LA0: $\text{altura Nil} = 0$ F

$\text{altura Nil} \overset{A0}{=} \overbrace{\text{foldAB } 0\ (\lambda i\ r\ d \to 1 + \max i\ d)}^{F}$ Nil

$\overset{F0}{=} 0$ □

LA1: $\forall i::AB\,a.\ \forall r::a.\ \forall d::AB\,a.\ \text{altura (Bin } i\ r\ d) = 1+\max(\text{altura } i)\ (\text{altura } d)$

$\text{altura (Bin } i\ r\ d) \overset{A0}{=} F(\text{Bin } i\ r\ d) \overset{F1}{=}$

$(\lambda r\ i\ x\ rd \to 1 + \max ri\ rd)\ (F\ i)\ r\ (F\ d) =_{\beta \times 3}$

$1 + \max (F\ i)\ (F\ d) \overset{A0}{=} 1 + \max (\text{altura } i)\ (\text{altura } d)$

$(\lambda x\ rd \to 1+\max (F\ d)\ x\ rd))\ r\ (F\ d)$

$P(t) = \text{altura}(t) \geq 0$

$P(\text{Nil}) = \text{altura Nil} \geq 0$

$\text{altura Nil} \overset{LA0}{=} 0 \geq 0$ (Int)

Caso Bin: $P(\text{Bin } i\ r\ d) = \text{altura (Bin } i\ r\ d) \geq 0$

$Hi: P(i) \wedge P(d)$

$\text{altura (Bin } i\ r\ d) \overset{LA1}{=} 1 + \max (\text{altura } i)\ (\text{altura } d) \geq_{Hi} 1 + 0 = 1 > 0$

$P(t) = \forall n::\text{Int}.\ n \geq 0 \Rightarrow \text{altura (truncar } t\ n) = \min n\ (\text{altura } t)$

$P(\text{Nil}) = \forall n::\text{Int}.\ n \geq 0 \Rightarrow \text{altura (truncar Nil } n) = \min n\ (\text{altura Nil})$

Si $n < 0$, vale $\Rightarrow$.

Suponemos $n \geq 0$.

$\text{altura (truncar Nil } n) \overset{T0}{=} \text{altura Nil} \overset{A0}{=} 0$

$\min n\ (\text{altura Nil}) \overset{A0}{=} \min n\ 0 \underset{n \geq 0}{=} 0$ □

$P(\text{Bin } i\ r\ d) = \forall n::\text{Int}.\ n \geq 0 \Rightarrow \underbrace{\text{altura (truncar (Bin } i\ r\ d)\ n)}_{A} = \underbrace{\min n\ (\text{altura (Bin } i\ r\ d))}_{B}$

$Hi: P(i) \wedge P(d)$

Si $n < 0$ vale

Sup. $n \geq 0$.

$\text{altura (truncar (Bin } i\ r\ d)\ n) \overset{T1}{=} \text{altura}(\text{if } n == 0 \text{ then Nil}$
$\text{else Bin (truncar } i\ (n-1))\ r\ (\text{truncar } d\ (n-1)))$

Por ext. Bool: $n == 0 = \text{True} \circ \text{False}$

qvq $A = B$

Caso $n==0$ = True

$* = \text{altura Nil} \overset{LA0}{=} 0 \} A$

$\text{min } 0 \text{ (altura (Bin i r d))} \overset{\text{Prop. anterior}}{=} 0 \} B$

caso $n==0$ = False

Caso $n>0$: (Porque para $n<0$ ya lo vimos)

$* = \text{altura (Bin (truncar i } (n-1)) \text{ r (truncar d } (n-1))) \overset{LA1}{=}$

$1+\max \text{ (altura (truncar i } (n-1))) \text{ (altura (truncar d } (n-1))) \overset{HI\ (n>0)}{=}$

$1+\max \text{ (min } (n-1) \text{ (altura i)) (min } (n-1) \text{ (altura d))} \overset{1.}{=}$

$1+\min (n-1) \text{ (max (altura i) (altura d))} \overset{2.}{=} \min \underbrace{(1+n-1)}_{n} \text{ (1+max (altura i)(altura d))}$

$\overset{LA1}{=} \underbrace{\text{min } n \text{ (altura (Bin i r d))}}_{B}$. $\square$

$\neg(p\wedge\sigma), p, \sigma \vdash p$ ax $\quad \neg(p\wedge\sigma), p, \sigma \vdash \sigma$ ax $\quad$ $\wedge i$

$\neg(p\wedge\sigma), p, \sigma \vdash p\wedge\sigma \quad \neg(p\wedge\sigma), p, \sigma \vdash \neg(p\wedge\sigma)$ ax $\quad$ $\neg e$

$\neg(p\wedge\sigma), p, \sigma \vdash \bot$ $\quad$ $\neg(p\wedge\sigma), p, \neg\sigma \vdash \neg\sigma$ ax

LEM $\quad$ $\bot e$ $\quad$ $\vee i2$

$\neg(p\wedge\sigma), p \vdash \sigma\vee\neg\sigma \quad \neg(p\wedge\sigma), p, \sigma \vdash \neg p\vee\neg\sigma \quad \neg(p\wedge\sigma), p, \neg\sigma \vdash \neg p\vee\neg\sigma$ $\quad$ $\vee e$

$\neg(p\wedge\sigma), \neg p \vdash \neg p$ ax

LEM $\quad$ $\vee i1$

$\neg(p\wedge\sigma) \vdash p\vee\neg p \quad \neg(p\wedge\sigma), p \vdash \neg p\vee\neg\sigma \quad \neg(p\wedge\sigma), \neg p \vdash \neg p\vee\neg\sigma$ $\quad$ $\vee e$

$\neg(p\wedge\sigma) \vdash \neg p\vee\neg\sigma$ $\quad$ $\Rightarrow i$

$\vdash \neg(p\wedge\sigma) \Rightarrow (\neg p\vee\neg\sigma)$

$\Gamma, \neg\sigma \vdash p\wedge\sigma$ ax

$\wedge e2$ $\quad$ ax

$\Gamma, \neg\sigma \vdash \sigma \quad \Gamma, \neg\sigma \vdash \neg\sigma$ $\quad$ $\neg e$

$\Gamma, \neg\sigma \vdash \bot$ $\quad$ $\bot e$

ax $\quad$ ax $\quad$ ax

$\neg p\vee\neg\sigma, p\wedge\sigma \vdash p\wedge\sigma \quad \Gamma \vdash \neg p\vee\neg\sigma \quad \Gamma, \neg p \vdash \neg p \quad \Gamma, \neg\sigma \vdash \neg p$

$\wedge e1$ $\quad$ $\vee e$

$\neg p\vee\neg\sigma, p\wedge\sigma \vdash p \quad \Gamma = \{\neg p\vee\neg\sigma, p\wedge\sigma\} \vdash \neg p$ $\quad$ $\vee e$ $\quad$ $\neg e$

$\neg p\vee\neg\sigma, p\wedge\sigma \vdash \bot$ $\quad$ $\neg i$

$\neg p\vee\neg\sigma \vdash \neg(p\wedge\sigma)$ $\quad$ $\Rightarrow i$

$\vdash (\neg p\vee\neg\sigma) \Rightarrow \neg(p\wedge\sigma)$

ax $\quad$ ax

$\Gamma, \neg p \vdash p\wedge\sigma$ $\quad$ $\Gamma, \neg\sigma \vdash p\wedge\sigma$ $\quad$ ax

$\wedge e1$ $\quad$ ax $\quad$ $\wedge e2$

$\Gamma, \neg p \vdash p \quad \Gamma, \neg p \vdash \neg p \quad \Gamma, \neg\sigma \vdash \sigma \quad \Gamma, \neg\sigma \vdash \neg\sigma$

ax $\quad$ $\neg e$ $\quad$ $\neg e$

$\Gamma = \{\neg p\vee\neg\sigma, p\wedge\sigma\} \vdash \neg p\vee\neg\sigma \quad \neg p\vee\neg\sigma, p\wedge\sigma, \neg p \vdash \bot \quad \neg p\vee\neg\sigma, p\wedge\sigma, \neg\sigma \vdash \bot$ $\quad$ $\vee e$

$\neg p\vee\neg\sigma, p\wedge\sigma \vdash \bot$ $\quad$ $\neg i$

$\neg p\vee\neg\sigma \vdash \neg(p\wedge\sigma)$ $\quad$ $\Rightarrow i$

$\vdash (\neg p\vee\neg\sigma) \Rightarrow \neg(p\wedge\sigma)$

$$\frac{}{\Gamma \vdash \langle\rangle_\sigma : Cola_\sigma}\ \text{Ax-Vacio}$$

$$\frac{\Gamma \vdash M : Cola_\sigma \quad \Gamma \vdash N : \sigma}{\Gamma \vdash M \bullet N : Cola_\sigma}\ \text{T-enc}$$

$$\frac{\Gamma \vdash M : Cola_\sigma}{\Gamma \vdash \text{Prox}(M) : \sigma}$$

$$\frac{\Gamma \vdash M : Cola_\sigma}{\Gamma \vdash \text{desencolar}(M) : Cola_\sigma}$$

$$\frac{\Gamma \vdash M : Cola_\tau \quad \Gamma \vdash N : \sigma \quad \Gamma \cup \{c : Cola_\tau, x : \tau\} \vdash O : \sigma}{\Gamma \vdash \text{case } M \text{ of } \langle\rangle \rightsquigarrow N \,;\, c \bullet x \rightsquigarrow O : \sigma}$$

$$V ::= \dots \mid \langle\rangle_\sigma \mid V \bullet V$$

$$\overline{\mathrm{Prox}(\langle\rangle_\sigma \cdot V_1) \to V_1}$$

$$\overline{\mathrm{Prox}((V_1 \bullet V_2) \bullet V_3) \to \mathrm{Prox}(V_1 \bullet V_2)}$$

$$\overline{\mathrm{desencolar}(\langle\rangle_\sigma \cdot V_1) \to \langle\rangle_\sigma}$$

$$\overline{\mathrm{desencolar}(V_1 \bullet V_2 \bullet V_3) \to \mathrm{desencolar}(V_1 \bullet V_2) \bullet V_3}$$

$$\overline{\mathrm{case}\ \langle\rangle_\sigma\ \mathrm{of}\ \langle\rangle \rightsquigarrow M ; c \bullet x \rightsquigarrow N \to M}$$

$$\overline{(\mathrm{case}\ V_1 \bullet V_2\ \mathrm{of}\ \langle\rangle \rightsquigarrow M ; c \bullet x \rightsquigarrow N)}$$ Cerz No $\emptyset$

$$\to N\{c := V_1 ; x := V_2\}$$

$\longrightarrow \text{isZero}(0) \longrightarrow \text{True}$
E-case
No 9

$\text{ultimo}_T := \lambda q : \text{Cola}_T .\ \text{case } q \text{ of}$
$\quad \langle\rangle \rightsquigarrow \text{Prox}(\langle\rangle_T)$
$\quad c \cdot v \rightsquigarrow v$

$\Gamma \vdash \text{ultimo}_T : \text{Cola}_T \to T$

$\{x : B \to B\} \vdash \left(\lambda f : B \to B \,.\, \lambda x : Nat \,.\, f\,(f\ \ iszero(x)) \right)\ x : N \to B$

$S = MGU\{(B \to B) \to N \to B \doteq X_6 \to X_7\} \to \{X_6 \doteq B \to B,\ X_7 \doteq N \to B\} = \{X_6 \leftarrow B \to B,\ X_7 \leftarrow N \to B\}$

$\emptyset \vdash \lambda f : B \to B \,.\, \lambda x : Nat .\ f\,(f\ \ iszero(x)) : (B \to B) \to N \to B$

$\{x : X_6\} \vdash x : X_6$

$\{f : B \to B\} \vdash \lambda x : Nat \,.\, f\,(f\ \ iszero(x)) : Nat \to Bool$

$\left\{\begin{matrix} f : B \to B, \\ x : Nat \end{matrix}\right\} \vdash f\ \ (f\ \ iszero(x)) : Bool$

$S = MGU\{X_4 \doteq X_3 \to X_5,\ X_4 \doteq Bool \to X_3\}$
$\to \{X_3 \to X_5 \doteq Bool \to X_3\} \to \{X_3 \doteq Bool,$
$X_5 \doteq X_3\} \to \ldots = \{X_3 \leftarrow Bool,\ X_5 \leftarrow Bool,$
$X_4 \leftarrow Bool \to Bool\}$

$\{f : X_4\} \vdash f : X_4$

$\left\{\begin{matrix} f : B \to X_3, \\ x : Nat \end{matrix}\right\} \vdash f\ \ \ iszero(x) : X_3$

$S = MGU\{X_2 \doteq Bool \to X_3\}$
$= \{X_2 \leftarrow Bool \to X_3\}$

$\{f : X_2\} \vdash f : X_2$

$\{x : Nat\} \vdash iszero(x) : Bool$

$S = MGU\{X_1 \doteq Nat\} = \{X_1 \leftarrow Nat\}$

$\{x : X_1\} \vdash x\ \ X_1$

$\boxed{\text{CBV}}$

$$V ::= \dots \mid \mathrm{left}(V) \mid \mathrm{right}(V)$$

$$\frac{\Gamma \vdash M \hookrightarrow V}{\Gamma \vdash \mathrm{left}(M) \hookrightarrow \mathrm{left}(V)} \qquad \frac{\Gamma \vdash M \hookrightarrow V}{\Gamma \vdash \mathrm{right}(M) \hookrightarrow \mathrm{right}(V)}$$

$$\frac{\Gamma \vdash M \hookrightarrow \mathrm{left}(V) \qquad \Gamma, x = V \vdash N \hookrightarrow V'}{\Gamma \vdash \mathrm{case}\ M\ \mathrm{of}\ \mathrm{left}(x) \rightsquigarrow N \parallel \mathrm{right}(y) \rightsquigarrow O \hookrightarrow V'}$$

$$\frac{\Gamma \vdash M \hookrightarrow \mathrm{right}(V) \qquad \Gamma, y = V \vdash O \hookrightarrow V'}{\Gamma \vdash \mathrm{case}\ M\ \mathrm{of}\ \mathrm{left}(x) \rightsquigarrow N \parallel \mathrm{right}(y) \rightsquigarrow O \hookrightarrow V'}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\vdash \mathrm{zero} \hookrightarrow \mathrm{zero}}}{\vdash \mathrm{succ}(\mathrm{zero}) \hookrightarrow \mathrm{succ}(\mathrm{zero})}}{\vdash \mathrm{succ}(\mathrm{succ}(\mathrm{zero})) \hookrightarrow \mathrm{succ}(\mathrm{succ}(\mathrm{zero}))}}{\vdash \mathrm{pred}(\underline{2}) \hookrightarrow \mathrm{succ}(\mathrm{zero})}}{\vdash \mathrm{left}(\mathrm{pred}(\underline{2})) \hookrightarrow \mathrm{left}(\underline{1})} \qquad \dfrac{x = \underline{1} \vdash x \hookrightarrow \underline{1}}{x = \underline{1} \vdash \mathrm{isZero}(x) \hookrightarrow \mathrm{false}}$$

$$\vdash \mathrm{case}\ \mathrm{left}(\mathrm{pred}(\underline{2}))\ \mathrm{of}\ \mathrm{left}\ x \rightsquigarrow \mathrm{isZero}(x) \parallel \mathrm{right}\ y \rightsquigarrow \mathrm{true} \hookrightarrow \mathrm{false}$$

$\boxed{\text{CBN}}$ $V ::= \ldots \mid \langle \mathrm{left}(M), \Gamma \rangle \mid \langle \mathrm{right}(M), \Gamma \rangle$

$$\frac{}{\Gamma \vdash \mathrm{left}(M) \Rightarrow \langle \mathrm{left}(M), \Gamma \rangle} \qquad \frac{}{\Gamma \vdash \mathrm{right}(M) \Rightarrow \langle \mathrm{right}(M), \Gamma \rangle}$$

$$\frac{\Gamma \vdash M \Rightarrow \langle \mathrm{left}(M'), \Gamma' \rangle \qquad \Gamma, x = \langle M', \Gamma' \rangle \vdash N \Rightarrow V}{\Gamma \vdash \mathrm{case}\ M\ \mathrm{of}\ \mathrm{left}\ x \mapsto N \parallel \mathrm{right}\ y \mapsto O \Rightarrow V}$$

$$\frac{\Gamma \vdash M \Rightarrow \langle \mathrm{right}(M'), \Gamma' \rangle \qquad \Gamma, y = \langle M', \Gamma' \rangle \vdash O \Rightarrow V}{\Gamma \vdash \mathrm{case}\ M\ \mathrm{of}\ \mathrm{left}\ x \mapsto N \parallel \mathrm{right}\ y \mapsto O \Rightarrow V}$$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{\vdash \mathrm{zero} \Rightarrow \mathrm{zero}}}{\vdash \mathrm{succ}(\mathrm{zero}) \Rightarrow \mathrm{succ}(\mathrm{zero})}}{\vdash \mathrm{succ}(\mathrm{succ}(\mathrm{zero})) \Rightarrow \mathrm{succ}(\mathrm{succ}(\mathrm{zero}))}}{\vdash \mathrm{pred}(\underline{2}) \Rightarrow \mathrm{succ}(\mathrm{zero})}}{x = \langle \mathrm{pred}(\underline{2}), \phi \rangle \vdash x \Rightarrow \underline{1}}$$

$$\bigstar \quad x = \langle \mathrm{pred}(\underline{2}), \phi \rangle \vdash \mathrm{isZero}(x) \Rightarrow \mathrm{false}$$

$$\frac{\frac{}{\vdash \mathrm{left}(\mathrm{pred}(\underline{2})) \Rightarrow \langle \mathrm{left}(\mathrm{pred}(\underline{2})), \phi \rangle} \qquad \bigstar}{\vdash \mathrm{case}\ \mathrm{left}(\mathrm{pred}(\underline{2}))\ \mathrm{of}\ \mathrm{left}\ x \mapsto \mathrm{isZero}(x) \parallel \mathrm{right}\ y \mapsto \mathrm{true} \Rightarrow \mathrm{false}}$$