Introduction to Partial Differential Equations – Exercises for Week 7 –

Our main reference is Chapter 6 of Rudin's book:

- W. Rudin. *Functional analysis*. Second. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, 1991, pp. xviii+424

There, you can find even more interesting exercises.

1. From last week

Exercise 1.1. Show that, if $A \in \mathscr{D}'(\Omega)$ has finite order N, then A extends as a continuous linear operator from $\mathscr{D}(\Omega)$ to $C^N(\Omega)$. $\Diamond$

2. Derivatives

Exercise 2.1. Show that, if $\alpha \in \mathbb{N}^n$, the function $\phi \mapsto \mathrm{D}^\alpha \phi$ is a continuous linear operator $\mathscr{D}(\Omega) \to \mathscr{D}(\Omega)$. $\Diamond$

Exercise 2.2. Let $f \in C^N(\Omega)$ and $\phi \in \mathscr{D}(\Omega)$. Show that, for every $\alpha \in \mathbb{N}^n$ with $|\alpha| \le N$,

$$\int_\Omega \mathrm{D}^\alpha f(x)\phi(x)\,\mathrm{d}x = (-1)^{|\alpha|}\int_\Omega f(x)\mathrm{D}^\alpha\phi(x)\,\mathrm{d}x. \tag{1}$$

In other words, $\mathrm{D}^\alpha A_f = A_{\mathrm{D}^\alpha f}$. $\Diamond$

Exercise 2.3. Show that, if $A \in \mathscr{D}'(\Omega)$, then $\mathrm{D}^\alpha\mathrm{D}^\beta A = \mathrm{D}^{\alpha+\beta}A = \mathrm{D}^\beta\mathrm{D}^\alpha A$ for all $\alpha, \beta \in \mathbb{N}^n$. $\Diamond$

Exercise 2.4. Show the following proposition:

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^n$ be open and $f \in C(\Omega)$ a continuous function. Suppose that, for every $j \in \{1,\dots,n\}$, there is a continuous function $g_j \in C(\Omega)$ such that $\mathrm{D}^j A_f = A_{g_j}$, i.e., $\mathrm{D}^j f = g_j$ in distributional sense. Then $f \in C^1(\Omega)$ and $\mathrm{D}^j f = g_j$.*

$\Diamond$

Exercise 2.5. Let $f : \mathbb{R} \to \mathbb{R}$ be a function with bounded variation. For instance, the Cantor staircase function. Show that $\mathrm{D}A_f = A_\mu$, where $\mu \in \mathtt{Rad}(\mathbb{R})$ is the measure defined by

$$\mu([a,b)) = f(b) - f(a) \tag{2}$$

for all $a, b \in \mathbb{R}$ with $a < b$. For instance, if f is the Cantor staircase function, then we know that, for almost every $x \in \mathbb{R}$, f is differentiable at x and $f'(x) = 0$. However, $\mathrm{D}A_f \neq 0$.

Hint: see [1, §6.14]. $\Diamond$

April 2, 2025. Last git commit: 615ea09 in branch: master

Exercise 2.6 (Generalized Leibniz Rule)**.** Show that, if $u \in \mathscr{D}'(\Omega)$ and $f \in C^\infty(\Omega)$, then, for every $\alpha \in \mathbb{N}^n$,

$$\mathrm{D}^\alpha(fu) = \sum_{\beta\le\alpha}\binom{\alpha}{\beta}\mathrm{D}^\beta f \cdot \mathrm{D}^{\alpha-\beta}u. \tag{3}$$

Hint: First of all, understand this formula when u is a smooth function. Then consider the case $|\alpha| = 1$ (just one derivative). $\Diamond$

3. Support

Exercise 3.1. Let $A \in \mathscr{D}'(\Omega)$ and $\mathscr{U}$ an open cover of Ω. Show that, if $\bar{A} \in \mathscr{D}'(\Omega)$ is such that $\bar{A} = A$ on ω, for every $\omega \in \mathscr{U}$, then $\bar{A} = A$. $\Diamond$

Exercise 3.2. Show the following statement: if $A \in \mathscr{D}'(\Omega)$ and $f \in C^\infty(\Omega)$ are such that $\mathrm{spt}A \subset \{f = 1\}$, then $fA = A$. $\Diamond$

Exercise 3.3. Show the following statement: if $A \in \mathscr{D}'(\Omega)$ and $f \in C^\infty(\Omega)$, then $\mathrm{spt}(fA) \subset \mathrm{spt}(f) \cap \mathrm{spt}(A)$. Is equality true?

Hint for question: Try with $A = \delta_0$. $\Diamond$

4. Convolution

Exercise 4.1 (Young's inequality)**.** Show that, if $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$, then $f * g \in L^p(\mathbb{R}^n)$ and $\|f * g\|_{L^p} \le \|f\|_{L^1}\|g\|_{L^p}$.

Hint. By Hölder inequality, with $\frac{1}{p} + \frac{1}{p'} = 1$, $\int |f(y)g(y-x)|\,\mathrm{d}y = \int |f(y)|^{1/p'} \cdot |f(y)|^{1/p}|g(y-x)|\,\mathrm{d}y \le (\int |f(y)|\,\mathrm{d}y)^{1/p'} \cdot (\int |f(y)||g(y-x)|^p\,\mathrm{d}y)^{1/p}$. Therefore, $\int (f * g(x))^p\,\mathrm{d}x \le (\int |f(y)|\,\mathrm{d}y)^{p/p'} \cdot \int\int |f(y)||g(y-x)|^p\,\mathrm{d}y\,\mathrm{d}x \le (\int |f(y)|\,\mathrm{d}y)^{p/p'} \cdot \int |g(y)|^p\,\mathrm{d}y \cdot \int |f(y)|\,\mathrm{d}y$. $\Diamond$

Exercise 4.2. Show that, if $f, g \in C^0(\mathbb{R}^n)$, then

$$\mathrm{spt}(f * g) \subset \mathrm{spt}(f) + \mathrm{spt}(g). \tag{4}$$

Can you find a case where equality holds? And where equality does not hold? $\Diamond$

Exercise 4.3. Show the relations

$$\tau_y\tau_z = \tau_{y+z}; \tag{5}$$

$$(\tau_x\phi)^\vee = \tau_{-x}\check{\phi}; \tag{6}$$

$$\tau_x(\mathrm{D}^\alpha\phi)^\vee = (-1)^{|\alpha|}\mathrm{D}^\alpha(\tau_x\check{\phi}). \tag{7}$$

$\Diamond$

Exercise 4.4. Show that, if $u \in \mathscr{D}'$ and $\phi \in \mathscr{D}$, then

$$u[\phi] = (u * \phi)(0). \tag{8}$$

$\Diamond$

Exercise 4.5. Show that, if $u \in \mathscr{D}'$ and $\phi \in \mathscr{D}$, then

$$\operatorname{spt}(u * \phi) \subset \operatorname{spt}(u) + \operatorname{spt}(\phi) = \{x + y : x \in \operatorname{spt}(u),\ y \in \operatorname{spt}(\phi)\}. \tag{9}$$

$\lozenge$

Exercise 4.6. Show that, if $u \in \mathscr{D}'$, $\phi \in \mathscr{D}$ and $v \in \mathbb{R}^n$, then

$$u * (\tau_v \phi) = \tau_v (u * \phi). \tag{10}$$

$\lozenge$

Exercise 4.7. Show that $\phi \mapsto u * \phi$ is linear. $\lozenge$

Exercise 4.8. Show that $\delta_0 * \phi = \phi$ for every $\phi \in \mathscr{D}$. What is $\delta_v * \phi$? $\lozenge$

5. Approximation of Lebesgue integral with Riemann sums

Exercise 5.1. In this exercise, you show that Riemann sums converge to the integral.

Let $f : \mathbb{R}^n \to \mathbb{C}$ be a continuous and integrable function. (Integrable: $\int_{\mathbb{R}^n} |f(z)| \, \mathrm{d}z < \infty$). For $h > 0$, define

$$F_h = \sum_{z \in \mathbb{Z}^n} h^n f(hz). \tag{11}$$

Show that $\lim_{h \to 0} F_h = \int_{\mathbb{R}^n} f(z) \, \mathrm{d}z$. $\lozenge$

Exercise 5.2. [To do while listening to Paganini's Caprice No. 24]. Variation over Exercise 5.1: Let $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{C}$ be a uniformly continuous and integrable function. Define $F : \mathbb{R}^n \to \mathbb{C}$ by

$$F(x) = \int_{\mathbb{R}^n} f(x, z) \, \mathrm{d}z. \tag{12}$$

For $h > 0$ and $x \in \mathbb{R}^n$, define

$$F_h(x) = \sum_{z \in \mathbb{Z}^n} h^n f(x, hz). \tag{13}$$

Show that $F_h \to F$ uniformly in x as $h \to 0$. $\lozenge$

6. Smooth approximation

Exercise 6.1. Let $\Omega \subset \mathbb{R}^n$ convex and $\phi \in C^1(\Omega)$ such that $L = \|\nabla \phi\|_{L^\infty} < \infty$ Show that, for every $x, y \in \Omega$, $|\phi(x) - \phi(y)| \le L|x - y|$.

Question: what happens if we drop the hypothesis of Ω being convex? $\lozenge$

Exercise 6.2. In class, I have rushed the proof of the following proposition. Try give the proof yourself.

Proposition 6.1. *Let $\{\rho_\epsilon\}_{\epsilon > 0}$ be an approximation of the identity on $\mathbb{R}^n$, $\phi \in \mathscr{D}$ and $u \in \mathscr{D}'$. Then*

$$\lim_{\epsilon \to 0} \phi * \rho_\epsilon = \phi \text{ in } \mathscr{D}, \tag{14}$$

$$\lim_{\epsilon \to 0} u * \rho_\epsilon = u \text{ in } \mathscr{D}'. \tag{15}$$

(16)

$\lozenge$

Exercise 6.3. Prove the following statement:

Proposition 6.2. *The space $C^\infty(\mathbb{R}^n)$ is dense in $\mathscr{D}'$ (with respect to the topology of $\mathscr{D}'$).*

$\lozenge$

Exercise 6.4. Show that, if $\Omega \subset \mathbb{R}^n$ is open, then the space $C^\infty(\Omega)$ is dense in $\mathscr{D}'(\Omega)$ (with respect to the topology of $\mathscr{D}'(\Omega)$). $\lozenge$

Exercise 6.5. Is $\mathscr{D}(\Omega)$ dense in $\mathscr{D}'(\Omega)$? (Try at least for $\Omega = \mathbb{R}^n$).

Hint: Take $A[\phi] = \int \phi \, \mathrm{d}x$ and try to approximate A with functions in $C^\infty_c(\Omega)$. $\lozenge$

References

[1] W. Rudin. *Functional analysis*. Second. International Series in Pure and Applied Mathematics. McGraw-Hill, Inc., New York, 1991, pp. xviii+424.