Differential Manifolds

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Definition 1.1. *For many differentiable atlases on one topological manifold M, the C^∞-compatibility is an equivalent relationship. Every equivalent class S is called a differential structure and (M, S) a differential manifold. In other words, we can also treat S as a maximum differential atlas, i.e., for any differentiable atlas S' compatible with S, we have $S' \subset S$.*

Example 1.1 (Torus)**.** *For independent $w_1, w_2 \in \mathbb{R}^2$, let $\Gamma = \{nw_1 + mw_2 : n, m \in \mathbb{Z}\}$. Define a relation $\sim$ on $\mathbb{R}^2$ by $p \sim q$ if and only if $p - q \in \Gamma$, then the torus $\mathbb{R}^2/\Gamma := \mathbb{R}^2/\sim$ is a 2-dimensional differential manifold.*

Proof. Easy to check a torus is Hausdorff and secondly countable, so we only have to equip an atlas on it. The natural projection $\pi : \mathbb{R}^2 \to \mathbb{R}^2/\Gamma$ is open since

$$\pi^{-1}\big(\pi(A)\big) = \bigcup_{x\in\Gamma} A + x.$$

Notice that $\{B_r(p) : p \in \mathbb{R}^2\}$ covers $\mathbb{R}^2$ for any $r > 0$, we prove $\pi|B_r(p) : B_r(p) \to U_p$ is a homeomorphism for small enough r where $U_p = \pi\big(B_r(p)\big)$. Suppose $\pi(x) = \pi(y)$, then $x - y \in \Gamma$, since $x - y \in B_{2r}(0)$ and $B_{2r}(0) \cap \Gamma = \{0\}$ for small enough r, we know $\pi|B_r(p)$ is injective and thus a homeomorphism. Now let $f_p = \big(\pi|B_r(p)\big)^{-1}$, we prove $f_p \circ f_q^{-1}$ is a translation and thus we did equip an atlas. Notice

$$f_p \circ f_q^{-1}(x) = x + z(x) \in B_r(p), \quad \forall x \in B_r(q)$$

for some $z(x) \in \Gamma$. Suppose $x + z_1$, $y + z_2$ are two such points, we know

$$\begin{aligned} |z_1 - z_2| &\le \big|z_1 + x - (z_2 + y)\big| + |x - y| \\ &\le |z_1 + x - p| + |z_2 + y - p| + |x - q| + |y - q| \\ &< 4r. \end{aligned}$$

Thus $z_1 - z_2 \in B_{4r}(0) \cap \Gamma = \{0\}$ for small enough r, which completes our proof. □