The Mordell-Faltings theorem

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2019 年 7 月 10 日

1 Some basics of algebraic number theory

Lemma 1.3

> Recall that $(\ ,\ )_{\mathrm{Tr}_{K/\mathbb{Q}}}$ is non-degenerate if the Gramm matrix with respect to one (and hence any) basis of L over F is invertible.

Proof. Almost trivial. Try to prove it. $\square$

Proposition 1.4

> Let $\{\beta_1, \cdots, \beta_n\}$ be the dual basis of $\{\alpha_1, \cdots, \alpha_n\}$ with respect to $(\ ,\ )_{\mathrm{Tr}_{K/\mathbb{Q}}}$. Then, for any $x \in O_K$, we have $x = (x, \alpha_1)_{\mathrm{Tr}_{K/\mathbb{Q}}}\beta_1 + \cdots + (x, \alpha_n)_{\mathrm{Tr}_{K/\mathbb{Q}}}\beta_n$.

Proof. Since the trace form $(\ ,\ )_{\mathrm{Tr}_{K/\mathbb{Q}}}$ is nondegenerate, $K \to K^*$ s.t. $x \mapsto (\cdot, x)_{\mathrm{Tr}_{K/\mathbb{Q}}}$ is a isomorphism. Let $p_i\colon K \to \mathbb{Q}$ be a projection map such that $p_i(x_1\alpha_1 + \cdots + x_n\alpha_n) = x_i$. Then, we set β_j the preimage of p_j.

$\square$

Lemma 1.7

> To see this, we take $t \in P(O_K)_P$ with $t \notin P^2(O_K)_P$.

remark. From Nakayama's lemma.

Adjacent to Lemma 1.8

> For a nonzero prime ideal P of O_K, we set $P \cap \mathbb{Z} = (p)$, where p is a prime of Z. Because O_K is a free Z-module of rank $[K : \mathbb{Q}]$, O_K/P is a finite extension of $\mathbb{Z}/(p)$ with degree at most $[K : \mathbb{Q}]$.

Proof. There is a canonical surjection $O_K/pO_K \to O_K/P$, so we get $\#(O_K/P) \leq \#(O_K/pO_K)$. But we obtain $O_K/pO_K \cong O_K \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$. Since O_K is a free $\mathbb{Z}$-module of rank $n = [K : \mathbb{Q}]$, we conclude $O_K/pO_K \cong (\mathbb{Z}/p\mathbb{Z})^n$. So, $\#(O_K/P) \leq \#(O_K/pO_K) = p^n$. $\square$

Lemma 1.8

> $$\bigoplus_{i=1}^{r} O_K/P_i^{e_i} = \bigoplus_{i=1}^{r} (O_K/P_i^{e_i})_{P_i}$$

Proof. Because $O_K/P_i^{e_i}$ is a local ring with maximal ideal $P_i/P_i^{e_i}$. $\square$

Adjacent to Theorem 1.9

> we consider the value $\sqrt{\det(\langle e_i, e_j\rangle)}$.

remark. Why we get $\det(\langle e_i, e_j\rangle)$? Apply Gram-Schmidt orthonormalization.

Adjacent to Theorem 1.9

> Then $\mathrm{vol}(M, \langle,\rangle)$ is equal to the volume of the n-dimensional parallelpiped Π spanned by $e_1, \cdots, e_n$,

Proof. Let $F\colon (V, \langle,\rangle) \to \mathbb{R}^n$ be an isometric isomorphism. Then, we generate

$$\begin{aligned}\mathrm{vol}(M, \langle,\rangle)^2 &= \det(\langle e_i, e_j\rangle)\\ &= \det(\langle Fe_i, Fe_j\rangle)\end{aligned}$$

We set $E = (Ee_1, \cdots, Fe_n)$. $E \in M_n(\mathbb{R})$. Then we get $(\langle Fe_i, Fe_j\rangle)_{i,j} = {}^tEE$, and $\mathrm{vol}(M, \langle,\rangle) = |\det E|$. From Yukie[3] Theorem 4.9.1, $|\det E| = \mathrm{vol}(\Pi)$. $\square$

Proposition 1.11

> The form $\langle,\rangle_K$ is an inner product on V.

remark. $\langle,\rangle_K$ is trivially an inner product on K. Why should we show this?

Let S be a $\mathbb{Q}$ vector space and $\langle,\rangle$ a inner product on S. Then, bilinear form extended to $S \otimes_{\mathbb{Q}} \mathbb{R}$ may not be an inner product. For example, set $S = \mathbb{Q}[\sqrt{2}]$ and $\langle x, y\rangle = xy$.

Lemma 1.12

> $\#(O_K/I)$ is finite. Then I is a free $\mathbb{Z}$-module of rank n.

Proof. $I \subset O_K$ is a free $\mathbb{Z}$-module. Since $\#(O_K/I)$ is finite, we get $\forall x \in K\ \exists n \in \mathbb{Z}$ s.t. $nx \in I$. So we obtain $I \otimes_{\mathbb{Z}} \mathbb{Q} = K$. The rank of I is n. $\square$

Lemma 1.16

> We have $[K' : K] = e_1 f_1 + \cdots + e_r f_r$.

Proof. See the proof of Prop 1.4. We obtain $O_{K'} \subset O_K\beta_1 \oplus \cdots \oplus O_K\beta_n$ for some $\beta_i \in K'$. That implies there is an injection such that $O_{K'} \to \bigoplus_i O_K$. Because localization is a flat module, we get $(O_{K'})_P \subset (O_K)_P\beta_1 \oplus \cdots \oplus (O_K)_P\beta_n$. Since $(O_K)_P$ is a PID, $(O_{K'})_P$ is a free $(O_K)_P$-module. The rank is $[K':K]$ because

$$(O_{K'})_P \otimes_{(O_K)_P} K = (O_{K'} \otimes_{O_K} (O_K)_P) \otimes_{(O_K)_P} K = O_{K'} \otimes_{O_K} K = K'.$$

Thus, as a O_K/P module,

$$\begin{aligned}
O_{K'}/PO_{K'} &\cong O_K/P \otimes_{O_K} O_{K'} \\
&\cong (O_K/P \otimes_{O_K} (O_K)_P \otimes_{(O_K)_P} (O_K)_P) \otimes_{O_K} O_{K'} \\
&\cong (O_K/P \otimes_{O_K} (O_K)_P) \otimes_{(O_K)_P} (O_{K'})_P \\
&\cong \bigoplus_{[K':K]} (O_K/P \otimes_{O_K} (O_K)_P) \\
&\cong \bigoplus_{[K':K]} O_K/P.
\end{aligned}$$

Then it follows that

$$\begin{aligned}
\#(O_K/P)^{[K':K]} &= \#(O_{K'}/PO_{K'}) \\
&= \prod_i \#(O_{K'}/{P'_i}^{e_i}) \\
&= \prod_i \#(O_{K'}/P'_i)^{e_i} \\
&= \prod_i \#(O_K/P)^{e_i f_i}.
\end{aligned}$$

Thus $[K':K] = \sum_i e_i f_i$. □

Adjacent to Lemma 1.17

> We take a integral basis $\{\omega_1, \cdots, \omega_n\}$ of O_K, we denote by $\{\beta_1, \cdots, \beta_n\}$ the dual basis with respect to $(\ ,\ )_{\mathrm{Tr}_{K/\mathbb{Q}}}$. Then we have $\mathcal{M} = \mathbb{Z}\beta_1 + \cdots + \mathbb{Z}\beta_n$.

Proof. See the note of Prop 1.4. □

Lemma 1.17

> Indeed, because $\#(O_K/\mathcal{D}_K) = \#(\mathcal{M}/O_K)$,

Proof. See Yukie[1] Proposition 1.8.6. □

Theorem 1.18

> Then we have
>
> $$\left|D_{K/\mathbb{Q}}\right| \leq \prod_{p\in S} p^{n-1+n\log_p(n)}.$$

Proof. We may assume that $S = \{p \in \mathbb{Z} \mid p \text{ is ramified}\}$. Set $B = O_K$ and $I = D_K$.

Step 1 Let $p \in \mathbb{Z}$ be a prime number. Then B_p and I_p are free $\mathbb{Z}_p$-module of rank n. So there is a matrix $C \in M_n(\mathbb{Z}_p) \cap GL_n(\mathbb{Q}_p)$ such that the following diagram

$$\begin{tikzcd} I_p \arrow[r] \arrow[d] & B_p \arrow[d] \\ \mathbb{Z}_p^n \arrow[r, "C"] & \mathbb{Z}_p^n \end{tikzcd}$$

commute. Then

$$\begin{aligned} \#(B/I \otimes_{\mathbb{Z}} \mathbb{Z}_p) &= \#(\operatorname{Coker} C) \\ &= \#(\mathbb{Z}_p/(\det C)\mathbb{Z}_p) \\ &= \#(\widehat{\mathbb{Z}}_p/(\det C)\widehat{\mathbb{Z}}_p) \qquad \text{(See Yukie[1] Proposition 1.2.13)} \\ &= \#(B/I \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}_p). \end{aligned}$$

Step 2 It follows that

$$\begin{aligned} B/I \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}_p &\cong B/I \otimes_B B \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}_p \\ &\cong B/I \otimes_B \bigoplus_i \widehat{B}_{P_i} \qquad \text{(See Yukie[1] Theorem 1.3.23)} \\ &\cong \bigoplus_i \widehat{B}_{P_i}/P_i^{\operatorname{ord}_{P_i}(I)}\widehat{B}_{P_i} \\ &\cong \bigoplus_i B/P_i^{\operatorname{ord}_{P_i}(I)} \end{aligned}$$

Step 3 Set $J = I \cap \mathbb{Z}$. Because B/I is finitely generated $\mathbb{Z}$-module, we get

$$\operatorname{Supp}_{\mathbb{Z}}(B/I) = V(\operatorname{ann}_{\mathbb{Z}}(B/I)) = V(J).$$

See Matsumura[4] adjacent to Theorem 4.4 if you do not understand the first equation. And for any prime number $p \in \mathbb{Z}$, then we obtain

$$\begin{aligned} p \notin \operatorname{Supp}_{\mathbb{Z}}(B/I) &\iff B/I \otimes_{\mathbb{Z}} \mathbb{Z}_p = 0 \\ &\iff \#(B/I \otimes_{\mathbb{Z}} \mathbb{Z}_p) = 1 \\ &\iff \forall i\ \#(B/P_i^{\operatorname{ord}_{P_i}(I)}) = 1 \\ &\iff \operatorname{ord}_{P_i}(I) = 0 \\ &\iff p \text{ is unramified} \end{aligned}$$

Thus we conclude $V(J) = \operatorname{Supp}_{\mathbb{Z}}(B/I) = S$.

Step 4 Then we get

$$\begin{aligned} \#(B/I \otimes_{\mathbb{Z}} \mathbb{Z}_p) &= \prod_i \#(B/P_i^{\operatorname{ord}_{P_i}(I)}) \\ &= \prod_i \#(B/P_i)^{\operatorname{ord}_{P_i}(I)} \\ &= \prod_i \#(\mathbb{Z}/p)^{f_i \operatorname{ord}_{P_i}(I)}. \end{aligned}$$

So we conclude $\log_p(\#(B/I \otimes_{\mathbb{Z}} \mathbb{Z}_p)) \leq n - 1 + n \log_p(n)$.

Step 5 Recall that $J = \mathrm{ann}_{\mathbb{Z}}(B/I)$. Then we get

$$\begin{aligned}
B/I &\cong (B/I)/J(B/I) \\
&\cong \bigoplus_{p\in S}(B/I)/p^e(B/I) && (e \text{ depends on } p) \\
&\cong \bigoplus_{p\in S} B/(p^eB+I) \\
&\cong \bigoplus_{p\in S} B/(p^eB+I)\otimes_{\mathbb{Z}}\mathbb{Z}_p \\
&\cong \bigoplus_{p\in S} B_p/(p^eB_p+I_p) \\
&\cong \bigoplus_{p\in S} B_p/(JB_p+I_p) \\
&\cong \bigoplus_{p\in S} B_p/I_p
\end{aligned}$$

Now we conclude that

$$\left|D_{K/\mathbb{Q}}\right| = \#(B/I) = \prod_{p\in S}\#(B_p/I_p) \leq \prod_{p\in S} p^{n-1+n\log_p(n)}.$$

□

2 Theory of heights

Theorem 2.3

> We set $n = [K : \mathbb{Q}]$. Let $\{\omega_1, \cdots, \omega_n\}$ be the integral basis of O_K. Then $\{x\omega_1, \cdots, x\omega_n\}$ is a basis of V.

Proof. There is a $c_{ij} \in \mathbb{Z}$ such that $x\omega_i = \sum_j c_{ij}\omega_i$. Set $C = (c_{ij}) \in M_n(\mathbb{Z})$. Then $\det C = N_{K/\mathbb{Q}}(x) \neq 0$, so we get $C \in GL_n(\mathbb{Q})$. And we obtain the assertion. $\square$

Proposition 2.5

$$h_K(x) \leq \sum_{\sigma \in K(\mathbb{C})} \log \left(\max_{1 \leq i \leq n} \{|x_i|_\sigma\} \right).$$

remark. **Misprint.** Add $1/[K : \mathbb{Q}]$ into the right.

Proposition 2.6

> for any $x \in \overline{\mathbb{Q}}^n$.

remark. **Misprint.** Exclude the case $x = 0$.

Proposition 2.8

> We consider two morphisms $\phi_1 \colon X \to \mathbb{P}^{m_1}$ and $\phi_2 \colon X \to \mathbb{P}^{m_2}$ over $\overline{\mathbb{Q}}$. If $\phi_1^*\mathcal{O}_{\mathbb{P}^{m_1}} \cong \phi_2^*\mathcal{O}_{\mathbb{P}^{m_2}}$, then there is a constant C such that, for any $x \in X(\overline{\mathbb{Q}})$,
> $$|h_{\phi_1}(x) - h_{\phi_2}(x)| \leq C.$$

Proof. Remark that $\mathcal{O}_{\mathbb{P}^{m_1}}(1)$ is a Serre's twisted sheaf. See Bosch[2] 9.2/Definition 3. or Hartshorne[6] section 2.5 Adjacent to Proposition 5.12. We set $L = \phi_1^*\mathcal{O}_{\mathbb{P}^{m_1}}$ and set $k = \overline{\mathbb{Q}}$ and $\mathscr{F} = \mathcal{O}_{\mathbb{P}^{m_1}}(1)$. Since X is a projective variety over k and L is an invertible sheaf on X, so $H^0(X, L) = \Gamma(X, L)$ is a k-vector space of finite dimension. (See Hartshorne[6] section 2.5 Theorem 5.19. and Hartshorne[6] section 2.4 Prop 4.10)

Let $\{t_0, \cdots, t_m\}$ be a basis of $H^0(X, L)$ and let $X_0, \cdots, X_{m_1}$ be the homogenous coordinates of $\mathbb{P}^{m_1}$. Note that each X_i is a global section of $\mathscr{F}$. And we set $s_i = \phi_1^* X_i \in H^0(X, L)$, where $\phi_1^* X_i$ is the image of $X_i \in H^0(\mathbb{P}^m, \mathscr{F})$ by the canonical map $\mathscr{F} \to \phi_{1*}\phi_1^*\mathscr{F}$. It follows from Hartshorne[6] section 2.7 Theorem 7.1 that $s_0, \cdots, s_{m_1}$ generate L. Because for any $x \in X$ the each germ $(s_i)_x$ is a linear combination of $(t_j)_x$, so $t_0, \cdots, t_m$ generate L.

There is a morphism $\phi\colon X \to \mathbb{P}^m$ such that $L \cong \phi^*\mathcal{O}_{\mathbb{P}^m}(1)$ and $s_i = \phi^* X_i$ under this isomorphism. See Hartshorne[6] section 2.7 Theorem 7.1(b). There is another explanation on what ϕ is. For any $x \in X$, we can consider the germ $(t_i)_x \in L_x$. Denote $(t_i)_x$ by $t_i(x)$. Since L is a line bundle, $L_x \cong \mathcal{O}_{X,x}$. Then we define the map $\phi\colon X \to \mathbb{P}^m$ by $\phi(x) = (t_0(x), \cdots, t_m(x))$. Note that there is a scalar ambiguity in choice of morphism $L_x \to k$. If $\forall i\ t_i(x) = 0$, then $(t_i)_x$ cannot generate L_x, which is a contradiction. Thus for any $x \in X$, there is an index i such that $t_i(x) \neq 0$.

Then, the rest of the proof is almost trivial. □

Theorem 2.9

First, suppose that L is globally generated.

remark. What "globally generated" means? We say L is globally generated iff there is an exact sequence $\bigoplus_I O_X \to L \to 0$. Even if L is an invertible sheaf, L is not necessarily globally generated. For example, set $X = \mathbb{P}^m$, $L = \mathcal{O}_{\mathbb{P}^m}(-1)$. Since $\Gamma(X, L) = 0$, L is not globally generated.

Theorem 2.9

Let

$$\phi_{|L|}\colon X \to \mathbb{P}(H^0(X, L))$$

be a morphism associated to the complete linear system $|L|$. We set $h_L = h_{\phi_{|L|}}$.

Proof. Note that we want to get $h_L \in \mathrm{Func}(X)/B(X)$, which is not contained in $\mathrm{Func}(X)$.

What is a $\mathbb{P}(H^0(X, L))$? I think it is isomorphic to $\mathbb{P}^m$ by taking a basis of $H^0(X, L)$.

Set $k = \overline{\mathbb{Q}}$. Since L be a globally generated line bundle on X, there is a basis $s_0, \cdots, s_m$ of $H^0(X, L)$ which generate L. Then we get a map $\phi_L\colon X \to \mathbb{P}^m$ such that $\phi_L^*\mathcal{O}_{\mathbb{P}^m}(1) \cong L$. We define h_L by $h_L = h_{\phi_L}$. □

Theorem 2.9

Then $s_i \otimes t_j$ induces a morphism $\phi\colon X \to \mathbb{P}^N$ such that $\phi^*(O_{\mathbb{P}^N(1)}) \cong L_1 \otimes L_2$.

remark. How $s_i \otimes t_j \in H^0(X, L_1) \otimes_k H^0(X, L_2)$ define an element of $H^0(X, L_1 \otimes L_2)$? Let $\mathscr{F}$ be a presheaf defined by $\mathscr{F}(U) = \Gamma(U, L_1) \otimes_{\mathcal{O}_X(U)} \Gamma(U, L_2)$. Then there is a canonical morphism $\mathscr{F} \to L_1 \otimes L_2$ since $L_1 \otimes L_2$ is the sheafification of $\mathscr{F}$. So we can see $s_i \otimes t_j \in H^0(X, L_1 \otimes L_2)$.

We denote the image of $s_i \otimes t_l$ by $s_i t_j \in H^0(X, L_1 \otimes L_2)$. Why $\{s_i t_j\}$ generate $L_1 \otimes L_2$? Take a stalk.

Theorem 2.9

> tell us that $L \otimes A^n$ is globally generated for any sufficiently large n.

remark. The ampleness of A implies that

$$\begin{aligned}
&\exists n_1 \ \text{ s.t. } \ n \geq n_1 \Rightarrow L \otimes A^n \text{ is globally generated} \\
&\exists n_2 \ \text{ s.t. } \ n \geq n_2 \Rightarrow A^n \text{ is globally generated}
\end{aligned}$$

Then we set $n = \max_i\{n_i\}$.

Theorem 2.9

> Then, modulo $B(X)$, we have
> $$h_{f^*(L)} = h_{f^*(C) \otimes f^*(C)^{-1}}$$

remark. See Görtz Wedhorn[5] Remark 7.10.

Theorem 2.9

> Then by (1), h_L must be equal to $h_{L_1} - h_{L_2}$ modulo $B(X)$.

remark. Let $\sigma\colon \{\text{line bundles }\} \to \mathrm{Func}(X)/B(X)$ be a map which satisfies the properties (1), (2), (3). By (3), for globally generated line bundle L, we get $\sigma_L = h_L$. Because $\Gamma(X, O_X) = k$, we obtain $\sigma_{O_X} = 0$. Thus (1) implies that $\sigma_L = h_L$ for general line bundle L.

Proposition 2.10

> Let B be the Zariski closed subset of X defined by the ideal sheaf
> $$\mathrm{Im}(H^0(X, L) \otimes L^{-1} \to O_X).$$

remark. What is the morphism $H^0(X, L) \otimes L^{-1} \to O_X$? Note that there is a canonical morphism $f^* f_* L \to L$ where $f\colon X \to \operatorname{Spec} k$ is a k-scheme structure. Note that $f_* L$ is isomorphic to $\widetilde{H^0(X, L)}$. We denote this canonical morphism $f^* f_* L \to L$ by

$$H^0(X, L) \otimes O_X \to L.$$

This is surjective if L is globally generated.

In general, we define $V \otimes_k O_X$ for k-module V, by setting

$$V \otimes_k O_k = f^{-1}\widetilde{V} \otimes_{f^{-1}O_{\operatorname{Spec} k}} O_X = f^*\widetilde{V}.$$

Then we get $H^0(X, L) \otimes L^{-1} \to O_X$ by tensoring L^{-1}.

Proposition 2.10

> Then $\{ss_i\}$ are linearly independent elements of $H^0(X,L)$.

remark. What are ss_i ? Note that there is a canonical morphism

$$H^0(X,L)\otimes H^0(X,L_2)\to H^0(X,L\otimes L_2)\cong H^0(X,L_1)$$

Thus I guess ss_i is the image of $s\otimes s_i$.

Moreover, why ss_i are linearly independent? It suffices to show that the morphism of k-module

$$s\colon H^0(X,L_2)\to H^0(X,L_1)$$

is injective.

We prepare the following lemma.

lemma. Let X be an integral scheme and let L be a line bundle on X. Assume that $s\in H^0(X,L)$ is not zero. Then for any $x\in X$, $s_x\neq 0$ in L_x.

Proof. Assume that there is a $z\in X$ such that $s_z=0$. We want to show $s=0\in H^0(X,L)$. Since L is invertible, there is an open affine covering $X=\bigcup_{i\in I}U_i$ such that

$$U_i=\operatorname{Spec}A_i,\ L|_{U_i}\cong\widetilde{A_i}$$

On the other hand, $s_z=0$ implies that there is an open subset $U\subset X$ such that $s|_U=0$ and $z\in U$. Since X is integral, $U\cap\operatorname{Spec}A_i\neq\emptyset$. Thus there is a $g_i\in A_i\setminus\{0\}$ such that $\emptyset\neq D(g_i)\subset U\cap\operatorname{Spec}A_i$. Then $s|_{D(g_i)}=0$ in $\Gamma(D(g_i),L)\cong A_{i g_i}$. Note that each A_i is an integral domain because X is integral. Thus we get $\forall i\ s|_{U_i}=0$ because $A_i\to A_{i g_i}$ is injective. It follows from the sheaf axiom that $s=0\in\Gamma(X,L)$. □

Then, we can prove the injectivity of $s\colon O_X\to L$. First, by the lemma, $0\to O_X\xrightarrow{s}L$ is exact. Since L_2 is flat, $0\to L_2\xrightarrow{s}L_1$ is exact. Since global section is left exact, we get $0\to H^0(X,L_2)\xrightarrow{s}H^0(X,L_1)$ is exact.

Proposition 2.10

> Let $s_1,\cdots,s_n$ be a basis of $H^0(X,L)$. $\cdots$
> Because $B=\{x\in X\mid s_1(x)=\cdots=s_n(x)=0\}$.

Proof. Why $B=\{x\in X\mid s_1(x)=\cdots=s_n(x)=0\}$? I guess

$$\begin{aligned}
B&=\operatorname{Supp}\operatorname{Coker}(H^0(X,L)\otimes L^{-1}\to O_X)\\
&=\operatorname{Supp}\operatorname{Coker}(H^0(X,L)\otimes O_X\to L)\otimes L^{-1} && \text{(right exactness of tensor)}\\
&=\operatorname{Supp}\operatorname{Coker}(H^0(X,L)\otimes O_X\to L)\cap\operatorname{Supp}L^{-1}\\
&=\operatorname{Supp}\operatorname{Coker}(H^0(X,L)\otimes O_X\to L)
\end{aligned}$$

Thus we get

$$\begin{aligned} x \in B &\iff \mathrm{Coker}(H^0(X,L)\otimes O_X \to L)_x \neq 0 \\ &\iff \forall s \in H^0(X,L)\ x_x \in \mathfrak{m}_x L_x \\ &\iff s(x) = 0 \\ &\iff s_1(x) = \cdots = s_n(x) = 0 \end{aligned}$$

□

Cor 2.25

> Let A be an abelian variety of dimension g over F.
>
> (1) For any integer n, $[n]\colon A \to A$ is a finite and flat morphism of degree n^{2g}.
>
> (2) The abelian group $A(\overline{F})$ is divisible, i.e., for any $x \in A(\overline{F})$ and for any positive integer n, there is a $y \in A(\overline{F})$ with $[n](y) = x$.

Proof.

(1) L を even かつ ample な line bundle とする。$[n]^*L = L^{n^2}$ より $[n]^*L$ も ample である。$[n]^*L$ が ample ということは、(閉とは限らない) 埋め込み

$$\psi_{|\widetilde{L}|}\colon A \to \mathbb{P}(\Gamma(A,\widetilde{L})^\vee)$$

がある。ただし $\widetilde{L} = [n]^*L$ であり、$^\vee$ は双対空間を表す。このとき次の図式は可換。

$$\begin{array}{ccc} A & \xrightarrow{\psi_{|\widetilde{L}|}} & \mathbb{P}(\Gamma(A,\widetilde{L})^\vee) \\ \uparrow & & \uparrow \\ \mathrm{Ker}[n] & \xrightarrow{\psi_{|\mathrm{Ker}[n]|}} & \mathbb{P}(\Gamma(\mathrm{Ker}[n],\widetilde{L}|_{\mathrm{Ker}[n]})^\vee) \end{array}$$

ここで $\widetilde{L}|_{\mathrm{Ker}[n]}$ は自明なので $\mathrm{Ker}[n]$ の既約性分への分解を $\mathrm{Ker}[n] = \coprod_{i\in I} P_i$ とすると、各成分 P_i の $\psi_{|\mathrm{Ker}[n]|}$ による像は $\mathbb{P}^0(F)$ に含まれる。つまり一点である。したがって、$\psi_{|\widetilde{L}|}$ は埋め込みなので各 P_i は一点である。よって $\dim \mathrm{Ker}[n] = 0$ である。

また $[n]$ は projective variety の間の射なので projective であり、とくに固有である。$\dim \mathrm{Ker}[n] = 0$ であることをいま示したが、$[n]$ は準同型なのですべての点の fiber の次元が等しい。ゆえに固有かつすべての点での fiber の次元がゼロなので $[n]$ は finite である。とくに fiber の次元が任意の点で等しいので $[n]$ は flat である。

次に degree について考える。ここでの degree は交点数を用いて定義される。ample line bundle L について

$$L^{\cdot g} = (L.\cdots.L)$$

と定義する。右辺の L は g 個ある。(Reference: 石井志保子「特異点入門」) 交点数は多重線形なので

$$([n]^*L)^{\cdot g} = (L^{n^2}.\cdots.L^{n^2}) = n^{2g}L^{\cdot g}$$

が従う。よって degree の定義から $\deg[n] = n^{2g}$ である。

(2) $[n] \colon A \to A$ は固有なのでその像 $[n](A)$ は閉部分多様体である。また $[n]$ は flat なので

$$\dim_0 A + \dim_0 \operatorname{Ker}[n] = \dim_0 [n](A)$$

が成り立つ。よって $\dim \operatorname{Ker}[n] = 0$ より $\dim A = \dim_0 A = \dim_0 [n](A) = \dim [n](A)$ である。真閉部分多様体は次元が落ちるはずなので $A = [n](A)$ がわかる。

□

Remark 2.27

> If D is an effective Cartier divisor on an abelian variety A, then $|2D|$ is base point free. In particular, D is nef.

remark. まず用語について解説する。D が base point free とは、rational map

$$\psi_{|D|} \colon A \dashrightarrow \mathbb{P}(\Gamma(A, D)^\vee)$$

が A 全体で定義されることである。言い換えれば、

$$\{x \in A \mid \forall 0 \neq s \in \Gamma(A, D)\ s(x) = 0\} = \emptyset$$

ということである。また D が nef(数値的正、ネフ) とは

$$\forall C \text{ irreducible curve } (D.C) \geq 0$$

(交点数がゼロ以上) として定義される。

さて effective な Cartier divisor D について、$2D$ が base point free ならば D が nef であることを確かめよう。$2D$ が nef ならあきらかに D も nef なので、はじめから D が base point free だと仮定して D が nef だといえばよい。

いま D は effective なので既約曲線 C が $C \not\subset D$ である限り、$(D.C) \geq 0$ となる。交点数は線形同値なもの同士を入れ替えても不変なので、どんな C についてもある D と線形同値な D' があって $C \not\subset D'$ となることをいえばよい。

ハイリホーで示す。ある既約曲線 C が存在して、すべての $D' \sim D$ なる D' について $C \subset D'$ であったとする。いま $\psi_{|D|}$ の定義から、任意の超曲面 $H \subset \mathbb{P}(\Gamma(A, D)^\vee)$ に対して $D \sim \psi_{|D|}^* H$ である。H はある大域切断 $s \in \Gamma(A, D)$ により $H = \{l \mid l(s) = 0\}$ と表せる。そこでこれを H_s とおく。このとき仮定から

$$\forall s\ C \subset \psi_{|D|}^* H_s$$

であるが、$\psi_{|D|}^* H_s = \{x \in A \mid s(x) = 0\}$ であったため、D が base point free であったことより $C = \emptyset$ となるしかない。これは矛盾である。

Remark 2.27

> For an $a \in A(\overline{F})$, we define the morphism $T_a \colon A_{\overline{F}} \to \mathrm{Pic}(A_{\overline{F}})$ by
> $$T_a \colon x \mapsto x + a.$$
> For any given line bundle L on $A_{\overline{F}}$, we define the map $\lambda_L \colon A(\overline{F}) \to \mathrm{Pic}(A_{\overline{F}})$ by
> $$\lambda \colon x \mapsto T_x^* L \otimes L^{-1}.$$
> Cor (Theorem of Square)
> As groups, the map λ_L is a homomorphism from $A(\overline{F})$ to $\mathrm{Pic}(A_{\overline{F}})$.

> If D is an effective Cartier divisor on an abelian variety A, then $|2D|$ is base point free. In particular, D is nef. Indeed, identifying D with the corresponding Weil divisor, we write $D+a$ and $D-a$ for $T_a(D)$ and $T_{-a}(D)$ respectively. For any $x \in A(\overline{F})$, we choose a point $a \notin \mathrm{Supp}(D-x) \cup \mathrm{Supp}(D+x)$. Then $x \notin \mathrm{Supp}(D-a) \cup \mathrm{Supp}(D+a)$. Further, the theorem of the square implies that $(D-a)+(D+a) \sim 2D$. Thus $|2D|$ is base point free.

remark. Hartshorne[6] section 2.6 命題 6.11 の、X が整分離的 Noether スキームで局所分解的なものとすると、Cartier divisor と Weil 因子が同型になるということを踏まえて同一視をする。ここで Abelian 多様体 A は smooth で、したがって局所環が regular であり、正則局所環は UFD であることから A は局所分解的であることに気を付ける。

閉点 $x \in A(\overline{F})$ が任意に与えられたとする。x を台に含まないような、$2D$ と線形同値な effective 因子の存在をいえば、$|2D|$ が base point free であることが従う。そういう因子を構成しよう。

点 $a \notin \mathrm{Supp}(D-x) \cup \mathrm{Supp}(-D+x)$ をとる。ここで因子 $D-x$ は像 $T_{-x}(D)$ を意味し、$-D+x$ は逆元をとる写像を i として $T_x(i(D))$ を意味する。このとき $a \notin \mathrm{Supp}(D-x)$ より $x \notin \mathrm{Supp}(D-a)$ であり、$a \notin \mathrm{Supp}(-D+x)$ より $x \notin \mathrm{Supp}(D+a)$ である。（本文には誤植がある）このとき Theorem of square により

$$\begin{aligned}(D-a)+(D+a) &= T_{-a}(D) + T_a(D) \\ &= \lambda_D(-a) + \lambda_D(a) + 2D \\ &= 2D\end{aligned}$$

だから $2D$ と $(D+a)+(D-a)$ は線形同値。$(D+a)+(D-a)$ はあきらかに effective なので $|2D|$ が base point free であることがいえた。

Cor 2.28

Let A be an abelian variety, and let D be an effective Cartier divisor on A. We set $L = O_X(D)$. (In particular, $|2D|$ is base point free) Then the following are equivalent.

(1) $\text{Ker}(\lambda_L)$ is a finite subgroup of $A(\overline{F})$.

(2) A morphism $\Phi\colon A \to \mathbb{P}(H^0(A, L^2))$ associated to the complete linear system $|2D|$ is a finite morphism onto its image.

(3) L is ample.

PROOF. Properties (2) and (3) hold over F if and only if those hold over $\overline{F}$, so we may assume that $F = \overline{F}$.

(1) $\Rightarrow$ (2) Suppose that Φ maps a projective curve C on A to a single point. We are going to deduce a contradiction by showing that $\text{Ker}(\lambda_L)$ contains $C - C = \{x_2 - x_1 \mid x_1, x_2 \in C(F)\}$. We write $D = \sum_{i=1}^{r} a_i D_i$ with $a_i > 0$ and prime divisors D_i on A.

We claim that either $(C + x) \cap D_i = \emptyset$ or $C + x \subset D_i$ for any i and for any $x \in A(F)$. Because D_i is an effective Cartier divisor on an abelian variety, it is nef by Remark 2.27. In particular, $(D_i \cdot C) \geq 0$. It follows from

$$0 = (D \cdot C) = a_1(D_1 \cdot C) + \cdots + a_r(D_r \cdot C),$$

that $(D_i \cdot C) = 0$. Further, because $C + x$ is algebraically equivalent to C, we have $(D_i \cdot C + x) = (D_i \cdot C) = 0$. Thus we obtain the claim.

Next, we claim that $D_i = D_i + x_1 - x_2$ for any $x_1, x_2 \in C(F)$. Indeed, if $y \in D_i$ is a closed point, then both $C - x_1 + y$ and D_i contain y. Thus $C - x_1 + y \subset D_i$ by the above argument. It follows that $x_2 - x_1 + y \in D_i$, so $y \in D_i + x_1 - x_2$. Thus $D_i \subset D_i + x_1 - x_2$. By switching x_1 and x_2 in the above argument, we have $D_i \supset D_i + x_1 - x_2$. Hence we obtain the claim.

We set $L_i = O_A(D_i)$. Then $L = L_1^{\otimes a_1} \otimes \cdots \otimes L_r^{\otimes a_r}$. By the above claim, we have $T_{x_2 - x_1}^*(L_i) = L_i$ for each i. It follows that

$$\begin{aligned} T_{x_2-x_1}^*(L) &= T_{x_2-x_1}^*(L_1)^{\otimes a_1} \otimes \cdots \otimes T_{x_2-x_1}^*(L_r)^{\otimes a_r} \\ &= L_1^{\otimes a_1} \otimes \cdots \otimes L_r^{\otimes a_r} \\ &= L. \end{aligned}$$

Thus $x_2 - x_1 \in \text{Ker}(\lambda_L)$, and so $C - C \subset \text{Ker}(\lambda_L)$. This is a contradiction.

(2)$\Rightarrow$(3) Let E be any coherent O_A-module on A. We are going to show that $E \otimes L^{2n}$ is globally generated for any sufficiently large $2n$. We write X for the image $\Phi(A)$ of Φ. Because $\Phi_*(E)$ is coherent on X and $O_X(1)$ is ample, there is an $n_0 > 0$ such that

$$H^0(X, \Phi_*(E) \otimes O_X(n)) \otimes O_X \to \Phi_*(E) \otimes O_X(n)$$

is surjective for any $n \geq n_0$ (See Hartshorne p.153). Pulling back by Φ, we find that

$$(*) \qquad H^0(X, \Phi_*(E) \otimes O_X(n)) \otimes O_A \to \Phi^*\Phi_*(E) \otimes \Phi^* O_X(n)$$

is also surjective.

On the other hand, it follows from $\Phi^* O_X(n) \cong L^{2n}$ and the projection formula that $\Phi_*(E \otimes L^{2n}) \cong \Phi_*(E) \otimes O_X(n)$. Further, as Φ is a finite morphism, the canonical morphism $\Phi^*\Phi_*(E) \to E$ is surjective. Thus the surjectivity of $(*)$ implies the surjectivity of

$$H^0(A, E \otimes L^{2n}) \otimes O_A \to E \otimes L^{2n}$$

for every $n \geq n_0$. Thus L^2 is ample, so L is ample.

(3)⇒(1) Let $p_i \colon A \times A \to A$ denote the i-th projection. First, we note the equality

$$\operatorname{Ker} \lambda_L = \left\{ x \in A(F) \;\middle|\; (m_A^* L^{-1} \otimes p_1^* L)|_{A \times \{x\}} \text{ is trivial} \right\}.$$

In particular, the seesaw theorem tells us that $\operatorname{Ker} \lambda_L$ endowed with the reduced induced scheme structure is regarded as a closed subgroup scheme of A. We denote by B the connected component of $\operatorname{Ker}(\lambda_L)$ containing the identity. Then B is an abelian subvariety of A. We are going to show that $\dim B = 0$.

We set $L' := (m_A^* L^{-1} \otimes p_1^* L \otimes p_2^* L)|_{B \times B}$ on $B \times B$. Because $L'|_{B \times \{x\}} = (m_A^* L^{-1} \otimes p_1^* L)|_{B \times \{x\}} \otimes p_2^* L|_{B \times \{x\}}$ is trivial for any $x \in B$ and $L'|_{\{0\} \times B}$ is trivial, the seesaw theorem implies that L' is trivial. Pulling back L' by $([1]_B, [-1]_B) \colon B \to B \times B$, we obtain that $L \otimes [-1]_A^* L$ is trivial on B. On the other hand, because L is ample on A, $L \otimes [-1]_A^* L$ is ample on A. Thus $\dim B = 0$, and we conclude that $\operatorname{Ker}(\lambda_L)$ is a finite set.

query.

(1) 「Properties (2) and (3) hold over F if and only if those hold over $\overline{F}$」とあります。これは何故なのでしょうか。

(2) (1)⇒(2) の証明で、背理法（対偶）が使われているようです。(2) が成り立たず、Φ が finite でなかったとして、$\operatorname{Ker} \lambda_L$ が無限集合であることを示す証明になっているように見えます。そこで Φ が finite でなければある x があって、$\Phi^{-1}(x)$ に含まれる projective な曲線 C があるということを使っているようですが、どうしてそういう C が存在するのでしょうか。
Φ のすべてのファイバーの次元が 0 だとすると、Φ は proper なので finite になります。よって Φ が finite でないという仮定から、Φ のある点でのファイバーの次元は 1 以上であり、したがってあるファイバーに含まれるような射影曲線 C がとれる、という証明で問題ないですか？

(3) 「Further, because $C + x$ is algebraically equivalent to C」とあります。algebraically equivalent とはどういう意味（定義）で、そしてどうしてそれが分かるのでしょうか。

(3)

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