

# JOURNEY FROM THE HUBBARD DIMER TO THE HUBBARD MODEL

ABHIRUP MUKHERJEE, DR. SIDDHARTHA LAL

DEPARTMENT OF PHYSICAL SCIENCES  
IISER KOLKATA

AUGUST 12, 2021

# OVERVIEW OF THE PROCESS

- Choose correlated Anderson model as the auxiliary model and perform URG analysis and extract zero mode to obtain Hubbard dimer as the effective low energy Hamiltonian.
- Translate this Hubbard dimer Hamiltonian to recreate a new/renormalized Hubbard model. This Hubbard model is assumed to be linked to the parent Hubbard model via a similarity transformation.
- Express equation between renormalized Hubbard and Hubbard dimers as relation between inverse Greens function matrix elements of full Hubbard model and those of the Hubbard dimer.
- Obtain Greens functions of the parent Hubbard model in terms of those of the Hubbard dimer.

# OVERVIEW OF THE PROCESS

- Choose correlated Anderson model as the auxiliary model and perform URG analysis and extract zero mode to obtain Hubbard dimer as the effective low energy Hamiltonian.
- Translate this Hubbard dimer Hamiltonian to recreate a new/renormalized Hubbard model. This Hubbard model is assumed to be linked to the parent Hubbard model via a similarity transformation.
- Express equation between renormalized Hubbard and Hubbard dimers as relation between inverse Greens function matrix elements of full Hubbard model and those of the Hubbard dimer.
- Obtain Greens functions of the parent Hubbard model in terms of those of the Hubbard dimer.

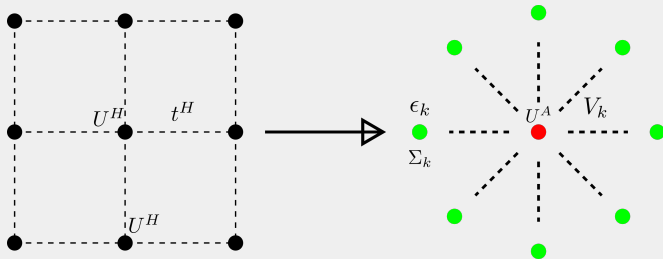
# OVERVIEW OF THE PROCESS

- Choose correlated Anderson model as the auxiliary model and perform URG analysis and extract zero mode to obtain Hubbard dimer as the effective low energy Hamiltonian.
- Translate this Hubbard dimer Hamiltonian to recreate a new/renormalized Hubbard model. This Hubbard model is assumed to be linked to the parent Hubbard model via a similarity transformation.
- Express equation between renormalized Hubbard and Hubbard dimers as relation between inverse Greens function matrix elements of full Hubbard model and those of the Hubbard dimer.
- Obtain Greens functions of the parent Hubbard model in terms of those of the Hubbard dimer.

# OVERVIEW OF THE PROCESS

- Choose correlated Anderson model as the auxiliary model and perform URG analysis and extract zero mode to obtain Hubbard dimer as the effective low energy Hamiltonian.
- Translate this Hubbard dimer Hamiltonian to recreate a new/renormalized Hubbard model. This Hubbard model is assumed to be linked to the parent Hubbard model via a similarity transformation.
- Express equation between renormalized Hubbard and Hubbard dimers as relation between inverse Greens function matrix elements of full Hubbard model and those of the Hubbard dimer.
- Obtain Greens functions of the parent Hubbard model in terms of those of the Hubbard dimer.

# CLUSTER-BATH APPROACH



$$H^H = -t^H \sum_{\sigma, \langle i,j \rangle} \left( c_{i\sigma}^\dagger c_{j\sigma} + \text{h.c.} \right) + U^H \sum_i \tau_{i\uparrow} \tau_{i\downarrow}$$

$$H^A = \sum_{k\sigma} \tilde{\epsilon}_k \tau_{k\sigma} + U^A \tau_{d\uparrow} \tau_{d\downarrow} + U_b \sum_{kk'} \hat{n}_k \hat{n}_{k'}$$

$$-t^A \sum_{k\sigma} \left( c_{d\sigma}^\dagger c_{k\sigma} + \text{h.c.} \right)$$

# RG ANALYSIS OF AUXILIARY SYSTEM

$$\sum_{k\sigma} \tilde{\epsilon}_k \tau_{k\sigma} + U \tau_{d\uparrow} \tau_{d\downarrow} + U_b \sum_{kk'} \hat{n}_k \hat{n}_{k'} - t \sum_{k\sigma} \left( c_{d\sigma}^\dagger c_{k\sigma} + \text{h.c.} \right)$$

$$\downarrow U H U^\dagger$$

$$\sum_{k\sigma}^* \left[ \tilde{\epsilon}_k \tau_{k\sigma} - t^* \left( c_{d\sigma}^\dagger c_{k\sigma} + \text{h.c.} \right) \right] + U^* \tau_{d\uparrow} \tau_{d\downarrow} + U^* \sum_{kk'}^* \hat{n}_k \hat{n}_{k'}$$

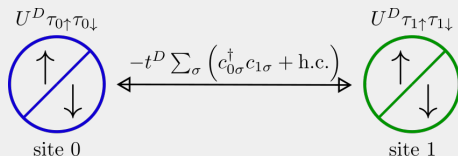
$$\downarrow \text{zero mode}$$

$$-t^* \sum_{\sigma} \left( c_{d\sigma}^\dagger c_{z\sigma} + \text{h.c.} \right) + U^* \tau_{d\uparrow} \tau_{d\downarrow} + U^* \tau_{z\uparrow} \tau_{z\downarrow}$$

# RESTORING TRANSLATIONAL INVARIANCE

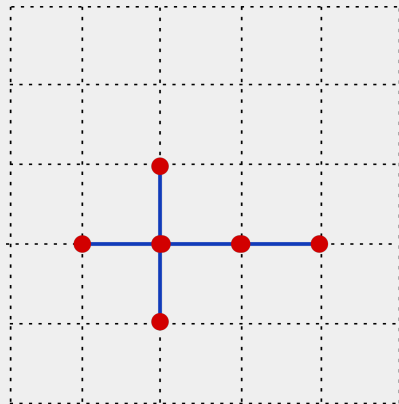
$$\tilde{H}(\tilde{t}, \tilde{U}) = \frac{2}{N_W} \sum_{\langle ij \rangle} H^D(i, j)$$

$$H^D = -t^D \sum_{\sigma} \left( c_{i\sigma}^{\dagger} c_{j\sigma} + \text{h.c.} \right) + U^D (\tau_{i\uparrow} \tau_{i\downarrow} + \tau_{j\uparrow} \tau_{j\downarrow})$$



$$\tilde{t} \equiv \frac{2}{N_W} t^D, \quad \tilde{U} \equiv \frac{2}{N_W} U^D$$

$$\tilde{H} = \mathcal{U} H^D \mathcal{U}^{-1} \quad \left[ \tilde{G}(\tilde{r}, \tilde{r}') = G^D(r, r') \right]$$





# INVERSE GREENS FUNCTION FROM HAMILTONIAN

**Write equation in terms of inverse Greens function  $G^{-1}(\omega) = \omega - H$**

$$\begin{aligned}\omega - \tilde{G}^{-1} &= \frac{2}{Nw} \sum_{\langle ij \rangle} [\omega - G_D^{-1}(\omega, i, j)] \\ \longrightarrow \tilde{G}^{-1} &= \frac{2}{Nw} \sum_{\langle ij \rangle} G_D^{-1}(\omega, i, j)\end{aligned}$$

- Take diagonal matrix element  $\langle \tilde{i} | G^{-1} | \tilde{i} \rangle$ . On the RHS, there are  $w$  terms that have the index  $i$ . Because of translational invariance, all such terms will be same.

$$(\tilde{G}^{-1})_{\tilde{i}, \tilde{i}} = (G^{-1})_{i, i} = \frac{2}{Nw} \times (G_D^{-1})_{ii} \times w = \frac{2}{N} (G_D^{-1})_{oo} \equiv g_o$$

**Write equation in terms of inverse Greens function  $G^{-1}(\omega) = \omega - H$**

$$\begin{aligned}\omega - \tilde{G}^{-1} &= \frac{2}{N_W} \sum_{\langle ij \rangle} [\omega - G_D^{-1}(\omega, i, j)] \\ \longrightarrow \tilde{G}^{-1} &= \frac{2}{N_W} \sum_{\langle ij \rangle} G_D^{-1}(\omega, i, j)\end{aligned}$$

- Take nearest-neighbour matrix element  $\langle \tilde{i} | \tilde{G}^{-1} | \tilde{j} \rangle$ . On the RHS, there's just one term that has both indices  $i$  and  $j$ .

$$(\tilde{G}^{-1})_{\tilde{i}\tilde{j}} = (G^{-1})_{i,j} = \frac{2}{N_W} \times (G_D^{-1})_{ij} = \frac{2}{N_W} (G_D^{-1})_{01} \equiv g_1$$

# INVERSE GREENS FUNCTION FROM HAMILTONIAN

Write equation in terms of inverse Greens function  $G^{-1}(\omega) = \omega - H$

$$\begin{aligned}\omega - \tilde{G}^{-1} &= \frac{2}{N_W} \sum_{\langle ij \rangle} [\omega - G_D^{-1}(\omega, i, j)] \\ \longrightarrow \tilde{G}^{-1} &= \frac{2}{N_W} \sum_{\langle ij \rangle} G_D^{-1}(\omega, i, j)\end{aligned}$$

- **All other matrix elements are zero**, because no term in the Hamiltonian scatters between non-nearest-neighbour sites  
→ **tri-diagonal inverse Greens function matrix**

# DIAGONALIZING THE INVERSE GREENS FUNCTION MATRIX

$$G^{-1} = \begin{pmatrix} g_0 & g_1 & 0 & \dots & g_1 \\ g_1 & g_0 & g_1 & 0 & \dots \\ 0 & g_1 & g_0 & g_1 & \dots \\ 0 & 0 & \dots & \dots & \dots \end{pmatrix} \xrightarrow{\text{Fourier transform to } k\text{-space}} g_0 + g_1 \begin{pmatrix} \xi_{\vec{k}_1} & 0 & 0 & \dots \\ 0 & \xi_{\vec{k}_2} & 0 & \dots \\ 0 & 0 & \xi_{\vec{k}_3} & \dots \\ 0 & 0 & \dots & \dots \end{pmatrix}$$

$$\xi_{\vec{k}} = \sum_{i=1}^w \cos(a_i q_i)$$

# DIAGONALIZING THE INVERSE GREENS FUNCTION MATRIX

$$G^{-1} = g_0 + g_1 \begin{pmatrix} \xi_{\vec{k}_1} & & \\ & \xi_{\vec{k}_2} & \\ & & \xi_{\vec{k}_3} \end{pmatrix} \xrightarrow{\text{invert the diagonal matrix}} G = \begin{pmatrix} G_{\vec{k}_1} & & \\ & G_{\vec{k}_2} & \\ & & G_{\vec{k}_3} \end{pmatrix}$$

**Space Greens function:** 
$$G_H(\vec{k}, \omega) = \frac{N}{2} \left\{ [G_D^{-1}(\omega)]_{00} + \frac{1}{W} [G_D^{-1}(\omega)]_{01} \xi_{\vec{k}} \right\}^{-1}$$

# COMPUTE SINGLE-PARTICLE GREENS FUNCTIONS AND OTHER STUFF

**$\vec{k}$ -space Greens function:**  $G_H(\vec{k}, \omega) = \frac{N}{2} \left\{ [G_D^{-1}(\omega)]_{00} + \frac{1}{W} [G_D^{-1}(\omega)]_{01} \xi_{\vec{k}} \right\}^{-1}$

**$\vec{r}$ -space Greens function:**  $G_H(\vec{r}, \omega) = \frac{1}{2} \sum_{\vec{k}} e^{i\vec{k} \cdot \vec{r}} \left\{ [G_D^{-1}(\omega)]_{00} + \frac{1}{W} [G_D^{-1}(\omega)]_{01} \xi_{\vec{k}} \right\}^{-1}$

# COMPUTE SINGLE-PARTICLE GREENS FUNCTIONS AND OTHER STUFF

**$\vec{k}$ -space Greens function:**  $G_H(\vec{k}, \omega) = \frac{N}{2} \left\{ [G_D^{-1}(\omega)]_{00} + \frac{1}{W} [G_D^{-1}(\omega)]_{01} \xi_{\vec{k}} \right\}^{-1}$

**$\vec{r}$ -space Greens function:**  $G_H(\vec{r}, \omega) = \frac{1}{2} \sum_{\vec{k}} e^{i\vec{k} \cdot \vec{r}} \left\{ [G_D^{-1}(\omega)]_{00} + \frac{1}{W} [G_D^{-1}(\omega)]_{01} \xi_{\vec{k}} \right\}^{-1}$

**self-energy:**  $\Sigma_H(\vec{k}, \omega) = \omega - g_0 + (t^H - g_1) \xi_{\vec{k}}$



# COMPUTE SINGLE-PARTICLE GREENS FUNCTIONS AND OTHER STUFF

**$\vec{k}$ -space Greens function:**  $G_H(\vec{k}, \omega) = \frac{N}{2} \left\{ [G_D^{-1}(\omega)]_{00} + \frac{1}{W} [G_D^{-1}(\omega)]_{01} \xi_{\vec{k}} \right\}^{-1}$

**$\vec{r}$ -space Greens function:**  $G_H(\vec{r}, \omega) = \frac{1}{2} \sum_{\vec{k}} e^{i\vec{k} \cdot \vec{r}} \left\{ [G_D^{-1}(\omega)]_{00} + \frac{1}{W} [G_D^{-1}(\omega)]_{01} \xi_{\vec{k}} \right\}^{-1}$

**self-energy:**  $\Sigma_H(\vec{k}, \omega) = \omega - g_0 + (t^H - g_1) \xi_{\vec{k}}$

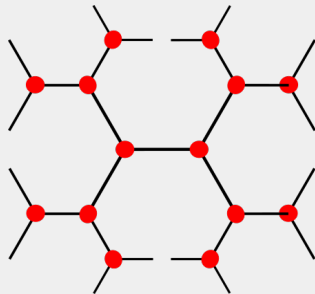
**Spectral function:**  $A_H(\vec{k}, \omega) = -\frac{1}{\pi} \text{Im}(G_H(\vec{k}, \omega)) = -\frac{1}{\pi} \text{Im} [(g_0 + g_1 \xi_{\vec{k}})^{-1}]$

# ON THE BETHE LATTICE ( $w \rightarrow \infty$ )

Hamiltonian scaling arguments suggest <sup>1</sup>

$$G_{ij} = G_{ii}\delta_{ij}$$

(**Greens function becomes local**)



Bethe lattice with  $w = 3$

---

<sup>1</sup>Vollhardt, Krzysztof and Marcus, Dynamical Mean-Field Theory, 2012, Springer Berlin Heidelberg

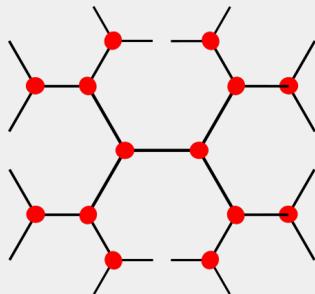
# ON THE BETHE LATTICE ( $w \rightarrow \infty$ )

Also emerges from this formulation:

$$G_{ii}^{-1} = \frac{2}{N} [G_D^{-1}]_{oo} \rightarrow \text{finite}$$

$$\mathbf{G}_{ij}^{-1} = \frac{2}{Nw} [\mathbf{G}_D^{-1}]_{oo} \rightarrow \mathbf{0} \quad \text{when } w \rightarrow \infty$$

$G_{ij}^{-1}$  becomes diagonal  $\longrightarrow$   $G_{ij}$  becomes diagonal



Bethe lattice with  $w = 3$