The Mordell-Faltings theorem

https://seasawher.github.io/kitamado/

@seasawher

2019 年 4 月 5 日

Some basics of algebraic number theory

Proposition 1.4

> **quotation.** Let $\{\beta_1, \cdots, \beta_n\}$ be the dual basis of $\{\alpha_1, \cdots, \alpha_n\}$ with respect to $(\ ,\ )_{\mathrm{Tr}_{K/\mathbb{Q}}}$. Then, for any $x \in O_K$, we have $x = (x, \alpha_1)_{\mathrm{Tr}_{K/\mathbb{Q}}}\beta_1 + \cdots + (x, \alpha_n)_{\mathrm{Tr}_{K/\mathbb{Q}}}\beta_n$.

Proof. Since the trace form $(\ ,\ )_{\mathrm{Tr}_{K/\mathbb{Q}}}$ is degenerate, $(\ , \alpha_i)_{\mathrm{Tr}_{K/\mathbb{Q}}}$ are linearly independent in $\mathrm{Hom}_{\mathbb{Q}}(K, \mathbb{Q}) = K^*$ and form $\mathbb{Q}$-basis of K^*. Let $p_i \colon K \to \mathbb{Q}$ be a projection map such that $p_i(x_1\alpha_1 + \cdots + x_n\alpha_n) = x_i$. There are $\beta_{ij} \in \mathbb{Q}$ such that

$$p_i = \sum_{j=1}^{n} (\ , \alpha_j)_{\mathrm{Tr}_{K/\mathbb{Q}}}\beta_{ij}.$$

This means $id_K = \sum_i \alpha_i p_i = \sum_j (\ , \alpha_j)_{\mathrm{Tr}_{K/\mathbb{Q}}} \sum_i \alpha_i\beta_{ij}$, then we get $O_K \subset \mathbb{Z}\beta_1 + \cdots + \mathbb{Z}\beta_n$ for $\beta_j = \sum_i \alpha_i\beta_{ij}$. Since $id_K = \sum_j (\ , \alpha_j)_{\mathrm{Tr}_{K/\mathbb{Q}}}\beta_j$, β_j are basis of K and $\mathbb{Z}\beta_1 + \cdots + \mathbb{Z}\beta_n$ is a free $\mathbb{Z}$-module. □

Lemma 1.16

> **quotation.** Because $(O_K)_P$ is a principal ideal domain, $(O_{K'})_P$ is a free $(O_K)_P$-module of rank $[K' : K]$.

Proof. It remain to be answered. □

Lemma 1.16

> **quotation.** Thus
>
> $$\begin{aligned} \dim_{O_K/P} O_{K'}/PO_{K'} &= \dim_{O_K/P} (O_{K'})_P/P(O_{K'})_P \\ &= \dim_{O_K/P} ((O_K)_P/P(O_K)_P) \otimes_{(O_K)_P} (O_{K'})_P \end{aligned}$$

Proof. We set $A = O_K, A' = O_{K'}$. Then we get

$$\begin{aligned}
A'/PA' &\cong A' \otimes_A A/P \\
&\cong A' \otimes_A \operatorname{Frac} A/P \\
&\cong A' \otimes_A \operatorname{Coker}(PA_P \to A_P) \\
&\cong \operatorname{Coker}(A' \otimes_A PA_P \to A' \otimes_A A_P) \\
&\cong (A')_P/P(A')_P \\
(A')_P/P(A')_P &\cong A' \otimes_A \operatorname{Coker}(PA_P \to A_P) \\
&\cong A' \otimes_A A_P/PA_P \\
&\cong (A' \otimes_A A_P) \otimes_{A_P} A_P/PA_P \\
&\cong (A')_P \otimes_{A_P} A_P/PA_P.
\end{aligned}$$

□

Adjacent to Lemma 1.17

> **quotation.** We take a integral basis $\{\omega_1, \cdots, \omega_n\}$ of O_K, we denote by $\{\beta_1, \cdots, \beta_n\}$ the dual basis with respect to $(\ ,\ )_{\mathrm{Tr}_{K/\mathbb{Q}}}$. Then we have $\mathcal{M} = \mathbb{Z}\beta_1 + \cdots + \mathbb{Z}\beta_n$.

Proof. It remain to be answered. □

Adjacent to Lemma 1.17

> **quotation.** The difference of K is defined by $\mathcal{D}_K = \mathcal{M}^{-1}$. Because $O_K \subset \mathcal{M}$, we have $\mathcal{D}_K \subset O_K$, so $\mathcal{D}_K$ is an ideal of O_K.

Proof. $O_K = \mathcal{M}\mathcal{M}^{-1} = \mathcal{D}_K\mathcal{M} \supset \mathcal{D}_K O_K \supset \mathcal{D}_K$. □