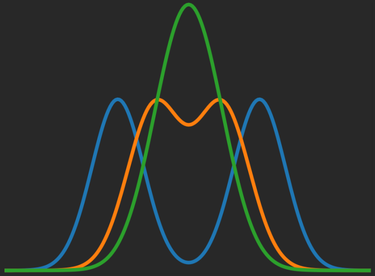
Unitary Renormalization Group Solution of the Single-Impurity Anderson model



Abhirup Mukherjee (18IP014)

Supervisor: Dr. Siddhartha Lal

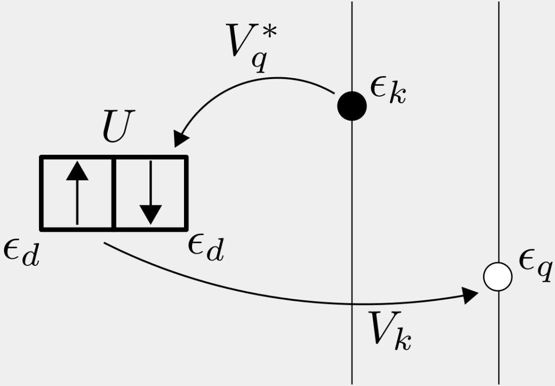
Department of Physical Sciences
IISER Kolkata

July 12, 2021

The Single-Impurity Anderson Model

The Single-Impurity Anderson Model

$$\mathcal{H} = \overbrace{\sum_{k\sigma} \epsilon_k \hat{n}_{k\sigma}}^{\text{conduction bath}} + \overbrace{\sum_{k\sigma} \left[V(k) c^\dagger_{k\sigma} c_{d\sigma} + \text{h.c.} \right]}^{\text{hybridisation}} + \overbrace{\epsilon_d \sum_{\sigma} \hat{n}_{d\sigma}}^{\text{impurity site energy}} + \overbrace{U \hat{n}_{d\uparrow} \hat{n}_{d\downarrow}}^{\text{d-d repulsion}}$$



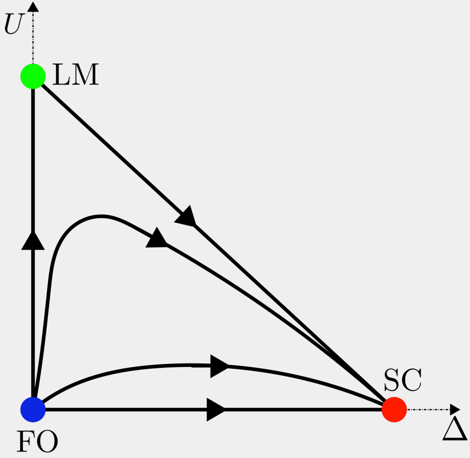
$$\rho(\epsilon) \approx \rho(\epsilon_F)$$

$$\Delta = \rho V^2$$

$$\epsilon_d = -\frac{1}{2} U \text{ (p-h symmetry)}$$

NRG Results - Symmetric Model

- the **free-orbital** fixed point ($U = \Delta = 0$) - unstable
- the **local moment** fixed point ($U = \infty, \Delta = 0$) - saddle point, and
- the **strong-coupling** fixed point ($\Delta = \infty, U =$ finite) - stable.



Krishna-murthy, Wilson, and Wilkins 1975.

Some Outstanding Questions

Some Outstanding Questions

- Is it possible to get **non-perturbative scaling equations** for the whole journey?

Some Outstanding Questions

- Is it possible to get **non-perturbative scaling equations** for the whole journey?
- What is the nature of the strong-coupling fixed point for a **finite system** where $V \neq \infty$?

Some Outstanding Questions

- Is it possible to get **non-perturbative scaling equations** for the whole journey?
- What is the nature of the strong-coupling fixed point for a **finite system** where $V \neq \infty$?
- Is it possible to show the **transfer of spectral weight** along the flow, possibly by tracking the spectral function?

Some Outstanding Questions

- Is it possible to get **non-perturbative scaling equations** for the whole journey?
- What is the nature of the strong-coupling fixed point for a **finite system** where $V \neq \infty$?
- Is it possible to show the **transfer of spectral weight** along the flow, possibly by tracking the spectral function?
- How does the renormalization affect the **many-particle entanglement** between the electrons?

Some Outstanding Questions

- Is it possible to get **non-perturbative scaling equations** for the whole journey?
- What is the nature of the strong-coupling fixed point for a **finite system** where $V \neq \infty$?
- Is it possible to show the **transfer of spectral weight** along the flow, possibly by tracking the spectral function?
- How does the renormalization affect the **many-particle entanglement** between the electrons?
- Are there any interesting **topological aspects** of the fixed points?

The Unitary Renormalization Group

Unitary Renormalization Group: Overview

The Short Version

Apply *unitary many-body transformations* to the Hamiltonian so as to successively *decouple* high energy states and hence obtain scaling equations.

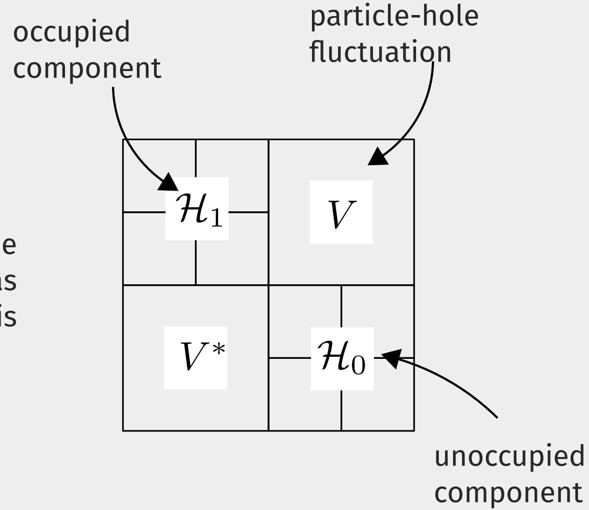
$$
\begin{gathered}
\mathcal{H}(D) \\
\Big\Downarrow \\
U\mathcal{H}(D)U^\dagger \\
\Big\Downarrow \\
H(D) + \mathcal{H}'(D - \delta D)
\end{gathered}
$$

Mukherjee and Lal 2020.

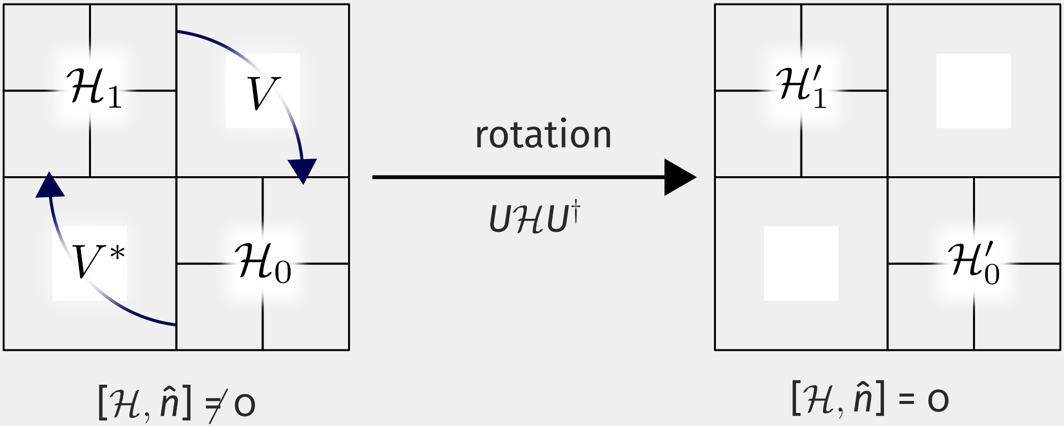
URG: Formalism

Step 1:

Start with the electrons farthest from the Fermi surface. Write the Hamiltonian as *diagonal and off-diagonal terms* in this basis.

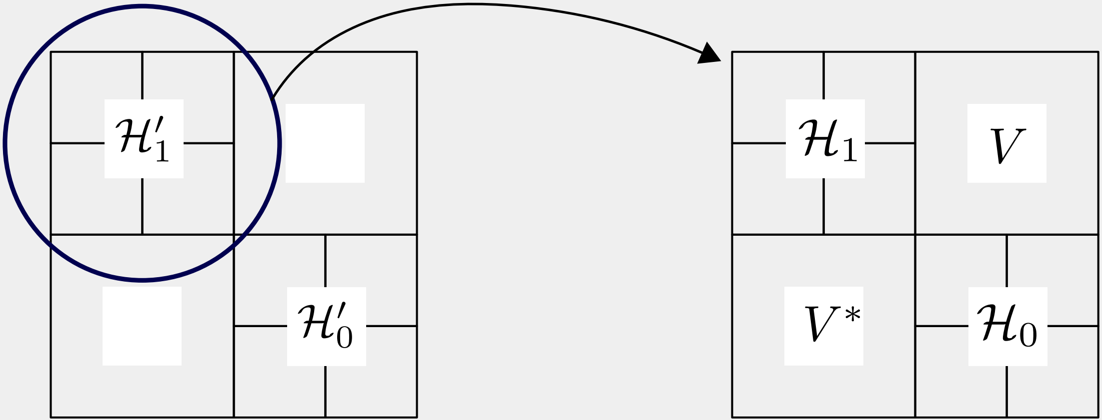


Step 2:
Rotate the Hamiltonian to kill the off-diagonal blocks.



Step 3:

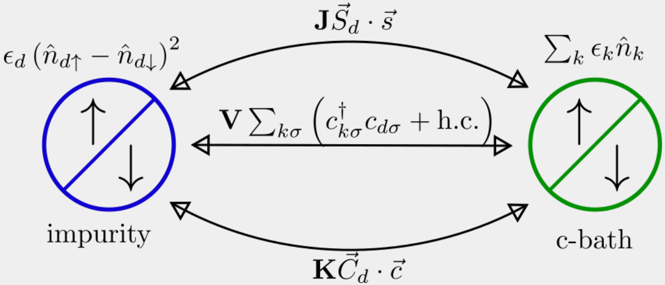
Repeat the process with the new blocks.



Generalized SIAM

Model: Generalized SIAM

$$H = H_{\text{SIAM}} + J\vec{S}_d \cdot \vec{s} + K\vec{C}_d \cdot \vec{c}$$



$$\vec{S}_d \equiv \frac{1}{2} \sum_{\alpha\beta} c^\dagger_{d\alpha} \vec{\sigma}_{\alpha\beta} c_{d\beta}$$

$$\vec{s} \equiv \frac{1}{2} \sum_{\alpha\beta} c^\dagger_{0\alpha} \vec{\sigma}_{\alpha\beta} c_{0\beta}$$

$$\vec{C}_d \equiv \frac{1}{2} \sum_{\alpha\beta} \psi^\dagger_{d\alpha} \vec{\sigma}_{\alpha\beta} \psi_{d\beta}$$

$$\vec{c} \equiv \frac{1}{2} \sum_{\alpha\beta} \psi^\dagger_{0\alpha} \vec{\sigma}_{\alpha\beta} \psi_{0\beta}$$

$$\vec{\psi}_d \equiv \begin{pmatrix} c_{d\uparrow} \\ c^\dagger_{d\downarrow} \end{pmatrix} \qquad \vec{\psi}_0 \equiv \sum_k \begin{pmatrix} c_{k\uparrow} \\ c^\dagger_{k\downarrow} \end{pmatrix}$$

Schrieffer and Wolff 1966.

RG Equations, Their Features and Fixed Points

RG Equations

$$\Delta U = 4|V|^2\left[\frac{1}{\omega - \frac{1}{2}D + \frac{U}{2} + \frac{1}{2}J} - \frac{1}{\omega - \frac{1}{2}D - \frac{U}{2} + \frac{1}{2}K}\right] + \sum_{k<\Lambda_j}\frac{3}{4}\frac{K^2 - J^2}{\omega - \frac{1}{2}D + \frac{1}{4}J + \frac{1}{4}K}$$

$$\Delta V = \frac{VK}{16}\left(\frac{1}{\omega - \frac{1}{2}D - \frac{U}{2} + \frac{1}{2}K} + \frac{1}{\omega - \frac{1}{2}D + \frac{1}{4}J + \frac{1}{4}K}\right) - \frac{3VJ}{4}\left(\frac{1}{\omega - \frac{1}{2}D + \frac{U}{2} + \frac{1}{2}J} + \frac{1}{\omega - \frac{1}{2}D + \frac{1}{4}J + \frac{1}{4}K}\right)$$

$$\Delta J = -J^2\left(\omega - \frac{1}{2}D + \frac{1}{4}J + \frac{1}{4}K\right)^{-1}$$

$$\Delta K = -K^2\left(\omega - \frac{1}{2}D + \frac{1}{4}J + \frac{1}{4}K\right)^{-1}$$

Passage to Poor Man's Scaling Results

Symmetric SIAM

- $J = 0, K = 0$
- $\omega = -\frac{D}{2}$
- $U = -\frac{\epsilon_d}{2} \ll D$

$$\longrightarrow \quad \delta U = \delta V = 0$$

Haldane 1978; Jefferson 1977.

Passage to Poor Man's Scaling Results

Asymmetric SIAM

- $J = 0, K = 0$
- $\omega = -\frac{D}{2}$
- $U \gg D \gg \epsilon_d$

$$\longrightarrow$$

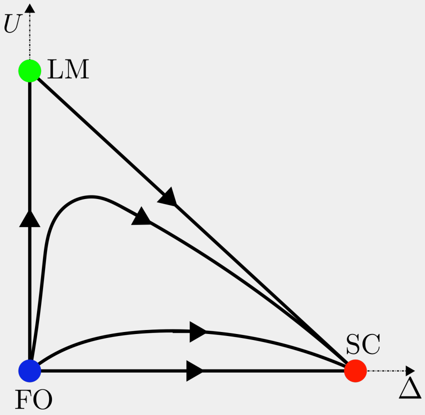
$$\delta U = \delta V = 0$$

$$\delta \epsilon_d = \frac{\Delta}{\pi} \delta \ln D$$

Haldane 1978; Jefferson 1977.

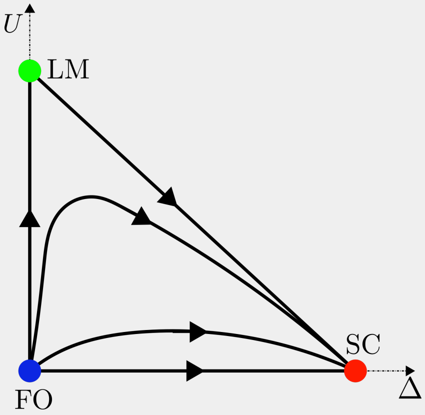
Fixed Points

- $J = K = 0 \longrightarrow \Delta V = 0$
- $J, K, V = 0^{+} \longrightarrow (V^{*}, J^{*}, K^{*}) = \text{large}, U^{*} = 0$
  - **strong-coupling fixed point**

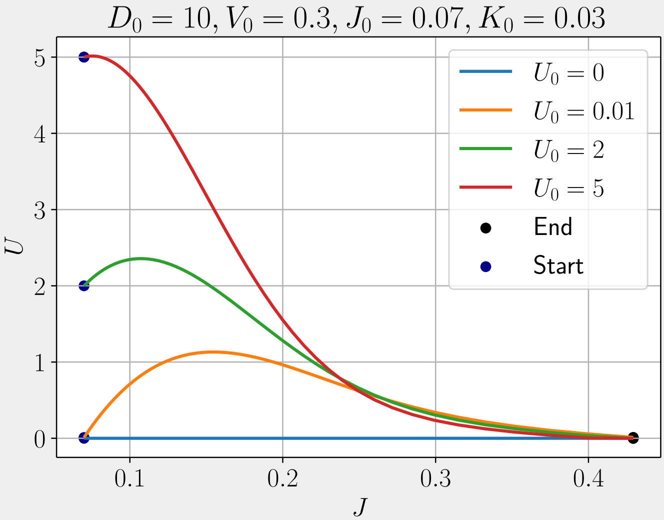


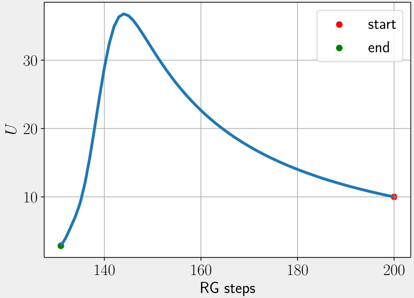
Fixed Points

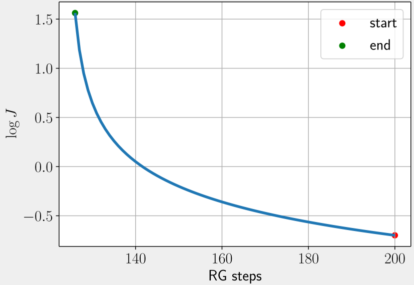
- $J = K = V = 0 \longrightarrow$ all couplings marginal
  - line of fixed points on y-axis
- $U = 0^{+} \longrightarrow$ **local moment fixed point**
  - ground-state is a decoupled impurity spin



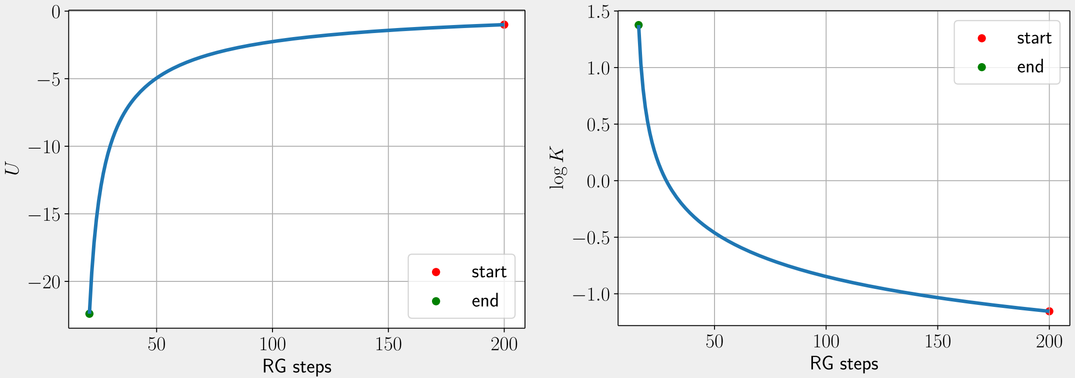
Results: $U > 0, J > K$





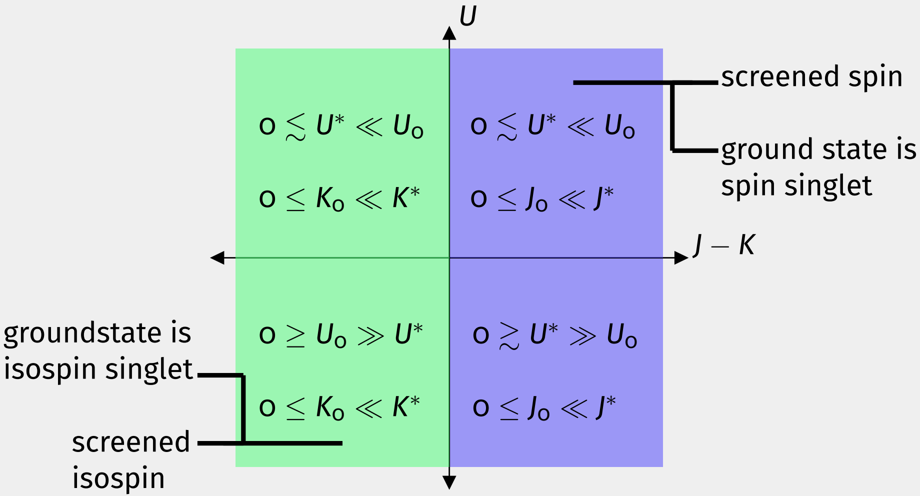


Results: $U < 0, J < K$



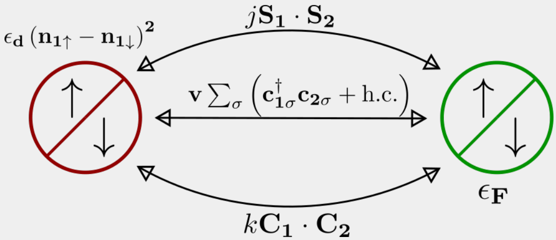
Low energy effective theory and ground state wavefunctions

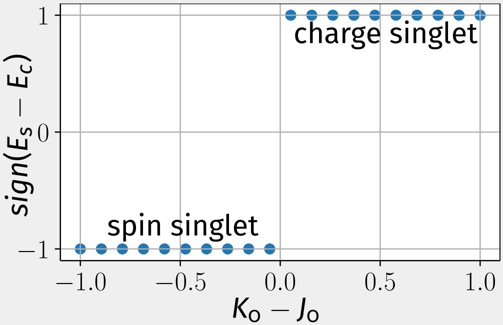
Results: Phase Diagram



Results: Effective Zero-mode Hamiltonian

$$H_{IR} = \epsilon_d^* \left(\hat{n}_{1\uparrow} - \hat{n}_{1\downarrow}\right)^2 + V^* \sqrt{N^*} \sum_\sigma \left(c_{1\sigma}^\dagger c_{2\sigma} + \text{h.c.}\right) + J^* N^* \vec{S}_1 \cdot \vec{S}_2 + K^* N^* \vec{C}_1 \cdot \vec{C}_2$$



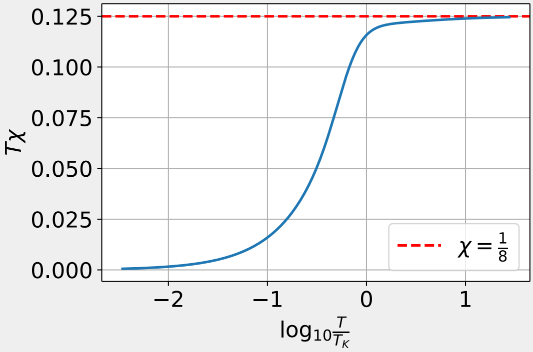


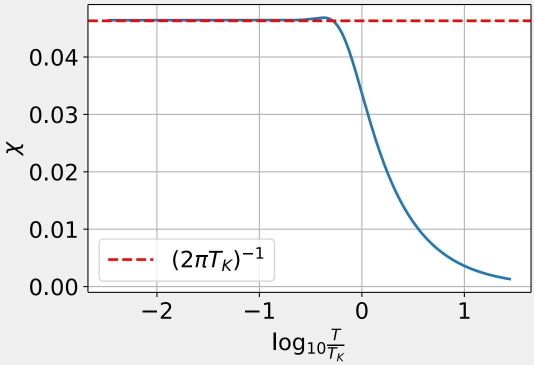
Wilson 1975; Krishna-murthy, Wilson, and Wilkins 1975; Taraphder and Coleman 1991.

Impurity Susceptibilities and Impurity spectral function

Results: Spin Susceptibility

$$\chi_s = \lim_{B \to 0} \frac{\partial m}{\partial B}$$



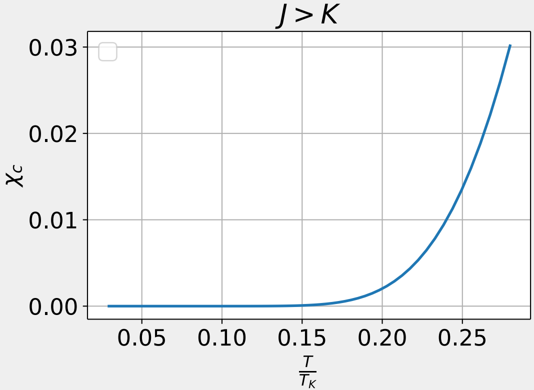


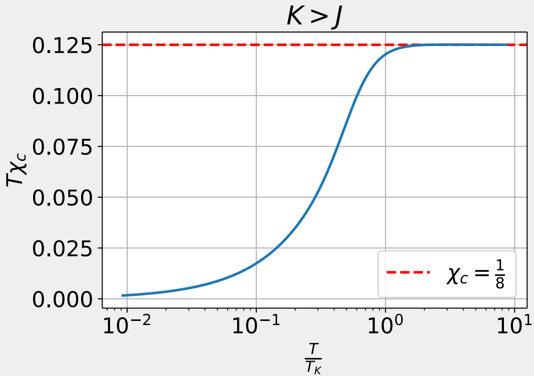
$\chi(T \to 0) = (2j)^{-1}$ $\qquad T_K \equiv \frac{2N^*}{\pi}(D^* - 2\omega)$ $\qquad (\chi \times T)(T \to \infty) = \frac{1}{8}$

Wilson 1975; Krishna-murthy, Wilson, and Wilkins 1975.

Results: Charge Susceptibility

$$\chi_c = \lim_{\mu \to 0} \frac{\partial N}{\partial \mu}$$



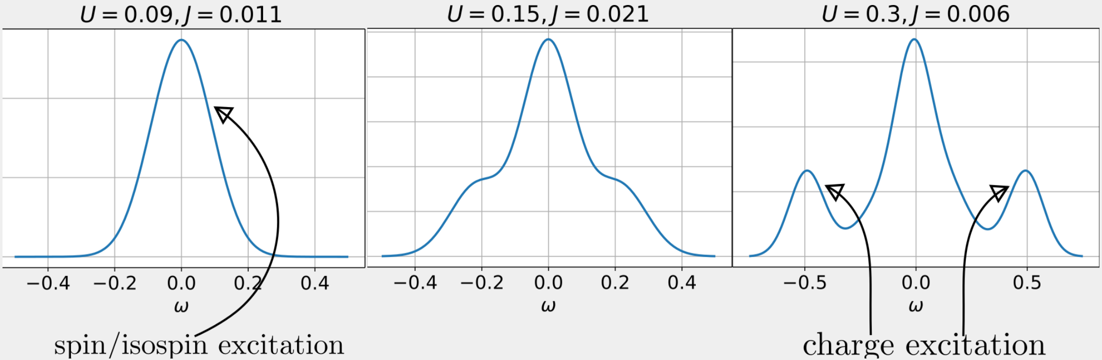


$$\chi_c(T \to 0)\Big|_{K>J} = \frac{1}{2k} \qquad (\chi_c \times T)(T \to 0)\Big|_{J>K} = 0 \qquad (\chi_c \times T)(T \to \infty) = \frac{1}{8}$$

Taraphder and Coleman 1991; Zitko and Bonca 2006.

Results: Impurity Spectral Function

$$\mathcal{A}(\omega) = -\frac{1}{\pi}\mathrm{Im}\,\left[G^{\sigma}_{dd}(\omega)\right] \qquad G^{\sigma}_{dd}(t) = -i\theta(t)\left\langle\left\{c_{d\sigma}(t), c^{\dagger}_{d\sigma}\right\}\right\rangle$$

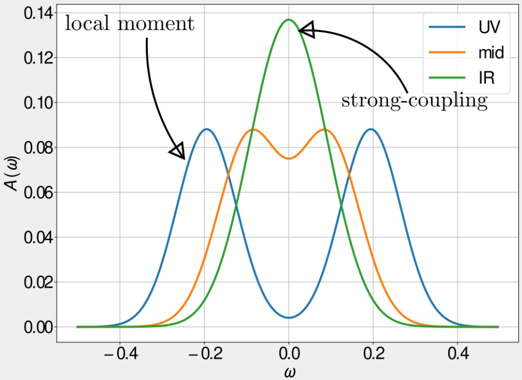


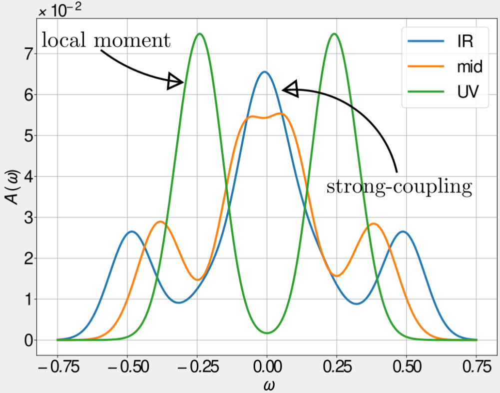
Hewson 1993; Bulla, Costi, and Pruschke 2008.

Results: Spectral Function Renormalization

$$\mathcal{A}(\omega) = -\frac{1}{\pi}\mathrm{Im}\ \left[G^{\sigma}_{dd}(\omega)\right]$$

$$G^{\sigma}_{dd}(t) = -i\theta(t)\left\langle\left\{c_{d\sigma}(t), c^{\dagger}_{d\sigma}\right\}\right\rangle$$



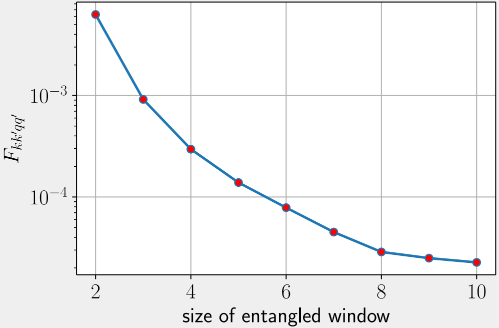


Entanglement measures and Topological Features of low energy theory

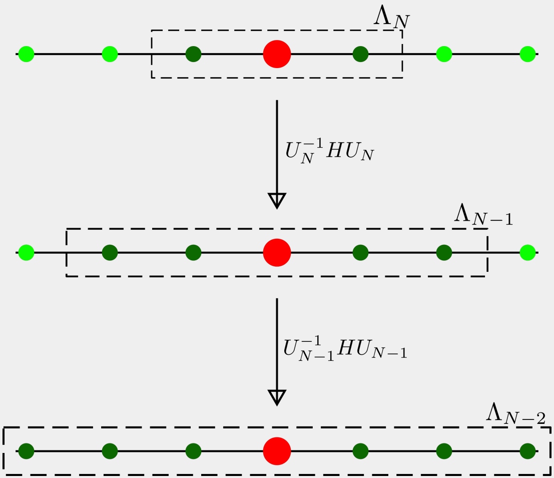
Results: Kondo Cloud Hamiltonian

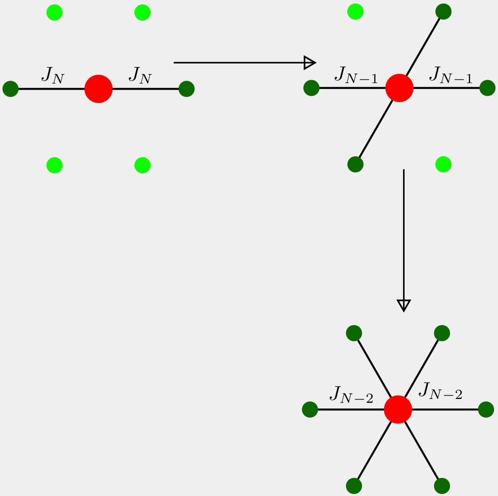
$$H^*(\mathrm{d, cloud}) \xrightarrow{\text{solve for bath Hamiltonian}} H^*_{\mathrm{cloud}}$$

$$H^*_{\mathrm{cloud}} = \overbrace{H^*_0}^{\text{kinetic energy}} + \overbrace{\sum_{kk'\sigma\sigma'} f_{kk'}\hat{n}_{k\sigma}\hat{n}_{k'\sigma'}}^{\text{Fermi liquid-type interaction}} + \overbrace{\sum_{kk'qq'} F_{kk'qq'} c^\dagger_{k\uparrow} c^\dagger_{k'\downarrow} c_{q\uparrow} c_{q'\downarrow}}^{\text{non-Fermi liquid-type interaction}}$$



Results: Reverse RG: Overview



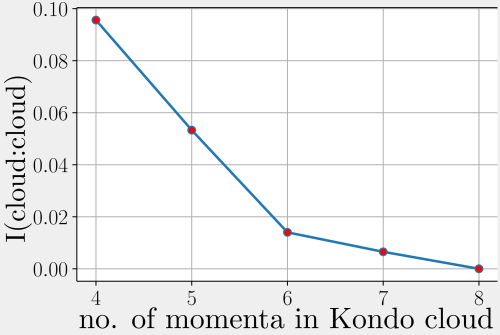


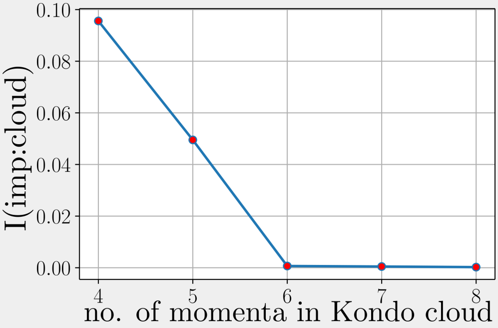
Mukherjee 2020.

Results: Reverse RG: Mutual Information

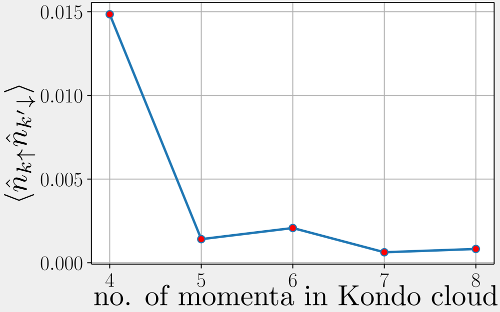
$$I(A:B) = S_A + S_B - S_{AB}$$

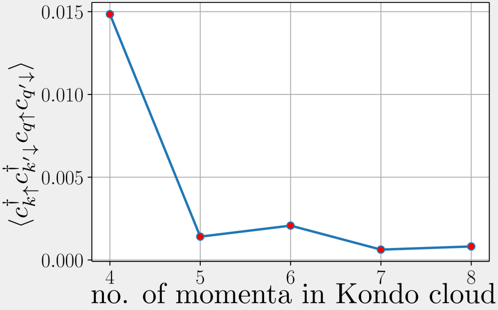
$$S_A = -\mathrm{Tr}\,\left[\rho_A \ln \rho_A\right]$$





Results: Reverse RG: Correlations



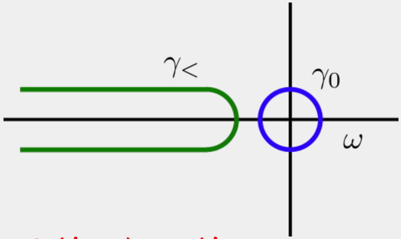


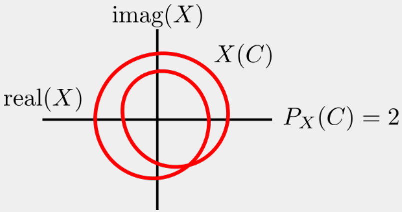
Results: Luttinger's Theorem

$$\overbrace{N}^{\text{total no. of particles}} = \overbrace{P_{\mathrm{Det}\,G_d}(\Gamma_<) + \frac{1}{2}P_{\mathrm{Det}\,G_d}(\Gamma_O)}^{\text{no. of poles of imp. Greens func.}} + \overbrace{V_L}^{\text{no. of poles of cbath Greens func}}$$

$$P_X(C) \equiv \frac{1}{2\pi i}\oint_C dz \frac{\partial \ln X}{\partial z} = \text{no. of poles of } X \text{ enclosed by curve } C$$

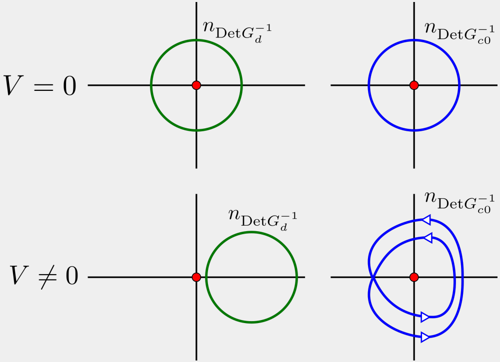
$$= \frac{1}{2\pi i}\oint_{X(C)} \frac{dX}{X} = \text{winding number of } X(C) \text{ around the origin}$$





Seki and Yunoki 2017.

Results: Luttinger's Theorem



$n_{\text{Det } G_d^{-1}} = 1$

$n_{\text{Det } G_d^{-1}} = 0$

$$V_L = V_L^0 + 1$$

Martin 1982.

Results: Local Fermi Liquid

$$H^* = \overbrace{J^* \vec{S_d} \cdot \vec{s} + K^* \vec{C_d} \cdot \vec{c} + V^* \left(c^\dagger_{d\sigma} c_{0\sigma} + \text{h.c.} \right)}^{\text{solve exactly}} + \overbrace{t \sum_{\langle i,j \rangle} c^\dagger_{i\sigma} c_{j\sigma}}^{\text{treat as perturbation}}$$

$$\Big\downarrow \; 4^{\text{th}} \text{ fourth order pert.}$$

$$E_1^{(4)} = -\frac{16t^4}{3J^{*3}}, E_2^{(4)} = -\frac{16t^4}{9J^{*3}}$$

$$H^* \sim J^* \vec{S_d} \cdot \vec{s} + K^* \vec{C_d} \cdot \vec{c} + V^* \left(c^\dagger_{d\sigma} c_{0\sigma} + \text{h.c.} \right) + \overbrace{\frac{t^4}{J^{*3}} \hat{n}_{1\uparrow} \hat{n}_{1\downarrow}}^{\text{local Fermi liquid}}$$

Nozières 1974.

Results: Wilson Ratio ($T = 0$)

thermal average: $\langle \hat{n}_{1\uparrow}\hat{n}_{1\downarrow} \rangle \xrightarrow[]{\text{mean field approximation}} \langle \hat{n}_{1\uparrow} \rangle \langle \hat{n}_{1\downarrow} \rangle$

$$\epsilon_{k\sigma} = \epsilon_k^0 + \sum_q f_{kq} \left\langle n_{q\bar{\sigma}} \right\rangle$$

- $f_{\uparrow\uparrow} = 0$
- $\chi_c(T \to 0) = 0$

$\longrightarrow$

- $C_v(T \to 0) = \rho_{\text{imp}} T$
- $\chi_s(T \to 0) = 2\rho_{\text{imp}}$

$$R = \frac{\chi_s}{\frac{C_v}{T}} = 2$$

Hewson 1994.

Results: Relation between R and ΔV_L

- particle-hole symmetry
- strong-coupling fixed-point
- $T = 0$

$\longrightarrow$

$$\frac{\chi_s}{C_v/T} = 1 + U\rho_{\text{imp}}(0)$$

$$\rho_{\text{imp}}(0) = (\pi\Delta)^{-1}\sin^2\delta(0)$$

$$R = 1 + \sin^2\delta(0)$$

Coleman 2015; Hewson 1993; Phillips 2003.

Results: Relation between R and ΔV_L

- particle-hole symmetry
- strong-coupling fixed-point
- $T = 0$

$$\longrightarrow \quad \frac{\chi_s}{C_v/T} = 1 + U\rho_{\mathrm{imp}}(0)$$

$$\rho_{\mathrm{imp}}(0) = (\pi\Delta)^{-1}\sin^2\delta(0)$$

$$R = 1 + \sin^2\delta(0)$$

- Friedel's sum rule
- scattering theory results

$$\longrightarrow \quad \frac{2}{\pi}\delta(0) = \tilde{N} = \Delta V_L$$

Coleman 2015; Hewson 1993; Phillips 2003.

Results: Relation between R and ΔV_L

- particle-hole symmetry
- strong-coupling fixed-point
- $T = 0$

$$\longrightarrow \quad \frac{\chi_s}{C_v/T} = 1 + U\rho_{\text{imp}}(0)$$

$$\rho_{\text{imp}}(0) = (\pi\Delta)^{-1}\sin^2\delta(0)$$

$$R = 1 + \sin^2\delta(0)$$

- Friedel's sum rule
- scattering theory results

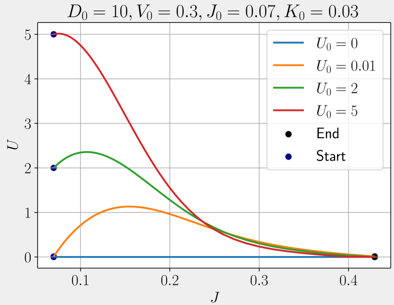
$$\longrightarrow \quad \frac{2}{\pi}\delta(0) = \tilde{N} = \Delta V_L$$

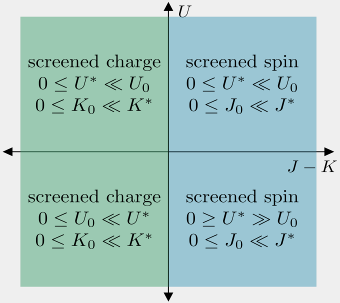
$$R = 1 + \sin^2\left(\frac{\pi}{2}\Delta V_L\right)$$

$$\Delta V_L = 1 \longrightarrow R = 2$$

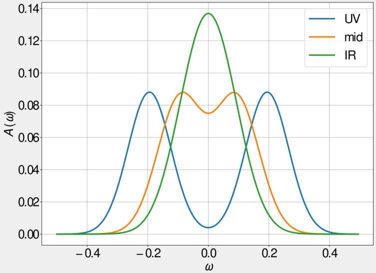
Coleman 2015; Hewson 1993; Phillips 2003.

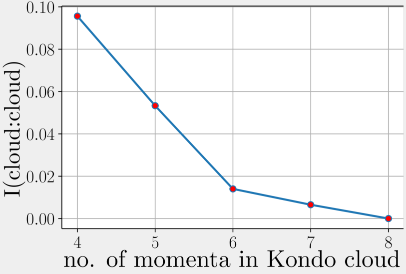
Summary of Results





$$H_{cloud} = H_0 + H_{FL} + H_{NFL}$$





$$R = 1 + \sin^2 \pi \Delta V_L = 2$$

$$\underset{\downarrow\ \text{SC}}{V_L} = \underset{\downarrow\ \text{LM}}{V_L^0} + 1$$

Future Directions

What's Next?

- Analytical expression for temperature-dependent Wilson ratio

What's Next?

- Analytical expression for temperature-dependent Wilson ratio
- Separating the contributions of various parts of the Kondo cloud to the spectral function

What's Next?

- Analytical expression for temperature-dependent Wilson ratio
- Separating the contributions of various parts of the Kondo cloud to the spectral function
- Suggested by the generalized double-bracket form of URG, we can try to see if URG can be used as an optimizer.

What's Next?

- Analytical expression for temperature-dependent Wilson ratio
- Separating the contributions of various parts of the Kondo cloud to the spectral function
- Suggested by the generalized double-bracket form of URG, we can try to see if URG can be used as an optimizer.
- Since the zero-mode low-energy theory is an Anderson molecule (which can be exactly solved), it would be interesting to see if there is a transformation which converts the Anderson molecule to the Hubbard molecule.

What's Next?

- Analytical expression for temperature-dependent Wilson ratio
- Separating the contributions of various parts of the Kondo cloud to the spectral function
- Suggested by the generalized double-bracket form of URG, we can try to see if URG can be used as an optimizer.
- Since the zero-mode low-energy theory is an Anderson molecule (which can be exactly solved), it would be interesting to see if there is a transformation which converts the Anderson molecule to the Hubbard molecule.
- We can also check how using more feature-full baths (with non-trivial self-energy) can change the phase diagram.

What's Next?

- Analytical expression for temperature-dependent Wilson ratio
- Separating the contributions of various parts of the Kondo cloud to the spectral function
- Suggested by the generalized double-bracket form of URG, we can try to see if URG can be used as an optimizer.
- Since the zero-mode low-energy theory is an Anderson molecule (which can be exactly solved), it would be interesting to see if there is a transformation which converts the Anderson molecule to the Hubbard molecule.
- We can also check how using more feature-full baths (with non-trivial self-energy) can change the phase diagram.
- extensions to the Kondo and Anderson lattices, and hence to the problem of heavy Fermions

Thanks for your attention!

Special thanks to Dr. Siddhartha Lal, Siddhartha Patra, Dr. Anirban Mukherjee and Mounica Mahankali for guidance and feedback. The support of IISER Kolkata through a junior research fellowship is acknowledged.

Anderson, P W (1970). “A poor man's derivation of scaling laws for the Kondo problem”. In: *Journal of Physics C: Solid State Physics* 3.12, pp. 2436–2441. DOI: 10.1088/0022-3719/3/12/008. URL: https://doi.org/10.1088/0022-3719/3/12/008.

Bulla, Ralf, Theo A. Costi, and Thomas Pruschke (2008). “Numerical renormalization group method for quantum impurity systems”. In: *Rev. Mod. Phys.* 80 (2), pp. 395–450. DOI: 10.1103/RevModPhys.80.395. URL: https://link.aps.org/doi/10.1103/RevModPhys.80.395.

Coleman, Piers (2015). *Introduction to Many-Body Physics*. Cambridge University Press.

Głazek, Stanisław D. and Kenneth G. Wilson (1993). “Renormalization of Hamiltonians”. In: *Phys. Rev. D* 48 (12), pp. 5863–5872. DOI: 10.1103/PhysRevD.48.5863. URL: https://link.aps.org/doi/10.1103/PhysRevD.48.5863.

Haldane, F. D. M. (1978). “Scaling Theory of the Asymmetric Anderson Model”. In: *Phys. Rev. Lett.* 40 (6), pp. 416–419. DOI: 10.1103/PhysRevLett.40.416. URL: https://link.aps.org/doi/10.1103/PhysRevLett.40.416.

Hewson, A. C. (1993). *The Kondo Problem to Heavy Fermions*. Cambridge University Press.

— (1994). “Renormalization group and Fermi liquid theory”. In: *Advances in Physics* 43.

Jefferson, J H (1977). “A renormalisation group approach to the mixed valence problem”. In: *Journal of Physics C: Solid State Physics* 10.18, pp. 3589–3599. DOI: 10.1088/0022-3719/10/18/023. URL: https://doi.org/10.1088/0022-3719/10/18/023.

Krishna-murthy, H. R., K. G. Wilson, and J. W. Wilkins (1975). “Temperature-Dependent Susceptibility of the Symmetric Anderson Model: Connection to the Kondo Model”. In: *Phys. Rev. Lett.* 35 (16), pp. 1101–1104. DOI: 10.1103/PhysRevLett.35.1101. URL: https://link.aps.org/doi/10.1103/PhysRevLett.35.1101.

Martin, Richard M (1982). “Fermi-surface sum rule and its consequences for periodic kondo and mixed-valence systems”. In: *Physical Review Letters* 48.5, p. 362.

Mukherjee, Anirban (2020). “Unitary renormalization group for correlated electrons”. PhD thesis. Indian Institute of Science Education and Research Kolkata.

Mukherjee, Anirban and Siddhartha Lal (2020). “Holographic unitary renormalization group for correlated electrons - I: A tensor network approach”. In: *Nuclear Physics B* 960. DOI: https://doi.org/10.1016/j.nuclphysb.2020.115170. URL: http://www.sciencedirect.com/science/article/pii/S055032132030256X.

Nozières, P. (1974). “A Fermi-Liquid Description of the Kondo Problem at Low Temperatures”. In: *Journal of Low Temperature Physics* 17.

Phillips, Philip (2003). *Advanced Solid State Physics*. Westview Press.

Schrieffer, J. R. and P. A. Wolff (1966). “Relation between the Anderson and Kondo Hamiltonians”. In: *Phys. Rev.* 149 (2), pp. 491–492. DOI: 10.1103/PhysRev.149.491. URL: https://link.aps.org/doi/10.1103/PhysRev.149.491.

Seki, Kazuhiro and Seiji Yunoki (2017). “Topological interpretation of the Luttinger theorem”. In: *Physical Review B* 96. DOI: 10.1103/physrevb.96.085124. URL: http://dx.doi.org/10.1103/PhysRevB.96.085124.

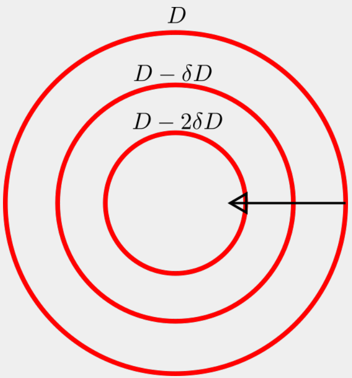
Taraphder, A. and P. Coleman (May 1991). “Heavy-fermion behavior in a negative-U Anderson model”. In: *Phys. Rev. Lett.* 66 (21), pp. 2814–2817. DOI: 10.1103/PhysRevLett.66.2814. URL: https://link.aps.org/doi/10.1103/PhysRevLett.66.2814.

Wegner, Franz (1994). “Flow-equations for Hamiltonians”. In: *Annalen der Physik* 506.2, pp. 77–91. DOI: https://doi.org/10.1002/andp.19945060203. eprint: https://onlinelibrary.wiley.com/doi/pdf/10.1002/andp.19945060203. URL: https://onlinelibrary.wiley.com/doi/abs/10.1002/andp.19945060203.

Wilson, K. G. (1975). “The Renormalization group: Critical phenomena and the Kondo Problem”. In: *Reviews of Modern Physics* 47.
Zitko, Rok and Janez Bonca (2006). “Spin-charge separation and simultaneous spin and charge Kondo effect”. In: *Phys. Rev. B* 74 (22), p. 224411. DOI: 10.1103/PhysRevB.74.224411. URL: https://link.aps.org/doi/10.1103/PhysRevB.74.224411.

URG: Relation to Poor Man's Scaling

$$H = H_0 + \underbrace{V_+ + V_-}_{\text{off-diagonal terms we want to remove}}$$



Philosophy of Poor Man's scaling:

- Successively eliminate high-energy energy shells
- Write high energy excitations as second-order correction to low-energy scatterings
- Typically perturbative

Anderson 1970.

URG: Relation to Poor Man's Scaling

$$H = H_0 + \underbrace{V_+ + V_-}_{\text{off-diagonal terms we want to remove}}$$

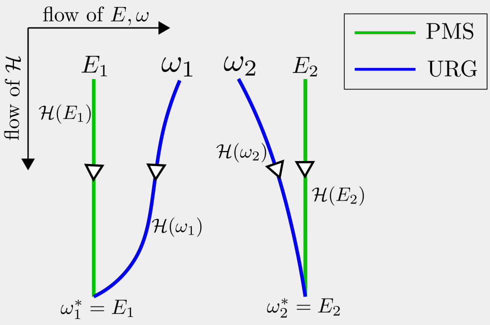
E = exact eigenvalue

ω = URG quantum fluctuation scale

$$\Delta H_{\text{PMS}} = V_- \frac{1}{E - H_0} V_+ + V_+ \frac{1}{E - H_0} V_-$$

$$\Big\downarrow E \to \omega$$

$$\Delta H_{\text{URG}} = V_- \frac{1}{\omega - H_0} V_+ + V_+ \frac{1}{\omega - H_0} V_-$$



URG: Relation to Continuous Unitary Transformation RG

$$H = \overbrace{H_d}^{\text{diagonal part}} + \overbrace{H_X}^{\text{off-diagonal part}}$$

$$\Delta H_{\text{CUT}} = \Delta l \left[\left[H_d(l), H_X(l) \right], H(l) \right]$$

$$V_{kq}(l) = V_{kq}(0) e^{(\epsilon_k - \epsilon_q) l}$$

- off-diagonal terms decay exponentially
- those that connect larger energy differences decay fastest

Głazek and Wilson 1993; Wegner 1994.

URG: Relation to Continuous Unitary Transformation RG

$$H = \overbrace{H_d}^{\text{diagonal part}} + \overbrace{H_X}^{\text{off-diagonal part}}$$

$$\Delta H_{\text{CUT}} = \Delta l \left[\left[H_d(l), H_X(l) \right], H(l) \right]$$

$$\Delta H_{\text{URG}} = \overbrace{\left[\left[H_d, \frac{1}{\omega_1 - \omega_0} \left(\hat{\omega} - H_d \right)^{-1} H_I \right], H \right]}^{\Delta H_0} - H^I$$

$$\Delta H_0 \xrightarrow{(\hat{\omega} - H_d)^{-1} \sim -H_d^{-1}} \Delta\lambda \times \left[\left[H_d, H_I \right], H \right]$$