The Mordell-Faltings theorem

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Some basics of algebraic number theory

Lemma 1.3

> **quotation.** Recall that $(\ ,\ )_{\mathrm{Tr}_{K/\mathbb{Q}}}$ is non-degenerate if the Gramm matrix with respect to one (and hence any) basis of L over F is invertible.

Proof. Almost trivial. Try to prove it. □

Proposition 1.4

> **quotation.** Let $\{\beta_1, \cdots, \beta_n\}$ be the dual basis of $\{\alpha_1, \cdots, \alpha_n\}$ with respect to $(\ ,\ )_{\mathrm{Tr}_{K/\mathbb{Q}}}$. Then, for any $x \in O_K$, we have $x = (x, \alpha_1)_{\mathrm{Tr}_{K/\mathbb{Q}}}\beta_1 + \cdots + (x, \alpha_n)_{\mathrm{Tr}_{K/\mathbb{Q}}}\beta_n$.

Proof. Since the trace form $(\ ,\ )_{\mathrm{Tr}_{K/\mathbb{Q}}}$ is degenerate, $(\ , \alpha_i)_{\mathrm{Tr}_{K/\mathbb{Q}}}$ are linearly independent in $\mathrm{Hom}_{\mathbb{Q}}(K, \mathbb{Q}) = K^*$ and form $\mathbb{Q}$-basis of K^*.

Let $p_i \colon K \to \mathbb{Q}$ be a projection map such that $p_i(x_1\alpha_1 + \cdots + x_n\alpha_n) = x_i$. There are $\beta_{ij} \in \mathbb{Q}$ such that

$$p_i = \sum_{j=1}^{n} (\ , \alpha_j)_{\mathrm{Tr}_{K/\mathbb{Q}}}\beta_{ij}.$$

This means $id_K = \sum_i \alpha_i p_i = \sum_j (\ , \alpha_j)_{\mathrm{Tr}_{K/\mathbb{Q}}} \sum_i \alpha_i\beta_{ij}$, then we get $O_K \subset \mathbb{Z}\beta_1 + \cdots + \mathbb{Z}\beta_n$ for $\beta_j = \sum_i \alpha_i\beta_{ij}$. Since $id_K = \sum_j (\ , \alpha_j)_{\mathrm{Tr}_{K/\mathbb{Q}}}\beta_j$, β_j are basis of K and $\mathbb{Z}\beta_1 + \cdots + \mathbb{Z}\beta_n$ is a free $\mathbb{Z}$-module.

We set $c_{ij} = (\alpha_i, \alpha_j)_{\mathrm{Tr}_{K/\mathbb{Q}}}$. And we get

$$\delta_{ik} = p_i(\alpha_k) = \sum_j \beta_{ij}c_{jk}.$$

That means $I = \beta c$ by setting $\beta = (\beta_{ij}), c = (c_{ij})$, so β is symmetric i.e. $\beta_{ij} = \beta_{ji}$.

Then, we get

$$\begin{aligned}
(\beta_j, \alpha_k)_{\mathrm{Tr}_{K/\mathbb{Q}}} &= \sum_i \beta_{ij}(\alpha_i, \alpha_k)_{\mathrm{Tr}_{K/\mathbb{Q}}} \\
&= \sum_i \beta_{ji}(\alpha_i, \alpha_k)_{\mathrm{Tr}_{K/\mathbb{Q}}} \\
&= p_j(\alpha_k) \\
&= \delta_{jk}.
\end{aligned}$$

This is suggestive of orthogonality. □

Lemma 1.16

> **quotation.** Because $(O_K)_P$ is a principal ideal domain, $(O_{K'})_P$ is a free $(O_K)_P$-module of rank $[K' : K]$.

Proof. It remains to be answered. □

Lemma 1.16

> **quotation.** Thus
>
> $$\begin{aligned} \dim_{O_K/P} O_{K'}/PO_{K'} &= \dim_{O_K/P}(O_{K'})_P/P(O_{K'})_P \\ &= \dim_{O_K/P}((O_K)_P/P(O_K)_P) \otimes_{(O_K)_P} (O_{K'})_P \end{aligned}$$

Proof. We set $A = O_K, A' = O_{K'}$. Then we get

$$\begin{aligned} A'/PA' &\cong A' \otimes_A A/P \\ &\cong A' \otimes_A \mathrm{Frac}\, A/P \\ &\cong A' \otimes_A \mathrm{Coker}(PA_P \to A_P) \\ &\cong \mathrm{Coker}(A' \otimes_A PA_P \to A' \otimes_A A_P) \\ &\cong (A')_P/P(A')_P \\ (A')_P/P(A')_P &\cong A' \otimes_A \mathrm{Coker}(PA_P \to A_P) \\ &\cong A' \otimes_A A_P/PA_P \\ &\cong (A' \otimes_A A_P) \otimes_{A_P} A_P/PA_P \\ &\cong (A')_P \otimes_{A_P} A_P/PA_P. \end{aligned}$$

□

Adjacent to Lemma 1.17

> **quotation.** We take a integral basis $\{\omega_1, \cdots, \omega_n\}$ of O_K, we denote by $\{\beta_1, \cdots, \beta_n\}$ the dual basis with respect to $(\ ,\ )_{\mathrm{Tr}_{K/\mathbb{Q}}}$. Then we have $\mathcal{M} = \mathbb{Z}\beta_1 + \cdots + \mathbb{Z}\beta_n$.

Proof. See the note of Prop 1.4. □

Adjacent to Lemma 1.17

> **quotation.** The difference of K is defined by $\mathcal{D}_K = \mathcal{M}^{-1}$. Because $O_K \subset \mathcal{M}$, we have $\mathcal{D}_K \subset O_K$, so $\mathcal{D}_K$ is an ideal of O_K.

Proof. $O_K = \mathcal{M}\mathcal{M}^{-1} = \mathcal{D}_K\mathcal{M} \supset \mathcal{D}_K O_K \supset \mathcal{D}_K$. □

Lemma 1.17

> **quotation.** Indeed, because $\#(O_K/\mathcal{D}_K) = \#(\mathcal{M}/O_K)$,

Proof. See Yukie[1] Proposition 1.8.6. □

Adjacent to 2.4

> **quotation.** In other words, the absolute (logarithmic) Weil height is not invariant under linear coordinate changes. This is why it is sometimes called the naive height.

query. Why they want the height to be invariant? Is there non-constant invariant height?

参考文献

[1] 雪江明彦『整数論 2 代数的整数論の基礎』(日本評論社, 2013)