# Derivation of Critical Gain Including Drag and Delay

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## 1 Relationship between Critical Gain and Scale

In this document, we present the full derivation of the critical gain equation in the presence of aerodynamic drag and time delay. First, we include the drag by modelling the acceleration of the robot with mass $m$ as:

$$\dot{V_z}^w = \frac{u_z + f_D}{m}, \tag{1}$$

with the drag force $f_D$:

$$f_D = \operatorname{sgn}(V_{\text{wind}} - V_z{}^w)\frac{1}{2}\rho C_D A\left(V_{\text{wind}} - V_z{}^w\right)^2, \tag{2}$$

where $\rho$ is the air density, $C_D$ is the drag coefficient, and $A$ is the reference area of the robot. Linearization of (1) gives:

$$\Delta\dot{V_z}^w = \frac{\Delta u_z}{m} - \operatorname{sgn}(V_{\text{wind}} - V_z{}^w)\frac{\rho C_D A}{m}\left(V_{\text{wind}} - V_z{}^w\right)\Delta V_z{}^w, \tag{3}$$

where $V_{\text{wind}}$ and $V_z{}^w$ are given the values at the linearization point. Hereafter, a constant $p$ will be used to avoid cluttering the formulas, which is defined as:

$$p := \operatorname{sgn}(V_{\text{wind}} - V_z{}^w)\frac{\rho C_D A}{m}\left(V_{\text{wind}} - V_z{}^w\right). \tag{4}$$

As a result, the linearized equation of motion including the drag gives the following continuous state space model:

$$\begin{aligned} A &= \begin{bmatrix} 0 & 1 \\ 0 & -p \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 1/m \end{bmatrix}, \\ C &= \begin{bmatrix} 0 & 1/\lambda \end{bmatrix}, \; D = [0]. \end{aligned} \tag{5}$$

Zero-order hold discretization with a sample time $T_s$ gives:

$$\begin{aligned} A_d &= \begin{bmatrix} 1 & \dfrac{\left(1 - e^{-pT_s}\right)}{p} \\ 0 & e^{-pT_s} \end{bmatrix}, \; B_d = \begin{bmatrix} \dfrac{T_s}{mp} - \dfrac{\left(1 - e^{-pT_s}\right)}{mp^2} \\ \dfrac{\left(1 - e^{-pT_s}\right)}{mp} \end{bmatrix}, \\ C_d &= \begin{bmatrix} 0 & 1/\lambda \end{bmatrix}, \qquad D_d = [0], \end{aligned} \tag{6}$$

which is equivalent to the following transfer function:

$$G(\sigma) = \frac{1 - e^{-pT_s}}{\lambda m p(\sigma - e^{-pT_s})}. \tag{7}$$

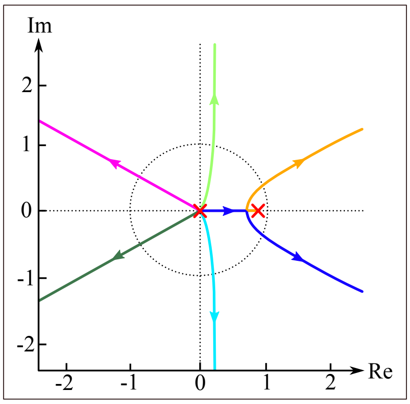


Figure 1: Root locus plot of the ZOH-model including drag and delay ($N = 5$).

Note that we use $\sigma$ as the Z-transform variable instead of $z$ to avoid confusion with the height $Z$.

Introducing a delay of $N$ sampling periods in (7) gives:

$$G(\sigma) = \frac{1 - e^{-pT_s}}{\lambda m p \sigma^N (\sigma - e^{-pT_s})}. \tag{8}$$

The closed-loop transfer function with a proportional control is thus given as:

$$H(\sigma) = \frac{K\left(1 - e^{-pT_s}\right)}{\lambda m p \sigma^N (\sigma - e^{-pT_s}) + K\left(1 - e^{-pT_s}\right)}. \tag{9}$$

Equation (9) shows that, for $K = 0$, the system has $N$ poles at $\sigma = 0$ and one additional pole at $\sigma = e^{-pT_s}$. This later pole is located on the positive part of the real axis inside the unit circle, as $p > 0$ and $T_s > 0$. Also, there are $N + 1$ zeros at equiangular infinities, creating asymptotes $2\pi/(N + 1)$ radians apart from each neighbor. An example of a root locus plot with $N = 5$ is shown in Fig. 1.

The critical gain $K_{cr}$ can be found by solving the following two equations:

$$\sigma = \cos\theta + j\sin\theta = e^{j\theta}, \tag{10}$$

$$\lambda m p \sigma^N \left(\sigma - e^{-pT_s}\right) + K_{cr}\left(1 - e^{-pT_s}\right) = 0, \tag{11}$$

where $\theta$ is the phase of the pole that intersects the unit circle. Given a specific $m$, $p$, $T_s$ and $N$, we can solve these two equations for the two unknowns $\theta$ and $K_{cr}$. In the following, we present the full derivation of the solution.

Substituting (10) into (11) gives:

$$\begin{aligned}
&\lambda m p \left(e^{j\theta}\right)^N \left(e^{j\theta} - e^{-pT_s}\right) + K_{cr}\left(1 - e^{-pT_s}\right) = 0,\\
&\lambda m p \left(e^{j(N+1)\theta} - e^{jN\theta} e^{-pT_s}\right) + K_{cr}\left(1 - e^{-pT_s}\right) = 0,\\
&\lambda m p \left[\cos\left((N+1)\theta\right) + j\sin\left((N+1)\theta\right) - e^{-pT_s}\left(\cos(N\theta) + j\sin(N\theta)\right)\right] + K_{cr}\left(1 - e^{-pT_s}\right) = 0.
\end{aligned} \tag{12}$$

The imaginary part of (12) gives:

$$\lambda mp\left[\sin\left(\left(N+1\right)\theta\right)-e^{-pT_s}\sin\left(N\theta\right)\right]=0,$$

which leads to:

$$e^{-pT_s}=\frac{\sin\left(\left(N+1\right)\theta\right)}{\sin\left(N\theta\right)}. \tag{13}$$

Likewise, the real part of (12) gives:

$$\lambda mp\left[\cos\left(\left(N+1\right)\theta\right)-e^{-pT_s}\cos\left(N\theta\right)\right]+K_{cr}\left(1-e^{-pT_s}\right)=0,$$

which can be rearranged into:

$$K_{cr}=-\frac{\lambda mp}{\left(1-e^{-pT_s}\right)}\left(\cos\left(\left(N+1\right)\theta\right)-e^{-pT_s}\cos\left(N\theta\right)\right). \tag{14}$$

Using (13), Equation (14) can be written as:

$$\begin{aligned}
K_{cr} &= -\frac{\lambda mp}{\left(1-e^{-pT_s}\right)}\left(\cos\left(\left(N+1\right)\theta\right)-\frac{\cos\left(N\theta\right)\sin\left(\left(N+1\right)\theta\right)}{\sin\left(N\theta\right)}\right)\\
&= -\frac{\lambda mp\left[\sin\left(N\theta\right)\cos\left(\left(N+1\right)\theta\right)-\cos\left(N\theta\right)\sin\left(\left(N+1\right)\theta\right)\right]}{\left(1-e^{-pT_s}\right)\sin\left(N\theta\right)}\\
&= -\frac{\lambda mp\ \sin\left(N\theta-(N+1)\theta\right)}{\left(1-e^{-pT_s}\right)\sin\left(N\theta\right)}\\
&= -\frac{\lambda mp\ \sin\left(-\theta\right)}{\left(1-e^{-pT_s}\right)\sin\left(N\theta\right)}\\
&= \frac{\lambda mp\ \sin\theta}{\left(1-e^{-pT_s}\right)\sin\left(N\theta\right)}.
\end{aligned} \tag{15}$$

Now, let us define $\alpha$ as follows:

$$\boxed{\alpha := \frac{K_{cr}}{\lambda}} \tag{16}$$

Using (15), $\alpha$ can be evaluated as:

$$\boxed{\alpha = \frac{mp\ \sin\theta}{\left(1-e^{-pT_s}\right)\sin\left(N\theta\right)}} \tag{17}$$

where $p$ is defined in (4). Note that the only unknown in (17) is $\theta$, and it can be found by solving (13) numerically.

## 2 Link to the MATLAB code

We provide a MATLAB script (`evaluate_alpha.m`) which computes $\alpha$ in (17) across different ranges of time delay $N$ and vertical velocity ${V_z}^w$ at linearization point. We find that:

1. While there are multiple solutions for $\theta$ that suffice (13), the poles that cross the unit circle in z-plane for the first time are those with the smallest magnitude of $\theta$ (around $\pm\frac{\pi}{(N+1)}$ rad). In Fig. 1, this is where orange and blue loci intersect the unit circle.

2. $\alpha$ decreases with an increase in time delay $N$.

3. The percentage deviation of $\alpha$ caused by different values of ${V_z}^w$ at different linearization points grow linearly with the time delay $N$. However, this effect is very small and can be neglected.

The last two findings imply that if we assume $m$, $T_s$, $\rho$, $C_D$, $A$ and $N$ to be constant, there is a unique linear relationship between the critical gain $K_{cr}$ and scale $\lambda$, characterized by a unique constant coefficient $\alpha$.