Assignment3

Q1

Exercise 3.20 Consider the high-wheel bicycle of Figure 3.26, in which the diameter of the front wheel is twice that of the rear wheel. Frames {a} and {b} are attached respectively to the centers of the wheels, and frame {c} is attached to the top of the front wheel. Assuming that the bike moves forward in the $\hat{y}$-direction, find T_{ac} as a function of the front wheel's rotation angle θ (assume $\theta = 0$ at the instant shown in the figure).

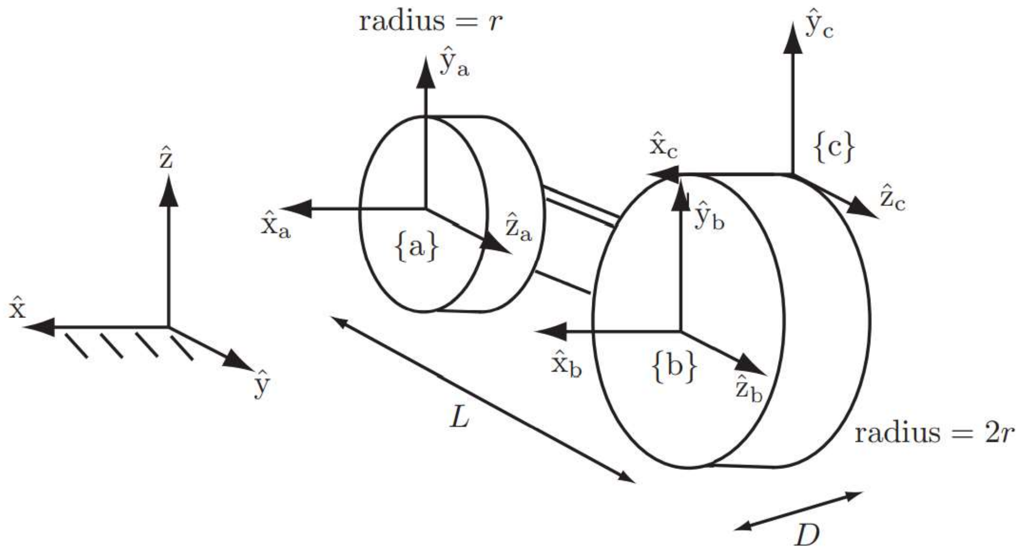


Figure 3.26: A high-wheel bicycle.

When the front wheel rotates θ, the rear wheel rotates 2θ. Assume there is a frame {b'} at the center of front wheel parallel to the world frame and a frame{a'} at the center of rear wheel parallel to the world frame.

$$
^{b'}T_c = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
^{a'}T_{b'} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & L \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
^{a}T_{a'} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cos(2\theta) & -sin(2\theta) & 0 \\ 0 & sin(2\theta) & cos(2\theta) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\begin{aligned}
^{a}T_c &= ^{a}T_{a'}^{a'}T_{b'}^{b'}T_c \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & cos(2\theta) & -sin(2\theta) & Lcos(2\theta) - 2rsin(2\theta) \\ 0 & sin(2\theta) & cos(2\theta) & Lsin(2\theta) + 2rcos(2\theta) \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

Q2

Exercise 3.23 Two toy cars are moving on a round table as shown in Figure 3.29. Car 1 moves at a constant speed v_1 along the circumference of the table, while car 2 moves at a constant speed v_2 along a radius; the positions of the two vehicles at $t = 0$ are shown in the figures.

(a) Find T_{01} and T_{02} as a function of t.

(b) Find T_{12} as a function of t.

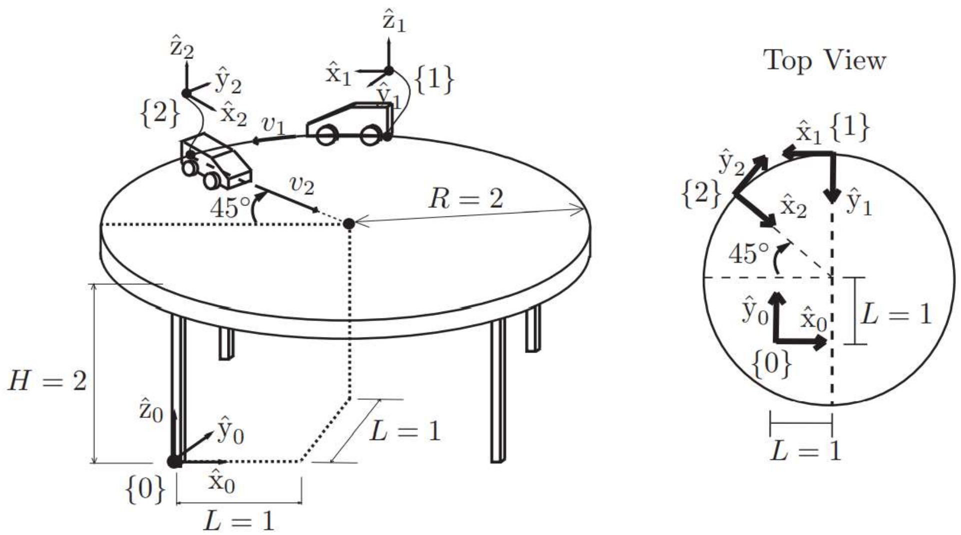


Figure 3.29: Two toy cars on a round table.

(1) $$ \begin{aligned} ^{0}T_{2} &=

$$
\begin{bmatrix} \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 1-\sqrt{2}+v_1t\frac{\sqrt{2}}{2} \\ \frac{-\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & 1+\sqrt{2}-v_1t\frac{\sqrt{2}}{2} \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

\

^{0}T_{1} &=

$$
\begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 3-v_2t \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

\

\end{aligned}

$$(2)$$

^{1}T_{2} = {^{1}T_{0}} {^{0}T_{2}}

$$And$$

^1T_{0} =

$$
\begin{bmatrix} {}^1R_0^T & -{}^1R_0{}^T{}^1p_0 \\ 0 & 1 \end{bmatrix}
$$

$$

Q3

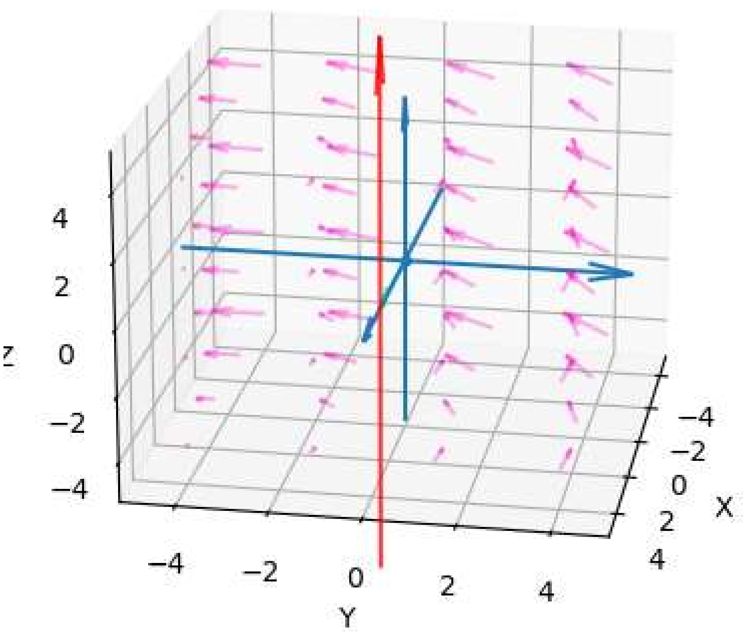
Exercise 3.26 Draw the screw axis for which $q = (3, 0, 0)$, $\hat{s} = (0, 0, 1)$, and $h = 2$.

In [ ]:

```
import numpy as np
import matplotlib.pyplot as plt
%matplotlib inline
q = np.array([3,0,0])
s = np.array([0,0,1])
h = 2
theta_d = 2
w = s*theta_d
v = -np.cross(s*theta_d,q)+h*s*theta_d
print('rotation direction={} in deg'.format(s*180/np.pi))
print('axis translation={}'.format(q))
print('screw pitch = {}'.format(h))
print('rotation speed={}'.format(theta_d))
L = 4
fig = plt.figure()
ax = plt.axes(projection='3d')
```

```
ax.grid()
ax.quiver(-5,0,0,1,0,0,normalize=True,length=10,arrow_length_ratio=0.1)
ax.quiver(0,-5,0,0,1,0,normalize=True,length=10,arrow_length_ratio=0.1)
ax.quiver(0,0,-5,0,0,1,normalize=True,length=10,arrow_length_ratio=0.1)
ax.quiver(0,0,0,q[0],q[1],q[2],normalize=True,length=np.linalg.norm(q),color=(0,
ax.quiver(q[0]-s[0]*2*L,q[1]-s[1]*2*L,q[2]-s[2]*2*L,s[0],s[1],s[2],
          normalize=True,length=4*L,arrow_length_ratio=0.1,color=(1, 0, 0, 0.8))
ax.scatter(0,0,0,'linewidth',10)
ax.set_xlabel('X')
ax.set_xlim3d(-5, 5)
ax.set_ylabel('Y')
ax.set_ylim3d(-5, 5)
ax.set_zlabel('Z')
ax.set_zlim3d(-5, 5)
x = np.linspace(-4,4,4)
y = np.linspace(-4,4,4)
z = np.linspace(-4,4,4)
x_,y_,z_ = np.meshgrid(x,y,z)
x_ = np.reshape(x_,(1,np.size(x_)))
y_ = np.reshape(y_,(1,np.size(y_)))
z_ = np.reshape(z_,(1,np.size(z_)))
p = np.vstack([x_,y_,z_])
v_ = np.zeros((3,np.size(x_)))
for i in range(np.size(x_)):
    v_[:,i] = v+np.cross(w,p[:,i])
    ax.quiver(x_[:,i],y_[:,i],z_[:,i],v_[0,i],v_[1,i],v_[2,i],normalize=True,ler
              color=(1,0,0.8,0.3),arrow_length_ratio=0.4)
ax.view_init(elev=20, azim=10)
plt.show()
```

```
rotation direction=[ 0.          0.          57.29577951] in deg
axis translation=[3 0 0]
screw pitch = 2
rotation speed=2
```



Q4

Exercise 3.28 Assume that the space-frame angular velocity is $\omega_s = (1, 2, 3)$

for a moving body with frame $\{b\}$ at

$$R = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix}$$

relative to the space frame $\{s\}$. Calculate the body's angular velocity ω_b in $\{b\}$.

ω_b is the angular velocity in {b} frame, and ω_s is the angular velocity expressed in {s} frame. The rotation matrix ${}^sR_b$ is given.

In [ ]:
```
R = np.array([[0,-1,0],[0,0,-1],[1,0,0]])
ws = np.array([1,2,3])
wb = np.matmul(R.T,ws.T)
print(wb)
```

```
[ 3 -1 -2]
```

Q5

Exercise 5.5 Referring to Figure 5.17, a rigid body, shown at the top right, rotates about the point (L, L) with angular velocity $\dot{\theta} = 1$.

(a) Find the position of point P on the moving body relative to the fixed reference frame {s} in terms of θ.

(b) Find the velocity of point P in terms of the fixed frame.

(c) What is T_{sb}, the configuration of frame {b}, as seen from the fixed frame {s}?

(d) Find the twist of T_{sb} in body coordinates.

(e) Find the twist of T_{sb} in space coordinates.

(f) What is the relationship between the twists from (d) and (e)?

(g) What is the relationship between the twist from (d) and $\dot{P}$ from (b)?

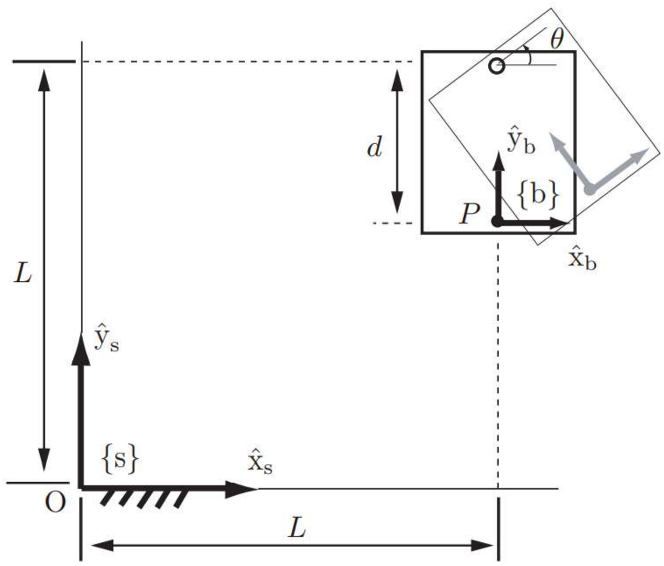


Figure 5.17: A rigid body rotating in the plane.

(h) What is the relationship between the twist from (e) and $\dot{P}$ from (b)?

(a)

Position of point P

$$^{s}p = ^{s}\overrightarrow{OP} = [L + dsin\theta, L - dcos\theta]^T$$

(b)

Velocity of point P

$$\frac{d^{s}p}{dt} = [dcos\theta, dsin\theta, 0]^T$$

(c)

The configuration of {b} seen from {s} is

$$
{}^sT_b = \begin{bmatrix} {}^sR_b & {}^sp_b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} c_\theta & -s_\theta & 0 & L + ds_\theta \\ s_\theta & c_\theta & 0 & L - dc_\theta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

(d)

Let the point r be the body fixed point in rotation axis.

The velocity of body fixed point corresponds to O_b is

$$
{}^bv_p = {}^b v_r + {}^b \omega \times {}^b \overrightarrow{ro_b} = [d, 0, 0]^T
$$

The twist of ${}^sT_b$ in {b} is

$$
{}^b\nu = [0, 0, 1, d, 0, 0]
$$

(e)

The velocity of body fixed point corresponds to O_s is

$$
{}^sv_{os} = {}^s v_r + {}^s \omega \times {}^s \overrightarrow{ro_s} = [L, -L, 0]^T
$$

The twist of ${}^sT_b$ in {s} is

$$
{}^s\nu = [0, 0, 1, L, -L, 0]
$$

(f)

$$
{}^s\omega = {}^sR_b {}^b\omega \tag{1}
$$

$$
{}^sv_{ob} = {}^sR_b {}^bv_{ob} + [{}^sp_b] {}^sR_b {}^b\omega \tag{2}
$$

(g)

The linear velocity part of twist ${}^b\nu$ is $\dot{p}$ expressed in {b} frame.

(h)

The linear velocity part of twist ${}^s\nu$ is $\dot{p} - {}^s\omega \times p$.

Q6

Exercise 5.6 Figure 5.18 shows a design for a new amusement park ride. A rider sits at the location indicated by the moving frame {b}. The fixed frame {s} is attached to the top shaft as shown. The dimensions indicated in the figure are $R = 10$ m and $L = 20$ m, and the two joints each rotate at a constant angular velocity of 1 rad/s.

(a) Suppose $t = 0$ at the instant shown in the figure. Find the linear velocity v_b and angular velocity ω_b of the rider as functions of time t. Express your answer in frame-{b} coordinates.

(b) Let p be the linear coordinates expressing the position of the rider in {s}. Find the linear velocity $\dot{p}(t)$.

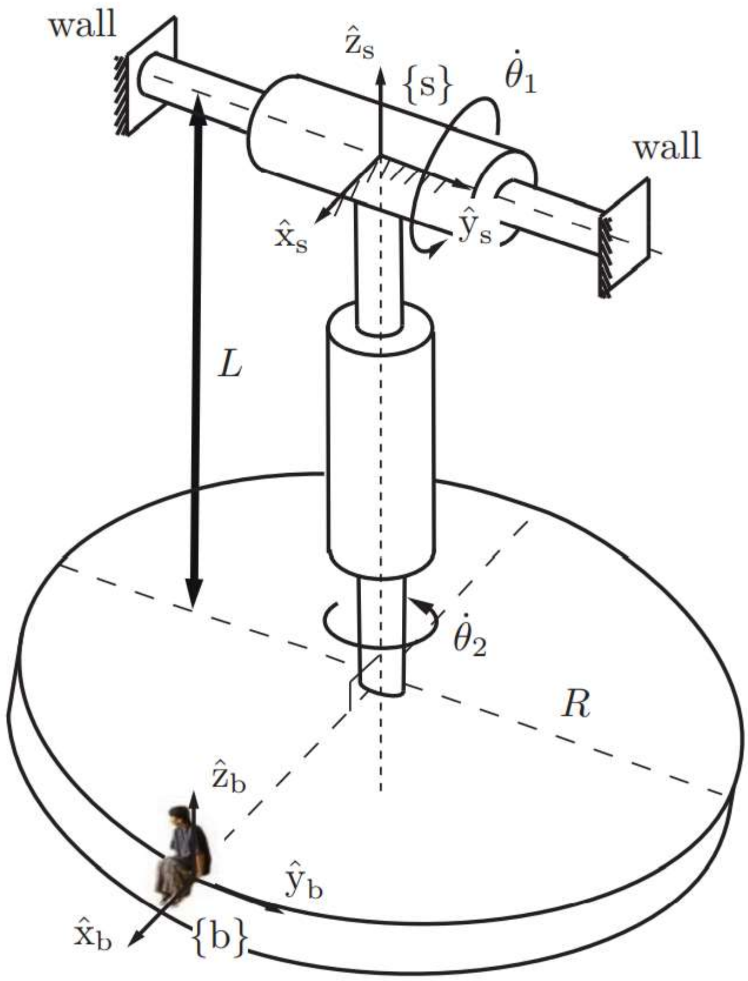


Figure 5.18: A new amusement park ride.

(1) The first step is to calculate the transformation matrix ${}^sT_b$. Assume at $t = 0$, $\hat{x_b}$ has the same direction with $\hat{x_s}$. And there is a coordinate system {c} at the center of the plate and fixed to the penperdicular shaft.

$$
{}^sT_c = \begin{bmatrix} cos(t) & 0 & sin(t) & -10sin(t) \\ 0 & 1 & 0 & 0 \\ -sin(t) & 0 & cos(t) & -10cos(t) \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

And

$$
{}^{c}T_b = \begin{bmatrix} cos(t) & -sin(t) & 0 & 20cos(t) \\ sin(t) & cos(t) & 0 & 20sin(t) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

Then

$$
{}^{s}T_b = {}^{s}T_c {}^{c}T_b \begin{bmatrix} cos^2(t) & -cos(t)sin(t) & sin(t) & 20cos^2(t) - 10sin(t) \\ sin(t) & cos(t) & 0 & 20sin(t) \\ -cos(t)sin(t) & sin^2(t) & cos(t) & -20sin(t) * cos(t) - 10cos(t) \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

And

$$
{}^{s}T_b^{-1} {}^{s}\dot{T}_b = \begin{bmatrix} [\omega_b] & v_b \\ 0 & 0 \end{bmatrix}
$$

$$
{}^{s}T_b^{-1} = \begin{bmatrix} c^2 & s & -cs & -20 \\ -cs & c & s^2 & 0 \\ s & 0 & c & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{s}\dot{T}_b = \begin{bmatrix} -2cs & s^2 - c^2 & c & -40cs - 10c \\ c & -s & 0 & 20c \\ s^2 - c^2 & 2cs & -s & 20s^2 - 20c^2 + 10s \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

$$
{}^{s}T_b^{-1} {}^{s}\dot{T}_b = \begin{bmatrix} 0 & -1 & c & -10c \\ 1 & 0 & -s & 10s + 20 \\ -c & s & 0 & -20c \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

In [ ]:

```
from sympy import*
t = symbols('t')
tsc = Matrix([[cos(t),0,sin(t),-10*sin(t)],[0,1,0,0],[-sin(t),0,cos(t),-10*cos(t
tcb = Matrix([[cos(t),-sin(t),0,20*cos(t)],[sin(t),cos(t),0,20*sin(t)],[0,0,1,0]
tsb = tsc.multiply(tcb)
print("Tsb=")
pprint(tsb)
tsb_inv = tsb.inv()
print("Tsb -1 = ")
pprint(simplify(tsb_inv))
diff_tsb = diff(tsb,t)
print("diff(Tsb) = ")
pprint(simplify(diff_tsb))
Vb = tsb_inv.multiply(diff_tsb)
print("Vb = ")
pprint(simplify(Vb))
```

Tsb=

$$\begin{bmatrix} \cos^{2}(t) & -\sin(t)\cdot\cos(t) & \sin(t) & -10\cdot\sin(t) + 20\cdot\cos^{2}(t) \\ \sin(t) & \cos(t) & 0 & 20\cdot\sin(t) \\ -\sin(t)\cdot\cos(t) & \sin^{2}(t) & \cos(t) & -20\cdot\sin(t)\cdot\cos(t) - 10\cdot\cos(t) \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Tsb -1 =

$$\begin{bmatrix} \cos^{2}(t) & \sin(t) & \frac{-\sin(2\cdot t)}{2} & -20 \\ \frac{-\sin(2\cdot t)}{2} & \cos(t) & \sin^{2}(t) & 0 \\ \sin(t) & 0 & \cos(t) & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

diff(Tsb) =

$$\begin{bmatrix} -\sin(2\cdot t) & -\cos(2\cdot t) & \cos(t) & -10\cdot(4\cdot\sin(t) + 1)\cdot\cos(t) \\ \cos(t) & -\sin(t) & 0 & 20\cdot\cos(t) \\ -\cos(2\cdot t) & \sin(2\cdot t) & -\sin(t) & 10\cdot\sin(t) - 20\cdot\cos(2\cdot t) \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Vb =

$$\begin{bmatrix} 0 & -1 & \cos(t) & -10\cdot\cos(t) \\ 1 & 0 & -\sin(t) & 10\cdot\sin(t) + 20 \\ -\cos(t) & \sin(t) & 0 & -20\cdot\cos(t) \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\omega_b = [sin(t), cos(t), 1]^T$$
$$v_b = [-10cos(t), 10sin(t) + 20, -20cos(t)]^T$$

(2) $\dot{p}$ can be calculated directly using ${}^sT_b$. Or we can use twist

$${}^s\dot{T}_b {}^sT_b^{-1} = \begin{bmatrix} [\omega_s] & v_s \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -cos(t) & 1 & 0 \\ cos(t) & 0 & -sin(t) & 0 \\ -1 & sin(t) & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

And

$$\omega_s = [sin(t), 1, cos(t)]^T v_s = [0, 0, 0]^T$$

The v_s is not the linear velocity, it represents the instantaneous velocity of the point on the body currently at the fixed-frame origin.

$$v_s = \dot{p} - \omega_s \times p = 0$$
$$\dot{p} = [\omega_s]p = [-40cs - 10c, 20c, -20c^2 + 20s^2 + 10s]$$

In [ ]:
```
Vs = diff_tsb.multiply(tsb_inv)
print("Vs = ")
pprint(simplify(Vs))
p = tsb[0:3,3]
print("p = ")
pprint(simplify(p))
diff_p = Vs[0:3,0:3].multiply(p)
print("diff_p = ")
pprint(simplify(diff_p))
```

```
Vs =
⎡  0     -cos(t)     1     0⎤
⎢                           ⎥
⎢cos(t)     0     -sin(t)  0⎥
⎢                           ⎥
⎢ -1     sin(t)      0     0⎥
⎢                           ⎥
⎣  0        0        0     0⎦
p =
⎡                     2    ⎤
⎢-10⋅sin(t) + 20⋅cos (t) ⎥
⎢                          ⎥
⎢      20⋅sin(t)          ⎥
⎢                          ⎥
⎣-10⋅sin(2⋅t) - 10⋅cos(t)⎦
diff_p =
⎡-10⋅(4⋅sin(t) + 1)⋅cos(t)⎤
⎢                          ⎥
⎢        20⋅cos(t)        ⎥
⎢                          ⎥
⎣ 10⋅sin(t) - 20⋅cos(2⋅t) ⎦
```