# Assignmet 1

In [30]:

```
import numpy as np
import matplotlib.pyplot as plt
```

## Lipschitz Continuity

(1) Please state the formal definition of continuous functions

(2) Please state the formal definitions of Lipschitz continuity and locally Lipschitz continuity.

### Ans:

(1)A Continuous function a function such that a small change of thr argument induces a continuous variation of the function value.

(2) Lipschitz continuity a strong form of uniform continuity. It limits how fast the function can change.(Lipschitz连续是一种特殊的连续) A real number function $f : R \Rightarrow R$ is called Lipschitz continuous if there exist a positive real constant $K$such that, for all real number $x_1$,$x_2$,

$$|f(x_1) - f(x_2)| \leq K|x_1 - x_2|$$

(一阶导数有限) A function is called locally Lipschitz continuous if for every $x$ in $X$ there exists a neighborhood $U$ of $x$ such that $f$ in $U$ is Lipschitz continuous. (在去心邻域内Lipschitz连续)

## Matrix Calculus

(1)

$$y = f : R^{n\times m} \Rightarrow R$$
$$[\frac{\partial y}{\partial X}]_{ij} = \frac{\partial y}{\partial x_{ij}}$$

(2)

$$y = tr(AX) = \sum_{j=1}^{n}\sum_{i=1}^{m} a_{ji}x_{ij}$$
$$[\frac{\partial tr(AX)}{\partial X}]_{ij} = \frac{\partial y}{\partial x_{ij}} = a_{ji}\frac{\partial tr(AX)}{\partial X} = A^T$$

(3)

$$y = f(x) = \sum_{j=1}^{n}\sum_{i=1}^{n} x_j x_i q_{ij} + \sum_{i=1}^{n} x_i^2$$
$$[\frac{\partial y}{\partial x}]_k = \frac{\partial y}{\partial x_k} = \sum_{i=1}^{n} x_i(q_{ik} + q_{ki}) + 2x_k$$
$$\frac{\partial y}{\partial x} = (Q^T + Q)x + 2x$$

## Inner Product

(a)

$$< x, y >= |x||y|cos(\theta)$$
$$\theta = arccos(\frac{< x, y >}{|x||y|})$$

(b)

$$\theta = arccos(\frac{tr(A^T B)}{tr(A^T A)tr(B^T B)})$$

In [37]:

```
A = np.array([[1,0,1],
              [0,1,0]])
B = np.array([[-1,2,1],
              [-1,0,1]])
inner_AB = np.trace(A.T.dot(B))
norm_A = np.trace(A.T.dot(A))
norm_B = np.trace(B.T.dot(B))
print(inner_AB)
print(norm_A)
print(norm_B)
theta = np.arccos(inner_AB/(norm_A*norm_B))
print(theta)
```

```
0
3
8
1.5707963267948966
```

## Linear Algebra

(a) $Ax$ is the linear combination of $A$, $b$ is in the column space of $A$

(b) $rank(A) = n - 2$ $Null(A) = Span([1, -1, 1, 0]^T, [1, 0, 1, -1]^T)$

(c) if $Ax = b$ is the projection of $y$ on $Col(A)$, then $y - b$ is in null space of $A$

$$A^T(y - Ax) = 0$$
$$x = (A^T A)^{-1} A^T y$$

## Ellipsoids

(a)If we assume $u$ is a point on unit circle, then

$$u_{new1} = Pu + x_c$$
$$u_{new2} = Au + x_c$$

so $A = P, b = x_c$

Expression1 means ellipsoid is first removed to the zero point and then change to a unit circle by $P^{-1}$, and then moves back to $x_c$.

Expression2 means the unit circle at the zero point is first changed to a Ellipsoids by matrix $A$ and then moves back to $x_c$.

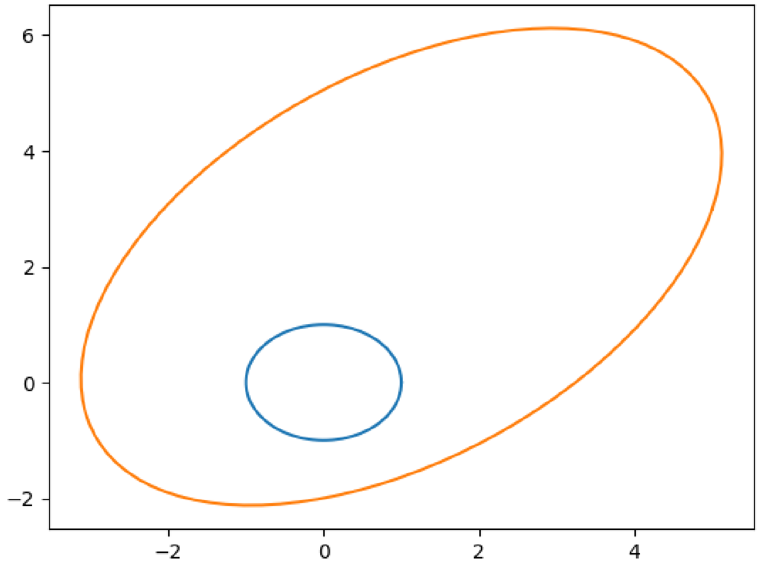
In [76]:

```
theta = np.linspace(0,2*np.pi,100)
fig = plt.figure
X = np.vstack([np.cos(theta),np.sin(theta)])
plt.plot(X[0,:],X[1,:])
P = np.array([[4,1],[1,4]])
xc = np.array([1,2])
X_new = np.zeros_like(X)
print(X_new.shape)
for i in range(np.size(X,1)):
    x = X[:,i].reshape(2,1)
    X_new[:,i] = x.T.dot(P)+xc.T
plt.plot(X_new[0,:],X_new[1,:])
```

```
(2, 100)
```

Out[76]:

```
[<matplotlib.lines.Line2D at 0x7f551f3fafd0>]
```




## Linear System Solution

By $||Ax_1 - Ax_2|| \leq ||A|| \cdot ||x_1 - x_2||$ (Cauthy-Schiwz),

$Ax(t)$ is Lipschitz continuous in x

when $u(t)$ is piecewise continuous in $t$

$f(x,t) = \dot{x}(t) = Ax(t) + Bu(t)$ is Lipschitz continuous in $x$ and piecewise continuous in t

The ODE equation has a unique solution.

If $x(t) = \cdots$ is a solution, it must satisfy I.C. condition and the system equation.

First, when $t = 0$, $x(t) = x_0$. So I.C is satisfied.

Second, with

$$e^{At} = I + At + \frac{A^2t^2}{2!} + \cdots$$

$$\frac{d}{dt}\int_0^t F(\tau)d\tau = F(t)$$

$$\begin{aligned}
\dot{x}(t) =& \frac{d}{dt}e^{At}x_0 + \frac{d}{dt}\int_0^t e^{A(t-\tau)}Bu(\tau)d\tau \\
=& \frac{d}{dt}(I + At + \frac{A^2t^2}{2!} + \cdots)x_0 + \frac{d}{dt}[e^{At}\int_0^t e^{-A\tau}Bu(\tau)d\tau] \\
=& Ae^{At}x_0 + Ae^{At}\int_0^t e^{-A\tau}Bu(\tau)d\tau + Bu(t) \\
=& A[e^{At}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau] + Bu(t) \\
=& Ax(t) + Bu(t)
\end{aligned}$$

So it is a unique solution.

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