "Quantum renormalization group and holography". In: *Journal of High Energy Physics* 2014.1, p. 76.
- Li, Weifei et al. (2006). "Scaling behavior of entanglement in two- and three-dimensional free-fermion systems". In: *Phys. Rev. B* 74 (7), p. 073103.
- Mukherjee, Anirban and Siddhartha Lal (2020a). "Unitary renormalisation group for correlated electrons-I: a tensor network approach". In: *Nuclear Physics B* 960, p. 115170.
- — (2020b). "Unitary renormalisation group for correlated electrons-II: insights on fermionic criticality". In: *Nuclear Physics B* 960, p. 115163.
- — (2022). "Superconductivity from repulsion in the doped 2D electronic Hubbard model: an entanglement perspective". In: *Journal of Physics: Condensed Matter* 34.27, p. 275601.
- Murciano, Sara, Paola Ruggiero, and Pasquale Calabrese (2020). "Symmetry resolved entanglement in two-dimensional systems via dimensional reduction". In: *Journal of Statistical Mechanics: Theory and Experiment* 2020.8, p. 083102.
- Nozaki, Masahiro, Shinsei Ryu, and Tadashi Takayanagi (2012). "Holographic geometry of entanglement renormalization in quantum field theories". In: *Journal of High Energy Physics* 2012.10, p. 193.

▶ Qi, Xiao-Liang (2013). "Exact holographic mapping and emergent space-time geometry". In: *arXiv preprint arXiv:1309.6282*.

▶ Ryu, Shinsei and Tadashi Takayanagi (2006a). "Aspects of holographic entanglement entropy". In: *Journal of High Energy Physics* 2006.08, p. 045.

▶ — (2006b). "Holographic derivation of entanglement entropy from the anti–de sitter space/conformal field theory correspondence". In: *Physical review letters* 96.18, p. 181602.

▶ Van Raamsdonk, Mark (2010). "Building up spacetime with quantum entanglement". In: *General Relativity and Gravitation* 42.10, pp. 2323–2329.

▶ Wolf, Michael M. (2006). "Violation of the Entropic Area Law for Fermions". In: *Phys. Rev. Lett.* 96 (1), p. 010404.

- Transformation to a different space

$$\tilde{\theta} = \frac{1}{1 - \sqrt{2}\theta}, \qquad \frac{d\tilde{\theta}}{dx_z} = \sqrt{2}\tilde{\theta}^2 \kappa$$

- Does generate θ^2 term

- Effective curvature is zero