Introduction to Partial Differential Equations – Exercises for Week 1 –

Contents

1. Exercise 1: Divergence theorem

The divergence of a vector field $v \in C^1(\bar{U};\mathbb{C}^n)$ is

$$\operatorname{div}(v) = \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j} \in C(U). \tag{1}$$

Exercise 1.1. Prove the following: If $f \in C^1(U)$ and $v \in C^1(U;\mathbb{C}^n)$, then

$$\operatorname{div}(fv) = \nabla f \cdot v + f \operatorname{div}(v). \tag{2}$$

Theorem 1.1 (Divergence Theorem: [Tre75, Lemma 10.1])**.** *Let $U \subset \mathbb{R}^n$ be an open set with C^1 boundary ∂U. We denote by $\nu : \partial U \to \mathbb{S}^{n-1}$ the outward normal. Let $v \in C^1(\bar{U};\mathbb{C}^n)$ be a vector field. Then*

$$\int_U \operatorname{div}(v)\,\mathrm{d}x = \int_{\partial U} v \cdot \nu \,\mathrm{d}S(x). \tag{3}$$

Use the divergence theorem to prove the following identities:

Exercise 1.2. Let $U \subset \mathbb{R}^n$ be an open set with C^1 boundary ∂U and outward normal $\nu : \partial U \to \mathbb{S}^{n-1}$. Show that, if $u, v \in C^1(\bar{U})$, then

$$\int_U \frac{\partial u}{\partial x_j} \cdot v \,\mathrm{d}x = -\int_U u \cdot \frac{\partial v}{\partial x_j}\,\mathrm{d}x + \int_{\partial U} uv\nu_i. \tag{4}$$

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Exercise 1.3. Let $U \subset \mathbb{R}^n$ be an open set with C^1 boundary ∂U and outward normal $\nu : \partial U \to \mathbb{S}^{n-1}$. Show that, if $u, v \in C^2(\bar{U})$, then

$$\int_U \triangle u \,\mathrm{d}x = \int_{\partial U} Du \cdot \nu \,\mathrm{d}S, \tag{5}$$

$$\int_U Du \cdot Dv \,\mathrm{d}x = -\int_U u\triangle v + \int_{\partial U} uDv \cdot \nu \,\mathrm{d}S, \tag{6}$$

$$\int_U (u\triangle v - v\triangle u)\,\mathrm{d}x = \int_{\partial U} (uDv - vDu)\cdot \nu \,\mathrm{d}S. \tag{7}$$

2. Exercise 2: Coarea formula

Theorem 2.1 (Coarea Formula:[AFP00, Theorem 2.93& Remark 2.94])**.** *Let $\Omega \subset \mathbb{R}^n$ be an open set and $F : \Omega \to \mathbb{R}^k$ a C^1-submersion, that is, a C^1-smooth map with surjective differential at each point. As a consequence, we have that $F(\Omega)$ is open in $\mathbb{R}^k$ and that, for every $y \in F(\Omega)$, the set $F^{-1}(y) \subset \Omega$ is an immersed submanifold of dimension $n-k$.*

Then, for every $u \in L^1(\Omega)$ with compact support,

$$\int_\Omega u(x)J(DF(x))\,\mathrm{d}x = \int_{F(\Omega)}\int_{F^{-1}(y)} u(x)\mathrm{d}S^{n-k}(x)\mathrm{d}y, \tag{8}$$

where

$$J(DF(x)) = \det(DF(x)\times DF(x)^T) = \sqrt{\sum_{B\in\{k\times k \text{ minors of } DF(x)\}} \det(B)^2}. \tag{9}$$

Exercise 2.1. Compute $J(DF)$ as in (9) when $k=1$ and when $k=n-1$.

Exercise 2.2. Show that, for $\Omega \subset \mathbb{R}^n$ open,

$$\int_\Omega u(x)\,\mathrm{d}x = \int_0^\infty \int_{\partial B(0,r)\cap\Omega} u(x)\,\mathrm{d}S^{n-1}(x)\,\mathrm{d}r \qquad \forall u \in C^0(\Omega). \tag{10}$$

From Theorem 2.1, we can deduce seemingly more general results. For instance, a coarea formula on the sphere:

Exercise 2.3. Show the following formulas. Let $\mathbb{S}^{n-1}$ the unit sphere in $\mathbb{R}^n$ centered at 0. If $f \in C^1(\mathbb{R}^n)$, then, for every $u \in C^0(\mathbb{S}^{n-1})$,

$$\int_{\mathbb{S}^{n-1}} u(x)|\nabla f(x) - (\nabla f(x)\cdot x)x|\,\mathrm{d}S^{n-1}(x) = \int_\mathbb{R}\int_{\mathbb{S}^{n-1}\cap\{f=z\}} u(x)\,\mathrm{d}S^{n-2}(x)\,\mathrm{d}z. \tag{11}$$

For example, if $f(x) = x_n$, then, for every $u \in C^0(\mathbb{S}^{n-1})$,

$$\int_{\mathbb{S}^{n-1}} u(x)\sqrt{1-x_n^2}\,\mathrm{d}S^{n-1}(x) = \int_{-1}^1 \int_{\mathbb{S}^{n-1}\cap\{f=z\}} u(x)\,\mathrm{d}S^{n-2}(x)\,\mathrm{d}z. \tag{12}$$

3. Exercise 3: Volume of balls

Exercise 3.1. Compute the volume of the ball $B(0,1)$ of radius 1 in $\mathbb{R}^n$, that is,

$$\alpha_n := \mathcal{L}^n(B(0,1)) = |B(0,1)|. \tag{13}$$

Exercise 3.2. Show that the surface measure of the sphere $\partial B(0,r)$ satisfies

$$S^{n-1}(\partial B(0,r)) = \frac{\mathrm{d}}{\mathrm{d}r}|B(0,r)| = n\alpha_n r^{n-1}. \tag{14}$$

Exercise 3.3. Show that, if $u \in C^0(\partial B(0,r))$ for some $r > 0$, then

$$\int_{\partial B(0,r)} u(x)\,\mathrm{d}S^{n-1}(x) = \int_{-r}^{r}\int_{\partial B(0,r)\cap\{x_3=z\}} u(x)\,\mathrm{d}S^{n-2}(x)\frac{1}{\sqrt{r^2-z^2}}\,\mathrm{d}z. \tag{15}$$

Exercise 3.4. For $k > 0$, compute the integral

$$\int_{B(0,R)} \frac{1}{|x|^k}\,\mathrm{d}x. \tag{16}$$

4. Exercise 4: Laplacian

Exercise 4.1. Let $u \in C^2(U)$, $O \in \mathsf{O}(n)$, $b \in \mathbb{R}^n$, $\lambda \neq 0$ real. Define $\bar{u}(y) = u(\lambda(Oy+b))$. Compute $\triangle\bar{u}$ in terms of $\triangle u$.

Exercise 4.2. Consider the differential operator on $\mathbb{R}^2$

$$P = 2\frac{\partial^2}{\partial x^2} + 2\frac{\partial^2}{\partial x\partial y} + \frac{\partial^2}{\partial y^2}. \tag{17}$$

Find coordinates on $\mathbb{R}^2$ such that, in the new coordinates, P is the Laplace operator.

Exercise 4.3. Describe harmonic polynomials of degree 3 in two variables.

Exercise 4.4. Compute the laplacian of the functions $\mathbb{R}^n \to \mathbb{C}$,

$$u_{v+iw}(x) = \exp(v\cdot x + iw\cdot x) \tag{18}$$

where $v, w \in \mathbb{R}^n$.

Exercise 4.5. (Try to) find nonzero solutions $u \in C^\infty(\mathbb{R}^n)$ to the PDE

$$-\triangle u = \lambda u \tag{19}$$

for $\lambda \in \mathbb{C}$. (These functions u are called *eigenfunctions* of the Laplacian. Not every λ gives a solution).

5. Exercise 5: Mollifiers

Exercise 5.1. Consider the function $\phi : \mathbb{R} \to \mathbb{R}$,

$$\phi(x) = \begin{cases} 0 & x \leq 0, \\ \exp(-1/x) & x > 0. \end{cases} \tag{20}$$

Show that $\phi \in C^\infty(\mathbb{R})$.

Exercise 5.2. Show that there exists $\phi \in C_c^\infty(\mathbb{R}^n)$ such that $\operatorname{spt}(\phi) \subset B(0,1)$, $\phi \geq 0$ and $\int_{\mathbb{R}^n}\phi(x)\,\mathrm{d}x = 1$.

Exercise 5.3 (Fundamental theorem of calculus of variations)**.** Let $U \subset \mathbb{R}^n$. Suppose that $f \in L^1_{\mathrm{loc}}(U)$ is such that

$$\int_U f(x)\phi(x)\,\mathrm{d}x = 0 \qquad \forall\phi \in C_c^\infty(U). \tag{21}$$

Show that $f = 0$ almost everywhere in U.

Exercise 5.4. Let $\psi \in C_c^\infty(\mathbb{R}^n)$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Define

$$f\star\psi(x) := \int_{\mathbb{R}^n} f(y)\psi(x-y)\,\mathrm{d}y \tag{22}$$

Prove the following:

(1) $f\star\psi(x) = \int_{\mathbb{R}^n} f(x-y)\psi(y)\,\mathrm{d}y$.
(2) $f\star\psi \in C^\infty(\mathbb{R}^n)$.
(3) For every $j \in \{1,\dots,n\}$, $\frac{\partial}{\partial x_j}(f\star\psi) = f\star\frac{\partial\psi}{\partial x_j}$.

Exercise 5.5. Let $\psi \in C_c^\infty(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n}\psi = 1$. For $\epsilon > 0$, define

$$\psi_\epsilon(x) := \frac{1}{\epsilon^n}\psi(x/\epsilon). \tag{23}$$

Prove the following:

(1) If $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then, for almost every $x \in \mathbb{R}^n$, $\lim_{\epsilon\to 0} f\star\psi_\epsilon(x) = f(x)$.
(2) If $p \in [1,\infty]$ and $f \in L^p(\mathbb{R}^n)$, then $\lim_{\epsilon\to 0}\|f\star\psi_\epsilon - f\|_{L^p} = 0$.
(3) If $f \in C^0(\mathbb{R}^n)$, then $f\star\psi_\epsilon \to f$ uniformly on compact sets, as $\epsilon \to 0$.
(4) If $k \geq 1$ and $f \in C^k(\mathbb{R}^n)$, then, for all $\alpha \in \mathbb{N}^n$ with $|\alpha| \leq k$, $D_x^\alpha(f\star\psi_\epsilon) \to D_x^\alpha f$ uniformly on compact sets, as $\epsilon \to 0$.

Exercise 5.6. Show that, if $K \Subset U \subset \mathbb{R}^n$, where K is compact and U is open, then there exists $\psi \in C^\infty(\mathbb{R}^n)$ such that $\phi(\mathbb{R}^n) \subset [0,1]$, $K \subset \{\psi = 1\}$ and $\operatorname{spt}(\psi) \subset U$.

References

[AFP00] Luigi Ambrosio, Nicola Fusco, and Diego Pallara, *Functions of bounded variation and free discontinuity problems*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 2000. MR 1857292

[Tre75] François Treves, *Basic linear partial differential equations*, Academic Press [A subsidiary of Harcourt Brace Jovanovich, Publishers], New York-London, 1975, Pure and Applied Mathematics, Vol. 62. MR 0447753