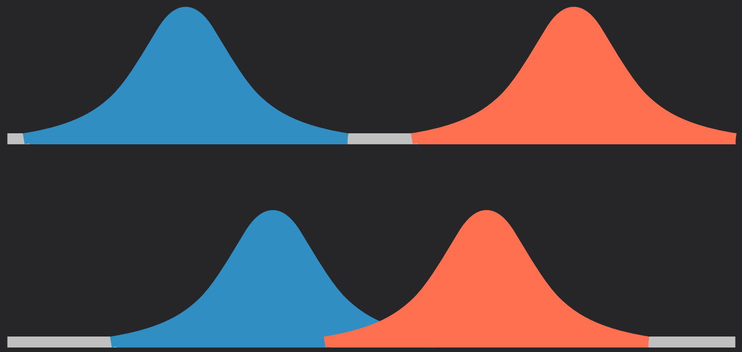
# EXACTLY SOLVABLE MODEL OF CORRELATED METAL-INSULATOR TRANSITION

## Insights on Non-Fermi Liquid and Mott Insulator



**ABHIRUP MUKHERJEE**

February 16, 2025

EPQM Seminar

# In A Nutshell

- An **exactly solvable** model that displays correlation-driven transition from a **non-Fermi liquid** to a **Mott insulator**.
- Analyse the non-Fermi liquid in this **controlled setting** to understand its features.
- Study the **superconducting instability** of this metal to figure out the nature of the superconductor arising from this exotic normal state.

# Where Does This Fit?

# Mott Insulators Are Different

Half-filled system is **metallic** in absence of interactions.


Dispersion away from band edge is **non-interacting**.

Mott (1968)

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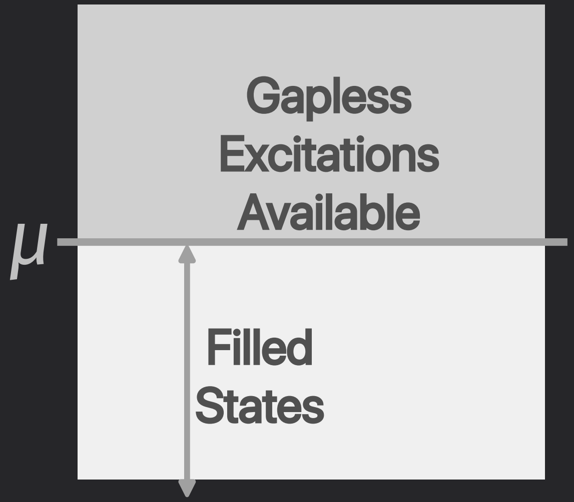
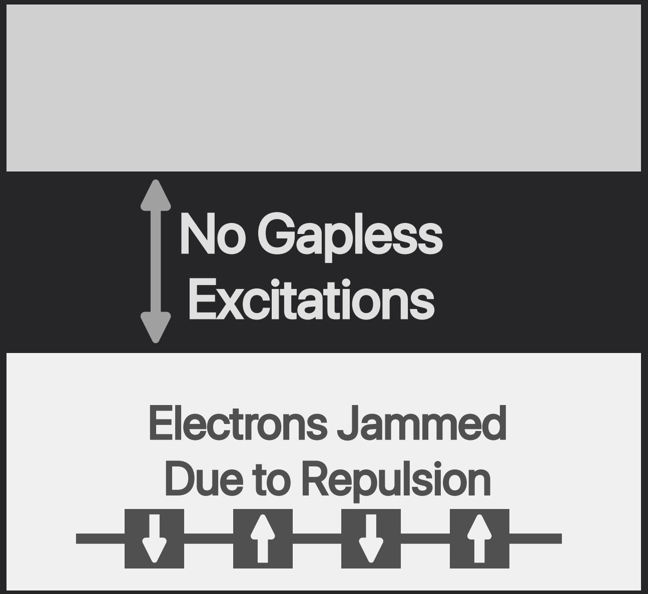



Dispersion away from band edge is **non-interacting**.

Add strong **interactions** - Mott Insulator!



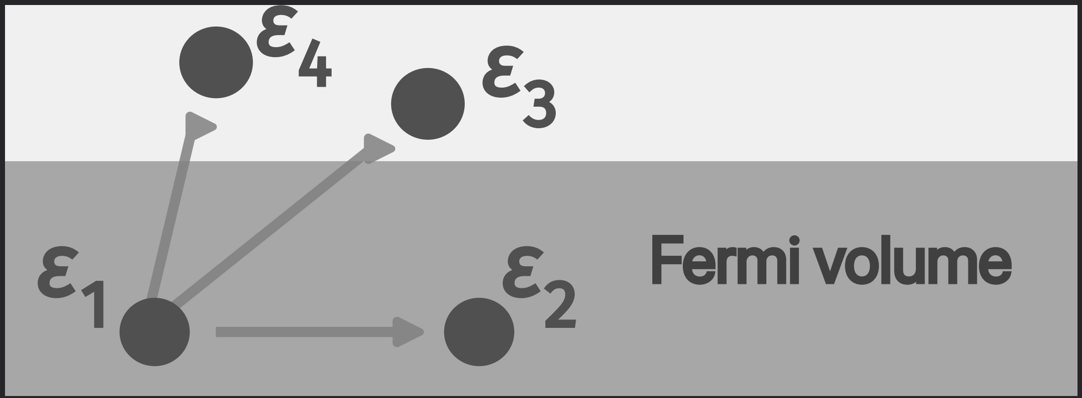

Gap opens inside the band: $\begin{pmatrix}\varepsilon & U \\ U & \varepsilon\end{pmatrix} \to \varepsilon \pm U$

Mott (1968)

# Non-Fermi Liquids Are Different

## Landau Fermi Liquid Theory (Postulates)

- Theory describing how **metals arise** in interacting systems
- Lack of scattering **phase space** at low-energies
- Fermi surface and low-lying electronic excitations **survive**.




$$\Gamma \sim \int d\varepsilon_4 d\varepsilon_3 d\varepsilon_2 \delta(\varepsilon - \varepsilon_2 - \varepsilon_3 - \varepsilon_4) \sim \varepsilon^2$$

$$\rho \sim T^2$$

# Non-Fermi Liquids Are Different

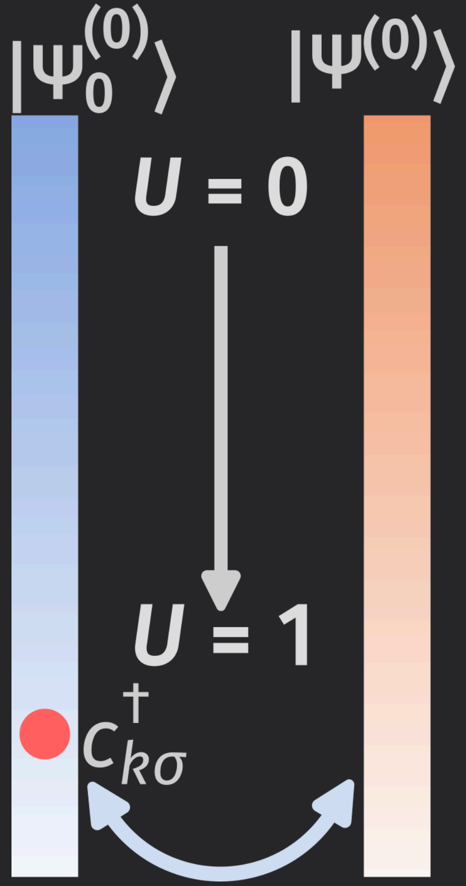
## Landau Fermi Liquid Theory (Quantification)

- **Self-energy** $\Sigma \sim i\omega^2$. Roughly equal to scattering rate. Vanishes very fast as $\omega \to 0$: essential for **quasiparticle** picture

- **Quasiparticle residue**: how similar are the true excitations to 1-particle excitations

$$Z = \langle \Psi | c^\dagger_{k\sigma} | \Psi_0 \rangle$$

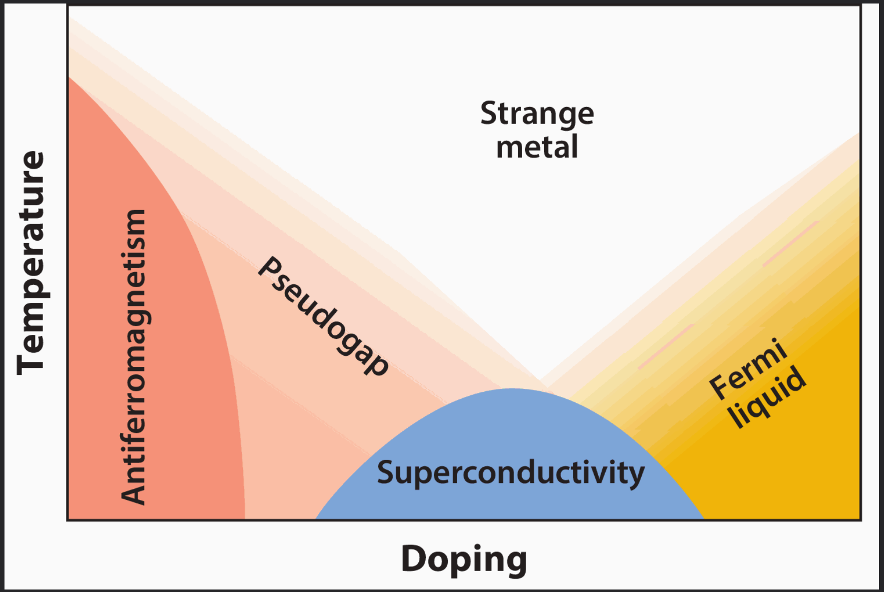
- $Z = \left(1 - \frac{\partial(\mathrm{Re}\,\Sigma)}{\partial\omega}\right)^{-1} \sim 1$. Must be non-zero for Landau Fermi liquid.



# Non-Fermi Liquids Are Different

## Violations Of Landau Fermi Liquid Theory

- **Tomonaga-Luttinger Liquid**: Interacting electrons in 1D
- **Overscreened** fixed points in Kondo models
- **Strange Metal**: Normal state of unconventional SCs in Cu oxides, heavy fermions, Fe pnictides, etc



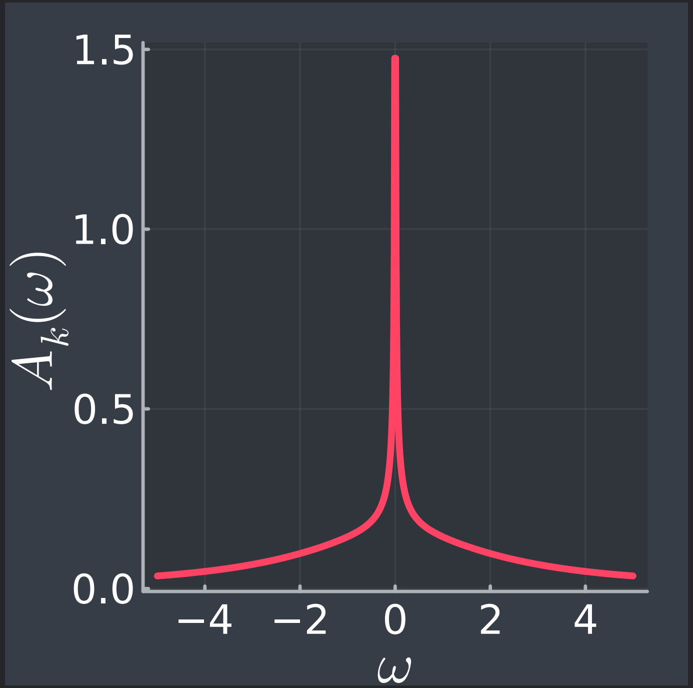

Custers et al. (2003); Doiron-Leyraud et al. (2009); Emery & Kivelson (1992); Haldane (1981); Keimer et al. (2015)

**The Marginal Fermi Liquid**

- **Phenomenological** explanation of normal state of cuprates: $\Sigma \sim \omega \log(|\omega|) - i\pi\ |\omega|$
- Quasiparticle residue **vanishes** at Fermi surface

$$Z^{-1} = 1 - \frac{\partial(\mathrm{Re}\ \Sigma)}{\partial \omega} \sim -\log(\omega) \to \infty$$

- Not accessible through **perturbative** corrections of Landau Fermi liquid




Mukherjee & Lal (2020); Varma et al. (1989)

# Non-Fermi Liquids Are Different

**Main Takeaways**

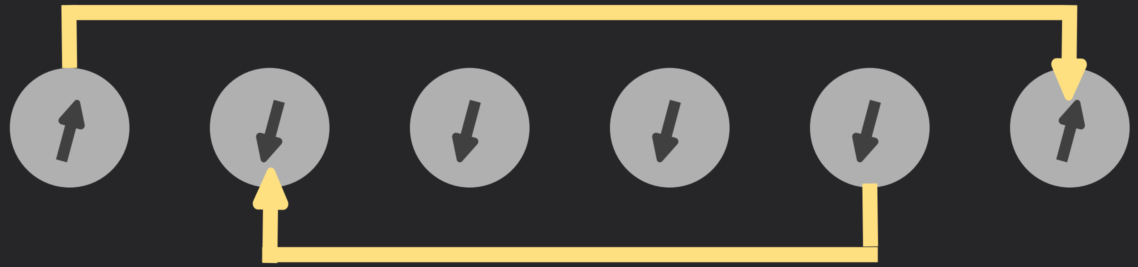
- Landau Fermi Liquid theory requires interacting eigenstates to be **adiabatically connected** to non-interacting eigenstates
- Non-Fermi liquids involve **vanishing quasiparticle residue**, signalling that the states are in fact not adiabatically connected.
- This typically means **non-perturbative** approaches are required to deal with such phases.
- The qualitatively different nature of excitations means that LFL and NFL correspond to **distinct fixed points** in the RG sense.

# An Exactly Solvable Model

# The Hatsugai-Kohmoto Model

Consider long-ranged interaction in real-space.

$$H = -t \sum_{<i,j>,\sigma} c^\dagger_{i,\sigma} c_{j,\sigma} + \frac{U}{L^d} \sum_{i_1,i_2,r} c^\dagger_{i_1+r,\uparrow} c^\dagger_{i_2-r,\downarrow} c_{i_2,\downarrow} c_{i_1,\uparrow}$$



Switch to momentum space, Hamiltonian **becomes local**!

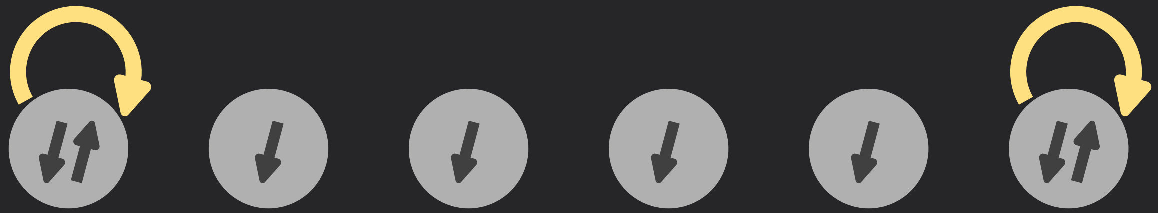
$$c^\dagger_r \sim \sum_k e^{-ikr} c^\dagger_k; \quad H = \sum_{\vec{k}} H_{\vec{k}}; \quad H_{\vec{k}} = \varepsilon_{\vec{k}} \sum_\sigma n_{\vec{k},\sigma} + U n_{\vec{k}\uparrow} n_{\vec{k}\downarrow}$$

Baskaran (1991); Hatsugai & Kohmoto (1992)

# The Hatsugai-Kohmoto Model

Contrast with the Hubbard interaction.

$$H_{\text{int}} \sim \sum_i n_{i,\uparrow} n_{i,\downarrow} = \sum_{k_1,k_2,q} c^\dagger_{k_1+q,\uparrow} c^\dagger_{k_2-q,\downarrow} c_{k_2,\downarrow} c_{k_1,\uparrow}$$



- local in real-space, highly **non-local** in $k$ –space
- HK model is $q = 0, k_1 = k_2$ (**zero mode**!) component of the Hubbard
- HKM is easier to solve than Hubbard (KE and PE **do not commute**)

# Spectrum

$$H = \sum_{\vec{k}} H_{\vec{k}}; \quad H_{\vec{k}} = \varepsilon_{\vec{k}} \sum_{\sigma} n_{\vec{k},\sigma} + U n_{\vec{k}\uparrow} n_{\vec{k}\downarrow}$$

Each $H_k$ can be diagonalised.

$$|0\rangle : E = 0, \quad |\sigma\rangle : E = \varepsilon_k, \quad |2\rangle : E = 2\varepsilon_k + U$$

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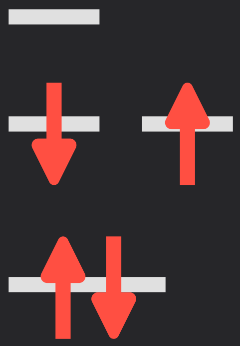

$E_0 = 0$

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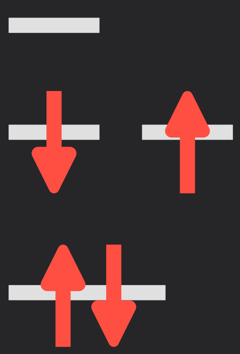


## Case Of Half-Filling

$$E(\mu) = E - \mu n_k, \quad \mu = \frac{U}{2}$$

$$E_0 = 0, \quad E_1 = \varepsilon_k - \frac{U}{2}, \quad E_2 = 2\varepsilon_k$$

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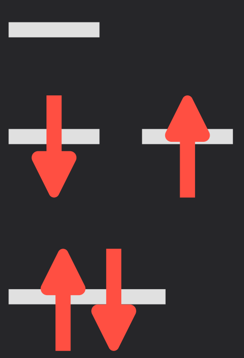

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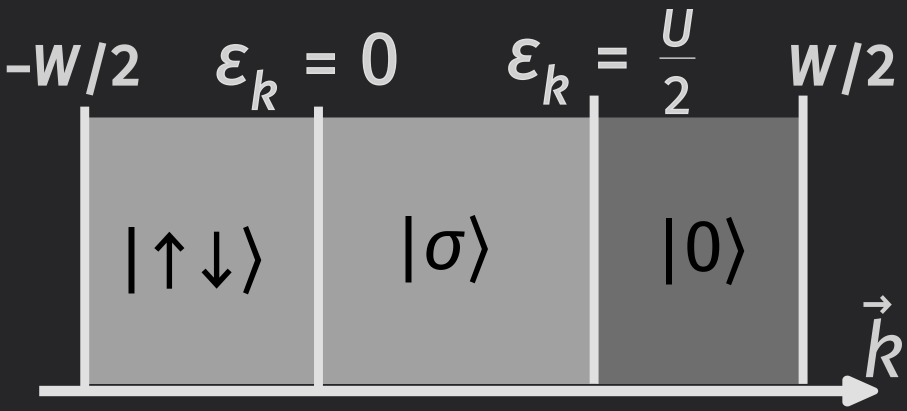


# Introduction to Greens Functions

Nature of **excitations** can be studied through Greens function

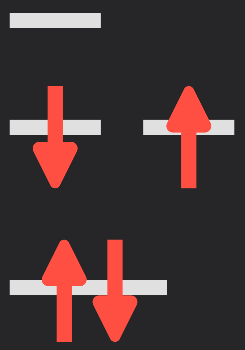
$$G_\nu(t) = -i\theta(t)\left\langle\left\{c_{\nu(t)}, c_\nu^\dagger\right\}\right\rangle$$

- Non-interacting system: $G_k(\omega + i\eta) = \frac{1}{\omega+i\eta-\varepsilon_k}$
- **Poles** of Greens function → one-particle excitations
- **Zeroes** of Greens function → destruction of one-particle excitations

# Exact Single-Particle Greens Function of the HKM

Greens function can be calculated as

$$G_{k\sigma}(\omega) = \frac{P_e(k\sigma)}{\omega - E_X} + \frac{P_h(k\sigma)}{\omega + E_X}$$



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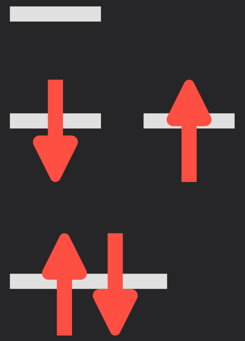

$E_0 = 0$

$E_1 = \varepsilon_k$

$E_2 = 2\varepsilon_k + U$

**If opposite spin is unoccupied**

- Particle addition: $E_X = \varepsilon_k$ $\quad |0\rangle \rightarrow |\sigma\rangle$
- Particle removal: $E_X = -\varepsilon_k$ $\quad |\sigma\rangle \rightarrow |0\rangle$

$$G \rightarrow \frac{1 - \langle n_{k\bar{\sigma}} \rangle}{\omega - \varepsilon_k}$$

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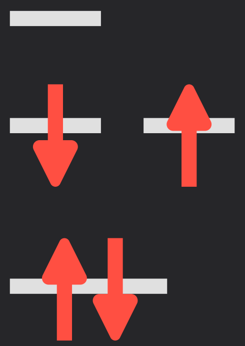

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**If opposite spin is occupied**

- Particle addition: $E_X = \varepsilon_k + U$ $\quad |0\rangle \rightarrow |\sigma\rangle$
- Particle removal: $E_X = -\varepsilon_k - U$ $\quad |\sigma\rangle \rightarrow |0\rangle$

$$G \rightarrow \frac{\langle n_{k\bar{\sigma}} \rangle}{\omega - \varepsilon_k - U}$$

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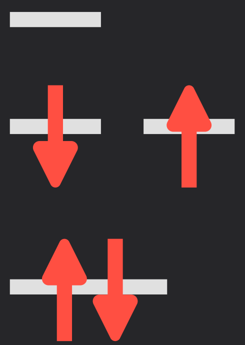

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**Total Greens Function**

$$G_{k\sigma} = \frac{1 - \langle n_{k\bar{\sigma}} \rangle}{\omega - \varepsilon_k} + \frac{\langle n_{k\bar{\sigma}} \rangle}{\omega - \varepsilon_k - U}$$

$$(\varepsilon_k \to \varepsilon_k - \mu)$$

# Correlated Metal-Insulator Transition

The Case Of **Half-Filling**: $2\mu = U, \langle n_{k\sigma} \rangle = \frac{1}{2}$

$$G_{k\sigma} = \frac{1}{2}\left[\left(\omega - \varepsilon_k + U/2\right)^{-1} + \left(\omega - \varepsilon_k - U/2\right)^{-1}\right]$$

Phillips et al. (2020)

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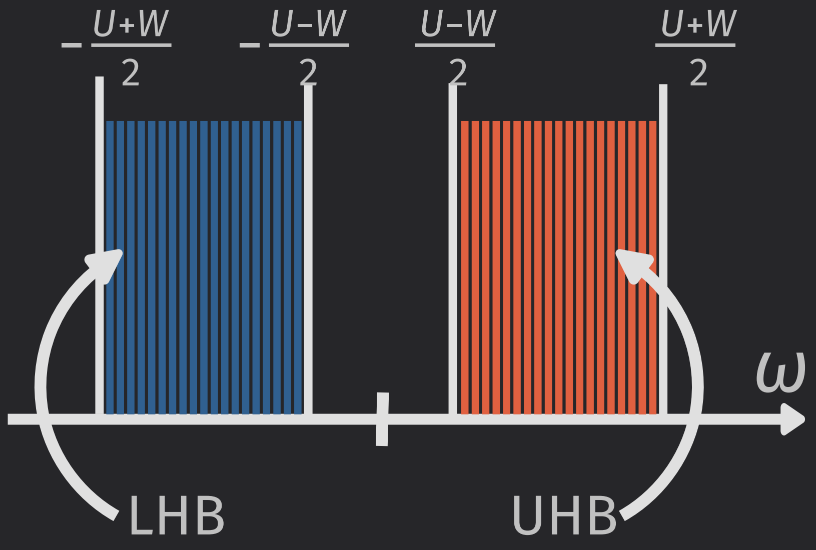
$U > W$ (**Mott Insulator**)

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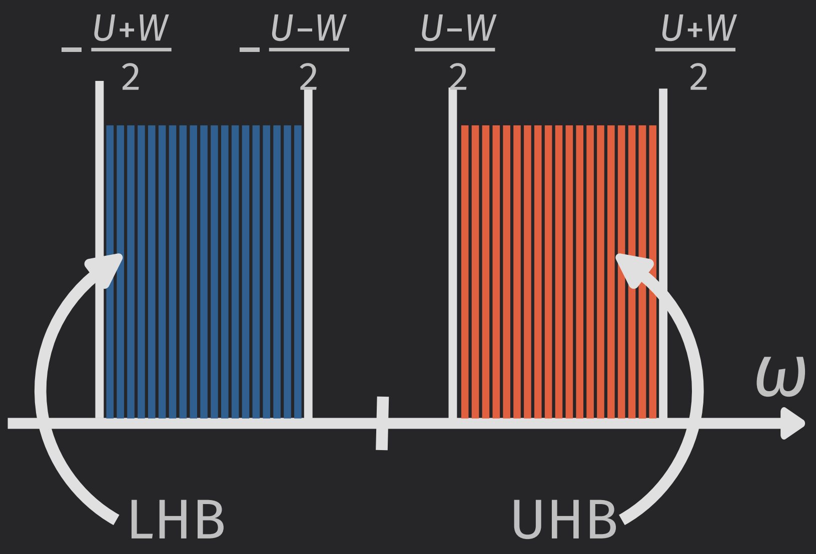
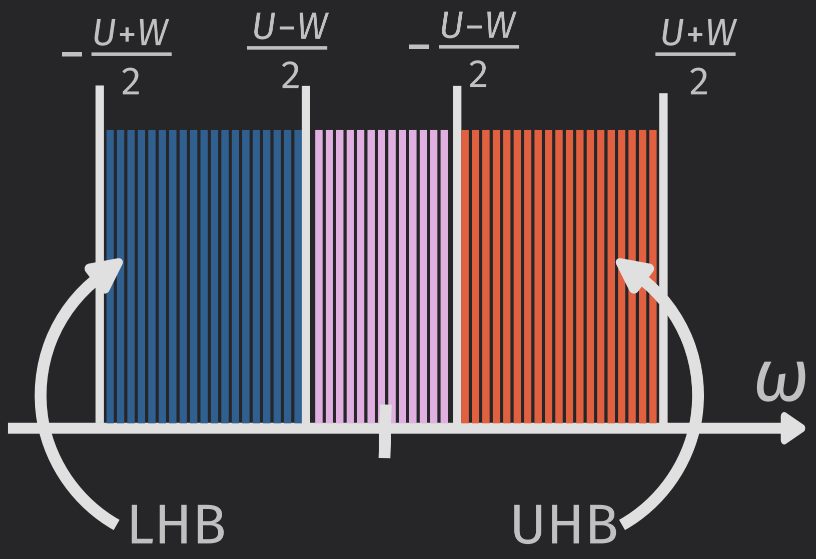



$U < W$ (Metal)




Phillips et al. (2020)

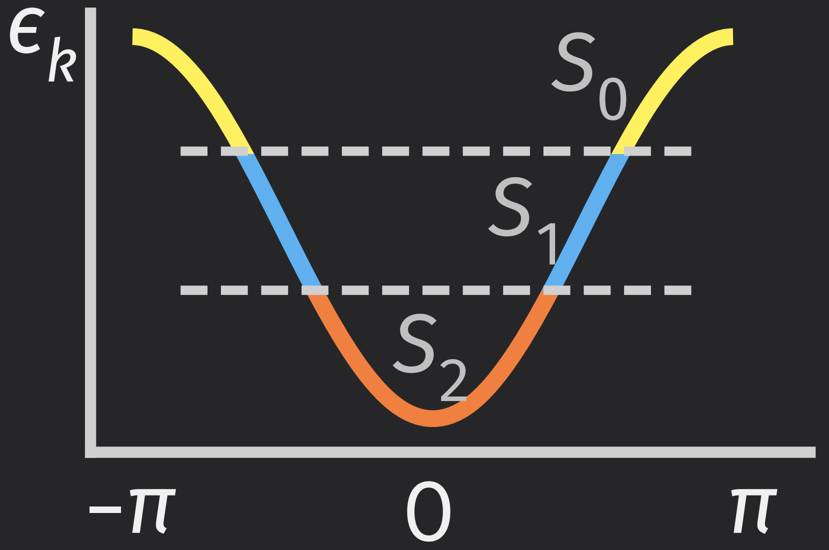
# Non-Fermi Liquid Signatures In Metallic Phase

# Signature I: Composite Gapless Excitations

$$G_{k\sigma} = \frac{1}{2}\left[\left(\omega - \varepsilon_k + U/2\right)^{-1} + \left(\omega - \varepsilon_k - U/2\right)^{-1}\right]$$

Momentum states classified into three groups:

- $S_2 : \varepsilon_k < -U/2$:
  **Both poles** below $\omega = 0$: $\langle n_k \rangle = 2$
- $S_1 : -U/2 < \varepsilon_k < U/2$:
  **One pole** below $\omega = 0$: $\langle n_k \rangle = 1$
- $S_0 : \varepsilon_k > U/2$:
  **No pole** below $\omega = 0$: $\langle n_k \rangle = 0$




Phillips et al. (2020)

# Signature I: Composite Gapless Excitations

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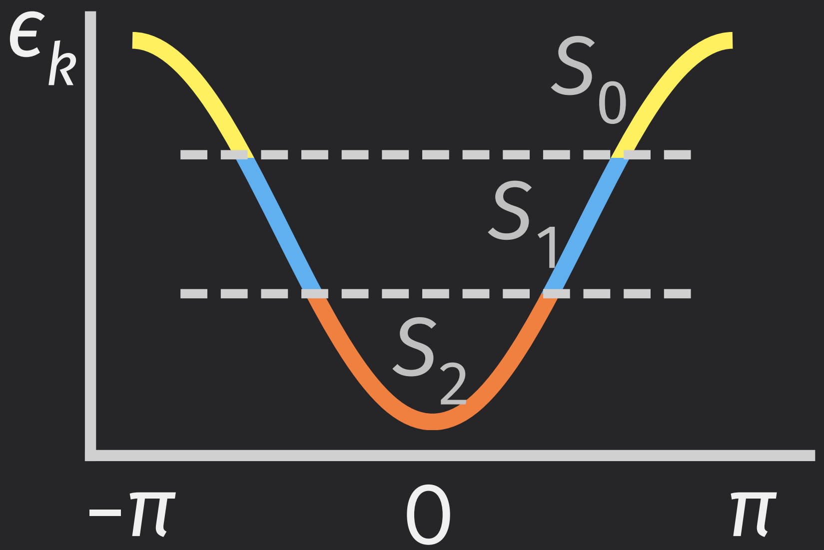
Momentum states classified into three groups:

Ground state is a **mixed** state.

- $k$ –states in $S_2$ are doubly-occupied.
- $k$ –states in $S_1$ are half-filled.
- $2^{N_1}$-fold degenerate.

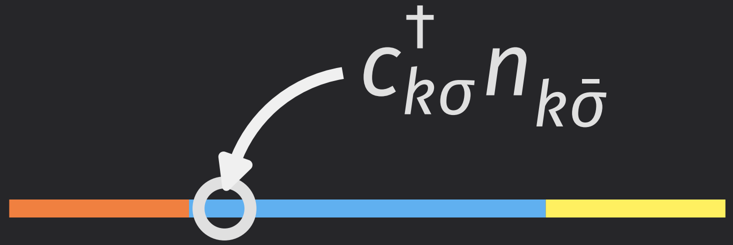



Phillips et al. (2020)

# Signature I: Composite Gapless Excitations

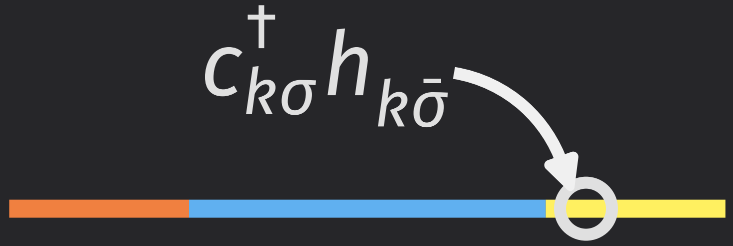
**Near $S_2 - S_1$ boundary** $\left(\varepsilon_k = -U/2\right)$

- Excitation operator: $c^\dagger_{k\sigma} n_{k\bar{\sigma}}$
- Excitation energy is $\varepsilon_k + U - \mu \to 0^+$




**Near $S_1 - S_0$ boundary** $\left(\varepsilon_k = U/2\right)$

- Excitation operator: $c^\dagger_{k\sigma}(1 - n_{k\bar{\sigma}})$
- Excitation energy is $\varepsilon_k - \mu \to 0^+$




**Projectors are needed** because the other excitations are gapped.

Baskaran (1991); Phillips et al. (2020); Zhao et al. (2024)

# Signature I: Composite Gapless Excitations

**Near $S_2 - S_1$ boundary**

$$c^\dagger_{k\sigma} n_{k\bar{\sigma}}$$

**Near $S_1 - S_0$ boundary**

$$c^\dagger_{k\sigma}(1 - n_{k\bar{\sigma}})$$

**Excitations are Non-Fermi Liquid In Nature!**

- Strong correlations lead to **composite** (hole/double) excitations
- Excitations are **not electronic** (do not satisfy $\{\cdot\}$ relations)
- **Breakdown** of quasiparticle picture, and hence of Fermi liquid theory

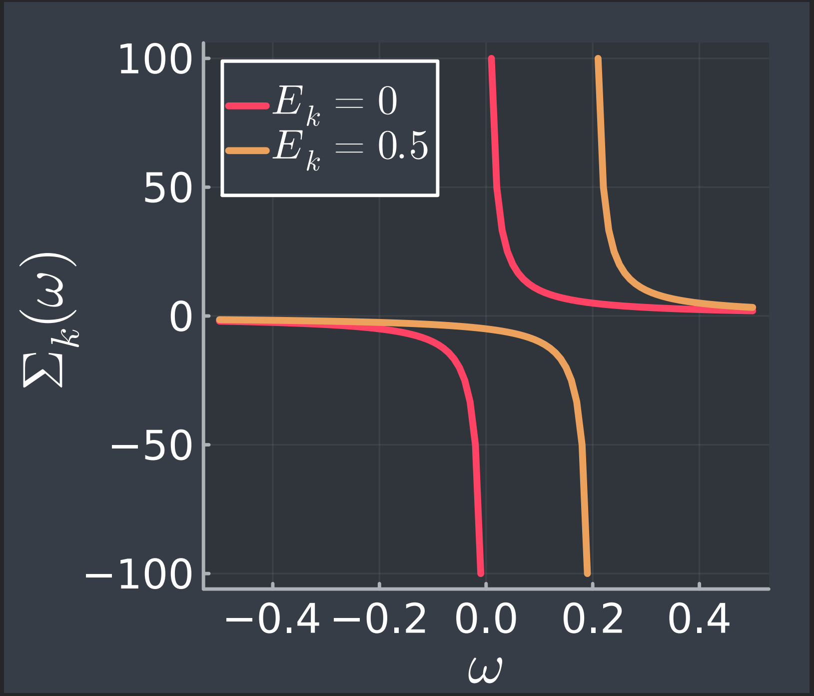
Baskaran (1991); Phillips et al. (2020); Zhao et al. (2024)

# Signature II: Divergence of Self-Energy

Greens function at 1/2-filling can be rewritten as $G_{k\sigma} = \frac{1}{\omega - \varepsilon_k + \frac{U^2/4}{\omega - \varepsilon_k}}$.

**Self-energy** is $\Sigma(\omega) = \frac{U^2/4}{\omega - \varepsilon_k}$

- **Diverges** along $\varepsilon_k = 0$ as $\omega \to 0$
- **Violates** Fermi Liquid Theory
- Leads to zeros of Greens function
- Death of Landau **quasiparticles**



# Signature II: Divergence of Self-Energy

**How Does A Diverging Self-Energy Leave The System Metallic?**

Greens functions for composite excitations do not have self-energy!

$$d^\dagger_{k\sigma} = c^\dagger_{k\sigma} n_{k\sigma}, \quad G_d = \frac{1}{\omega - \varepsilon_k - \frac{U}{2}}$$

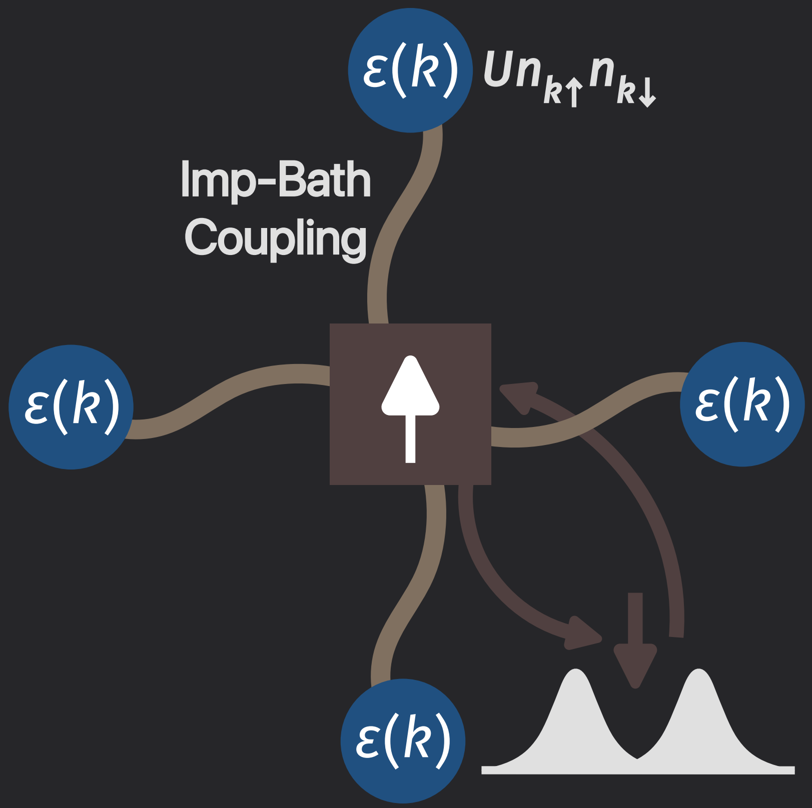
$$h^\dagger_{k\sigma} = c^\dagger_{k\sigma}(1 - n_{k\sigma}), \quad G_h = \frac{1}{\omega - \varepsilon_k + \frac{U}{2}}$$

These can therefore propagate with **long lifetimes**.

# Summary of Main Ideas

# Avenues for Futher Investigation

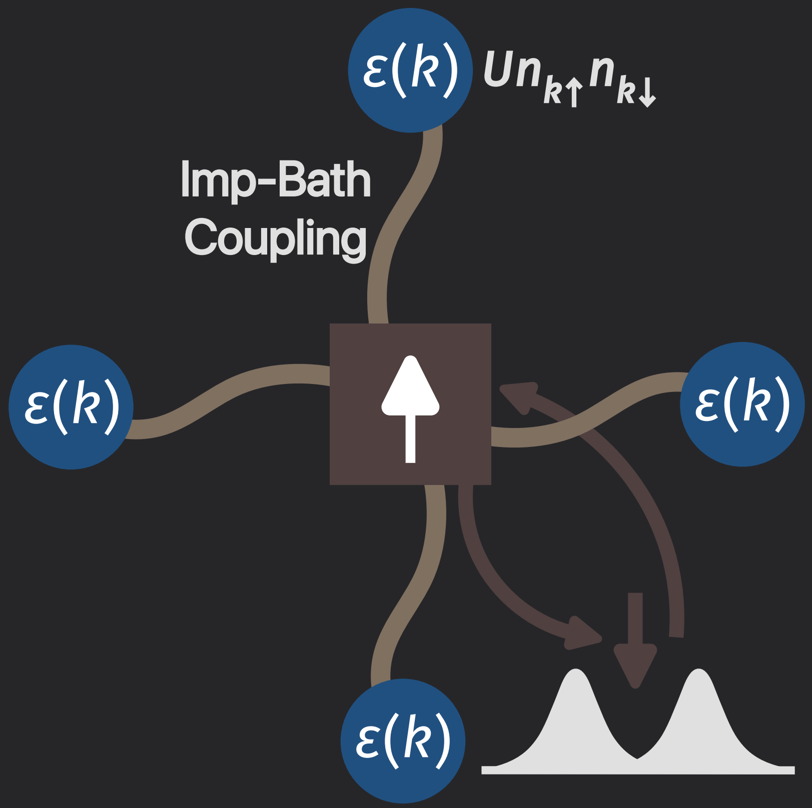
# Kondo Screening in Hatsugai-Kohmoto Model




Consider local moment hybridising with HK Model

$$H = H_{\text{Kondo}} + H_{\text{HKM}}$$

# Kondo Screening in Hatsugai-Kohmoto Model

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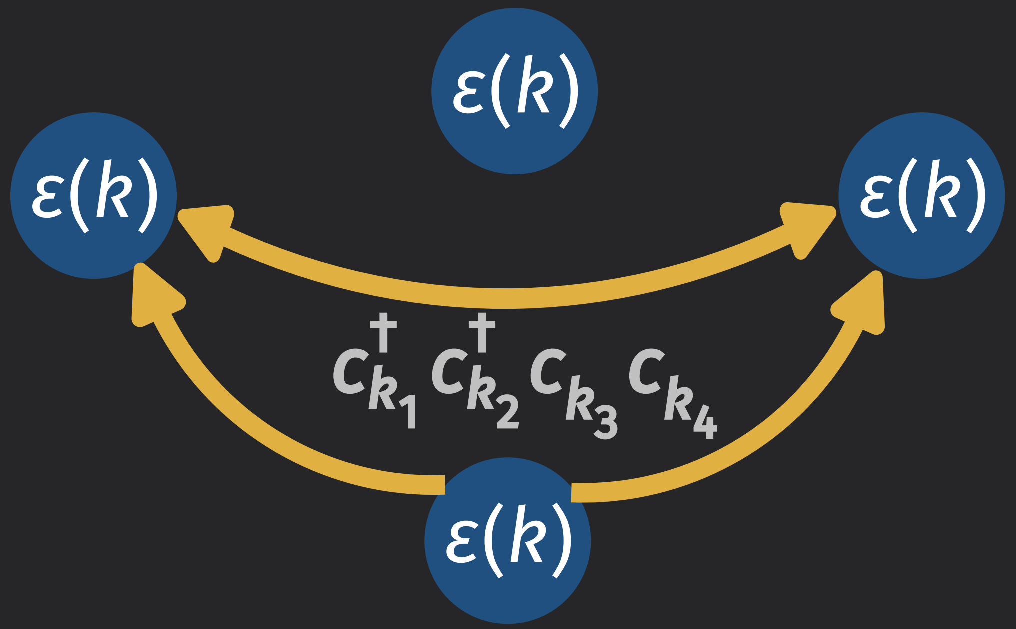
How does **absence** of quasiparticles affect Kondo screening?

# Toy Model for Thermalisation and Many-Body Scars




Consider HK Model perturbed by Hubbard interaction

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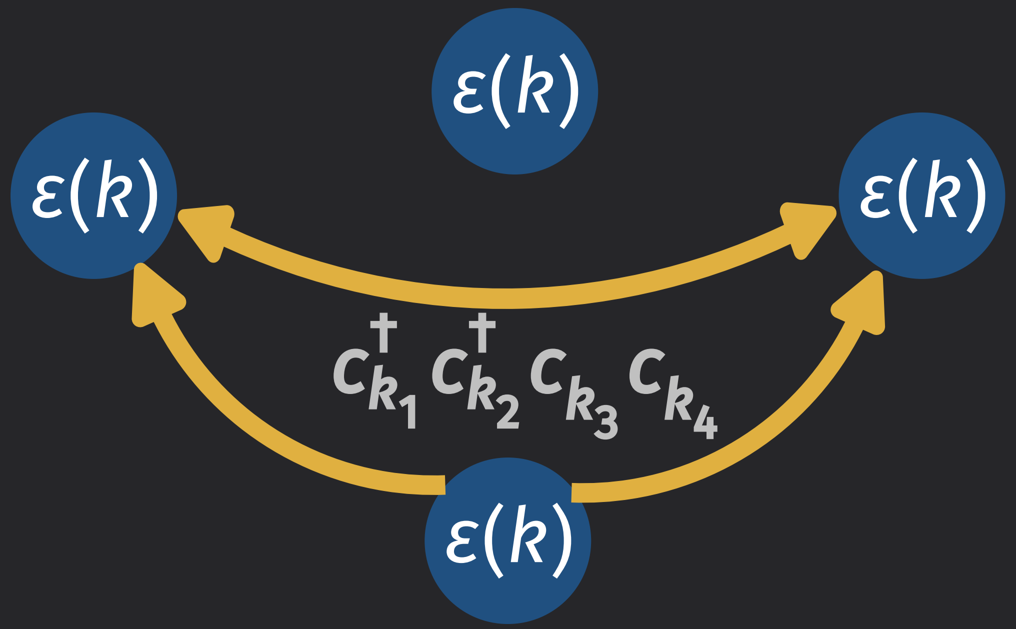


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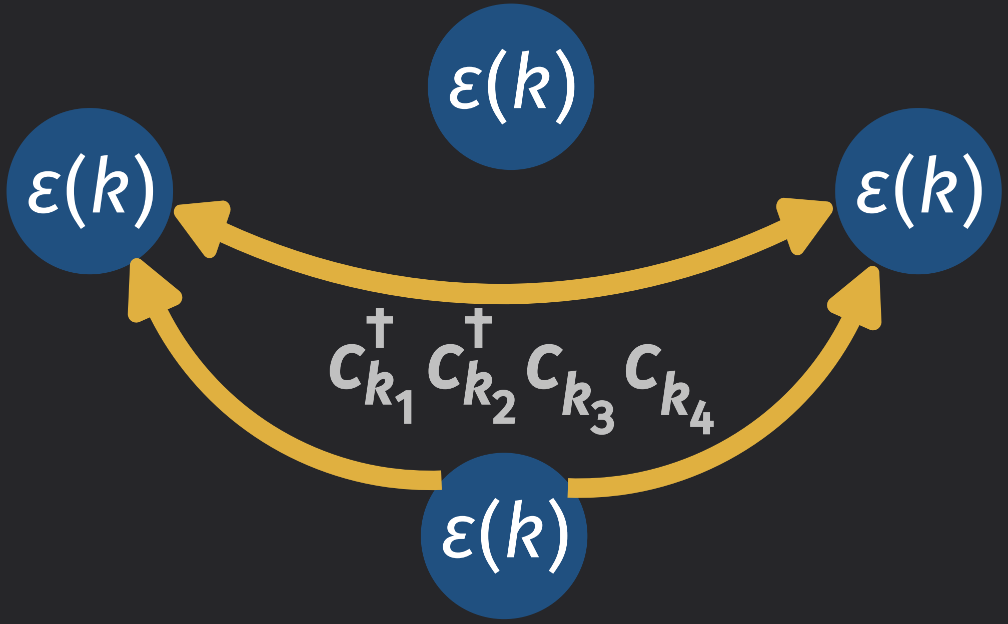
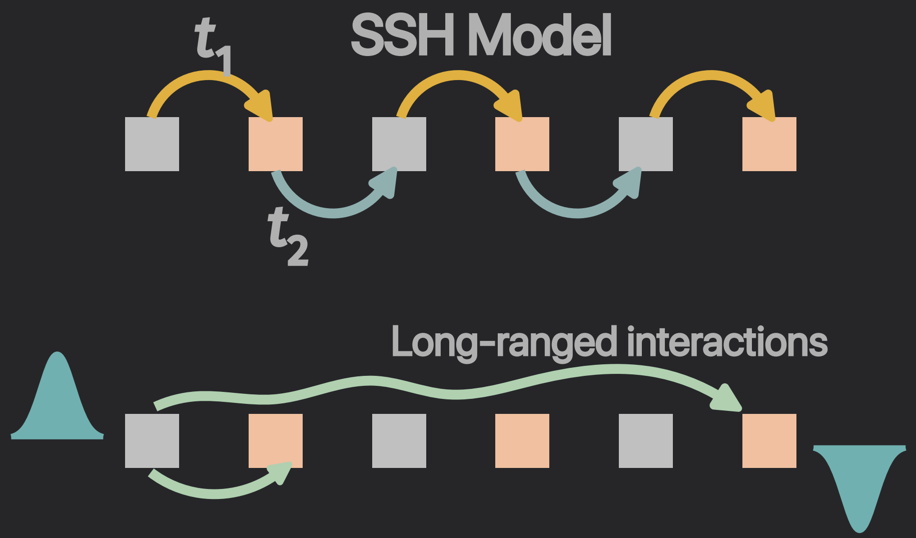



Consider HK Model perturbed by Hubbard interaction

$$H = H_{\mathrm{HKM}} + P_v H_{\mathrm{Hub}} P_v$$

- $H_{\mathrm{Hub}}$ allows **thermalisation** of $k$ –states
- $P_v$ will preserve certain sectors. **Scars**?

# Effect of Hatsugai-Kohmoto Interaction on Edge Modes of SSH Chain

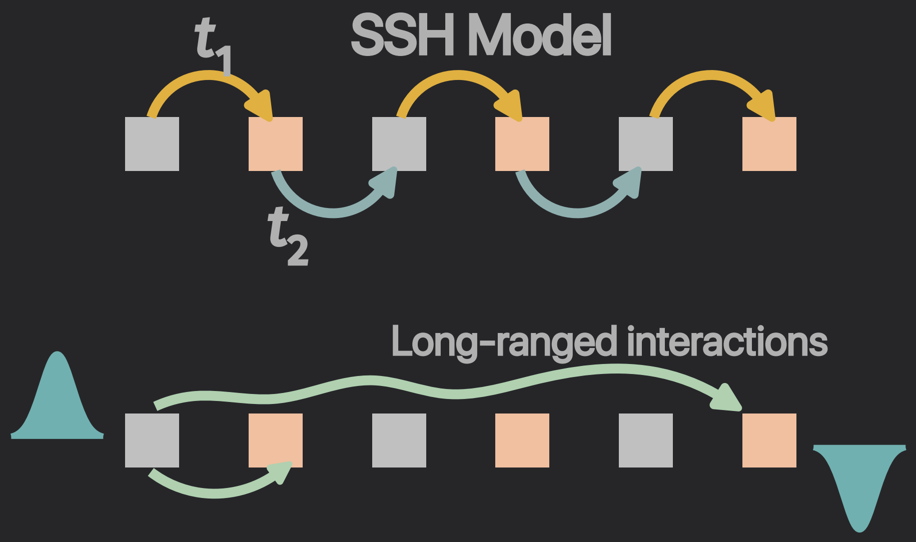



Add long-ranged HKM interactions to SSH chain.

$$H_{\mathrm{SSH}} = -t_1 c_1^\dagger c_2 - t_2 c_2^\dagger c_3 + \ldots$$

$$H = H_{\mathrm{HKM}} + H_{\mathrm{SSH}}$$

# Effect of Hatsugai-Kohmoto Interaction on Edge Modes of SSH Chain




Add long-ranged HKM interactions to SSH chain.

$$H_{\mathrm{SSH}} = -t_1 c_1^\dagger c_2 - t_2 c_2^\dagger c_3 + \dots$$

$$H = H_{\mathrm{HKM}} + H_{\mathrm{SSH}}$$

- Fate of topological **edge modes**?

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