Differential Topology Homework 1

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Problem 1. *Describe an embedding $S^1 \times S^1 \hookrightarrow \mathbb{R}^3$ explicitly using elementary functions; more generally show that $S^p \times S^q$ embeds in $\mathbb{R}^{p+q+1}$. (Hint: show that $S^p \times \mathbb{R}^{q+1}$ embeds in $\mathbb{R}^{p+q+1}$ inductively.)*

Proof. Denote $f : S^1 \times S^1 \to \mathbb{R}^3$ by

$$x = \big(1 + \frac{c}{r}\big)a,\ y = \big(1 + \frac{c}{r}\big)b,\ z = d,$$

where $a^2 + b^2 = r^2,\ r > 1,\ c^2 + d^2 = 1$. To Show f is differentiable, we need an atlas over $S^1 \times S^1$. Let $S_t^+ = \{(a, b) : a^2 + b^2 = t^2, b > 0\}$, then

$$\varphi : S_r^+ \times S_1^+ \to (-r, r) \times (-1, 1),\ \varphi(a, b, c, d) = (a, c)$$

is a chart and $f \circ \varphi^{-1} : (-r, r) \times (-1, 1) \to \mathbb{R}^3$ is given by

$$x = \big(1 + \frac{c}{r}\big)a,\ y = \big(1 + \frac{c}{r}\big)\sqrt{1 - a^2},\ z = \sqrt{1 - c^2},$$

which is smooth. Similarly, it is true for all other charts and thus f is smooth. Now we show $X := f(S^1 \times S^1)$ is a submanifold. Since

$$X = \{(x, y, z) \in \mathbb{R}^3 : \big(\sqrt{x^2 + y^2} - r\big)^2 + z^2 = 1\},$$

we define $g(x, y, z) = \big(\sqrt{x^2 + y^2} - r\big)^2 + z^2 - 1$ near X, then $g(X) \subset \mathbb{R}^2 \times 0$. Notice that

$$g_x = 8\Big(1 - \frac{r}{\sqrt{x^2 + y^2}}\Big)x,\ g_y = 8\Big(1 - \frac{r}{\sqrt{x^2 + y^2}}\Big)y,\ g_z = 2z,$$

easy to check g_x, g_y, g_z cannot vanish simultaneously on X, thus there must be at least one map around every point on X, such that the rank is constantly 2. For example, we let $\phi(x, y, z) = \big(x, y, g(x, y, z)\big)$ or $\phi(x, y, z) = \big(g(x, y, z), y, z\big)$, then by the constant-rank level set theorem, X is a submanifold. Finally we show $f : S^1 \times S^1 \to X$ is a diffeomorphism, notice $f^{-1} : X \to S^1 \times S^1$ is given by

$$a = \frac{rx}{\sqrt{x^2 + y^2}},\ b = \frac{ry}{\sqrt{x^2 + y^2}},\ c = \sqrt{x^2 + y^2} - r,\ d = z,$$

we know $\varphi \circ f^{-1}$ is smooth and so is f^{-1}. □

Problem 2. *Let $x = [x_0, \dots, x_n]$ be the homogeneous coordinates of points in $\mathbb{R}P^n$. Show that*

$$f : \mathbb{R}P^n \times \mathbb{R}P^m \to \mathbb{R}P^{mn+m+n}$$
$$([x_0, \dots, x_n], [y_0, \dots, y_m]) \mapsto [x_0, y_0, x_0y_1, \dots, x_iy_j, \dots, x_ny_m]$$

is an embedding.

Problem 3. *Let M be a differential manifold, $C^\infty(M)$ the algebra of differentiable functions on M. For a point $p \in M$, let $\mathfrak{M}_p = \{\phi \in C^\infty(M) : \phi(p) = 0\}$. Show that*

(a) *$\mathfrak{M}_p$ is a maximal ideal of $C^\infty(M)$.*

(b) *If M is compact and $\mathfrak{M} \subsetneq C^\infty(M)$ is a maximal ideal, then there exists some $p \in M$ such that $\mathfrak{M} = \mathfrak{M}_p$.*

Proof. For (a), suppose $\mathfrak{M}_p \subsetneq \mathfrak{M} \subset C^\infty(M)$ where $\mathfrak{M}$ is an ideal, then there is some smooth function $f \in \mathfrak{M}$ with $f(p) \neq 0$. Since the constant function $a/f(p)$ is smooth for all $a \in \mathbb{R}$, $g_a := af/f(p) \in \mathfrak{M}$ with $g_a(p) = a$. Now for any smooth function h, we have $h - g_{h(p)} \in \mathfrak{M}_p$ and thus $h \in \mathfrak{M}$, which is to say $\mathfrak{M} = C^\infty(M)$.

For (*b*), first we show there must be some $p \in M$ such that $f(p) = 0$, $\forall f \in \mathfrak{M}$. If not, then for all $p \in M$, there is some open neighborhood $p \in U_p$ and $f_p \in \mathfrak{M}$ such that f_p does not have any zeros on U_p. Since all these U_p-s cover M and M is compact, we get a finite cover $U_{p_1}, \dots, U_{p_n}$. Easy to see $0 < f := f_{p_1}^2 + \cdots + f_{p_n}^2 \in \mathfrak{M}$ and thus $1 \in \mathfrak{M}$, $\mathfrak{M} = C^\infty(M)$ since $1/f$ is smooth, which is a contradiction. Now let $\mathfrak{M}_p \subset \mathfrak{M}$ for some p, then by (a) $\mathfrak{M}_p = \mathfrak{M}$ follows obviously. □

Problem 4. *Let $\phi : S^n \to \mathbb{R}$ be a differentiable function. Show that there are two different points $p, q \in S^n$ such that ϕ_{*p} and ϕ_{*q} are both zero.*

Proof. Suppose ϕ is not a constant function, since S^n is compact and ϕ is continuous, there must be two different points where ϕ gets its extreme values. Let p be such a point and $(0 \in U \subset \mathbb{R}^n, f)$ be a chart where $f(0) = p$, then $\phi \circ f : U \to \mathbb{R}$ gets its extreme value at p, so

$$0 = \big(\phi \circ f\big)_{*p} = \phi_{*p} \circ f_{*0}.$$

Notice f is invertible on U, so $\phi_{*p} = 0$. □

Problem 5. *Let $X = \mathbb{R} \sqcup \mathbb{R}$ be the disjoint union of two real lines, $f : X \to \mathbb{R}^2$ be the map, which on the first connected component is $x \mapsto (x, 0)$, on the second connected component is $y \mapsto (0, \exp(y))$. Show that f is an injective immersion, but not an embedding. Draw a sketch of the image.*

Proof. We call the first component $\mathbb{R}_1$ and the second $\mathbb{R}_2$, then

$$\begin{aligned} \operatorname{rank}_x(f) &= \operatorname{rank}_x(1,0)^T = 1, \quad \forall x \in \mathbb{R}_1, \\ \operatorname{rank}_y(f) &= \operatorname{rank}_y(0, e^y)^T = 1, \quad \forall y \in \mathbb{R}_2, \end{aligned}$$

which tells us f is an immersion. Now suppose f is an embedding and let

$$Y := f(X) = (\mathbb{R} \times 0) \cup (0 \times \mathbb{R}_+),$$

then there is some chart (φ, U) around $(0,0)$ such that $\varphi(U \cap Y) = \varphi(U) \cap (\mathbb{R} \times 0)$. For some small $\epsilon > 0$ with $0 \in B_0(\epsilon) \subset U$, since φ is a homeomorphism, we know $\varphi\big(B_0(\epsilon) \cap Y\big) = (a, b)$ for some $a < 0 < b$. This is impossible since $\varphi\big(B_0(\epsilon) \cap Y \backslash (0,0)\big) = (a, c) \cup (c, b)$ leads to that the domain has three connected components while the image has two. Thus we finally proved f is not an embedding. □

Problem 6. *Let $f : \mathbb{R} \sqcup S^1 \to \mathbb{C}$ be the map, which on the first connected component is $t \mapsto (1 + e^t) \cdot e^{it}$, on the second connected component is $e^{it} \mapsto e^{it}$. Show that f is an injective immersion, but not an embedding. Draw a sketch of the image.*