C Y B E R N E T I C S

Course Content

Theory and applications

1.- Information theory: information in communication

2.- Control theory: information in regulation and control

Information in communication:

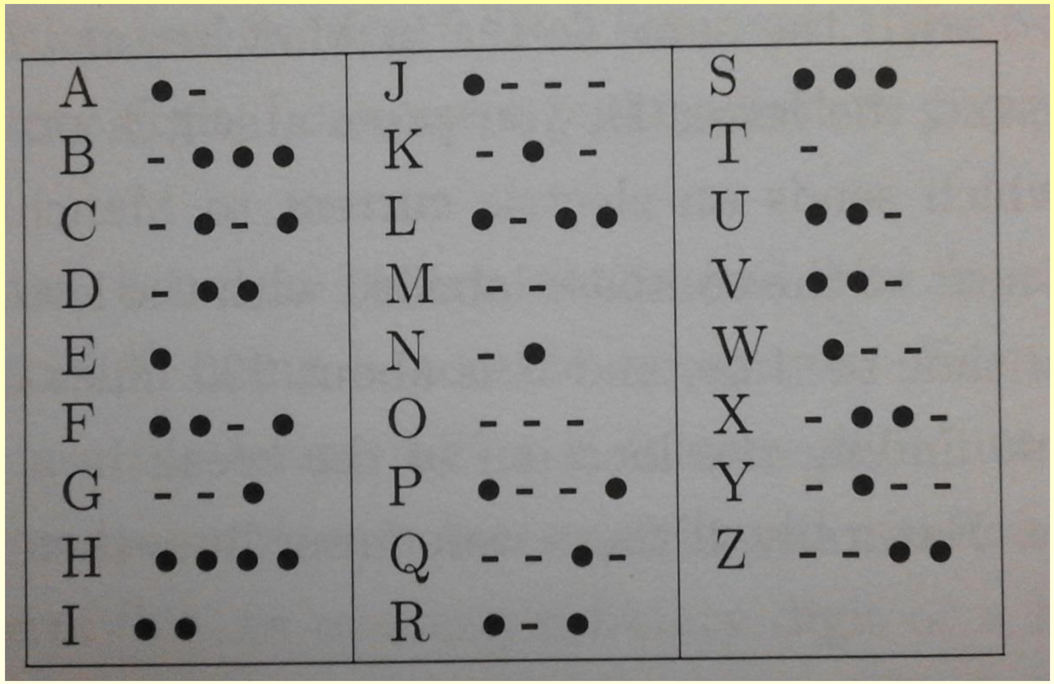
Shannon Information Theory

Object of Information Theory

To provide a mathematical approach to the acquisition, coding and communication of information

What is information?

Information coding – Telegraphy, Morse code



A	• –	J	• – – –	S	• • •
B	– • • •	K	– • –	T	–
C	– • – •	L	• – • •	U	• • –
D	– • •	M	– –	V	• • –
E	•	N	– •	W	• – –
F	• • – •	O	– – –	X	– • • –
G	– – •	P	• – – •	Y	– • – –
H	• • • •	Q	– – • –	Z	– – • •
I	• •	R	• – •		

26 letters are encoded in 26 *codewords*

Frequent letters → shorter codewords

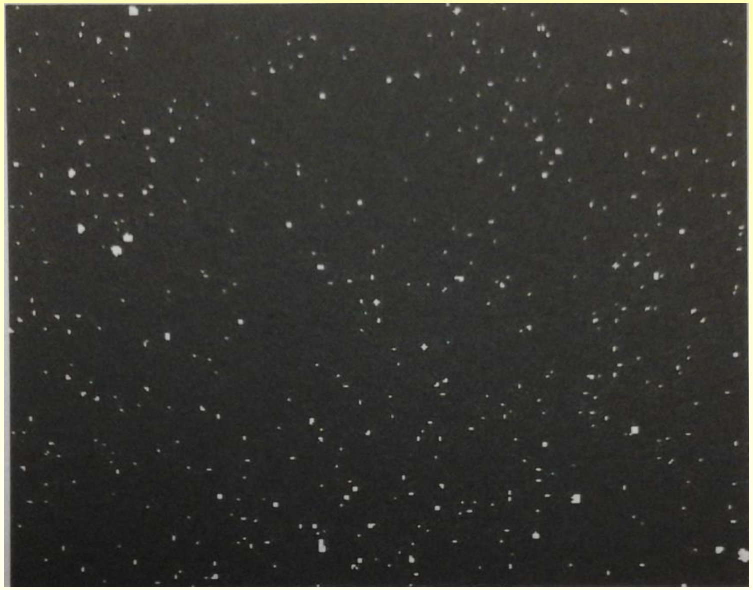
Infrequent letters → longer codewords

→ Optimization of transmission!

Fundamental question of information theory

How can we tell if a communication channel is being used as efficiently as possible?

Black and white image of stars
(100 x 100 pixels)



Coding method 1:

100 x 100 binary digits
black → 0
white → 1

10,000 bits

Very inefficient coding
for this image!

Fundamental question of information theory

How can we tell if a communication channel is being used as efficiently as possible?

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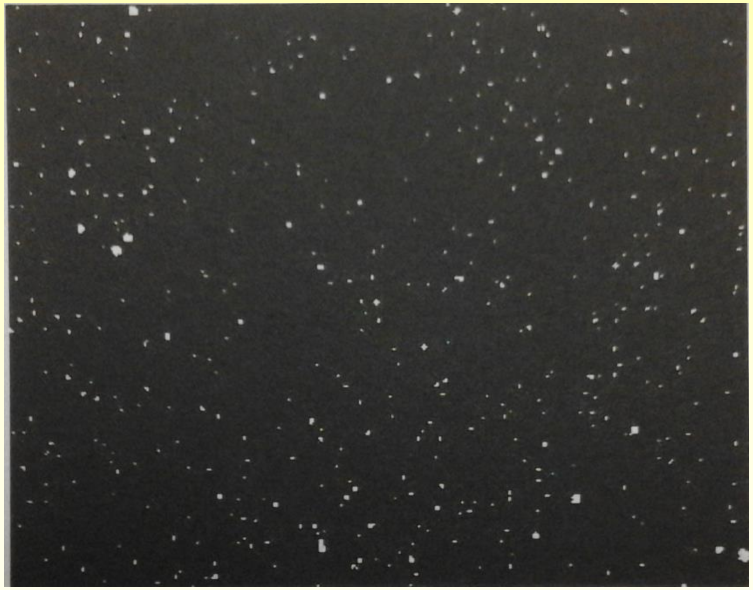

Coding method 2:

Send location of white pixels

Code: [(19,13),(22,30),...]

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Black and white image of stars
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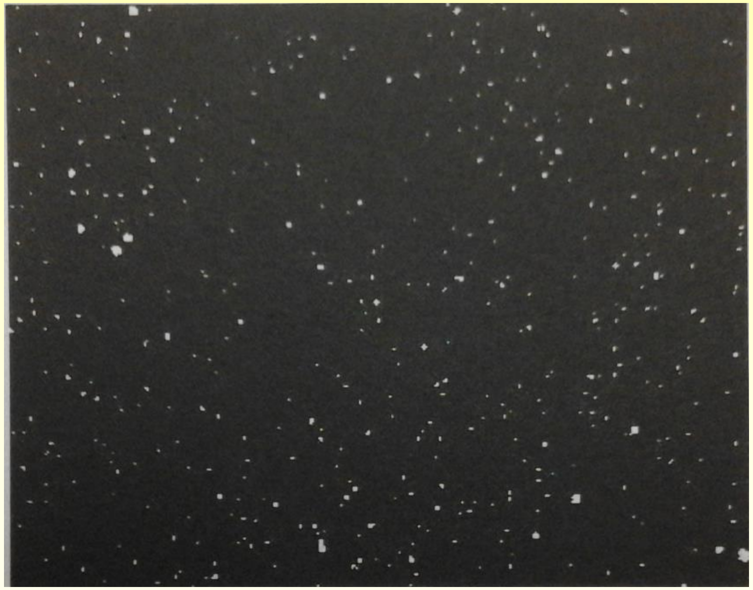


Coding method 3:

Concatenate the rows

Send the position of the white pixels

Code: [13,9,...]

The choice of the best method will depend of the type of image

Fundamental question of information theory

How can we tell if a communication channel is being used as efficiently as possible?

Black and white image
(100 x 100 pixels)

Coding method 3:

Concatenate the rows

Send the position of the white pixels

Too many consecutive white pixels!

Better to send the positions where the color changes

Fundamental question of information theory

How can we tell if a communication channel is being used as efficiently as possible?

Grey-level image
(100 x 100 pixels)

Color coded in 256 grey levels

$log_2 256 = 8$ bits/pixel

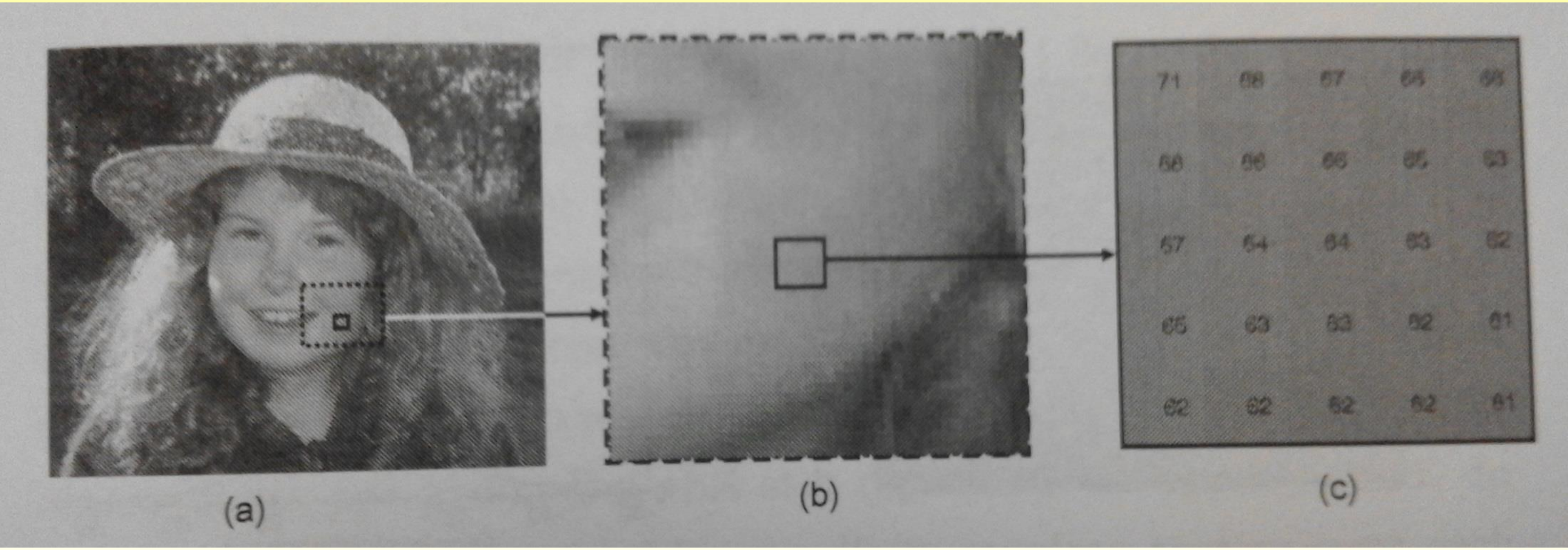
80,000 bits per image!

Can we make it better?

Notice the *redundancy* in the image

Fundamental question of information theory

How can we tell if a communication channel is being used as efficiently as possible?

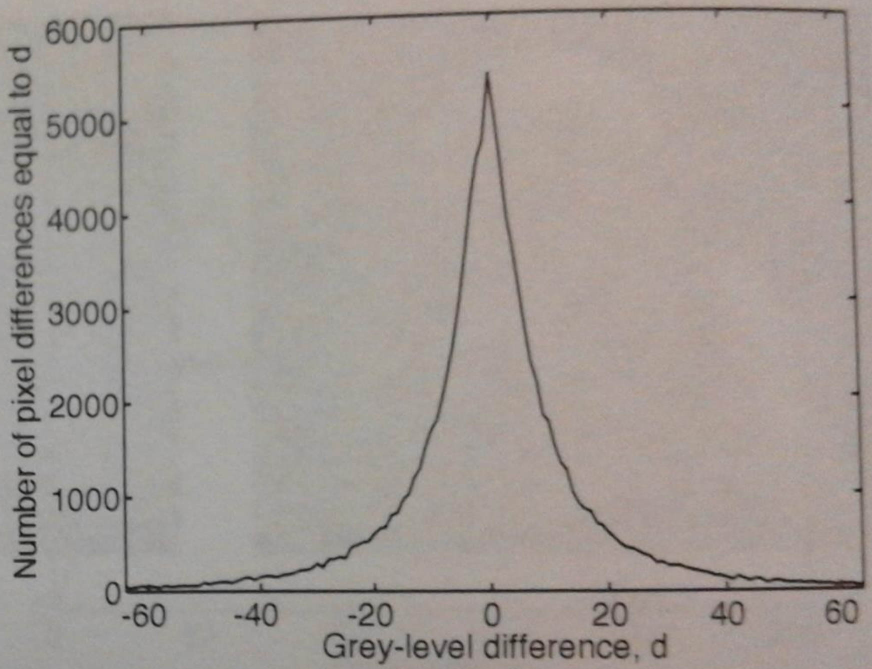


Redundancy in the color levels:
Most grey levels in contiguous pixels are not independent

Fundamental question of information theory

How can we tell if a communication channel is being used as efficiently as possible?

Image reconstructed from the differences in grey level of contiguous pixels

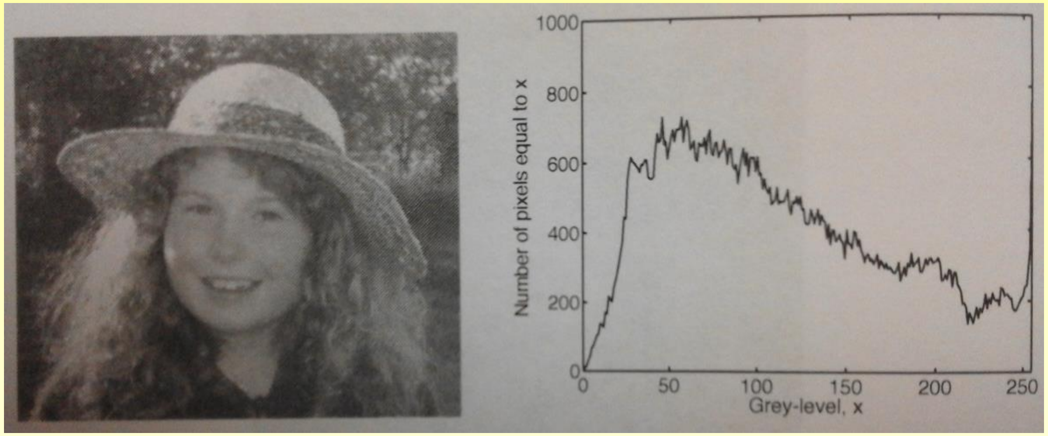


Most values of grey level differences are in a narrow interval

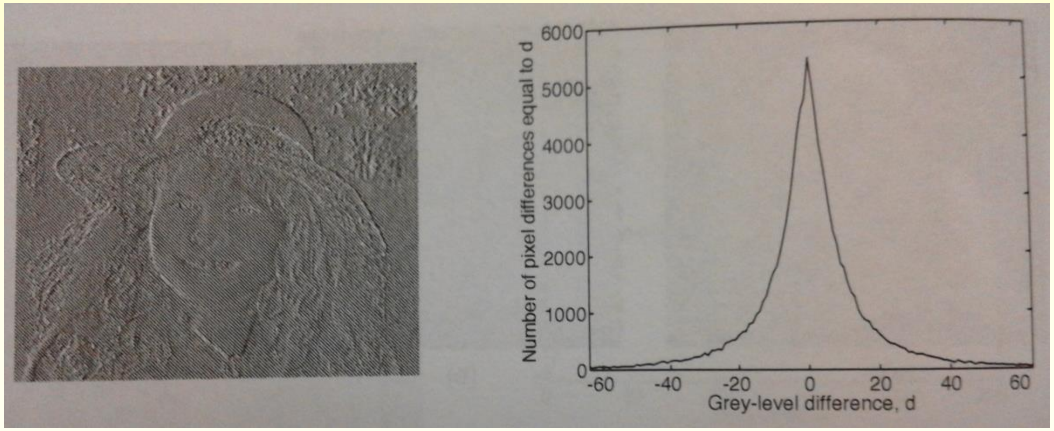
Most information can be encoded in 127 values → 7 bits/pixel

Fundamental question of information theory

How can we tell if a communication channel is being used as efficiently as possible?



$log_2 256 = 8$ bits/pixel



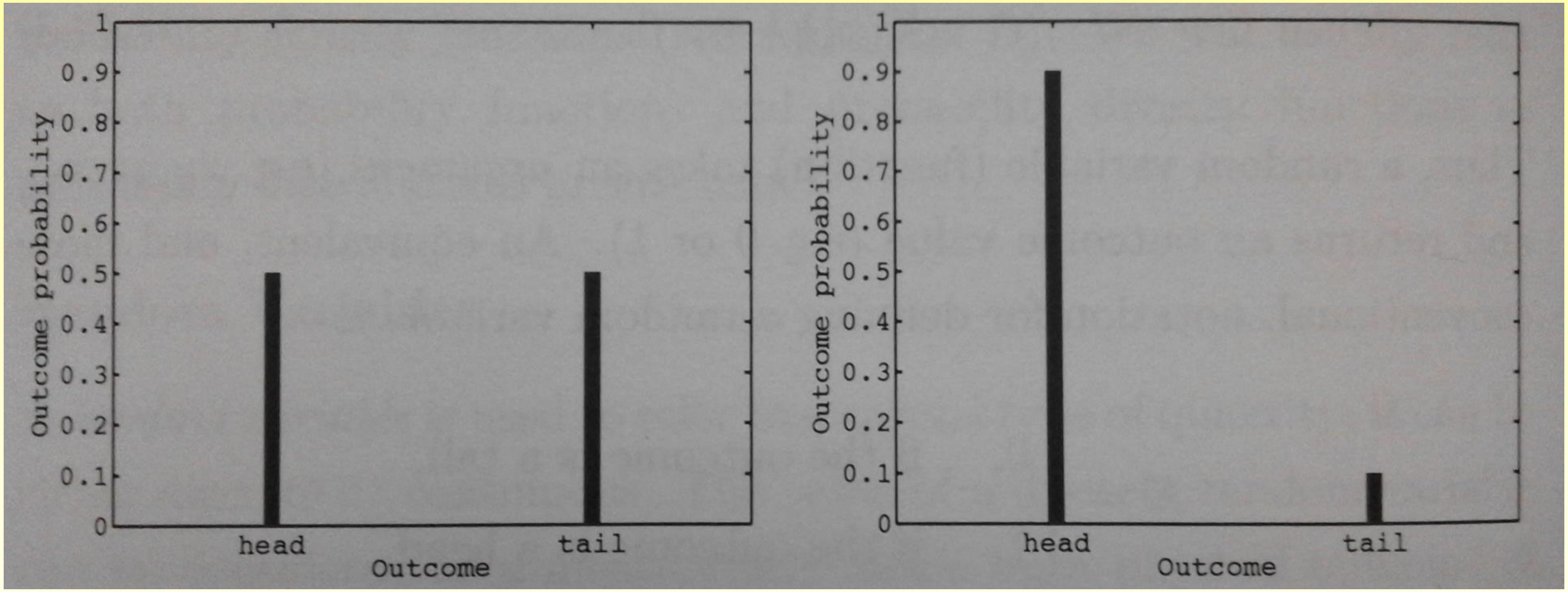
$log_2 127 = 7$ bits/pixel

How much actual <u>information</u> does each pixel contain?

How to measure information?

Flipping coins:

Unbiased coin (50-50) Biased coin (90-10)



Unexpected → Informative Expected → Not informative

Information should be inversely proportional to the expectancy:
$h(x) \sim 1/p(x)$

How to measure information?

Mathematical properties of Shannon Information:

- **Continuity**: continuous function of the probability of possible outcomes

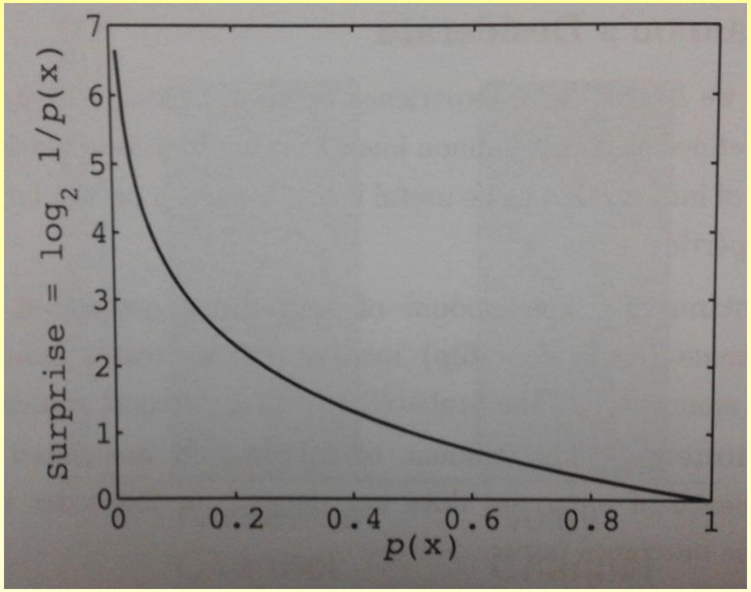
- **Additive**: the information associated with a set of outcomes is obtained by adding the information of individual outcomes

- **Symmetry**: the information associated with a sequence of outcomes does not depend on the order in which those outcomes occur

- **Maximal value**: information is maximal for outcomes that occur with equal probability

How to measure information?

Properties of Shannon Information



$$h(x) \sim 1/p(x) \rightarrow h(x) = log_2(1/p(x)) \text{ bits}$$
$$= -log_2(p(x)) \text{ bits}$$

But we are actually interested in the average information contained in a set of possible values

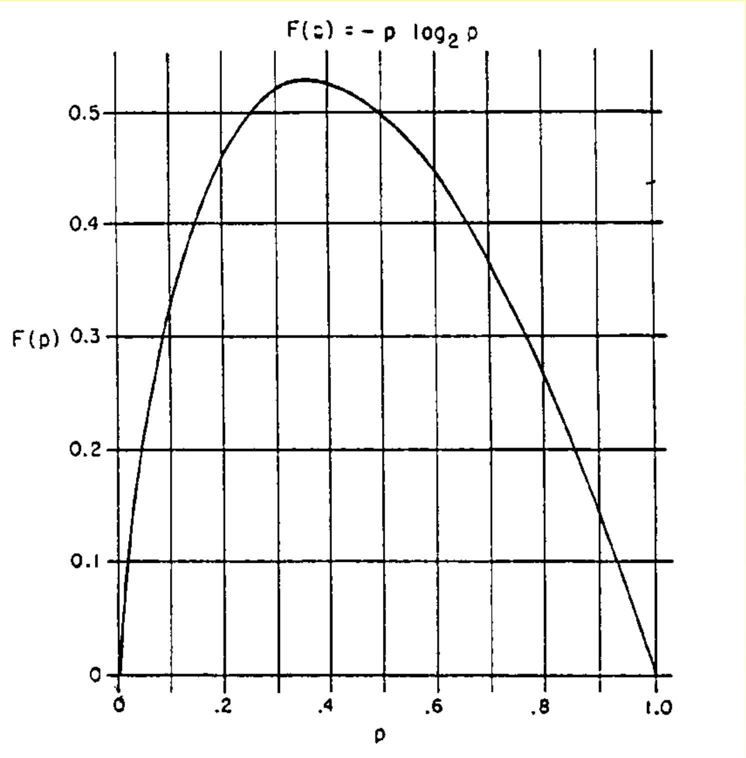
$$H(X) = \sum_{i=1}^{n} p(x_i) log_2 \frac{1}{p(x_i)}$$

Shannon Entropy

How to measure information?

Properties of Shannon Entropy:

$$H(X) = \sum_{i=1}^{n} p(x_i) log_2 \frac{1}{p(x_i)}$$



Entropy is always be larger than or equal to zero

An event with small probability has small contribution to total uncertainty

The entropy of an experiment is only zero if one of the probabilities equals 1

For the case of equally probable events, the Shannon entropy reduces to Hartley's entropy

The uncertainty is largest for events with equal probability: $p(x_i) = 1/n$; $H_{max} = \log n$

Entropy can be interpreted as the average surprise value of the different outcomes

Entropy

Information theory:

$$H(X) = \sum_{i=1}^{n} p(x_i) log_2 \frac{1}{p(x_i)}$$ (Shannon)

in case of equal probabilities:

H = log n (Hartley)

Uncertainty is related to number of possibilities

Thermodynamics: S = k log W (Boltzmann)

(S = entropy, W number of possible microscopic states, k Boltzmann constant)

Entropy is related to number of different possible states

In thermodynamics as well as in information theory:

Entropy is related to disorder, uncertainty, number of possible states

Entropy

Uncertainty of experiment with 2 possible outcomes as function of p

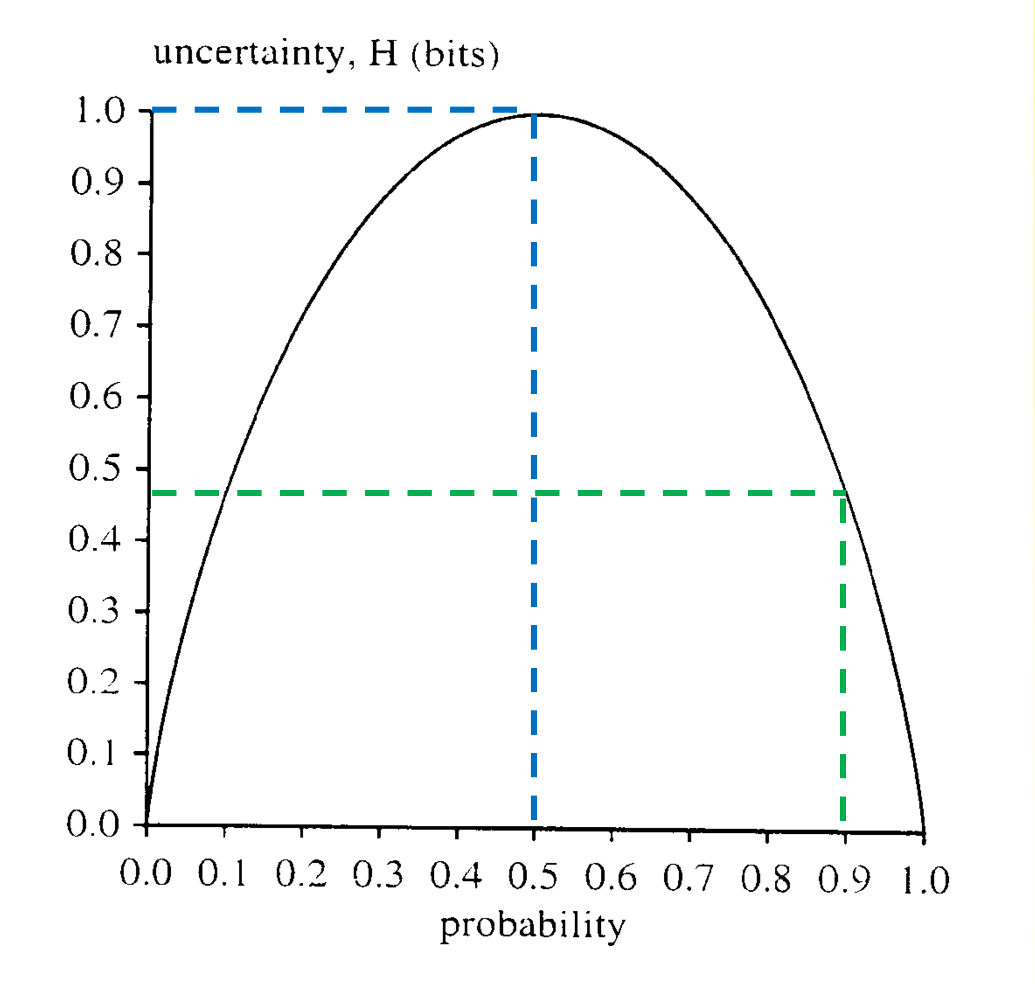
Unbiased coin

$H = -0.5\log_2 0.5 - 0.5\log_2 0.5$
$H = 1$ bit

Biased coin

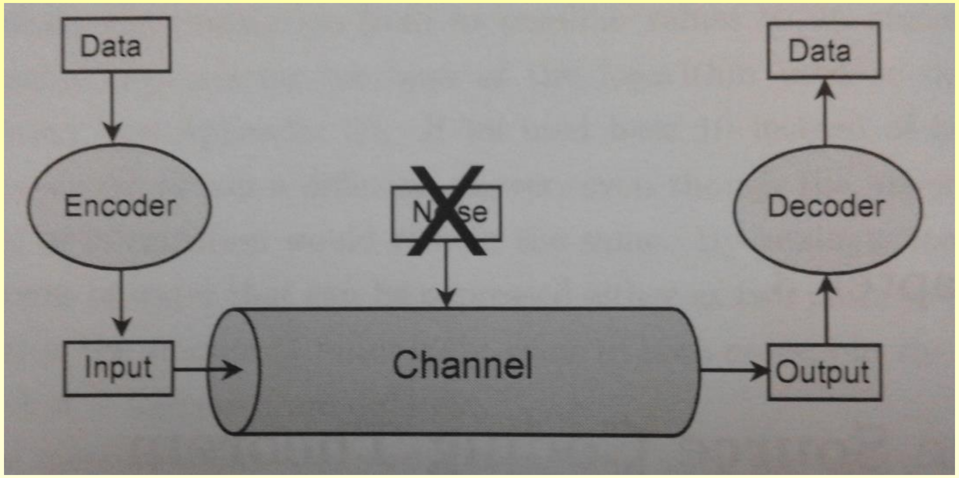
$H = -0.9\log_2 0.9 - 0.1\log_2 0.1$
$H = 0.469$ bit

The biased coin is like an unbiased coin with $2^{0.469} = 1.38$ sides



Fundamental question of information theory

How can we tell if a communication channel is being used as efficiently as possible?



Most natural signals contain information in a diluted form
Example: contiguous “pixels” tend to have similar values

For efficient communication (coding):
- Inputs should be transformed to signals with independent values
- The transformed signal should have a distribution optimized for the particular channel

Shannon Source Coding Theorem

Channel capacity: *C* = # of information bits communicated per second

But! The amount of bits that can be encoded in binary digits depend on the coding method

The **Shannon Source Coding Theorem** states that:
For every channel there is a <u>coding method</u> for which it is possible to transmit at an average of C/H - ε symbols per second, where ε is arbitrarily small.

Shannon Source Coding Theorem

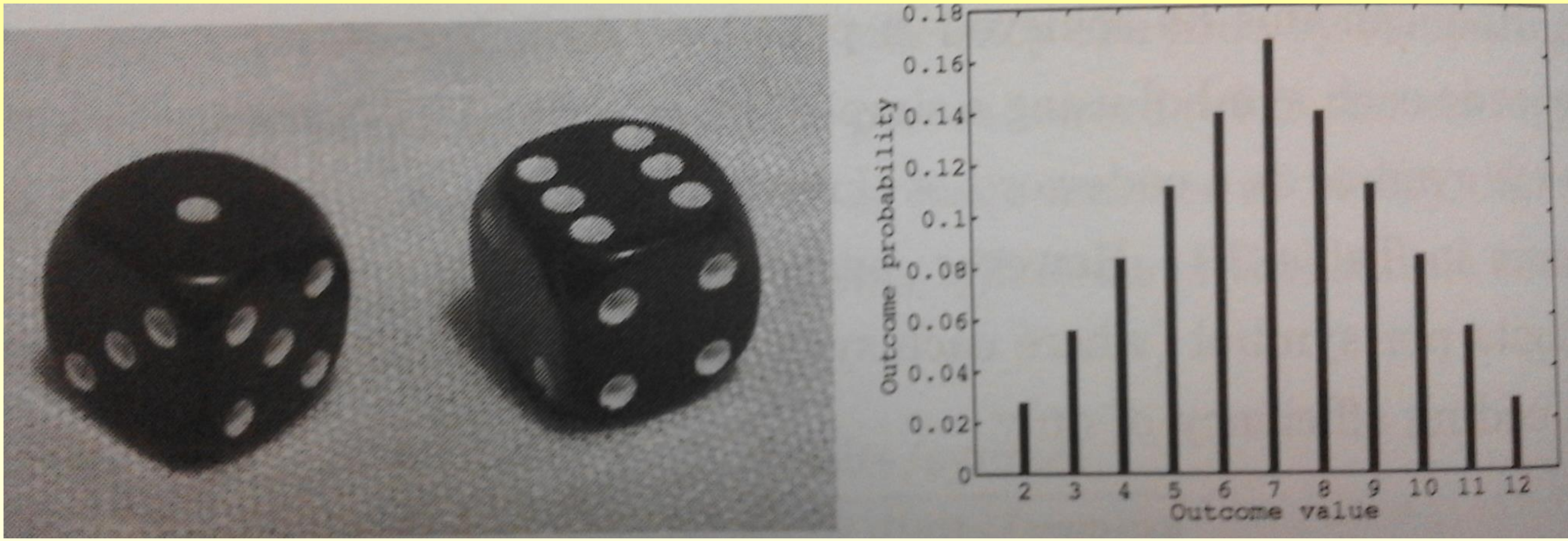
Channel capacity: C = # of information bits communicated per second

But! The amount of bits that can be encoded in binary digits depend on the coding method

The full capacity of a channel is utilized if the source is encoded in such a way that each transmitted binary digit represents an average of one bit of information

Data compression

Throw of 2 6-sided dice



H = 3.27 bits/symbol

3 binary digits are not enough to code all outputs

4 binary digits are too many and give a coding efficiency
H/Lenght = 0.818 bits/binary digit

Data compression

Throw of 2 6-sided dice

Symbol	Sum	Dice	Freq	p	h	Code x
s_1	2	1:1	1	0.03	5.17	10000
s_2	3	1:2, 2:1	2	0.06	4.17	0110
s_3	4	1:3, 3:1, 2:2	3	0.08	3.59	1001
s_4	5	2:3, 3:2, 1:4, 4:1	4	0.11	3.17	001
s_5	6	2:4, 4:2, 1:5, 5:1, 3:3	5	0.14	2.85	101
s_6	7	3:4, 4:3, 2:5, 5:2, 1:6, 6:1	6	0.17	2.59	111
s_7	8	3:5, 5:3, 2:6, 6:2, 4:4	5	0.14	2.85	110
s_8	9	3:6, 6:3, 4:5, 5:4	4	0.11	3.17	010
s_9	10	4:6, 6:4, 5:5	3	0.08	3.59	000
s_{10}	11	5:6, 6:5	2	0.06	4.17	0111
s_{11}	12	6:6	1	0.03	5.17	10001

$\langle L \rangle = \text{Sum } p(x_i)L(x_i) = 3.31$

Then the coding efficiency: H/L = 3.27/3.31 bits/binary digit

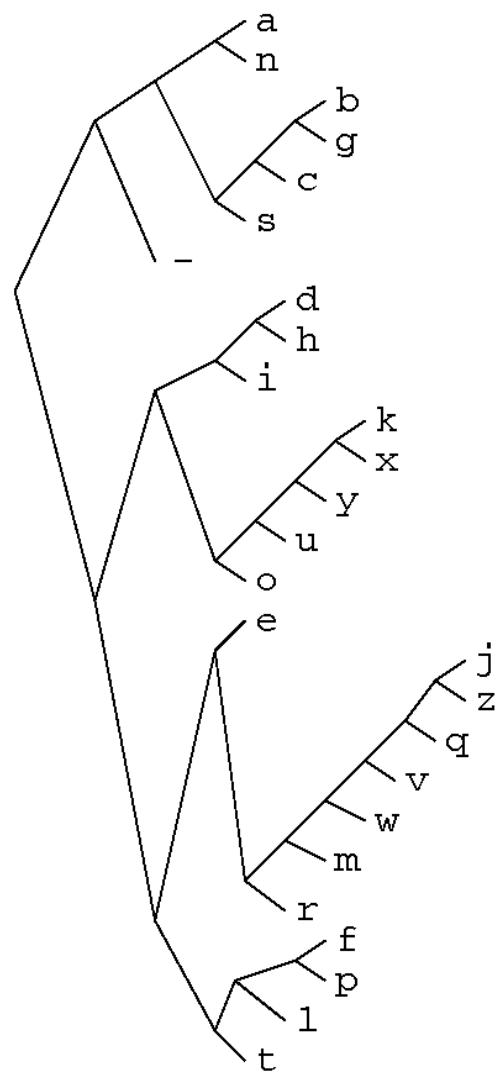
= 0.99 bits/binary digit

Optimal prefix code: Huffman code
Principle of Huffman code

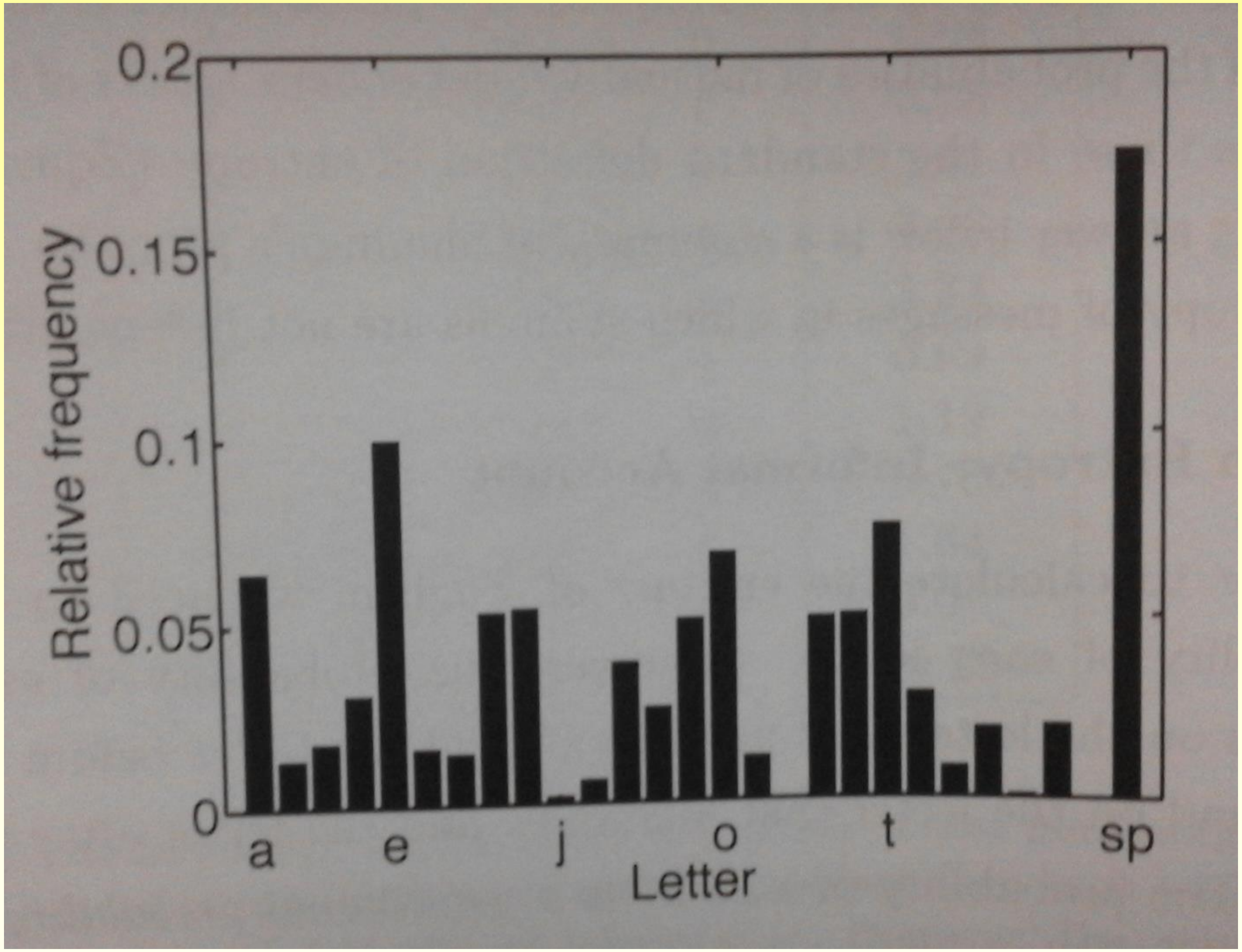
Source Character	$P(a_i)$	$P(a_i')$	$P(a_i'')$	$P(a_i''')$	Code Word
a_1	0.3	0.3	0.45	1 0.55	11
a_2	0.25	0.25	1 0.3	0 0.45	10
a_3	0.25	1 0.25	0 0.25		01
a_4	1 0.1	0 0.2			001
a_5	0 0.1				000
E.g. a_4	1	0	-	0	= 001

Huffman code for English

a_i	p_i	$\log_2 \frac{1}{p_i}$	l_i	$c(a_i)$
a	0.0575	4.1	4	0000
b	0.0128	6.3	6	001000
c	0.0263	5.2	5	00101
d	0.0285	5.1	5	10000
e	0.0913	3.5	4	1100
f	0.0173	5.9	6	111000
g	0.0133	6.2	6	001001
h	0.0313	5.0	5	10001
i	0.0599	4.1	4	1001
j	0.0006	10.7	10	1101000000
k	0.0084	6.9	7	1010000
l	0.0335	4.9	5	11101
m	0.0235	5.4	6	110101
n	0.0596	4.1	4	0001
o	0.0689	3.9	4	1011
p	0.0192	5.7	6	111001
q	0.0008	10.3	9	110100001
r	0.0508	4.3	5	11011
s	0.0567	4.1	4	0011
t	0.0706	3.8	4	1111
u	0.0334	4.9	5	10101
v	0.0069	7.2	8	11010001
w	0.0119	6.4	7	1101001
x	0.0073	7.1	7	1010001
y	0.0164	5.9	6	101001
z	0.0007	10.4	10	1101000001
–	0.1928	2.4	2	01



Huffman code for English



Uncertainties in English

H_0	H_1	H_2	H_3	H_5	H_8	H_∞
4.75	4.03	3.32	3.10	2.16	1.86	1.33 bit

A character in English texts thus contains not much more than 1 bit of information!

The redundancy $R \simeq 72\%$

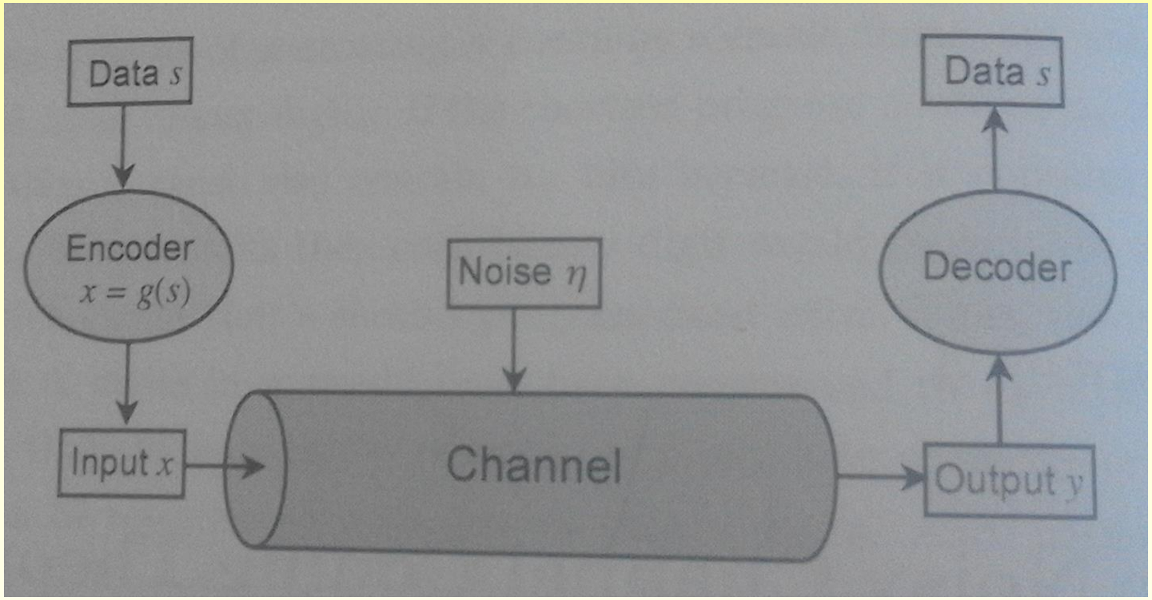
Th_r_ _s _nly _n_ w_y t_ f_ll _n th_ v_w_ls _n th_s s_nt_nc_.

Information in communication:

Shannon Information Theory II

The Noisy Channel Coding Theory

Communication of information in the presence of noise



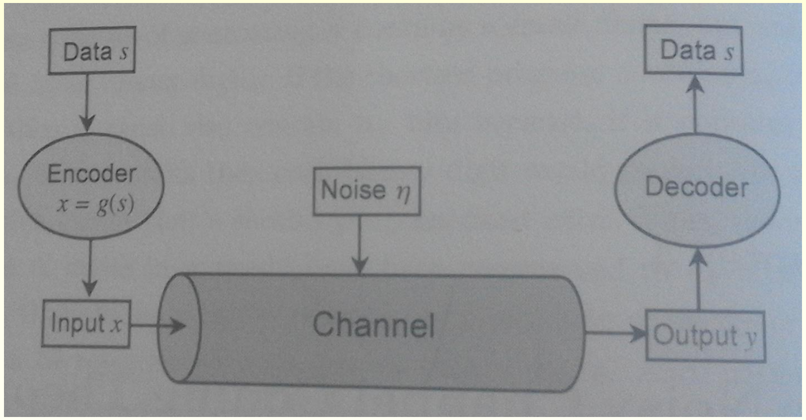
The concept of Mutual Information

- Is a general measure of association between two variables: (input and output)

- For the variables X and Y, the mutual information I(X,Y) is:

The average information we gain about Y after knowing a single value of X, (x_i)

- Symmetrical: I(X,Y) = I(Y,X)

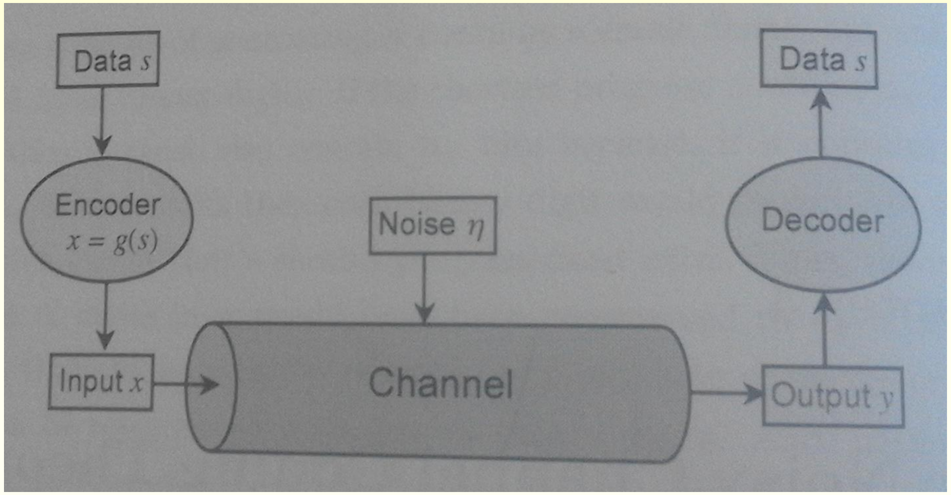


The concept of Mutual Information

- I(X,Y) is the average reduction in <u>uncertainty</u> about Y, H(Y), after knowing a value of X, (x_i) and vice versa

H(Y) → reading X → residual uncertainty about Y: H(Y|X)

H(Y|X) is called *conditional entropy*



Because Y = X + η

then H(Y|X) = H(η)

H(η) is the entropy of a *joint distribution*

Error correcting codes

Input Output

Why do we still see the image,
even in the presence of a lot of noise?

The image has a lot of Redundancy

Redundancy could be used to correct for transmission errors

Error correcting codes

s = [1101001101011000]

Arrange in a grid:

s =

1	1	0	1
0	0	1	1
0	1	0	1
1	0	0	0

Add binary digit for parity check

x =

1	1	0	1	1
0	0	1	1	0
0	1	0	1	0
1	0	0	0	1
0	0	1	1	-

Error correcting codes

s = [1101001101011000]

Error detection:

$\mathbf{y} =$

0*	1	0	1	1
0	0	1	1	0
0	1	0	1	0
1	0	0	0	1
0	0	1	1	-

This allows detecting 1 error in 4x4 + 2x4 binary digits

If using an nxn grid we can detect 1 error in $n^2 + 2n$ binary digits

This increases the number of binary digits in a factor:
$(n^2 + 2n)/n^2 = 1+2/n$ binary digits

The investment in parity digits improves with n, but allows correcting only 1 binary digit

Redundancy: Good and Bad

Input Output

Redundant data is more resistance to errors

but

large data processing to recover a small amount of original information

Capacity of a Noisy Channel

$$\boldsymbol{C} = \max_{\boldsymbol{p}(\boldsymbol{X})} \boldsymbol{I}(\boldsymbol{X}, \boldsymbol{Y})$$

$$\boldsymbol{I}(\boldsymbol{X}, \boldsymbol{Y}) = \mathbf{H}(\mathbf{X}) - \mathbf{H}(\mathbf{X}|\mathbf{Y})$$

Shannon's theorem for noisy channels

For a channel with capacity C and source of entropy H,

- If H < C there is a code allowing transmission with an arbitrarily small error
- If H > C there is a code allowing transmission with an error close to H – C

- It is possible to communicate information with a low error at a rate close to the channel capacity.
- It is not possible to communicate with no error at a rate higher than C

Information capacity in the nervous system I

Spread of electric signals: passive vs. active propagation

Passive propagation:

- “Basic” electrical properties of cells
- Conductance is voltage-independent
- Decays with space and time
- Important biological function!

Active propagation:

- “Special” electrical properties of cells
- Conductances are voltage-dependent
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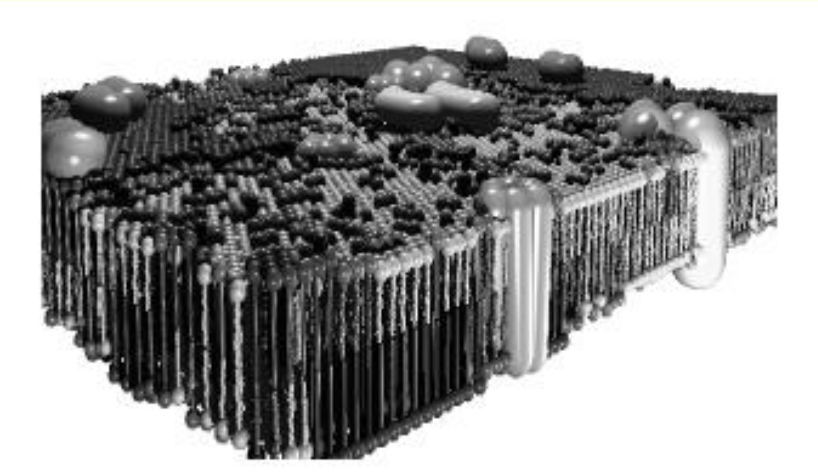
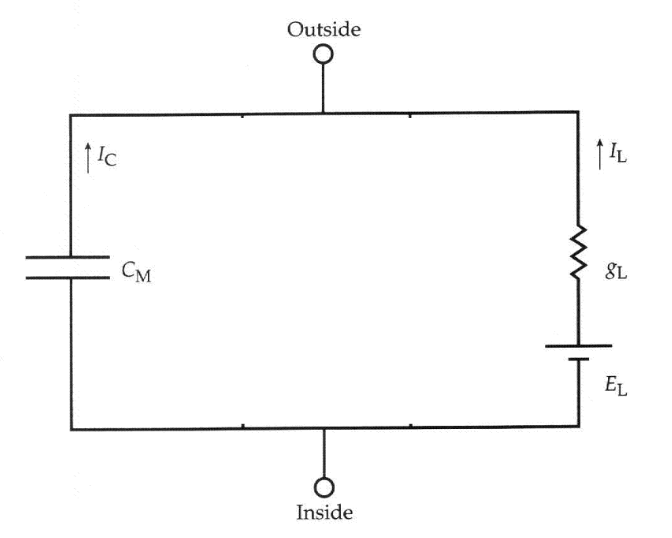


Membrane model:



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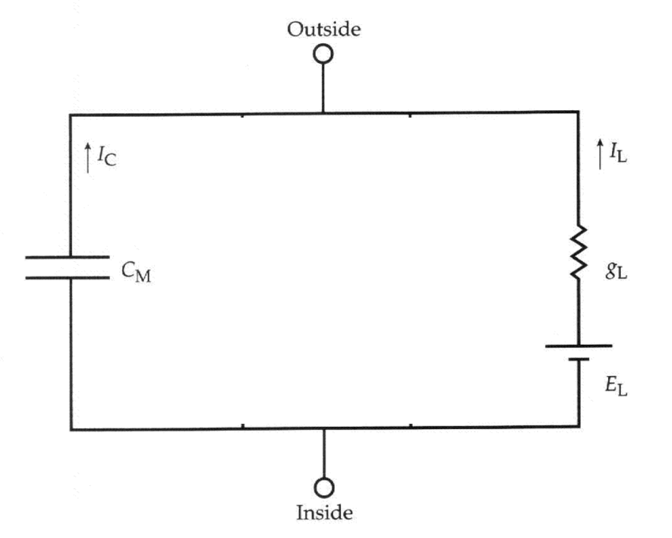
$$I_{STIMULUS} = C\frac{dV}{dt} + \frac{(V - V_{REST})}{R}$$

$$R(V,t) = R$$

$$V(t) = V_{REST} + I_{STIMULUS} R\left(1 - \mathrm{e}^{-t/RC}\right)$$

Membrane model:



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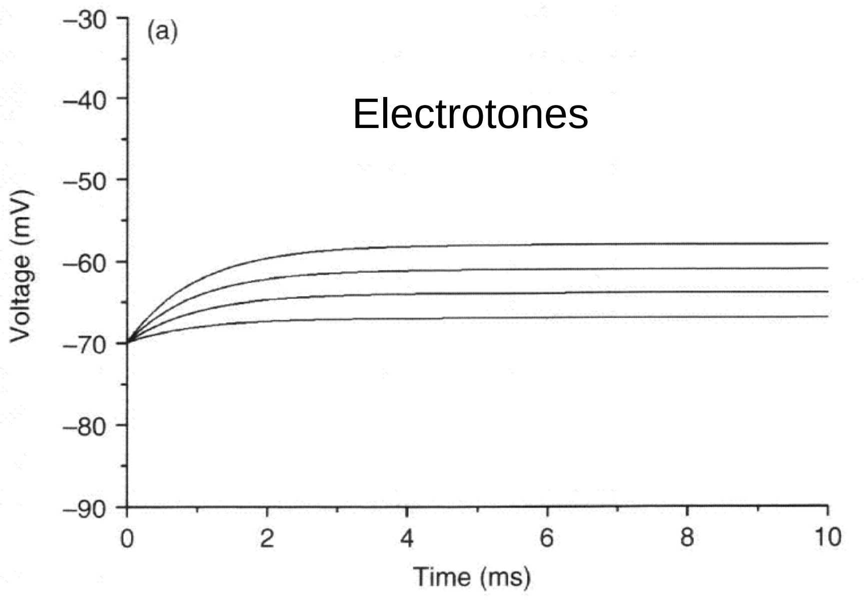

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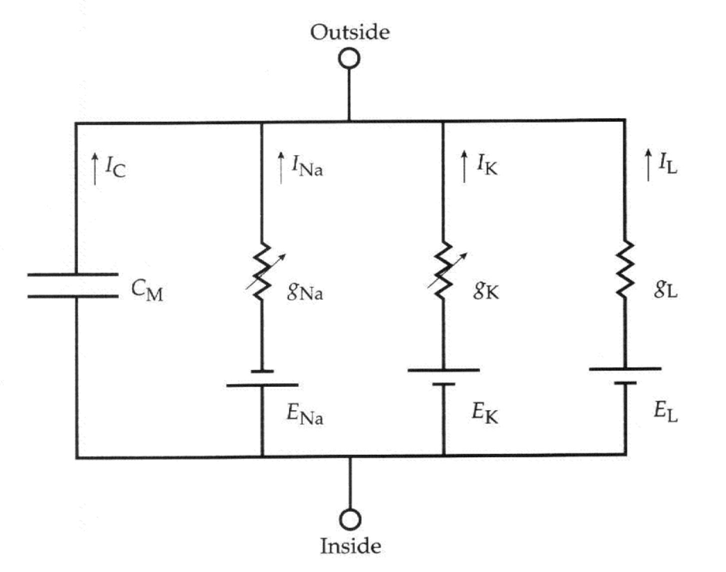
Active properties

Membrane model:



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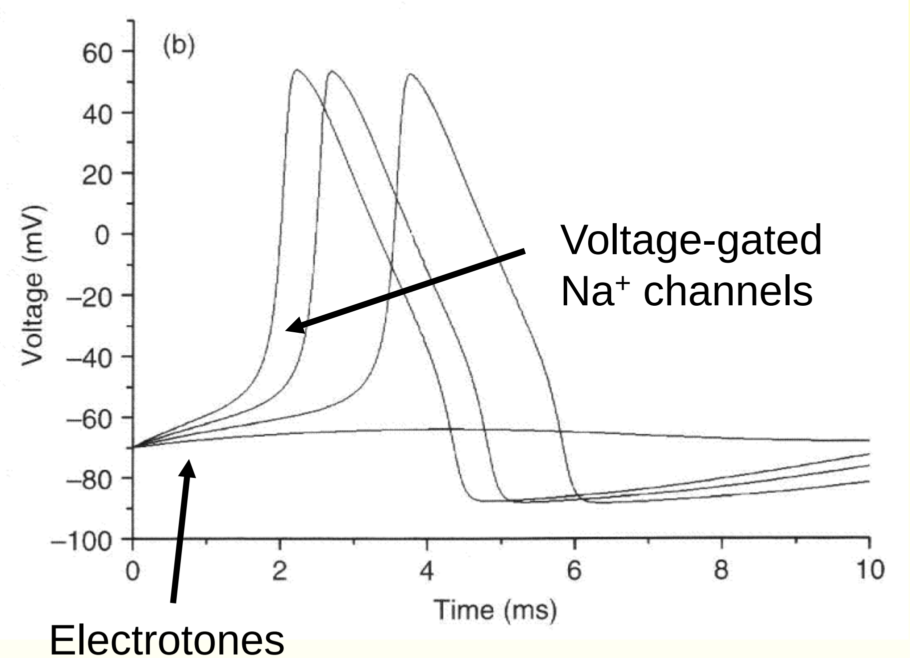
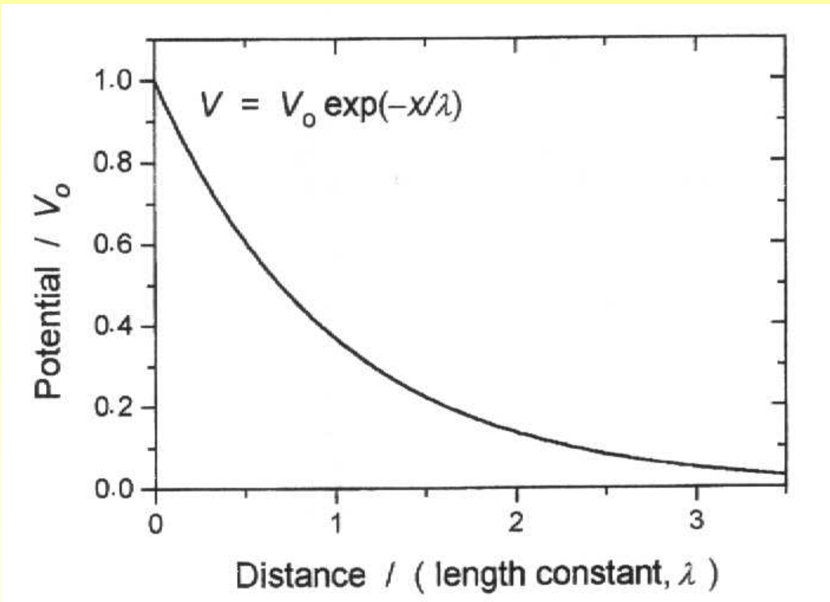
Active properties



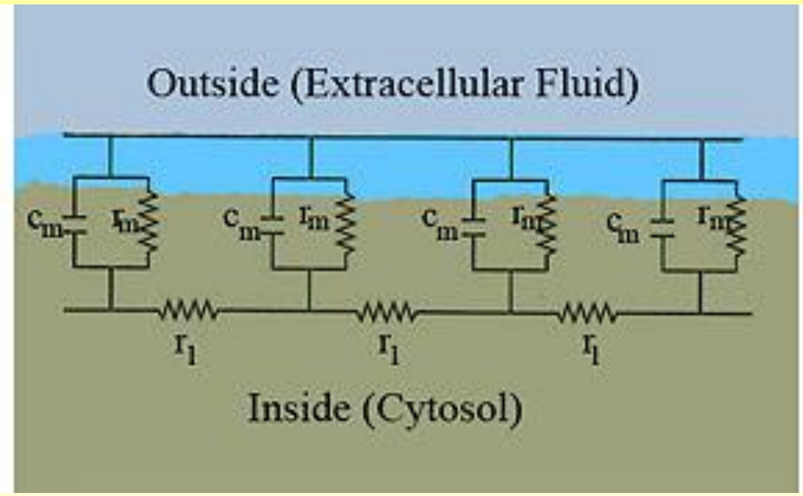
Electrotones

Conduction along the axons: Cable theory

The Cable equation: Steady-state



Axon equivalent circuit:



$$V(x) = \frac{r_m}{r_i}\frac{\partial^2 V}{\partial x^2}$$

$$V(x=0) = V_0$$

$$V(x=\infty) = 0$$

$$V(x) = V_0 e^{-x/\lambda}$$

$$\lambda = \sqrt{r_m / r_i}$$

→ Space constant $\lambda = \sqrt{\dfrac{a\rho_m}{2\rho_i}}$

Conduction along the axons: Cable theory

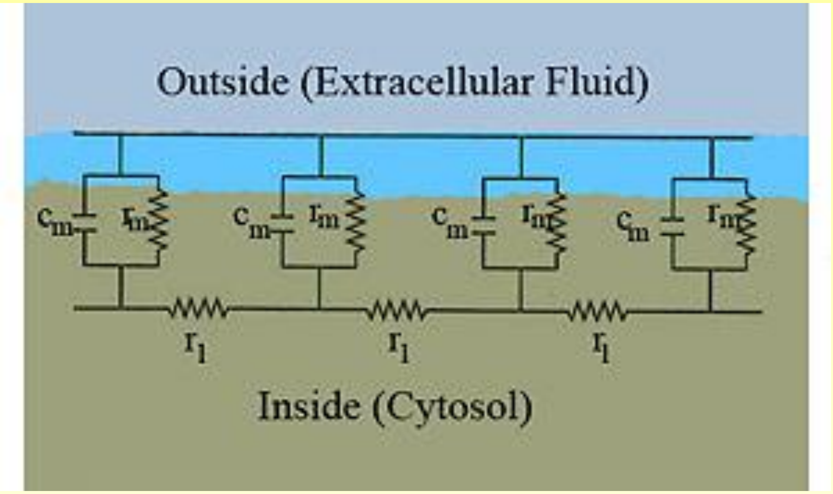
The Cable equation: Time dependency

$$V(x,t) = \frac{r_m}{r_i}\frac{\partial^2 V}{\partial x^2} - r_m c_m \frac{\partial V}{\partial t}$$

$$V(x,t) = \lambda^2 \frac{\partial^2 V}{\partial x^2} - \tau_m \frac{\partial V}{\partial t}$$

$$X = \frac{x}{\lambda} \qquad T = \frac{t}{\tau_m} = \frac{t}{r_m c_m}$$

Axon equivalent circuit:



Initial condition:

$$V(x;t=0)=0$$

Boundary conditions:

$$V(x=0;t)=V_0$$

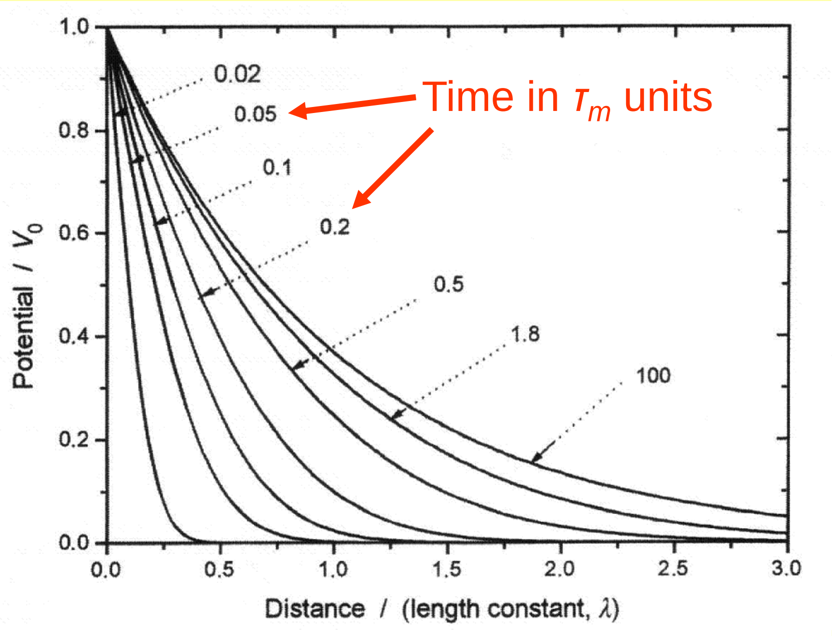
$$V(x=\infty;t)=0$$

$$V(x,t)=\frac{1}{2}V_0\left\{e^{-X}Erfc\left(\frac{X}{2\sqrt{T}}-\sqrt{T}\right)+e^{X}Erfc\left(\frac{X}{2\sqrt{T}}+\sqrt{T}\right)\right\}$$

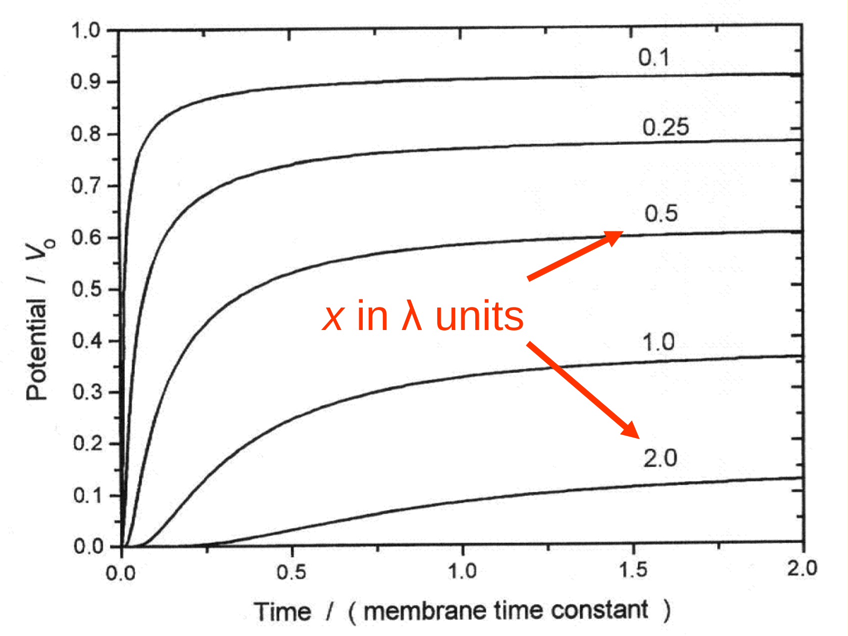
$$Erfc(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-u^2}du$$

Conduction along the axons: Cable theory

The Cable equation: Time dependency



Axon equivalent circuit:

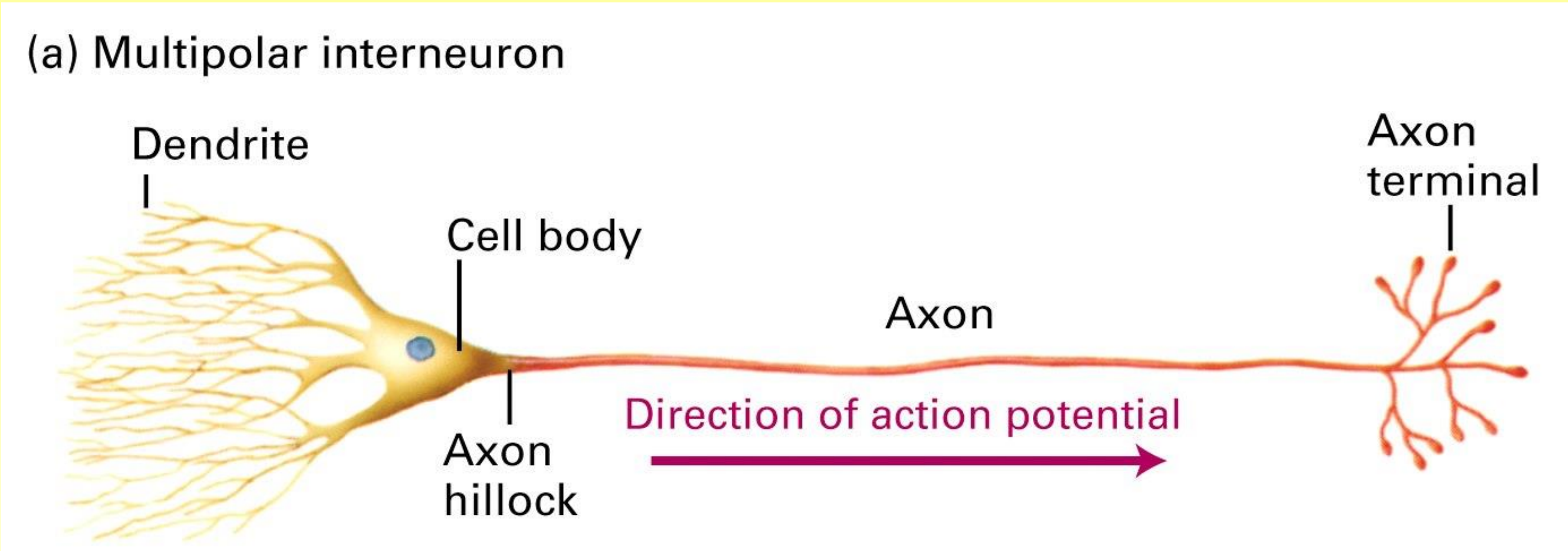


$$V(x,t) = \frac{1}{2} V_0 \left\{ e^{-X} Erfc\left(\frac{X}{2\sqrt{T}} - \sqrt{T} \right) + e^{X} Erfc\left(\frac{X}{2\sqrt{T}} + \sqrt{T} \right) \right\}$$

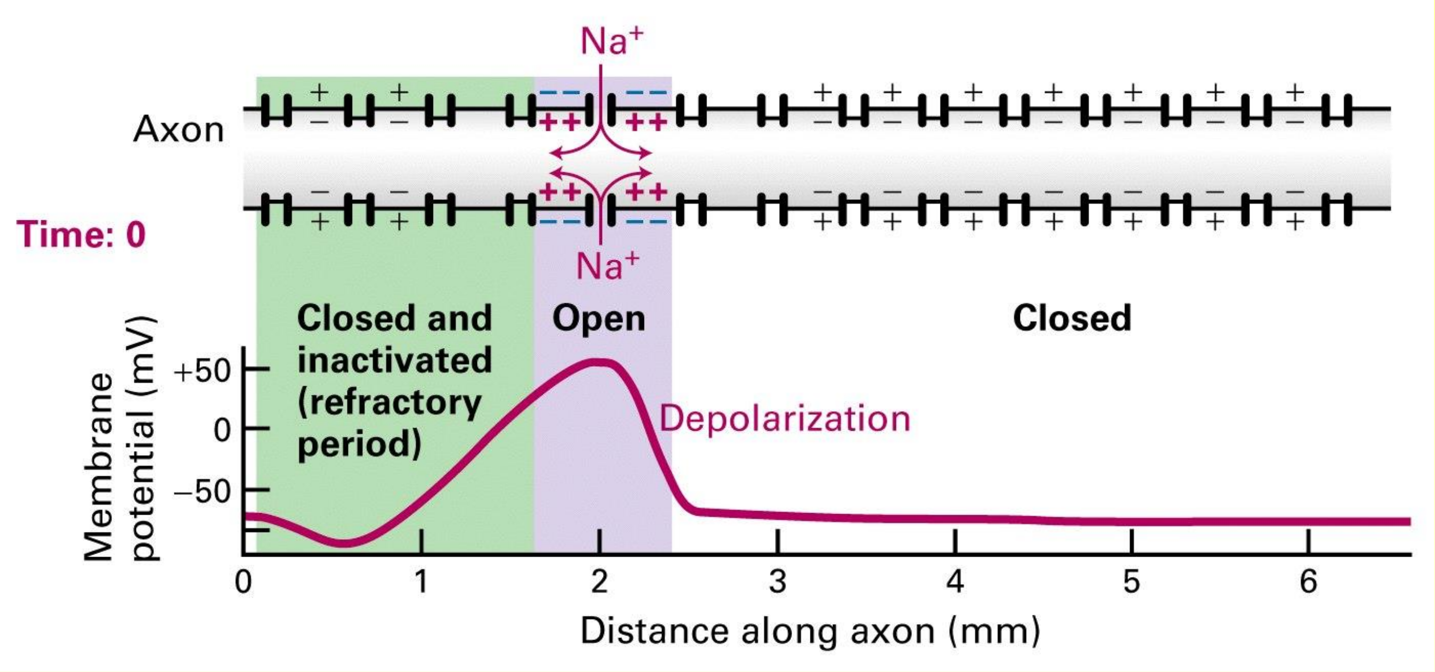
$$Erfc(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-u^2} du$$

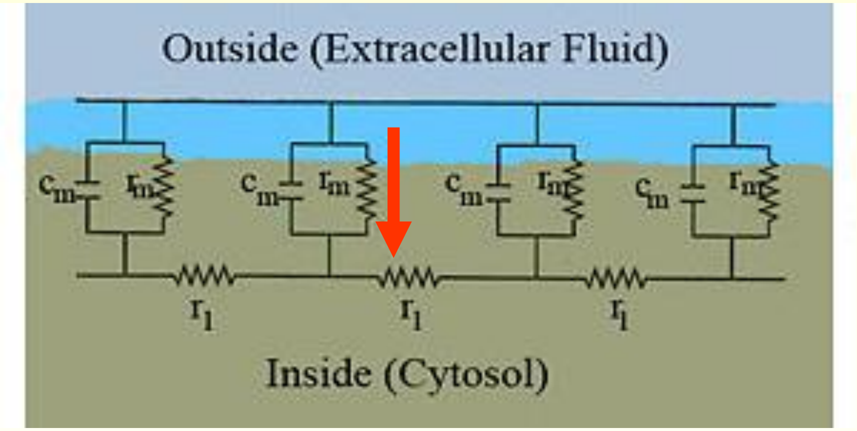
Conduction along the axons: Cable theory

Conduction of the Nerve Action Potential

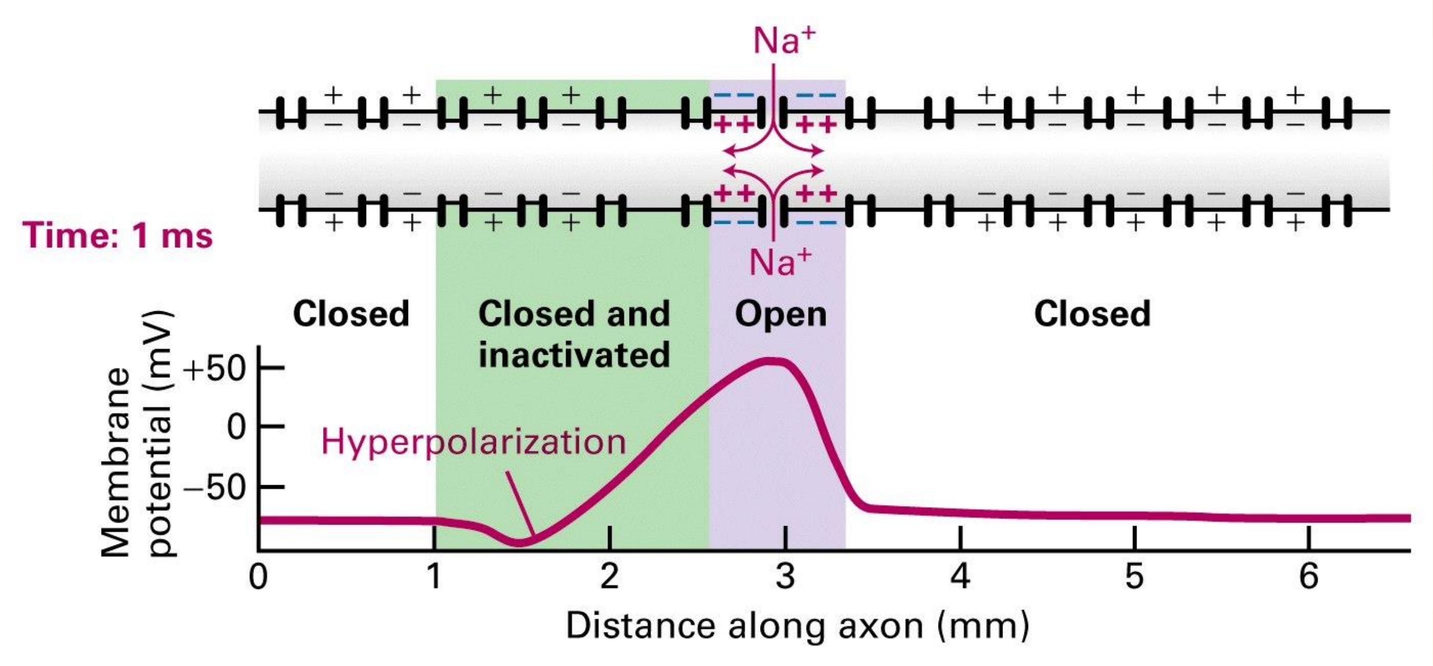


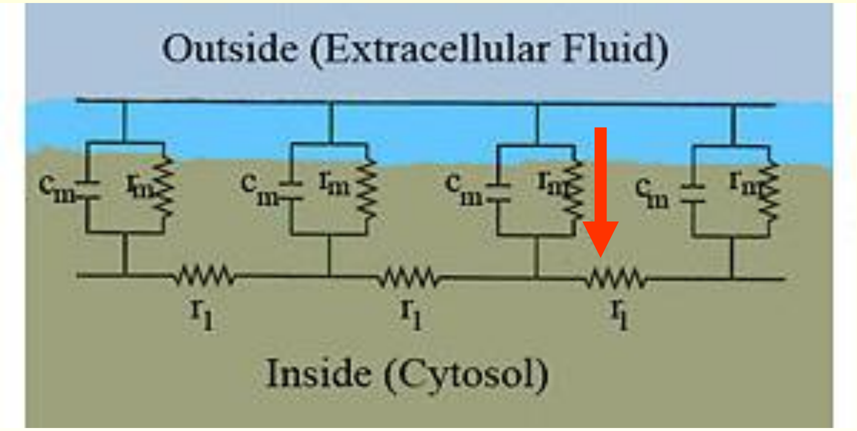
Conduction along the axons: Cable theory



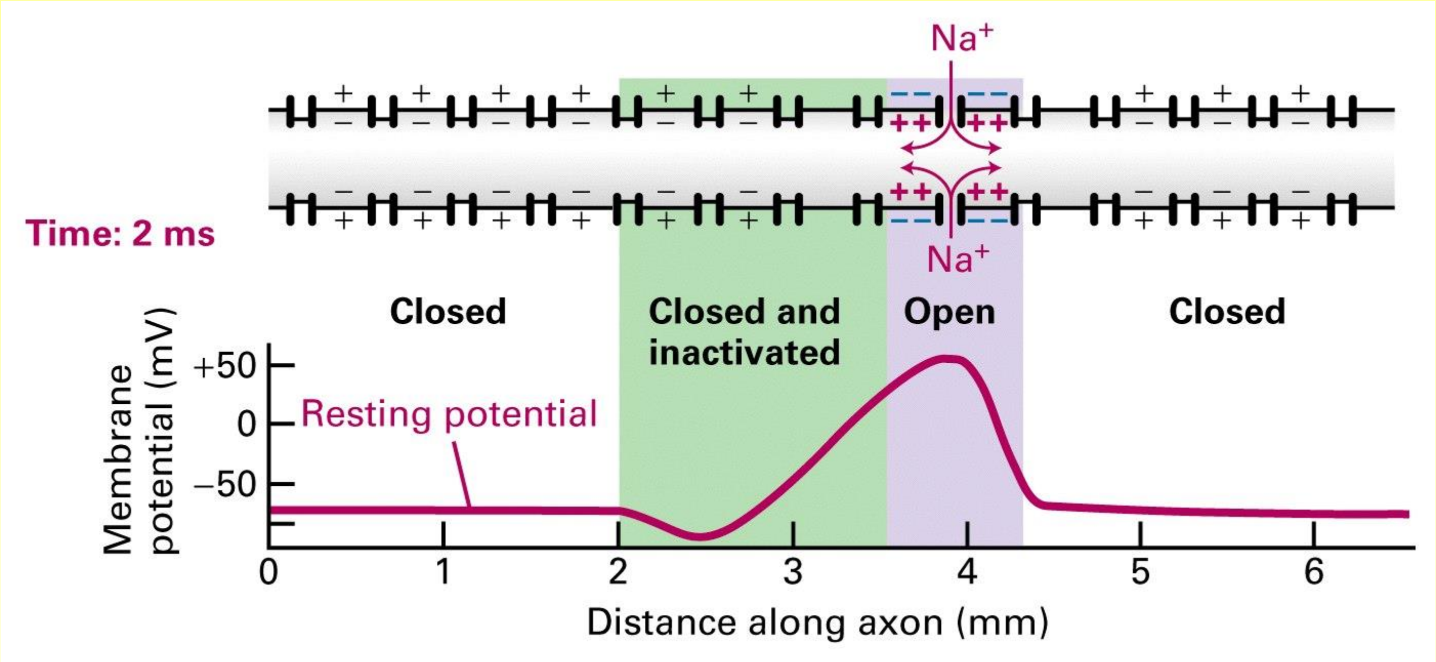


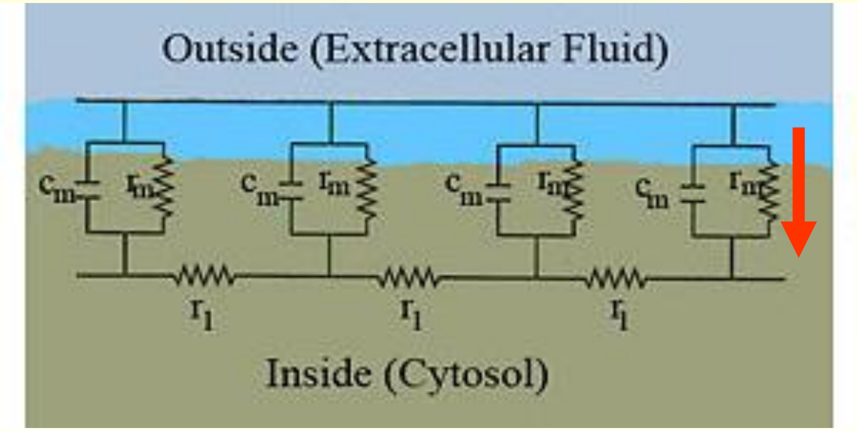
Conduction along the axons: Cable theory





Conduction along the axons: Cable theory



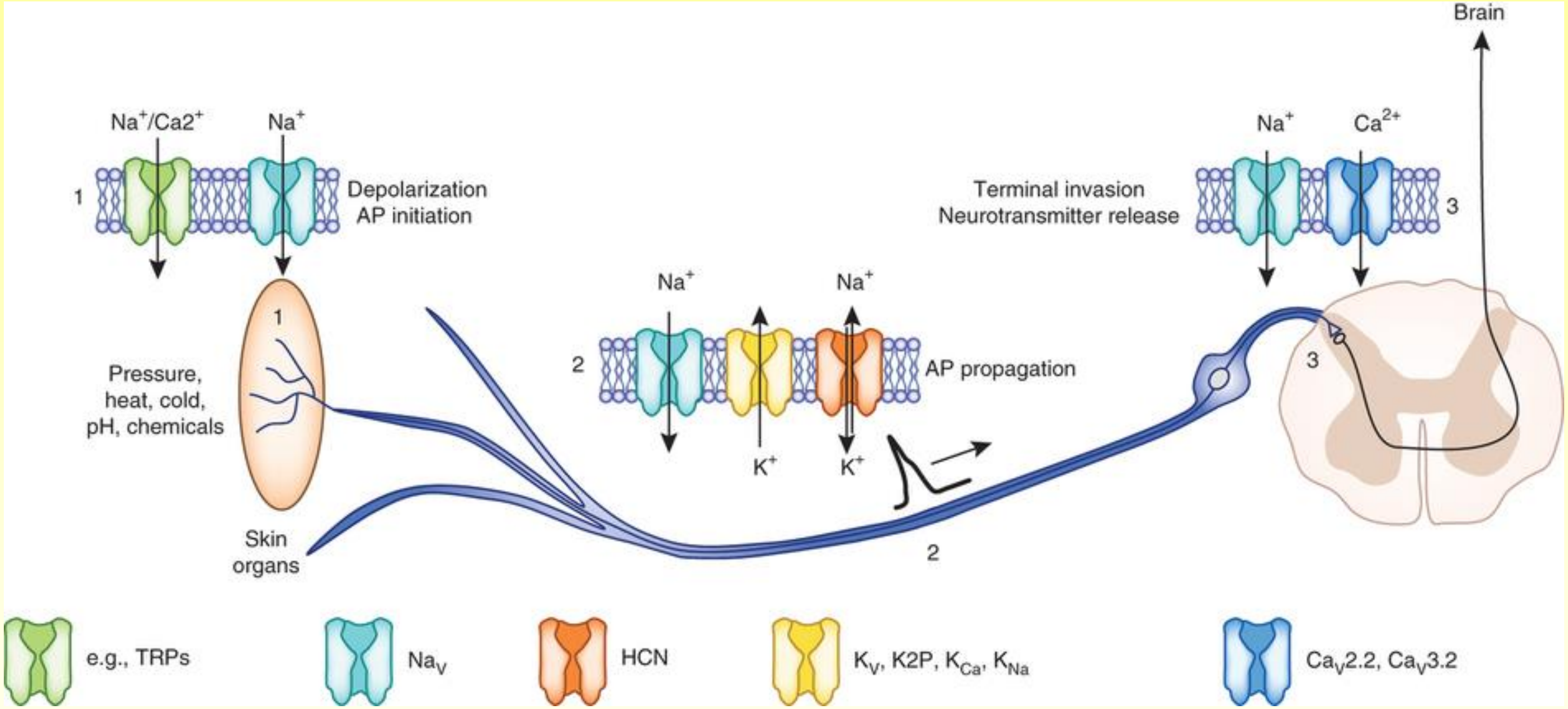


Sodium channel gating

$$C \leftrightarrow O \leftrightarrow I$$

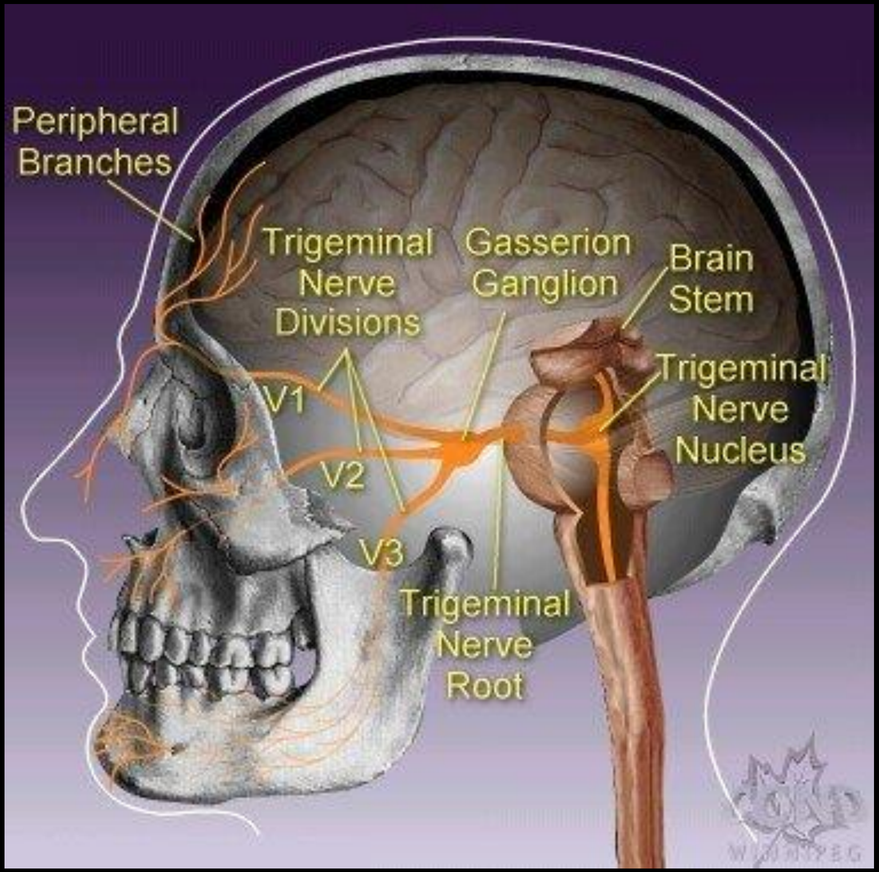
Refactory period

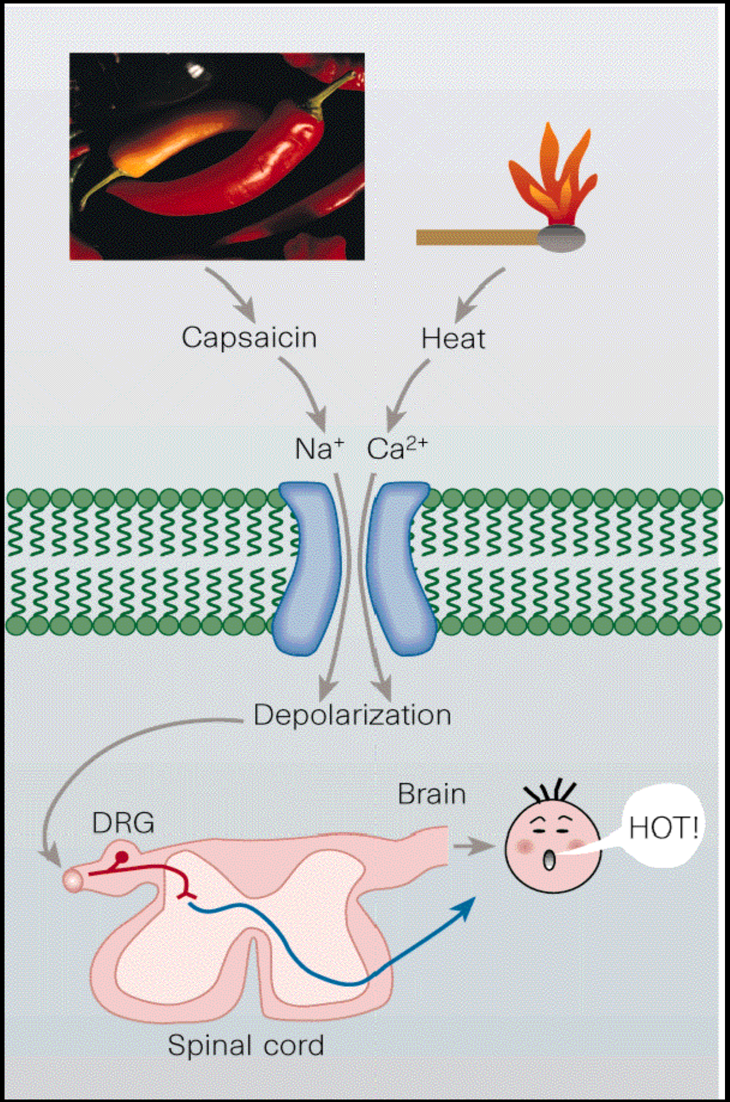
Generation of action potentials in sensory nerve endings



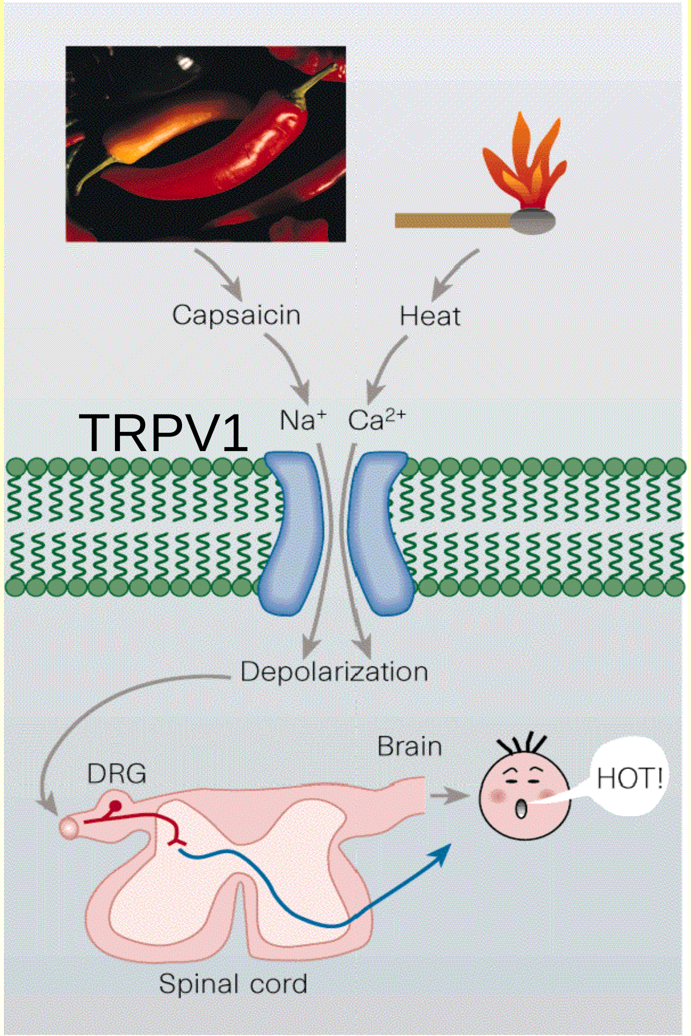
TRPV1: an excitatory channel in the pain pathway

- Capsaicin receptor
- Activated by noxious heat and acidosis

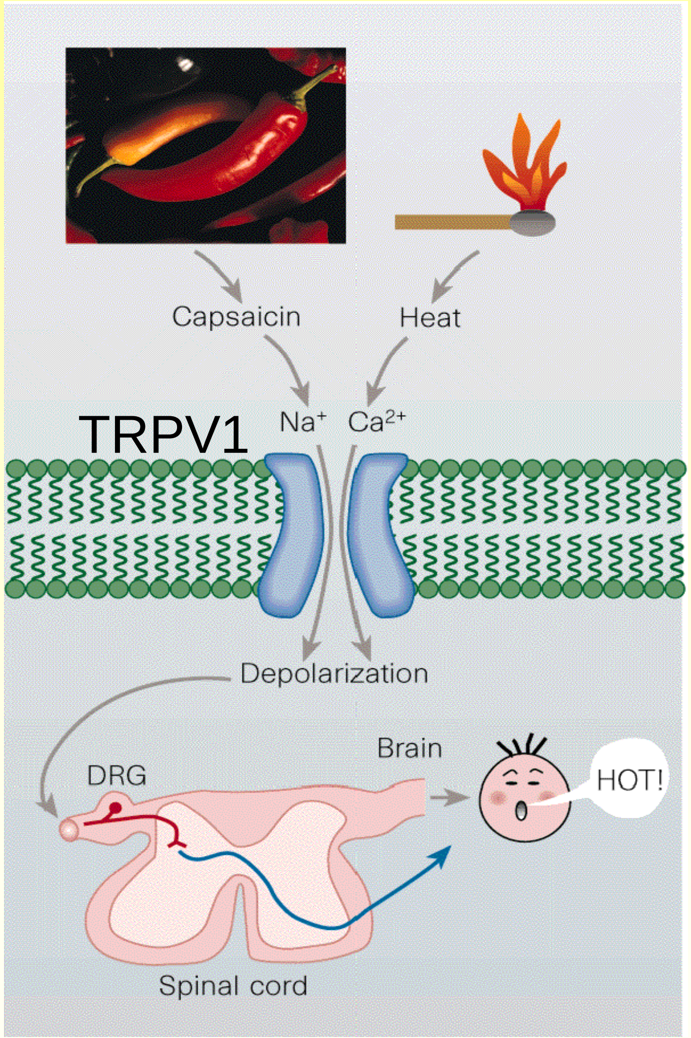


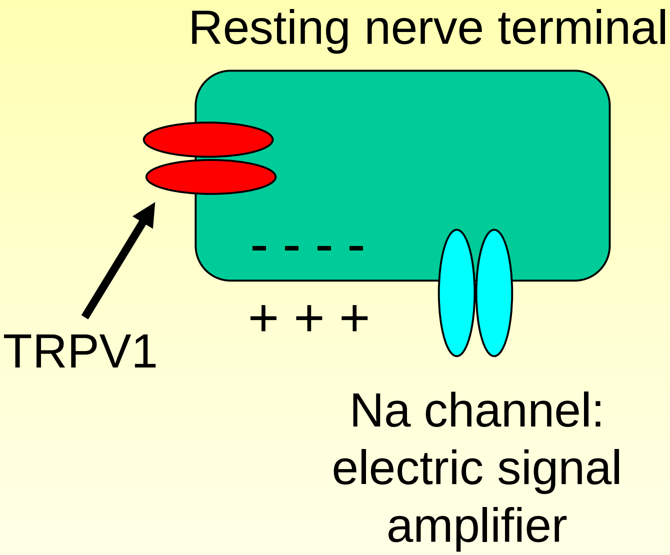


Ion channels as key molecular sensors

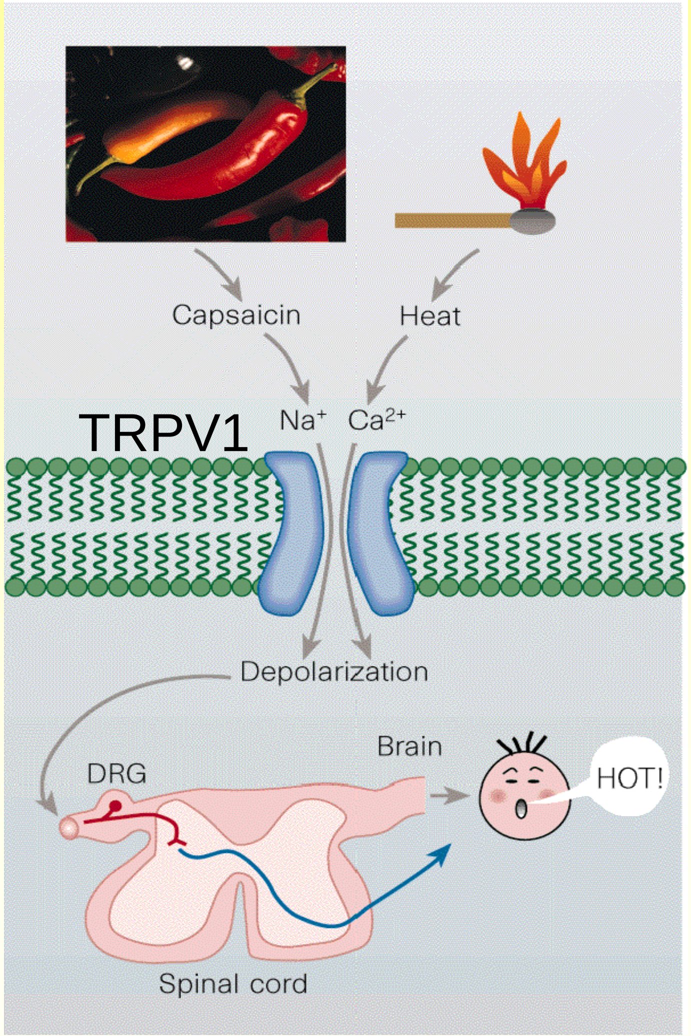


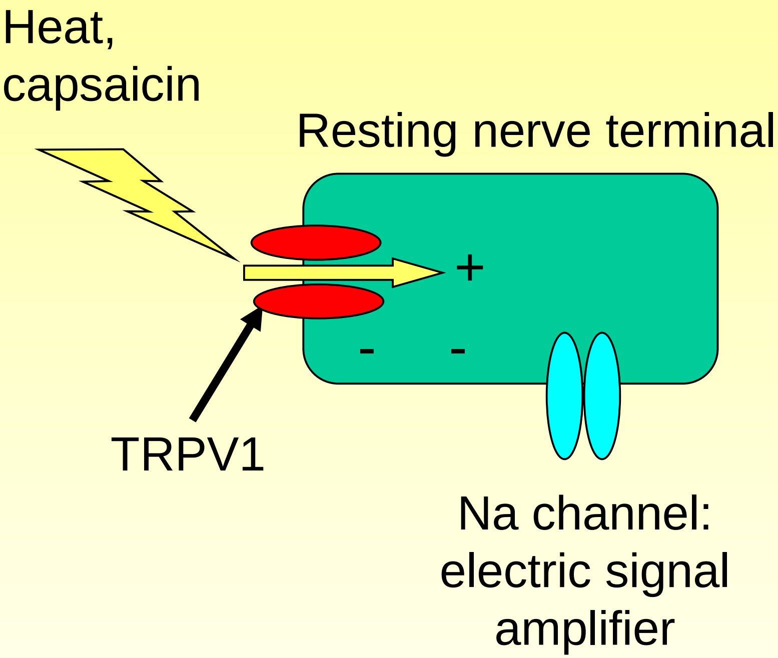
Ion channels as key molecular sensors



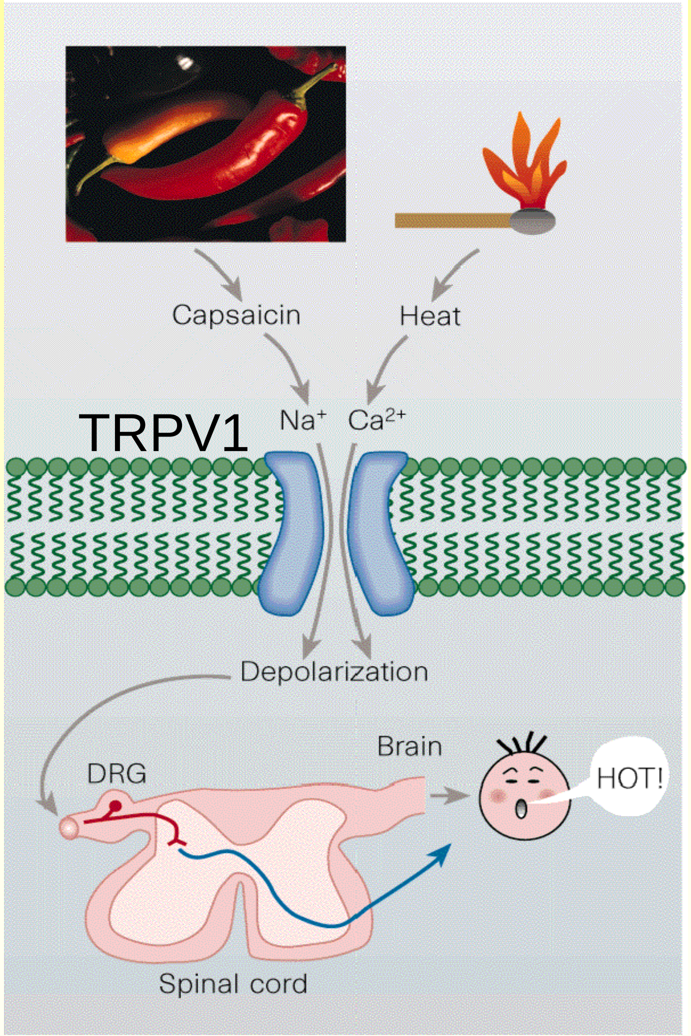


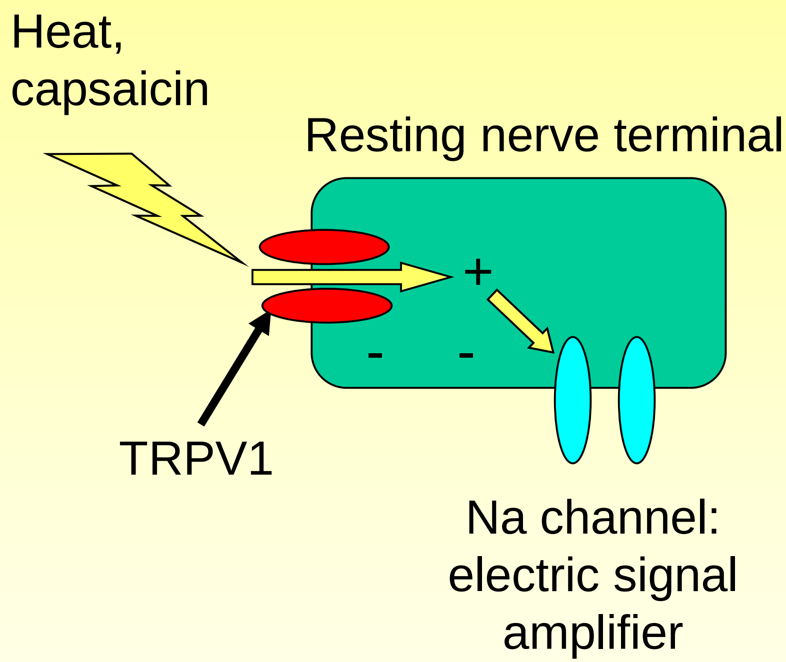
Ion channels as key molecular sensors



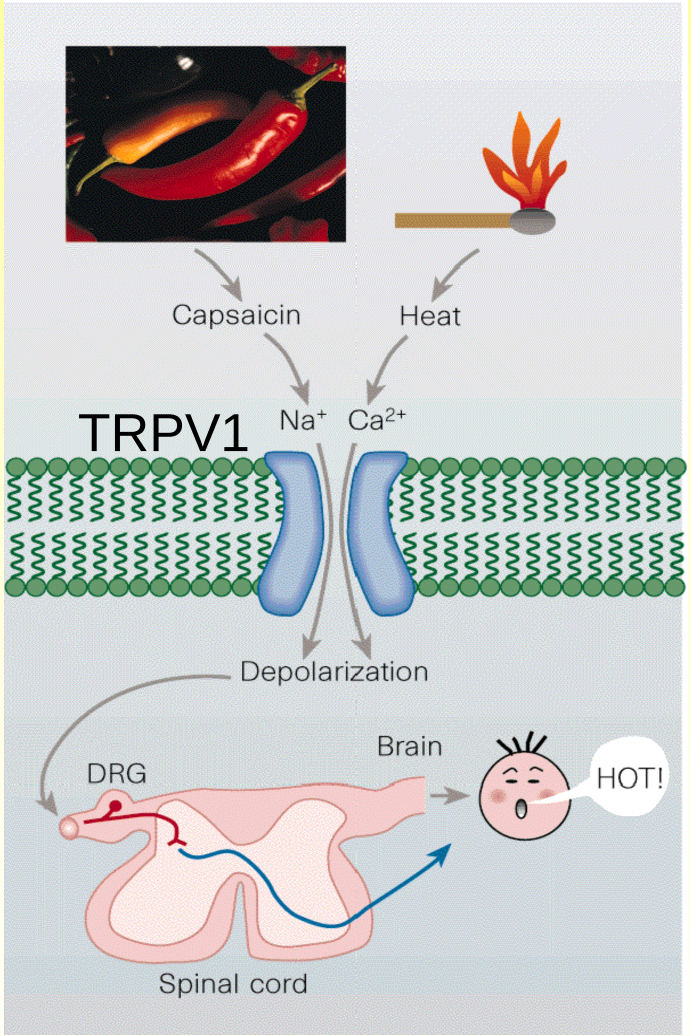


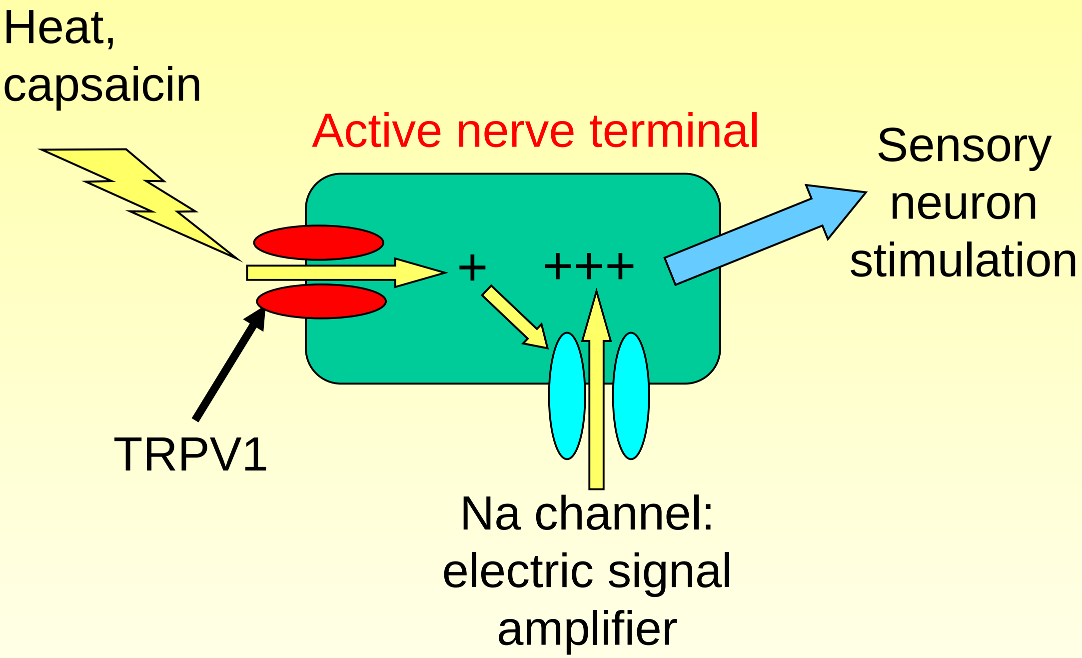
Ion channels as key molecular sensors





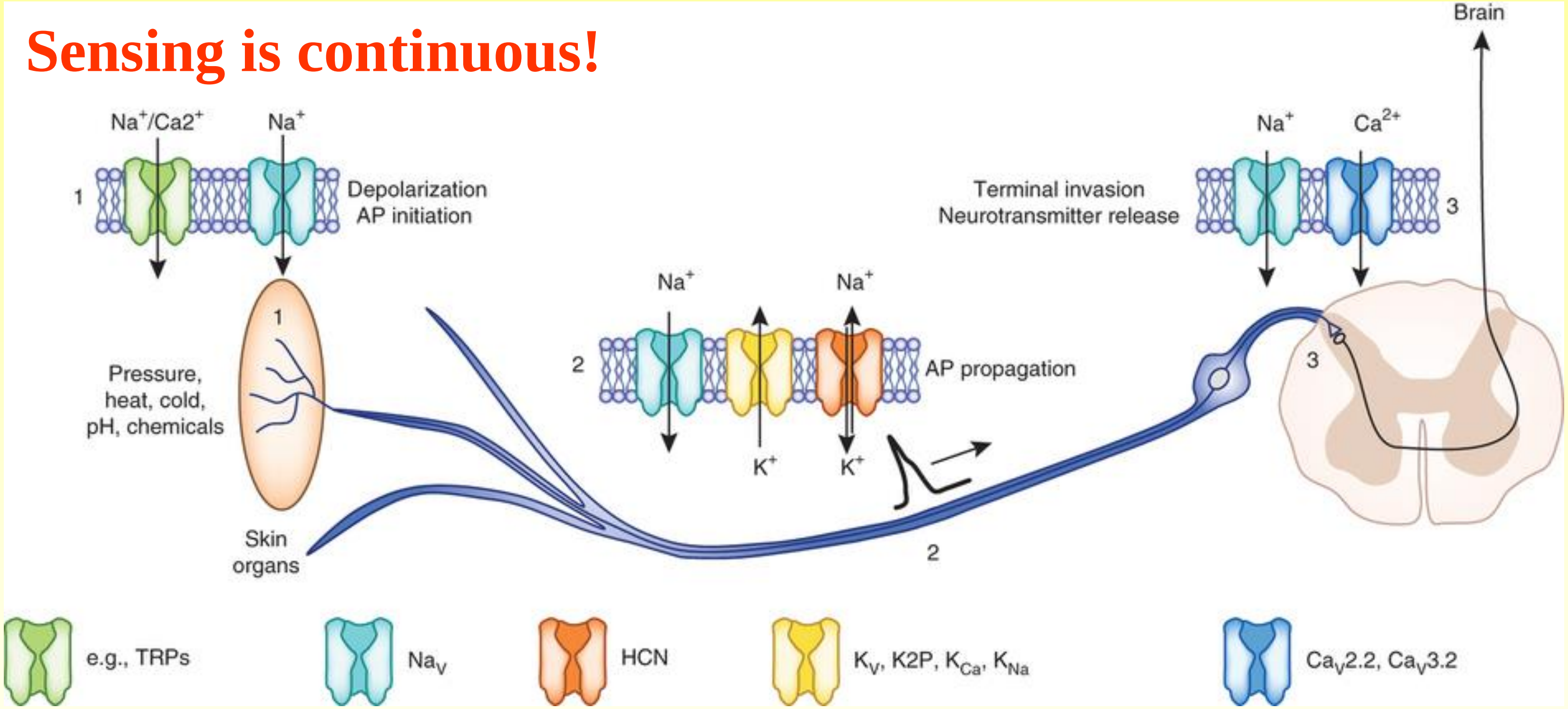
Ion channels as key molecular sensors





Sensing versus Conduction

Sensing is continuous!



Conduction is discrete!

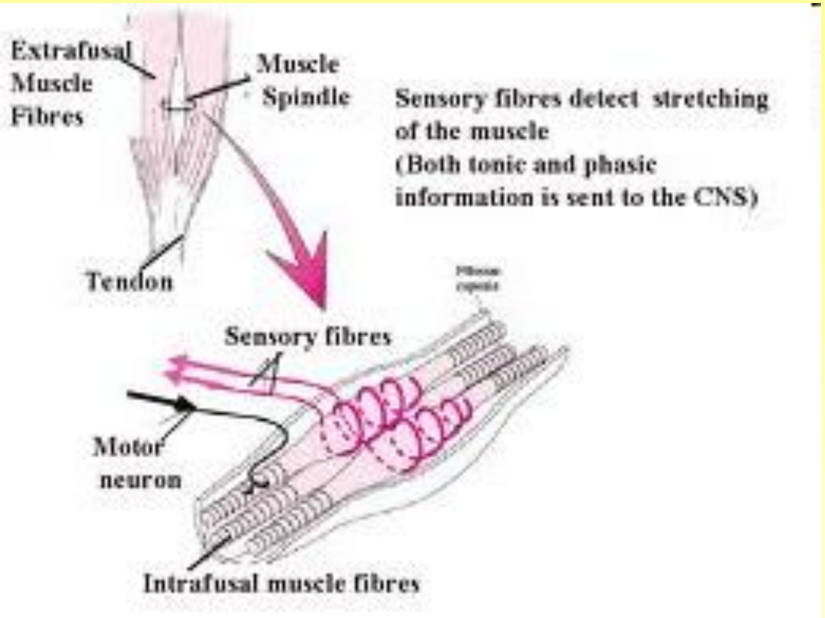
Information capacity in the nervous system II

Channel capacity of muscle spindle

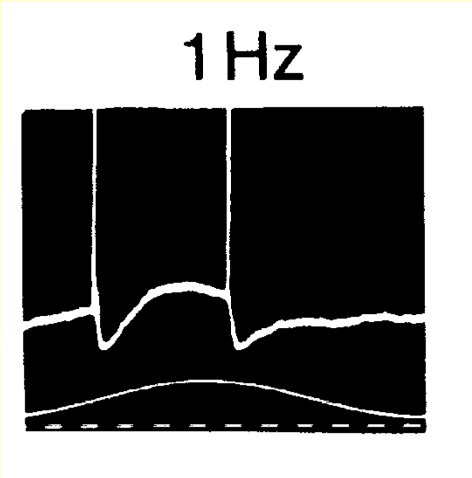
Principle of measurement of information capacity in the muscle spindle

Experimental procedure:

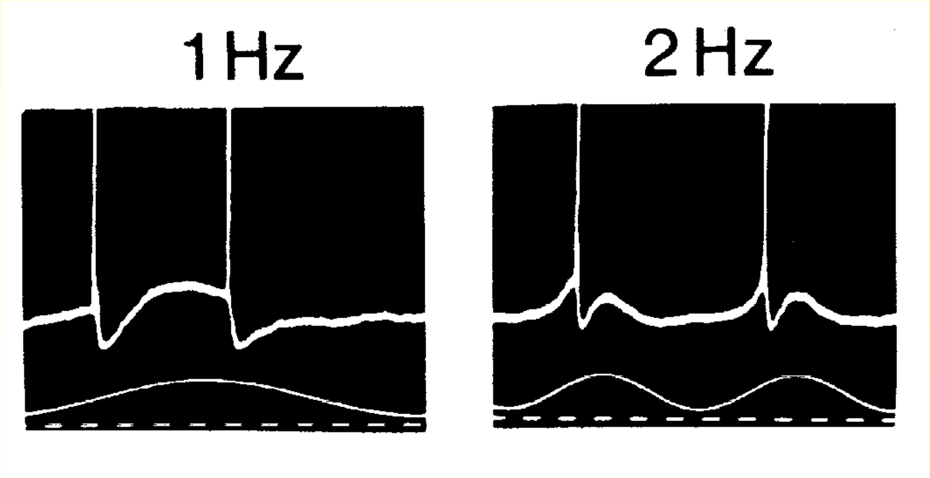
- Mechanical stretch to frog *musculus extensor digitorum longus IV* with different types mechanical stimuli



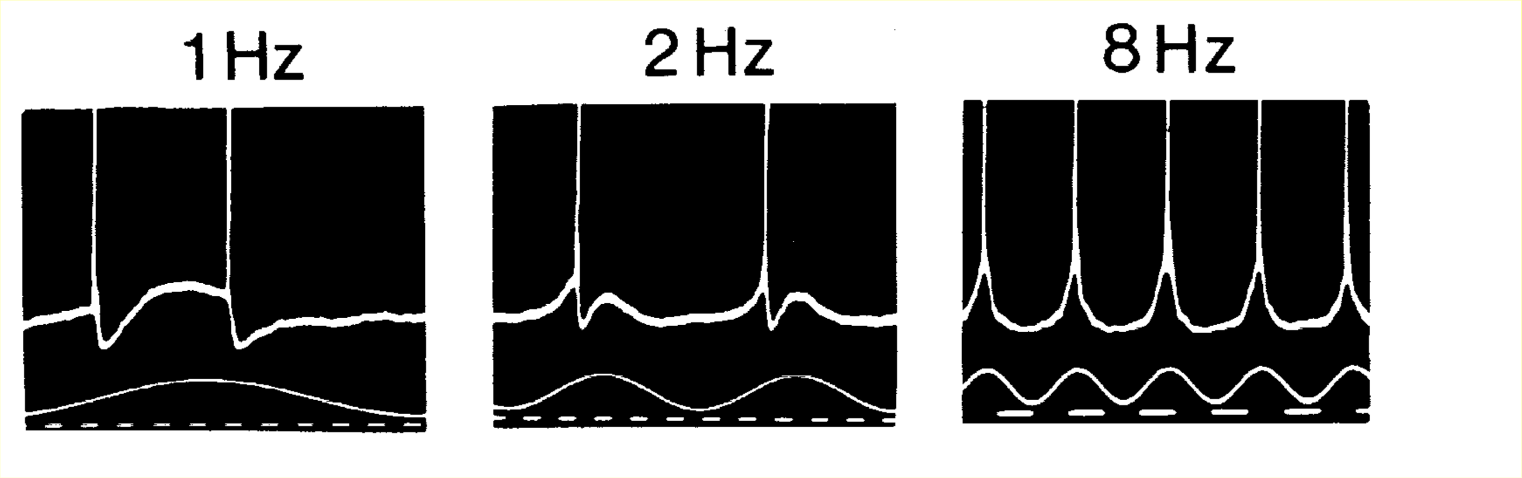
Response of the muscle spindle at different frequencies



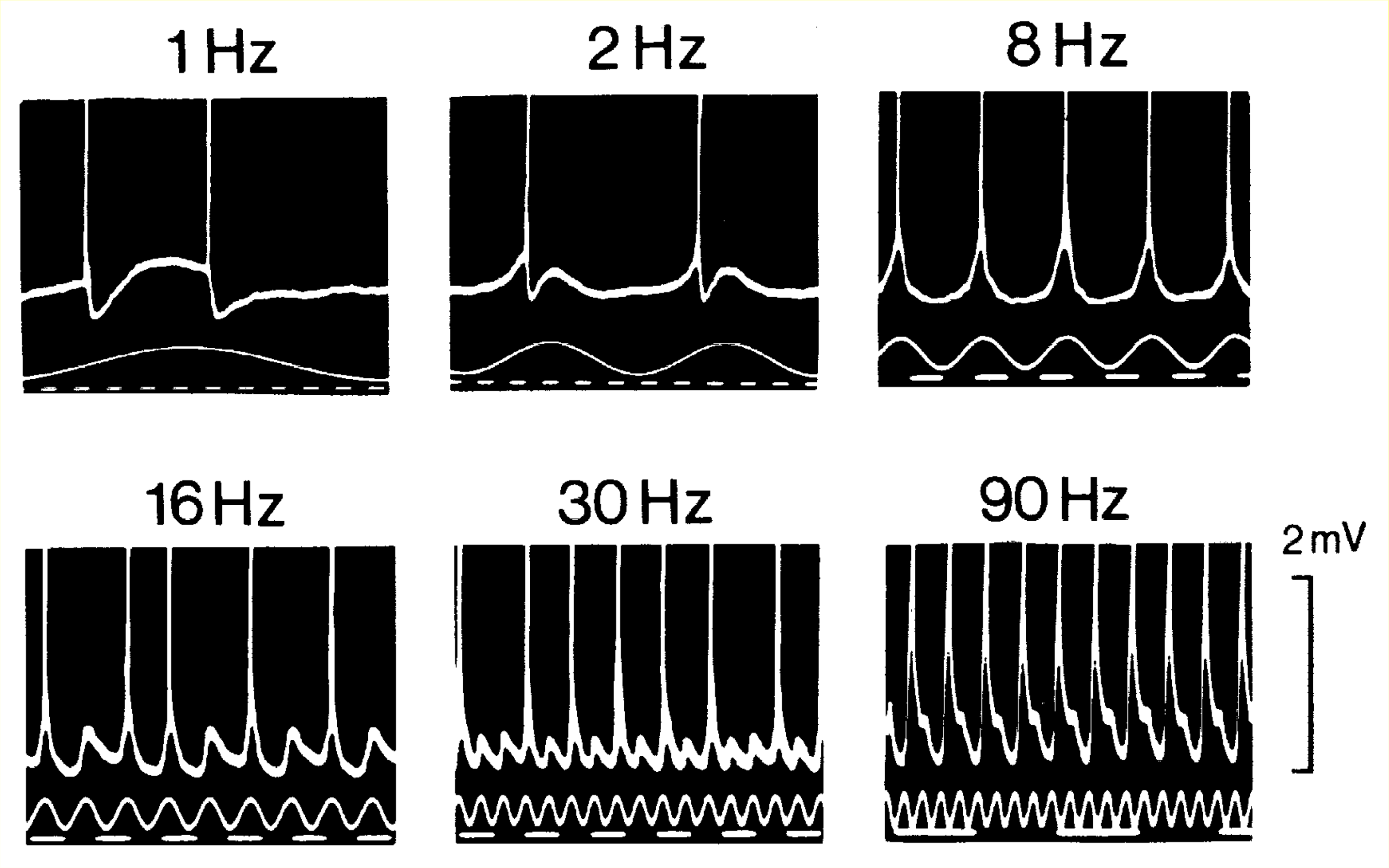
Response of the muscle spindle at different frequencies



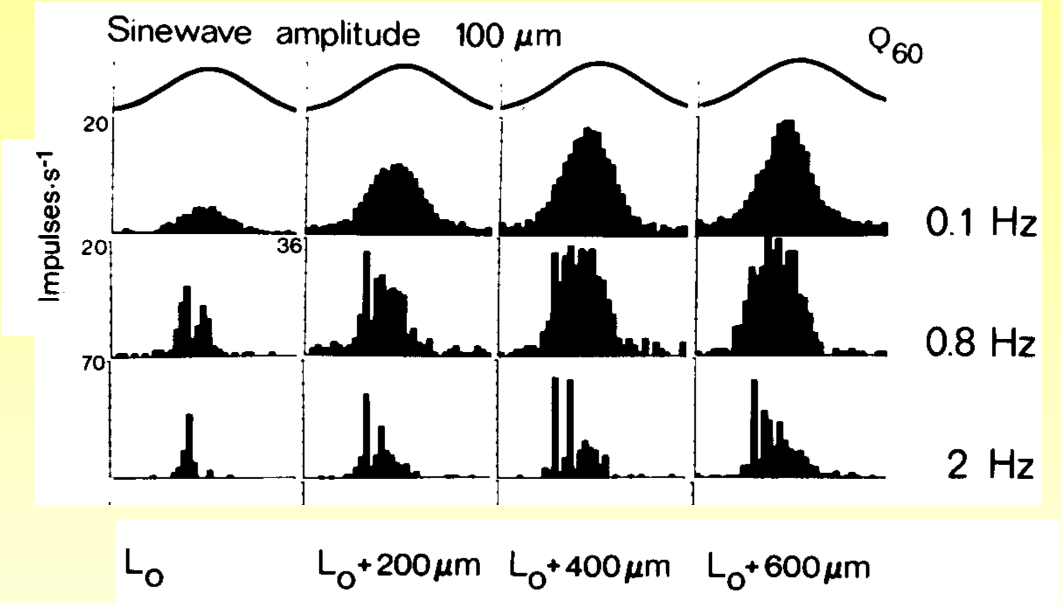
Response of the muscle spindle at different frequencies



Response of the muscle spindle at different frequencies



Cycle histograms of average response at different frequencies and different base stretch



At f ≤ 2Hz

- Linear response
- 5 < C < 15 bit /s
- Pre-stretch increases C by < 10 bit /s

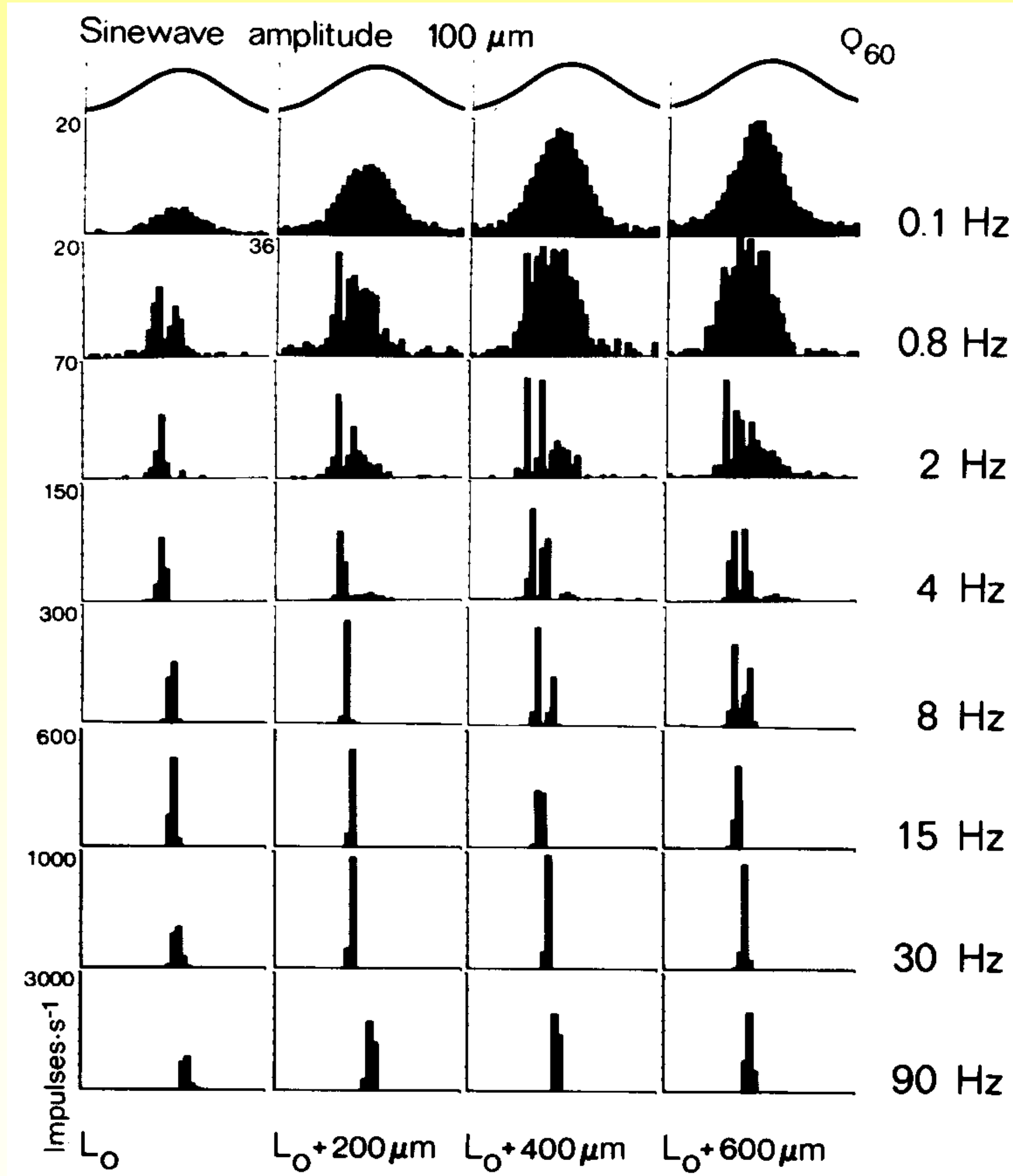
Cycle histograms of average response at different frequencies and different base stretch

At f ≤ 2Hz

- Linear response
- 5 < C < 15 bit /s
- Pre-stretch increases C by < 10 bit /s

At f > 2Hz

- Non-linear response
- Centered at peak stimulus (phase-locked)



Channel capacity of spiking neurons
Theoretical estimation

Estimated neuronal channel capacity as function of the maximum allowable time interval between 2 impulses

$$C = L\log n = \frac{\log n}{\tau_m}$$

n = number of coding levels

τ_m = average time between action potentials

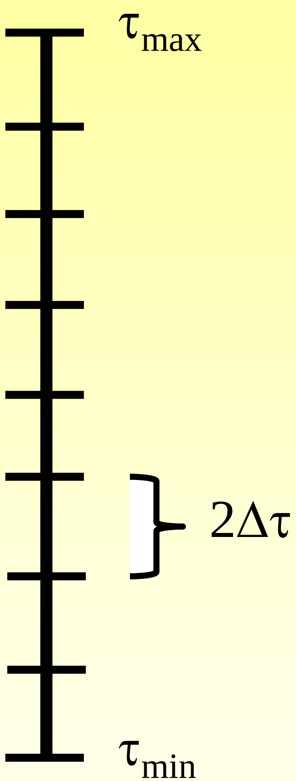
$$n = \left(\tau_{\max} - \tau_{\min}\right)/2\Delta\tau + 1$$

$\Delta\tau$ = uncertainty in the interval determination

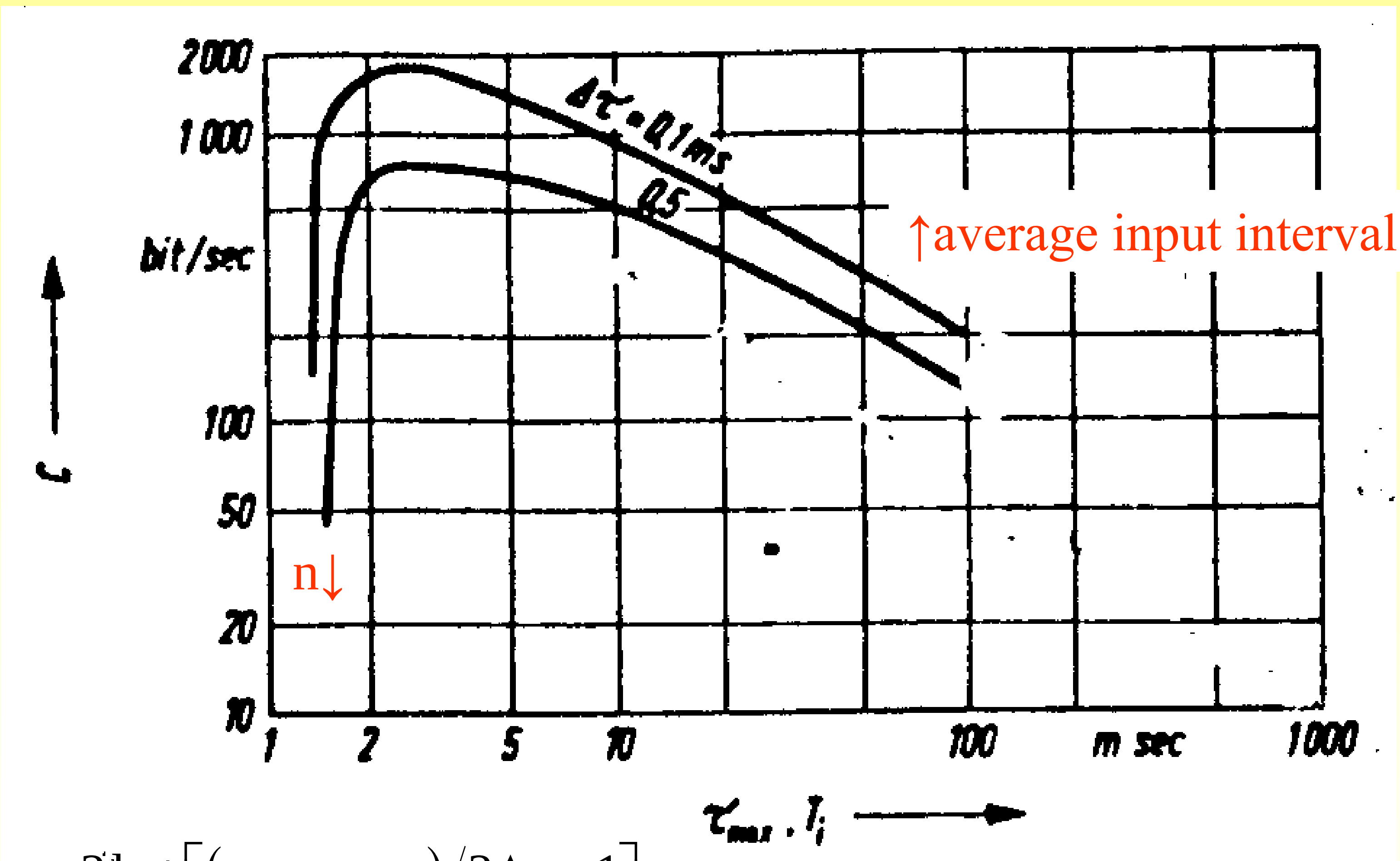
$$\tau_m = \frac{\left(\tau_{\max} + \tau_{\min}\right)}{2}$$

τ_{max} = maximum waiting time
τ_{min} = minimum waiting time

$$C = \frac{2\log\left[\left(\tau_{\max} - \tau_{\min}\right)/2\Delta\tau + 1\right]}{\tau_{\max} + \tau_{\min}}$$



Estimated neuronal channel capacity as function of the maximum allowable time interval between 2 impulses



$$C = \frac{2\log\left[\left(\tau_{\max} - \tau_{\min}\right)/2\Delta\tau + 1\right]}{\tau_{\max} + \tau_{\min}}$$

Channel capacity of spiking neurons
Experimental estimation

Third class neurons of the lobular plate of the fly

Results:

C = 300 bit/s

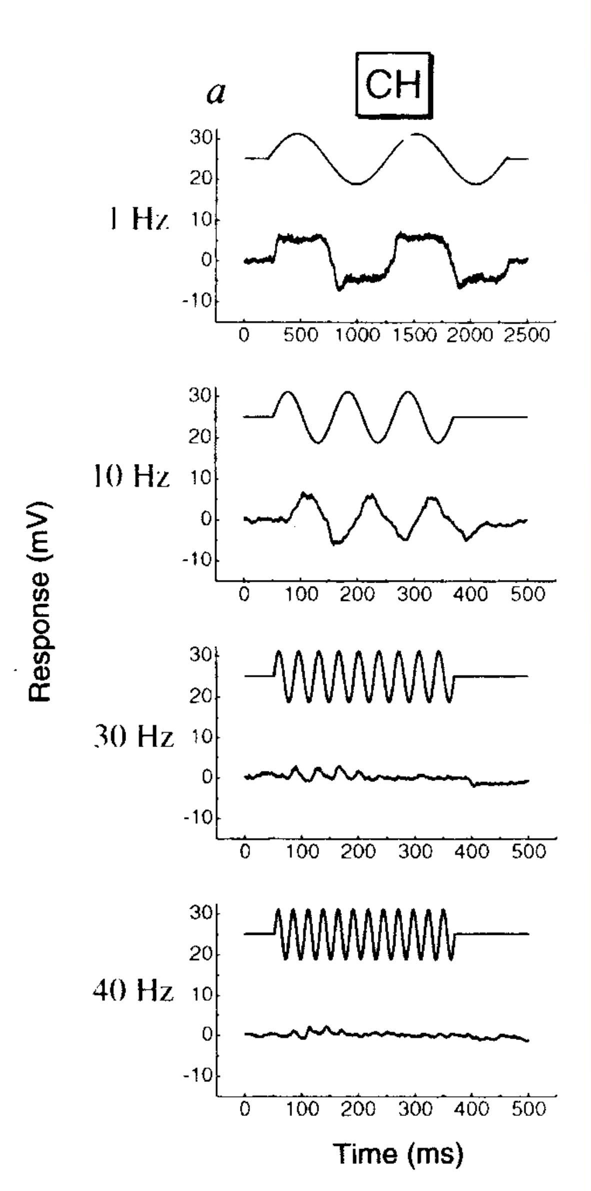
(5 times lower than 1650 bit/s
estimated for non-spiking neurons)

With average AP frequency of 100 per s:

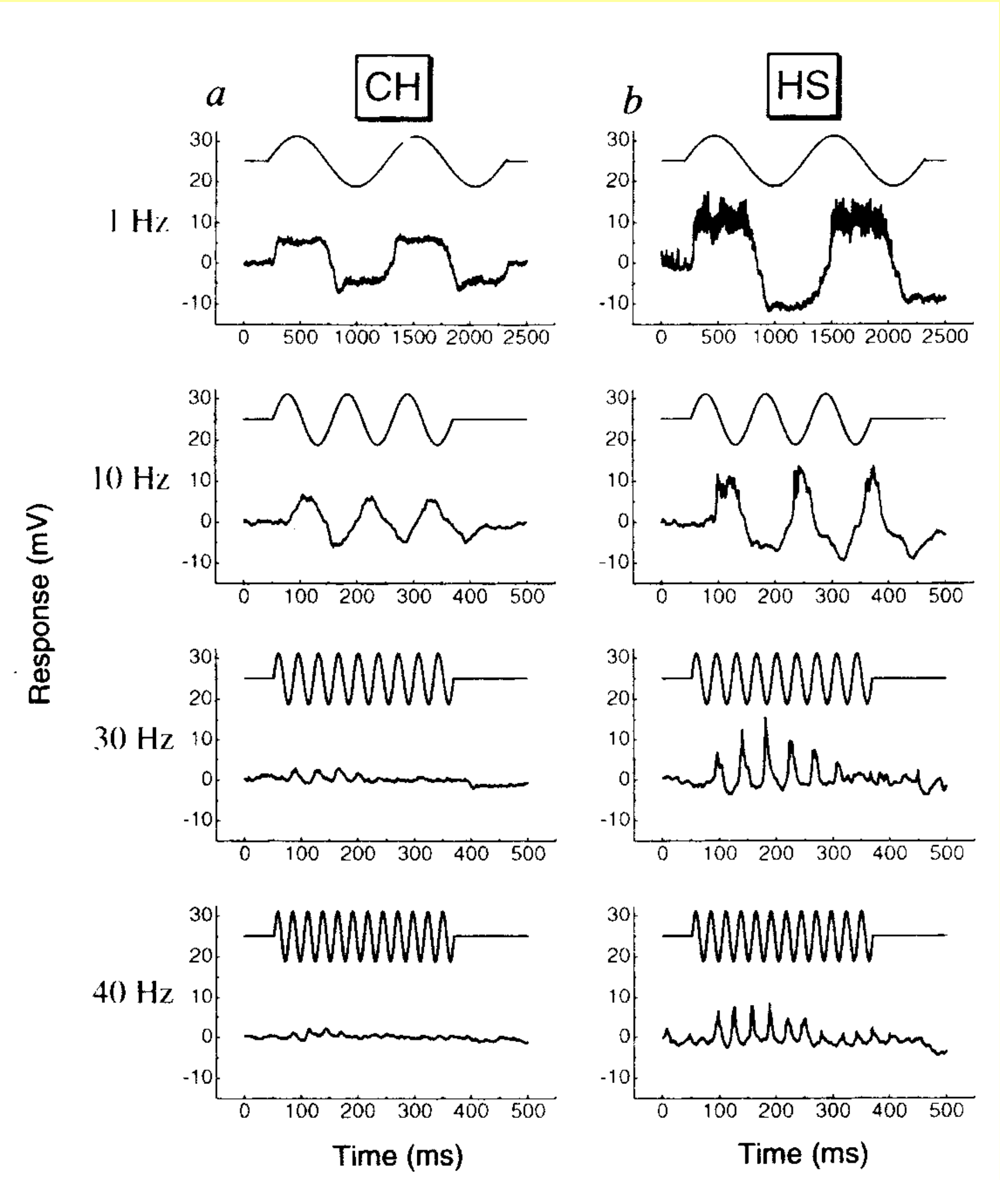
- the interval between action potentials contains about 3 bit information
- the interval can thus code 8 different levels

Comparison of spiking and non-spiking neurons (Activity in dendrites)

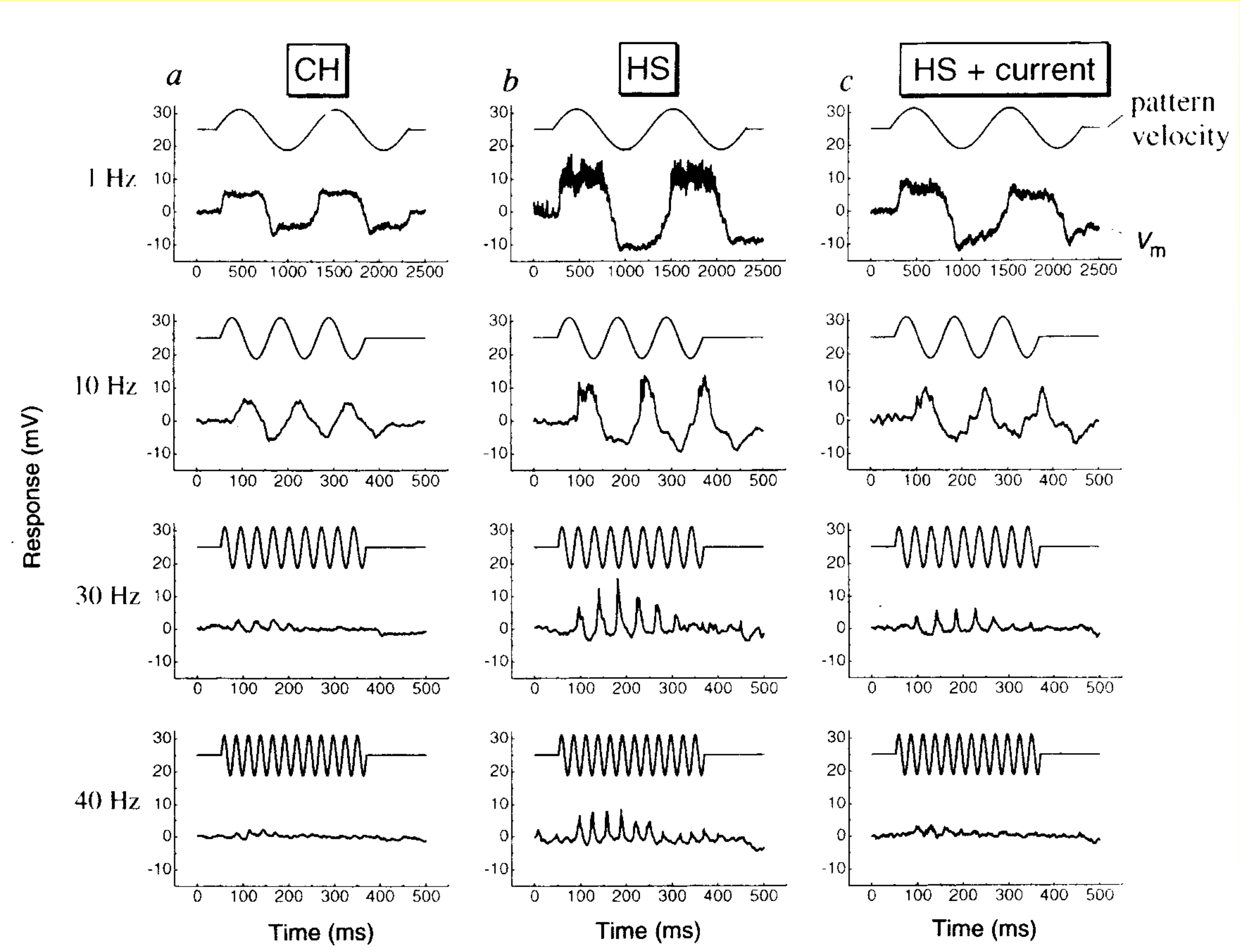
Response of CH en HS cells to sinusoidal movement stimuli in control and during hyperpolarizing current



Response of CH en HS cells to sinusoidal movement stimuli in control and during hyperpolarizing current



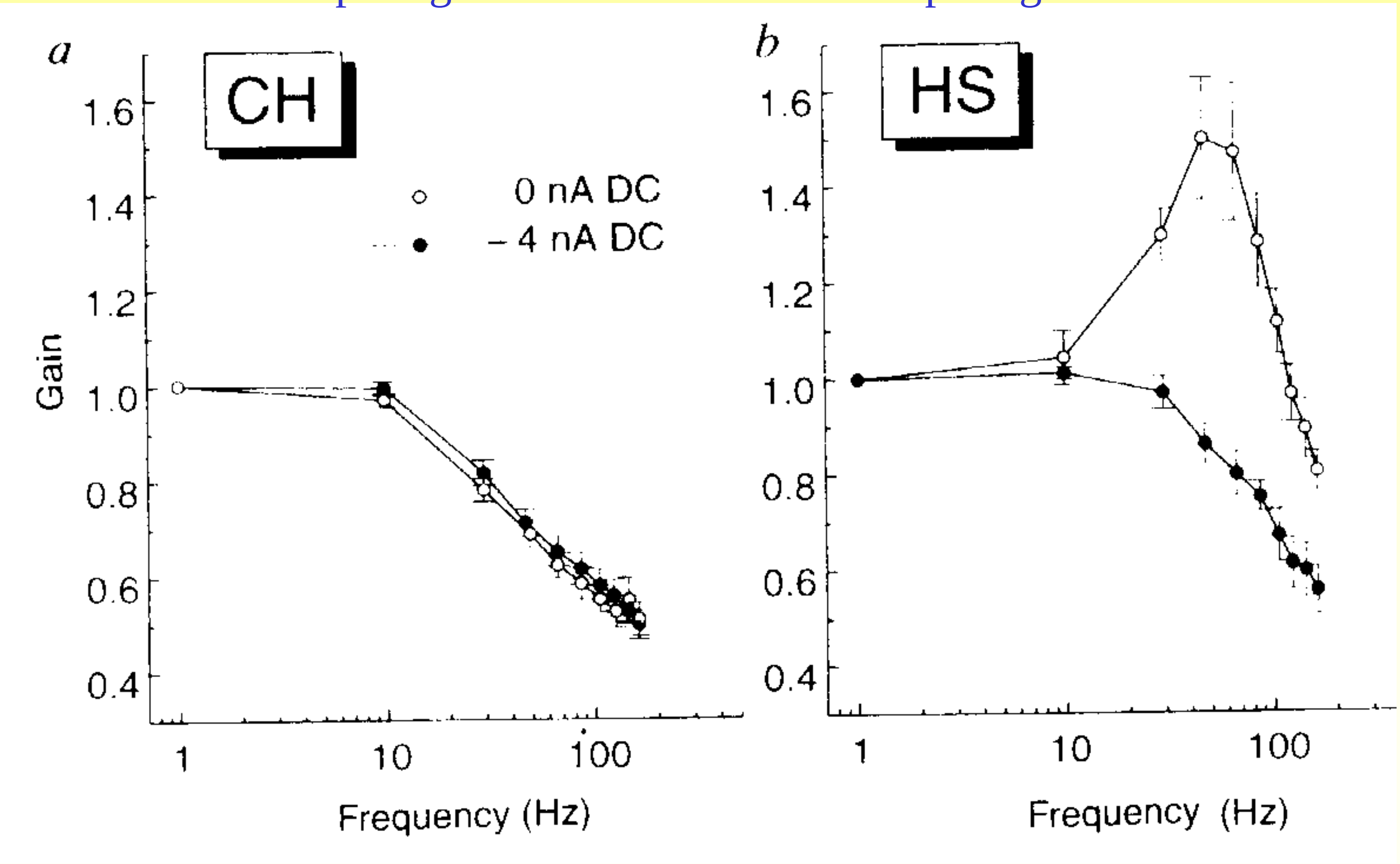
Response of CH en HS cells to sinusoidal movement stimuli in control and during hyperpolarizing current



Bode diagram of CH en HS cells (motion-sensitive visual neurons) in control and during hyperpolarizing current

Non-spiking dendrites

Spiking dendrites



Significance of spiking

Receptors: “analog” signaling

- High channel capacity
- Accurate representation sensory signals
- Attenuation of signal with distance

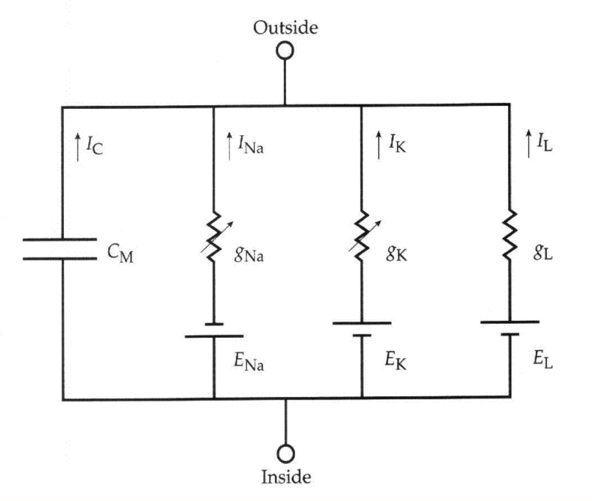
Spiking neurons: “digital” signaling

- No attenuation of signal with distance
- Small channel capacity

Channel information capacity less important:

– parallel processing
– convergence of information
– lossy data compression

Membrane model:



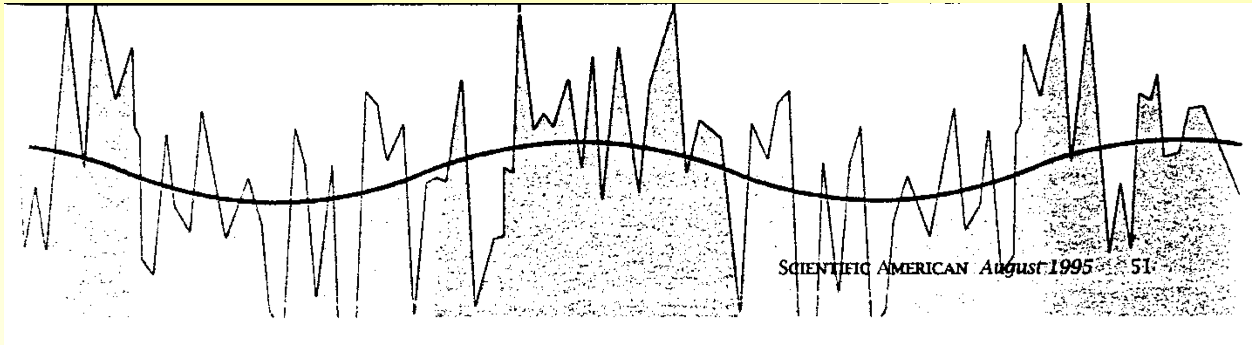
$$I_{STIMULUS} = C\frac{dV}{dt} + \frac{(V - V_{REST})}{R(V,t)}$$

Active properties

Stochastic resonance

Stochastic resonance

Noise can increase the channel capacity
by enhancing signal-to-noise ratio in non-linear systems



Stochastic resonance in biological systems

- Crayfish mechanoreceptor system
- Human hearing: cochlear implants
- Human muscle spindle
- Human tactile sensitivity: vibrating gel insoles
- Human contrast detection
- Binocular rivalry
- Potential-dependent ion channels
- Hippocampal CA3-CA1 recall
- Calcium dynamics in cells (hepatocytes)

Stochastic resonance in human hearing

Discrimination conversation in noisy environment

Discrimination gets worse
- by frequent exposure to very loud sounds
- with age

Loss of discrimination due to death outer hair cells cochlea

Experiment:

• Determination perception threshold of particular frequency in the presence of noise.
• Conclusion: Threshold is minimal in the presence of a certain amount of noise
• Application: cochlear implants with added noise

Chatterjee et al. 2005:
"*Noise improves modulation detection by cochlear implant listeners*"

Stochastic resonance in human tactile sensitivity

Small amounts or random noise increase tactile sensitivity

Elderly people easily loose balance and become wobbly

due to decreased sensitivity to changes in foot pressure

Experiment:

Platform with hundreds of randomly vibrating nylon rods

Balance tested of elder volunteers with balance problems

blindfolded and barefoot on platform

with vibration amplitude set below detection threshold

Conclusion:

Stochastic resonance can improve stability

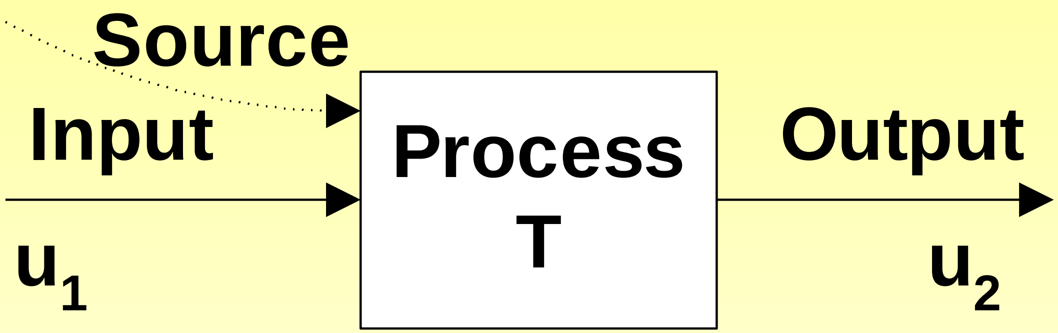
Application:

Vibrating gel insoles

Control theory

System description

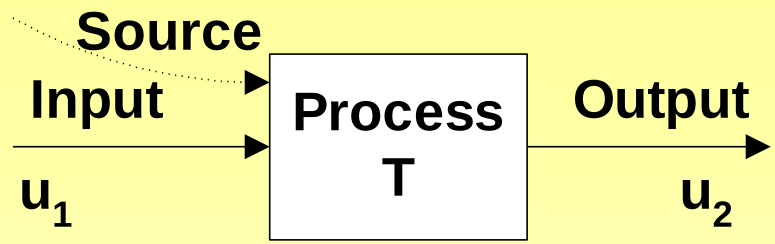
System diagram



Example:
Source: potential energy of water
Process: change of flow by valve
Input: position of valve
Output: water flow in tube

The transfer function of a process

$$u_2 = T u_1 \quad \Rightarrow \quad T = \frac{u_2}{u_1}$$

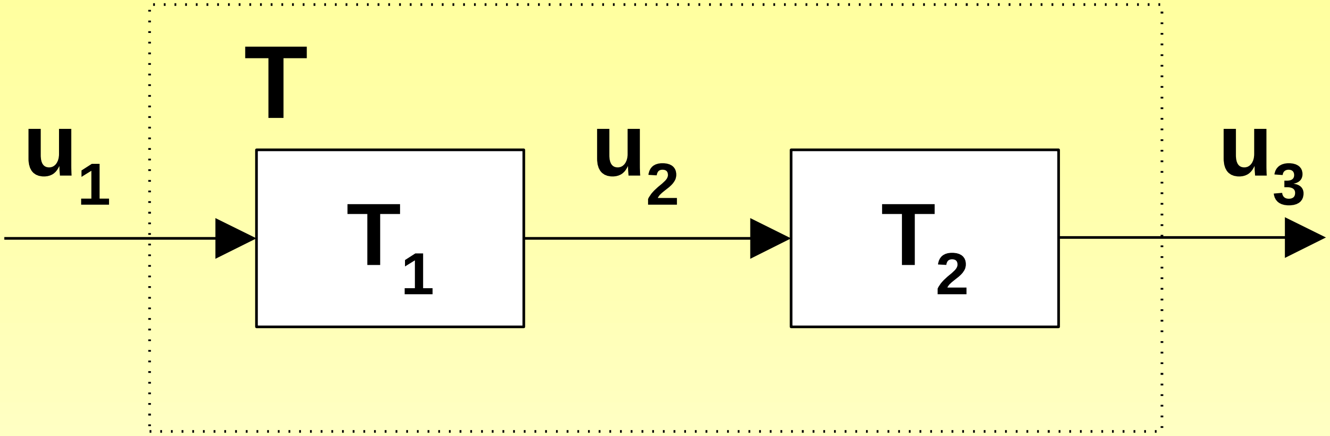


For linear time-independent systems:

$$a_n \frac{d^n u_2}{dt^n} + a_{n-1} \frac{d^{n-1} u_2}{dt^{n-1}} + \cdots + a_0 u_2 = u_1$$

$$T = \frac{u_2}{u_1} = \frac{1}{a_n D^n + a_{n-1} D^{n-1} + \cdots + a_0}$$

Systems in series



$$u_2 = T_1 u_1$$

$$u_3 = T u_1$$

$$u_3 = T_2 u_2$$

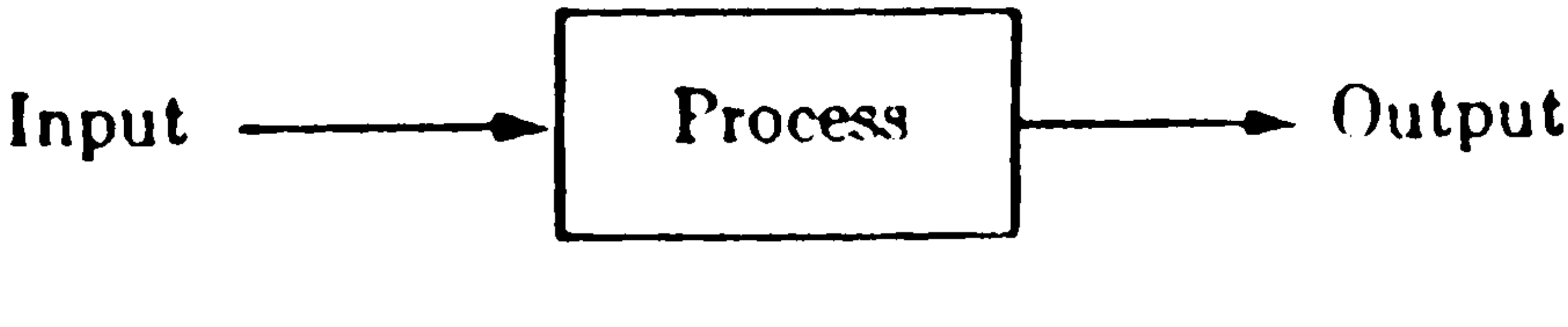
$$u_3 = T u_1 = T_2 u_2 = T_2(T_1 u_1) = T_2 T_1 u_1$$

$$T = T_2 T_1$$

$$T_2 T_1 \neq T_1 T_2$$

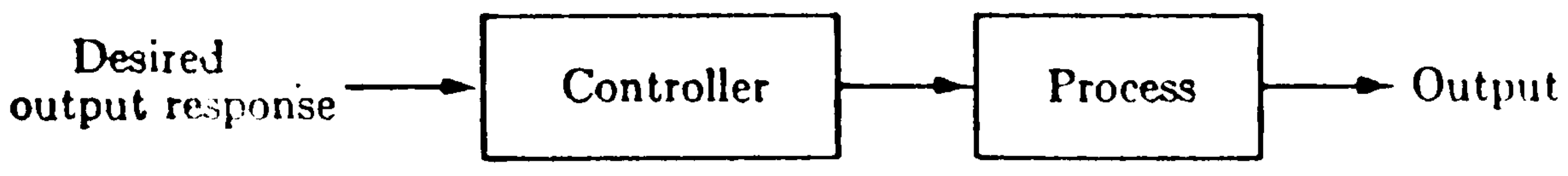
Closed loop systems

Process



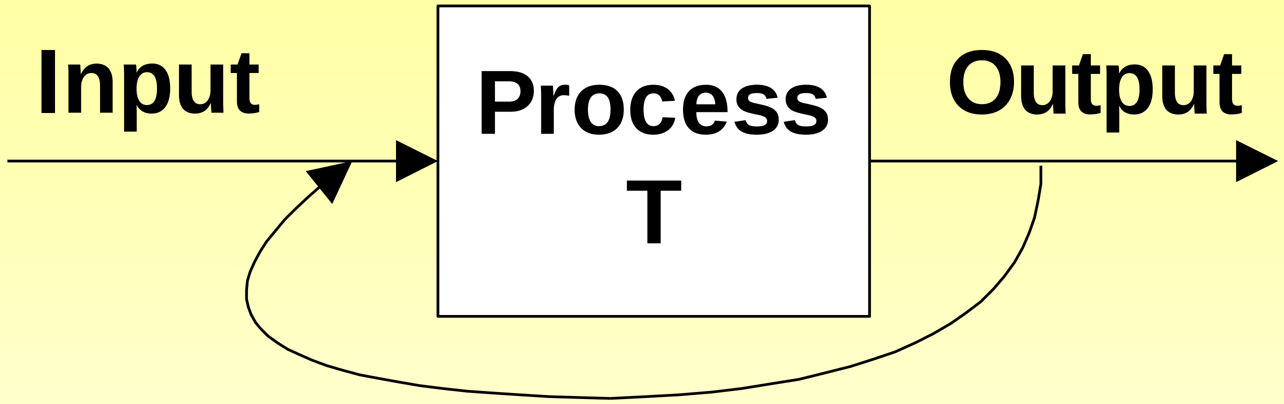
Process to be controlled.

Open loop control system



Open-loop control system.

Closed system



Negative feedback: stable levels

E.g., thermoregulation

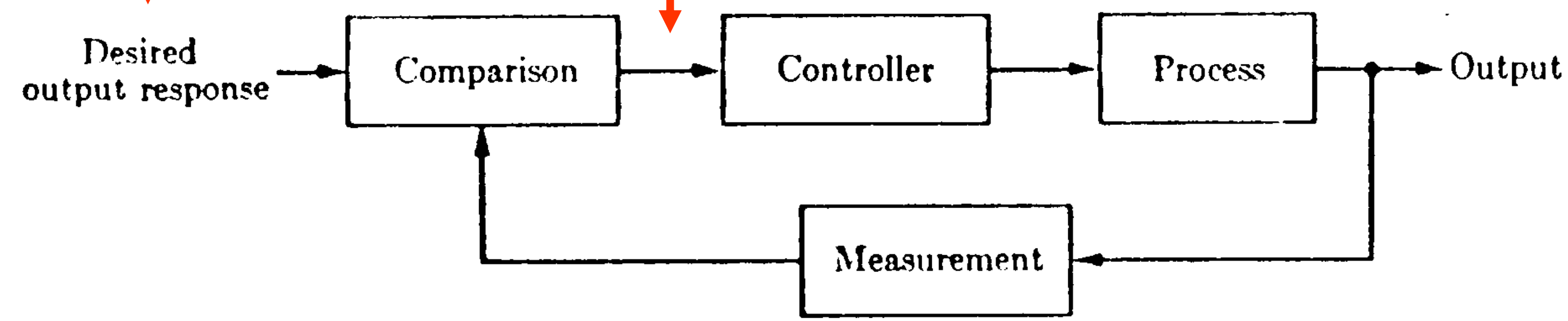
Positive feedback: on-off

E.g. action potential firing, cell division, cell death, blood coagulation, micturition

Closed loop feedback system

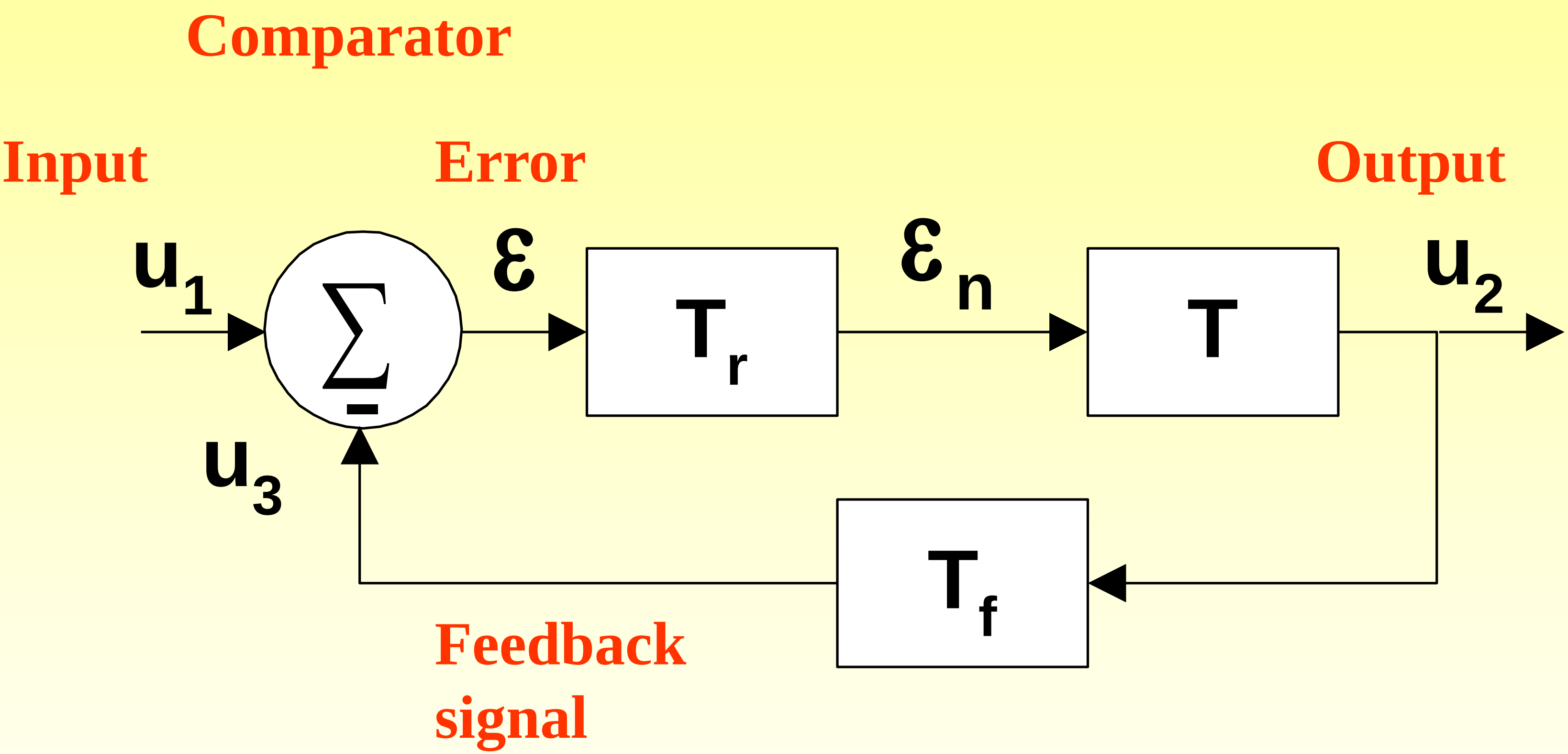
Reference signal
Command Signal
Input Signal

Difference:
Error Signal



Closed-loop feedback control system.

Representation of a control system

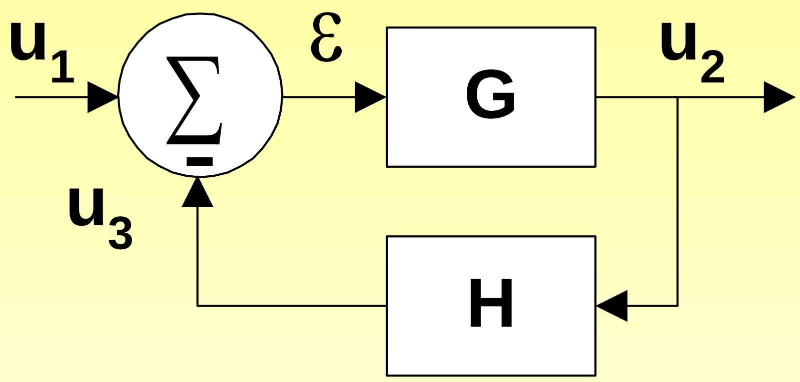


Properties of control systems

$$u_2 = G\varepsilon$$

$$\varepsilon = u_1 - u_3$$

$$u_3 = \mathrm{H}\, u_2$$



$$\varepsilon = \frac{1}{1 + GH} u_1$$

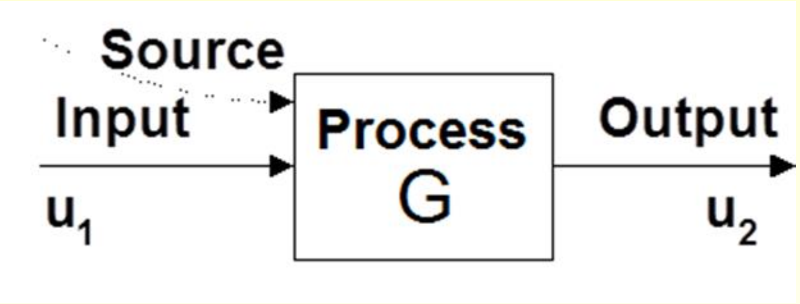
$$u_2 = \frac{G}{1 + GH} u_1$$

The gain of a closed system is smaller than that of the open system

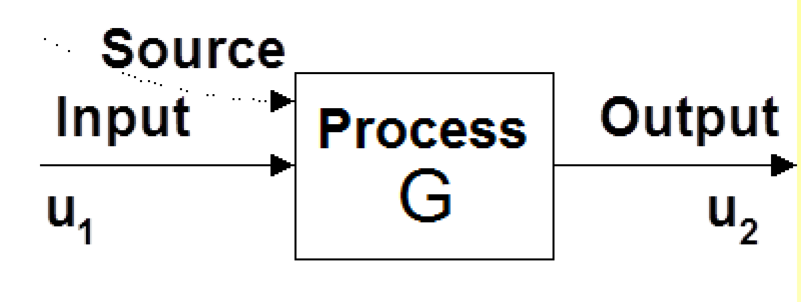
Sensitivity analysis

Definition of sensitivity

$$S_x = \frac{\frac{\delta u_2}{u_2}}{\frac{\delta x}{x}}$$



Sensitivity of open system to G



$$u_2 = Gu_1 \rightarrow \delta u_2 = \delta G u_1$$

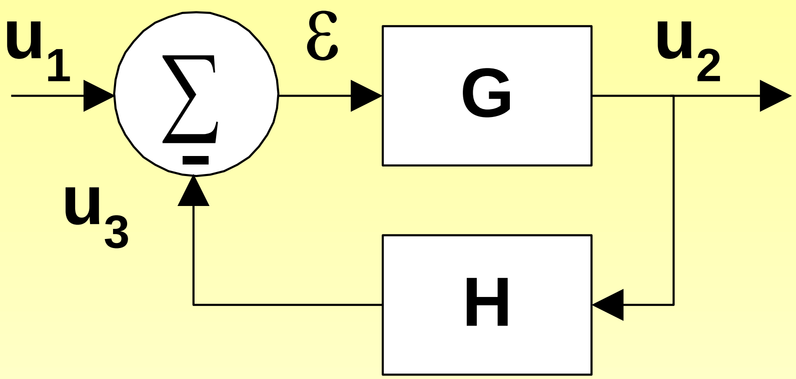
$$\frac{\delta u_2}{u_2} = \frac{\delta G u_1}{u_2} = \frac{\delta G u_1}{G u_1} = \frac{\delta G}{G}$$

$$S_G = \frac{\frac{\delta u_2}{u_2}}{\frac{\delta G}{G}} = 1$$

High sensitivity to variations in the process G

Sensitivity of closed system to G

$$u_2 = \frac{G}{1+GH} u_1$$



$$\delta u_2 = \frac{\delta G(1+GH) - GH\delta G}{(1+GH)^2} u_1$$

$$\delta u_2 = \frac{\delta G}{(1+GH)^2} u_1$$

$$\frac{\delta u_2}{u_2} = \frac{1}{1+GH} \frac{\delta G}{G}$$

$$S_G = \frac{1}{1+GH}$$

for GH >> 1 $\Rightarrow$ $S_G \cong 1/GH$

Low sensitivity to variations in the process G