The Mordell-Faltings theorem

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1 Some basics of algebraic number theory

Lemma 1.3

> **quotation.** Recall that $(\ ,\ )_{\mathrm{Tr}_{K/\mathbb{Q}}}$ is non-degenerate if the Gramm matrix with respect to one (and hence any) basis of L over F is invertible.

Proof. Almost trivial. Try to prove it. $\square$

Proposition 1.4

> **quotation.** Let $\{\beta_1, \cdots, \beta_n\}$ be the dual basis of $\{\alpha_1, \cdots, \alpha_n\}$ with respect to $(\ ,\ )_{\mathrm{Tr}_{K/\mathbb{Q}}}$. Then, for any $x \in O_K$, we have $x = (x, \alpha_1)_{\mathrm{Tr}_{K/\mathbb{Q}}}\beta_1 + \cdots + (x, \alpha_n)_{\mathrm{Tr}_{K/\mathbb{Q}}}\beta_n$.

Proof. Since the trace form $(\ ,\ )_{\mathrm{Tr}_{K/\mathbb{Q}}}$ is nondegenerate, $K \to K^*$ s.t. $x \mapsto (\cdot, x)_{\mathrm{Tr}_{K/\mathbb{Q}}}$ is a isomorphism. Let $p_i\colon K \to \mathbb{Q}$ be a projection map such that $p_i(x_1\alpha_1 + \cdots + x_n\alpha_n) = x_i$. Then, we set β_j the preimage of p_j. $\square$

Lemma 1.7

> **quotation.** To see this, we take $t \in P(O_K)_P$ with $t \notin P^2(O_K)_P$.

remark. From Nakayama's lemma.

Adjacent to Lemma 1.8

> **quotation.** For a nonzero prime ideal P of O_K, we set $P \cap \mathbb{Z} = (p)$, where p is a prime of Z. Because O_K is a free Z-module of rank $[K : \mathbb{Q}]$, O_K/P is a finite extension of $\mathbb{Z}/(p)$ with degree at most $[K : \mathbb{Q}]$.

Proof. There is a canonical surjection $O_K/pO_K \to O_K/P$, so we get $\#(O_K/P) \leq \#(O_K/pO_K)$. But we obtain $O_K/pO_K \cong O_K \otimes_{\mathbb{Z}} \mathbb{Z}/p\mathbb{Z}$. Since O_K is a free $\mathbb{Z}$-module of rank $n = [K : \mathbb{Q}]$, we conclude $O_K/pO_K \cong (\mathbb{Z}/p\mathbb{Z})^n$. So, $\#(O_K/P) \leq \#(O_K/pO_K) = p^n$. $\square$

Lemma 1.8

> **quotation.**
> $$\bigoplus_{i=1}^{r} O_K/P_i^{e_i} = \bigoplus_{i=1}^{r} (O_K/P_i^{e_i})_{P_i}$$

Proof. Because $O_K/P_i^{e_i}$ is a local ring with maximal ideal $P_i/P_i^{e_i}$. $\square$

Adjacent to Theorem 1.9

> **quotation.** we consider the value $\sqrt{\det(\langle e_i, e_j \rangle)}$.

remark. Why we get $\det(\langle e_i, e_j \rangle)$? Apply Gram-Schmidt orthonormalization.

Adjacent to Theorem 1.9

> **quotation.** Then $\mathrm{vol}(M, \langle , \rangle)$ is equal to the volume of the n-dimensional parallelpiped Π spanned by $e_1, \cdots, e_n$,

Proof. Let $F\colon (V, \langle , \rangle) \to \mathbb{R}^n$ be an isometric isomorphism. Then, we generate

$$\begin{aligned} \mathrm{vol}(M, \langle , \rangle)^2 &= \det(\langle e_i, e_j \rangle) \\ &= \det(\langle Fe_i, Fe_j \rangle) \end{aligned}$$

We set $E = (Ee_1, \cdots, Fe_n)$. $E \in M_n(\mathbb{R})$. Then we get $(\langle Fe_i, Fe_j \rangle)_{i,j} = {}^tEE$, and $\mathrm{vol}(M, \langle , \rangle) = |\det E|$. From Yukie[3] Theorem 4.9.1, $|\det E| = \mathrm{vol}(\Pi)$. $\square$

Proposition 1.11

> **quotation.** The form $\langle , \rangle_K$ is an inner product on V.

remark. $\langle , \rangle_K$ is trivially an inner product on K. Why should we show this?

Let S be a $\mathbb{Q}$ vector space and $\langle , \rangle$ a inner product on S. Then, bilinear form extended to $S \otimes_{\mathbb{Q}} \mathbb{R}$ may not be an inner product. For example, set $S = \mathbb{Q}[\sqrt{2}]$ and $\langle x, y \rangle = xy$.

Lemma 1.12

> **quotation.** $\#(O_K/I)$ is finite. Then I is a free $\mathbb{Z}$-module of rank n.

Proof. $I \subset O_K$ is a free $\mathbb{Z}$-module. Since $\#(O_K/I)$ is finite, we get $\forall x \in K\ \exists n \in \mathbb{Z}$ s.t. $nx \in I$. So we obtain $I \otimes_{\mathbb{Z}} \mathbb{Q} = K$. The rank of I is n. □

Lemma 1.16

> **quotation.** Because $(O_K)_P$ is a principal ideal domain, $(O_{K'})_P$ is a free $(O_K)_P$-module of rank $[K' : K]$.

Proof. See the proof of Prop 1.4. We obtain $O_{K'} \subset O_K\beta_1 \oplus \cdots \oplus O_K\beta_n$ for some $\beta_i \in K'$. Taking a localization, we get $(O_{K'})_P \subset (O_K)_P\beta_1 \oplus \cdots \oplus (O_K)_P\beta_n$. Since $(O_K)_P$ is a PID, $(O_{K'})_P$ is a free $(O_K)_P$-module. The rank is not lower than $[K' : K]$ because integral basis generate K' over K. □

Lemma 1.16

> **quotation.** Thus
>
> $$\begin{aligned} \dim_{O_K/P} O_{K'}/PO_{K'} &= \dim_{O_K/P}(O_{K'})_P/P(O_{K'})_P \\ &= \dim_{O_K/P}((O_K)_P/P(O_K)_P) \otimes_{(O_K)_P} (O_{K'})_P \end{aligned}$$

Proof. We set $A = O_K, A' = O_{K'}$. Then we get

$$\begin{aligned} A'/PA' &\cong A' \otimes_A A/P \\ &\cong A' \otimes_A \mathrm{Frac}\, A/P \\ &\cong A' \otimes_A \mathrm{Coker}(PA_P \to A_P) \\ &\cong \mathrm{Coker}(A' \otimes_A PA_P \to A' \otimes_A A_P) \\ &\cong (A')_P/P(A')_P \\ (A')_P/P(A')_P &\cong A' \otimes_A \mathrm{Coker}(PA_P \to A_P) \\ &\cong A' \otimes_A A_P/PA_P \\ &\cong (A' \otimes_A A_P) \otimes_{A_P} A_P/PA_P \\ &\cong (A')_P \otimes_{A_P} A_P/PA_P. \end{aligned}$$

□

Adjacent to Lemma 1.17

> **quotation.** We take a integral basis $\{\omega_1, \cdots, \omega_n\}$ of O_K, we denote by $\{\beta_1, \cdots, \beta_n\}$ the dual basis with respect to $(\ ,\ )_{\mathrm{Tr}_{K/\mathbb{Q}}}$. Then we have $\mathcal{M} = \mathbb{Z}\beta_1 + \cdots + \mathbb{Z}\beta_n$.

Proof. See the note of Prop 1.4. □

Adjacent to Lemma 1.17

> **quotation.** The difference of K is defined by $\mathcal{D}_K = \mathcal{M}^{-1}$. Because $O_K \subset \mathcal{M}$, we have $\mathcal{D}_K \subset O_K$, so $\mathcal{D}_K$ is an ideal of O_K.

Proof. $O_K = \mathcal{M}\mathcal{M}^{-1} = \mathcal{D}_K\mathcal{M} \supset \mathcal{D}_K O_K \supset \mathcal{D}_K$. □

Lemma 1.17

> **quotation.** Indeed, because $\#(O_K/\mathcal{D}_K) = \#(\mathcal{M}/O_K)$,

Proof. See Yukie[1] Proposition 1.8.6. □

Theorem 1.18

> **quotation.** Lemma 1.17 (3) gives
>
> $$\log_p(\#(((O_K)_P/(\mathcal{D}_K)_P)) = \sum_i \mathrm{ord}_{P_i}(\mathcal{D}_K)_{f_i}$$

Proof. **It remains to be solved.** □

Theorem 1.18

> **quotation.** Because $\#(O_K/\mathcal{D}_K) = \prod_{p\in S} \#(((O_K)_P/(\mathcal{D}_K)_P)$, we obtain the assertion.

Proof. See Yukie[1] Prop1.8.9. □

2 Theory of heights

Proposition 2.8

> **quotation.** If $\phi_1^*(O_{\mathbb{P}^{m_1}}(1)) \cong \phi_2^*(O_{\mathbb{P}^{m_2}}(1))$,

remark. What is a $O_{\mathbb{P}^{m_1}}(1)$? I think it is a Serre's twisted sheaf. See Bosch[2] 9.2/Definition 3. **It remains to be learned.**

参考文献

[1] 雪江明彦『整数論 2 代数的整数論の基礎』(日本評論社, 2013)

[2] Siegfried Bosch『Algebraic Geometry and Commutative Algebra』(Springer, 2013)

[3] 雪江明彦『線形代数学概説』(培風館, 2006)