Introduction to Partial Differential Equations – Exercises for Week 3 –

For this week, I do not have many exercises to propose you. If you want more, Evans book as a very interesting section of problems: by now, you should be able to solve all of them. Go to page 84 of [1].

Exercise 0.1. Show the following statement: *Let $U \subset \mathbb{R}^n$ be open, bounded and connected, $T > 0$, and $u \in C^{2;1}(U_T;\mathbb{R}) \cap C^0(\bar{U}_T;\mathbb{R})$ such that $(\partial_t - \triangle)u = 0$ in U_T. If there exists $(x,t) \in U_T$ such that $u(x,t) = \max_{\bar{U}_T} u$, then u is constant on $\bar{U}_t$.*

[This exercise turned out to be more difficult than I expected: It is proven by Evans using the mean value formula for the heat equation.]

Exercise 0.2. Prove the *strong minimum principle* for the heat operator.

Exercise 0.3. Prove the strong maximum principle for the heat operator using the mean-value property; see [1].

Exercise 0.4. State and prove the strong minimum principle for the unbounded Cauchy problem.

Exercise 0.5. Find a counterexample to the uniqueness for the unbounded Cauchy problem

Hint: See [2, chapter 7].

Exercise 0.6. Fix Evans proof of smoothness. See `https://math.stackexchange.com/questions/5044460/smoothness-of-heat-equation-in-evans-partial-differential-equations-again` for details.

References

[1] L. C. Evans. *Partial differential equations.* Second. Vol. 19. Graduate Studies in Mathematics. American Mathematical Society, Providence, RI, 2010, pp. xxii+749.

[2] F. John. *Partial differential equations. 4th ed.* English. Vol. 1. Appl. Math. Sci. Springer, Cham, 1982.

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