The Mordell-Faltings theorem

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1 Some basics of algebraic number theory

Lemma 1.3

> Recall that $(\ , \ )_{\mathrm{Tr}_{K/\mathbb{Q}}}$ is non-degenerate if the Gramm matrix with respect to one (and hence any) basis of L over F is invertible.

Proof. Almost trivial. Try to prove it. □

Proposition 1.4

> Let $\{\beta_1, \cdots, \beta_n\}$ be the dual basis of $\{\alpha_1, \cdots, \alpha_n\}$ with respect to $(\ , \ )_{\mathrm{Tr}_{K/\mathbb{Q}}}$. Then, for any $x \in O_K$, we have $x = (x, \alpha_1)_{\mathrm{Tr}_{K/\mathbb{Q}}}\beta_1 + \cdots + (x, \alpha_n)_{\mathrm{Tr}_{K/\mathbb{Q}}}\beta_n$.

Proof. Since the trace form $(\ , \ )_{\mathrm{Tr}_{K/\mathbb{Q}}}$ is nondegenerate, $K \to K^*$ s.t. $x \mapsto (\cdot, x)_{\mathrm{Tr}_{K/\mathbb{Q}}}$ is a isomorphism. Let $p_i \colon K \to \mathbb{Q}$ be a projection map such that $p_i(x_1\alpha_1 + \cdots + x_n\alpha_n) = x_i$. Then, we set β_j the preimage of p_j.

□

Lemma 1.7

> To see this, we take $t \in P(O_K)_P$ with $t \notin P^2(O_K)_P$.

remark. From Nakayama's lemma.

Adjacent to Lemma 1.8

> For a nonzero prime ideal P of O_K, we set $P \cap \mathbb{Z} = (p)$, where p is a prime of Z. Because O_K is a free Z-module of rank $[K : \mathbb{Q}]$, O_K/P is a finite extension of $\mathbb{Z}/(p)$ with degree at most $[K : \mathbb{Q}]$.

Proof. There is a canonical surjection $O_K/pO_K \to O_K/P$, so we get $\#(O_K/P) \leq \#(O_K/pO_K)$. But we obtain $O_K/pO_K \cong O_K \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$. Since O_K is a free $\mathbb{Z}$-module of rank $n = [K : \mathbb{Q}]$, we conclude $O_K/pO_K \cong (\mathbb{Z}/p\mathbb{Z})^n$. So, $\#(O_K/P) \leq \#(O_K/pO_K) = p^n$. □

Lemma 1.8

> $$\bigoplus_{i=1}^{r} O_K/P_i^{e_i} = \bigoplus_{i=1}^{r} (O_K/P_i^{e_i})_{P_i}$$

Proof. Because $O_K/P_i^{e_i}$ is a local ring with maximal ideal $P_i/P_i^{e_i}$. $\square$

Adjacent to Theorem 1.9

> we consider the value $\sqrt{\det(\langle e_i, e_j\rangle)}$.

remark. Why we get $\det(\langle e_i, e_j\rangle)$? Apply Gram-Schmidt orthonormalization.

Adjacent to Theorem 1.9

> Then $\mathrm{vol}(M, \langle,\rangle)$ is equal to the volume of the n-dimensional parallelpiped Π spanned by $e_1, \cdots, e_n$,

Proof. Let $F\colon (V, \langle,\rangle) \to \mathbb{R}^n$ be an isometric isomorphism. Then, we generate

$$\begin{aligned}\mathrm{vol}(M, \langle,\rangle)^2 &= \det(\langle e_i, e_j\rangle)\\ &= \det(\langle Fe_i, Fe_j\rangle)\end{aligned}$$

We set $E = (Ee_1, \cdots, Fe_n)$. $E \in M_n(\mathbb{R})$. Then we get $(\langle Fe_i, Fe_j\rangle)_{i,j} = {}^tEE$, and $\mathrm{vol}(M, \langle,\rangle) = |\det E|$. From Yukie[3] Theorem 4.9.1, $|\det E| = \mathrm{vol}(\Pi)$. $\square$

Proposition 1.11

> The form $\langle,\rangle_K$ is an inner product on V.

remark. $\langle,\rangle_K$ is trivially an inner product on K. Why should we show this?

Let S be a $\mathbb{Q}$ vector space and $\langle,\rangle$ a inner product on S. Then, bilinear form extended to $S \otimes_{\mathbb{Q}} \mathbb{R}$ may not be an inner product. For example, set $S = \mathbb{Q}[\sqrt{2}]$ and $\langle x, y\rangle = xy$.

Lemma 1.12

> $\#(O_K/I)$ is finite. Then I is a free $\mathbb{Z}$-module of rank n.

Proof. $I \subset O_K$ is a free $\mathbb{Z}$-module. Since $\#(O_K/I)$ is finite, we get $\forall x \in K\ \exists n \in \mathbb{Z}$ s.t. $nx \in I$. So we obtain $I \otimes_{\mathbb{Z}} \mathbb{Q} = K$. The rank of I is n. $\square$

Lemma 1.16

> We have $[K' : K] = e_1 f_1 + \cdots + e_r f_r$.

Proof. See the proof of Prop 1.4. We obtain $O_{K'} \subset O_K\beta_1 \oplus \cdots \oplus O_K\beta_n$ for some $\beta_i \in K'$. That implies there is an injection such that $O_{K'} \to \bigoplus_i O_K$. Because localization is a flat module, we get $(O_{K'})_P \subset (O_K)_P\beta_1 \oplus \cdots \oplus (O_K)_P\beta_n$. Since $(O_K)_P$ is a PID, $(O_{K'})_P$ is a free $(O_K)_P$-module. The rank is $[K' : K]$ because

$$(O_{K'})_P \otimes_{(O_K)_P} K = (O_{K'} \otimes_{O_K} (O_K)_P) \otimes_{(O_K)_P} K = O_{K'} \otimes_{O_K} K = K'.$$

Thus, as a O_K/P module,

$$\begin{aligned} O_{K'}/PO_{K'} &\cong O_K/P \otimes_{O_K} O_{K'} \\ &\cong (O_K/P \otimes_{O_K} (O_K)_P \otimes_{(O_K)_P} (O_K)_P) \otimes_{O_K} O_{K'} \\ &\cong (O_K/P \otimes_{O_K} (O_K)_P) \otimes_{(O_K)_P} (O_{K'})_P \\ &\cong \bigoplus_{[K':K]} (O_K/P \otimes_{O_K} (O_K)_P) \\ &\cong \bigoplus_{[K':K]} O_K/P. \end{aligned}$$

Then it follows that

$$\begin{aligned} \#(O_K/P)^{[K':K]} &= \#(O_{K'}/PO_{K'}) \\ &= \prod_i \#(O_{K'}/P_i'^{e_i}) \\ &= \prod_i \#(O_{K'}/P_i')^{e_i} \\ &= \prod_i \#(O_K/P)^{e_i f_i}. \end{aligned}$$

Thus $[K' : K] = \sum_i e_i f_i$. □

Adjacent to Lemma 1.17

> We take a integral basis $\{\omega_1, \cdots, \omega_n\}$ of O_K, we denote by $\{\beta_1, \cdots, \beta_n\}$ the dual basis with respect to $(\ ,\ )_{\mathrm{Tr}_{K/\mathbb{Q}}}$. Then we have $\mathcal{M} = \mathbb{Z}\beta_1 + \cdots + \mathbb{Z}\beta_n$.

Proof. See the note of Prop 1.4. □

Lemma 1.17

> Indeed, because $\#(O_K/\mathcal{D}_K) = \#(\mathcal{M}/O_K)$,

Proof. See Yukie[1] Proposition 1.8.6. □

Theorem 1.18

> Then we have
> $$|D_{K/\mathbb{Q}}| \le \prod_{p \in S} p^{n-1+n\log_p(n)}.$$

Proof. We may assume that $S = \{p \in \mathbb{Z} \mid p \text{ is ramified}\}$. Set $B = O_K$ and $I = D_K$.

Step 1 Let $p \in \mathbb{Z}$ be a prime number. Then B_p and I_p are free $\mathbb{Z}_p$-module of rank n. So there is a matrix $C \in M_n(\mathbb{Z}_p) \cap GL_n(\mathbb{Q}_p)$ such that the following diagram

$$\begin{tikzcd} I_p \arrow[r] \arrow[d] & B_p \arrow[d] \\ \mathbb{Z}_p^n \arrow[r, "C"] & \mathbb{Z}_p^n \end{tikzcd}$$

commute. Then

$$\begin{aligned} \#(B/I \otimes_{\mathbb{Z}} \mathbb{Z}_p) &= \#(\operatorname{Coker} C) \\ &= \#(\mathbb{Z}_p/(\det C)\mathbb{Z}_p) \\ &= \#(\widehat{\mathbb{Z}}_p/(\det C)\widehat{\mathbb{Z}}_p) && \text{(See Yukie[1] Proposition 1.2.13)} \\ &= \#(B/I \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}_p). \end{aligned}$$

Step 2 It follows that

$$\begin{aligned} B/I \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}_p &\cong B/I \otimes_B B \otimes_{\mathbb{Z}} \widehat{\mathbb{Z}}_p \\ &\cong B/I \otimes_B \bigoplus_i \widehat{B}_{P_i} && \text{(See Yukie[1] Theorem 1.3.23)} \\ &\cong \bigoplus_i \widehat{B}_{P_i}/P_i^{\operatorname{ord}_{P_i}(I)}\widehat{B}_{P_i} \\ &\cong \bigoplus_i B/P_i^{\operatorname{ord}_{P_i}(I)} \end{aligned}$$

Step 3 Set $J = I \cap \mathbb{Z}$. Because B/I is finitely generated $\mathbb{Z}$-module, we get

$$\operatorname{Supp}_{\mathbb{Z}}(B/I) = V(\operatorname{ann}_{\mathbb{Z}}(B/I)) = V(J).$$

See Matsumura[4] adjacent to Theorem 4.4 if you do not understand the first equation. And for any prime number $p \in \mathbb{Z}$, then we obtain

$$\begin{aligned} p \notin \operatorname{Supp}_{\mathbb{Z}}(B/I) &\iff B/I \otimes_{\mathbb{Z}} \mathbb{Z}_p = 0 \\ &\iff \#(B/I \otimes_{\mathbb{Z}} \mathbb{Z}_p) = 1 \\ &\iff \forall i\ \#(B/P_i^{\operatorname{ord}_{P_i}(I)}) = 1 \\ &\iff \operatorname{ord}_{P_i}(I) = 0 \\ &\iff p \text{ is unramified} \end{aligned}$$

Thus we conclude $V(J) = \operatorname{Supp}_{\mathbb{Z}}(B/I) = S$.

Step 4 Then we get

$$\begin{aligned} \#(B/I \otimes_{\mathbb{Z}} \mathbb{Z}_p) &= \prod_i \#(B/P_i^{\operatorname{ord}_{P_i}(I)}) \\ &= \prod_i \#(B/P_i)^{\operatorname{ord}_{P_i}(I)} \\ &= \prod_i \#(\mathbb{Z}/p)^{f_i \operatorname{ord}_{P_i}(I)}. \end{aligned}$$

So we conclude $\log_p(\#(B/I \otimes_{\mathbb{Z}} \mathbb{Z}_p)) \leq n - 1 + n\log_p(n)$.

Step 5 Recall that $J = \mathrm{ann}_{\mathbb{Z}}(B/I)$. Then we get

$$\begin{aligned}
B/I &\cong (B/I)/J(B/I) \\
&\cong \bigoplus_{p \in S} (B/I)/p^e(B/I) && (e \text{ depends on } p) \\
&\cong \bigoplus_{p \in S} B/(p^e B + I) \\
&\cong \bigoplus_{p \in S} B/(p^e B + I) \otimes_{\mathbb{Z}} \mathbb{Z}_p \\
&\cong \bigoplus_{p \in S} B_p/(p^e B_p + I_p) \\
&\cong \bigoplus_{p \in S} B_p/(J B_p + I_p) \\
&\cong \bigoplus_{p \in S} B_p/I_p
\end{aligned}$$

Now we conclude that

$$\left|D_{K/\mathbb{Q}}\right| = \#(B/I) = \prod_{p \in S} \#(B_p/I_p) \leq \prod_{p \in S} p^{n-1+n \log_p(n)}.$$

□

2 Theory of heights

Theorem 2.3

We set $n = [K : \mathbb{Q}]$. Let $\{\omega_1, \cdots, \omega_n\}$ be the integral basis of O_K. Then $\{x\omega_1, \cdots, x\omega_n\}$ is a basis of V.

Proof. There is a $c_{ij} \in \mathbb{Z}$ such that $x\omega_i = \sum_j c_{ij}\omega_i$. Set $C = (c_{ij}) \in M_n(\mathbb{Z})$. Then $\det C = N_{K/\mathbb{Q}}(x) \neq 0$, so we get $C \in GL_n(\mathbb{Q})$. And we obtain the assertion. □

Proposition 2.5

$$h_K(x) \leq \sum_{\sigma \in K(\mathbb{C})} \log \left(\max_{1 \leq i \leq n} \{|x_i|_\sigma\} \right).$$

remark. **Misprint.** Add $1/[K : \mathbb{Q}]$ into the right.

Proposition 2.6

for any $x \in \overline{\mathbb{Q}}^n$.

remark. **Misprint.** Exclude the case $x = 0$.

Proposition 2.8

We consider two morphisms $\phi_1 \colon X \to \mathbb{P}^{m_1}$ and $\phi_2 \colon X \to \mathbb{P}^{m_2}$ over $\overline{\mathbb{Q}}$. If $\phi_1^*\mathcal{O}_{\mathbb{P}^{m_1}} \cong \phi_2^*\mathcal{O}_{\mathbb{P}^{m_2}}$, then there is a constant C such that, for any $x \in X(\overline{\mathbb{Q}})$,

$$|h_{\phi_1}(x) - h_{\phi_2}(x)| \leq C.$$

Proof. Remark that $\mathcal{O}_{\mathbb{P}^{m_1}}(1)$ is a Serre's twisted sheaf. See Bosch[2] 9.2/Definition 3. or Hartshorne[6] section 2.5 Adjacent to Proposition 5.12. We set $L = \phi_1^*\mathcal{O}_{\mathbb{P}^{m_1}}$ and set $k = \overline{\mathbb{Q}}$ and $\mathscr{F} = \mathcal{O}_{\mathbb{P}^{m_1}}(1)$. Since X is a projective variety over k and L is an invertible sheaf on X, so $H^0(X, L) = \Gamma(X, L)$ is a k-vector space of finite dimension. (See Hartshorne[6] section 2.5 Theorem 5.19. and Hartshorne[6] section 2.4 Prop 4.10)

Let $\{t_0, \cdots, t_m\}$ be a basis of $H^0(X, L)$ and let $X_0, \cdots, X_{m_1}$ be the homogenous coordinates of $\mathbb{P}^{m_1}$. Note that each X_i is a global section of $\mathscr{F}$. And we set $s_i = \phi_1^*X_i \in H^0(X, L)$, where $\phi_1^*X_i$ is the image of $X_i \in H^0(\mathbb{P}^m, \mathscr{F})$ by the canonical map $\mathscr{F} \to \phi_{1*}\phi_1^*\mathscr{F}$. It follows from Hartshorne[6] section 2.7 Theorem 7.1 that $s_0, \cdots, s_{m_1}$ generate L. Because for any $x \in X$ the each germ $(s_i)_x$ is a linear combination of $(t_j)_x$, so $t_0, \cdots, t_m$ generate L.

There is a morphism $\phi\colon X \to \mathbb{P}^m$ such that $L \cong \phi^*\mathcal{O}_{\mathbb{P}^m}(1)$ and $s_i = \phi^* X_i$ under this isomorphism. See Hartshorne[6] section 2.7 Theorem 7.1(b). There is another explanation on what ϕ is. For any $x \in X$, we can consider the germ $(t_i)_x \in L_x$. Denote $(t_i)_x$ by $t_i(x)$. Since L is a line bundle, $L_x \cong \mathcal{O}_{X,x}$. Then we define the map $\phi\colon X \to \mathbb{P}^m$ by $\phi(x) = (t_0(x), \cdots, t_m(x))$. Note that there is a scalar ambiguity in choice of morphism $L_x \to k$. If $\forall i\ t_i(x) = 0$, then $(t_i)_x$ cannot generate L_x, which is a contradiction. Thus for any $x \in X$, there is an index i such that $t_i(x) \neq 0$.

Then, the rest of the proof is almost trivial. □

Theorem 2.9

First, suppose that L is globally generated.

remark. What "globally generated" means? We say L is globally generated iff there is an exact sequence $\bigoplus_I O_X \to L \to 0$. Even if L is an invertible sheaf, L is not necessarily globally generated. For example, set $X = \mathbb{P}^m$, $L = \mathcal{O}_{\mathbb{P}^m}(-1)$. Since $\Gamma(X, L) = 0$, L is not globally generated.

Theorem 2.9

Let

$$\phi_{|L|}\colon X \to \mathbb{P}(H^0(X, L))$$

be a morphism associated to the complete linear system $|L|$. We set $h_L = h_{\phi_{|L|}}$.

Proof. Note that we want to get $h_L \in \mathrm{Func}(X)/B(X)$, which is not contained in $\mathrm{Func}(X)$.

What is a $\mathbb{P}(H^0(X, L))$? I think it is isomorphic to $\mathbb{P}^m$ by taking a basis of $H^0(X, L)$.

Set $k = \overline{\mathbb{Q}}$. Since L be a globally generated line bundle on X, there is a basis $s_0, \cdots, s_m$ of $H^0(X, L)$ which generate L. Then we get a map $\phi_L\colon X \to \mathbb{P}^m$ such that $\phi_L^*\mathcal{O}_{\mathbb{P}^m}(1) \cong L$. We define h_L by $h_L = h_{\phi_L}$. □

Theorem 2.9

Then $s_i \otimes t_j$ induces a morphism $\phi\colon X \to \mathbb{P}^N$ such that $\phi^*(O_{\mathbb{P}^N(1)}) \cong L_1 \otimes L_2$.

remark. How $s_i \otimes t_j \in H^0(X, L_1) \otimes_k H^0(X, L_2)$ define an element of $H^0(X, L_1 \otimes L_2)$? Let $\mathscr{F}$ be a presheaf defined by $\mathscr{F}(U) = \Gamma(U, L_1) \otimes_{\mathcal{O}_X(U)} \Gamma(U, L_2)$. Then there is a canonical morphism $\mathscr{F} \to L_1 \otimes L_2$ since $L_1 \otimes L_2$ is the sheafification of $\mathscr{F}$. So we can see $s_i \otimes t_j \in H^0(X, L_1 \otimes L_2)$.

We denote the image of $s_i \otimes t_l$ by $s_i t_j \in H^0(X, L_1 \otimes L_2)$. Why $\{s_i t_j\}$ generate $L_1 \otimes L_2$? Take a stalk.

Theorem 2.9

> tell us that $L \otimes A^n$ is globally generated for any sufficiently large n.

remark. The ampleness of A implies that

$$\exists n_1 \text{ s.t. } n \geq n_1 \Rightarrow L \otimes A^n \text{ is globally generated}$$
$$\exists n_2 \text{ s.t. } n \geq n_2 \Rightarrow A^n \text{ is globally generated}$$

Then we set $n = \max_i\{n_i\}$.

Theorem 2.9

> Then, modulo $B(X)$, we have
> $$h_{f^*(L)} = h_{f^*(C)\otimes f^*(C)^{-1}}$$

remark. See Görtz Wedhorn[5] Remark 7.10.

Theorem 2.9

> Then by (1), h_L must be equal to $h_{L_1} - h_{L_2}$ modulo $B(X)$.

remark. Let $\sigma\colon$ {line bundles } $\to$ Func$(X)/B(X)$ be a map which satisfies the properties (1), (2), (3). By (3), for globally generated line bundle L, we get $\sigma_L = h_L$. Because $\Gamma(X, O_X) = k$, we obtain $\sigma_{O_X} = 0$. Thus (1) implies that $\sigma_L = h_L$ for general line bundle L.

Proposition 2.10

> Let B be the Zariski closed subset of X defined by the ideal sheaf
> $$\mathrm{Im}(H^0(X, L) \otimes L^{-1} \to O_X).$$

remark. What is the morphism $H^0(X, L) \otimes L^{-1} \to O_X$? Note that there is a canonical $\overbrace{\text{morphism}}$ $f^*f_*L \to L$ where $f\colon X \to \operatorname{Spec} k$ is a k-scheme structure. Note that f_*L is isomorphic to $\widetilde{H^0(X, L)}$. We denote this canonical morphism $f^*f_*L \to L$ by

$$H^0(X, L) \otimes O_X \to L.$$

This is surjective if L is globally generated.

In general, we define $V \otimes_k O_X$ for k-module V, by setting

$$V \otimes_k O_k = f^{-1}\widetilde{V} \otimes_{f^{-1}O_{\operatorname{Spec} k}} O_X = f^*\widetilde{V}.$$

Then we get $H^0(X, L) \otimes L^{-1} \to O_X$ by tensoring L^{-1}.

Proposition 2.10

Then $\{ss_i\}$ are linearly independent elements of $H^0(X, L)$.

remark. What are ss_i ? Note that there is a canonical morphism

$$H^0(X, L) \otimes H^0(X, L_2) \to H^0(X, L \otimes L_2) \cong H^0(X, L_1)$$

Thus I guess ss_i is the image of $s \otimes s_i$.

Moreover, why ss_i are linearly independent? It suffices to show that the morphism of k-module

$$s\colon H^0(X, L_2) \to H^0(X, L_1)$$

is injective.

We prepare the following lemma.

lemma. Let X be an integral scheme and let L be a line bundle on X. Assume that $s \in H^0(X, L)$ is not zero. Then for any $x \in X$, $s_x \neq 0$ in L_x.

Proof. Assume that there is a $z \in X$ such that $s_z = 0$. We want to show $s = 0 \in H^0(X, L)$. Since L is invertible, there is an open affine covering $X = \bigcup_{i \in I} U_i$ such that

$$U_i = \operatorname{Spec} A_i,\ L|_{U_i} \cong \widetilde{A_i}$$

On the other hand, $s_z = 0$ implies that there is an open subset $U \subset X$ such that $s|_U = 0$ and $z \in U$. Since X is integral, $U \cap \operatorname{Spec} A_i \neq \emptyset$. Thus there is a $g_i \in A_i \setminus \{0\}$ such that $\emptyset \neq D(g_i) \subset U \cap \operatorname{Spec} A_i$. Then $s|_{D(g_i)} = 0$ in $\Gamma(D(g_i), L) \cong A_{i g_i}$. Note that each A_i is an integral domain because X is integral. Thus we get $\forall i\ s|_{U_i} = 0$ because $A_i \to A_{i g_i}$ is injective. It follows from the sheaf axiom that $s = 0 \in \Gamma(X, L)$. $\square$

Then, we can prove the injectivity of $s\colon O_X \to L$. First, by the lemma, $0 \to O_X \xrightarrow{s} L$ is exact. Since L_2 is flat, $0 \to L_2 \xrightarrow{s} L_1$ is exact. Since global section is left exact, we get $0 \to H^0(X, L_2) \xrightarrow{s} H^0(X, L_1)$ is exact.

Proposition 2.10

Let $s_1, \cdots, s_n$ be a basis of $H^0(X, L)$. $\cdots$

Because $B = \{x \in X \mid s_1(x) = \cdots = s_n(x) = 0\}$.

Proof. Why $B = \{x \in X \mid s_1(x) = \cdots = s_n(x) = 0\}$? I guess

$$\begin{aligned}
B &= \operatorname{Supp} \operatorname{Coker}(H^0(X, L) \otimes L^{-1} \to O_X) \\
&= \operatorname{Supp} \operatorname{Coker}(H^0(X, L) \otimes O_X \to L) \otimes L^{-1} && \text{(right exactness of tensor)} \\
&= \operatorname{Supp} \operatorname{Coker}(H^0(X, L) \otimes O_X \to L) \cap \operatorname{Supp} L^{-1} \\
&= \operatorname{Supp} \operatorname{Coker}(H^0(X, L) \otimes O_X \to L)
\end{aligned}$$

Thus we get

$$\begin{aligned} x \in B &\iff \operatorname{Coker}(H^0(X,L) \otimes O_X \to L)_x \neq 0 \\ &\iff \forall s \in H^0(X,L)\ x_x \in \mathfrak{m}_x L_x \\ &\iff s(x) = 0 \\ &\iff s_1(x) = \cdots = s_n(x) = 0 \end{aligned}$$

□

Lemma 2.11

> Then the definition of h^+ gives $h^+(\tau(x)) = h^+(x)$.

remark. Note that $\operatorname{ord}_P(\tau(x)) = \operatorname{ord}_{\tau^{-1}(P)}(x)$ and the residue degree at P_i is independent of i since $K/\mathbb{Q}$ is a Galois extension. (See Yukie[1] Number 2 Cor 1.3,27)

Lemma 2.17

> Because X is projective, p is a closed morphism

remark. $X \to \operatorname{Spec} F$ は projective なので proper であり、とくに universally closed である。したがって p は閉写像。

Lemma 2.19

> As the natural homomorphism $p^*M \to L$ coincides on each fiber with the natural homomorphism $\cdots$ we obtain that $p^*M \cong L$.

remark. X, Y を F スキームとする。このとき L_1, L_2 が $X \times_F Y$ 上のラインバンドルであり、任意の $y \in Y$ に対して

$$L_1|_{X \times \{y\}} \cong L_2|_{X \times \{y\}}$$

であったとしても $L_1 \cong L_2$ であるとは限らない。

例を挙げよう。$p \colon X \times_F Y \to Y$ を射影とし、M を Y 上のラインバンドルとする。このとき $p^*M|_{X \times \{y\}} \cong O_{X \times Y}|_{X \times \{y\}}$ だが、必ずしも $p^*M \cong O_{X \times Y}$ ではない。たとえば $X = \operatorname{Spec} F$, $Y = \mathbb{P}^m_F$, $M = \mathcal{O}_{\mathbb{P}}(1)$ とすればよい。

Lemma 2.21

> Let X be a projective variety, and let x_0 and x_1 be closed points of X. Then there is a projective curve C on X passing through x_0 and x_1.

Proof. $x_0 = x_1$ ならこの点を通る直線 f に対して $X \cap V(f)$ を考えることを繰り返し、次元を 1 ずつ下げることができるのであきらか。(既約性をどう保証するのか？) $x_0 \neq x_1$ とする。$\dim X = 1$ ならあきらかなので $\dim X \geq 2$ とする。$\dim X$ についての帰納法で示そう。

$\pi\colon \widetilde{X} \to X$ を x_0, x_1 に沿う X の blowup とする。X が射影的なので $\widetilde{X}$ も射影的。よって閉埋め込み $\widetilde{X} \to \mathbb{P}^N$ がある。ここで N はそのような性質を満たす最小のものをとることにする。

ここで x_0, x_1 は閉点なので、各 i について x_i を含み x_{1-i} を含まないようなアファイン近傍 $U_i = \operatorname{Spec} A_i$ がある。閉点 x_i は A_i の極大イデアル $\mathfrak{m}_i$ に対応しているとしよう。このとき $\pi^{-1}(x_i)$ の次元がどうなっているかを考えたい。X が smooth なら $\pi^{-1}(x_i) \cong \mathbb{P}^{\dim X - 1}$ であるが、ここではそれは使えないので次のように計算する。

$$\begin{aligned}
\pi^{-1}(x_i) &= \pi^{-1}(U_i) \times_{U_i} x_i \\
&\cong \operatorname{Proj}(\oplus_{d \geq 0} \mathfrak{m}_i^d) \times \operatorname{Spec} A_i/\mathfrak{m}_i \\
&\cong \operatorname{Proj}((\oplus_{d \geq 0} \mathfrak{m}_i^d) \otimes \operatorname{Spec} A_i/\mathfrak{m}_i) \\
&\cong \operatorname{Proj}(\oplus_{d \geq 0} \mathfrak{m}_i^d/\mathfrak{m}_i^{d+1}) \\
&\cong \operatorname{Proj}(\operatorname{gr}_{\mathfrak{m}_i} A_i)
\end{aligned}$$

である。$\dim X \geq 2$ という仮定より

$$\begin{aligned}
\dim \pi^{-1}(x_i) &= \dim \operatorname{gr}_{\mathfrak{m}_i} A_i - 1 \\
&= \dim A_i - 1 \\
&\geq 1
\end{aligned}$$

がわかる。したがって $\mathbb{P}^N$ の hyperplane H をとると、あきらかに $\dim H = N - 1$ なので $\dim H + \dim \pi^{-1}(x_i) \geq N$ が従う。ゆえに射影次元定理 (Hartshorne[6] section 1-7.2) よりすべての i について $H \cap \pi^{-1}(x_i) \neq \emptyset$ が成り立つ。いま Bertini の定理により H が一般の hyperplane とするとき $\widetilde{Y} = \widetilde{X} \cap H$ は既約。よって $Y = \pi(\widetilde{Y})$ とすれば Y は X の部分多様体であり、$H \cap \pi^{-1}(x_i) \neq \emptyset$ より Y は x_0, x_1 を含む。かつ N の最小性より (最小性を使わなくても一般の超平面で切っているのだから当然に思えるが)$\widetilde{Y} \neq \widetilde{X}$ なので、$\dim Y \leq \dim \widetilde{Y} < \dim \widetilde{X} = \dim X$ である。よってこの Y に帰納法が適用できて証明が回る。 □

Cor 2.25

> Let A be an abelian variety of dimension g over F.
>
> (1) For any integer n, $[n]\colon A \to A$ is a finite and flat morphism of degree n^{2g}.
>
> (2) The abelian group $A(\overline{F})$ is divisible, i.e., for any $x \in A(\overline{F})$ and for any positive integer n, there is a $y \in A(\overline{F})$ with $[n](y) = x$.

Proof.

(1) L を even かつ ample な line bundle とする。$[n]^*L = L^{n^2}$ より $[n]^*L$ も ample である。$[n]^*L$ が ample ということは、(閉とは限らない) 埋め込み

$$\psi_{|\widetilde{L}|}\colon A \to \mathbb{P}(\Gamma(A, \widetilde{L})^\vee)$$

がある。ただし $\widetilde{L}=[n]^*L$ であり、$^\vee$ は双対空間を表す。このとき次の図式は可換。

$$\begin{tikzcd}
A \arrow[r, "\psi_{|\widetilde{L}|}"] & \mathbb{P}(\Gamma(A,\widetilde{L})^\vee) \\
\mathrm{Ker}[n] \arrow[u] \arrow[r, "\psi_{|\mathrm{Ker}[n]|}"] & \mathbb{P}(\Gamma(\mathrm{Ker}[n],\widetilde{L}|_{\mathrm{Ker}[n]})^\vee) \arrow[u]
\end{tikzcd}$$

ここで $\widetilde{L}|_{\mathrm{Ker}[n]}$ は自明なので $\mathrm{Ker}[n]$ の既約性分への分解を $\mathrm{Ker}[n]=\coprod_{i\in I}P_i$ とすると、各成分 P_i の $\psi_{|\mathrm{Ker}[n]|}$ による像は $\mathbb{P}^0(F)$ に含まれる。つまり一点である。したがって、$\psi_{|\widetilde{L}|}$ は埋め込みなので各 P_i は一点である。よって $\dim\mathrm{Ker}[n]=0$ である。

また $[n]$ は projective variety の間の射なので projective であり、とくに固有である。$\dim\mathrm{Ker}[n]=0$ であることをいま示したが、$[n]$ は準同型なのですべての点の fiber の次元が等しい。ゆえに固有かつすべての点での fiber の次元がゼロなので $[n]$ は finite である。とくに fiber の次元が任意の点で等しいので $[n]$ は flat である。

次に degree について考える。ここでの degree は交点数を用いて定義される。ample line bundle L について

$$L^{\cdot g}=(L.\cdots.L)$$

と定義する。右辺の L は g 個ある。(Reference: 石井志保子「特異点入門」) 交点数は多重線形なので

$$([n]^*L)^{\cdot g}=(L^{n^2}.\cdots.L^{n^2})=n^{2g}L^{\cdot g}$$

が従う。よって degree の定義から $\deg[n]=n^{2g}$ である。

(2) $[n]\colon A\to A$ は固有なのでその像 $[n](A)$ は閉部分多様体である。また $[n]$ は flat なので

$$\dim_0 A+\dim_0\mathrm{Ker}[n]=\dim_0[n](A)$$

が成り立つ。よって $\dim\mathrm{Ker}[n]=0$ より $\dim A=\dim_0 A=\dim_0[n](A)=\dim[n](A)$ である。真閉部分多様体は次元が落ちるはずなので $A=[n](A)$ がわかる。

□

Remark 2.27

> If D is an effective Cartier divisor on an abelian variety A, then $|2D|$ is base point free. In particular, D is nef.

remark. まず用語について解説する。D が base point free とは、rational map

$$\psi_{|D|}\colon A\dashrightarrow\mathbb{P}(\Gamma(A,D)^\vee)$$

が A 全体で定義されることである。言い換えれば、

$$\{x\in A\mid\forall 0\neq s\in\Gamma(A,D)\ s(x)=0\}=\emptyset$$

ということである。また D が nef(数値的正、ネフ) とは

$$\forall C\text{ irreducible curve }(D.C)\geq 0$$

(交点数がゼロ以上) として定義される。

さて effective な Cartier divisor D について、$2D$ が base point free ならば D が nef であることを確かめよう。$2D$ が nef ならあきらかに D も nef なので、はじめから D が base point free だと仮定して D が nef だといえばよい。

いま D は effective なので既約曲線 C が $C \not\subset D$ である限り、$(D.C) \geq 0$ となる。交点数は線形同値なもの同士を入れ替えても不変なので、どんな C についてもある D と線形同値な D' があって $C \not\subset D'$ となることをいえばよい。

ハイリホーで示す。ある既約曲線 C が存在して、すべての $D' \sim D$ なる D' について $C \subset D'$ であったとする。いま $\psi_{|D|}$ の定義から、任意の超曲面 $H \subset \mathbb{P}(\Gamma(A, D)^\vee)$ に対して $D \sim \psi^*_{|D|}H$ である。H はある大域切断 $s \in \Gamma(A, D)$ により $H = \{l \mid l(s) = 0\}$ と表せる。そこでこれを H_s とおく。このとき仮定から

$$\forall s\ C \subset \psi^*_{|D|}H_s$$

であるが、$\psi^*_{|D|}H_s = \{x \in A \mid s(x) = 0\}$ であったため、D が base point free であったことより $C = \emptyset$ となるしかない。これは矛盾である。

Remark 2.27

> For an $a \in A(\overline{F})$, we define the morphism $T_a \colon A_{\overline{F}} \to \mathrm{Pic}(A_{\overline{F}})$ by
> $$T_a \colon x \mapsto x + a.$$
> For any given line bundle L on $A_{\overline{F}}$, we define the map $\lambda_L \colon A(\overline{F}) \to \mathrm{Pic}(A_{\overline{F}})$ by
> $$\lambda \colon x \mapsto T_x^* L \otimes L^{-1}.$$
> Cor (Theorem of Square)
> As groups, the map λ_L is a homomorphism from $A(\overline{F})$ to $\mathrm{Pic}(A_{\overline{F}})$.

> If D is an effective Cartier divisor on an abelian variety A, then $|2D|$ is base point free. In particular, D is nef. Indeed, identifying D with the corresponding Weil divisor, we write $D+a$ and $D-a$ for $T_a(D)$ and $T_{-a}(D)$ respectively. For any $x \in A(\overline{F})$, we choose a point $a \notin \mathrm{Supp}(D-x) \cup \mathrm{Supp}(D+x)$. Then $x \notin \mathrm{Supp}(D-a) \cup \mathrm{Supp}(D+a)$. Further, the theorem of the square implies that $(D-a)+(D+a) \sim 2D$. Thus $|2D|$ is base point free.

remark. Hartshorne[6] section 2.6 命題 6.11 の、X が整分離的 Noether スキームで局所分解的なものとすると、Cartier divisor と Weil 因子が同型になるということを踏まえて同一視をする。ここで Abelian 多様体 A は smooth で、したがって局所環が regular であり、正則局所環は UFD であることから A は局所分解的であることに気を付ける。

閉点 $x \in A(\overline{F})$ が任意に与えられたとする。x を台に含まないような、$2D$ と線形同値な effective 因子の存在をいえば、$|2D|$ が base point free であることが従う。そういう因子を構成しよう。

点 $a \notin \mathrm{Supp}(D-x) \cup \mathrm{Supp}(-D+x)$ をとる。ここで因子 $D-x$ は像 $T_{-x}(D)$ を意味し、$-D+x$ は逆元をとる写像を i として $T_x(i(D))$ を意味する。このとき $a \notin \mathrm{Supp}(D-x)$ より $x \notin \mathrm{Supp}(D-a)$ であり、$a \notin \mathrm{Supp}(-D+x)$ より $x \notin \mathrm{Supp}(D+a)$ である。（本文には誤植がある）このとき Theorem of square により

$$\begin{aligned}(D-a)+(D+a) &= T_{-a}(D) + T_a(D) \\ &= \lambda_D(-a) + \lambda_D(a) + 2D \\ &= 2D\end{aligned}$$

だから $2D$ と $(D+a)+(D-a)$ は線形同値。$(D+a)+(D-a)$ はあきらかに effective なので $|2D|$ が base point free であることがいえた。

Cor 2.28

Let A be an abelian variety, and let D be an effective Cartier divisor on A. We set $L = O_X(D)$. (In particular, $|2D|$ is base point free) Then the following are equivalent.

(1) $\mathrm{Ker}(\lambda_L)$ is a finite subgroup of $A(\overline{F})$.

(2) A morphism $\Phi\colon A \to \mathbb{P}(H^0(A, L^2))$ associated to the complete linear system $|2D|$ is a finite morphism onto its image.

(3) L is ample.

PROOF. Properties (2) and (3) hold over F if and only if those hold over $\overline{F}$, so we may assume that $F = \overline{F}$.

(1) $\Rightarrow$ (2) Suppose that Φ maps a projective curve C on A to a single point. We are going to deduce a contradiction by showing that $\mathrm{Ker}(\lambda_L)$ contains $C - C = \{x_2 - x_1 \mid x_1, x_2 \in C(F)\}$. We write $D = \sum_{i=1}^{r} a_i D_i$ with $a_i > 0$ and prime divisors D_i on A.
We claim that either $(C + x) \cap D_i = \emptyset$ or $C + x \subset D_i$ for any i and for any $x \in A(F)$. Because D_i is an effective Cartier divisor on an abelian variety, it is nef by Remark 2.27. In particular, $(D_i \cdot C) \geq 0$. It follows from

$$0 = (D \cdot C) = a_1(D_1 \cdot C) + \cdots + a_r(D_r \cdot C),$$

that $(D_i \cdot C) = 0$. Further, because $C + x$ is algebraically equivalent to C, we have $(D_i \cdot C + x) = (D_i \cdot C) = 0$. Thus we obtain the claim.
Next, we claim that $D_i = D_i + x_1 - x_2$ for any $x_1, x_2 \in C(F)$. Indeed, if $y \in D_i$ is a closed point, then both $C - x_1 + y$ and D_i contain y. Thus $C - x_1 + y \subset D_i$ by the above argument. It follows that $x_2 - x_1 + y \in D_i$, so $y \in D_i + x_1 - x_2$. Thus $D_i \subset D_i + x_1 - x_2$. By switching x_1 and x_2 in the above argument, we have $D_i \supset D_i + x_1 - x_2$. Hence we obtain the claim.
We set $L_i = O_A(D_i)$. Then $L = L_1^{\otimes a_1} \otimes \cdots \otimes L_r^{\otimes a_r}$. By the above claim, we have $T^*_{x_2 - x_1}(L_i) = L_i$ for each i. It follows that

$$\begin{aligned} T^*_{x_2-x_1}(L) &= T^*_{x_2-x_1}(L_1)^{\otimes a_1} \otimes \cdots \otimes T^*_{x_2-x_1}(L_r)^{\otimes a_r} \\ &= L_1^{\otimes a_1} \otimes \cdots \otimes L_r^{\otimes a_r} \\ &= L. \end{aligned}$$

Thus $x_2 - x_1 \in \mathrm{Ker}(\lambda_L)$, and so $C - C \subset \mathrm{Ker}(\lambda_L)$. This is a contradiction.

(2)$\Rightarrow$(3) Let E be any coherent O_A-module on A. We are going to show that $E \otimes L^{2n}$ is globally generated for any sufficiently large $2n$. We write X for the image $\Phi(A)$ of Φ. Because $\Phi_*(E)$ is coherent on X and $O_X(1)$ is ample, there is an $n_0 > 0$ such that

$$H^0(X, \Phi_*(E) \otimes O_X(n)) \otimes O_X \to \Phi_*(E) \otimes O_X(n)$$

is surjective for any $n \geq n_0$ (See Hartshorne p.153). Pulling back by Φ, we find that

$$(*) \qquad H^0(X, \Phi_*(E) \otimes O_X(n)) \otimes O_A \to \Phi^*\Phi_*(E) \otimes \Phi^* O_X(n)$$

is also surjective.

On the other hand, it follows from $\Phi^* O_X(n) \cong L^{2n}$ and the projection formula that $\Phi_*(E \otimes L^{2n}) \cong \Phi_*(E) \otimes O_X(n)$. Further, as Φ is a finite morphism, the canonical morphism $\Phi^*\Phi_*(E) \to E$ is surjective. Thus the surjectivity of $(*)$ implies the surjectivity of

$$H^0(A, E \otimes L^{2n}) \otimes O_A \to E \otimes L^{2n}$$

for every $n \geq n_0$. Thus L^2 is ample, so L is ample.

(3)⇒(1) Let $p_i \colon A \times A \to A$ denote the i-th projection. First, we note the equality

$$\operatorname{Ker} \lambda_L = \left\{ x \in A(F) \mid (m_A^* L^{-1} \otimes p_1^* L)|_{A \times \{x\}} \text{ is trivial} \right\}.$$

In particular, the seesaw theorem tells us that $\operatorname{Ker} \lambda_L$ endowed with the reduced induced scheme structure is regarded as a closed subgroup scheme of A. We denote by B the connected component of $\operatorname{Ker}(\lambda_L)$ containing the identity. Then B is an abelian subvariety of A. We are going to show that $\dim B = 0$.

We set $L' := (m_A^* L^{-1} \otimes p_1^* L \otimes p_2^* L)|_{B \times B}$ on $B \times B$. Because $L'|_{B \times \{x\}} = (m_A^* L^{-1} \otimes p_1^* L)|_{B \times \{x\}} \otimes p_2^* L|_{B \times \{x\}}$ is trivial for any $x \in B$ and $L'|_{\{0\} \times B}$ is trivial, the seesaw theorem implies that L' is trivial. Pulling back L' by $([1]_B, [-1]_B) \colon B \to B \times B$, we obtain that $L \otimes [-1]_A^* L$ is trivial on B. On the other hand, because L is ample on A, $L \otimes [-1]_A^* L$ is ample on A. Thus $\dim B = 0$, and we conclude that $\operatorname{Ker}(\lambda_L)$ is a finite set.

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