Estimadores Insesgados

Si se utiliza un estimador varias veces en una misma situación, queremos que la media del valor del estimador, sea el valor que se quiere estimar

Un estimador puntual $\hat{\theta}$ del parámetro θ es insesgado si $E_\theta(\hat{\theta}) = \theta \quad \forall\theta$

$$\text{sesgo de } \hat{\theta} = b(\hat{\theta}) = E_\theta(\hat{\theta}) - \theta$$

Estimador Asintóticamente insesgado: $\hat{\theta}$ es asintóticamente insesgado si

$$E_\theta(\hat{\theta}) \xrightarrow[n \to \infty]{} \theta \quad \forall\theta$$

Estimador Consistente

Otra propiedad que podemos pedir a un estimador es que tienda al valor que se quiere estimar, cuando el tamaño muestral crece

Sea $X_1, \dots, X_n$ una muestra aleatoria de distribución $F_\theta(x)$ y sea $\hat{\theta}_n$ un estimador puntual de θ. Diremos que la sucesión $\{\hat{\theta}_n\}$ es una sucesión consistente (o que $\hat{\theta}_n$ es un estimador consistente de θ) si

$$\hat{\theta}_n \xrightarrow{P} \theta \qquad \left(\forall \varepsilon > 0,\ P(|\hat{\theta}_n - \theta| > \varepsilon) \xrightarrow[n \to \infty]{} 0\right)$$

Error Cuadrático Medio

Permite comparar estimadores

$$ECM(\hat{\theta}) = E_\theta\left[(\hat{\theta} - \theta)^2\right] = Var(\hat{\theta}) + b_\theta^2(\hat{\theta})$$

Ejemplos

1) $f(x,\theta) = \frac{1}{\theta} x^{\left(\frac{1}{\theta}-1\right)} I_{(0,1)}(x) \qquad \theta > 0$

- EMV :

$$\hat{\theta}_{MV} = -\frac{1}{n}\sum_{i=1}^{n} \ln(x_i)$$

- Insesgado:

$$E(\hat{\theta}_{MV}) = E\left(-\frac{1}{n}\sum_{i=1}^{n}\ln(x_i)\right) = -\frac{1}{n}E\left(\sum_{i=1}^{n}\ln(x_i)\right)$$

$$= -\frac{1}{n}\sum_{i=1}^{n}E(\ln(x_i)) \underset{\text{i.i.d}}{=} -\frac{1}{\cancel{n}}\cancel{n}E(\ln(x_1))$$

Calculo $E(\ln(x_1))$

$$E(\ln(x_1)) = \int_{-\infty}^{+\infty} \ln(x) f(x;\theta)\,dx = \int_0^1 \ln(x) \cdot \frac{1}{\theta} x^{\frac{1}{\theta}-1}\,dx$$

$f = \ln(x) \rightarrow f' = 1/x$

$g' = \frac{1}{\theta}x^{1/\theta - 1} \rightarrow g = x^{1/\theta}$

$$= x^{\frac{1}{\theta}}\ln(x)\Big|_0^1 - \int_0^1 \frac{1}{x} x^{\frac{1}{\theta}}\,dx$$

$$= \underbrace{x^{1/\theta}\ln(x)}_{} \Big|_0^1 - \theta x^{1/\theta}\Big|_0^1 = 0 - 0 - (\theta . 1^{1/\theta} - \theta . 0) = -\theta$$

por L'Hôpital, $\lim_{x\to 0^+} x^{1/\theta}\cdot \ln(x) = 0$

$\Rightarrow E(\hat{\theta}_{MV}) = -E(\ln(x_1)) = -(-\theta) = \theta \longrightarrow$ Es insesgado

- Consistencia

Llamo $Y_i = \ln(x_i)$. Entonces

$$\hat{\theta}_{MV} = -\frac{1}{n}\sum_{i=1}^{n}\ln(x_i) = -\frac{1}{n}\sum_{i=1}^{n} Y_i = -\overline{Y_n}$$

Por L.G.N

$$\overline{Y_n} \xrightarrow{P} E(Y) = E(\ln(x)) = -\theta$$

$\Rightarrow \hat{\theta}_{MV} = -\overline{Y_n} \xrightarrow{P} -(-\theta) = \theta \longrightarrow$ Es consistente.

2) Sea $f(y;\theta) = \frac{3y^2}{\theta} e^{-y^3/\theta}$ para $y>0$. Consideremos $\hat{\theta} = \sum_{i=1}^{n} Y_i^3/n$. Calcular el sesgo.

$$E(Y^3) = \int_0^{+\infty} y^3 f(y,\theta) = \int_0^{+\infty} y^3 \cdot \frac{3y^2}{\theta} e^{-\frac{y^3}{\theta}} dy = ?$$

Trabajo con la integral indefinida

$$\int y^3 \cdot \frac{3y^2}{\theta} \cdot e^{-\frac{y^3}{\theta}} dy \underset{\uparrow}{=} \int \theta t e^{-t} dt = \theta \int t e^{-t} dt$$

$t = y^3/\theta \Rightarrow \theta t = y^3$

$dt = \frac{3y^2}{\theta} dy$

Ahora aplico partes:

Si $f = t \Rightarrow f' = 1$

$g' = e^{-t} \quad g = -e^{-t}$

$$\Rightarrow \int t e^{-t} dt = -te^{-t} - \int 1 \cdot (-e^{-t}) dt$$
$$= -te^{-t} + \int e^{-t} dt$$
$$= -te^{-t} - e^{-t}$$

Luego $\int y^3 \frac{3y^2}{\theta} e^{-\frac{y^3}{\theta}} dy = \theta\left[-te^{-t} - e^{-t}\right] = -e^{-t}\theta\left[t-1\right]$

$$= -\theta e^{-y^3/\theta}\left(\frac{y^3}{\theta} - 1\right)$$

$$E(Y^3) = \int_0^{+\infty} y^3 \frac{3y^2}{\theta} e^{-y^3/\theta} dy = -\theta e^{-y^3/\theta}\left(\frac{y^3}{\theta} - 1\right)\Big|_0^{+\infty}$$

$$= \underbrace{\lim_{t \to +\infty} -\theta e^{-t^3/\theta}\left(\frac{t^3}{\theta} - 1\right)}_{=0} - \left(\underbrace{-\theta e^0 (0-1)}_{=-\theta}\right) = \theta$$

$$\lim_{t \to +\infty} \frac{\frac{t^3}{\theta} - 1}{\theta e^{t^3/\theta}} \overset{\frac{\infty}{\infty}}{\underset{\text{L'Hopital}}{=}} \lim_{t \to +\infty} \frac{\cancel{3t^2}/\theta}{\theta \cancel{3t^2}/\theta \, e^{t^3/\theta}} = \lim_{t \to +\infty} \frac{1}{e^{t^3/\theta}} = 0$$

3) Estudiar el comportamiento de $\hat{\theta}_1 = 2\bar{x}$ y de $\hat{\theta}_2 = máx\,(x_1, \dots, x_n)$, estimadores de θ de una variable aleatoria $X \sim \mathcal{U}(0,\theta)$.

• Sesgo • $E(\hat{\theta}_1) = E(2\bar{X}_n) = 2E(\bar{X}_n) = 2 \cdot \frac{\theta}{2} = \theta \rightarrow$ Insesgado

• $E(\hat{\theta}_2) = ?$

Necesito la distribución del máximo: si $U = max(X_1, \dots, X_n)$

$$P(U \le t) = P(X_1 \le t, X_2 \le t, \dots, X_n \le t) \overset{i.i.d}{=} [P(X_1 \le t)]^n = [F_X(t)]^n$$

derivando, queda que $f_U(t) = n(F_X(t))^{n-1} f_X(t)$

Como $X \sim \mathcal{U}(0,\theta)$,

$$f_X(t) = \frac{1}{\theta} I_{[0,\theta]}^{(t)} \quad y \quad F_X(t) = \begin{cases} 0 & t < 0 \\ \frac{t}{\theta} & 0 \le t \le \theta \\ 1 & t > \theta \end{cases}$$

Luego $f_U(t) = n\left(\frac{t}{\theta}\right)^{n-1} \frac{1}{\theta} = n\frac{t^{n-1}}{\theta^n}$ para $0 < t < \theta$

$$E(\hat{\theta}_2) = \int_0^\theta t\, n \frac{t^{n-1}}{\theta^n} dt = \frac{n}{\theta^n} \int_0^\theta t^n dt = \frac{n}{\theta^n} \frac{t^{n+1}}{n+1} \Big|_0^\theta$$

$$\Rightarrow E(\hat{\theta}_2) = \frac{n}{\theta^n} \frac{\theta^{n+1}}{n+1} = \frac{n}{n+1} \cdot \theta \xrightarrow[n \to +\infty]{} \theta$$ - Asintoticamente insesgado

• Por L.G.N, $\bar{X}_n \xrightarrow{P} E(X) = \frac{\theta}{2}$

$$\Rightarrow \hat{\theta}_1 = 2\bar{X}_n \xrightarrow{P} 2 \cdot \frac{\theta}{2} = \theta$$ → Consistente

• Para ver consistencia y calcular el ECM, calculo la varianza de $\hat{\theta}_2$

$$E(\hat{\theta}_2^2) = \frac{n}{n+2}\theta^2 \quad \left(\text{cuenta similar a la de } E(\hat{\theta}_2)\right)$$

$$\text{sesgo}(\hat{\theta}_2) = E(\hat{\theta}_2) - \theta = \frac{n}{n+1}\theta - \theta = \frac{n\theta - \theta(n+1)}{n+1} = \frac{-\theta}{n+1}$$

$$Var(\hat{\theta}_2) = \frac{n}{n+2}\theta^2 - \frac{n^2}{(n+1)^2}\theta^2 = \theta^2\left(\frac{n(n+1)^2 - n^2(n+2)}{(n+2)(n+1)^2}\right)$$

$$= \frac{\theta^2\left(n(n^2+2n+1) - n^2(n+2)\right)}{(n+2)(n+1)^2}$$

$$Var(\hat{\theta}_2) = \frac{n}{(n+2)(n+1)^2}\cdot\theta^2$$

Como $E(\hat{\theta}_2) \xrightarrow[n\to\infty]{} \theta$ y $Var(\hat{\theta}_2) \xrightarrow[n\to\infty]{} 0$

entonces, $\hat{\theta}_2$ es consistente de θ.

ECM: • $ECM(\hat{\theta}_1) = Var(\hat{\theta}_1) = 4\,Var(\bar{X}_n) = 4\,\frac{Var(X)}{n}$

$$= \frac{4}{n}\cdot\frac{(\theta-0)^2}{12} = \frac{\theta^2}{3n}$$

• $ECM(\hat{\theta}_2) = Var(\hat{\theta}_2) + \left(b(\hat{\theta}_2)\right)^2$

$$= \frac{n\theta^2}{(n+2)(n+1)^2} + \frac{\theta^2}{n+1} = \frac{2\theta^2}{(n+2)(n+1)}$$

$ECM(\hat{\theta}_2) < ECM(\hat{\theta}_1)$ ya que $\frac{2\theta^2}{(n+2)(n+1)} < \frac{\theta^2}{3n}$ #