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Ramanujan's Lost Notebook

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Mathematics Subject Classification: 14H42, 33D15, 33C75, 40A15

Library of Congress Control Number: 2005923547

ISBN-10: 0-387-25529-X Printed on acid-free paper.
ISBN-13: 978-0387-25529-3

Printed in the United States of America. (MVY)

9 8 7 6 5 4 3 2 1

springeronline.com

Readers will learn in the introduction to this volume that mathematicians owe a huge debt to R.A. Rankin and J.M. Whittaker for their efforts in preserving Ramanujan's "Lost Notebook." If it were not for them, Ramanujan's lost notebook likely would have been permanently lost. Rankin was born in Garlieston, Scotland, in October 1915 and died in Glasgow in January 2001. For several years he was professor of Mathematics at the University of Glasgow. An account of his life and work has been given by B.C. Berndt, W. Kohnen, and K. Ono in [79]. Whittaker was born in March 1905 in Cambridge and died in Sheffield in January 1984. At his retirement, he was vice-chancellor of Sheffield University. A description of Whittaker's life and work has been written by W.K. Hayman [150].

Through long lapse of time,
This knowledge was lost.
But now, as you are devoted to truth,
I will reveal the supreme secret.

Bhagavad Gita, IV.2 & IV.3

Preface

This volume is the first of approximately four volumes devoted to the examination of all claims made by Srinivasa Ramanujan in *The Lost Notebook and Other Unpublished Papers*. This publication contains Ramanujan's famous lost notebook; copies of unpublished manuscripts in the Oxford library, in particular, his famous unpublished manuscript on the partition function and the tau-function; fragments of both published and unpublished papers; miscellaneous sheets; and Ramanujan's letters to G.H. Hardy, written from nursing homes during Ramanujan's final two years in England. This volume contains accounts of 442 entries (counting multiplicities) made by Ramanujan in the aforementioned publication. The present authors have organized these claims into eighteen chapters, containing anywhere from two entries in Chapter 13 to sixty-one entries in Chapter 17.

Contents

Introduction

Finding the Lost Notebook

In the spring of 1976, G.E. Andrews visited Trinity College Library at Cambridge University. Dr. Lucy Slater had suggested to him that there were materials deposited there from the estate of the late G.N. Watson that might contain some work on q-series. In one box of materials from Watson's estate, Andrews found several items written by Srinivasa Ramanujan. The most interesting item in this box was a manuscript of more than one hundred pages written on 138 sides in Ramanujan's distinctive handwriting. The sheets contained over six hundred mathematical formulas listed consecutively without proofs. Although technically not a notebook, and although technically not "lost," as we shall see later, it was natural in view of the fame of Ramanujan's notebooks [227] to name this manuscript *Ramanujan's lost notebook*. Almost surely, this manuscript, or at least most of it, was written during the last year of Ramanujan's life, after his return to India from England. We do not possess a bona fide proof of this claim, but we shall later present considerable evidence for it.

The manuscript contains no introduction or covering letter. In fact, there are hardly any words in the manuscript. There are a few marks evidently made by a cataloguer, and there are also a few remarks in the handwriting of G.H. Hardy. Undoubtedly, the most famous objects examined in the lost notebook are the *mock theta functions*, about which more will be said later. Concerning this manuscript, Ms. Rosemary Graham, manuscript cataloguer of the Trinity College Library, remarked, "...the notebook and other material was discovered among Watson's papers by Dr. J.M. Whittaker, who wrote the obituary of Professor Watson for the Royal Society. He passed the papers to Professor R.A. Rankin of Glasgow University, who, in December 1968, offered them to Trinity College so that they might join the other Ramanujan manuscripts already given to us by Professor Rankin on behalf of Professor Watson's widow." Since her late husband had been a fellow and scholar at Trinity College and had had an abiding, lifelong affection for Trinity College, Mrs. Watson agreed with Rankin's suggestion that the library at Trinity College would be the best place to preserve her husband's papers. Since Ramanujan had also been a fellow at Trinity College, Rankin's suggestion was even more appropriate.

The natural, burning question now is, *How did this manuscript of Ramanujan come into Watson's possession?* We think that the manuscript's history can be traced.

History of the Lost Notebook

After Ramanujan died on April 26, 1920, his notebooks and unpublished papers were given by his widow, Janaki, to the University of Madras. Also at that time, Hardy strongly advocated bringing together all of Ramanujan's manuscripts, both published and unpublished, for publication. On August 30, 1923, Francis Dewsbury, the registrar at the University of Madras, wrote to Hardy informing him that [81, p. 266]:

> I have the honour to advise despatch to-day to your address per registered and insured parcel post of the four manuscript note-books referred to in my letter No. 6796 of the 2nd idem.
> I also forward a packet of miscellaneous papers which have not been copied. It is left to you to decide whether any or all of them should find a place in the proposed memorial volume. Kindly preserve them for ultimate return to this office.

(The notebooks were returned to Madras, but Hardy evidently kept all the miscellaneous papers.) Although no accurate record of this material exists, the amount sent to Hardy was doubtless substantial. It is therefore highly likely that this "packet of miscellaneous papers" contained the aforementioned "lost notebook." Rankin, in fact, opines [230], [82, p. 124]:

> It is clear that the long MS represents work of Ramanujan subsequent to January 1920 and there can therefore be little doubt that it constitutes the whole or part of the miscellaneous papers dispatched to Hardy from Madras on 30 August 1923.

Further details can be found in Rankin's accounts of Ramanujan's unpublished manuscripts [230], [81, pp. 120–123], [82, pp. 117–142].

In 1934, Hardy passed on to Watson a considerable amount of his material on Ramanujan. However, it appears that either Watson did not possess the "lost" notebook in 1936 and 1937 when he published his papers [289], [290] on mock theta functions, or he had not examined it thoroughly. In any event, Watson [289, p. 61], [81, p. 330] writes that he believes that Ramanujan was unaware of certain third order mock theta functions and their transformation formulas. But, in his lost notebook, Ramanujan did indeed examine these functions and their transformation formulas. Watson's interest in Ramanujan's mathematics waned in the late 1930s, and Hardy died in 1947. In conclusion, sometime between 1934 and 1947 and probably closer to 1947, Hardy gave Watson the manuscript we now call the "lost notebook." More will be said in the sequel about further contents of the lost notebook.

Watson devoted about 10 to 15 years of his research to Ramanujan's work, with over 30 papers having their genesis in Ramanujan's mathematics, in particular, his notebooks and the letters he wrote to Hardy from India. Watson was Mason professor of pure mathematics at the University of Birmingham for most of his career, retiring in 1951. He died in 1965 at the age of 79. Rankin, who succeeded Watson as Mason professor of pure mathematics in Birmingham but who had since become professor of mathematics at the University of Glasgow, was asked to write an obituary of Watson for the London Mathematical Society. Rankin writes [230], [82, p. 120]:

> For this purpose I visited Mrs Watson on 12 July 1965 and was shown into a fair-sized room devoid of furniture and almost knee-deep in manuscripts covering the floor area. In the space of one day I had time only to make a somewhat cursory examination, but discovered a number of interesting items. Apart from Watson's projected and incomplete revision of Whittaker and Watson's *Modern Analysis* in five or more volumes, and his monograph on *Three decades of midland railway locomotives*, there was a great deal of material relating to Ramanujan, including copies of Notebooks 1 and 2, his work with B.M. Wilson on the Notebooks and much other material. ...
> In November 19 1965 Dr J.M. Whittaker who had been asked by the Royal Society to prepare an obituary notice [293], paid a similar visit and unearthed a second batch of Ramanujan material. A further batch was given to me in April 1969 by Mrs Watson and her son George.

A more colorful rendition of Whittaker's visit with Mrs. Watson was described in a letter of August 15, 1979, to Andrews [81, p. 304]:

> When the Royal Society asked me to write G.N. Watson's obituary memoir I wrote to his widow to ask if I could examine his papers. She kindly invited me to lunch and afterwards her son took me upstairs to see them. They covered the floor of a fair sized room to a depth of about a foot, all jumbled together, and were to be incinerated in a few days. One could only make lucky dips and, as Watson never threw away anything, the result might be a sheet of mathematics but more probably a receipted bill or a draft of his income tax return for 1923. By an extraordinary stroke of luck one of my dips brought up the Ramanujan material which Hardy must have passed on to him when he proposed to edit the earlier notebooks.

(That Watson's papers "were to be incinerated in a few days" seems fanciful.) Rankin dispatched Watson's and Ramanujan's papers to Trinity College in three batches on November 2, 1965; December 26, 1968; and December 30, 1969, with the Ramanujan papers being in the second shipment. Rankin did not realize the importance of Ramanujan's papers, and so when he wrote Watson's obituary [229] for the *Journal of the London Mathematical Society*, he did not mention any of Ramanujan's manuscripts. Thus, for almost eight years, Ramanujan's "lost notebook" and some fragments of papers by Ramanujan lay in the library at Trinity College, known only to a few of the library's cataloguers, Rankin, Mrs. Watson, Whittaker, and perhaps a few others. The 138-page manuscript waited there until Andrews found it and brought it before the mathematical public in the spring of 1976. It was not until the centenary of Ramanujan's birth on December 22, 1987, that Narosa Publishing House in New Delhi published in photocopy form Ramanujan's lost notebook and his other unpublished papers [228].

The Origin of the Lost Notebook

Having detailed the probable history of Ramanujan's lost notebook, we return now to our earlier claim that the lost notebook emanates from the last year of Ramanujan's life. On February 17, 1919, Ramanujan returned to India after almost five years in England, the last two being confined to nursing homes. Despite the weakening effects of his debilitating illness, Ramanujan continued to work on mathematics. Of this intense mathematical activity, up to the discovery of the lost notebook, the mathematical community knew only of the mock theta functions. These functions were described in Ramanujan's last letter to Hardy, dated January 12, 1920 [226, pp. xxix–xxx, 354–355], [81, pp. 220–223], where he wrote:

> I am extremely sorry for not writing you a single letter up to now .... I discovered very interesting functions recently which I call "Mock" ϑ-functions. Unlike the "False" ϑ-functions (studied partially by Prof. Rogers in his interesting paper) they enter into mathematics as beautifully as the ordinary theta functions. I am sending you with this letter some examples.

In this letter, Ramanujan defines four third order mock theta functions, ten fifth order functions, and three seventh order functions. He also includes three identities satisfied by the third order functions and five identities satisfied by his first five fifth order functions. He states that the other five fifth order functions also satisfy similar identities. In addition to the definitions and formulas stated by Ramanujan in his last letter to Hardy, the lost notebook contains further discoveries of Ramanujan about mock theta functions. In particular, it contains the five identities for the second family of fifth order functions that were only mentioned but not stated in the letter.

We hope that we have made the case for our assertion that the lost notebook was composed during the last year of Ramanujan's life, when, by his own words, he discovered the mock theta functions. In fact, only a fraction (perhaps 5%) of the notebook is devoted to the mock theta functions themselves.

The Content of the Lost Notebook

The next fundamental question is, *What is in Ramanujan's lost notebook besides mock theta functions?* A majority of the results fall under the purview of q-series. These include mock theta functions, theta functions, partial theta function expansions, false theta functions, identities connected with the Rogers–Fine identity, several results in the theory of partitions, Eisenstein series, modular equations, the Rogers–Ramanujan continued fraction, other q-continued fractions, asymptotic expansions of q-series and q-continued fractions, integrals of theta functions, integrals of q-products, and incomplete elliptic integrals. Other continued fractions, other integrals, infinite series identities, Dirichlet series, approximations, arithmetic functions, numerical calculations, Diophantine equations, and elementary mathematics are some of the further topics examined by Ramanujan in his lost notebook.

The Narosa edition [228] contains further unpublished manuscripts, fragments of both published and unpublished papers, letters to Hardy written from nursing homes, and scattered sheets and fragments. The three most famous of these unpublished manuscripts are those on the partition function and Ramanujan's tau function, forty identities for the Rogers–Ramanujan functions, and the unpublished remainder of Ramanujan's published paper on highly composite numbers [222], [226, pp. 78–128].

This Volume on the Lost Notebook

This volume is the first of approximately four volumes devoted to providing statements, proofs, and discussions of all the claims made by Ramanujan in his lost notebook and all his other manuscripts and letters published with the lost notebook in [228]. For simplicity, we shall sometimes refer to the entire volume [228] as the lost notebook, even though only 138 pages of this work constitute what was originally the lost notebook. We have attempted to arrange all this disparate material into chapters. Doubtless, we have inadvertently misplaced entries.

With the statement of each entry from Ramanujan's lost notebook, we provide the page number(s) in the lost notebook where the entry can be found. Almost all of Ramanujan's claims are given the designation "Entry," although a few of them have the appellation "Corollary." Results in this volume named theorems, corollaries (except in the aforementioned few cases), and lemmas are not due to Ramanujan. We emphasize that Ramanujan's claims always have page numbers from the lost notebook attached to them.

However, the format of Chapter 10, in which Ramanujan's empirical evidence for the Rogers–Ramanujan identities is discussed, is different. Here we quote Ramanujan from pages 358–361 in the lost notebook and then prove and discuss his claims.

So that readers can more readily find where a certain entry is discussed, we place at the conclusion of each volume a *Location Guide* to where entries can be found in that particular volume. Thus, if a reader wants to know whether a certain identity on page 172 of the Narosa edition [228] can be found in a particular volume, she can turn to this index and determine where in that volume identities on page 172 are discussed.

Following the Location Guide, we provide a *Provenance* indicating the sources from which we have drawn in preparing significant portions of the given chapters. We emphasize that in the Provenance we do not list all papers in which results from a given chapter are established. For example, the content of Chapter 6 has generated dozens of papers. In the chapter itself we have attempted to cite all relevant papers known to us, but in the Provenance we list only those papers from which we have drawn our exposition. On the other hand, almost all chapters contain material previously unpublished. For example, except for the combinatorial proofs, none of the material in Chapter 9 has been previously published.

We now describe the contents of each of the eighteen chapters constituting this first volume. Most, but not all, of the results have been established earlier in the literature, often by Andrews; or Berndt, usually in collaboration with some of his former or current graduate students; or other mathematicians, including the aforementioned students.

An enormous amount of material in the lost notebook is on the Rogers–Ramanujan continued fraction, $R(q)$, clearly one of Ramanujan's favorite functions. From (1.1.2) of Chapter 1, we observe that the Rogers–Ramanujan continued fraction can be represented as a quotient of theta functions. Hence, $R(q)$ lives in the realms of elliptic functions and modular forms, and so the vast machineries of these two fruitful fields can be employed to produce a plethora of theorems. Chapter 1 focuses on identities, modular equations, and representations for $R(q)$ arising from the theory of theta functions and modular equations. Ramanujan evaluated in closed form $R(\pm e^{-\pi\sqrt{n}})$, for certain rational values of n, with many of these values found in his lost notebook. However, in several cases, Ramanujan indicated only that he could find certain values without explicitly providing them. Chapter 2 is devoted to explicit evaluations of $R(\pm e^{-\pi\sqrt{n}})$. Published with the lost notebook is a fragment summarizing some of Ramanujan's findings on the Rogers–Ramanujan continued fraction and on his cubic continued fraction; this brief fragment is examined in Chapter 3. Partition-theoretic implications of the Rogers–Ramanujan continued fraction are contained in Chapter 4. Ramanujan obtained several interesting series representations for $R(q)$, especially one for $R^3(q)$, all of which can also be found in Chapter 4. Chapter 5 is devoted to finite Rogers–Ramanujan continued fractions and other finite continued fractions of the same sort. Some are connected with class invariants.

After these five chapters on the Rogers–Ramanujan continued fraction, we examine other q-continued fractions. Chapter 6 contains some beautiful general theorems followed by many elegant special cases found by Ramanujan. Chapter 7 is in a different vein and is devoted to some asymptotic formulas for continued fractions. One of Ramanujan's most engaging continued fractions is his continued fraction for $(q^2; q^3)_\infty/(q; q^3)_\infty$, the topic of Chapter 8. In contrast to the Rogers–Ramanujan continued fraction, which arises as a special case of general theorems in Chapter 6, this continued fraction does not. One of Ramanujan's most fascinating theorems in the lost notebook is the seemingly enigmatic formula (8.1.2) arising out of the theory of $(q^2; q^3)_\infty/(q; q^3)_\infty$, a theory much different from that of $R(q)$.

The Rogers–Fine identity is one of the most useful theorems in the subject of q-series. Although not explicitly given in his notebooks or lost notebook, Ramanujan clearly was familiar with it and found many applications for it in the lost notebook. More than two dozen identities associated with the Rogers–Fine identity are proved in Chapter 9, some by combinatorial means.

The Rogers–Ramanujan continued fraction is intimately associated with the Rogers–Ramanujan identities, which appear at various places in the first five chapters. In Chapter 10, we examine a fragment on these identities giving empirical evidence for the truth of the identities, and so evidently written before Ramanujan found proofs for them. This chapter is followed by a chapter on other identities of this sort.

Although mock theta functions will not be examined until a further volume, certain partial fraction expansions, the topic of Chapter 12, have intimate associations with mock theta functions.

Chapter 13 is devoted to the study of two of the most enigmatic formulas in the lost notebook. Both are product expansions. One is for a function prominent in the theory of the Rogers–Ramanujan identities. The other is for a quasi-theta function and so can be considered to be an analogue of the Jacobi triple product identity. Although some elements of our proofs might reflect Ramanujan's thinking, we are clearly in the dark about what led Ramanujan ever to think that such formulas might even exist.

One of the most intriguing identities in the lost notebook is a formula relating a character analogue of the Dedekind eta function, an integral of eta functions, and a value of a Dirichlet L-series. This wonderful formula and other integrals of theta functions are the subject of Chapter 14. In Chapter 15, we again examine integrals of eta functions, but these are much different and are related to incomplete elliptic integrals of the first kind. As with so much of the work in Ramanujan's lost notebook, there are no other results of this kind in the literature. The brief Chapter 16 is devoted to five integrals of q-products.

It is difficult to organize Ramanujan's modular equations into one chapter, because they are frequently employed to prove other entries; for example, many new modular equations can be found in Chapter 1. Consigned to Chapter 17 are discussions of one page in the lost notebook and two fragments published with the lost notebook on modular equations.

The last chapter, Chapter 18, is devoted to two fragments on Lambert series, which are also prominent in Chapter 4.

Acknowledgments

The second author is grateful to several of his current and former graduate students for their contributions to this volume, either solely or in collaboration with him. These include Heng Huat Chan, Song Heng Chan, Sen–Shan Huang, Soon–Yi Kang, Wen–Chin Liaw, Jaebum Sohn, Seung Hwan Son, Jinhee Yi, and Liang–Cheng Zhang. He also thanks Ae Ja Yee for her many collaborations during her three postdoctoral years at the University of Illinois, as well as his colleague Alexandru Zaharescu for fruitful collaborations. We are particularly grateful to Sohn and S.H. Chan for carefully reading several chapters, uncovering misprints and more serious errors, and offering many useful suggestions. Our thanks are also extended to Michael D. Hirschhorn for several helpful suggestions.

We thank Springer editor Ina Lindemann for her encouragement and patience. Springer-Verlag's technical editors, Fred Bartlett and Frank Ganz, answered a myriad of questions about LaTeXfor us, and we are very grateful for their advice. We also thank Brandt Kronholm for composing the index and David Kramer for uncovering several stylistic inconsistencies and some further typographical errors as copy editor of our book.

The first author thanks the National Science Foundation, and the second author thanks the John Simon Guggenheim Memorial Foundation, the National Security Agency, and the University of Illinois Research Board for their support.

1

The Rogers–Ramanujan Continued Fraction and Its Modular Properties

1.1 Introduction

The Rogers–Ramanujan continued fraction, defined by

$$R(q) := \frac{q^{1/5}}{1} \, {}_{+} \, \frac{q}{1} \, {}_{+} \, \frac{q^2}{1} \, {}_{+} \, \frac{q^3}{1} \, {}_{+} \cdots, \qquad |q| < 1, \tag{1.1.1}$$

first appeared in a paper by L.J. Rogers [234] in 1894. Using the Rogers–Ramanujan identities, established for the first time in [234], Rogers proved that

$$R(q) = q^{1/5} \frac{(q;q^5)_\infty (q^4;q^5)_\infty}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \tag{1.1.2}$$

Here and in the sequel we employ the customary q-product notation. Thus, set $(a)_0 := (a;q)_0 := 1$, and, for $n \geq 1$, let

$$(a)_n := (a;q)_n := \prod_{k=0}^{n-1} (1 - aq^k). \tag{1.1.3}$$

Furthermore, set

$$(a)_\infty := (a;q)_\infty := \prod_{k=0}^{\infty} (1 - aq^k), \quad |q| < 1.$$

If the base q is understood, we use $(a)_n$ and $(a)_\infty$ instead of $(a;q)_n$ and $(a;q)_\infty$, respectively.

In his first two letters to G.H. Hardy [226, pp. xxvii, xxviii], [81, pp. 29, 57], Ramanujan communicated several theorems on $R(q)$. He also briefly mentioned the more general continued fraction

$$R(a,q) := \frac{1}{1} \, {}_{+} \, \frac{aq}{1} \, {}_{+} \, \frac{aq^2}{1} \, {}_{+} \, \frac{aq^3}{1} \, {}_{+} \cdots, \qquad |q| < 1, \tag{1.1.4}$$

now called the generalized Rogers–Ramanujan continued fraction , and further generalizations. Hardy was intrigued by Ramanujan's theorems on this continued fraction, and on 26 March 1913 (the day on which Paul Erdős was born) wrote [81, pp. 77–78]:

> What I should like above all is a definite proof of some of your results concerning continued fractions of the type
>
> $$\frac{x}{1}\,+\,\frac{x^2}{1}\,+\,\frac{x^3}{1}\,+\cdots;$$
>
> and I am quite sure that the wisest thing you can do, in your own interests, is to let me have one as soon as possible.

Later, in another letter, probably written on 24 December 1913, Hardy further exhorted [81, p. 87]

> If you will send me your proof written out carefully (so that it is easy to follow), I will (assuming that I agree with it—of which I have very little doubt) try to get it published for you in England. Write it in the form of a paper "On the continued fraction
>
> $$\frac{x}{1}\,+\,\frac{x^2}{1}\,+\,\frac{x^3}{1}\,+\cdots,"$$
>
> giving a full proof of the principal and most remarkable theorem, viz. that the fraction can be expressed in finite terms when $x = e^{-\pi\sqrt{n}}$, when $\underline{n}$ is rational.

However, Ramanujan never followed Hardy's advice.

In his notebooks [227], Ramanujan offered many beautiful theorems on $R(q)$. In particular, see (1.1.10) and (1.1.11) below, K.G. Ramanathan's papers [215]–[218], the *Memoir* by Andrews, Berndt, L. Jacobsen, and R.L. Lamphere [39], and Berndt's book [63, Chapter 32].

Ramanujan's lost notebook [228] contains a large number of beautiful, surprising, and remarkable results on the Rogers–Ramanujan continued fraction. In this opening chapter, we prove many theorems arising from modular properties of the Rogers–Ramanujan continued fraction. Papers containing proofs of results proved in this opening chapter include those by Berndt, S.–S. Huang, J. Sohn, and S.H. Son [78], S.–Y. Kang [171], [172], Ramanathan [215], Sohn [253], and Son [254]. But as we emphasized in the Introduction, succeeding chapters also contain theorems about the Rogers–Ramanujan continued fraction. Chapter 2 contains explicit evaluations of $R(q)$ found in the lost notebook. Chapter 3 focuses on a fragment on the Rogers–Ramanujan continued fraction and the cubic continued fraction, which is not found in the lost notebook but was published with the lost notebook. Chapter 4 is devoted to relations connecting $R(q)$ with Lambert series and partitions. Finite Rogers–Ramanujan continued fractions are featured in Chapter 5. Chapter 6 contains theorems in the lost notebook on generalizations (such as (1.1.4)), various analogues, and other q-continued fractions. A survey describing many of Ramanujan's discoveries about the Rogers–Ramanujan continued fraction, especially those found in the lost notebook, can be found in [71].

We now provide notation that will be used throughout the chapter. Recall Ramanujan's general theta function $f(a,b)$, namely,

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \qquad |ab| < 1. \tag{1.1.5}$$

The most important special cases of $f(a,b)$ are defined by (in Ramanujan's notation)

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = (-q;q^2)_\infty^2 (q^2;q^2)_\infty = \frac{(-q;-q)_\infty}{(q;-q)_\infty}, \quad |q| < 1, \tag{1.1.6}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \qquad |q| < 1, \tag{1.1.7}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty, \qquad |q| < 1, \tag{1.1.8}$$

where the latter equality is Euler's pentagonal number theorem. The product representations in (1.1.6)–(1.1.8) follow from Jacobi's triple product identity, given in Lemma 1.2.2 below. Lastly, define

$$\chi(-q) := (q;q^2)_\infty. \tag{1.1.9}$$

Two of the most important formulas for $R(q)$ are given by

$$\frac{1}{R(q)} - 1 - R(q) = \frac{f(-q^{1/5})}{q^{1/5} f(-q^5)} \tag{1.1.10}$$

and

$$\frac{1}{R^5(q)} - 11 - R^5(q) = \frac{f^6(-q)}{q f^6(-q^5)}. \tag{1.1.11}$$

These equalities were found by G.N. Watson [286], [287] in Ramanujan's notebooks and proved by him [286] in order to establish claims about the Rogers–Ramanujan continued fraction communicated by Ramanujan in the aforementioned two letters to Hardy. The proof of (1.1.10) given by Watson [286] is identical to the one given by Ramanujan in his unpublished manuscript on the partition and tau functions, which was published with his lost notebook [228, pp. 135–177, 238–243]; in particular, see page 238. The manuscript was published with proofs and commentary by Berndt and K. Ono [80]. With revised and more extensive commentary, the manuscript will be reproduced in the present authors' third volume on the lost notebook [38]. Different proofs of (1.1.10) and (1.1.11) can be found in Berndt's book [61, pp. 265–267].

We now briefly describe some of the results proved in this chapter.

Our first theorem is remarkable. Ramanujan found three related identities in two variables, two of which contain (1.1.10) and (1.1.11) as special cases. Section 1.2 is devoted to Son's elegant proofs [254].

On page 48 in his lost notebook, Ramanujan offers two further formulas akin to (1.1.10) and (1.1.11). These formulas are "between" (1.1.10) and (1.1.11) in that they involve $R^2(q)$ and $R^3(q)$. Statements and proofs of these identities can be found in Section 1.3.

On the other hand, on page 206 in his lost notebook, Ramanujan claims that (1.1.10) and (1.1.11) can be refined by factoring each side into two factors and then equating appropriate factors on each side, giving four equalities. It is amazing that factoring in this way actually leads to identities, which are proved in Section 1.4.

In his first letter to Hardy [226, p. xxvii], [81, p. 29], Ramanujan claimed that $R^5(q)$ is a particular quotient of quartic polynomials in $R(q^5)$. This was first proved in print by Rogers [236] in 1920, while Watson [286] gave another proof nine years later. At scattered places in his notebooks [227], Ramanujan also gave modular equations relating $R(q)$ with $R(-q)$, $R(q^2)$, $R(q^3)$, and $R(q^4)$. In the publication of his lost notebook [228], these results are conveniently summarized by Ramanujan on page 365; in this book they can be found in Chapter 3. Proofs of most of these modular relations can be found in the *Memoir* [39, Entries 6, 20, 21, 24–26, pp. 11, 27, 28, 31–37], and in Berndt's book [63, Chapter 32, Entries 1–6]. Rogers [236] found modular equations relating $R(q)$ with $R(q^n)$, for $n = 2$, 3, 5, and 11; the latter equation is not found in Ramanujan's work. J. Yi [299] has found a modular equation for $n = 7$, while also devising simpler proofs for degrees 3 and 11. H.H. Chan and V. Tan [118] discovered a modular equation of degree 19 and devised another proof of Rogers's modular equation of degree 11. On page 205 in his lost notebook [228], Ramanujan offers two modular equations relating the Rogers–Ramanujan continued fraction at *three* arguments. These are proved in Section 1.5. The results described in the last three sections were first proved in the paper by Berndt, Huang, Sohn, and Son [78].

In the next four sections we establish several beautiful identities involving the Rogers–Ramanujan continued fraction and some elegant associated theta-function identities. These results were first proved by Kang [171]. In Section 1.6 we prove some theta-function identities of degree 5, in other words, modular equations of degree 5. In the following Section 1.7, we first establish some factorizations, which involve $R(q)$, of the identities in Section 1.6. The next theorem also provides factorizations, and these are in the same spirit as the factorizations of (1.1.10) and (1.1.11) in Section 1.4. In the following Section 1.8, we introduce Ramanujan's parameters $k := R(q)R^2(q^2)$, $\mu := R(q)R(q^4)$, and $\nu := R^2(q^{1/2})R(q)/R(q^2)$, and prove several elegant identities for $R(q)$, $\varphi(q)$, and $\psi(q)$ in terms of these parameters. Section 1.9 gives further identities arising from the parameter k.

In Section 1.10, we prove some formulas for $R(q)$, $R(q^2)$, and $R(q^3)$, each in terms of one of the others, arising from (1.1.11). These proofs are published here for the first time and are taken from Sohn's doctoral thesis [253].

1.2 Two-Variable Generalizations of (1.1.10) and (1.1.11)

On page 207 in his lost notebook [228], Ramanujan listed three identities,

$$P - Q = 1 + \frac{f(-q^{1/5}, -\lambda q^{2/5})}{q^{1/5} f(-\lambda^{10}q^5, -\lambda^{15}q^{10})}, \tag{1.2.1}$$

$$PQ = 1 - \frac{f(-\lambda, -\lambda^4 q^3) f(-\lambda^2 q, -\lambda^3 q^2)}{f^2(-\lambda^{10}q^5, -\lambda^{15}q^{10})}, \tag{1.2.2}$$

and

$$P^5 - Q^5 = 1 + 5PQ + 5P^2Q^2 + \frac{f(-q, -\lambda^5 q^2) f^5(-\lambda^2 q, -\lambda^3 q^2)}{q\, f^6(-\lambda^{10}q^5, -\lambda^{15}q^{10})}, \tag{1.2.3}$$

without specifying the functions P and Q. In this section, the functions P and Q are determined, and the identities, which are remarkable generalizations of (1.1.10) and (1.1.11), are proved.

We shall need several lemmas.

Lemma 1.2.1. *We have*

$$f(-1, a) = 0 \tag{1.2.4}$$

and, if n is an integer,

$$f(a, b) = a^{n(n+1)/2} b^{n(n-1)/2} f\left(a(ab)^n, b(ab)^{-n}\right). \tag{1.2.5}$$

For proofs of these elementary properties, see [61, p. 34, Entry 18].

Lemma 1.2.2 (Jacobi's Triple Product Identity). *If $f(a,b)$ is defined by* (1.1.5)*, then*

$$f(a, b) = (-a; ab)_\infty (-b; ab)_\infty (ab; ab)_\infty.$$

For a proof, see [61, p. 35, Entry 19].

Corollary 1.2.1.

$$f(-q, -q^4) f(-q^2, -q^3) = f(-q) f(-q^5).$$

This follows immediately from Lemma 1.2.2 and (1.1.8). See also [61, p. 44, Corollary].

Lemma 1.2.3. *Let* $U_n = a^{n(n+1)/2}b^{n(n-1)/2}$ *and* $V_n = a^{n(n-1)/2}b^{n(n+1)/2}$. *Then*

$$f(U_1, V_1) = \sum_{r=0}^{n-1} U_r f\left(\frac{U_{n+r}}{U_r}, \frac{V_{n-r}}{U_r}\right).$$

For a proof of Lemma 1.2.3, see [61, p. 48, Entry 31].

The next entry is Ramanujan's version of the quintuple product identity, and it is found on page 207 of his lost notebook, the same page as the identities for P and Q given above. Although Ramanujan undoubtedly used the quintuple product many times in proving results offered in his notebooks, this is the only instance where he recorded the quintuple product identity. For a proof along the lines that Ramanujan might have used and for references to other proofs, see [61, pp. 80–83].

Entry 1.2.1 (Quintuple Product Identity; p. 207). *For* $|\lambda x^3| < 1$,

$$f(-\lambda^2 x^3, -\lambda x^6) + x f(-\lambda, -\lambda^2 x^9) = \frac{f(-x^2, -\lambda x) f(-\lambda x^3)}{f(-x, -\lambda x^2)}. \tag{1.2.6}$$

To prove (1.2.3), we need instances of the following general product formula, which is due to Son [254]. Special cases of this lemma can be found in Ramanujan's notebooks [227]; see Berndt's books [61, pp. 264, 307, 346, 348], [62, pp. 142, 145, 188, 192].

Lemma 1.2.4. *Let* $|ab| < 1$, *let* p *be an odd prime, let* j *and* k *be integers with* $(j,k) \not\equiv (0,0) \pmod{p}$, *let* $\zeta := \exp(2\pi i/p)$, *and let* $x = s$, $0 \le x < p$, *be the solution of*

$$(j+k)x + j \equiv 0 \pmod{p}$$

when p *does not divide* $j+k$. *Then*

$$\prod_{n=1}^{p} f(\zeta^{jn} a, \zeta^{kn} b) \tag{1.2.7}$$
$$= \begin{cases} \dfrac{f^p(a^{s+1}b^s, a^{p-s-1}b^{p-s}) f(a^p, b^p)}{f(a^{p(s+1)}b^{ps}, a^{p(p-s-1)}b^{p(p-s)})}, & \textit{if} \quad j+k \not\equiv 0 \pmod{p}, \\ f^p(-ab) \dfrac{f(a^p, b^p)}{f(-a^p b^p)}, & \textit{if} \quad j+k \equiv 0 \pmod{p}. \end{cases}$$

Proof. Let

$$C := \prod_{n=1}^{p} f(-\zeta^{jn} a, -\zeta^{kn} b).$$

By the Jacobi triple product identity, Lemma 1.2.2,

$$C = \prod_{n=1}^{p} (\zeta^{jn}a; \zeta^{(j+k)n}ab)_\infty (\zeta^{kn}b; \zeta^{(j+k)n}ab)_\infty (\zeta^{(j+k)n}ab; \zeta^{(j+k)n}ab)_\infty$$
$$= C_1C_2C_3, \tag{1.2.8}$$

where

$$C_1 := \prod_{\ell=1}^{p} (\zeta^{j\ell}a; \zeta^{(j+k)\ell}ab)_\infty,$$
$$C_2 := \prod_{\ell=1}^{p} (\zeta^{k\ell}b; \zeta^{(j+k)\ell}ab)_\infty,$$

and

$$C_3 := \prod_{\ell=1}^{p} (\zeta^{(j+k)\ell}ab; \zeta^{(j+k)\ell}ab)_\infty.$$

First suppose that $j+k \not\equiv 0 \pmod p$. Then

$$\begin{aligned}
C_1 &= \prod_{\substack{n=0\\ n\equiv s \,(\mathrm{mod}\, p)}}^{\infty} \big(1-a(ab)^n\big)^p \prod_{\substack{n=0\\ n\not\equiv s \,(\mathrm{mod}\, p)}}^{\infty} \big(1-a^p(ab)^{pn}\big)\\
&= \prod_{n=0}^{\infty}\big(1-a(ab)^{pn+s}\big)^p \prod_{n=0}^{\infty}\big(1-a^p(ab)^{pn}\big) \Big/ \prod_{\substack{n=0\\ n\equiv s \,(\mathrm{mod}\, p)}}^{\infty} \big(1-a^p(ab)^{pn}\big)\\
&= (a^{s+1}b^s; a^pb^p)_\infty^p \frac{(a^p; a^pb^p)_\infty}{(a^{p(s+1)}b^{ps}; a^{p^2}b^{p^2})_\infty}.
\end{aligned}$$

Similarly, since $p-s-1$ is a solution of $(j+k)x+k \equiv 0 \pmod p$,

$$C_2 = (a^{p-s-1}b^{p-s}; a^pb^p)_\infty^p \frac{(b^p; a^pb^p)_\infty}{(a^{p(p-s-1)}b^{p(p-s)}; a^{p^2}b^{p^2})_\infty},$$

and since $p-1$ is a solution of $(j+k)x+(j+k) \equiv 0 \pmod p$,

$$C_3 = (a^pb^p; a^pb^p)_\infty^p \frac{(a^pb^p; a^pb^p)_\infty}{(a^{p^2}b^{p^2}; a^{p^2}b^{p^2})_\infty}.$$

Hence, by (1.2.8) and the Jacobi triple product identity, Lemma 1.2.2,

$$\begin{aligned}
C &= C_1C_2C_3\\
&= \big\{(a^{s+1}b^s; a^pb^p)_\infty (a^{p-s-1}b^{p-s}; a^pb^p)_\infty (a^pb^p; a^pb^p)_\infty\big\}^p\\
&\quad \times \frac{(a^p; a^pb^p)_\infty (b^p; a^pb^p)_\infty (a^pb^p; a^pb^p)_\infty}{(a^{p(s+1)}b^{ps}; a^{p^2}b^{p^2})_\infty (a^{p(p-s-1)}b^{p(p-s)}; a^{p^2}b^{p^2})_\infty (a^{p^2}b^{p^2}; a^{p^2}b^{p^2})_\infty}\\
&= f^p(-a^{s+1}b^s, -a^{p-s-1}b^{p-s}) \frac{f(-a^p,-b^p)}{f(-a^{p(s+1)}b^{ps}, -a^{p(p-s-1)}b^{p(p-s)})},
\end{aligned}$$

which, after $-a$ and $-b$ are replaced by a and b, respectively, establishes Lemma 1.2.4 in the case that $j+k \not\equiv 0 \pmod p$.

Second, if $j+k \equiv 0 \pmod p$,

$$C_1 = \prod_{n=0}^{\infty}\left(1 - a^p(ab)^{pn}\right) = (a^p; a^p b^p)_\infty.$$

Similarly,

$$C_2 = (b^p; a^p b^p)_\infty,$$

and, by (1.1.8),

$$C_3 = (ab; ab)_\infty^p = f^p(-ab).$$

Hence, by (1.2.8) and the Jacobi triple product identity, Lemma 1.2.2, we deduce that

$$C = C_1C_2C_3 = f^p(-ab)(a^p; a^p b^p)_\infty (b^p; a^p b^p)_\infty = f^p(-ab)\frac{f(-a^p, -b^p)}{f(-a^p b^p)},$$

and so the proof is complete after $(-a, -b)$ is replaced by (a, b). $\square$

We are now ready to give Son's proofs [254] of the mysterious identities on page 207 of the lost notebook [228].

Entry 1.2.2 (p. 207). *If*

$$P = \frac{f(-\lambda^{10}q^7, -\lambda^{15}q^8) + \lambda q f(-\lambda^5 q^2, -\lambda^{20}q^{13})}{q^{1/5} f(-\lambda^{10}q^5, -\lambda^{15}q^{10})} \tag{1.2.9}$$

and

$$Q = \frac{\lambda f(-\lambda^5 q^4, -\lambda^{20}q^{11}) - \lambda^3 q f(-q, -\lambda^{25}q^{14})}{q^{-1/5} f(-\lambda^{10}q^5, -\lambda^{15}q^{10})}, \tag{1.2.10}$$

then (1.2.1)*,* (1.2.2)*, and* (1.2.3) *hold.*

Proof. In Lemma 1.2.3, let $a = -q^{1/5}$, $b = -\lambda q^{2/5}$, and $n = 5$, and then employ Lemma 1.2.1 to obtain (1.2.1).

By (1.2.9) and (1.2.10), the identity (1.2.2) is equivalent to the identity,

$$\begin{aligned} S :&= f(-\lambda, -\lambda^4 q^3) f(-\lambda^2 q, -\lambda^3 q^2) \\ &= f(-\lambda^{10}q^5, -\lambda^{15}q^{10}) f(-\lambda^{10}q^5, -\lambda^{15}q^{10}) \\ &\quad - \lambda f(-\lambda^5 q^4, -\lambda^{20}q^{11}) f(-\lambda^{10}q^7, -\lambda^{15}q^8) \\ &\quad - \lambda^2 q f(-\lambda^5 q^4, -\lambda^{20}q^{11}) f(-\lambda^5 q^2, -\lambda^{20}q^{13}) \\ &\quad + \lambda^3 q f(-q, -\lambda^{25}q^{14}) f(-\lambda^{10}q^7, -\lambda^{15}q^8) \\ &\quad + \lambda^4 q^2 f(-q, -\lambda^{25}q^{14}) f(-\lambda^5 q^2, -\lambda^{20}q^{13}). \end{aligned} \tag{1.2.11}$$

Then

$$S=\sum_{u=-\infty}^{\infty}\sum_{v=-\infty}^{\infty}h(u,v),$$

where

$$h(u,v):=(-1)^{u+v}\lambda^{(5u^2+5v^2-u-3v)/2}q^{(3u^2+3v^2-u-3v)/2}.$$

We now subdivide this sum into five sums according to

$$2u+v\equiv k\,(\mathrm{mod}\,5),\qquad 0\le k\le 4.$$

Then

$$5u=2(2u+v)+(u-2v)\equiv 0(\mathrm{mod}\ 5),$$

which implies that $u-2v\equiv -2k\,(\mathrm{mod}\,5)$. Write

$$S=S_0+S_1+S_2+S_3+S_4, \tag{1.2.12}$$

where S_k denotes the sum for $2u+v\equiv k\,(\mathrm{mod}\,5)$, $0\le k\le 4$. Let $2u+v=5m$ and $u-2v=-5n$. Then $u=2m-n$, $v=m+2n$, and

$$\begin{aligned}h(u,v)&=h(2m-n,m+2n)\\&=(-1)^{(3m+n)}\lambda^{5(5m^2+5n^2-m-n)/2}q^{5(3m^2+3n^2-m-n)/2}.\end{aligned}$$

Therefore,

$$\begin{aligned}S_0&=\sum_{\substack{u,v\\2u+v\equiv 0\,(\mathrm{mod}\,5)}}h(u,v)\\&=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}h(2m-n,m+2n)\\&=\sum_{m=-\infty}^{\infty}\sum_{n=-\infty}^{\infty}(-1)^{(3m+n)}\lambda^{5(5m^2+5n^2-m-n)/2}q^{5(3m^2+3n^2-m-n)/2}\\&=\sum_{m=-\infty}^{\infty}(-1)^m(\lambda^{25}q^{15})^{m^2/2}(\lambda^{-5}q^{-5})^{m/2}\\&\quad\times\sum_{n=-\infty}^{\infty}(-1)^n(\lambda^{25}q^{15})^{n^2/2}(\lambda^{-5}q^{-5})^{n/2}\\&=f(-\lambda^{10}q^5,-\lambda^{15}q^{10})f(-\lambda^{10}q^5,-\lambda^{15}q^{10}).\end{aligned} \tag{1.2.13}$$

Similarly,

$$S_1=-\lambda f(-\lambda^5q^4,-\lambda^{20}q^{11})f(-\lambda^{10}q^7,-\lambda^{15}q^8), \tag{1.2.14}$$

$$S_2=-\lambda^2qf(-\lambda^5q^4,-\lambda^{20}q^{11})f(-\lambda^5q^2,-\lambda^{20}q^{13}), \tag{1.2.15}$$

$$S_3=\lambda^3qf(-q,-\lambda^{25}q^{14})f(-\lambda^{10}q^7,-\lambda^{15}q^8), \tag{1.2.16}$$

and

$$S_4 = \lambda^4 q^2 f(-q, -\lambda^{25} q^{14}) f(-\lambda^5 q^2, -\lambda^{20} q^{13}). \tag{1.2.17}$$

Substituting (1.2.13)–(1.2.17) in (1.2.12) and then using (1.2.11), we complete the proof of (1.2.2).

In (1.2.1), replace $q^{1/5}$ by $\zeta^n q^{1/5}$, where ζ is a primitive fifth root of unity and $n = 1, 2, 3, 4, 5$, and then multiply the five identities. Thus, we find that

$$\prod_{n=1}^{5} \left(\frac{P}{\zeta^n} - \zeta^n Q - 1 \right) = \frac{1}{q f^5(-\lambda^{10} q^5, -\lambda^{15} q^{10})} \prod_{n=1}^{5} f(-\zeta^n q^{1/5}, -\zeta^{2n} \lambda q^{2/5}). \tag{1.2.18}$$

Simplifying the left side of (1.2.18) yields

$$P^5 - Q^5 - 1 - 5PQ - 5P^2 Q^2. \tag{1.2.19}$$

Now in Lemma 1.2.4, let $a = -q^{1/5}$, $b = -\lambda q^{2/5}$, $p = 5$, $j = 1$, and $k = 2$. Then $s = 3$ is a solution of $3x + 1 \equiv 0 \pmod 5$, and so

$$\prod_{n=1}^{5} f(-\zeta^n q^{1/5}, -\zeta^{2n} \lambda q^{2/5}) = \frac{f(-q, -\lambda^5 q^2) f^5(-\lambda^2 q, -\lambda^3 q^2)}{f(-\lambda^{10} q^5, -\lambda^{15} q^{10})}. \tag{1.2.20}$$

Using (1.2.19) and (1.2.20) in (1.2.18), we finish the proof of (1.2.3). □

Now we shall show that (1.1.10) and (1.1.11) are special cases of (1.2.1) and (1.2.3).

Proof of (1.1.10) *and* (1.1.11). Let $\lambda = 1$ in (1.2.1) and (1.2.3). Then by applying the quintuple product identity, Entry 1.2.1, with $(x, \lambda) = (q, q^2)$ and (q^2, q^{-1}), respectively, we see that by Lemma 1.2.1, Lemma 1.2.2, and (1.1.2),

$$P = \frac{f(-q^7, -q^8) + q f(-q^2, -q^{13})}{q^{1/5} f(-q^5)} = \frac{f(-q^2, -q^3)}{q^{1/5} f(-q, -q^4)} = \frac{1}{R(q)} \tag{1.2.21}$$

and

$$Q = \frac{f(-q^4, -q^{11}) - q f(-q, -q^{14})}{q^{-1/5} f(-q^5)} = \frac{q^{1/5} f(-q, -q^4)}{f(-q^2, -q^3)} = R(q). \tag{1.2.22}$$

Since $PQ = 1$, (1.2.1) and (1.2.3) reduce to (1.1.10) and (1.1.11), respectively. □

1.3 Hybrids of (1.1.10) and (1.1.11)

Entry 1.3.1 (p. 48). *If* $f(-q)$ *is defined by* (1.1.8), *then*

$$\sum_{n=-\infty}^{\infty}(-1)^n(10n+3)q^{(5n+3)n/2}=\left(\frac{3}{R^2(q)}+R^3(q)\right)q^{2/5}f^3(-q^5) \qquad (1.3.1)$$

and

$$\sum_{n=-\infty}^{\infty}(-1)^n(10n+1)q^{(5n+1)n/2}=\left(\frac{1}{R^3(q)}-3R^2(q)\right)q^{3/5}f^3(-q^5). \qquad (1.3.2)$$

Proof. The key to our proofs is Jacobi's identity [61, p. 39, Entry 24(ii)],

$$f^3(-q)=\sum_{n=-\infty}^{\infty}(-1)^n nq^{n(n+1)/2}. \qquad (1.3.3)$$

By (1.1.10),

$$\left(\frac{1}{R(q)}-1-R(q)\right)^3=\frac{f^3(-q^{1/5})}{q^{3/5}f^3(-q^5)},$$

from which it follows that

$$q^{3/5}f^3(-q^5)\left\{5-\left(\frac{3}{R^2(q)}+R^3(q)\right)+\left(\frac{1}{R^3(q)}-3R^2(q)\right)\right\}=f^3(-q^{1/5}). \qquad (1.3.4)$$

If we expand the left side of (1.3.4) as a power series in q, we find that the exponents of q in

$$5q^{3/5}f^3(-q^5) \qquad (1.3.5)$$

are congruent to $\frac{3}{5}\pmod 1$, the exponents in

$$-q^{3/5}f^3(-q^5)\left(\frac{3}{R^2(q)}+R^3(q)\right) \qquad (1.3.6)$$

are congruent to $\frac{1}{5}\pmod 1$, and the exponents in

$$q^{3/5}f^3(-q^5)\left(\frac{1}{R^3(q)}-3R^2(q)\right) \qquad (1.3.7)$$

are integers.

By Jacobi's identity (1.3.3),

$$\begin{aligned}f^3(-q^{1/5})&=\sum_{n=-\infty}^{\infty}(-1)^n nq^{n(n+1)/10} \qquad (1.3.8)\\&=\sum_{n=-\infty}^{\infty}(-1)^{5n}(5n)q^{5n(5n+1)/10}\\&\quad+\sum_{n=-\infty}^{\infty}(-1)^{5n+1}(5n+1)q^{(5n+1)(5n+2)/10}\end{aligned}$$

$$+ \sum_{n=-\infty}^{\infty} (-1)^{5n+2}(5n+2)q^{(5n+2)(5n+3)/10}$$
$$+ \sum_{k=-\infty}^{\infty} (-1)^{5k+3}(5k+3)q^{(5k+3)(5k+4)/10}$$
$$+ \sum_{k=-\infty}^{\infty} (-1)^{5k+4}(5k+4)q^{(5k+4)(5k+5)/10}.$$

Letting $k = -n-1$, we obtain

$$\sum_{k=-\infty}^{\infty} (-1)^{5k+3}(5k+3)q^{(5k+3)(5k+4)/10} = -\sum_{n=-\infty}^{\infty} (-1)^{n}(5n+2)q^{(5n+2)(5n+1)/10} \tag{1.3.9}$$

and

$$\sum_{k=-\infty}^{\infty} (-1)^{5k+4}(5k+4)q^{(5k+4)(5k+5)/10} = \sum_{n=-\infty}^{\infty} (-1)^{n}(5n+1)q^{(5n+1)(5n)/10}. \tag{1.3.10}$$

Therefore, substituting (1.3.9) and (1.3.10) in (1.3.8), we find that

$$\begin{aligned} f^3(-q^{1/5}) &= \sum_{n=-\infty}^{\infty} (-1)^n \Big(5n + (5n+1)\Big) q^{n(5n+1)/2} \qquad (1.3.11)\\ &\quad - q^{1/5} \sum_{n=-\infty}^{\infty} (-1)^n \Big((5n+1) + (5n+2)\Big) q^{(5n+3)n/2} \\ &\quad + q^{3/5} \left(5 \sum_{n=-\infty}^{\infty} (-1)^n n (q^5)^{(n+1)n/2} + 2 \sum_{n=-\infty}^{\infty} (-1)^n (q^5)^{(n+1)n/2} \right). \end{aligned}$$

Since by (1.2.4),

$$\sum_{n=-\infty}^{\infty} (-1)^n (q^5)^{(n+1)n/2} = 0$$

and by (1.3.3),

$$\sum_{n=-\infty}^{\infty} (-1)^n n (q^5)^{n(n+1)/2} = f^3(-q^5),$$

we find that by (1.3.11),

$$
\begin{aligned}
f^3(-q^{1/5}) = &\sum_{n=-\infty}^{\infty} (-1)^n (10n+1) q^{n(5n+1)/2} \\
&- q^{1/5} \sum_{n=-\infty}^{\infty} (-1)^n (10n+3) q^{(5n+3)n/2} \\
&+ 5q^{3/5} f^3(-q^5).
\end{aligned} \tag{1.3.12}
$$

The powers of q in the first sum on the right side of (1.3.12) are integers, the powers of q in the second expression are congruent to $\frac{1}{5} \pmod 1$, and the powers of q in the last expression on the right side of (1.3.12) are congruent to $\frac{3}{5} \pmod 1$. Therefore, from our observations about the powers of q in (1.3.5)–(1.3.7) and our observations about the powers of q in (1.3.12), we conclude that

$$
-q^{3/5} f^3(-q^5) \left(\frac{3}{R^2(q)} + R^3(q) \right) = -q^{1/5} \sum_{n=-\infty}^{\infty} (-1)^n (10n+3) q^{(5n+3)n/2}
$$

and

$$
q^{3/5} f^3(-q^5) \left(\frac{1}{R^3(q)} - 3R^2(q) \right) = \sum_{n=-\infty}^{\infty} (-1)^n (10n+1) q^{n(5n+1)/2}.
$$

The identities (1.3.1) and (1.3.2) now follow, respectively, from the last two equalities. ☐

1.4 Factorizations of (1.1.10) and (1.1.11)

It had been thought that Ramanathan [215] published the first proof of the factorization theorems below. However, possibly due to an attempt to be brief, the argument for a key step is absent. This important step, an application of an addition theorem for theta functions due to Ramanujan and found in Ramanujan's notebooks [227], is perhaps the most difficult part of the proof.

Throughout this section, we set

$$
\alpha = \frac{1-\sqrt{5}}{2} \qquad \text{and} \qquad \beta = \frac{1+\sqrt{5}}{2}.
$$

Entry 1.4.1 (p. 206). *If $t = R(q)$, then*

$$
\frac{1}{\sqrt{t}} - \alpha\sqrt{t} = \frac{1}{q^{1/10}} \sqrt{\frac{f(-q)}{f(-q^5)}} \prod_{n=1}^{\infty} \frac{1}{1 + \alpha q^{n/5} + q^{2n/5}}, \tag{1.4.1}
$$

$$
\frac{1}{\sqrt{t}} - \beta\sqrt{t} = \frac{1}{q^{1/10}} \sqrt{\frac{f(-q)}{f(-q^5)}} \prod_{n=1}^{\infty} \frac{1}{1 + \beta q^{n/5} + q^{2n/5}}, \tag{1.4.2}
$$

$$\left(\frac{1}{\sqrt{t}}\right)^5 - \left(\alpha\sqrt{t}\right)^5 = \frac{1}{q^{1/2}}\sqrt{\frac{f(-q)}{f(-q^5)}}\prod_{n=1}^{\infty}\frac{1}{(1+\alpha q^n+q^{2n})^5}, \tag{1.4.3}$$

$$\left(\frac{1}{\sqrt{t}}\right)^5 - \left(\beta\sqrt{t}\right)^5 = \frac{1}{q^{1/2}}\sqrt{\frac{f(-q)}{f(-q^5)}}\prod_{n=1}^{\infty}\frac{1}{(1+\beta q^n+q^{2n})^5}. \tag{1.4.4}$$

It is not difficult to verify that by multiplying (1.4.1) by (1.4.2) we obtain (1.1.10), and by multiplying (1.4.3) by (1.4.4) we obtain (1.1.11). Therefore, (1.4.1) and (1.4.3) are equivalent to (1.4.2) and (1.4.4), respectively, and so it suffices to establish (1.4.1) and (1.4.3).

Lemma 1.4.1. *If* $\zeta = e^{2\pi i/5}$*, then*

$$f(-q^2,-q^3) - \alpha q^{1/5} f(-q,-q^4) = f(-\zeta^2,-\zeta^3 q^{1/5})/(1-\zeta^2) \tag{1.4.5}$$

and

$$f(-q^2,-q^3) - \beta q^{1/5} f(-q,-q^4) = f(-\zeta,-\zeta^4 q^{1/5})/(1-\zeta). \tag{1.4.6}$$

Proof. By Lemma 1.2.3 with $n=5$, $a=-\zeta^2$, and $b=-\zeta^3 q^{1/5}$,

$$\begin{aligned} f(-\zeta^2,-\zeta^3 q^{1/5}) &= f(-q^2,-q^3) - \zeta^2 f(-q^3,-q^2) + \zeta^4 q^{1/5} f(-q^4,-q) \\ &\quad - \zeta q^{3/5} f(-q^5,-1) + \zeta^3 q^{6/5} f(-q^6,-q^{-1}) \\ &= (1-\zeta^2) f(-q^2,-q^3) - (\zeta^3-\zeta^4) q^{1/5} f(-q,-q^4), \end{aligned}$$

since $f(-q^5,-1)=0$ and $f(-q^6,-q^{-1}) = -q^{-1} f(-q,-q^4)$ by Lemma 1.2.1, with $a=-q^{-1}$, $b=-q^6$, and $n=1$ in (1.2.5). Finally, (1.4.5) follows easily by noting that $\alpha = -(\zeta+\zeta^{-1})$; and so $\zeta^3-\zeta^4 = \alpha(1-\zeta^2)$.

By Lemma 1.2.3 with $n=5$, $a=-\zeta$, and $b=-\zeta^4 q^{1/5}$, and the observations made above,

$$\begin{aligned} f(-\zeta,-\zeta^4 q^{1/5}) &= f(-q^2,-q^3) - \zeta f(-q^3,-q^2) + \zeta^2 q^{1/5} f(-q^4,-q) \\ &\quad - \zeta^3 q^{3/5} f(-q^5,-1) + \zeta^4 q^{6/5} f(-q^6,-q^{-1}) \\ &= (1-\zeta)\big(f(-q^2,-q^3) - \beta q^{1/5} f(-q,-q^4)\big), \end{aligned}$$

since $\zeta^2+\zeta^3 = -\beta$. This proves (1.4.6). □

Lemma 1.4.2. *Let* n *be a positive integer not divisible by* 5*, and set* $\zeta = e^{2\pi i/5}$*. Then*

$$\prod_{j=0}^{4}(1+\alpha\zeta^{nj}q^{n/5}+\zeta^{2nj}q^{2n/5}) = (1-q^n)^2.$$

Proof. First, recall that $\alpha = -(\zeta+\zeta^{-1})$. Then,

$$\prod_{j=0}^{4}(1+\alpha\zeta^{nj}q^{n/5}+\zeta^{2nj}q^{2n/5}) = \prod_{j=0}^{4}\left(1-(\zeta+\zeta^{-1})\zeta^{nj}q^{n/5}+\zeta^{2nj}q^{2n/5}\right)$$
$$= \left\{\prod_{j=0}^{4}(1-\zeta^{nj-1}q^{n/5})\right\}\left\{\prod_{j=0}^{4}(1-\zeta^{nj+1}q^{n/5})\right\}.$$

Since n is not divisible by 5, ζ^{nj} runs through all the fifth roots of unity when j runs through 0, 1, 2, 3, 4. Therefore, the last two products are both equal to

$$\prod_{j=0}^{4}(1-\zeta^{j}q^{n/5}) = 1-q^{n}.$$

This completes the proof. □

Proof of Entry 1.4.1. Let ζ denote $e^{2\pi i/5}$. By (1.1.2), (1.4.5), and Corollary 1.2.1,

$$\frac{1}{\sqrt{t}}-\alpha\sqrt{t} = \frac{f(-q^2,-q^3)-\alpha q^{1/5}f(-q,-q^4)}{q^{1/10}\sqrt{f(-q,-q^4)f(-q^2,-q^3)}}$$
$$= \frac{f(-\zeta^2,-\zeta^3q^{1/5})/(1-\zeta^2)}{q^{1/10}\sqrt{f(-q)f(-q^5)}}. \tag{1.4.7}$$

By Lemma 1.2.2 and (1.1.8),

$$f(-\zeta^2,-\zeta^3q^{1/5})/(1-\zeta^2) = (\zeta^2q^{1/5};q^{1/5})_\infty(\zeta^3q^{1/5};q^{1/5})_\infty(q^{1/5};q^{1/5})_\infty$$
$$= \frac{f(-q)}{(\zeta q^{1/5};q^{1/5})_\infty(\zeta^4q^{1/5};q^{1/5})_\infty}$$
$$= \frac{f(-q)}{\prod_{n=1}^{\infty}(1+\alpha q^{n/5}+q^{2n/5})}. \tag{1.4.8}$$

Substituting (1.4.8) in (1.4.7), we complete the proof of (1.4.1).

It remains to prove (1.4.3). This can be done by using (1.4.1). For each $j=0,1,2,3,4$, we obtain an identity by replacing $q^{1/5}$ with $\zeta^jq^{1/5}$ in (1.4.1). Note that t is then replaced by ζ^jt. Multiplying these five identities together, we deduce that

$$\prod_{j=0}^{4}\left\{\frac{1}{\sqrt{\zeta^jt}}-\alpha\sqrt{\zeta^jt}\right\}$$
$$= \prod_{j=0}^{4}\left\{\frac{1}{(\zeta^jq^{1/5})^{1/2}}\sqrt{\frac{f(-q)}{f(-q^5)}}\prod_{n=1}^{\infty}\frac{1}{1+\alpha(\zeta^jq^{1/5})^n+(\zeta^jq^{1/5})^{2n}}\right\},$$

which can be easily reduced to

$$\left(\frac{1}{\sqrt{t}}\right)^5 - \left(\alpha\sqrt{t}\right)^5 = \frac{1}{q^{1/2}}\sqrt{\frac{f^5(-q)}{f^5(-q^5)}}\prod_{j=0}^{4}\prod_{n=1}^{\infty}\frac{1}{1+\alpha(\zeta^j q^{1/5})^n+(\zeta^j q^{1/5})^{2n}}. \tag{1.4.9}$$

Furthermore, the double product in (1.4.9) equals

$$\begin{aligned}
&\left\{\prod_{j=0}^{4}\prod_{5|n}\frac{1}{1+\alpha(\zeta^j q^{1/5})^n+(\zeta^j q^{1/5})^{2n}}\right\}\\
&\quad\times\left\{\prod_{j=0}^{4}\prod_{5\nmid n}\frac{1}{1+\alpha(\zeta^j q^{1/5})^n+(\zeta^j q^{1/5})^{2n}}\right\}\\
&=\left\{\prod_{k=1}^{\infty}\frac{1}{(1+\alpha q^k+q^{2k})^5}\right\}\left\{\prod_{5\nmid n}\prod_{j=0}^{4}\frac{1}{1+\alpha(\zeta^j q^{1/5})^n+(\zeta^j q^{1/5})^{2n}}\right\}\\
&=\left\{\prod_{k=1}^{\infty}\frac{1}{(1+\alpha q^k+q^{2k})^5}\right\}\left\{\prod_{5\nmid n}\frac{1}{(1-q^n)^2}\right\}\\
&=\left\{\prod_{k=1}^{\infty}\frac{1}{(1+\alpha q^k+q^{2k})^5}\right\}\frac{f^2(-q^5)}{f^2(-q)},
\end{aligned}$$

where the penultimate equality follows from Lemma 1.4.2. Therefore, (1.4.9) becomes

$$\left(\frac{1}{\sqrt{t}}\right)^5 - \left(\alpha\sqrt{t}\right)^5 = \frac{1}{q^{1/2}}\sqrt{\frac{f(-q)}{f(-q^5)}}\prod_{k=1}^{\infty}\frac{1}{(1+\alpha q^k+q^{2k})^5}.$$

This completes the proof of Entry 1.4.1. □

Alternatively, Entry 1.4.1 can be proved without the help of (1.1.10) and (1.1.11). Indeed, by using (1.4.6) instead of (1.4.5), we can prove (1.4.2) and then (1.4.4) in a similar manner. By doing so, we discover a new proof for the two remarkable identities (1.1.10) and (1.1.11).

1.5 Modular Equations

Recall that $R(q)$ is defined in (1.1.1). Following Ramanujan, set

$$u = R(q), \qquad u' = -R(-q), \qquad v = R(q^2), \qquad \text{and} \qquad w = R(q^4).$$

Entry 1.5.1 (p. 205). *We have*

$$uw = \frac{w-u^2v}{w+v^2} \tag{1.5.1}$$

and

$$uu'v^2 = \frac{uu' - v}{u' - u}. \tag{1.5.2}$$

Proof. First recall that

$$uv^2 = \frac{v - u^2}{v + u^2}. \tag{1.5.3}$$

This modular equation is found in Ramanujan's notebooks [227, vol. 2, p. 326]; the first proof was given in [39, p. 31, Entry 24(i)] and later reproduced in [63, Chapter 32, Entry 1, p. 12]. It is also given in a fragment with the publication of his lost notebook [228, p. 365, Entry (10)(a)]. In this book, it can be found in Entry 3.2.10 of Chapter 3. Replacing q by q^2 in (1.5.3), we find that

$$vw^2 = \frac{w - v^2}{w + v^2}. \tag{1.5.4}$$

Rewriting (1.5.3) and (1.5.4) in the forms

$$uv^3 + u^3v^2 - v + u^2 = 0, \tag{1.5.5}$$

$$w^2v^3 + v^2 + w^3v - w = 0, \tag{1.5.6}$$

respectively, we eliminate the constant terms in this pair of cubic equations in v by multiplying (1.5.5) by w and (1.5.6) by u^2 and then adding the resulting equalities. Accordingly,

$$\begin{aligned} v&\left((uw + u^2w^2)v^2 + (u^3w + u^2)v + w(u^2w^2 - 1)\right) \\ &= v(1 + uw)\left(uwv^2 + u^2v + w(uw - 1)\right) = 0. \end{aligned}$$

Since for $0 < q < 1$, $v(1 + uw) \neq 0$, we conclude that

$$uwv^2 + u^2v + w(uw - 1) = 0. \tag{1.5.7}$$

A rearrangement of (1.5.7) yields (1.5.1).

Secondly, replace q by $-q$ in (1.5.3) to deduce that

$$-u'v^2 = \frac{v - u'^2}{v + u'^2}. \tag{1.5.8}$$

Rewriting (1.5.3) and (1.5.8) in the forms

$$v - u^2 = uv^2(v + u^2), \tag{1.5.9}$$

$$v - u'^2 = -u'v^2(v + u'^2), \tag{1.5.10}$$

respectively, we multiply (1.5.9) by u', multiply (1.5.10) by u, and add the resulting equations to eliminate the cubic term in v. Thus,

$$v(u + u') - uu'(u + u') = uu'v^2(u^2 - u'^2) = uu'v^2(u + u')(u - u').$$

Since $u + u' \neq 0$, for $0 < q < 1$,

$$v - uu' = uu'v^2(u - u'). \tag{1.5.11}$$

We now immediately deduce (1.5.2) from (1.5.11). $\square$

1.6 Theta-Function Identities of Degree 5

The results in the next four sections were first proved by Kang [171].

Entry 1.6.1 (p. 56). *With* $\varphi(q)$, $\psi(q)$, *and* $f(-q)$ *defined in* (1.1.6), (1.1.7), *and* (1.1.8), *respectively,*

$$\text{(i)} \qquad \frac{f^3(-q)}{f^3(-q^5)} = \frac{\psi(q)}{\psi(q^5)} \times \frac{\psi^2(q) - 5q\psi^2(q^5)}{\psi^2(q) - q\psi^2(q^5)},$$

$$\text{(ii)} \qquad \frac{f^6(-q^2)}{f^6(-q^{10})} = \frac{\psi^4(q)}{\psi^4(q^5)} \times \frac{\psi^2(q) - 5q\psi^2(q^5)}{\psi^2(q) - q\psi^2(q^5)},$$

$$\text{(iii)} \qquad \frac{f^3(-q^2)}{qf^3(-q^{10})} = \frac{\varphi(q)}{\varphi(q^5)} \times \frac{5\varphi^2(q^5) - \varphi^2(q)}{\varphi^2(q) - \varphi^2(q^5)},$$

$$\text{(iv)} \qquad \frac{f^6(-q)}{qf^6(-q^5)} = \frac{\varphi^4(-q)}{\varphi^4(-q^5)} \times \frac{5\varphi^2(-q^5) - \varphi^2(-q)}{\varphi^2(-q^5) - \varphi^2(-q)}.$$

Proof. Using the identities

$$f^3(-q) = \varphi^2(-q)\psi(q) \tag{1.6.1}$$

and

$$f^3(-q^2) = \varphi(-q)\psi^2(q) \tag{1.6.2}$$

in [61, p. 39, Entries 24(ii), (iv)], we find that (i) and (ii) reduce to

$$\frac{\varphi^2(-q)}{\varphi^2(-q^5)} = \frac{\psi^2(q) - 5q\psi^2(q^5)}{\psi^2(q) - q\psi^2(q^5)}, \tag{1.6.3}$$

and (iii) and (iv) reduce to

$$\frac{\psi^2(q)}{q\psi^2(q^5)} = \frac{\varphi^2(-q) - 5\varphi^2(-q^5)}{\varphi^2(-q) - \varphi^2(-q^5)}. \tag{1.6.4}$$

Hence (i)–(iv) are all equivalent identities, because (1.6.3) and (1.6.4) are simply rearrangements of each other. Let us prove (1.6.4).

Rearranging (1.6.4), we see that it suffices to prove that

$$\psi^2(q)\varphi^2(-q) - \psi^2(q)\varphi^2(-q^5) - q\varphi^2(-q)\psi^2(q^5) + 5q\psi^2(q^5)\varphi^2(-q^5) = 0. \tag{1.6.5}$$

By some further elementary theta-function identities in Ramanujan's second notebook [61, p. 262, Entry 10(iv), (v)], we find that

$$\varphi^2(q) - \varphi^2(q^5) = 4qf(q,q^9)f(q^3,q^7) \tag{1.6.6}$$

and

$$\psi^2(q) - q\psi^2(q^5) = f(q,q^4)f(q^2,q^3). \tag{1.6.7}$$

Using (1.6.7), (1.6.6), Jacobi's triple product identity (Lemma 1.2.2), (1.1.6), (1.1.7), and Euler's identity ,

$$(-q;q)_\infty = \frac{1}{(q;q^2)_\infty}, \tag{1.6.8}$$

on the left-hand side of (1.6.5), we deduce that

$$\begin{aligned}
&[\psi^2(q) - q\psi^2(q^5)][\varphi^2(-q) - \varphi^2(-q^5)] + 4q\psi^2(q^5)\varphi^2(-q^5) \\
&= (f(q,q^4)f(q^2,q^3))(-4qf(-q,-q^9)f(-q^3,-q^7)) + 4q\psi^2(q^5)\varphi^2(-q^5) \\
&= \left(\frac{(-q;q)_\infty}{(-q^5;q^5)_\infty}(q^5;q^5)_\infty^2\right)\left(-4q\frac{(q;q^2)_\infty}{(q^5;q^{10})_\infty}(q^{10};q^{10})_\infty^2\right) \\
&\quad + 4q\psi^2(q^5)\varphi^2(-q^5) \\
&= -4q(q^5;q^5)_\infty^2(q^{10};q^{10})_\infty^2 + 4q\left(\frac{(q^{10};q^{10})_\infty^2}{(q^5;q^{10})_\infty^2}\frac{(q^5;q^5)_\infty^2}{(-q^5;q^5)_\infty^2}\right) \\
&= -4q(q^5;q^5)_\infty^2(q^{10};q^{10})_\infty^2 + 4q(q^5;q^5)_\infty^2(q^{10};q^{10})_\infty^2 \\
&= 0,
\end{aligned}$$

which proves (1.6.5). □

In the following theorem, we state identities for $\varphi^2(q)-5\varphi^2(q^5)$ and $\psi^2(q)-5q\psi^2(q^5)$, analogous to (1.6.6) and (1.6.7), but which do not appear in the lost notebook. These identities will be needed in Section 1.7.

Theorem 1.6.1. *If* $\chi(q)$ *is defined by* (1.1.9)*, then*

(i) $$\varphi^2(q) - 5\varphi^2(q^5) = -4f^2(-q^2)\frac{\chi(q^5)}{\chi(q)},$$

(ii) $$\psi^2(q) - 5q\psi^2(q^5) = f^2(-q)\frac{\chi(-q)}{\chi(-q^5)}.$$

Proof of (i)*.* From (1.6.4),

$$\varphi^2(-q) - 5\varphi^2(-q^5) = (\varphi^2(-q) - \varphi^2(-q^5))\frac{\psi^2(q)}{q\psi^2(q^5)}.$$

Using (1.6.6) and Jacobi's triple product identity (Lemma 1.2.2) in the first and the second equalities below, respectively, we obtain, by (1.1.7), (1.1.8), and (1.1.9),

$$\begin{aligned}
\varphi^2(-q) - 5\varphi^2(-q^5) &= -4f(-q,-q^9)f(-q^3,-q^7)\frac{(q^2;q^2)_\infty^2(q^5;q^{10})_\infty^2}{(q;q^2)_\infty^2(q^{10};q^{10})_\infty^2} \\
&= -4\frac{(q;q^2)_\infty}{(q^5;q^{10})_\infty}\frac{(q^2;q^2)_\infty^2(q^5;q^{10})_\infty^2}{(q;q^2)_\infty^2}
\end{aligned}$$

$$= -4f^2(-q^2)\frac{(q^5;q^{10})_\infty}{(q;q^2)_\infty}$$
$$= -4f^2(-q^2)\frac{\chi(-q^5)}{\chi(-q)}.$$

The identity (i) now follows by replacing q by $-q$ above. □

Proof of (ii). The proof is similar to that for (i) but uses (1.6.3) and (1.6.7) instead of (1.6.4) and (1.6.6). □

Entry 1.6.2 (p. 50). *We have*

$$\text{(i)} \qquad 16qf^2(-q^2)f^2(-q^{10}) = (\varphi^2(q)-\varphi^2(q^5))(5\varphi^2(q^5)-\varphi^2(q))$$

and

$$\text{(ii)} \qquad f^2(-q)f^2(-q^5) = (\psi^2(q)-q\psi^2(q^5))(\psi^2(q)-5q\psi^2(q^5)).$$

Proof. These follow immediately from (1.6.6), (1.6.7), and Theorem 1.6.1 by using (1.1.8), (1.1.9), Lemma 1.2.2, and (1.6.8). □

1.7 Refinements of the Previous Identities

On the same page of the lost notebook as Entry 1.6.1, Ramanujan gives factorizations of (1.6.6) and (1.6.7), which we state in the following entry.

Entry 1.7.1 (p. 56). *Recalling that $R(q)$ is defined in* (1.1.1)*, we have*

$$\text{(i)} \qquad \varphi(q)+\varphi(q^5) = 2q^{4/5}f(q,q^9)R^{-1}(q^4),$$
$$\text{(ii)} \qquad \varphi(q)-\varphi(q^5) = 2q^{1/5}f(q^3,q^7)R(q^4),$$
$$\text{(iii)} \qquad \psi(q^2)+q\psi(q^{10}) = q^{1/5}f(q^2,q^8)R^{-1}(q),$$
$$\text{(iv)} \qquad \psi(q^2)-q\psi(q^{10}) = q^{-1/5}f(q^4,q^6)R(q),$$
$$\text{(v)} \qquad \psi(q^2)+q\psi(q^{10}) = \frac{f(-q^{10})}{(q;q^{10})_\infty(-q^3;q^{10})_\infty(-q^7;q^{10})_\infty(q^9;q^{10})_\infty}.$$

Proof. By (1.6.6), (i) and (ii) are equivalent, and so are (iii) and (iv) by (1.6.7). Also, the right hand side of (v) is a rearrangement of that of (iii) by (1.1.2) and Lemma 1.2.2.

Assume that (i) is true. Replacing q by $-q$ in (i) and subtracting the result from (i), we find that

$$(\varphi(q)-\varphi(-q)) + \left(\varphi(q^5)-\varphi(-q^5)\right) = 2q^{4/5}R^{-1}(q^4)\left(f(q,q^9)-f(-q,-q^9)\right).$$

With the use of [61, p. 40, Entry 25 (ii)]

$$\varphi(q) - \varphi(-q) = 4q\psi(q^8) \tag{1.7.1}$$

and the definition of $f(a,b)$ in (1.1.5), the equation above can be rewritten in the form

$$4q\psi(q^8) + 4q^5\psi(q^{40}) = 4q^{9/5}R^{-1}(q^4)\sum_{n=-\infty}^{\infty} q^{20n^2+12n}.$$

We now deduce (iii) from the equation above by dividing both sides by $4q$ and then replacing q by $q^{1/4}$. So it suffices to prove (i).

By (1.1.6) and Jacobi's triple product identity, Lemma 1.2.2,

$$\begin{aligned}
\varphi(-q) + \varphi(-q^5) &= \frac{(q;q)_\infty}{(-q;q)_\infty} + \frac{(q^5;q^5)_\infty}{(-q^5;q^5)_\infty} \\
&= \frac{(q;q)_\infty}{(-q;q)_\infty}\left\{1 + \frac{(-q;q)_\infty}{(q;q)_\infty}\frac{(q^5;q^5)_\infty}{(-q^5;q^5)_\infty}\right\} \\
&= \frac{(q;q)_\infty}{(-q;q)_\infty}\left\{1 + \frac{(-q;q^5)_\infty(-q^2;q^5)_\infty(-q^3;q^5)_\infty(-q^4;q^5)_\infty}{(q;q^5)_\infty(q^2;q^5)_\infty(q^3;q^5)_\infty(q^4;q^5)_\infty}\right\} \\
&= \frac{(q;q)_\infty}{(-q;q)_\infty}\left\{1 + \frac{f(q,q^4)f(q^2,q^3)}{f(-q,-q^4)f(-q^2,-q^3)}\right\} \\
&= \frac{(q;q)_\infty}{(-q;q)_\infty}\left\{\frac{f(-q,-q^4)f(-q^2,-q^3) + f(q,q^4)f(q^2,q^3)}{f(-q,-q^4)f(-q^2,-q^3)}\right\}.
\end{aligned}$$

Appealing to a further entry in Ramanujan's second notebook [61, p. 45, Entry 29(i)], we find that

$$\varphi(-q) + \varphi(-q^5) = 2\frac{(q;q)_\infty}{(-q;q)_\infty}\frac{f(q^3,q^7)f(q^4,q^6)}{f(-q,-q^4)f(-q^2,-q^3)}. \tag{1.7.2}$$

Using Jacobi's triple product identity, Lemma 1.2.2, and Euler's identity (1.6.8), we find that (1.7.2) takes the form

$$\varphi(-q) + \varphi(-q^5) = 2f(-q,-q^9)\frac{(q^8;q^{20})_\infty(q^{12};q^{20})_\infty}{(q^4;q^{20})_\infty(q^{16};q^{20})_\infty},$$

which is equivalent to (i) with q replaced by $-q$, by (1.1.2). $\square$

On page 56 in his lost notebook, Ramanujan factored the identities in Theorem 1.6.1, as he did in Entry 1.7.1 for (1.6.6) and (1.6.7). The factorizations are given below in Entry 1.7.2 with a misprint corrected.

Entry 1.7.2 (p. 56). *If* $\alpha = \dfrac{1-\sqrt{5}}{2}$ *and* $\beta = \dfrac{1+\sqrt{5}}{2}$, *then*

(i) $$\varphi(q)+\sqrt{5}\varphi(q^5)=\frac{(1+\sqrt{5})f(-q^2)}{\prod\limits_{n\ \text{odd}}\left(1+\alpha q^n+q^{2n}\right)\prod\limits_{n\ \text{even}}\left(1-\beta q^n+q^{2n}\right)},$$

(ii) $$\varphi(q)-\sqrt{5}\varphi(q^5)=\frac{(1-\sqrt{5})f(-q^2)}{\prod\limits_{n\ \text{even}}\left(1-\alpha q^n+q^{2n}\right)\prod\limits_{n\ \text{odd}}\left(1+\beta q^n+q^{2n}\right)},$$

(iii) $$\psi(q^2)+q\sqrt{5}\psi(q^{10})=\frac{f(-q^2)}{\prod\limits_{n\ \text{odd}}\left(1+\alpha q^n+q^{2n}\right)\prod\limits_{n\ \text{odd}}\left(1-\beta q^n+q^{2n}\right)},$$

(iv) $$\psi(q^2)-q\sqrt{5}\psi(q^{10})=\frac{f(-q^2)}{\prod\limits_{n\ \text{odd}}\left(1-\alpha q^n+q^{2n}\right)\prod\limits_{n\ \text{odd}}\left(1+\beta q^n+q^{2n}\right)}.$$

Proof of (i). Let $\zeta=\exp(2\pi i/5)$. Then $\zeta+\zeta^4=-\alpha$ and $\zeta^2+\zeta^3=-\beta$. Hence,

$$1-\alpha q^n+q^{2n}=(1+\zeta q^n)(1+\zeta^4q^n), \tag{1.7.3}$$
$$1-\beta q^n+q^{2n}=(1+\zeta^2 q^n)(1+\zeta^3q^n). \tag{1.7.4}$$

Since $\beta-\alpha=\sqrt{5}$, by (1.1.6),

$$\begin{aligned}
\varphi(-q)+\sqrt{5}\varphi(-q^5)&=\frac{(q;q)_\infty}{(-q;q)_\infty}+(\beta-\alpha)\frac{(q^5;q^5)_\infty}{(-q^5;q^5)_\infty}\\
&=\frac{(q;q)_\infty}{(-q;q)_\infty}\left(1+(\zeta+\zeta^4-\zeta^2-\zeta^3)\prod_{j=1}^{4}\frac{(\zeta^jq;q)_\infty}{(-\zeta^jq;q)_\infty}\right)\\
&=\frac{(q;q)_\infty}{(-q;q)_\infty}\left(1+\frac{\zeta(1-\zeta)(1-\zeta^2)\prod\limits_{j=1}^{4}(\zeta^jq;q)_\infty}{\prod\limits_{j=1}^{4}(-\zeta^jq;q)_\infty}\right)\\
&=\frac{(q;q)_\infty}{(-q;q)_\infty}\left(1+\frac{\zeta(\zeta;q)_\infty(\zeta^2;q)_\infty(\zeta^3q;q)_\infty(\zeta^4q;q)_\infty}{(-\zeta q;q)_\infty(-\zeta^2q;q)_\infty(-\zeta^3q;q)_\infty(-\zeta^4q;q)_\infty}\right).
\end{aligned}\tag{1.7.5}$$

Multiplied by $\dfrac{(1+\zeta)(1+\zeta^2)}{(1+\zeta)(1+\zeta^2)}$, the right side of (1.7.5) becomes

$$\begin{aligned}
&\frac{(q;q)_\infty}{(-q;q)_\infty}\left(1+\frac{\zeta(1+\zeta)(1+\zeta^2)(\zeta;q)_\infty(\zeta^2;q)_\infty(\zeta^3q;q)_\infty(\zeta^4q;q)_\infty}{(-\zeta;q)_\infty(-\zeta^2;q)_\infty(-\zeta^3q;q)_\infty(-\zeta^4q;q)_\infty}\right)\\
&=\frac{(q;q)_\infty}{(-q;q)_\infty}\frac{f(\zeta,\zeta^4q)f(\zeta^2,\zeta^3q)+(\zeta+\zeta^2+\zeta^3+\zeta^4)f(-\zeta,-\zeta^4q)f(-\zeta^2,-\zeta^3q)}{f(\zeta,\zeta^4q)f(\zeta^2,\zeta^3q)}\\
&=\frac{(q;q)_\infty}{(-q;q)_\infty}\frac{f(\zeta,\zeta^4q)f(\zeta^2,\zeta^3q)-f(-\zeta,-\zeta^4q)f(-\zeta^2,-\zeta^3q)}{f(\zeta,\zeta^4q)f(\zeta^2,\zeta^3q)},
\end{aligned}$$

by Jacobi's triple product identity, Lemma 1.2.2, and the fact $\alpha+\beta=1$. From [61, p. 45, Entry 29 (ii)] and Lemma 1.2.2, we see that

$$\begin{aligned}\varphi(-q)+\sqrt{5}\varphi(-q^5) &= \frac{(q;q)_\infty}{(-q;q)_\infty}\frac{2\zeta f(\zeta^2 q,\zeta^3 q)f(\zeta,\zeta^4 q^2)}{f(\zeta,\zeta^4 q)f(\zeta^2,\zeta^3 q)} \\ &= 2\zeta\frac{(q;q)_\infty(q^2;q^2)_\infty^2}{(-q;q)_\infty(q;q)_\infty^2}\frac{(-\zeta;q^2)_\infty(-\zeta^2 q;q^2)_\infty(-\zeta^3 q;q^2)_\infty(-\zeta^4 q^2;q^2)_\infty}{(-\zeta;q)_\infty(-\zeta^2;q)_\infty(-\zeta^3 q;q)_\infty(-\zeta^4 q;q)_\infty} \\ &= 2\frac{\zeta(q^2;q^2)_\infty}{(1+\zeta^2)}\frac{(-\zeta q^2;q^2)_\infty(-\zeta^2 q;q^2)_\infty(-\zeta^3 q;q^2)_\infty(-\zeta^4 q^2;q^2)_\infty}{(-\zeta q;q)_\infty(-\zeta^2 q;q)_\infty(-\zeta^3 q;q)_\infty(-\zeta^4 q;q)_\infty}.\end{aligned}$$

Since $\zeta/(1+\zeta^2)=(\zeta+\zeta^{-1})^{-1}=(\zeta+\zeta^4)^{-1}=-1/\alpha$, we find that

$$\begin{aligned}&\varphi(-q)+\sqrt{5}\varphi(-q^5) \\ &\quad= -\frac{2}{\alpha}\frac{(q^2;q^2)_\infty}{(-\zeta q;q^2)_\infty(-\zeta^2 q^2;q^2)_\infty(-\zeta^3 q^2;q^2)_\infty(-\zeta^4 q;q^2)_\infty} \\ &\quad= \frac{(1+\sqrt{5})f(-q^2)}{(-\zeta q;q^2)_\infty(-\zeta^4 q;q^2)_\infty(-\zeta^2 q^2;q^2)_\infty(-\zeta^3 q^2;q^2)_\infty} \\ &\quad= \frac{(1+\sqrt{5})f(-q^2)}{\prod\limits_{n\text{ odd}}(1-\alpha q^n+q^{2n})\prod\limits_{n\text{ even}}(1-\beta q^n+q^{2n})},\end{aligned}\tag{1.7.6}$$

by (1.7.3) and (1.7.4). We complete the proof of (i) by replacing q by $-q$ on both sides. □

Proof of (ii). By Euler's identity (1.6.8),

$$\frac{(-q^5;q^{10})_\infty}{(-q;q^2)_\infty}=\frac{(q;-q)_\infty}{(q^5;-q^5)_\infty}=\frac{1}{(\zeta q;-q)_\infty(\zeta^2 q;-q)_\infty(\zeta^3 q;-q)_\infty(\zeta^4 q;-q)_\infty}.\tag{1.7.7}$$

Using (1.7.6) with q replaced by $-q$, (1.7.7), (1.7.3), and (1.7.4), we deduce from Theorem 1.6.1(i) and (1.1.9) that

$$\begin{aligned}\varphi(q)-\sqrt{5}\varphi(q^5) &= -4f^2(-q^2)\frac{(-q^5;q^{10})_\infty}{(-q;q^2)_\infty}\left(\frac{1}{\varphi(q)+\sqrt{5}\varphi(q^5)}\right) \\ &= (1-\sqrt{5})f(-q^2)\frac{(\zeta q;q^2)_\infty(-\zeta^2 q^2;q^2)_\infty(-\zeta^3 q^2;q^2)_\infty(\zeta^4 q;q^2)_\infty}{(\zeta q;-q)_\infty(\zeta^2 q;-q)_\infty(\zeta^3 q;-q)_\infty(\zeta^4 q;-q)_\infty} \\ &= \frac{(1-\sqrt{5})f(-q^2)}{(-\zeta q^2;q^2)_\infty(\zeta^2 q;q^2)_\infty(\zeta^3 q;q^2)_\infty(-\zeta^4 q^2;q^2)_\infty} \\ &= \frac{(1-\sqrt{5})f(-q^2)}{\prod\limits_{n\text{ even}}(1-\alpha q^n+q^{2n})\prod\limits_{n\text{ odd}}(1+\beta q^n+q^{2n})}.\end{aligned}$$

Proof of (iii). Using (1.7.1) and subtracting (1.7.6) from (i) yields

$$\begin{aligned}
&4q\psi(q^8)+4\sqrt{5}q^5\psi(q^{40})=(\varphi(q)-\varphi(-q))+\sqrt{5}(\varphi(q^5)-\varphi(-q^5))\\
&\quad=\frac{(1+\sqrt{5})f(-q^2)}{(-\zeta^2q^2;q^2)_\infty(-\zeta^3q^2;q^2)_\infty}\\
&\quad\quad\times\left(\frac{1}{(\zeta q;q^2)_\infty(\zeta^4q;q^2)_\infty}-\frac{1}{(-\zeta q;q^2)_\infty(-\zeta^4q;q^2)_\infty}\right)\\
&\quad=\frac{2\beta(q^2;q^2)_\infty}{(-\zeta^2q^2;q^2)_\infty(-\zeta^3q^2;q^2)_\infty}\\
&\quad\quad\times\left(\frac{(-\zeta q;q^2)_\infty(-\zeta^4q;q^2)_\infty-(\zeta q;q^2)_\infty(\zeta^4q;q^2)_\infty}{(\zeta^2q^2;q^4)_\infty(\zeta^3q^2;q^4)_\infty}\right)\\
&\quad\quad\frac{2\beta}{(-\zeta^2q^2;q^2)_\infty(-\zeta^3q^2;q^2)_\infty}\left(\frac{f(\zeta q,\zeta^4q)-f(-\zeta q,-\zeta^4q)}{(\zeta^2q^2;q^4)_\infty(\zeta^3q^2;q^4)_\infty}\right),
\end{aligned}$$

by Jacobi's triple product identity, Lemma 1.2.2. Thus using Jacobi's triple product identity in the second equality below, we deduce from [61, p. 46, Entry 30 (iii)] that

$$\begin{aligned}
&4q\psi(q^8)+4\sqrt{5}q^5\psi(q^{40})\\
&=\frac{2\beta}{(-\zeta^2q^2;q^2)_\infty(-\zeta^3q^2;q^2)_\infty}\left(\frac{2\zeta qf(\zeta^3,\zeta^2q^8)}{(\zeta^2q^2;q^4)_\infty(\zeta^3q^2;q^4)_\infty}\right)\\
&=\frac{4\beta\zeta q(-\zeta^3;q^8)_\infty(-\zeta^2q^8;q^8)_\infty(q^8;q^8)_\infty}{(-\zeta^2q^2;q^2)_\infty(-\zeta^3q^2;q^2)_\infty(\zeta^2q^2;q^4)_\infty(\zeta^3q^2;q^4)_\infty}\\
&=\frac{4\beta\zeta q(1+\zeta^3)(-\zeta^3q^8;q^8)_\infty(-\zeta^2q^8;q^8)_\infty(q^8;q^8)_\infty}{(-\zeta^2q^2;q^4)_\infty(-\zeta^2q^4;q^4)_\infty(-\zeta^3q^2;q^4)_\infty(-\zeta^3q^4;q^4)_\infty(\zeta^2q^2;q^4)_\infty(\zeta^3q^2;q^4)_\infty}\\
&=\frac{4q(-\zeta^3q^8;q^8)_\infty(-\zeta^2q^8;q^8)_\infty(q^8;q^8)_\infty}{(\zeta q^4;q^8)_\infty(-\zeta^2q^4;q^4)_\infty(-\zeta^3q^4;q^4)_\infty(\zeta^4q^4;q^8)_\infty},
\end{aligned}$$

since $\zeta+\zeta^4=-\alpha$ and $\alpha\beta=-1$. Dividing both sides by $4q$, replacing q by $q^{1/4}$, and using (1.7.3) and (1.7.4), we deduce that

$$\begin{aligned}
\psi(q^2)+q\sqrt{5}\psi(q^{10})&=\frac{(-\zeta^3q^2;q^2)_\infty(-\zeta^2q^2;q^2)_\infty(q^2;q^2)_\infty}{(\zeta q;q^2)_\infty(-\zeta^2q;q)_\infty(-\zeta^3q;q)_\infty(\zeta^4q;q^2)_\infty}\\
&=\frac{(q^2;q^2)_\infty}{(\zeta q;q^2)_\infty(-\zeta^2q;q^2)_\infty(-\zeta^3q;q^2)_\infty(\zeta^4q;q^2)_\infty}\\
&=\frac{f(-q^2)}{\prod\limits_{n\ \text{odd}}(1+\alpha q^n+q^{2n})\prod\limits_{n\ \text{odd}}(1-\beta q^n+q^{2n})},
\end{aligned}$$

which proves (iii). ∎

Proof of (iv). The proof of (iv) is similar to that of (ii). Use Theorem 1.6.1(ii) with q replaced by q^2, (1.1.9), (iii), (1.7.3), and (1.7.4) to find that

$$\begin{aligned}\psi(q^2) - q\sqrt{5}\psi(q^{10}) &= f^2(-q^2)\frac{(q^2;q^4)_\infty}{(q^{10};q^{20})_\infty}\left(\frac{1}{\psi(q^2)+q\sqrt{5}\psi(q^{10})}\right)\\ &= f(-q^2)\frac{(\zeta q;q^2)_\infty(-\zeta^2 q;q^2)_\infty(-\zeta^3 q;q^2)_\infty(\zeta^4 q;q^2)_\infty}{(\zeta q^2;q^4)_\infty(\zeta^2 q^2;q^4)_\infty(\zeta^3 q^2;q^4)_\infty(\zeta^4 q^2;q^4)_\infty}\\ &= \frac{f(-q^2)}{(-\zeta q;q^2)_\infty(\zeta^2 q;q^2)_\infty(\zeta^3 q;q^2)_\infty(-\zeta^4 q;q^2)_\infty}\\ &= \frac{f(-q^2)}{\prod\limits_{n\text{ odd}}(1-\alpha q^n+q^{2n})\prod\limits_{n\text{ odd}}(1+\beta q^n+q^{2n})},\end{aligned}$$

as desired. □

1.8 Identities Involving the Parameter $k = R(q)R^2(q^2)$

Recall again that $R(q)$ denotes the Rogers–Ramanujan continued fraction. In his notebooks [227, p. 362], Ramanujan introduced the parameter

$$k := R(q)R^2(q^2)$$

and asserted that

$$R^5(q) = k\left(\frac{1-k}{1+k}\right)^2 \qquad \text{and} \qquad R^5(q^2) = k^2\left(\frac{1+k}{1-k}\right). \tag{1.8.1}$$

For proofs of (1.8.1), see [39, Entry 24] or [63, pp. 12–14, Entry 1(i)]. See also Entry 2.6.2 in Chapter 2. To prove the several identities involving k stated by Ramanujan in his lost notebook, we need the following relations between the Rogers–Ramanujan continued fraction and theta functions.

Entry 1.8.1 (p. 26). *Let* $\mu := \mu(q) := R(q)R(q^4)$ *and* $\nu := \nu(q) := R^2(q^{1/2})R(q)/R(q^2)$. *Then*

(i) $$\frac{\varphi(q)}{\varphi(q^5)} = \frac{1+\mu}{1-\mu},$$

(ii) $$\frac{\psi(q)}{\sqrt{q}\psi(q^5)} = \frac{1+\nu}{1-\nu}.$$

Only the first identity is in the lost notebook; the second is its analogue, but it is not found in Ramanujan's work.

Proof of (i). From Entry 1.7.1(i), (ii) and (1.1.2), we have

$$\begin{aligned} S := \frac{\varphi(q)-\varphi(q^5)}{\varphi(q)+\varphi(q^5)} &= \frac{2qf(q^3,q^7)}{2f(q,q^9)}\frac{(q^4;q^{20})_\infty^2(q^{16};q^{20})_\infty^2}{(q^8;q^{20})_\infty^2(q^{12};q^{20})_\infty^2} \\ &= q\frac{(-q^3;q^{10})_\infty(-q^7;q^{10})_\infty(q^{10};q^{10})_\infty(q^4;q^{20})_\infty^2(q^{16};q^{20})_\infty^2}{(-q;q^{10})_\infty(-q^9;q^{10})_\infty(q^{10};q^{10})_\infty(q^8;q^{20})_\infty^2(q^{12};q^{20})_\infty^2}, \end{aligned}$$

where we have used Jacobi's triple product identity, Lemma 1.2.2, in the last equality above. For convenience, define

$$(a_1,a_2,\dots,a_n;q)_\infty := \prod_{k=1}^{n}(a_k;q)_\infty.$$

Then, from above

$$\begin{aligned} S &= q\frac{(q^4,q^{16};q^{20})_\infty}{(q^8,q^{12};q^{20})_\infty}\frac{(-q^3,-q^7;q^{10})_\infty(-q^2,q^2;q^{10})_\infty(-q^8,q^8;q^{10})_\infty}{(-q,-q^9;q^{10})_\infty(-q^4,q^4;q^{10})_\infty(-q^6,q^6;q^{10})_\infty} \\ &= q\frac{(q^4,q^{16};q^{20})_\infty}{(q^8,q^{12};q^{20})_\infty}\frac{(-q^2;q^5)_\infty(-q^3;q^5)_\infty(q^2;q^{10})_\infty(q^8;q^{10})_\infty}{(-q;q^5)_\infty(-q^4;q^5)_\infty(q^4;q^{10})_\infty(q^6;q^{10})_\infty}. \end{aligned}$$

Multiplying both the numerator and denominator above by $(q;q)_\infty$, we find that

$$\begin{aligned} S &= q\frac{(q^4,q^{16};q^{20})_\infty}{(q^8,q^{12};q^{20})_\infty}\frac{(q,q^2,q^3,q^4;q^5)_\infty(-q^2,-q^3;q^5)_\infty(q^2,q^8;q^{10})_\infty}{(q,q^2,q^3,q^4;q^5)_\infty(-q,-q^4;q^5)_\infty(q^4,q^6;q^{10})_\infty} \\ &= q\frac{(q^4,q^{16};q^{20})_\infty}{(q^8,q^{12};q^{20})_\infty}\frac{(q;q^5)_\infty(q^4;q^5)_\infty(q^4;q^{10})_\infty(q^6;q^{10})_\infty(q^2;q^{10})_\infty(q^8;q^{10})_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty(q^2;q^{10})_\infty(q^8;q^{10})_\infty(q^4;q^{10})_\infty(q^6;q^{10})_\infty} \\ &= q\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}\frac{(q^4;q^{20})_\infty(q^{16};q^{20})_\infty}{(q^8;q^{20})_\infty(q^{12};q^{20})_\infty} \\ &= \frac{q^{1/5}(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}\frac{q^{4/5}(q^4;q^{20})_\infty(q^{16};q^{20})_\infty}{(q^8;q^{20})_\infty(q^{12};q^{20})_\infty} \\ &= R(q)R(q^4) \\ &= \mu. \end{aligned}$$

Using the last equality above and the definition of S, after some very elementary algebra, we easily deduce (i). ☐

Proof of (ii). Similarly, using Lemma 1.2.2 in the second equality below, we deduce from Entry 1.7.1(iii), (iv) and (1.1.2) that

$$\begin{aligned} \frac{\psi(q^2)-q\psi(q^{10})}{\psi(q^2)+q\psi(q^{10})} &= \frac{f(q^4,q^6)}{f(q^2,q^8)}\frac{(q;q^5)_\infty^2(q^4;q^5)_\infty^2}{(q^2;q^5)_\infty^2(q^3;q^5)_\infty^2} \\ &= \frac{(q;q^5)_\infty^2(q^4;q^5)_\infty^2}{(q^2;q^5)_\infty^2(q^3;q^5)_\infty^2}\frac{(-q^4;q^{10})_\infty(-q^6;q^{10})_\infty}{(-q^2;q^{10})_\infty(-q^8;q^{10})_\infty} \end{aligned}$$

$$
\begin{aligned}
&= \frac{(q;q^5)_\infty^2 (q^4;q^5)_\infty^2}{(q^2;q^5)_\infty^2 (q^3;q^5)_\infty^2} \frac{(q^2;q^{10})_\infty (q^8;q^{10})_\infty}{(q^4;q^{10})_\infty (q^6;q^{10})_\infty} \frac{(q^8;q^{20})_\infty (q^{12};q^{20})_\infty}{(q^4;q^{20})_\infty (q^{16};q^{20})_\infty} \\
&= \frac{R^2(q)R(q^2)}{R(q^4)} \\
&= \nu(q^2).
\end{aligned}
$$

This last equality is equivalent to (ii) with q replaced by q^2. □

Entry 1.8.2 (p. 56). *If* $k \le \sqrt{5} - 2$, *then*

(i)
$$\frac{\varphi^2(-q)}{\varphi^2(-q^5)} = \frac{1 - 4k - k^2}{1 - k^2},$$

(ii)
$$\frac{\psi^2(q)}{q\psi^2(q^5)} = \frac{1 + k - k^2}{k}.$$

Proof. The condition $k \le \sqrt{5} - 2$ is set in order to ensure that $1 - 4k - k^2 \ge 0$. By (1.8.1) and (1.1.11), we find that

$$
\begin{aligned}
\frac{f^6(-q)}{qf^6(-q^5)} &= \frac{1}{k}\left(\frac{1+k}{1-k}\right)^2 - 11 - k\left(\frac{1-k}{1+k}\right)^2 \\
&= \frac{(1 + k - k^2)(1 - 4k - k^2)^2}{k(1 - k^2)^2} \\
&= \left(\frac{1 + k - k^2}{k}\right)\left(\frac{1 - 4k - k^2}{1 - k^2}\right)^2.
\end{aligned}
$$

If we set $K = (1 + k - k^2)/k$, the last equality above can be written in the form

$$\frac{f^6(-q)}{qf^6(-q^5)} = K\left(\frac{K-5}{K-1}\right)^2. \tag{1.8.2}$$

On the other hand, by (1.6.1) and (1.6.3),

$$\frac{f^6(-q)}{qf^6(-q^5)} = \frac{\psi^2(q)}{q\psi^2(q^5)} \frac{\varphi^4(-q)}{\varphi^4(-q^5)} = \frac{\psi^2(q)}{q\psi^2(q^5)} \left(\frac{\psi^2(q) - 5q\psi^2(q^5)}{\psi^2(q) - q\psi^2(q^5)}\right)^2. \tag{1.8.3}$$

Let $\lambda = \psi^2(q)/(q\psi^2(q^5))$. Then (1.8.3) may be rewritten as

$$\frac{f^6(-q)}{qf^6(-q^5)} = \lambda\left(\frac{\lambda-5}{\lambda-1}\right)^2. \tag{1.8.4}$$

So, from (1.8.2) and (1.8.4), we conclude that $\lambda = K$, and so we have proved both (i) and (ii), because

$$\frac{\varphi^2(-q)}{\varphi^2(-q^5)} = \frac{\lambda-5}{\lambda-1} = \frac{K-5}{K-1} = \frac{1 - 4k - k^2}{1 - k^2}. \tag{1.8.5}$$

□

In the following two entries, with some minor errors of Ramanujan corrected, we express $R(q^{1/2})$ and $R(q^4)$ in terms of k.

Set $x = R(q^{1/2})$, $u = R(q)$, $v = R(q^2)$, and $w = R(q^4)$. Then $k = uv^2$.

Entry 1.8.3 (p. 56). *If* $k \leq \sqrt{5} - 2$, *then*

$$R(q^{1/2}) = \frac{k^{1/10}(1+k)^{4/5}(1-k)^{1/5}}{\sqrt{k} + \sqrt{1+k-k^2}}.$$

Proof. Entry 1.8.1(ii) and Entry 1.8.2(ii) imply that

$$\frac{1+k-k^2}{k} = \left(\frac{v+x^2u}{v-x^2u}\right)^2.$$

Solving this equality for x^2 using the quadratic formula, we obtain

$$x^2 = \frac{uv(1+2k-k^2) \pm 2uv\sqrt{k(1+k-k^2)}}{(1-k^2)u^2}.$$

Using (1.8.1), we deduce that

$$x^2 = k^{1/5}\frac{\{(1+k-k^2)+k\} \pm 2\sqrt{k(1+k-k^2)}}{(1+k)^{2/5}(1-k)^{8/5}}.$$

Thus,

$$x = k^{1/10}\frac{\sqrt{1+k-k^2} \pm \sqrt{k}}{(1+k)^{1/5}(1-k)^{4/5}}, \tag{1.8.6}$$

since $k < 1$. But

$$u = R(q) = \frac{q^{1/5}}{1} \, \frac{}{+} \, \frac{q}{1} \, \frac{}{+} \cdots \approx \frac{q^{1/5}}{1+q} \approx q^{1/5}(1-q), \tag{1.8.7}$$

for small values of q. Hence,

$$x = R(q^{1/2}) \approx q^{1/10}(1-\sqrt{q}) \tag{1.8.8}$$

and

$$k = uv^2 \approx q(1-q) \tag{1.8.9}$$

for small values of q. Thus, by (1.8.6) and (1.8.9), we find that

$$x = R(q^{1/2}) \approx k^{1/10}(1 \pm \sqrt{k}) \approx q^{1/10}(1-q)^{1/10}(1 \pm \sqrt{q}(1-q)^{1/2})$$

for small values of q. Therefore, we conclude, by (1.8.6) and (1.8.8) that

$$x = k^{1/10}\frac{\sqrt{1+k-k^2} - \sqrt{k}}{(1+k)^{1/5}(1-k)^{4/5}} = \frac{k^{1/10}(1+k)^{4/5}(1-k)^{1/5}}{\sqrt{1+k-k^2} + \sqrt{k}},$$

which completes the proof. □

Entry 1.8.4 (p. 56). *We have*

$$R(q^4) = \left(\frac{1-k}{1+k}\right)^{1/10} \frac{2k^{4/5}}{(\sqrt{1-k^2} + \sqrt{1-4k-k^2})}.$$

In the lost notebook, the factor $((1-k)/(1+k))^{1/10}$ is missing.

Proof. Recall from (1.5.1) that

$$\frac{w - u^2 v}{w + v^2} = uw. \tag{1.8.10}$$

Since $k = uv^2$, solving (1.8.10) for w yields

$$w = \frac{(1-k) \pm \sqrt{(1-k)^2 - 4u^3 v}}{2u}. \tag{1.8.11}$$

By (1.8.1), the equality (1.8.11) becomes

$$\begin{aligned} w = R(q^4) &= \frac{(1-k)\{1 \pm \sqrt{1 - 4k(1-k^2)^{-1}}\}}{2k^{1/5}(1-k)^{2/5}(1+k)^{-2/5}} \\ &= \frac{1 \pm \sqrt{1 - 4k(1-k^2)^{-1}}}{2k^{1/5}(1-k)^{-3/5}(1+k)^{-2/5}} \\ &= \frac{\sqrt{1-k^2} \pm \sqrt{(1-k^2) - 4k}}{2k^{1/5}(1-k)^{-1/10}(1+k)^{1/10}} \\ &= \frac{(1-k)^{1/10}}{(1+k)^{1/10}} \frac{2k^{4/5}}{\sqrt{1-k^2} \mp \sqrt{1-4k-k^2}}. \end{aligned} \tag{1.8.12}$$

Since both $R(q^4)$ and k approach 0 as q does, we have to take the positive sign in the denominator on the far right side of (1.8.12). This completes the proof. ∎

Entry 1.8.5 (p. 53). *If $k \leq \sqrt{5} - 2$, then*

(i) $$\frac{k}{1-k^2}\left(\frac{1+k-k^2}{1-4k-k^2}\right)^5 = q(-q;q)_\infty^{24},$$

(ii) $$\left(\frac{k}{1-k^2}\right)^5 \frac{1+k-k^2}{1-4k-k^2} = q^5(-q^5;q^5)_\infty^{24}.$$

Let $\Delta(\tau)$ denote the discriminant function defined by

$$\Delta(\tau) = q(q;q)_\infty^{24},$$

where $q = e^{2\pi i\tau}$ and $\operatorname{Im} \tau > 0$. Using the definition of Δ, we can easily see that the identities in Entry 1.8.5 are representations of certain quotients of Δ's in terms of k, namely,

$$\frac{k}{1-k^2}\left(\frac{1+k-k^2}{1-4k-k^2}\right)^5=\frac{\Delta(2\tau)}{\Delta(\tau)}$$

and

$$\left(\frac{k}{1-k^2}\right)^5\frac{1+k-k^2}{1-4k-k^2}=\frac{\Delta(10\tau)}{\Delta(5\tau)},$$

respectively, where $q=e^{2\pi i\tau}$ and $\operatorname{Im}\tau>0$.

Proof. Let $k_*=k(-q)$. Then Entry 1.8.2(i) implies that

$$\frac{\varphi^2(q)}{\varphi^2(q^5)}=1-\frac{4k_*}{1-k_*^2}.$$

Thus, from Ramanujan's notebooks [61, p. 288, Entry 14(ii)], when $\sqrt{\alpha}$ and $\sqrt{\beta}$ denote moduli in a modular equation of degree 5,

$$4\alpha(1-\alpha)=\frac{-2k_*}{1-k_*^2}\left(\frac{2+2k_*/(1-k_*^2)}{1-4k_*/(1-k_*^2)}\right)^5 \tag{1.8.13}$$

and

$$4\beta(1-\beta)=\left(\frac{-2k_*}{1-k_*^2}\right)^5\left(\frac{2+2k_*/(1-k_*^2)}{1-4k_*/(1-k_*^2)}\right). \tag{1.8.14}$$

Simplifying (1.8.13) and (1.8.14), we arrive at

$$-\frac{1}{16}\alpha(1-\alpha)=\frac{k_*}{1-k_*^2}\left(\frac{1+k_*-k_*^2}{1-4k_*-k_*^2}\right)^5 \tag{1.8.15}$$

and

$$-\frac{1}{16}\beta(1-\beta)=\left(\frac{k_*}{1-k_*^2}\right)^5\left(\frac{1+k_*-k_*^2}{1-4k_*-k_*^2}\right). \tag{1.8.16}$$

Using the equality

$$\psi^8(-q)=\frac{\varphi^8(q)}{16q}\alpha(1-\alpha),$$

easily derived from results in Ramanujan's notebooks [61, p. 122, Entry 10(i); p. 123, Entry 11(ii)], the fact that

$$\chi^3(q)=\frac{\varphi(q)}{\psi(-q)}, \tag{1.8.17}$$

which is also found in the notebooks [61, p. 39, Entry 24(iii)], (1.1.9), and Euler's identity (1.6.8), we find that

$$\frac{1}{16}\alpha(1-\alpha)=q\left(\frac{\psi(-q)}{\varphi(q)}\right)^8=q\chi(q)^{-24}=q(q;-q)_\infty^{24} \tag{1.8.18}$$

and, similarly,

$$\frac{1}{16}\beta(1-\beta) = q^5(q^5;-q^5)_\infty^{24}. \tag{1.8.19}$$

Therefore, combining (1.8.15) and (1.8.18) yields (i), and combining (1.8.16) and (1.8.19) creates (ii) with q replaced by $-q$ in both cases. □

In a similar way, we can derive an analogue to the entry above for $R(q)R(q^4)$. This result is not found in the lost notebook.

Theorem 1.8.1. *If* $\mu = R(q)R(q^4)$, *as in Entry* 1.8.1, *and* $q = e^{2\pi i\tau}$, *where* $\operatorname{Im}\tau > 0$, *then*

$$\frac{\mu}{(1-\mu)^2}\left(\frac{1-3\mu+\mu^2}{1+2\mu+\mu^2}\right)^5 = \frac{q}{(-q;q^2)_\infty^{24}} = -\frac{\Delta(2\tau)}{\Delta(\frac{1}{2}+\tau)},$$
$$\left(\frac{\mu}{(1-\mu)^2}\right)^5\frac{1-3\mu+\mu^2}{1+2\mu+\mu^2} = \frac{q^5}{(-q^5;q^{10})_\infty^{24}} = -\frac{\Delta(10\tau)}{\Delta(\frac{1}{2}+5\tau)}.$$

Proof. By Entry 1.8.1(i),

$$\frac{\varphi^2(q)}{\varphi^2(q^5)} = 1 + \frac{4\mu}{(1-\mu)^2}.$$

The remainder of the proof is similar to that for Entry 1.8.5. □

1.9 Other Representations of Theta Functions Involving $R(q)$

The identities in the following two entries are not related to the parameter k. However, we use properties of k proved in the previous section to prove these theorems. The identities in Entry 1.9.1 are modified and completed forms of Ramanujan's incomplete statements on page 209 in the lost notebook.

Entry 1.9.1 (p. 209). *We have*

$$\frac{\varphi(-q^{1/5})}{\varphi(-q^5)} = 1 + U_1 + V_1, \tag{1.9.1}$$

where for $\lambda_1 = \varphi(-q)/\varphi(-q^5)$,

$$U_1 = \frac{\lambda_1 - 1}{R(q^4)} \tag{1.9.2}$$

$$= \frac{\lambda_1^2 + 1 - \sqrt{\lambda_1^4 - 2\lambda_1^2 + 5}}{2}\frac{1}{R^2(q^2)} \tag{1.9.3}$$

$$= \frac{\lambda_1^2 - 3 - \sqrt{\lambda_1^4 - 2\lambda_1^2 + 5}}{2}R(q) \tag{1.9.4}$$

and

$$V_1 = (\lambda_1 + 1)R(q^4) \tag{1.9.5}$$
$$= \frac{\lambda_1^2 + 1 + \sqrt{\lambda_1^4 - 2\lambda_1^2 + 5}}{2} R^2(q^2) \tag{1.9.6}$$
$$= \frac{-(\lambda_1^2 - 3) - \sqrt{\lambda_1^4 - 2\lambda_1^2 + 5}}{2} \frac{1}{R(q)}. \tag{1.9.7}$$

Moreover, $U_1 V_1 = \lambda_1^2 - 1$ *and*

$$U_1^5 + V_1^5 = (\lambda_1^2 - 1)^3 - 2\lambda_1^2(\lambda_1^2 - 1) + 10(\lambda_1^2 - 1). \tag{1.9.8}$$

Proof. We begin with a result from Ramanujan's notebooks [61, p. 265, Entry 11(i)], namely,

$$\varphi(-q^{1/5}) = \varphi(-q^5) + \mu^{1/5} + \nu^{1/5}, \tag{1.9.9}$$

where $\mu^{1/5} = -2q^{1/5} f(-q^3, -q^7)$ and $\nu^{1/5} = 2q^{4/5} f(-q, -q^9)$. Dividing both sides of (1.9.9) by $\varphi(-q^5)$ yields

$$\frac{\varphi(-q^{1/5})}{\varphi(-q^5)} = 1 + \frac{-2q^{1/5} f(-q^3, -q^7)}{\varphi(-q^5)} + \frac{2q^{4/5} f(-q, -q^9)}{\varphi(-q^5)}.$$

Hence, there exist functions U_1 and V_1 such that

$$\frac{\varphi(-q^{1/5})}{\varphi(-q^5)} = 1 + U_1 + V_1,$$

and moreover, we have shown that

$$U_1 = \frac{\mu^{1/5}}{\varphi(-q^5)} = \frac{-2q^{1/5} f(-q^3, -q^7)}{\varphi(-q^5)} \tag{1.9.10}$$

and

$$V_1 = \frac{\nu^{1/5}}{\varphi(-q^5)} = \frac{2q^{4/5} f(-q, -q^9)}{\varphi(-q^5)}. \tag{1.9.11}$$

Utilizing Entry 1.7.1(ii) in (1.9.10), we find that

$$U_1 = R^{-1}(q^4) \frac{\varphi(-q) - \varphi(-q^5)}{\varphi(-q^5)} = \frac{\lambda_1 - 1}{R(q^4)},$$

and this completes the proof of (1.9.2).

Next, Entry 1.8.2(i) implies that

$$\lambda_1^2 = \frac{1 - 4k - k^2}{1 - k^2}, \tag{1.9.12}$$

and solving (1.9.12) for k, we deduce that

$$k = \frac{2 - \sqrt{\lambda_1^4 - 2\lambda_1^2 + 5}}{\lambda_1^2 - 1}. \tag{1.9.13}$$

To resolve the sign ambiguity in deducing (1.9.13), we used the facts that k and λ_1 approach 0 and 1, respectively, as q approaches 0. Now applying Entry 1.8.4 and (1.9.12) to (1.9.2), we find that

$$\begin{aligned}
U_1 &= (\lambda_1 - 1)\frac{(1+k)^{1/10}\left(\sqrt{1-k^2} + \sqrt{1-4k-k^2}\right)}{2k^{4/5}(1-k)^{1/10}} \\
&= (\lambda_1^2 - 1)\frac{(1+k)^{1/10}\sqrt{1-k^2}}{2k^{4/5}(1-k)^{1/10}} \\
&= (\lambda_1^2 - 1)\frac{(1+k)^{3/5}(1-k)^{2/5}}{2k^{4/5}} \\
&= \frac{(\lambda_1^2 - 1)(1+k)}{2}\left(k^2\left(\frac{1+k}{1-k}\right)\right)^{-2/5} \\
&= \frac{(\lambda_1^2 - 1)(1+k)}{2R^2(q^2)},
\end{aligned} \tag{1.9.14}$$

where the last equality above is deduced from (1.8.1). Using (1.9.12), we now easily deduce (1.9.3) from (1.9.14). Furthermore, we find that

$$U_1 = R(q)k^{-1}\frac{\lambda_1^2 + 1 - \sqrt{\lambda_1^4 - 2\lambda_1^2 + 5}}{2}, \tag{1.9.15}$$

by (1.9.3) and the definition of k. Hence, by (1.9.13), (1.9.4) immediately follows from (1.9.15).

For the formulas for V_1, first apply Entry 1.7.1(i) to (1.9.11). Then we find that

$$V_1 = R(q^4)\frac{\varphi(-q) + \varphi(-q^5)}{\varphi(-q^5)} = (\lambda_1 + 1)R(q^4),$$

which proves (1.9.5). Therefore, upon multiplying (1.9.2) and (1.9.5), we find that

$$U_1V_1 = \lambda_1^2 - 1. \tag{1.9.16}$$

Dividing (1.9.3) and (1.9.4) by $\lambda_1^2 - 1$ and using (1.9.16), we obtain (1.9.6) and (1.9.7), respectively. So it remains to prove (1.9.8). By (1.9.10), (1.9.11), and a formula for $\mu + \nu$ found in Ramanujan's notebooks [61, p. 265, Entry 11(i)], namely,

$$\mu + \nu = \frac{\varphi^2(q) - \varphi^2(q^5)}{\varphi(q^5)}\left\{\varphi^4(q) - 4\varphi^2(q)\varphi^2(q^5) + 11\varphi^4(q^5)\right\},$$

we find that

$$\begin{aligned}
U_1^5+V_1^5 &= \frac{\mu+\nu}{\varphi^5(-q^5)} \\
&= \frac{\varphi^2(-q)-\varphi^2(-q^5)}{\varphi^6(-q^5)}(\varphi^4(-q)-4\varphi^2(-q)\varphi^2(-q^5)+11\varphi^4(-q^5)) \\
&= \left\{\frac{\varphi^2(-q)-\varphi^2(-q^5)}{\varphi^2(-q^5)}\right\}\left\{\frac{\varphi^4(-q)-4\varphi^2(-q)\varphi^2(-q^5)+11\varphi^4(-q^5)}{\varphi^4(-q^5)}\right\} \\
&= (\lambda_1^2-1)(\lambda_1^4-4\lambda_1^2+11) \\
&= (\lambda_1^2-1)((\lambda_1^2-1)^2-2\lambda_1^2+10) \\
&= (\lambda_1^2-1)^3-2\lambda_1^2(\lambda_1^2-1)+10(\lambda_1^2-1).
\end{aligned}$$

□

Entry 1.9.2 (p. 209). *We have*

$$\frac{\psi(q^{1/5})}{q^{3/5}\psi(q^5)} = 1+U_2+V_2, \tag{1.9.17}$$

where for $\lambda_2 = \psi(q)/\sqrt{q}\psi(q^5)$,

$$U_2 = \frac{\lambda_2-1}{R(q^{1/2})} \tag{1.9.18}$$

$$= \frac{\lambda_2^2+1+\sqrt{\lambda_2^4-2\lambda_2^2+5}}{2}R^2(q) \tag{1.9.19}$$

$$= \frac{\lambda_2^2-3+\sqrt{\lambda_2^4-2\lambda_2^2+5}}{2}R(q^2) \tag{1.9.20}$$

and

$$V_2 = (\lambda_2+1)R(q^{1/2}) \tag{1.9.21}$$

$$= \frac{\lambda_2^2+1-\sqrt{\lambda_2^4-2\lambda_2^2+5}}{2}\frac{1}{R^2(q)} \tag{1.9.22}$$

$$= \frac{-(\lambda_2^2-3)+\sqrt{\lambda_2^4-2\lambda_2^2+5}}{2}\frac{1}{R(q^2)}. \tag{1.9.23}$$

Moreover, $U_2V_2 = \lambda_2^2-1$ *and*

$$U_2^5+V_2^5 = (\lambda_2^2-1)^3-2\lambda_2^2(\lambda_2^2-1)+10(\lambda_2^2-1). \tag{1.9.24}$$

Proof. We only sketch the proofs of the identities in this entry, since they are similar to the proofs of the identities of Entry 1.9.1.

Use the identity [61, p. 265, Entry 11(ii)]

$$q^{1/40}\psi(q^{1/5}) = q^{5/8}\psi(q^5) + \mu^{1/5} + \nu^{1/5}, \tag{1.9.25}$$

where $\mu^{1/5} = q^{1/40}f(q^2, q^3)$ and $\nu^{1/5} = q^{9/40}f(q, q^4)$, to prove that

$$\frac{\psi(q^{1/5})}{q^{3/5}\psi(q^5)} = 1 + U_2 + V_2,$$

where

$$U_2 = \frac{f(q^2, q^3)}{q^{3/5}\psi(q^5)} \qquad \text{and} \qquad V_2 = \frac{f(q, q^4)}{q^{2/5}\psi(q^5)}. \tag{1.9.26}$$

To obtain (1.9.18) and (1.9.21), we apply Entries 1.7.1(iv), (iii), with q replaced by $q^{1/2}$, to the equalities of (1.9.26), respectively.

Next,

$$\lambda_2^2 = \frac{1 + k - k^2}{k}, \tag{1.9.27}$$

which follows from Entry 1.8.2(ii). Solving (1.9.27) for k, we find that

$$k = \frac{(1 - \lambda_2^2) + \sqrt{\lambda_2^4 - 2\lambda_2^2 + 5}}{2}, \tag{1.9.28}$$

since $k > 0$ for $q > 0$. Utilize Entry 1.8.3, (1.9.27), (1.8.1), and (1.9.28), in the given order, to obtain (1.9.19) from (1.9.18). From (1.8.1), we easily deduce that

$$R^2(q) = R(q^2)\frac{1 - k}{1 + k}. \tag{1.9.29}$$

Equality (1.9.20) is deduced from (1.9.19) by applying (1.9.29) and (1.9.28). Upon multiplying (1.9.18) and (1.9.21), we see that $U_2V_2 = \lambda_2^2 - 1$. Therefore, (1.9.22) and (1.9.23) follow from (1.9.19) and (1.9.20), respectively. Lastly, (1.9.24) is another form of an identity from Ramanujan's notebooks [61, p. 265, Entry 11(ii)]. □

Entry 1.9.3 (p. 53). *If $u = R(q)$ and $v = R(q^2)$, then*

$$\frac{\psi(q^{1/5})}{q^{3/5}\psi(q^5)} = \frac{1 - uv^2}{uv} + \frac{1 + uv^2}{v} + 1.$$

Proof. We find from (1.9.20) and (1.9.23) in Entry 1.9.2 that

$$\frac{\psi(q^{1/5})}{q^{3/5}\psi(q^5)} = 1 + U_2 + V_2. \tag{1.9.30}$$

By (1.9.28),

$$\frac{1 - k}{k} = \frac{\lambda_2^2 - 3 + \sqrt{\lambda_2^4 - 2\lambda_2^2 + 5}}{2} \tag{1.9.31}$$

and

$$1+k=\frac{-(\lambda_2^2-3)+\sqrt{\lambda_2^4-2\lambda_2^2+5}}{2}. \tag{1.9.32}$$

Since $k=uv^2$, from (1.9.20) and (1.9.31) we find that

$$U_2=\frac{1-k}{k}v=\frac{1-uv^2}{uv},$$

and also, by (1.9.23) and (1.9.32), we find that

$$V_2=\frac{1+k}{v}=\frac{1+uv^2}{v}.$$

Using the last two equalities in (1.9.30), we complete the proof. □

1.10 Explicit Formulas Arising from (1.1.11)

In this last section of the chapter, we prove two formulas for $R(q)$, one in terms of $R(q^2)$, the other in terms of $R(q^3)$, found on page 205 in the lost notebook. These formulas were stated without proof in [78] and proved for the first time by Sohn [253].

Entry 1.10.1 (p. 205). *Let* $\omega=\exp(2\pi i/3)$, $u=R(q)$, *and* $v=R(q^2)$. *If*

$$R:=\frac{f^3(-q)}{\sqrt{q}f^3(-q^5)}=\sqrt{\frac{1}{u^5}-11-u^5}, \tag{1.10.1}$$

then

$$-3v=u^2+\omega\left(u^6+18u+3iu\sqrt{3}\ R\right)^{1/3}+\omega^2\left(u^6+18u-3iu\sqrt{3}\ R\right)^{1/3}. \tag{1.10.2}$$

If

$$R:=\frac{f^3(-q^2)}{qf^3(-q^{10})}=\sqrt{\frac{1}{v^5}-11-v^5}, \tag{1.10.3}$$

then

$$\begin{aligned}-3u=\frac{1}{v^2}&+\omega\left(\frac{1}{v^6}-\frac{18}{v}+\frac{3(\omega-\omega^2)}{v}R\right)^{1/3}\\&+\omega^2\left(\frac{1}{v^6}-\frac{18}{v}-\frac{3(\omega-\omega^2)}{v}R\right)^{1/3}.\end{aligned} \tag{1.10.4}$$

Entry 1.10.2 (p. 205). *Let* $\omega = \exp(2\pi i/3)$, $u = R(q)$, *and* $v = R(q^3)$. *If*

$$R := \frac{f^2(-q^3)}{qf^2(-q^{15})} = \left(\frac{1}{v^5} - 11 - v^5\right)^{1/3}, \tag{1.10.5}$$

then

$$4u = -\frac{1}{v^3} - \sqrt{\frac{1}{v^6} - \frac{8+4R}{v}} + \sqrt{\frac{1}{v^6} - \frac{8+4R\omega}{v}} + \sqrt{\frac{1}{v^6} - \frac{8+4R\omega^2}{v}}. \tag{1.10.6}$$

If

$$R := \frac{f^2(-q)}{q^{1/3}f^2(-q^5)} = \left(\frac{1}{u^5} - 11 - u^5\right)^{1/3}, \tag{1.10.7}$$

then

$$4v = u^3 - \sqrt{u^6 + u(8+4R)} + \sqrt{u^6 + u(8+4R\omega)} + \sqrt{u^6 + u(8+4R\omega^2)}. \tag{1.10.8}$$

Lemma 1.10.1. *If* R *is defined by* (1.10.5), *then*

$$\begin{aligned}
&\sqrt{\frac{1}{v^6} - \frac{8+4R\omega}{v}} \pm \sqrt{\frac{1}{v^6} - \frac{8+4R\omega^2}{v}} \\
&= \sqrt{\frac{2}{v^6} - \frac{16}{v} + \frac{4R}{v} \pm 2\sqrt{\left(\frac{1}{v^6} - \frac{8}{v} - \frac{4R\omega}{v}\right)\left(\frac{1}{v^6} - \frac{8}{v} - \frac{4R\omega^2}{v}\right)}} \qquad (1.10.9)\\
&= \sqrt{\frac{2}{v^6} - \frac{16}{v} + \frac{4R}{v} \pm 2\sqrt{\left(\frac{1}{v^6} - \frac{8}{v}\right)^2 + \left(\frac{1}{v^6} - \frac{8}{v}\right)\left(\frac{4R}{v}\right) + \frac{16R^2}{v^2}}} \qquad (1.10.10)\\
&= \sqrt{\frac{2}{v^6} - \frac{16}{v} + \frac{4R}{v} \pm \left(\frac{2}{v^3}\sqrt{\frac{1}{v^6} - \frac{4R}{v} - \frac{8}{v}} - \frac{8(1-R+2v^5)}{v\sqrt{1-8v^5-4Rv^5}}\right)}. \qquad (1.10.11)
\end{aligned}$$

Proof. To prove the first equality, we use the property $\sqrt{a} \pm \sqrt{b} = \sqrt{a+b\pm 2\sqrt{ab}}$. Next, (1.10.10) is just a rewritten form of (1.10.9). Thirdly, we can verify (after a long and tedious calculation) that

$$\begin{aligned}
&\left(\frac{2}{v^3}\sqrt{\frac{1}{v^6} - \frac{4R}{v} - \frac{8}{v}} - \frac{8(1-R+2v^5)}{v\sqrt{1-8v^5-4Rv^5}}\right)^2 \\
&\qquad = \left(2\sqrt{\left(\frac{1}{v^6} - \frac{8}{v}\right)^2 + \left(\frac{1}{v^6} - \frac{8}{v}\right)\left(\frac{4R}{v}\right) + \frac{16R^2}{v^2}}\right)^2.
\end{aligned}$$

In a neighborhood of the origin, the dominant term on each side is $4/v^{12}$, and so upon taking the square root of each side, we find that

$$\frac{2}{v^3}\sqrt{\frac{1}{v^6}-\frac{4R}{v}-\frac{8}{v}}-\frac{8(1-R+2v^5)}{v\sqrt{1-8v^5-4Rv^5}}$$
$$=2\sqrt{\left(\frac{1}{v^6}-\frac{8}{v}\right)^2+\left(\frac{1}{v^6}-\frac{8}{v}\right)\left(\frac{4R}{v}\right)+\frac{16R^2}{v^2}},$$

which proves (1.10.11). □

Lemma 1.10.2. *Let a, b, c, and d be any real numbers, and let ω denote a primitive cube root of unity. If $\sqrt{a+b\omega}=c+di$, then $\sqrt{a+b\omega^2}=c-di$.*

Proof. Let $\omega=\exp(2\pi i/3)$. Then

$$\sqrt{a+b\omega}=\sqrt{a-\frac{b}{2}+\frac{b\sqrt{3}i}{2}}=c+di.$$

So

$$a-\frac{b}{2}+\frac{b\sqrt{3}}{2}i=(c+di)^2=c^2-d^2+2cdi.$$

Hence $a-\dfrac{b}{2}=c^2-d^2$ and $\dfrac{b\sqrt{3}}{2}=2cd$. Now

$$c-di=\sqrt{c^2-d^2-2cdi}=\sqrt{a-\frac{b}{2}-\frac{b\sqrt{3}}{2}i}=\sqrt{a+b\omega^2},$$

which proves the second equality. The proof is similar if $\omega=\exp(4\pi i/3)$. □

Lemma 1.10.3. *Let a, b, c, and d be any real numbers and assume that we consider only principal arguments. If $\sqrt[3]{a+bi}=c+di$, then $\sqrt[3]{a-bi}=c-di$.*

Proof. Let $\sqrt[3]{a+bi}=c+di$. Then

$$a+bi=(c+di)^3=c^3-3cd^2+\left(3c^2d-d^3\right)i.$$

Hence $a=c^3-3cd^2$ and $b=3c^2d-d^3$. Therefore

$$a-bi=c^3-3cd^2-\left(3c^2d-d^3\right)i=(c-di)^3.$$

Since we consider only the principal argument, $\sqrt[3]{a-bi}=c-di$, which proves the lemma. □

Proof of Entry 1.10.2. To prove (1.10.6) and (1.10.8), we use Ramanujan's modular equation relating $u=R(q)$ and $v=R(q^3)$, namely,

$$(v-u^3)(1+uv^3)=3u^2v^2, \tag{1.10.12}$$

which is found on page 321 in Ramanujan's second notebook [227]; see [39, p. 27, Entry 20] and [63, p. 17, Entry 3]. It is also on page 365 in the publication of his lost notebook [228]; see Entry 3.2.11 in Chapter 3 of this book. The only two proofs in the literature are due to Rogers [236] and Yi [299]. Observe that (1.10.12) is quartic in each of u and v. We thus use Ferrari's method [277, pp. 94–96] to solve for each of u and v. From (1.10.12),

$$v^3u^4 + u^3 + 3v^2u^2 - v^4u - v = 0, \tag{1.10.13}$$

$$uv^4 - u^4v^3 - 3u^2v^2 + v - u^3 = 0. \tag{1.10.14}$$

Considering (1.10.13) as a quartic equation in u, we rewrite it in the form

$$u^4 + \frac{1}{v^3}u^3 + \frac{3}{v}u^2 - vu - \frac{1}{v^2} = 0. \tag{1.10.15}$$

First, we briefly explain Ferrari's method. To solve the quartic equation

$$x^4 + px^3 + qx^2 + rx + s = 0, \tag{1.10.16}$$

we first determine a, b, and k such that

$$x^4 + px^3 + qx^2 + rx + s + (ax+b)^2 = \left(x^2 + \frac{p}{2}x + k\right)^2. \tag{1.10.17}$$

The determination of a, b, and k is accomplished by equating the coefficients of like powers of x in the first and second members of (1.10.17). This leads to the relations

$$\begin{cases} a^2 + q &= 2k + \dfrac{p^2}{4}, \\ 2ab + r &= kp, \\ b^2 + s &= k^2. \end{cases} \tag{1.10.18}$$

Hence,

$$(kp - r)^2 = 4a^2b^2 = 4\left(2k + \frac{p^2}{4} - q\right)(k^2 - s),$$

or

$$k^3 - \frac{q}{2}k^2 + \frac{1}{4}(pr - 4s)k + \frac{1}{8}(4qs - p^2s - r^2) = 0. \tag{1.10.19}$$

We find k by solving the equation (1.10.19), and then determining a and b by substituting this value k in (1.10.18). Note that it is not necessary to find all the roots of (1.10.19), since any one will suffice. Now upon adding $(ax+b)^2$ to both sides of (1.10.16), an equation is obtained in which both members are perfect squares. More precisely,

$$\left(x^2 + \frac{p}{2}x + k\right)^2 = (ax+b)^2.$$

Therefore,

$$x^2 + \frac{p}{2}x + k = ax + b \qquad \text{or} \qquad x^2 + \frac{p}{2}x + k = -ax - b, \tag{1.10.20}$$

and the four roots of (1.10.16) can be found by solving the quadratic equations (1.10.20). From (1.10.20),

$$x^2 + \left(\frac{p}{2} - a\right)x + k - b = 0 \qquad \text{or} \qquad x^2 + \left(\frac{p}{2} + a\right)x + k + b = 0.$$

The solutions are, respectively,

$$\frac{-p + 2a \pm \sqrt{p^2 - 4pa + 4a^2 - 16k + 16b}}{4}$$

and

$$\frac{-p - 2a \pm \sqrt{p^2 + 4pa + 4a^2 - 16k - 16b}}{4}.$$

Therefore, the solutions of (1.10.15) have the forms,

$$\begin{cases} 4u_1 &= -p + 2a + \sqrt{p^2 + 4a^2 - 16k - 4pa + 16b}, \\ 4u_2 &= -p + 2a - \sqrt{p^2 + 4a^2 - 16k - 4pa + 16b}, \\ 4u_3 &= -p - 2a + \sqrt{p^2 + 4a^2 - 16k + 4pa - 16b}, \\ 4u_1 &= -p - 2a - \sqrt{p^2 + 4a^2 - 16k + 4pa - 16b}. \end{cases} \tag{1.10.21}$$

Now we solve the quartic equation (1.10.15) by using the same steps as explained above. In (1.10.16),

$$p = \frac{1}{v^3}, \quad q = \frac{3}{v}, \quad r = -v, \quad \text{and} \quad s = -\frac{1}{v^2}. \tag{1.10.22}$$

First, determine k, which must satisfy (1.10.19), i.e.,

$$\begin{aligned} &k^3 - \frac{3}{2v}k^2 + \frac{1}{4}\left(-\frac{1}{v^2} + \frac{4}{v^2}\right)k + \frac{1}{8}\left(-\frac{12}{v^3} + \frac{1}{v^8} - v^2\right) \\ &= k^3 - \frac{3}{2v}k^2 + \frac{3}{4v^2}k + \frac{1}{8}\left(-\frac{12}{v^3} + \frac{1}{v^8} - v^2\right) = 0. \end{aligned} \tag{1.10.23}$$

To solve the cubic equation (1.10.23), we use Cardan's formulas, i.e., if

$$x^3 + cx^2 + dx + f = 0, \tag{1.10.24}$$

then (1.10.24) has the three roots,

$$\begin{cases} \theta_1 &= -\frac{1}{3}c + \sqrt[3]{A} + \sqrt[3]{B}, \\ \theta_2 &= -\frac{1}{3}c + \omega\sqrt[3]{A} + \omega^2\sqrt[3]{B}, \\ \theta_3 &= -\frac{1}{3}c + \omega^2\sqrt[3]{A} + \omega\sqrt[3]{B}, \end{cases} \tag{1.10.25}$$

where $\omega = \exp(2\pi i/3)$,

$$A = -\frac{g}{2} + \sqrt{\frac{g^2}{4} + \frac{h^3}{27}}, \quad \text{and} \quad B = -\frac{g}{2} - \sqrt{\frac{g^2}{4} + \frac{h^3}{27}}, \tag{1.10.26}$$

with

$$g = f - \frac{cd}{3} + \frac{2c^3}{27} \quad \text{and} \quad h = d - \frac{c^2}{3}. \tag{1.10.27}$$

Thus, from (1.10.23), in the notation (1.10.24),

$$c = -\frac{3}{2v}, \quad d = \frac{3}{4v^2}, \quad \text{and} \quad f = \frac{1}{8}\left(-\frac{12}{v^3} + \frac{1}{v^8} - v^2\right). \tag{1.10.28}$$

Then from (1.10.27),

$$h = 0 \quad \text{and} \quad g = \frac{1}{8}\left(-\frac{11}{v^3} + \frac{1}{v^8} - v^2\right). \tag{1.10.29}$$

Thus, from (1.10.26), $A = 0$ and $B = -g$. Therefore, from (1.10.25) and (1.10.29),

$$\begin{cases} k_1 &= -\frac{c}{3} + \sqrt[3]{B} = \frac{1}{2v} - \frac{1}{2}\left(-\frac{11}{v^3} + \frac{1}{v^8} - v^2\right)^{1/3}, \\ k_2 &= -\frac{c}{3} + \omega^2\sqrt[3]{B} = \frac{1}{2v} - \frac{1}{2}\omega^2\left(-\frac{11}{v^3} + \frac{1}{v^8} - v^2\right)^{1/3}, \\ k_3 &= -\frac{c}{3} + \omega\sqrt[3]{B} = \frac{1}{2v} - \frac{1}{2}\omega\left(-\frac{11}{v^3} + \frac{1}{v^8} - v^2\right)^{1/3}. \end{cases} \tag{1.10.30}$$

Now we can take k to be either k_1, k_2, or k_3. Take $k = k_1$. Recalling the definition of R in (1.10.5), we have, by (1.10.30),

$$k = k_1 = \frac{1}{2v} - \frac{R}{2v}. \tag{1.10.31}$$

Now determine a and b by using (1.10.18). From the first equation of (1.10.18), (1.10.31), and (1.10.22),

$$a^2 = 2k + \frac{p^2}{4} - q = 2\left(\frac{1}{2v} - \frac{R}{2v}\right) + \frac{1}{4}\left(\frac{1}{v^6}\right) - \frac{3}{v} = \frac{1}{4v^6} - \frac{R}{v} - \frac{2}{v}.$$

Choose

$$a = \sqrt{\frac{1}{4v^6} - \frac{R}{v} - \frac{2}{v}}. \tag{1.10.32}$$

(We can choose a to be either positive or negative, but once the sign of a is determined, the sign of b should satisfy (1.10.18).) From the second equation of (1.10.18), (1.10.31), (1.10.22), and (1.10.32),

$$
\begin{aligned}
b = \frac{kp-r}{2a} &= \frac{\left(\dfrac{1}{2v}-\dfrac{R}{2v}\right)\dfrac{1}{v^3}+v}{2\sqrt{\dfrac{1}{4v^6}-\dfrac{R}{v}-\dfrac{2}{v}}} \\
&= \frac{\dfrac{1}{2v^4}-\dfrac{R}{2v^4}+\dfrac{2v^5}{2v^4}}{\sqrt{\dfrac{1}{v^6}-\dfrac{4R}{v}-\dfrac{8}{v}}} \\
&= \frac{1-R+2v^5}{2v\sqrt{1-8v^5-4Rv^5}}. \qquad (1.10.33)
\end{aligned}
$$

Now return to (1.10.21). By (1.10.32), (1.10.22), (1.10.31), and (1.10.33),

$$
2a = \sqrt{\frac{1}{v^6}-\frac{4R}{v}-\frac{8}{v}}, \qquad (1.10.34)
$$

$$
p^2+4a^2-16k = \frac{2}{v^6}-\frac{16}{v}+\frac{4R}{v}, \qquad (1.10.35)
$$

$$
4pa-16b = \frac{2}{v^3}\sqrt{\frac{1}{v^6}-\frac{4R}{v}-\frac{8}{v}}-\frac{8(1-R+2v^5)}{v\sqrt{1-8v^5-4Rv^5}}. \qquad (1.10.36)
$$

Hence, from (1.10.21), (1.10.22), (1.10.34), (1.10.35), and (1.10.36),

$$
\begin{aligned}
4u_1 = &-\frac{1}{v^3}+\sqrt{\frac{1}{v^6}-\frac{8}{v}-\frac{4R}{v}} \\
&+\sqrt{\frac{2}{v^6}-\frac{16}{v}+\frac{4R}{v}-\frac{2}{v^3}\sqrt{\frac{1}{v^6}-\frac{8}{v}-\frac{4R}{v}}+\frac{8(1-R+2v^5)}{v\sqrt{1-8v^5-4Rv^5}}}, \\
4u_2 = &-\frac{1}{v^3}+\sqrt{\frac{1}{v^6}-\frac{8}{v}-\frac{4R}{v}} \\
&-\sqrt{\frac{2}{v^6}-\frac{16}{v}+\frac{4R}{v}-\frac{2}{v^3}\sqrt{\frac{1}{v^6}-\frac{8}{v}-\frac{4R}{v}}+\frac{8(1-R+2v^5)}{v\sqrt{1-8v^5-4Rv^5}}}, \\
4u_3 = &-\frac{1}{v^3}-\sqrt{\frac{1}{v^6}-\frac{8}{v}-\frac{4R}{v}} \\
&+\sqrt{\frac{2}{v^6}-\frac{16}{v}+\frac{4R}{v}+\frac{2}{v^3}\sqrt{\frac{1}{v^6}-\frac{8}{v}-\frac{4R}{v}}-\frac{8(1-R+2v^5)}{v\sqrt{1-8v^5-4Rv^5}}}, \\
4u_4 = &-\frac{1}{v^3}-\sqrt{\frac{1}{v^6}-\frac{8}{v}-\frac{4R}{v}} \\
&-\sqrt{\frac{2}{v^6}-\frac{16}{v}+\frac{4R}{v}+\frac{2}{v^3}\sqrt{\frac{1}{v^6}-\frac{8}{v}-\frac{4R}{v}}-\frac{8(1-R+2v^5)}{v\sqrt{1-8v^5-4Rv^5}}}.
\end{aligned}
$$

After applying Lemma 1.10.1 above, we find that

$$\begin{cases} 4u_1 &= -\dfrac{1}{v^3} + \sqrt{\dfrac{1}{v^6} - \dfrac{8}{v} - \dfrac{4R}{v}} + \sqrt{\dfrac{1}{v^6} - \dfrac{8+4R\omega}{v}} - \sqrt{\dfrac{1}{v^6} - \dfrac{8+4R\omega^2}{v}}, \\ 4u_2 &= -\dfrac{1}{v^3} + \sqrt{\dfrac{1}{v^6} - \dfrac{8}{v} - \dfrac{4R}{v}} - \sqrt{\dfrac{1}{v^6} - \dfrac{8+4R\omega}{v}} + \sqrt{\dfrac{1}{v^6} - \dfrac{8+4R\omega^2}{v}}, \\ 4u_3 &= -\dfrac{1}{v^3} - \sqrt{\dfrac{1}{v^6} - \dfrac{8}{v} - \dfrac{4R}{v}} + \sqrt{\dfrac{1}{v^6} - \dfrac{8+4R\omega}{v}} + \sqrt{\dfrac{1}{v^6} - \dfrac{8+4R\omega^2}{v}}, \\ 4u_4 &= -\dfrac{1}{v^3} - \sqrt{\dfrac{1}{v^6} - \dfrac{8}{v} - \dfrac{4R}{v}} - \sqrt{\dfrac{1}{v^6} - \dfrac{8+4R\omega}{v}} - \sqrt{\dfrac{1}{v^6} - \dfrac{8+4R\omega^2}{v}}. \end{cases} \tag{1.10.37}$$

Now consider the discriminant of the quartic equation (1.10.15). By (1.10.30) and (1.10.5),

$$\begin{cases} k_1 &= \dfrac{1}{2v} - \dfrac{R}{2v}, \\ k_2 &= \dfrac{1}{2v} - \dfrac{R}{2v}\omega^2, \\ k_3 &= \dfrac{1}{2v} - \dfrac{R}{2v}\omega. \end{cases} \tag{1.10.38}$$

Hence, after a long calculation, we see that the discriminant D is given by

$$D = 64(k_1-k_2)^2(k_1-k_3)^2(k_2-k_3)^2 = -\frac{27R^6}{v^6},$$

a negative real number. Therefore, we know that there are two distinct real roots and two conjugate complex roots. Hence, from (1.10.37), after applying Lemma 1.10.2 with $\sqrt{\dfrac{1}{v^6} - \dfrac{8+4R\omega}{v}} = x + iy$, we find that

$$\begin{cases} 4u_1 &= -\dfrac{1}{v^3} + \sqrt{\dfrac{1}{v^6} - \dfrac{8}{v} - \dfrac{4R}{v}} + 2iy, \\ 4u_2 &= -\dfrac{1}{v^3} + \sqrt{\dfrac{1}{v^6} - \dfrac{8}{v} - \dfrac{4R}{v}} - 2iy, \\ 4u_3 &= -\dfrac{1}{v^3} - \sqrt{\dfrac{1}{v^6} - \dfrac{8}{v} - \dfrac{4R}{v}} + 2x, \\ 4u_4 &= -\dfrac{1}{v^3} - \sqrt{\dfrac{1}{v^6} - \dfrac{8}{v} - \dfrac{4R}{v}} - 2x. \end{cases} \tag{1.10.39}$$

Now it is obvious that u_1 and u_2 are the conjugate imaginary roots and u_3 and u_4 are the two real roots. Thus, one of u_3 and u_4 is the solution. Now observe that as $q \to 0^+$,

$$u = O(q^{1/5}), \qquad v = O(q^{3/5}), \qquad R = O(q^{-1}), \quad \text{and} \quad \frac{R}{v} = O(q^{-8/5}).$$

Therefore, as $q \to 0^+$, we find that $u \to 0^+, v \to 0^+$, and $R/v \to +\infty$. Hence, if $\omega = \exp(2\pi i/3)$, then $1/v^6 - (8+4R\omega)/v$, is in the fourth quadrant. Therefore, if we take the principal argument, i.e., $-\pi \le \theta < \pi$, then

$$\sqrt{\frac{1}{v^6} - \frac{8+4R\omega}{v}} = x + iy$$

lies in the fourth quadrant, but $1/v^6$ is the dominant term, and so as $q \to 0^+$, $x \to +\infty$ and $y \to 0^-$. Now $u \to 0$, but from (1.10.39), the expression for $4u_4$ approaches $-\infty$ as $q \to 0^+$. Hence u_4 is not a solution. Therefore u_3 must be the correct solution, as Ramanujan claimed.

Now we establish the second part of Entry 1.10.2. To prove (1.10.8), write (1.10.14) as

$$v^4 - u^3v^3 - 3uv^2 + \frac{v}{u} - u^2 = 0. \tag{1.10.40}$$

Observe that (1.10.40) can be obtained by replacing u by v, and v by $-1/u$ in (1.10.15). Hence, from (1.10.37), if R is defined by (1.10.7), we find that the four roots of (1.10.40) are

$$\begin{aligned}
4v_1 &= u^3 + \sqrt{u^6+8u+4uR} + \sqrt{u^6+4u(2+R\omega)} - \sqrt{u^6+4u(2+R\omega^2)},\\
4v_2 &= u^3 + \sqrt{u^6+8u+4uR} - \sqrt{u^6+4u(2+R\omega)} + \sqrt{u^6+4u(2+R\omega^2)},\\
4v_3 &= u^3 - \sqrt{u^6+8u+4uR} + \sqrt{u^6+4u(2+R\omega)} + \sqrt{u^6+4u(2+R\omega^2)},\\
4v_4 &= u^3 - \sqrt{u^6+8u+4uR} - \sqrt{u^6+4u(2+R\omega)} - \sqrt{u^6+4u(2+R\omega^2)}.
\end{aligned}$$

Using (1.10.38), replacing u by v, and v by $-1/u$, we find the discriminant D of (1.10.40) to be

$$D = 64(k_1-k_2)^2(k_1-k_3)^2(k_2-k_3)^2 = -27u^6R^6,$$

a negative real number. Therefore, we know that there are two distinct real roots and two conjugate imaginary roots. If we apply Lemma 1.10.2 with $\sqrt{u^6+4u(2+R\omega)} = x+iy$, then the roots above take the shapes

$$\begin{cases}
4v_1 &= u^3 + \sqrt{u^6+8u+4uR} + 2iy,\\
4v_2 &= u^3 + \sqrt{u^6+8u+4uR} - 2iy,\\
4v_3 &= u^3 - \sqrt{u^6+8u+4uR} + 2x,\\
4v_4 &= u^3 - \sqrt{u^6+8u+4uR} - 2x.
\end{cases} \tag{1.10.41}$$

Therefore, from (1.10.41), v_3 and v_4 are the real roots and v_1 and v_2 are the conjugate imaginary roots. So one of v_3, v_4 is a solution. Once again, observe that as $q \to 0^+$,

$$u = O(q^{1/5}), \quad v = O(q^{3/5}), \quad R = O(q^{-1/3}), \quad \text{and} \quad uR = O(q^{-2/15}).$$

Thus, as $q \to 0^+$, we find that $u \to 0^+$, $v \to 0^+$, and $uR \to +\infty$. Hence if $\omega = \exp(2\pi i/3)$, then $u^6+8u+4uR\omega$ is a value in the second quadrant as $q \to 0^+$. Therefore, if we consider the principal argument, then $\sqrt{u^6+8u+4uR\omega} = x+iy$ is a value in the first quadrant. So as $q \to 0^+$, we see that $x \to +\infty$ and

$y \to +\infty$. (actually $\sqrt{3}x \approx y$.) Now $v \to 0$, but in (1.10.41) the expression for $4v_4$ approaches $-\infty$ as $q \to 0^+$. So v_4 is not a solution. Therefore v_3 is the desired solution. Ramanujan actually claimed that v_4 is the solution. If we consider the argument $\pi \leq \theta < 3\pi$, then his claim is correct. $\square$

Proof of Entry 1.10.1. To prove (1.10.2) and (1.10.4), we use Ramanujan's modular equation relating $u = R(q)$ and $v = R(q^2)$, given in (1.5.3). Rewrite (1.5.3) in the forms

$$uv^3 + u^3v^2 - v + u^2 = 0, \tag{1.10.42}$$

$$v^2u^3 + u^2 + v^3u - v = 0. \tag{1.10.43}$$

Note that (1.10.42) is cubic in v and (1.10.43) is cubic in u. To prove (1.10.2), we use Cardan's method [277, pp. 84–86] to solve for u in terms of v, and we similarly employ Cardan's method to prove (1.10.4). Considering (1.10.42) as a cubic equation in v, we rewrite it in the form

$$v^3 + u^2v^2 - \frac{1}{u}v + u = 0, \tag{1.10.44}$$

and we rewrite (1.10.25) as

$$\begin{cases} -3\theta_1 & = c + \sqrt[3]{-27A} + \sqrt[3]{-27B}, \\ -3\theta_2 & = c + \omega\sqrt[3]{-27A} + \omega^2\sqrt[3]{-27B}, \\ -3\theta_3 & = c + \omega^2\sqrt[3]{-27A} + \omega\sqrt[3]{-27B}. \end{cases} \tag{1.10.45}$$

By (1.10.26),

$$-27A = \frac{27g}{2} - \sqrt{\frac{27g^2}{4} + (3h)^3} \quad \text{and} \quad -27B = \frac{27g}{2} + \sqrt{\frac{27g^2}{4} + (3h)^3}. \tag{1.10.46}$$

The coefficients of (1.10.44) are, in the notation (1.10.24),

$$c = u^2, \qquad d = -\frac{1}{u}, \qquad \text{and} \qquad f = u. \tag{1.10.47}$$

By (1.10.27) and (1.10.47),

$$3h = 3d - c^2 = -\frac{3}{u} - u^4 \tag{1.10.48}$$

and

$$\frac{27g}{2} = \frac{27f}{2} - \frac{9cd}{2} + c^3 = \frac{27f - 9cd + 2c^3}{2} = \frac{27u + 9u + 2u^6}{2} = u^6 + 18u. \tag{1.10.49}$$

Hence by (1.10.46), (1.10.48), (1.10.49), and (1.10.1),

$$-27A = u^6 + 18u - \sqrt{(u^6 + 18u)^2 + \left(-\frac{3}{u} - u^4\right)^3}$$

$$
\begin{aligned}
&= u^6 + 18u - i\sqrt{-(u^6+18u)^2 + \left(\frac{3}{u} + u^4\right)^3} \\
&= u^6 + 18u - i\sqrt{\frac{27}{u^3} - 297u^2 - 27u^7} \\
&= u^6 + 18u - iu3\sqrt{3}\sqrt{\frac{1}{u^5} - 11 - u^5} \\
&= u^6 + 18u - iu3\sqrt{3}R.
\end{aligned}
$$

Similarly, by (1.10.46), (1.10.48), (1.10.49), and (1.10.1),

$$-27B = u^6 + 18u + iu3\sqrt{3}R.$$

Therefore, from (1.10.45) and the calculations above, we find that

$$
\begin{cases}
-3v_1 &= u^2 + \sqrt[3]{u^6+18u-iu3\sqrt{3}R} + \sqrt[3]{u^6+18u+iu3\sqrt{3}R}, \\
-3v_2 &= u^2 + \omega\sqrt[3]{u^6+18u-iu3\sqrt{3}R} + \omega^2\sqrt[3]{u^6+18u+iu3\sqrt{3}R}, \\
-3v_3 &= u^2 + \omega^2\sqrt[3]{u^6+18u-iu3\sqrt{3}R} + \omega\sqrt[3]{u^6+18u+iu3\sqrt{3}R}.
\end{cases}
\tag{1.10.50}
$$

Hence, after applying Lemma 1.10.3 with $\sqrt[3]{u^6+18u+iu3\sqrt{3}R} = x+iy$, by (1.10.50), we find that

$$
\begin{aligned}
-3v_1 &= u^2 + 2x, \\
-3v_2 &= u^2 - x + \sqrt{3}y, \\
-3v_3 &= u^2 - x - \sqrt{3}y.
\end{aligned}
$$

Observe that as $q \to 0^+$,

$$u = O(q^{1/5}), \quad v = O(q^{2/5}), \quad R = O(q^{-1/2}), \quad \text{and} \quad uR = O(q^{-3/10}).$$

If $\omega = \exp(2\pi i/3)$, then $u \to 0^+$, $v \to 0^+$, $R \to +\infty$, and $uR \to +\infty$ as $q \to 0^+$. Consequently, $u^6 + 18u + i3\sqrt{3}uR$ tends to a value on the positive imaginary axis. Hence, $\sqrt[3]{u^6+18u+iu3\sqrt{3}R} = x+iy$ lies in the first quadrant when we consider the principal argument. Therefore, as $q \to 0^+$, we find that $x \to +\infty$ and $y \to +\infty$. (actually $x \approx \sqrt{3}y$.) Now the expression for $-3v_1$ approaches $+\infty$, and the expression for $-3v_3$ approaches $-\infty$. Hence v_2 is the correct solution. But Ramanujan claimed that v_1 is the correct answer. If we consider the argument $3\pi \le \theta < 5\pi$ instead of the principal argument, then his claim is correct.

To prove (1.10.4), we rewrite (1.10.43) as

$$u^3 + \frac{1}{v^2}u^2 + vu - \frac{1}{v} = 0. \tag{1.10.51}$$

Observe that (1.10.51) can be obtained by replacing v by u and u by $-1/v$ in (1.10.44). Hence, from (1.10.50), with R defined by (1.10.3), we find that the roots of (1.10.51) are given by

$$-3u_1 = \frac{1}{v^2} + \sqrt[3]{\frac{1}{v^6} - \frac{18}{v} + \frac{3(\omega-\omega^2)R}{v}} + \sqrt[3]{\frac{1}{v^6} - \frac{18}{v} - \frac{3(\omega-\omega^2)R}{v}},$$
$$-3u_2 = \frac{1}{v^2} + \omega\sqrt[3]{\frac{1}{v^6} - \frac{18}{v} + \frac{3(\omega-\omega^2)R}{v}} + \omega^2\sqrt[3]{\frac{1}{v^6} - \frac{18}{v} - \frac{3(\omega-\omega^2)R}{v}},$$
$$-3u_3 = \frac{1}{v^2} + \omega^2\sqrt[3]{\frac{1}{v^6} - \frac{18}{v} + \frac{3(\omega-\omega^2)R}{v}} + \omega\sqrt[3]{\frac{1}{v^6} - \frac{18}{v} - \frac{3(\omega-\omega^2)R}{v}}.$$

If we apply Lemma 1.10.3 with

$$\sqrt[3]{\frac{1}{v^6} - \frac{18}{v} + \frac{3(\omega-\omega^2)R}{v}} = x + iy,$$

then the roots above assume the shapes

$$-3u_1 = \frac{1}{v^2} + 2x,$$
$$-3u_2 = \frac{1}{v^2} - x - \sqrt{3}y,$$
$$-3u_3 = \frac{1}{v^2} - x + \sqrt{3}y.$$

Observe that as $q \to 0^+$,

$$\frac{1}{v^6} = O(q^{-12/5}), \qquad R = O(q^{-1}), \qquad \text{and} \qquad \frac{R}{v} = O(q^{-7/5}).$$

If $\omega = \exp(2\pi i/3)$, then as $q \to 0^+$, we see that $u \to 0^+, v \to 0^+, R \to +\infty$, and $R/v \to +\infty$. So

$$\frac{1}{v^6} - \frac{18}{v} + \frac{3(\omega-\omega^2)R}{v}$$

lies in the first quadrant. Hence,

$$\sqrt[3]{\frac{1}{v^6} - \frac{18}{v} + \frac{3(\omega-\omega^2)R}{v}} = x + iy$$

resides in the first quadrant when we consider the principal argument. Now $1/v^6$ is the dominant term, and so as $q \to 0^+$, we find that $x \to +\infty$ and $y \to 0^+$. The expression for $-3u_1$ approaches $+\infty$. Hence, $-3u_1$ is not a solution. Next, we verify that $1/v^2 - x \to 0^+$ as $q \to 0^+$. Now $-3u_3 \to 0^-$, and the right side of $-3u_3$ approaches 0^+ as $q \to 0^+$. Hence u_2 is the proper solution, as Ramanujan claimed. $\square$

Explicit Evaluations of the Rogers–Ramanujan Continued Fraction

2.1 Introduction

Recall that for $|q| < 1$, the Rogers–Ramanujan continued fraction $R(q)$ is defined by

$$R(q) := \frac{q^{1/5}}{1} \underset{+}{} \frac{q}{1} \underset{+}{} \frac{q^2}{1} \underset{+}{} \frac{q^3}{1} \underset{+}{} \cdots. \tag{2.1.1}$$

Also define

$$S(q) := -R(-q). \tag{2.1.2}$$

In this chapter, we concentrate on explicit evaluations of the Rogers–Ramanujan continued fraction. In particular, we establish the evaluations on pages 46, 204, and 210 in the lost notebook. On page 210, Ramanujan recorded a table of arguments and values for $R(q)$, but most of the values were, in fact, omitted by Ramanujan. Evidently, Ramanujan knew that he could indeed find these values, but perhaps because of his terminal illness and the desire to discover further theorems, he did not work out the details.

We note at the outset that two evaluations are elementary, namely,

$$R(1) = \frac{\sqrt{5}-1}{2} \qquad \text{and} \qquad S(1) = \frac{\sqrt{5}+1}{2}.$$

Ramanujan and I. Schur independently proved that $R(q)$ converges at primitive nth roots of unity if n is not a multiple of 5; if n is a multiple of 5, they proved that $R(q)$ diverges. Furthermore, in the cases of convergence, they explicitly evaluated $R(q)$. See [63, pp. 35–36] for details and references.

In his first letter to Hardy [226, p. xxvii], [81, p. 29], Ramanujan gave the first nonelementary evaluations of $R(q)$, namely,

$$R(e^{-2\pi}) = \sqrt{\frac{5+\sqrt{5}}{2}} - \frac{\sqrt{5}+1}{2} \tag{2.1.3}$$

and

$$S(e^{-\pi}) = \sqrt{\frac{5-\sqrt{5}}{2}} - \frac{\sqrt{5}-1}{2}. \tag{2.1.4}$$

In his second letter letter to Hardy [226, p. xxviii], [81, p. 57], Ramanujan further asserted that

$$R(e^{-2\pi\sqrt{5}}) = \frac{\sqrt{5}}{1+\sqrt[5]{5^{3/4}\left(\frac{\sqrt{5}-1}{2}\right)^{5/2}-1}} - \frac{\sqrt{5}+1}{2}.$$

In both letters, Ramanujan claimed [226, p. xxvii], [81, pp. 29, 57], "It is always possible to find exactly the value of $R(e^{-\pi\sqrt{n}})$." All of the evaluations that we establish in this chapter are of this form or of the form $S(e^{-\pi\sqrt{n}})$. Moreover, we shall provide a meaning for Ramanujan's statement about $R(e^{-\pi\sqrt{n}})$. Much of the content of this chapter can be found in a paper by Berndt, H.H. Chan, and L.–C. Zhang [73].

The first attempt to find a "uniform" method for evaluating $R(q)$ was made by K.G. Ramanathan [218]. By studying the ideal class groups of imaginary quadratic fields with the property that each genus contains a single class, Ramanathan was able to compute $R(e^{-2\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$ for several rational numbers n using Kronecker's limit formula. This method is quite limited in its applications, and not all of the values claimed by Ramanujan can be verified in this manner. Furthermore, Ramanujan probably did not know Kronecker's limit formula, and so used different methods.

In this chapter we present two closely related methods for explicitly determining values of the Rogers–Ramanujan continued fraction. The first uses modular equations, more precisely, eta-function identities discovered by Ramanujan, and was first used by Berndt and Chan [67], [63, pp. 20–30] to establish some particular values of $R(q)$ found in Ramanujan's first notebook. The second method, which is found in the paper by Berndt, Chan, and Zhang [73], also uses Ramanujan's eta-function identities, but goes further in offering general formulas for evaluating $R(e^{-2\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$ in terms of class invariants. Modular equations, in particular eta-function identities, were systematically employed by J. Yi [297], [298] not only to prove many of the evaluations found in the lost notebook but also to establish many new values for $R(q)$ as well. We think that the theorems of the aforementioned authors provide meaning to the general claim made by Ramanujan in each of his first two letters to Hardy; if the requisite class invariants are known, then the values of $R(e^{-2\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$ can be explicitly determined.

Generally, as illustrated in (2.1.3) and (2.1.4), if we can evaluate $R(e^{-2\pi\sqrt{n}})$, then we can also evaluate $S(e^{-\pi\sqrt{n}})$, and conversely. There is another sense in which values come in pairs. If $\alpha, \beta > 0$ and $\alpha\beta = \pi^2$, then, in his second letter to Hardy, Ramanujan [226, p. xxviii], [81, p. 57] claimed that

$$\left(\frac{1+\sqrt{5}}{2} + R(e^{-2\alpha})\right)\left(\frac{1+\sqrt{5}}{2} + R(e^{-2\beta})\right) = \frac{5+\sqrt{5}}{2}. \tag{2.1.5}$$

Thus, if we know the value of $R(e^{-2\alpha})$ for a certain number α, then by using (2.1.5) we can also determine $R(e^{-2\pi^2/\alpha})$.

We do not know Ramanujan's methods for evaluating $R(q)$ and $S(q)$. However, we conjecture that he indeed did use modular equations, in particular, eta-function identities. In his lost notebook, Ramanujan actually states some general formulas for $R(q)$ and $R(q^2)$, which, in principal, can be used to explicitly evaluate $R(e^{-2\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$. However, these formulas appear cumbersome to apply in most evaluations. In Section 2.5, we establish some requisite theta-function identities from the lost notebook, and in Section 2.6, we prove these general formulas. Proofs of these theorems were first published by S.–Y. Kang [172].

2.2 Explicit Evaluations Using Eta-Function Identities

In this section we show how Ramanujan's eta-function identities can be used to explicitly evaluate $R(q)$ and $S(q)$, defined, respectively, in (2.1.1) and (2.1.2).

Entry 2.2.1 (p. 46). *We have*

$$S(e^{-\pi\sqrt{3}}) = \frac{-(3+\sqrt{5})+\sqrt{6(5+\sqrt{5})}}{4}. \tag{2.2.1}$$

The first proof of (2.2.1) was given by Ramanathan [220], who used Kronecker's limit formula. The proof we give here is due to Chan [112]. J. Yi [298, Corollary 3.18] has also found an elegant proof.

We need to first recall some definitions. Let

$$f(-q) := (q;q)_\infty, \qquad |q| < 1, \tag{2.2.2}$$

where

$$(a;q)_\infty = \prod_{n=0}^{\infty}(1-aq^n), \qquad |q|<1.$$

The function f satisfies the well-known transformation formula [61, p. 43, Chapter 16, Entry 27(iv)]

$$e^{-\pi z/24} f(e^{-\pi z}) = \frac{1}{\sqrt{z}} e^{-\pi/(24z)} f(e^{-\pi/z}), \qquad \operatorname{Re} z > 0. \tag{2.2.3}$$

One of the most fundamental and useful properties of the Rogers–Ramanujan continued fraction is the following equation for $R(q)$, which is due to Ramanujan and recorded by him in Chapter 19 of his second notebook [227], [61, p. 84, equation (39.1)]. An especially simple and short proof has been given by M.D. Hirschhorn [158].

Lemma 2.2.1. *For* $|q| < 1$,

$$\frac{1}{R(q)} - R(q) - 1 = \frac{f(-q^{1/5})}{q^{1/5} f(-q^5)}. \tag{2.2.4}$$

If we rewrite (2.2.4) in terms of $S(q)$, we find that

$$\frac{1}{S(q)} - S(q) + 1 = \frac{f(q^{1/5})}{q^{1/5} f(q^5)}. \tag{2.2.5}$$

We also need the following modular equation of Ramanujan [62, p. 221, Chapter 25, Entry 62].

Lemma 2.2.2. *Let*

$$P := \frac{f(q)}{q^{1/12} f(q^3)} \qquad \textit{and} \qquad Q := \frac{f(q^5)}{q^{5/12} f(q^{15})}.$$

Then

$$(PQ)^2 + \frac{9}{(PQ)^2} = \left(\frac{Q}{P}\right)^3 - \left(\frac{P}{Q}\right)^3 + 5. \tag{2.2.6}$$

Proof of Entry 2.2.1. Let $q = e^{-\pi/\sqrt{3}}$ in Lemma 2.2.2. By (2.2.3), we deduce that

$$P = \frac{f(e^{-\pi/\sqrt{3}})}{e^{-\pi/12\sqrt{3}} f(e^{-\sqrt{3}\pi})} = 3^{1/4}. \tag{2.2.7}$$

Similarly,

$$Q = \frac{f(e^{-5\pi/\sqrt{3}})}{e^{-5\pi/12\sqrt{3}} f(e^{-5\sqrt{3}\pi})} = \frac{3^{1/4}}{\sqrt{5}} e^{\sqrt{3}\pi/5} \frac{f(e^{-\sqrt{3}\pi/5})}{f(e^{-5\sqrt{3}\pi})}. \tag{2.2.8}$$

If we let

$$B := e^{\sqrt{3}\pi/5} \frac{f(e^{-\sqrt{3}\pi/5})}{f(e^{-5\sqrt{3}\pi})}, \tag{2.2.9}$$

then, by (2.2.7)–(2.2.9),

$$PQ = \sqrt{\frac{3}{5}} B \qquad \text{and} \qquad \frac{Q}{P} = \frac{B}{\sqrt{5}}. \tag{2.2.10}$$

Substituting (2.2.10) into Lemma 2.2.2, we find that

$$\frac{3}{5} B^2 + \frac{15}{B^2} = \left(\frac{B}{\sqrt{5}}\right)^3 - \left(\frac{\sqrt{5}}{B}\right)^3 + 5,$$

which may be rewritten in the form

$$3\left(\left(\frac{B}{\sqrt{5}}-\frac{\sqrt{5}}{B}\right)^2+2\right)=\left(\frac{B}{\sqrt{5}}-\frac{\sqrt{5}}{B}\right)^3+3\left(\frac{B}{\sqrt{5}}-\frac{\sqrt{5}}{B}\right)+5,$$

which implies that

$$\left(\left(\frac{B}{\sqrt{5}}-\frac{\sqrt{5}}{B}\right)-1\right)^3=0. \tag{2.2.11}$$

Solving (2.2.11), we deduce that

$$B=\frac{5+\sqrt{5}}{2}. \tag{2.2.12}$$

From (2.2.5) it follows that

$$\frac{1}{S(e^{-\sqrt{3}\pi})}-S(e^{-\sqrt{3}\pi})+1=e^{\sqrt{3}\pi/5}\frac{f(e^{-\sqrt{3}\pi/5})}{f(e^{-5\sqrt{3}\pi})}. \tag{2.2.13}$$

By (2.2.9) and (2.2.12), we conclude that the right-hand side of (2.2.13) equals $(5+\sqrt{5})/2$. Solving the quadratic equation (2.2.13), we obtain (2.2.1). This completes the proof. $\square$

For the next four evaluations, we shall employ another fundamental result about $R(q)$ from Chapter 19 of Ramanujan's second notebook [61, pp. 270–271]. (In particular, see equation (12.13) on page 270 and the definitions of μ and ν given on page 271.)

Lemma 2.2.3. *For* $|q|<1$,

$$\frac{1}{R^5(q)}-11-R^5(q)=\frac{f^6(-q)}{qf^6(-q^5)}. \tag{2.2.14}$$

Rewriting (2.2.14) in terms of $S(q)$, we find that

$$\frac{1}{S^5(q)}+11-S^5(q)=\frac{f^6(q)}{qf^6(q^5)}. \tag{2.2.15}$$

Our next proof employs another eta-function identity of Ramanujan [62, p. 236, Entry 71].

Lemma 2.2.4. *Let*

$$P:=\frac{f(q)}{q^{1/4}f(q^7)} \qquad \textit{and} \qquad Q:=\frac{f(q^5)}{q^{5/4}f(q^{35})}.$$

Then

$$(PQ)^2+5+\frac{49}{(PQ)^2}=\left(\frac{Q}{P}\right)^3+5\left(\frac{Q}{P}\right)^2+5\left(\frac{P}{Q}\right)^2-\left(\frac{P}{Q}\right)^3. \tag{2.2.16}$$

Entry 2.2.2 (pp. 204, 210). *We have*

$$S(e^{-\pi/\sqrt{35}}) = \left(5\sqrt{5} - 7 + \sqrt{35(5-2\sqrt{5})}\right)^{1/5}.$$

Proof of Entry 2.2.2. Setting

$$2u := 11 - \frac{f^6(q)}{qf^6(q^5)} \tag{2.2.17}$$

and solving (2.2.15) for $S(q)$, we readily find that

$$S(q) = \left(u + \sqrt{u^2+1}\right)^{1/5}, \tag{2.2.18}$$

where we took the positive root of the quadratic equation in $S^5(q)$, because $S(q) > 0$. If $q = e^{-\pi/\sqrt{35}}$, we thus see from (2.2.15) that it suffices to determine

$$e^{\pi/\sqrt{35}} \frac{f^6(e^{-\pi/\sqrt{35}})}{f^6(e^{-\pi\sqrt{5/7}})}.$$

To determine the quotient above, we employ Lemma 2.2.4 with $q = e^{-\pi/\sqrt{35}}$. Then

$$P = e^{\pi/(4\sqrt{35})} \frac{f(e^{-\pi/\sqrt{35}})}{f(e^{-\pi\sqrt{7/5}})} \qquad \text{and} \qquad Q = e^{5\pi/(4\sqrt{35})} \frac{f(e^{-\pi\sqrt{5/7}})}{f(e^{-\pi\sqrt{35}})}. \tag{2.2.19}$$

Setting $z = \sqrt{35}$ and then $z = \sqrt{7/5}$ in the transformation formula (2.2.3), we find that, respectively,

$$f(e^{-\pi\sqrt{35}}) = (35)^{-1/4} e^{17\pi/(12\sqrt{35})} f(e^{-\pi/\sqrt{35}}) \tag{2.2.20}$$

and

$$f(e^{-\pi\sqrt{7/5}}) = (5/7)^{1/4} e^{\pi/(12\sqrt{35})} f(e^{-\pi\sqrt{5/7}}). \tag{2.2.21}$$

Then, by (2.2.19)–(2.2.21),

$$\begin{aligned} PQ &= \frac{e^{\pi/(4\sqrt{35})} f(e^{-\pi/\sqrt{35}})}{(5/7)^{1/4} e^{\pi/(12\sqrt{35})} f(e^{-\pi\sqrt{5/7}})} \frac{e^{5\pi/(4\sqrt{35})} f(e^{-\pi\sqrt{5/7}})}{(35)^{-1/4} e^{17\pi/(12\sqrt{35})} f(e^{-\pi/\sqrt{35}})} \\ &= \sqrt{7} \end{aligned} \tag{2.2.22}$$

and

$$\frac{Q}{P} = \sqrt{5} \left(e^{-\pi/(6\sqrt{35})} \frac{f(e^{-\pi\sqrt{5/7}})}{f(e^{-\pi/\sqrt{35}})} \right)^2 =: \sqrt{5} A^2. \tag{2.2.23}$$

Substituting (2.2.22) and (2.2.23) into (2.2.16), we deduce that

$$19 = (\sqrt{5}A^2)^3 + 5(\sqrt{5}A^2)^2 + 5(\sqrt{5}A^2)^{-2} - (\sqrt{5}A^2)^{-3}.$$

Setting $x = \sqrt{5}A^2 - (\sqrt{5}A^2)^{-1}$, we can rewrite the foregoing equation in the form

$$19 = x^3 + 3x + 5x^2 + 10,$$

or

$$(x-1)(x+3)^2 = 0.$$

It is not difficult to see that x is positive. Thus, $x = 1$ is the only viable root. Solving the resulting equation

$$(\sqrt{5}A^2)^2 - (\sqrt{5}A^2) - 1 = 0,$$

we find that

$$A^2 = \frac{5+\sqrt{5}}{10}.$$

Hence, with $q = e^{-\pi/\sqrt{35}}$, it follows that

$$\frac{f^6(q)}{qf^6(q^5)} = \left(\frac{5+\sqrt{5}}{10}\right)^{-3}. \tag{2.2.24}$$

Thus, by (2.2.17),

$$2u = 11 - \left(\frac{10}{5+\sqrt{5}}\right)^3 = 11 - \frac{25}{5+2\sqrt{5}} = -14 + 10\sqrt{5}.$$

Using this value for u in (2.2.18), we conclude that

$$S(e^{-\pi/\sqrt{35}}) = \left(5\sqrt{5} - 7 + \sqrt{(5\sqrt{5}-7)^2 + 1}\right)^{1/5},$$

which, upon simplification, yields Entry 2.2.2. □

Entry 2.2.3 (p. 210). *We have*

$$S(e^{-\pi\sqrt{7/5}}) = \left(-5\sqrt{5} - 7 + \sqrt{35(5+2\sqrt{5})}\right)^{1/5}. \tag{2.2.25}$$

Proof. We shall provide only a sketch of the proof, since the details are very similar to those in the proof of Entry 2.2.2.

Let $q = e^{-\pi\sqrt{7/5}}$. Then from (2.2.15), (2.2.17), and (2.2.18), we see that it suffices to evaluate

$$e^{\pi\sqrt{7/5}} \frac{f^6(e^{-\pi\sqrt{7/5}})}{f^6(e^{-\pi\sqrt{35}})}.$$

However, from (2.2.20), (2.2.21), and (2.2.24),

$$e^{\pi\sqrt{7/5}}\frac{f^6(e^{-\pi\sqrt{7/5}})}{f^6(e^{-\pi\sqrt{35}})} = 125e^{-\pi/\sqrt{35}}\frac{f^6(e^{-\pi\sqrt{5/7}})}{f^6(e^{-\pi/\sqrt{35}})} = 125\left(\frac{10}{5+\sqrt{5}}\right)^{-3}.$$

Thus, by (2.2.17),

$$2u = 11 - 125\left(\frac{10}{5+\sqrt{5}}\right)^{-3} = -14 - 10\sqrt{5}.$$

The remainder of the proof follows in exactly the same way as before. □

Ramanathan [218] employed more recondite ideas to establish Entries 2.2.2 and 2.2.3, although only Entry 2.2.3 is explicitly stated by him. Yi's elegant proof [298, Corollary 4.3] of these two entries employs eta-function identities.

Entry 2.2.4 (p. 210). *We have*

$$S(e^{-\pi/\sqrt{15}}) = \left(\frac{5\sqrt{5} - 3 + \sqrt{30(5-\sqrt{5})}}{4}\right)^{1/5}. \tag{2.2.26}$$

Proof. To prove Entry 2.2.4, by (2.2.15), it suffices to determine

$$e^{\pi/\sqrt{15}}\frac{f^6(e^{-\pi/\sqrt{15}})}{f^6(e^{-\pi\sqrt{3/5}})}.$$

Set $q = e^{-\pi/\sqrt{15}}$, so that

$$P = e^{\pi/(12\sqrt{15})}\frac{f(e^{-\pi/\sqrt{15}})}{f(e^{-\pi\sqrt{3/5}})} \qquad \text{and} \qquad Q = e^{5\pi/(12\sqrt{15})}\frac{f(e^{-\pi\sqrt{5/3}})}{f(e^{-\pi\sqrt{15}})}.$$

By (2.2.3), with $z = \sqrt{15}$ and then $z = \sqrt{3/5}$, we find that, respectively,

$$f(e^{-\pi\sqrt{15}}) = (15)^{-1/4}e^{14\pi/(24\sqrt{15})}f(e^{-\pi/\sqrt{15}}) \tag{2.2.27}$$

and

$$f(e^{-\pi\sqrt{3/5}}) = (5/3)^{1/4}e^{-2\pi/(24\sqrt{15})}f(e^{-\pi/\sqrt{5/3}}). \tag{2.2.28}$$

It follows that upon simplification,

$$PQ = \sqrt{3} \tag{2.2.29}$$

and

$$\frac{Q}{P} = \sqrt{5}\left(e^{-\pi/(6\sqrt{15})}\frac{f(e^{-\pi\sqrt{5/3}})}{f(e^{-\pi/\sqrt{15}})}\right)^2 =: \sqrt{5}A^2. \tag{2.2.30}$$

Employing (2.2.29) and (2.2.30) in (2.2.6), we deduce that

$$5\sqrt{5}A^6 - \frac{1}{5\sqrt{5}A^6} = 1.$$

Solving for A^6, we find that

$$\frac{qf^6(q^5)}{f^6(q)} = A^6 = \frac{1+\sqrt{5}}{10\sqrt{5}}, \tag{2.2.31}$$

since $A^6 > 0$. Using this value in (2.2.17), we deduce that

$$2u = 11 - \frac{10\sqrt{5}}{1+\sqrt{5}} = -\frac{3}{2} + \frac{5\sqrt{5}}{2}$$

and

$$u^2 + 1 = \frac{150 - 30\sqrt{5}}{16}.$$

Using these calculations in (2.2.18), we complete the proof. □

Entry 2.2.5 (p. 210). *We have*

$$S(e^{-\pi\sqrt{3/5}}) = \left(\frac{-5\sqrt{5} - 3 + \sqrt{30(5+\sqrt{5})}}{4}\right)^{1/5}. \tag{2.2.32}$$

Proof. By (2.2.15), we need to calculate

$$e^{\pi\sqrt{3/5}}\frac{f^6(e^{-\pi\sqrt{3/5}})}{f^6(e^{-\pi\sqrt{15}})}.$$

A brief calculation with the use of (2.2.27) and (2.2.28) shows that

$$e^{\pi\sqrt{3/5}}\frac{f^6(e^{-\pi\sqrt{3/5}})}{f^6(e^{-\pi\sqrt{15}})} = 125e^{-\pi\sqrt{15}}\frac{f^6(e^{-\pi\sqrt{5/3}})}{f^6(e^{-\pi/\sqrt{15}})}.$$

Hence, from (2.2.17) and (2.2.31),

$$2u = 11 - 125\frac{1+\sqrt{5}}{10\sqrt{5}} = -\frac{3}{2} - \frac{5\sqrt{5}}{2},$$

and the remainder of the proof is exactly the same as that for Entry 2.2.4. □

By using Kronecker's limit formula, Ramanathan [218] established both Entries 2.2.4 and 2.2.5. These entries were also proved by Yi [298, Corollary 4.12], who used eta-function identities. Entry 2.2.5 was also elegantly established by N.D. Baruah [52], who used explicit values of theta functions.

Entries 2.2.4 and 2.2.5 are not explicitly stated on page 210 of the lost notebook [228]; Ramanujan merely implies that he is able to calculate the values of these two continued fractions.

2.3 General Formulas for Evaluating $R(e^{-2\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$

In this section, we derive some general formulas for evaluating the Rogers–Ramanujan continued fraction. The key relations that we shall use are (2.2.4), (2.2.5), (2.2.14), and (2.2.15). In the next section, we shall then use these formulas to complete the table on page 210 of the lost notebook. The contents of these two sections are derived from a paper by Berndt, Chan, and Zhang [73].

For the theory in this section and Section 2.5, we need to define two theta functions, the function χ, and the important class invariants G_n and g_n, upon which our theory rests. Recall from Chapter 1 that the theta functions φ and ψ are defined by

$$\varphi(q) := \sum_{k=-\infty}^{\infty} q^{k^2} = \frac{(-q;-q)_\infty}{(q;-q)_\infty} \tag{2.3.1}$$

and

$$\psi(q) := \sum_{k=0}^{\infty} q^{k(k+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}. \tag{2.3.2}$$

(The product representations for φ and ψ are consequences of the Jacobi triple product identity, Lemma 1.2.2 in Chapter 1.) Ramanujan's function χ is defined by

$$\chi(-q) = (q;q^2)_\infty. \tag{2.3.3}$$

If $q = e^{-\pi\sqrt{n}}$, where n is a rational number, then Ramanujan's class invariants (or the Ramanujan–Weber class invariants) , G_n and g_n, are defined by

$$G_n := 2^{-1/4}e^{\pi\sqrt{n}/24}\chi(e^{-\pi\sqrt{n}}) \quad \text{and} \quad g_n := 2^{-1/4}e^{\pi\sqrt{n}/24}\chi(-e^{-\pi\sqrt{n}}). \tag{2.3.4}$$

For Ramanujan's extensive contributions to class invariants, see [63, Chapter 34]. If

$$q = \exp\left(-\pi \frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-\alpha)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\alpha)} \right),$$

where ${}_2F_1$ denotes the ordinary hypergeometric function, then [61, p. 124, Entries 12(v), (vi)]

$$\chi(q) = 2^{1/6}\{\alpha(1-\alpha)/q\}^{-1/24} \quad \text{and} \quad \chi(-q) = 2^{1/6}(1-\alpha)^{1/12}(\alpha/q)^{-1/24}. \tag{2.3.5}$$

It follows from (2.3.4) and (2.3.5) that

$$G_n = \{4\alpha_n(1-\alpha_n)\}^{-1/24} \qquad \text{and} \qquad g_n = 2^{-1/12}(1-\alpha_n)^{1/12}\alpha_n^{-1/24}. \tag{2.3.6}$$

In view of (2.2.4) and (2.2.5), in order to compute $R(e^{-2\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$, it suffices to evaluate

$$A := e^{2\pi\sqrt{n}/5}\frac{f(-e^{-2\pi\sqrt{n}/5})}{f(-e^{-10\pi\sqrt{n}})} \tag{2.3.7}$$

and

$$A_1 := e^{\pi\sqrt{n}/5}\frac{f(e^{-\pi\sqrt{n}/5})}{f(e^{-5\pi\sqrt{n}})}, \tag{2.3.8}$$

respectively.

Theorem 2.3.1. *Let*

$$V := \sqrt{\frac{G_{25n}}{G_{n/25}}} \tag{2.3.9}$$

and

$$U := \sqrt{\frac{g_{25n}}{g_{n/25}}}. \tag{2.3.10}$$

(i) *If A is defined by* (2.3.7), *then*

$$\frac{A}{\sqrt{5}\,V} - \frac{\sqrt{5}\,V}{A} = (V - V^{-1})^2\left(\frac{V - V^{-1}}{\sqrt{5}} + \frac{\sqrt{5}}{V - V^{-1}}\right) \tag{2.3.11}$$

and

$$\frac{A}{\sqrt{5}\,U} + \frac{\sqrt{5}\,U}{A} = (U + U^{-1})^2\left(\frac{U + U^{-1}}{\sqrt{5}} - \frac{\sqrt{5}}{U + U^{-1}}\right). \tag{2.3.12}$$

(ii) *If A_1 is defined by* (2.3.8), *then*

$$\frac{A_1\,V}{\sqrt{5}} - \frac{\sqrt{5}}{A_1\,V} = (V - V^{-1})^2\left(\frac{V - V^{-1}}{\sqrt{5}} + \frac{\sqrt{5}}{V - V^{-1}}\right). \tag{2.3.13}$$

Proof. Let

$$q^{1/5} = \exp\left(-\pi\frac{{}_2F_1(\frac12,\frac12;1;1-\alpha)}{{}_2F_1(\frac12,\frac12;1;\alpha)}\right) \tag{2.3.14}$$

and

$$q^5 = \exp\left(-\pi\frac{{}_2F_1(\frac12,\frac12;1;1-\beta)}{{}_2F_1(\frac12,\frac12;1;\beta)}\right), \tag{2.3.15}$$

so that β is of degree 25 over α. Then [61, p. 291, Entries 15(i), (ii)]

$$\left(\frac{\beta}{\alpha}\right)^{1/8} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/8} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8} - 2\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/12} = \sqrt{m\,m'} \tag{2.3.16}$$

and

$$\left(\frac{\alpha}{\beta}\right)^{1/8}+\left(\frac{1-\alpha}{1-\beta}\right)^{1/8}-\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/8}-2\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/12}=\frac{5}{\sqrt{m\,m'}}, \tag{2.3.17}$$

where $\sqrt{m\,m'}=\varphi(q^{1/5})/\varphi(q^5)$. From (2.3.16) and (2.3.17), we deduce that, respectively,

$$\begin{aligned}&\frac{(\beta(1-\alpha))^{1/8}+(\alpha(1-\beta))^{1/8}}{(\alpha(1-\alpha))^{1/8}}\\&\qquad=\sqrt{m\,m'}+\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8}+2\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/12}\end{aligned} \tag{2.3.18}$$

and

$$\begin{aligned}&\frac{(\beta(1-\alpha))^{1/8}+(\alpha(1-\beta))^{1/8}}{(\beta(1-\beta))^{1/8}}\\&\qquad=\frac{5}{\sqrt{m\,m'}}+\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/8}+2\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/12}.\end{aligned} \tag{2.3.19}$$

Eliminating $(\beta(1-\alpha))^{1/8}+(\alpha(1-\beta))^{1/8}$ from (2.3.18) and (2.3.19), we arrive at

$$\begin{aligned}&\sqrt{m\,m'}+\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8}+2\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/12}\\&=\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8}\left(\frac{5}{\sqrt{m\,m'}}+\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/8}+2\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/12}\right).\end{aligned} \tag{2.3.20}$$

From (2.3.5), we have

$$q^{-1/120}\chi(q^{1/5})=2^{1/6}\{\alpha(1-\alpha)\}^{-1/24} \tag{2.3.21}$$

and

$$q^{-5/24}\chi(q^5)=2^{1/6}\{\beta(1-\beta)\}^{-1/24}. \tag{2.3.22}$$

Hence, we can rewrite (2.3.20) as

$$\begin{aligned}&q^{-2/5}\frac{\varphi(q^{1/5})}{\varphi(q^5)}\left(\frac{\chi(q^5)}{\chi(q^{1/5})}\right)^2+q^{1/5}\frac{\chi(q^{1/5})}{\chi(q^5)}+2\\&=q^{-1/5}\frac{\chi(q^5)}{\chi(q^{1/5})}\left(5q^{2/5}\frac{\varphi(q^5)}{\varphi(q^{1/5})}\left(\frac{\chi(q^{1/5})}{\chi(q^5)}\right)^2+q^{-1/5}\frac{\chi(q^5)}{\chi(q^{1/5})}+2\right).\end{aligned} \tag{2.3.23}$$

From the product representations of $f(-q)$, $\varphi(q)$, and $\chi(q)$ given in (2.2.2), (2.3.1), and (2.3.3), we deduce that

$$q^{-2/5}\frac{f(-q^{2/5})}{f(-q^{10})}=q^{-2/5}\frac{\varphi(q^{1/5})}{\varphi(q^5)}\left(\frac{\chi(q^5)}{\chi(q^{1/5})}\right)^2. \tag{2.3.24}$$

Substituting (2.3.24) into (2.3.23), and setting $q=e^{-\pi\sqrt{n}}$, we deduce that

$$A+V^{-2}+2=5A^{-1}V^2+V^4+2V^2, \tag{2.3.25}$$

by (2.3.4), (2.3.7), and (2.3.9). Rearranging (2.3.25), we obtain (2.3.11).

To prove (2.3.12), we first replace q by $-q$ in (2.3.23) and (2.3.24). Next, set $q=e^{-\pi\sqrt{n}}$. By (2.3.4), (2.3.7), and (2.3.10), we see that

$$A-U^{-2}+2=-5A^{-1}U^2+U^4-2U^2. \tag{2.3.26}$$

Rearranging (2.3.26), we deduce (2.3.12).

In order to prove (2.3.13), we first observe that from (2.3.7)–(2.3.9),

$$A_1=e^{\pi\sqrt{n}/5}\frac{f(e^{-\pi\sqrt{n}/5})}{f(e^{-5\pi\sqrt{n}})}=e^{2\pi\sqrt{n}/5}\frac{f(-e^{-2\pi\sqrt{n}/5})}{f(-e^{-10\pi\sqrt{n}})}V^{-2}=AV^{-2}. \tag{2.3.27}$$

Substituting (2.3.27) into (2.3.11), we arrive at (2.3.13). This completes the proof of Theorem 2.3.1. □

Proposition 2.3.1. *Let A and A_1 be defined by* (2.3.7) *and* (2.3.8), *respectively. Then*

(i) *if $2c=A+1$, then $R(e^{-2\pi\sqrt{n}})=\sqrt{c^2+1}-c$,*

(ii) *if $2c=A_1-1$, then $S(e^{-\pi\sqrt{n}})=\sqrt{c^2+1}-c$.*

Proof. Solve the quadratic equations (2.2.4) and (2.2.5). This proves the proposition. □

We now return to the fundamental relations (2.2.14) and (2.2.15). We see that in order to compute $R(e^{-2\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$, it suffices to evaluate

$$A':=e^{2\pi\sqrt{n}/6}\frac{f(-e^{-2\pi\sqrt{n}})}{f(-e^{-10\pi\sqrt{n}})} \tag{2.3.28}$$

and

$$A_1':=e^{\pi\sqrt{n}/6}\frac{f(e^{-\pi\sqrt{n}})}{f(e^{-5\pi\sqrt{n}})}, \tag{2.3.29}$$

respectively.

Theorem 2.3.2. *Let*

$$V'=\frac{G_{25n}}{G_n} \tag{2.3.30}$$

and

$$U'=\frac{g_{25n}}{g_n}. \tag{2.3.31}$$

(i) *If A' is defined by* (2.3.28), *then*

$$\frac{A'^2}{\sqrt{5}\,V'} - \frac{\sqrt{5}\,V'}{A'^2} = \frac{1}{\sqrt{5}}\left(V'^3 - V'^{-3}\right) \tag{2.3.32}$$

and

$$\frac{A'^2}{\sqrt{5}\,U'} + \frac{\sqrt{5}\,U'}{A'^2} = \frac{1}{\sqrt{5}}\left(U'^3 + U'^{-3}\right). \tag{2.3.33}$$

(ii) *If A'_1 is defined by* (2.3.29), *then*

$$\frac{A_1'^2\,V'}{\sqrt{5}} - \frac{\sqrt{5}}{A_1'^2\,V'} = \frac{1}{\sqrt{5}}\left(V'^3 - V'^{-3}\right). \tag{2.3.34}$$

Proof. Let

$$q = \exp\left(-\pi\frac{{}_2F_1(\frac12,\frac12;1;1-\alpha)}{{}_2F_1(\frac12,\frac12;1;\alpha)}\right) \tag{2.3.35}$$

and

$$q^5 = \exp\left(-\pi\frac{{}_2F_1(\frac12,\frac12;1;1-\beta)}{{}_2F_1(\frac12,\frac12;1;\beta)}\right), \tag{2.3.36}$$

so that β is of degree 5 over α. Then [61, pp. 281–282, Entry 13(xii)]

$$m = \left(\frac{\beta}{\alpha}\right)^{1/4} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/4} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4} \tag{2.3.37}$$

and

$$\frac{5}{m} = \left(\frac{\alpha}{\beta}\right)^{1/4} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/4} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/4}, \tag{2.3.38}$$

where $m = \varphi^2(q)/\varphi^2(q^5)$. From (2.3.37) and (2.3.38), we find that

$$(\alpha(1-\alpha))^{1/4}\left(m + \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4}\right) = (\beta(1-\beta))^{1/4}\left(\frac{5}{m} + \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/4}\right). \tag{2.3.39}$$

Using (2.2.2), (2.3.21) (with $q^{1/5}$ replaced by q), (2.3.22), and the resulting equality

$$\frac{f(-q^2)}{f(-q^{10})} = \frac{\varphi(q)}{\varphi(q^5)}\left(\frac{\chi(q^5)}{\chi(q)}\right)^2, \tag{2.3.40}$$

we can rewrite (2.3.39) in the form

$$\left(\frac{f(-q^2)}{f(-q^{10})}\right)^2\left(\frac{\chi(q)}{\chi(q^5)}\right)^4 - 5q\left(\frac{f(-q^{10})}{f(-q^2)}\right)^2\left(\frac{\chi(q)}{\chi(q^5)}\right)^2 = 1 - q\left(\frac{\chi(q)}{\chi(q^5)}\right)^6. \tag{2.3.41}$$

Next, set $q = e^{-\pi\sqrt{n}}$. By (2.3.41), (2.3.4), (2.3.28), and (2.3.30), we find that

$$\frac{A'^2}{V'} - 5\frac{V'}{A'^2} = V'^3 - V'^{-3}. \tag{2.3.42}$$

Rearranging, we deduce (2.3.32).

To prove (2.3.33), we simply replace q by $-q$ in (2.3.41) and set $q = e^{-\pi\sqrt{n}}$. By (2.3.4), (2.3.28), and (2.3.31), we deduce that

$$\frac{A'^2}{U'} + 5\frac{U'}{A'^2} = U'^3 + U'^{-3}, \tag{2.3.43}$$

which gives (2.3.33) after rearrangement.

Finally, we observe that by (2.3.28), (2.3.29), (2.3.30), and (2.3.4),

$$A'_1 = \frac{A'}{V'}. \tag{2.3.44}$$

Substituting (2.3.44) into (2.3.32), we deduce (2.3.34). This completes the proof of Theorem 2.3.2. □

Proposition 2.3.2. *Let A' and A'_1 be defined by* (2.3.28) *and* (2.3.29), *respectively. Then*

(i) *if* $2c = A'^6 + 11$, *then* $R^5(e^{-2\pi\sqrt{n}}) = \sqrt{c^2+1} - c$,

(ii) *if* $2c = A_1'^6 - 11$, *then* $S^5(e^{-\pi\sqrt{n}}) = \sqrt{c^2+1} - c$.

Proof. Solve the quadratic equations (2.2.14) and (2.2.15). □

2.4 Page 210 of Ramanujan's Lost Notebook

On page 210 of his lost notebook [228], Ramanujan defined $S(q)$ and constructed a table of values for $S(e^{-\pi\sqrt{n/5}})$ and $S(e^{-\pi/\sqrt{5n}})$ for odd integers n between 1 and 15. The table is incomplete, and only three of the fourteen values are actually given, namely, for $S(e^{-\pi/\sqrt{5}})$, $S(e^{-\pi/\sqrt{35}})$, and $S(e^{-\pi\sqrt{7/5}})$. In this section we complete the table using Theorem 2.3.2 and some results proved in [74].

On page 364 in his lost notebook, Ramanujan offered a reciprocity theorem for $R^5(e^{-2\alpha})$ like that given in (2.1.5). A proof is given in Entry 3.2.9 of Chapter 3. Ramanathan [215] proved an analogue for $S(q)$. If n is any positive number, then

$$\left\{\left(\frac{\sqrt{5}-1}{2}\right)^5 + S^5(e^{-\pi\sqrt{n/5}})\right\}\left\{\left(\frac{\sqrt{5}-1}{2}\right)^5 + S^5(e^{-\pi/\sqrt{5n}})\right\}$$

$$= 5\sqrt{5}\left(\frac{\sqrt{5}-1}{2}\right)^5. \tag{2.4.1}$$

In view of (2.4.1), it suffices to evaluate either $S(e^{-\pi\sqrt{n/5}})$ or $S(e^{-\pi/\sqrt{5n}})$. We choose to compute the former.

Entry 2.4.1 (p. 210). *We have*

$$S^5(e^{-\pi/\sqrt{5}}) = \sqrt{\left(\frac{5\sqrt{5}-11}{2}\right)^2+1} - \frac{5\sqrt{5}-11}{2}.$$

Proof. Let $n = 1/5$ in (2.3.30). Then, since $G_n = G_{1/n}$,

$$V' = \frac{G_5}{G_{1/5}} = 1.$$

By Theorem 2.3.2(ii), we have

$$A_1'^2 = \sqrt{5}. \tag{2.4.2}$$

We complete the proof upon substituting (2.4.2) into Proposition 2.3.2(ii). □

Entry 2.4.1 was first proved by Ramanathan [217]. After the proof of Berndt, Chan, and Zhang [73], a third proof was found by Yi [298]. A fourth proof was given by Baruah [52].

Second Proof of Entry 2.2.5. Let $n = 3/5$ in Theorem 2.3.2(ii). From Weber's table [291, p. 721] or from [63, p. 190], we have

$$G_{15} = 2^{-1/12}(1+\sqrt{5})^{1/3}.$$

Using one of Ramanujan's modular equations of degree 5 [61, p. 282, Entry 13(xiv)], we deduce that

$$G_{3/5} = 2^{-1/12}(\sqrt{5}-1)^{1/3}.$$

Hence,

$$V' = \left(\frac{\sqrt{5}+1}{\sqrt{5}-1}\right)^{1/3} \tag{2.4.3}$$

and

$$V'^3 - V'^{-3} = \sqrt{5}. \tag{2.4.4}$$

Substituting (2.4.3) and (2.4.4) into (2.3.34) and solving for A_1', we deduce that

$$A_1'^6 = \frac{25+5\sqrt{5}}{2}.$$

We may now complete the proof using Proposition 2.3.2(ii). □

Note that for $n = 5$, the value of $S(e^{-\pi})$ is given by (2.1.4).

We cannot deduce the value of $S^5(e^{-\pi\sqrt{7/5}})$ from Theorem 2.3.2, since we do not have a simple expression for $G_{35}/G_{7/5}$. However, recall that we established its value in Entry 2.2.3.

Entry 2.4.2 (p. 210). *Let* $a = 2\sqrt{15}$ *and* $b = 3\sqrt{5} - 1$. *If*

$$2c = \frac{a+b}{a-b} 5\sqrt{5} - 11,$$

then

$$S^5(e^{-\pi\sqrt{9/5}}) = \sqrt{c^2+1} - c.$$

Proof. Let $n = 9/5$ in Theorem 2.3.2(ii). From [73, Theorem 1], we deduce that

$$V' = \left(\frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}}\right)^{1/3}. \tag{2.4.5}$$

Hence,

$$V'^3 - V'^{-3} = \frac{\sqrt{5}+\sqrt{3}}{\sqrt{5}-\sqrt{3}} - \frac{\sqrt{5}-\sqrt{3}}{\sqrt{5}+\sqrt{3}} = 2\sqrt{15}. \tag{2.4.6}$$

Substituting (2.4.5) and (2.4.6) into Theorem 2.3.2(ii), we find that

$$A_1'^6 = \frac{2\sqrt{15}+3\sqrt{5}-1}{2\sqrt{15}-3\sqrt{5}+1} 5\sqrt{5},$$

after some simplification. Thus, by Proposition 2.3.2(ii), we deduce Entry 2.4.2. □

Entry 2.4.3 (p. 210). *If*

$$A_1'^2 = \frac{\sqrt{3\sqrt{5}+7} - \sqrt{3\sqrt{5}-1}}{\sqrt{9\sqrt{5}+27} - \sqrt{9\sqrt{5}+19}} \sqrt{5}$$

and $2c = A_1'^6 - 11$, *then*

$$S^5(e^{-\pi\sqrt{11/5}}) = \sqrt{c^2+1} - c.$$

Proof. It is known that [63, p. 192]

$$G_{55} = 2^{1/4}(\sqrt{5}+2)^{1/6}\left(\sqrt{\frac{7+\sqrt{5}}{8}} + \sqrt{\frac{\sqrt{5}-1}{8}}\right). \tag{2.4.7}$$

Using (2.4.7) along with one of Ramanujan's modular equations of degree 5 [61, p. 282, Entry 13(xiv)], we find that

$$G_{11/5} = 2^{1/4}(\sqrt{5}+2)^{1/6}\left(\sqrt{\frac{7+\sqrt{5}}{8}} - \sqrt{\frac{\sqrt{5}-1}{8}}\right). \tag{2.4.8}$$

Recall that V' is defined by (2.3.30). Hence, by (2.3.30) with $n = 11/5$, (2.4.7), and (2.4.8),

$$V' = \frac{\sqrt{7+\sqrt{5}} + \sqrt{\sqrt{5}-1}}{\sqrt{7+\sqrt{5}} - \sqrt{\sqrt{5}-1}} = \sqrt{\frac{3\sqrt{5}+7}{8}} + \sqrt{\frac{3\sqrt{5}-1}{8}} \tag{2.4.9}$$

and

$$V' - V'^{-1} = \sqrt{\frac{3\sqrt{5}-1}{2}}. \tag{2.4.10}$$

Now, by (2.4.10), we have

$$\begin{aligned} \frac{1}{\sqrt{5}}\left(V'^3 - V'^{-3}\right) &= \frac{1}{\sqrt{5}}\left(V' - V'^{-1}\right)\left(\left(V' - V'^{-1}\right)^2 + 3\right) \\ &= \frac{1}{\sqrt{5}}\left(\sqrt{\frac{3\sqrt{5}-1}{2}}\right)\frac{3\sqrt{5}+5}{2} \\ &= \sqrt{\frac{19+9\sqrt{5}}{2}}. \end{aligned} \tag{2.4.11}$$

Substituting (2.4.9) and (2.4.11) into Theorem 2.3.2(ii) and simplifying, we deduce Entry 2.4.3. □

Entry 2.4.4 (p. 210). *If*

$$A_1'^2 = \frac{\sqrt{\sqrt{65}+7} - \sqrt{\sqrt{65}-1}}{\sqrt{\sqrt{65}+9} - \sqrt{\sqrt{65}+7}}\frac{\sqrt{5}}{2}$$

and $2c = A_1'^6 - 11$, *then*

$$S^5(e^{-\pi\sqrt{13/5}}) = \sqrt{c^2+1} - c.$$

Proof. From [74] or from [63, p. 192],

$$V' = \frac{G_{65}}{G_{13/5}} = \sqrt{\frac{\sqrt{65}+7}{8}} + \sqrt{\frac{\sqrt{65}-1}{8}}. \tag{2.4.12}$$

Using calculations similar to those in the proof of Entry 2.4.3, we deduce Entry 2.4.4. □

To complete Ramanujan's table mentioned in the beginning of this section, it remains to evaluate $S(e^{-\pi\sqrt{3}})$. We determined $S(e^{-\pi\sqrt{3}})$ in Entry 2.2.1, but we cannot deduce the value from Theorem 2.3.2(ii), because we do not know the requisite class invariants.

It is likely that Ramanujan has a misprint in his very last entry on page 210, for he asks for the value of $S(e^{-\pi\sqrt{3/25}})$. The previously listed unrecorded value is for $S(e^{-\pi/\sqrt{75}})$, which would imply that the companion value is for $S(e^{-\pi\sqrt{3}})$. The value of $S(e^{-\pi\sqrt{3}/5})$ can indeed be determined by using the value of $S(e^{-\pi\sqrt{3}})$ from Entry 2.2.1 along with a famous modular equation connecting $R(q^5)$ with $R(q)$ found in Entry 14 on page 365 of Ramanujan's lost notebook [228], or in his second notebook [61, pp. 19–20]. We do not record the value here, because it is not particularly elegant.

Berndt, Chan, and Zhang [73] also determined the values of $S(e^{-\pi\sqrt{29/5}})$, $S(e^{-\pi\sqrt{41/5}})$, $S(e^{-\pi\sqrt{53/5}})$, and $S(e^{-\pi\sqrt{101/5}})$. Chan and V. Tan [118] determined $S(e^{-\pi\sqrt{11}})$ and $S(e^{-\pi\sqrt{19}})$ using modular equations satisfied by $R(q)$ and $R(q^n)$ for $n = 11$ and 19, respectively. Yi [298], also using modular equations, determined values, among others, for $R(e^{-\pi})$, $R(e^{-\pi/2})$, $R(e^{-2\pi/3})$, $R(e^{-2\pi/5})$, $R(e^{-2\pi\sqrt{7/5}})$, $R(e^{-2\pi/\sqrt{35}})$, $S(e^{-\pi/5})$, and $S(e^{-\sqrt{3}\pi/9})$.

S.–Y. Kang [172] has recorded a table of all known values of the Rogers–Ramanujan continued fraction up until the time her paper was written in 1999.

The most extensive computations of $R(e^{-\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$ were made by Yi [297] in her doctoral dissertation; see also her paper [298]. She not only found different proofs for most of the evaluations in the lost notebook, but she also explicitly determined many new values as well, as we indicated above. Her proofs rest on a systematic exploitation of eta-function identities, several of which are originally due to her.

Baruah [52] has also found several values for $R(e^{-\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$; his proofs are somewhat different from those cited above, in that he primarily used values of theta functions. K.R. Vasuki and M.S. Mahadeva Naika [280] used values of quotients of eta functions to determine several values of $R(q)$ and $S(q)$.

Observant readers will have noticed that the values of $R(q)$ and $S(q)$ that we have established in this chapter are units. Indeed, Berndt, Chan, and Zhang [73] have proved that for any rational number n, $R(e^{-\pi\sqrt{n}})$ and $S(e^{-\pi\sqrt{n}})$ are units.

2.5 Some Theta-Function Identities

Entry 2.5.1 (p. 46). *Let*

$$t_1 := t_1(q) := q^{1/6}\frac{\chi(-q)}{\chi(-q^5)} \qquad \textit{and} \qquad s_1 := s_1(q) := \frac{\varphi(-q)}{\varphi(-q^5)}. \tag{2.5.1}$$

Then

$$\text{(i)}\quad \frac{f(-q)}{q^{1/6}f(-q^5)}=\frac{s_1}{t_1},\quad \text{(ii)}\quad \frac{f(-q^2)}{q^{1/3}f(-q^{10})}=\frac{s_1}{t_1^2},\quad \text{(iii)}\quad \frac{\psi(q)}{\sqrt{q}\psi(q^5)}=\frac{s_1}{t_1^3},$$

$$\text{(iv)}\quad s_1^2=\frac{1}{2}\left((1+t_1^6)+\sqrt{(1+t_1^6)^2-20t_1^6}\right).$$

Instead of (iv), Ramanujan actually stated

$$s_1=\frac{1}{2}\left(\sqrt{1+2\sqrt{5}t_1^3+t_1^6}+\sqrt{1-2\sqrt{5}t_1^3+t_1^6}\right) \tag{2.5.2}$$

with a slight misprint. But in applications, it is more convenient to use the equality in (iv) instead of (2.5.2).

Proof of (i). Set $t=t_1$ and $s=s_1$ throughout the proof. By (2.3.3), we have

$$t=q^{1/6}\frac{(q;q^2)_\infty}{(q^5;q^{10})_\infty}. \tag{2.5.3}$$

Using the definition of $f(-q)$ in (2.2.2), Euler's identity,

$$(-q;q)_\infty=\frac{1}{(q;q^2)_\infty}, \tag{2.5.4}$$

(2.3.1), and (2.5.3), we deduce that

$$\begin{aligned}\frac{f(-q)}{q^{1/6}f(-q^5)}&=\frac{(q;q)_\infty}{q^{1/6}(q^5;q^5)_\infty}\\&=\frac{(q;q)_\infty}{(-q;q)_\infty}\frac{(-q^5;q^5)_\infty}{(q^5;q^5)_\infty}\frac{(q^5;q^{10})_\infty}{q^{1/6}(q;q^2)_\infty}\\&=\frac{\varphi(-q)}{\varphi(-q^5)}\frac{\chi(-q^5)}{q^{1/6}\chi(-q)}=\frac{s}{t},\end{aligned}$$

which completes the proof of (i). ∎

Proof of (ii). Using in turn (2.2.2), Euler's identity (2.5.4), (2.3.1), and (2.5.3), we find that

$$\begin{aligned}\frac{f(-q^2)}{q^{1/3}f(-q^{10})}&=\frac{(q^2;q^2)_\infty}{q^{1/3}(q^{10};q^{10})_\infty}\\&=\frac{(q;q)_\infty}{(-q;q)_\infty}\frac{(-q^5;q^5)_\infty}{(q^5;q^5)_\infty}\frac{(q^5;q^5)_\infty(q^5;q^{10})_\infty}{(q;q)_\infty(q;q^2)_\infty}\frac{(q^2;q^2)_\infty}{q^{1/3}(q^{10};q^{10})_\infty}\\&=\frac{\varphi(-q)}{\varphi(-q^5)}\frac{(q^5;q^{10})_\infty^2}{q^{1/3}(q;q^2)_\infty^2}=\frac{s}{t^2}.\end{aligned}$$

∎

Proof of (iii). Applying (2.3.2), Euler's identity (2.5.4), (2.3.1), and (2.5.3), we have

$$\begin{aligned}
\frac{\psi(q)}{q^{1/2}\psi(q^5)} &= \frac{(q^2;q^2)_\infty}{q^{1/2}(q;q^2)_\infty}\frac{(q^5;q^{10})_\infty}{(q^{10};q^{10})_\infty}\\
&= \frac{(q;q)_\infty}{(-q;q)_\infty}\frac{(-q^5;q^5)_\infty}{(q^5;q^5)_\infty}\frac{(q^5;q^5)_\infty(q^5;q^{10})_\infty}{(q;q)_\infty(q;q^2)_\infty}\\
&\quad\times\frac{(q^2;q^2)_\infty}{q^{1/2}(q;q^2)_\infty}\frac{(q^5;q^{10})_\infty}{(q^{10};q^{10})_\infty}\\
&= \frac{\varphi(-q)}{\varphi(-q^5)}\frac{(q^5;q^{10})^3_\infty}{q^{1/2}(q;q^2)^3_\infty} = \frac{s}{t^3}.
\end{aligned}$$

□

Proof of (iv). In the sequel, set

$$P_1 := P_1(q) := \frac{f(-q)}{q^{1/6}f(-q^5)} \quad \text{and} \quad Q_1 := Q_1(q) := \frac{f(-q^2)}{q^{1/3}f(-q^{10})}. \tag{2.5.5}$$

Recall another eta-function identity of Ramanujan [62, p. 206, Entry 53],

$$P_1Q_1 + \frac{5}{P_1Q_1} = \left(\frac{P_1}{Q_1}\right)^3 + \left(\frac{Q_1}{P_1}\right)^3.$$

Since $P_1 = s/t$ and $Q_1 = s/t^2$ by (i) and (ii), respectively, the equation above can be simplified to

$$s^4 - (1+t^6)s^2 + 5t^6 = 0.$$

Thus (iv) follows immediately from the equation above by an application of the quadratic formula. □

Theorem 2.5.1. *Let*

$$t_2 := t_2(q) := q^{1/5}\frac{\chi(-q^{1/5})}{\chi(-q^5)} \qquad \textit{and} \qquad s_2 := s_2(q) := \frac{\varphi(-q^{1/5})}{\varphi(-q^5)}.$$

Then

$$\text{(i)}\quad \frac{f(-q^{1/5})}{q^{1/5}f(-q^5)} = \frac{s_2}{t_2}, \quad \text{(ii)}\quad \frac{f(-q^{2/5})}{q^{2/5}f(-q^{10})} = \frac{s_2}{t_2^2}, \quad \text{(iii)}\quad \frac{\psi(q^{1/5})}{q^{3/5}\psi(q^5)} = \frac{s_2}{t_2^3},$$

$$\text{(iv)}\quad s_2 = \frac{1-2t_2-2t_2^2+t_2^3+\sqrt{1-4t_2-10t_2^3-4t_2^5+t_2^6}}{2}.$$

Proof. Set $t = t_2$ and $s = s_2$ throughout the proof. From (2.3.3), we find that

$$t = q^{1/5}\frac{(q^{1/5};q^{2/5})_\infty}{(q^5;q^{10})_\infty}. \tag{2.5.6}$$

The proofs of (i), (ii), and (iii) are similar to those of Entry 2.5.1(i), (ii), and (iii), respectively.

To prove (iv), set

$$P_2 := P_2(q) = \frac{f(-q^{1/5})}{q^{1/5}f(-q^5)} \qquad \text{and} \qquad Q_2 := \frac{f(-q^{2/5})}{q^{2/5}f(-q^{10})}. \tag{2.5.7}$$

Then by another eta-function identity of Ramanujan [62, p. 212, Entry 58], we find that

$$P_2Q_2 + \frac{25}{P_2Q_2} = \left(\frac{Q_2}{P_2}\right)^3 - 4\left(\frac{Q_2}{P_2}\right)^2 - 4\left(\frac{P_2}{Q_2}\right)^2 + \left(\frac{P_2}{Q_2}\right)^3. \tag{2.5.8}$$

Since $P_2 = s/t$ and $Q_2 = s/t^2$ from (i) and (ii), respectively, we find that

$$\frac{s^2}{t^3} + 25\frac{t^3}{s^2} = \frac{1}{t^3} - 4\frac{1}{t^2} - 4t^2 + t^3. \tag{2.5.9}$$

Multiply both sides of (2.5.9) by s^2t^3 to deduce that

$$s^4 - (1 - 4t - 4t^5 + t^6)s^2 + 25t^6 = 0. \tag{2.5.10}$$

The solutions of this equation in s are given by

$$s = \frac{1 - 2t - 2t^2 + t^3 \pm \sqrt{1 - 4t - 10t^3 - 4t^5 + t^6}}{2}$$

and

$$s = \frac{-1 + 2t + 2t^2 - t^3 \pm \sqrt{1 - 4t - 10t^3 - 4t^5 + t^6}}{2}.$$

But since t and s approach 0 and 1, respectively, as q approaches 0, the appropriate solution for s is

$$s = \frac{1 - 2t - 2t^2 + t^3 + \sqrt{1 - 4t - 10t^3 - 4t^5 + t^6}}{2}.$$

This completes the proof. □

Theorem 2.5.1 is analogous to Entry 2.5.1, but it evidently was not stated anywhere by Ramanujan. We shall use Theorem 2.5.1 to prove one of Ramanujan's formulas in the next section.

Kang [172] showed that Entry 2.5.1 and Theorem 2.5.1 can be utilized to give alternative proofs of the main theorems in Section 2.3. She also showed that these theorems easily lead to the explicit evaluations of certain quotients of theta functions.

2.6 Ramanujan's General Explicit Formulas for the Rogers–Ramanujan Continued Fraction

Entry 2.6.1 (p. 208). *Let* t_2 *be given in Theorem* 2.5.1. *Then*

$$\text{(i)} \qquad R(q) = \frac{1}{4t_2}\left(\left(1+t_2\frac{\sqrt{5}+1}{2}\right)\sqrt{1-t_2}\right.$$
$$\left. -\sqrt{(1-t_2)\left(1+t_2\frac{\sqrt{5}+1}{2}\right)^2 - 2t_2(\sqrt{5}+1)}\right)$$
$$\times\left(-\left(1-t_2\frac{\sqrt{5}-1}{2}\right)\sqrt{1-t_2}\right.$$
$$\left. +\sqrt{(1-t_2)\left(1-t_2\frac{\sqrt{5}-1}{2}\right)^2 + 2t_2(\sqrt{5}-1)}\right),$$

$$\text{(ii)} \qquad R(q^2) = \frac{1}{4t_2^2}\left(\left(1-t_2\frac{\sqrt{5}+1}{2}\right)\sqrt{1-t_2}\right.$$
$$\left. -\sqrt{(1-t_2)\left(1+t_2\frac{\sqrt{5}+1}{2}\right)^2 - 2t_2(\sqrt{5}+1)}\right)$$
$$\times\left(-\left(1+t_2\frac{\sqrt{5}-1}{2}\right)\sqrt{1-t_2}\right.$$
$$\left. +\sqrt{(1-t_2)\left(1-t_2\frac{\sqrt{5}-1}{2}\right)^2 + 2t_2(\sqrt{5}-1)}\right).$$

Proof of (i). Set $t = t_2$ throughout the proof. From (2.2.4) and Theorem 2.5.1(i), (iv), we have

$$\frac{1}{R(q)} - 1 - R(q) = \frac{1-2t-2t^2+t^3+\sqrt{1-4t-10t^3-4t^5+t^6}}{2t},$$

which is equivalent to

$$\frac{1}{R(q)} - R(q) = \frac{1-2t^2+t^3+\sqrt{1-4t-10t^3-4t^5+t^6}}{2t}. \tag{2.6.1}$$

Motivated by the fact that $R(q)$ is a unit when $q = e^{-\pi\sqrt{n}}$ [73], let us assume that $R(q)$ can be written as a product of two expressions of the form

$$R(q) = (\sqrt{a+1} - \sqrt{a})(\sqrt{b} - \sqrt{b-1}). \tag{2.6.2}$$

Then

$$\frac{1}{R(q)} - R(q) = 2\left(\sqrt{ab} + \sqrt{(a+1)(b-1)}\right). \tag{2.6.3}$$

From (2.6.1) and (2.6.3), we may set

$$\sqrt{ab} = \sqrt{\frac{(1-2t^2+t^3)^2}{16t^2}} \tag{2.6.4}$$

and

$$\sqrt{(a+1)(b-1)} = \sqrt{\frac{1-4t-10t^3-4t^5+t^6}{16t^2}}. \tag{2.6.5}$$

Solving (2.6.4) and (2.6.5) yields

$$a = \left(\frac{\sqrt{5}+1}{2}\right)\left(\frac{1-t}{4t}\right)\left(1-t\frac{\sqrt{5}-1}{2}\right)^2 \tag{2.6.6}$$

and

$$b = \left(\frac{\sqrt{5}-1}{2}\right)\left(\frac{1-t}{4t}\right)\left(1+t\frac{\sqrt{5}+1}{2}\right)^2. \tag{2.6.7}$$

Hence (i) follows from (2.6.2), (2.6.6), and (2.6.7). □

Proof of (ii). The proof of the formula for $R(q^2)$ is similar to that for $R(q)$. By (2.2.4) and Theorem 2.5.1(ii), (iv),

$$\frac{1}{R(q^2)} - R(q^2) = \frac{1-2t+t^3+\sqrt{1-4t-10t^3-4t^5+t^6}}{2t^2}. \tag{2.6.8}$$

As before, let

$$R(q^2) = (\sqrt{a+1} - \sqrt{a})(\sqrt{b} - \sqrt{b-1}). \tag{2.6.9}$$

Then

$$\frac{1}{R(q^2)} - R(q^2) = 2\left(\sqrt{ab} + \sqrt{(a+1)(b-1)}\right). \tag{2.6.10}$$

From (2.6.8) and (2.6.10), we may set

$$\sqrt{ab} = \sqrt{\frac{(1-2t+t^3)^2}{16t^4}} \tag{2.6.11}$$

and

$$\sqrt{(a+1)(b-1)} = \sqrt{\frac{1-4t-10t^3-4t^5+t^6}{16t^4}}. \tag{2.6.12}$$

Then solving (2.6.11) and (2.6.12), we deduce that

$$a = \left(\frac{\sqrt{5}+1}{2}\right)\left(\frac{1-t}{4t^2}\right)\left(1+t\frac{\sqrt{5}-1}{2}\right)^2 \tag{2.6.13}$$

and

$$b = \left(\frac{\sqrt{5}-1}{2}\right)\left(\frac{1-t}{4t^2}\right)\left(1-t\frac{\sqrt{5}+1}{2}\right)^2. \tag{2.6.14}$$

We complete the proof by utilizing (2.6.13) and (2.6.14) in (2.6.9). □

Kang used Entry 2.6.1 to determine $R(e^{-2\pi})$ and $S(e^{-\pi})$, but even for these two simple values, the computations are quite laborious. Thus, it does not appear that this theorem is very useful for finding explicit values.

In his notebooks [227, p. 362], Ramanujan introduced a parameter n and recorded some beautiful modular equations involving n [39, Entry 24], [63, Entry 1, pp. 12–13]. Ramanujan returns to this parameter in the lost notebook but uses k instead of n. The parameter k is defined by

$$k = R(q)R^2(q^2). \tag{2.6.15}$$

In the next entry, we give Ramanujan's formulas for k and $(1-k)/(1+k)$ in terms of the function χ. See Section 1.8 of Chapter 1 for many further identities involving k.

Entry 2.6.2 (p. 208). *Let t_1 be given in Entry* 2.5.1*, and let k be defined by* (2.6.15)*. Then*

$$R(q) = k^{1/5}\left(\frac{1-k}{1+k}\right)^{2/5} \qquad \text{and} \qquad R(q^2) = k^{2/5}\left(\frac{1+k}{1-k}\right)^{1/5}. \tag{2.6.16}$$

Furthermore,

$$\begin{aligned} k = \frac{1}{4t_1^6}&\left(\sqrt{1-t_1^6} - \sqrt{1-t_1^6\left(\frac{\sqrt{5}+1}{2}\right)^6}\right) \\ &\times\left(\sqrt{1-t_1^6\left(\frac{\sqrt{5}-1}{2}\right)^6} - \sqrt{1-t_1^6}\right) \end{aligned} \tag{2.6.17}$$

and

$$\begin{aligned} \frac{1-k}{1+k} = \frac{1}{4}&\left(\sqrt{\left(\frac{\sqrt{5}+1}{2}\right)^6 - t_1^6} - \sqrt{1-t_1^6}\right) \\ &\times\left(\sqrt{\left(\frac{\sqrt{5}-1}{2}\right)^6 - t_1^6} + \sqrt{1-t_1^6}\right). \end{aligned} \tag{2.6.18}$$

Proof. For brevity, we set $t = t_1$ throughout the proof. Equalities (2.6.16) are the identities of Ramanujan to which we alluded above and were first proved by Andrews, Berndt, Jacobsen, and Lamphere [39, Entry 24], [63, Entry 1, pp. 12–13]. So it suffices to prove (2.6.17) and (2.6.18).

Utilizing (2.6.16) in Lemma 2.2.3, we see that

$$\begin{aligned} \frac{f^6(-q)}{qf^6(-q^5)} &= \frac{1}{k}\left(\frac{1+k}{1-k}\right)^2 - 11 - k\left(\frac{1-k}{1+k}\right)^2 \\ &= \left(\frac{1+k-k^2}{k}\right)\left(\frac{1-4k-k^2}{1-k^2}\right)^2 . \end{aligned}$$

Hence, by Entry 2.5.1(i),

$$\frac{s_1^6}{t^6} = \left(\frac{1+k-k^2}{k}\right)\left(\frac{1-4k-k^2}{1-k^2}\right)^2,$$

or

$$t^6 = \left(\frac{k}{1+k-k^2}\right)\left(\frac{1-k^2}{1-4k-k^2}\right)^2\left(\frac{\varphi(-q)}{\varphi(-q^5)}\right)^6 .$$

But from another entry of the lost notebook [228, p. 56], established in Entry 1.8.2 of Chapter 1,

$$\frac{\varphi^2(-q)}{\varphi^2(-q^5)} = \frac{1-4k-k^2}{1-k^2},$$

we obtain

$$t^6 = \frac{k(1-4k-k^2)}{(1-k^2)(1+k-k^2)}. \tag{2.6.19}$$

Rearranging (2.6.19), we find that

$$t^6k^4 + (1-t^6)k^3 + (4-2t^6)k^2 - (1-t^6)k + t^6 = 0,$$

which can be expressed as

$$t^6\left(\frac{1}{k}-k\right)^2 - (1-t^6)\left(\frac{1}{k}-k\right) + 4 = 0. \tag{2.6.20}$$

By the quadratic formula,

$$\frac{1}{k} - k = 2\left(\frac{(1-t^6)+\sqrt{t^{12}-18t^6+1}}{4t^6}\right). \tag{2.6.21}$$

As in the proof of Entry 2.6.1, let

$$k = (\sqrt{a+1}-\sqrt{a})(\sqrt{b}-\sqrt{b-1}). \tag{2.6.22}$$

Then

$$\frac{1}{k} - k = 2\left(\sqrt{ab} + \sqrt{(a+1)(b-1)}\right).$$

Comparing this with (2.6.21), we may set

$$\sqrt{ab} = \sqrt{\frac{(1-t^6)^2}{16t^{12}}}$$

and

$$\sqrt{(a+1)(b-1)} = \sqrt{\frac{t^{12} - 18t^6 + 1}{16t^{12}}}.$$

Hence we can conclude that

$$a = \left(\frac{1+\sqrt{5}}{2}\right)^3 \frac{1-t^6}{4t^6} \tag{2.6.23}$$

and

$$b = \left(\frac{\sqrt{5}-1}{2}\right)^3 \frac{1-t^6}{4t^6}. \tag{2.6.24}$$

Formula (2.6.17) now follows from (2.6.22), (2.6.23), and (2.6.24).

We can establish (2.6.18) in a similar way. Let $u = (1-k)/(1+k)$. Substituting $k = (1-u)/(1+u)$ in (2.6.20), we find that

$$4t^6 - (1-t^6)\left(\frac{1}{u} - u\right) + \left(\frac{1}{u} - u\right)^2 = 0,$$

and hence, by the quadratic formula,

$$\frac{1}{u} - u = 2\left(\frac{(1-t^6) + \sqrt{t^{12} - 18t^6 + 1}}{4}\right). \tag{2.6.25}$$

Proceeding as in the proof above, if we set

$$u = (\sqrt{a+1} - \sqrt{a})(\sqrt{b} - \sqrt{b-1}), \tag{2.6.26}$$

then

$$\frac{1}{u} - u = 2\left(\sqrt{ab} + \sqrt{(a+1)(b-1)}\right).$$

By (2.6.25), we then see that we may take

$$\sqrt{ab} = \sqrt{\frac{(1-t^6)^2}{16}}$$

and

$$\sqrt{(a+1)(b-1)} = \sqrt{\frac{t^{12} - 18t^6 + 1}{16}}.$$

Solving these identities, we deduce that

$$a = \left(\frac{\sqrt{5}-1}{2}\right)^3 \frac{1-t^6}{4}$$

and

$$b = \left(\frac{1+\sqrt{5}}{2}\right)^3 \frac{1-t^6}{4}.$$

We complete the proof of (2.6.18) by substituting these values into (2.6.26). ☐

Second Proof of (2.6.17). We recall from Entry 1.8.5 in Chapter 1 two identities from the lost notebook [228, p. 53]. If $k \leq \sqrt{5} - 2$, then

$$\frac{k}{1-k^2} \left(\frac{1+k-k^2}{1-4k-k^2}\right)^5 = q(-q;q)_\infty^{24} \tag{2.6.27}$$

and

$$\left(\frac{k}{1-k^2}\right)^5 \frac{1+k-k^2}{1-4k-k^2} = q^5(-q^5;q^5)_\infty^{24}. \tag{2.6.28}$$

Divide (2.6.28) by (2.6.27) to deduce that

$$\left(\frac{k(1-4k-k^2)}{(1-k^2)(1+k-k^2)}\right)^4 = \left(q^{1/6} \frac{(-q^5;q^5)_\infty}{(-q;q)_\infty}\right)^{24}.$$

Taking fourth roots of both sides yields (2.6.19). The remainder of the proof is the same as above. ☐

3

A Fragment on the Rogers–Ramanujan and Cubic Continued Fractions

3.1 Introduction

Published with Ramanujan's lost notebook [228, pp. 363–366] is a fragment entitled "Additional Results." This fragment comprises a summary of some of Ramanujan's theorems on the Rogers–Ramanujan and cubic continued fractions. Most likely, these results were compiled before Ramanujan left India in March 1914, or shortly after he arrived in Cambridge. Most of the theorems can be found in Ramanujan's notebooks, but four of them have evidently not been proved in print before. On the last page, after stating several theorems on the cubic continued fraction, Ramanujan wrote, "...and many results analogous to the previous continued fraction." Evidently, Ramanujan implied that there exists a theory for the cubic continued fraction that parallels that for the Rogers–Ramanujan continued fraction. Motivated by Ramanujan's declaration, H.H. Chan [112] developed a beautiful theory for the cubic continued fraction.

In this chapter we shall state all of the theorems contained in this fragment, provide citations to the literature where proofs of the known theorems can be found, give proofs for the aforementioned new theorems, relate most of Chan's paper [112], and describe some explicit evaluations of the cubic continued fraction from a paper by Berndt, Chan, and L.–C. Zhang [72]. Many further evaluations of the cubic continued fraction can be found in J. Yi's doctoral dissertation [297, Chapter 6].

We shall shorten the statements of Ramanujan's claims by introducing notation and employing summation notation. Define three versions of the Rogers–Ramanujan continued fraction by

$$R(q) := \frac{q^{1/5}}{1} \, \underset{+}{} \, \frac{q}{1} \, \underset{+}{} \, \frac{q^2}{1} \, \underset{+}{} \, \frac{q^3}{1} \, \underset{+}{} \, \underset{\cdots}{}, \qquad |q| < 1, \tag{3.1.1}$$

$$F(q) := q^{-1/5}R(q), \qquad \text{and} \qquad C(q) := 1/F(q). \tag{3.1.2}$$

The famous Rogers–Ramanujan functions $G(q)$ and $H(q)$ are defined by

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} \qquad \text{and} \qquad H(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n}, \tag{3.1.3}$$

where, as customary,

$$(a;q)_n := \prod_{k=0}^{n-1}(1-aq^k) \qquad \text{and} \qquad (a;q)_\infty = \lim_{n\to\infty}(a;q)_n, \qquad |q|<1.$$

The closely related functions $G_1(q)$ and $H_1(q)$ are defined by

$$G_1(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^4;q^4)_n} \qquad \text{and} \qquad H_1(q) := \sum_{n=0}^{\infty} \frac{q^{n(n+2)}}{(q^4;q^4)_n}. \tag{3.1.4}$$

We shall follow Ramanujan's lead and define

$$f(-q) := (q;q)_\infty. \tag{3.1.5}$$

The cubic continued fraction $G(q)$ is defined by

$$G(q) = \frac{q^{1/3}}{1} \, {}_{+} \, \frac{q+q^2}{1} \, {}_{+} \, \frac{q^2+q^4}{1} \, {}_{+} \, \frac{q^3+q^6}{1} \, {}_{+} \cdots, \qquad |q|<1. \tag{3.1.6}$$

The notation in (3.1.6) conflicts with that in (3.1.3), but there should be no cause for confusion in the sequel. The notation (3.1.3) is used in the first two sections of this chapter, while the notation (3.1.6) is used only in the last two sections of the chapter.

3.2 The Rogers–Ramanujan Continued Fraction

Entry 3.2.1 (p. 363). *With* $G_1(q)$ *and* $H_1(q)$ *defined by* (3.1.4) *and* $F(q)$ *defined by* (3.1.2),

$$\frac{G_1(q)}{H_1(q)} = F(q).$$

With the use of the Rogers–Ramanujan functions $G(q)$ and $H(q)$ and the identities given in the next two entries, Entry 3.2.1 translates into a very famous theorem initially proved by L.J. Rogers [234]. It is found in Ramanujan's notebooks as Entry 38(iii) in Chapter 16 [227], [61, p. 79]. There now exist many proofs of Entry 3.2.1; for references see [61, pp. 30–31, 79].

We next offer the Rogers–Ramanujan identities in two forms. In the first entry, which is not found in this fragment but which is found nearby on page 347, they are presented as they usually are written. The formulations in the second entry are found in the fragment.

Entry 3.2.2 (Rogers–Ramanujan Identities; p. 347). *If $G(q)$ and $H(q)$ are defined by* (3.1.3), *then*

$$G(q) = \frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty} \qquad \textit{and} \qquad H(q) = \frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty}.$$

Entry 3.2.3 (p. 363). *With $G_1(q)$ and $H_1(q)$ defined by* (3.1.4),

$$G_1(q) = \frac{(q^2;q^4)_\infty}{(q;q^5)_\infty(q^4;q^5)_\infty} \qquad \textit{and} \qquad H_1(q) = \frac{(q^2;q^4)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}.$$

When he recorded these identities, Ramanujan did not possess proofs, since he prefaces their statements with the words [228, p. 363], "I have found empirically that." Rogers [234] proved that

$$G_1(q) = (q^2;q^4)_\infty G(q) \qquad \text{and} \qquad H_1(q) = (q^2;q^4)_\infty H(q). \tag{3.2.1}$$

Fortunately, Ramanujan preserved his empirical thoughts, and an account of them is the subject of Chapter 10 of this volume. The history of these famous identities is now well known; see, for example, Hardy's book [148, pp. 90–99], Andrews's text [21, Chapter 7], or Berndt's book [61, pp. 77–79]. Many proofs of the identities now exist; a description and classification of all known proofs up to 1989 can be found in Andrews's paper [30]. It is interesting that only in this fragment did Ramanujan express Entries 3.2.1 and 3.2.3 in terms of $G_1(q)$ and $H_1(q)$. Elsewhere, Ramanujan expressed versions of Entries 3.2.1 and 3.2.3 in terms of $G(q)$ and $H(q)$. Clearly, Ramanujan had also discovered (3.2.1).

Entry 3.2.4 (p. 363). *If $C(q)$ is defined by* (3.1.2), *then*

$$5q\frac{d}{dq}C(q) = \left(1 - \frac{f^5(-q)}{f(-q^5)}\right)C(q).$$

This result is equivalent to Entry 9(v) in Chapter 19 of Ramanujan's second notebook [227], [61, p. 258], and a proof can be found in [61, pp. 260–261].

The next two entries appear to be new. We are grateful to Chan for supplying the following proofs.

Entry 3.2.5 (p. 363). *If*

$$v := R(q^5), \tag{3.2.2}$$

then

$$q\left(v + \frac{1}{v}\right)\frac{f^5(-q^5)}{f(-q)} = 1 + \sum_{n=1}^{\infty}\left(\frac{nq^n}{1-q^n} - \frac{25nq^{25n}}{1-q^{25n}}\right).$$

To prove Entry 3.2.5, we need two lemmas.

Lemma 3.2.1. *Let* $f(-q)$ *be defined by* (3.1.5). *Then*

$$\begin{aligned}1+\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}-25\sum_{n=1}^{\infty}\frac{nq^{25n}}{1-q^{25n}}&\\ =\frac{f^5(-q^5)}{f(-q)f(-q^{25})}&\left\{f^2(-q)+2qf(-q)f(-q^{25})+5q^2f^2(-q^{25})\right\}^{1/2}.\end{aligned}$$

Lemma 3.2.1 is the same as Entry 7(iii) in Chapter 21 of Ramanujan's second notebook [227], [61, p. 475].

Lemma 3.2.2. *Let* v *be defined by* (3.2.2). *Then*

$$q\left(-v+\frac{1}{v}\right)=q+\frac{f(-q)}{f(-q^{25})}.$$

Lemma 3.2.2 is a famous formula for $R(q^5)$ due to Ramanujan in his notebooks; see [61, p. 267, equation (11.5)]. The first proof was given by Watson [286].

Proof of Entry 3.2.5. By Lemmas 3.2.2 and 3.2.1,

$$\begin{aligned}\left(q\left(v+\frac{1}{v}\right)\frac{f^5(-q^5)}{f(-q)}\right)^2&=\left(q^2\left(-v+\frac{1}{v}\right)^2+4q^2\right)\frac{f^{10}(-q^5)}{f^2(-q)}\\&=5q^2\frac{f^{10}(-q^5)}{f^2(-q)}+2q\frac{f^{10}(-q^5)}{f(-q)f(-q^{25})}+\frac{f^{10}(-q^5)}{f^2(-q^{25})}\\&=\left(1+\sum_{n=1}^{\infty}\frac{nq^n}{1-q^n}-\sum_{n=1}^{\infty}\frac{25nq^{25n}}{1-q^{25n}}\right)^2.\end{aligned}$$

This completes the proof of Entry 3.2.5. ∎

Entry 3.2.6 (p. 364). *If* $v:=R(q)$, *then*

$$q\left(\frac{1}{v^5}+v^5\right)\frac{f^5(-q^5)}{f(-q)}=1+6\sum_{n=1}^{\infty}\left(\frac{nq^n}{1-q^n}-\frac{5nq^{5n}}{1-q^{5n}}\right).$$

We shall again need two lemmas.

Lemma 3.2.3. *We have*

$$\begin{aligned}1+6\sum_{n=1}^{\infty}&\left(\frac{nq^n}{1-q^n}-\frac{5nq^{5n}}{1-q^{5n}}\right)\\&=\frac{1}{f(-q)f(-q^5)}\left\{f^{12}(-q)+22qf^6(-q)f^6(-q^5)+125q^2f^{12}(-q^5)\right\}^{1/2}.\end{aligned}$$

Lemma 3.2.3 is recorded in Entry 4(i) in Chapter 21 of Ramanujan's second notebook [61, p. 463]. Like Lemma 3.2.2, Lemma 3.2.4 is another famous result of Ramanujan found in his notebooks [61, p. 267, equation (11.6)]. See also (1.1.11) in Chapter 1 and Lemma 2.2.3 in Chapter 2.

Lemma 3.2.4. *If* $v := R(q)$, *then*

$$q\left(\frac{1}{v^5} - v^5\right) = 11q + \frac{f^6(-q)}{f^6(-q^5)}.$$

Proof of Entry 3.2.6. By Lemmas 3.2.4 and 3.2.3,

$$\begin{aligned}\left(q\left(\frac{1}{v^5}+v^5\right)\frac{f^5(-q^5)}{f(-q)}\right)^2 &= \left(\left(11q+\frac{f^6(-q)}{f^6(-q^5)}\right)^2+4q^2\right)\frac{f^{10}(-q^5)}{f^2(-q)}\\ &= \frac{f^{10}(-q)}{f^2(-q^5)} + 22qf^4(-q)f^4(-q^5)\\ &\quad + 125q^2\frac{f^{10}(-q^5)}{f^2(-q)}\\ &= \left(1+6\sum_{n=1}^{\infty}\left(\frac{nq^n}{1-q^n}-\frac{5nq^{5n}}{1-q^{5n}}\right)\right)^2.\end{aligned}$$

Entry 3.2.6 now easily follows. □

Entry 3.2.7 (p. 364). *If*

$$2u := 11 + \frac{f^6(-q)}{qf^6(-q^5)}$$

and

$$2v := 1 + \frac{f(-q^{1/5})}{q^{1/5}f(-q^5)},$$

then

$$\sqrt[5]{\sqrt{u^2+1}-u} = \sqrt{v^2+1}-v = R(q).$$

Entry 3.2.7 is identical to Entry 11(iii) in Chapter 19 of Ramanujan's second notebook [61, pp. 265–266]. (Ramanujan inadvertently wrote $f(-q)$ for $f(-q^5)$ in the definition of v.)

Entry 3.2.8 (p. 364). *If*

$$2u := 11 + 125q\frac{f^6(-q^5)}{f^6(-q)} \tag{3.2.3}$$

and

$$2v := 1 + 5q\frac{f(-q^{25})}{f(-q)}, \tag{3.2.4}$$

then

$$\frac{\sqrt{5}}{1+\frac{\sqrt{5}-1}{2}\sqrt[5]{\sqrt{u^2+1}-u}} = \frac{\sqrt{5}}{1+\frac{\sqrt{5}-1}{2}\left(\sqrt{v^2+1}-v\right)}$$
$$= \frac{1+\sqrt{5}}{2}+R(q^5). \tag{3.2.5}$$

Proof of Entry 3.2.8. We shall use ideas that we employed in proving the results in Section 12 of Chapter 19 in the second notebook [61, p. 270]. Replace q by Q in (3.2.3)–(3.2.5) and suppose that the positive variables q and Q satisfy the equality

$$5\log(1/Q)\log(1/q) = 4\pi^2. \tag{3.2.6}$$

Then, using the transformation formula for $f(-q)$, we showed that [61, p. 270, equation (12.9)]

$$\frac{f(-q)}{q^{1/6}f(-q^5)} = \sqrt{5}Q^{1/6}\frac{f(-Q^5)}{f(-Q)}.$$

Hence, condition (3.2.3) (with q replaced by Q) translates into the equality

$$2u = 11 + \frac{f^6(-q)}{qf^6(-q^5)}. \tag{3.2.7}$$

Again, using the transformation formula for $f(-q)$, we also showed that [61, p. 270, equation (12.10)]

$$\frac{f(-q^{1/5})}{q^{1/5}f(-q^5)} = 5Q\frac{f(-Q^{25})}{f(-Q)}.$$

Hence, (3.2.4) takes the equivalent form

$$2v = 1 + \frac{f(-q^{1/5})}{q^{1/5}f(-q^5)}. \tag{3.2.8}$$

Now let $q = e^{-2\alpha}$ and $Q^5 = e^{-2\beta}$, where $\alpha, \beta > 0$. Then, by (3.2.6), $\alpha\beta = \pi^2$. Under these conditions on α and β, in his second letter to Hardy, Ramanujan [226, p. xxviii], [81, p. 57] claimed that

$$\left(\frac{1+\sqrt{5}}{2}+R(e^{-2\alpha})\right)\left(\frac{1+\sqrt{5}}{2}+R(e^{-2\beta})\right) = \frac{5+\sqrt{5}}{2}, \tag{3.2.9}$$

which was first proved in print by Watson [287]. This result is also recorded as Entry 39(i) in Chapter 16 of Ramanujan's second notebook; see [61, pp. 84–85] for another proof and further references. Thus, from (3.2.9),

$$\frac{1+\sqrt{5}}{2}+R(e^{-2\beta}) = \frac{\frac{5+\sqrt{5}}{2}}{\frac{1+\sqrt{5}}{2}+R(e^{-2\alpha})} = \frac{\sqrt{5}}{1+\frac{\sqrt{5}-1}{2}R(e^{-2\alpha})}.$$

In view of Entry 3.2.7 and the fact that $e^{-2\beta} = Q^5$, we have completed the proof of Entry 3.2.8. $\square$

Entry 3.2.9 (p. 364). *If* $\alpha\beta = \pi^2/5$, *then*

$$\left(\left(\frac{\sqrt{5}+1}{2}\right)^5 + R^5(e^{-2\alpha})\right)\left(\left(\frac{\sqrt{5}+1}{2}\right)^5 + R^5(e^{-2\beta})\right) = 5\sqrt{5}\left(\frac{\sqrt{5}+1}{2}\right)^5. \tag{3.2.10}$$

Observe that Entry 3.2.9 is an analogue of (3.2.9). Ramanathan first noticed (3.2.10) in the lost notebook.

Proof. We shall use Lemma 3.2.4 twice. Thus, with α and β as given in Entry 3.2.9,

$$\left(\frac{1}{R^5(e^{-2\alpha})} - R^5(e^{-2\alpha}) - 11\right)\left(\frac{1}{R^5(e^{-2\beta})} - R^5(e^{-2\beta}) - 11\right) = \frac{f^6(-e^{-2\alpha})}{e^{-2\alpha}f^6(-e^{-10\alpha})}\frac{f^6(-e^{-2\beta})}{e^{-2\beta}f^6(-e^{-10\beta})}. \tag{3.2.11}$$

Recall the transformation formula for $f(-q)$ [61, p. 43]. If $ab = \pi^2$, then

$$e^{-a/12}\sqrt[4]{a}f(-e^{-2a}) = e^{-b/12}\sqrt[4]{b}f(-e^{-2b}). \tag{3.2.12}$$

Applying (3.2.12) twice in (3.2.11), the first with $a = \alpha$, $b = 5\beta$, and the second with $a = \beta$, $b = 5\alpha$, we find that

$$\left(\frac{1}{R^5(e^{-2\alpha})} - R^5(e^{-2\alpha}) - 11\right)\left(\frac{1}{R^5(e^{-2\beta})} - R^5(e^{-2\beta}) - 11\right) = 125. \tag{3.2.13}$$

For brevity, set $A = R^5(e^{-2\alpha})$ and $B = R^5(e^{-2\beta})$. Then (3.2.13) takes the form

$$(A^2 + 11A - 1)(B^2 + 11B - 1) = 125AB. \tag{3.2.14}$$

By a straightforward calculation and (3.2.14), we find that

$$\begin{aligned}\left(AB + \frac{11}{2}(A+B) - 1\right)^2 &= (A^2 + 11A - 1)(B^2 + 11B - 1) \\ &\quad + \frac{125}{4}A^2 + \frac{125}{4}B^2 - \frac{125}{2}AB \\ &= 125AB + \frac{125}{4}A^2 + \frac{125}{4}B^2 - \frac{125}{2}AB \\ &= \frac{125}{4}(A+B)^2.\end{aligned}$$

As $q \to 0+, A, B \to 0$. Thus, taking the square root of each side above, we deduce that

$$AB + \frac{11}{2}(A+B) - 1 = -\frac{5\sqrt{5}}{2}(A+B). \tag{3.2.15}$$

Hence, by (3.2.15),

$$\begin{aligned}
&\left(\left(\frac{\sqrt{5}+1}{2}\right)^5 + A\right)\left(\left(\frac{\sqrt{5}+1}{2}\right)^5 + B\right) \\
&\qquad = \left(\frac{1}{2}(11+5\sqrt{5}) + A\right)\left(\frac{1}{2}(11+5\sqrt{5}) + B\right) \\
&\qquad = \frac{123}{2} + \frac{55}{2}\sqrt{5} + \frac{1}{2}(11+5\sqrt{5})(A+B) + AB \\
&\qquad = \frac{123}{2} + \frac{55}{2}\sqrt{5} + 1 \\
&\qquad = 5\sqrt{5}\left(\frac{\sqrt{5}+1}{2}\right)^5.
\end{aligned}$$

This completes the proof. □

Entry 3.2.10 (p. 365). *If*

$$u := U^{1/5} := \frac{q^{1/5}}{1} \, \underset{+}{} \, \frac{q}{1} \, \underset{+}{} \, \frac{q^2}{1} \, \underset{+}{} \, \frac{q^3}{1} \, \underset{+}{} \cdots$$

and

$$v := V^{1/5} := \frac{q^{2/5}}{1} \, \underset{+}{} \, \frac{q^2}{1} \, \underset{+}{} \, \frac{q^4}{1} \, \underset{+}{} \, \frac{q^6}{1} \, \underset{+}{} \cdots,$$

then

(a) $$\frac{v-u^2}{v+u^2} = uv^2,$$

(b) $$UV^2(U^2+V) + U^2 - V + 10UV(UV - U + V + 1) = 0,$$

(c) $$U = t\left(\frac{1-t}{1+t}\right)^2 \quad \textit{and} \quad V = t^2\frac{1+t}{1-t},$$

where $t \leq \sqrt{5} - 2$.

Entry 3.2.11 (p. 365). *If*

$$u := \frac{q^{1/5}}{1} \, \underset{+}{} \, \frac{q}{1} \, \underset{+}{} \, \frac{q^2}{1} \, \underset{+}{} \, \frac{q^3}{1} \, \underset{+}{} \cdots$$

and

$$v := \frac{q^{3/5}}{1} \, \underset{+}{} \, \frac{q^3}{1} \, \underset{+}{} \, \frac{q^6}{1} \, \underset{+}{} \, \frac{q^9}{1} \, \underset{+}{} \cdots,$$

then

$$(v-u^3)(1+uv^3) = 3u^2v^2.$$

Entry 3.2.12 (p. 365). *If*

$$u := \frac{q^{1/5}}{1} \, \frac{}{+} \, \frac{q}{1} \, \frac{}{+} \, \frac{q^2}{1} \, \frac{}{+} \, \frac{q^3}{1} \, \frac{}{+} \cdots$$

and

$$v := \frac{q^{1/5}}{1} \, \frac{}{-} \, \frac{q}{1} \, \frac{}{+} \, \frac{q^2}{1} \, \frac{}{-} \, \frac{q^3}{1} \, \frac{}{+} \cdots,$$

then

$$uv(u-v)^4 - u^2v^2(u-v)^2 + 2u^3v^3 + (u-v)(1+u^5v^5) = 0. \tag{3.2.16}$$

Entry 3.2.13 (p. 365). *If*

$$u := \frac{q^{1/5}}{1} \, \frac{}{+} \, \frac{q}{1} \, \frac{}{+} \, \frac{q^2}{1} \, \frac{}{+} \, \frac{q^3}{1} \, \frac{}{+} \cdots$$

and

$$v := \frac{q^{4/5}}{1} \, \frac{}{+} \, \frac{q^4}{1} \, \frac{}{+} \, \frac{q^8}{1} \, \frac{}{+} \, \frac{q^{12}}{1} \, \frac{}{+} \cdots,$$

then

$$(u^5 + v^5)(uv - 1) + u^5v^5 + uv = 5u^2v^2(uv-1)^2.$$

Entry 3.2.14 (p. 365). *Let*

$$u := \frac{q^{1/5}}{1} \, \frac{}{+} \, \frac{q}{1} \, \frac{}{+} \, \frac{q^2}{1} \, \frac{}{+} \, \frac{q^3}{1} \, \frac{}{+} \cdots$$

and

$$v := \frac{q}{1} \, \frac{}{+} \, \frac{q^5}{1} \, \frac{}{+} \, \frac{q^{10}}{1} \, \frac{}{+} \, \frac{q^{15}}{1} \, \frac{}{+} \cdots.$$

Then

$$u^5 = v\frac{1 - 2v + 4v^2 - 3v^3 + v^4}{1 + 3v + 4v^2 + 2v^3 + v^4}.$$

The last five entries provide modular equations satisfied by the Rogers–Ramanujan continued fraction. All of them can be found at scattered places in Ramanujan's notebooks [227]. See Berndt's book [63, pp. 12–20] for proofs and references, or the introduction of Chapter 1 of this book for references.

Except for Entries 3.2.10(b), (c), the last five entries are also recorded by Ramanujan in a one-page fragment with the lost notebook [228, p. 348].

We close this section with an entry not found in the fragment, but it is found nearby on page 347. Also, it is given in Entry 11(iii) of Chapter 19 in Ramanujan's second notebook [227], [61, pp. 265–266].

Entry 3.2.15 (p. 347). *If*

$$2u := 11 + \frac{(q;q)_\infty^6}{q(q^5;q^5)_\infty^6}$$

and

$$2v := 11 + \frac{(q^{1/5};q^{1/5})_\infty}{q^{1/5}(q^5;q^5)_\infty},$$

then

$$\sqrt[5]{\sqrt{u^2+1}-u} = \sqrt{v^2+1}-v = q^{1/5}\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}.$$

3.3 The Theory of Ramanujan's Cubic Continued Fraction

In the next entry, which is devoted to several results on the cubic continued fraction, Ramanujan prefaces his statements by writing, "I have also found empirically the following result." Maybe Ramanujan had a different meaning for "empirical" than we have, for it would seem that in order to write down these entries, he would necessarily have had proofs. We remind readers of the definitions of Ramanujan's theta functions:

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} \qquad \text{and} \qquad \psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2}, \qquad |q|<1. \tag{3.3.1}$$

Entry 3.3.1 (p. 366). *If*

$$v = \frac{q^{1/3}}{1} + \frac{q+q^2}{1} + \frac{q^2+q^4}{1} + \frac{q^3+q^6}{1} + \cdots, \qquad |q|<1,$$

then

(a)
$$v = q^{1/3}\frac{(q;q^2)_\infty}{(q^3;q^6)_\infty^3},$$

(b)
$$\frac{1}{v} = \frac{\psi(q^{1/3})}{q^{1/3}\psi(q^3)} - 1 = \sqrt[3]{\frac{\psi^4(q)}{q\psi^4(q^3)}} - 1,$$

(c)
$$2v = 1 - \frac{\varphi(-q^{1/3})}{\varphi(-q)} = \sqrt[3]{1 - \frac{\varphi^4(-q)}{\varphi^4(-q^3)}},$$

(d)
$$\frac{1}{v} + 4v^2 = 3 + \frac{f^3(-q^{1/3})}{q^{1/3}f^3(-q^3)}$$
$$= \sqrt[3]{27 + \frac{f^{12}(-q)}{qf^{12}(-q^3)}}.$$

In the first part of (b), Ramanujan mistakenly wrote $\psi(q)$ for $\psi(q^3)$. Parts (a) and (b) are contained in Entry 1(i), and the first part of (c) is part of Entry 1(ii) in Chapter 20 of the second notebook [61, p. 345]. The second part of (c) was derived in the course of proving other results in Entry 1; see [61, p. 347]. Both parts of (d) are found in Entry 1(iv) of Chapter 20 [61, p. 345].

At the end of the fragment, Ramanujan claims, "and many results analogous to the previous continued fraction." He then closes with the explicit value of one particular cubic continued fraction. In the remainder of this section we present Chan's [112] theory of the cubic continued fraction.

As usual, set
$$\chi(-q) = (q; q^2)_\infty. \tag{3.3.2}$$

Theorem 3.3.1. *Let $G(q)$ be defined by* (3.1.6)*, which is the same as v in Entry* 3.3.1*. Then*

$$G(q) + G(-q) + 2G^2(-q)G^2(q) = 0, \tag{3.3.3}$$
$$G^2(q) + 2G^2(q^2)G(q) - G(q^2) = 0, \tag{3.3.4}$$

and
$$G^3(q) = G(q^3)\frac{1 - G(q^3) + G^2(q^3)}{1 + 2G(q^3) + 4G^2(q^3)}. \tag{3.3.5}$$

Proof. We first prove (3.3.3). Let $v := G(q)$ and $u := G(-q)$. From Entry 3.3.1(a) and (3.3.2), $G(q)$ and $G(-q)$ have the representations

$$v = q^{1/3}\frac{\chi(-q)}{\chi^3(-q^3)} \qquad \text{and} \qquad u = -q^{1/3}\frac{\chi(q)}{\chi^3(q^3)}. \tag{3.3.6}$$

We shall employ some of Ramanujan's modular equations of degree 3. When β has degree 3 over α, it follows from [61, p. 124, Entries 12(v), (vi)] that

$$v = 2^{-1/3}\frac{(1-\alpha)^{1/12}\beta^{1/8}}{(1-\beta)^{1/4}\alpha^{1/24}} \qquad \text{and} \qquad u = -2^{-1/3}\frac{(\beta(1-\beta))^{1/8}}{(\alpha(1-\alpha))^{1/24}},$$

from which we observe that

$$\frac{u}{v} = -\left(\frac{(1-\beta)^3}{1-\alpha}\right)^{1/8} \qquad \text{and} \qquad vu^2 = \frac{1}{2}\left(\frac{\beta^3}{\alpha}\right)^{1/8}. \tag{3.3.7}$$

Furthermore [61, p. 230, Entry 5(i)], we find that

$$1=\left(\frac{(1-\beta)^3}{1-\alpha}\right)^{1/8}-\left(\frac{\beta^3}{\alpha}\right)^{1/8}. \tag{3.3.8}$$

Hence, using (3.3.7) in (3.3.8), we deduce that

$$-\frac{u}{v}-2vu^2=1.$$

Simplifying, we obtain (3.3.3).

We next prove (3.3.4). Recall that χ is defined in (3.3.2). From the identities given in (3.3.6), we find that

$$-G(q)G(-q)=q^{2/3}\frac{\chi(-q)\chi(q)}{\chi^3(-q^3)\chi^3(q^3)}=q^{2/3}\frac{\chi(-q^2)}{\chi^3(-q^6)}=G(q^2). \tag{3.3.9}$$

If we multiply (3.3.3) by $G(q)$ and invoke (3.3.9), we obtain (3.3.4).

Lastly, we prove (3.3.5). Let $w:=G(q^3)$ and $v:=G(q)$, as in the proof of (3.3.3). From Entries 3.3.1(c), (c), and (d), respectively,

$$\frac{\varphi(-q^3)}{\varphi(-q^{1/3})}=\frac{1}{1-2v}, \tag{3.3.10}$$

$$\frac{\varphi^4(-q)}{\varphi^4(-q^3)}=1-8v^3, \tag{3.3.11}$$

and

$$3+\frac{f^3(-q^{1/3})}{q^{1/3}f^3(-q^3)}=\frac{1}{v}+4v^2=\left(27+\frac{f^{12}(-q)}{qf^{12}(-q^3)}\right)^{1/3}, \tag{3.3.12}$$

where $f(-q)$ and $\varphi(q)$ are defined in (3.1.5) and (3.3.1), respectively. Using (3.3.10)–(3.3.12), with q replaced by q^3 and v replaced by w, we obtain

$$\frac{\varphi(-q^9)}{\varphi(-q)}=\frac{1}{1-2w}, \tag{3.3.13}$$

$$\frac{\varphi^4(-q^3)}{\varphi^4(-q^9)}=1-8w^3, \tag{3.3.14}$$

and

$$3+\frac{f^3(-q)}{qf^3(-q^9)}=\frac{1}{w}+4w^2=\left(27+\frac{f^{12}(-q^3)}{q^3f^{12}(-q^9)}\right)^{1/3}. \tag{3.3.15}$$

To prove (3.3.5), we also require the identity [61, p. 345, Entry 1(iv)]

$$1+9q\frac{f^3(-q^9)}{f^3(-q)}=\left(1+27q\frac{f^{12}(-q^3)}{f^{12}(-q)}\right)^{1/3}. \tag{3.3.16}$$

We first establish an identity that relates v and w. Now, from the second equality of (3.3.12), we find that

$$\frac{1}{27}\left(\left(\frac{1}{v}+4v^2\right)^3-27\right)=\frac{1}{27}\frac{f^{12}(-q)}{qf^{12}(-q^3)}. \tag{3.3.17}$$

By (3.3.16) and (3.3.15), we deduce that

$$\begin{aligned}\frac{1}{27}\frac{f^{12}(-q)}{qf^{12}(-q^3)}&=\left(\left(1+9q\frac{f^3(-q^9)}{f^3(-q)}\right)^3-1\right)^{-1} \\ &=\left(\left(1+9\left(\frac{1}{w}+4w^2-3\right)^{-1}\right)^3-1\right)^{-1} \\ &=\frac{(1-3w+4w^3)^3}{(1+6w+4w^3)^3-(1-3w+4w^3)^3}.\end{aligned} \tag{3.3.18}$$

Hence, by (3.3.17) and (3.3.18), we deduce that

$$\frac{1}{27}\left(\left(\frac{1}{v}+4v^2\right)^3-27\right)=\frac{(1-3w+4w^3)^3}{(1+6w+4w^3)^3-(1-3w+4w^3)^3}. \tag{3.3.19}$$

From Entry 24(iii) in Chapter 16 of Ramanujan's second notebook [61, p. 39],

$$\chi(-q)=\frac{\varphi(-q)}{f(-q)}. \tag{3.3.20}$$

Using (3.3.6) and (3.3.20), we deduce that

$$v=q^{1/3}\frac{\varphi(-q)}{f(-q)}\frac{f^3(-q^3)}{\varphi^3(-q^3)} \qquad \text{and} \qquad w=q\frac{\varphi(-q^3)}{f(-q^3)}\frac{f^3(-q^9)}{\varphi^3(-q^9)}.$$

Thus,

$$\frac{w}{v}=q^{2/3}\frac{\varphi^4(-q^3)}{\varphi^4(-q^9)}\frac{\varphi(-q^9)}{\varphi(-q)}\frac{f^3(-q^9)}{f^3(-q)}\frac{f^4(-q)}{f^4(-q^3)}. \tag{3.3.21}$$

By (3.3.21), (3.3.14), (3.3.13), the first equality of (3.3.15), and the second equality of (3.3.12), we find that

$$\begin{aligned}\frac{w}{v}&=q^{2/3}\left(1+2w+4w^2\right)\left(q\left(\frac{1}{w}+4w^2-3\right)\right)^{-1} \\ &\quad\times q^{1/3}\left(\left(\frac{1}{v}+4v^2\right)^3-27\right)^{1/3} \\ &=\frac{(1+2w+4w^2)w}{1+4w^3-3w}\left(\left(\frac{1}{v}+4v^2\right)^3-27\right)^{1/3}.\end{aligned} \tag{3.3.22}$$

Finally, we cube both sides of (3.3.22) and use (3.3.19) to arrive at

$$v^3=\frac{(1+6w+4w^3)^3-(1-3w+4w^3)^3}{(3(1+2w+4w^2))^3}. \tag{3.3.23}$$

Simplifying the right-hand side of (3.3.23), we deduce (3.3.5). □

C. Adiga, T. Kim, M.S. Mahadeva Naika and H.S. Madhusudhan [4] gave a simpler proof of (3.3.5) by eliminating $\psi(q)$, $\psi(q^3)$, and $\psi(q^9)$ among the identities in Entries 1(i) and 1(ii) of Chapter 20 in Ramanujan's second notebook [61, p. 345]. They also showed that

$$w = \frac{1-T}{2+T},$$

where

$$T = \left(\frac{1-8v^3}{1+v^3}\right)^{1/3}.$$

This is similar to the triplication formula satisfied by the cubic singular modulus [117].

N.D. Baruah [53] has also given an alternative proof of (3.3.5). He has also established modular equations connecting $G(q)$ with $G(q^5)$ and $G(q^7)$, respectively. Further modular equations for $G(q)$ have been found by Mahadeva Naika [191].

Theorem 3.3.2. *If $\alpha\beta = 1$, then*

$$\left(4G^2(e^{-2\pi\alpha})+\frac{1}{G(e^{-2\pi\alpha})}-3\right)\left(4G^2(e^{-2\pi\beta})+\frac{1}{G(e^{-2\pi\beta})}-3\right) = 27, \tag{3.3.24}$$

$$\left(1-2G(-e^{-\pi\alpha})\right)\left(1-2G(-e^{-\pi\beta})\right) = 3, \tag{3.3.25}$$

and

$$\left(1+G(e^{-\sqrt{2}\pi\alpha})\right)\left(1+G(e^{-\sqrt{2}\pi\beta})\right) = \frac{3}{2}. \tag{3.3.26}$$

Proof. We first prove (3.3.24). From the first equality of (3.3.12), we observe that

$$\begin{aligned}\left(4G^2(e^{-2\pi\alpha})+\frac{1}{G(e^{-2\pi\alpha})}-3\right)&\left(4G^2(e^{-2\pi\beta})+\frac{1}{G(e^{-2\pi\beta})}-3\right)\\ &= \frac{f^3(-e^{-2\pi\alpha/3})}{e^{-2\pi\alpha/3}f^3(-e^{-6\pi\alpha})}\frac{f^3(-e^{-2\pi\beta/3})}{e^{-2\pi\beta/3}f^3(-e^{-6\pi\beta})}.\end{aligned} \tag{3.3.27}$$

From the transformation formula (3.2.12) with $a = \pi\alpha/3$ and $b = 3\pi\beta$, we deduce that

$$e^{-\pi\alpha/12}f^3(-e^{-2\pi\alpha/3}) = \left(\frac{3}{\alpha}\right)^{3/2} e^{-3\pi\beta/4}f^3(-e^{-6\pi\beta}), \tag{3.3.28}$$

where $\alpha\beta = 1$. Similarly, we find that

$$e^{-\pi\beta/12} f^3(-e^{-2\pi\beta/3}) = \left(\frac{3}{\beta}\right)^{3/2} e^{-3\pi\alpha/4} f^3(-e^{-6\pi\alpha}). \tag{3.3.29}$$

Using (3.3.28) and (3.3.29), we can rewrite the right-hand side of (3.3.27) as

$$\frac{e^{\pi\alpha/12}e^{-3\pi\beta/4}}{e^{-2\pi\alpha/3}}\left(\frac{3}{\alpha}\right)^{3/2}\frac{e^{\pi\beta/12}e^{-3\pi\alpha/4}}{e^{-2\pi\beta/3}}\left(\frac{3}{\beta}\right)^{3/2} = 27,$$

as required. This completes the proof of (3.3.24).

We next prove (3.3.25). Using (3.3.10), we have

$$\begin{aligned}\left(1-2G(-e^{-\pi\alpha})\right)\left(1-2G(-e^{-\pi\beta})\right) &= \frac{\varphi(e^{-\pi\alpha/3})}{\varphi(e^{-3\pi\alpha})}\frac{\varphi(e^{-\pi\beta/3})}{\varphi(e^{-3\pi\beta})}\\ &= \frac{\varphi(e^{-\pi\alpha/3})}{\varphi(e^{-3\pi/\beta})}\frac{\varphi(e^{-\pi\beta/3})}{\varphi(e^{-3\pi/\alpha})},\end{aligned} \tag{3.3.30}$$

since $\alpha\beta = 1$. Recall the transformation formula for $\varphi(q)$ [61, p. 43, Entry 27(i)], namely,

$$\varphi(e^{-\pi z}) = \frac{1}{\sqrt{z}}\varphi(e^{-\pi/z}), \qquad \operatorname{Re} z > 0. \tag{3.3.31}$$

If we set $z = \beta/3$ and $z = \alpha/3$, respectively, in (3.3.31), then

$$\varphi(e^{-\pi\beta/3}) = \sqrt{\frac{3}{\beta}}\varphi(e^{-3\pi/\beta}) \quad \text{and} \quad \varphi(e^{-\pi\alpha/3}) = \sqrt{\frac{3}{\alpha}}\varphi(e^{-3\pi/\alpha}). \tag{3.3.32}$$

Using (3.3.32) and the condition $\alpha\beta = 1$, we find from (3.3.30) that

$$\left(1-2G(-e^{-\pi\alpha})\right)\left(1-2G(-e^{-\pi\beta})\right) = \sqrt{\frac{3}{\beta}}\sqrt{\frac{3}{\alpha}}\frac{\varphi(e^{-3\pi/\alpha})}{\varphi(e^{-3\pi/\beta})}\frac{\varphi(e^{-3\pi/\beta})}{\varphi(e^{-3\pi/\alpha})} = 3,$$

as required.

Lastly, we establish (3.3.26). Recall from Entry 3.3.1(b) that

$$1 + \frac{1}{G(q)} = \frac{\psi(q^{1/3})}{q^{1/3}\psi(q^3)}. \tag{3.3.33}$$

We shall need the transformation formula for $\psi(q)$ [61, p. 43, Entry 27(ii)], namely,

$$e^{-\pi z/8}\psi(e^{-\pi z}) = \frac{1}{\sqrt{2z}}\varphi(-e^{-2\pi/z}), \qquad \operatorname{Re} z > 0. \tag{3.3.34}$$

If we let $q = e^{-\sqrt{2}\pi\alpha}$ in (3.3.33) and invoke (3.3.34), we find that

$$1 + \frac{1}{G(e^{-\sqrt{2}\pi\alpha})} = \frac{\psi(e^{-\sqrt{2}\pi\alpha/3})}{e^{-\sqrt{2}\pi\alpha/3}\psi(e^{-3\sqrt{2}\pi\alpha})} = 3\frac{\varphi(-e^{-3\sqrt{2}\pi/\alpha})}{\varphi(-e^{-\sqrt{2}\pi/3\alpha})}. \tag{3.3.35}$$

On the other hand, by (3.3.10) with $q = e^{-\sqrt{2}\pi/\alpha}$, we deduce that

$$\frac{\varphi(-e^{-3\sqrt{2}\pi/\alpha})}{\varphi(-e^{-\sqrt{2}\pi/3\alpha})} = \frac{1}{1 - 2G(e^{-\sqrt{2}\pi/\alpha})} = \frac{1}{1 - 2G(e^{-\sqrt{2}\pi\beta})}, \tag{3.3.36}$$

since $\alpha\beta = 1$. Combining (3.3.35) and (3.3.36), we conclude our proof of (3.3.26) after some simplifications. □

Adiga, Kim, Mahadeva Naika, and Madhusudhan [4] have obtained three additional reciprocity theorems for $G(q)$ on repeated applications of the theta function transformation in Entry 27(ii) of Chapter 16 in Ramanujan's second notebook [61, p. 43].

3.4 Explicit Evaluations of $G(q)$

We first use Theorem 3.3.2 to easily deduce some specific values for the cubic continued fraction. Secondly, we employ one of Ramanujan's modular equations to determine $G(-e^{-\sqrt{5}\pi})$. Thirdly, we present a general method from [72] for evaluating $G(\pm q)$. Fourthly, we use this method to establish the one specific value of $G(q)$ recorded by Ramanujan at the conclusion of this fragment.

Theorem 3.4.1. *We have*

$$G(-e^{-\pi}) = \frac{1 - \sqrt{3}}{2}, \tag{3.4.1}$$

$$G(e^{-\pi}) = \frac{(1 + \sqrt{3})(-(1 + \sqrt{3}) + \sqrt{6\sqrt{3}})}{4}, \tag{3.4.2}$$

$$G(e^{-2\pi}) = \frac{-(1 + \sqrt{3}) + \sqrt{6\sqrt{3}}}{4}, \tag{3.4.3}$$

$$G(e^{-\sqrt{2}\pi}) = \frac{-2 + \sqrt{6}}{2}, \tag{3.4.4}$$

$$G^3(e^{-\sqrt{2}\pi/3}) = \frac{G(e^{-\sqrt{2}\pi})}{2}. \tag{3.4.5}$$

Proof. We first establish (3.4.1). If $\alpha = \beta = 1$ in (3.3.25), then

$$(1 - 2G(-e^{-\pi}))^2 = 3,$$

and this proves (3.4.1), since $G(-e^{-\pi}) < 0$.

Set $x = G(e^{-\pi})$. Then, from (3.3.3),

$$2G^2(-e^{-\pi})x^2 + x + G(-e^{-\pi}) = 0.$$

Using (3.4.1) and solving for x, we deduce (3.4.2).

Using (3.3.9), (3.4.1), and (3.4.2), we deduce that

$$G(e^{-2\pi}) = -G(-e^{-\pi})G(e^{-\pi}) = \frac{-(1+\sqrt{3})+\sqrt{6\sqrt{3}}}{4},$$

and so (3.4.3) is established.

For simplicity, let $A := G(e^{-\sqrt{2}\pi})$. If we set $\alpha = \beta = 1$ in (3.3.26), then we obtain

$$(A+1)^2 = \frac{3}{2}.$$

Solving for A yields (3.4.4).

We substitute (3.4.4) into the right side of (3.3.5) to obtain (3.4.5). □

Theorem 3.4.2. *We have*

$$G(-e^{-\sqrt{5}\pi}) = \frac{(\sqrt{5}-3)(\sqrt{5}-\sqrt{3})}{4}. \tag{3.4.6}$$

To prove (3.4.6), we require the following identity of Ramanujan.

Lemma 3.4.1. *Let*

$$P := \frac{\varphi(q)}{\varphi(q^5)} \qquad \textit{and} \qquad Q := \frac{\varphi(q^3)}{\varphi(q^{15})}.$$

Then

$$PQ + \frac{5}{PQ} = \left(\frac{Q}{P}\right)^2 + 3\frac{Q}{P} + 3\frac{P}{Q} - \left(\frac{P}{Q}\right)^2.$$

For a proof of Lemma 3.4.1, see [62, p. 235, Entry 67].

Proof of Theorem 3.4.2. Let $q = e^{-\pi/\sqrt{5}}$ and invoke (3.3.31) to deduce that

$$P = \frac{\varphi(e^{-\pi/\sqrt{5}})}{\varphi(e^{-\sqrt{5}\pi})} = 5^{1/4}$$

and

$$Q = \frac{\varphi(e^{-3\pi/\sqrt{5}})}{\varphi(e^{-3\sqrt{5}\pi})} = \frac{5^{1/4}}{\sqrt{3}}\frac{\varphi(e^{-\sqrt{5}\pi/3})}{\varphi(e^{-3\sqrt{5}\pi})}.$$

If we let

$$C := \frac{\varphi(e^{-\sqrt{5}\pi/3})}{\varphi(e^{-3\sqrt{5}\pi})}, \tag{3.4.7}$$

then

$$PQ = \sqrt{\frac{5}{3}}C \qquad \text{and} \qquad \frac{P}{Q} = \frac{\sqrt{3}}{C}. \tag{3.4.8}$$

Substituting (3.4.8) into Lemma 3.4.1, we deduce that

$$\sqrt{\frac{5}{3}}C + \frac{\sqrt{15}}{C} = \left(\frac{C}{\sqrt{3}}\right)^2 + 3\frac{C}{\sqrt{3}} + 3\frac{\sqrt{3}}{C} - \left(\frac{\sqrt{3}}{C}\right)^2,$$

which may be rewritten as

$$\sqrt{5}\left(\frac{C}{\sqrt{3}} + \frac{\sqrt{3}}{C}\right) = \left(\frac{C}{\sqrt{3}} - \frac{\sqrt{3}}{C}\right)\left(\frac{C}{\sqrt{3}} + \frac{\sqrt{3}}{C}\right) + 3\left(\frac{C}{\sqrt{3}} + \frac{\sqrt{3}}{C}\right).$$

Since

$$\left(\frac{C}{\sqrt{3}} + \frac{\sqrt{3}}{C}\right) \neq 0,$$

we conclude that

$$\sqrt{5} = \left(\frac{C}{\sqrt{3}} - \frac{\sqrt{3}}{C}\right) + 3. \tag{3.4.9}$$

Solving the quadratic equation (3.4.9), we find that

$$C = \frac{-3\sqrt{3} + \sqrt{15} + 3(\sqrt{5} - 1)}{2}. \tag{3.4.10}$$

From (3.3.10), we know that

$$1 - 2G(-e^{-\sqrt{5}\pi}) = \frac{\varphi(e^{-\sqrt{5}\pi/3})}{\varphi(e^{-3\sqrt{5}\pi})}. \tag{3.4.11}$$

Thus, (3.4.6) follows from (3.4.7), (3.4.10), and (3.4.11). □

A completely different proof of (3.4.10) can be found in [61, p. 210, eq. (23.5)].

A very general method for calculating explicit values of $G(q)$ was given by Berndt, Chan, and Zhang in [72]. We now present this method and illustrate it with another proof of Theorem 3.4.2 and a proof of Ramanujan's last claim in this fragment. Further evaluations of $G(q)$ may be found in [72].

We need to define Ramanujan's class invariants G_n and g_n. If $q = \exp(-\pi\sqrt{n})$, where n is any positive rational number, define

$$G_n := 2^{-1/4}q^{-1/24}\chi(q) \qquad \text{and} \qquad g_n := 2^{-1/4}q^{-1/24}\chi(-q), \tag{3.4.12}$$

where $\chi(q)$ is defined by (3.3.2). The following two theorems were proved in [72]; see also [63, pp. 205–208, Theorems 3.1, 3.2]. The latter theorem is found in Ramanujan's first notebook [227, p. 318].

Theorem 3.4.3. *Let*

$$p = G_n^4 + G_n^{-4}. \tag{3.4.13}$$

Then, for $n \geq 1$,

$$G_{9n} = G_n\left(p + \sqrt{p^2 - 1}\right)^{1/6} \tag{3.4.14}$$
$$\times \left\{\sqrt{\frac{p^2 - 2 + \sqrt{(p^2-1)(p^2-4)}}{2}} + \sqrt{\frac{p^2 - 4 + \sqrt{(p^2-1)(p^2-4)}}{2}}\right\}^{1/3}.$$

Theorem 3.4.4. *Let*

$$p = g_n^4 - g_n^{-4}. \tag{3.4.15}$$

Then, for $n > 0$,

$$g_{9n} = g_n \left(p + \sqrt{p^2+1}\right)^{1/6} \tag{3.4.16}$$
$$\times \left\{ \sqrt{\frac{p^2+4+\sqrt{(p^2+1)(p^2+4)}}{2}} + \sqrt{\frac{p^2+2+\sqrt{(p^2+1)(p^2+4)}}{2}} \right\}^{1/3}.$$

Theorem 3.4.5. *Let* $G(q)$ *be defined by* (3.1.6), *and let* p *be defined by* (3.4.13). *Then*

$$G(-e^{-\pi\sqrt{n}}) = -\sqrt{\frac{p-\sqrt{p^2-1}}{p+\sqrt{p^2-4}}}$$
$$\times \left(\sqrt{\frac{p^2-2+\sqrt{(p^2-1)(p^2-4)}}{2}} - \sqrt{\frac{p^2-4+\sqrt{(p^2-1)(p^2-4)}}{2}} \right).$$

Proof. Recall from Entry 3.3.1(a) that

$$G(q) = q^{1/3} \frac{\chi(-q)}{\chi^3(-q^3)}, \tag{3.4.17}$$

where χ is defined in (3.3.2). For $q = -e^{-\pi\sqrt{n}}$, we find that by (3.4.17) and (3.4.12),

$$\begin{aligned} G(-e^{-\pi\sqrt{n}}) &= -\, e^{-\pi\sqrt{n}/3} \frac{\chi(e^{-\pi\sqrt{n}})}{\chi^3(e^{-3\pi\sqrt{n}})} \\ &= -\frac{1}{\sqrt{2}} \frac{G_n}{G_{9n}^3} = -\frac{1}{\sqrt{2}} G_n^{-2} \left(\frac{G_n}{G_{9n}}\right)^3. \end{aligned} \tag{3.4.18}$$

If

$$u(p) := \frac{p^2-4+\sqrt{(p^2-1)(p^2-4)}}{2},$$

then, from (3.4.14) and (3.4.18), we find that

$$G(-e^{-\pi\sqrt{n}}) = -\frac{1}{\sqrt{2}} G_n^{-2} \sqrt{p-\sqrt{p^2-1}} \left(\sqrt{u(p)+1} - \sqrt{u(p)}\right). \tag{3.4.19}$$

But an easy calculation from (3.4.13) yields

$$G_n^{-2} = \left(\frac{p+\sqrt{p^2-4}}{2}\right)^{-1/2}. \tag{3.4.20}$$

Substituting (3.4.20) into (3.4.19), we complete the proof. □

Second Proof of (3.4.1). Let $n = 1$. Then, trivially, $G_1 = 1$ and $p = 2$. Thus, by Theorem 3.4.5,

$$G(-e^{-\pi}) = -\left(\frac{2-\sqrt{3}}{2}\right)^{1/2} = \frac{1-\sqrt{3}}{2}.$$

□

Second Proof of Theorem 3.4.2. Let $n = 5$. From Weber's treatise [291, p. 721] or from Berndt's book [63, p. 189],

$$G_5 = \left(\frac{1+\sqrt{5}}{2}\right)^{1/4}.$$

It easily follows that $p = \sqrt{5}$, and so

$$G(-e^{-\pi\sqrt{5}}) = -\sqrt{\frac{\sqrt{5}-2}{\sqrt{5}+1}}\left(\sqrt{\frac{5}{2}} - \sqrt{\frac{3}{2}}\right) = \frac{(\sqrt{5}-3)(\sqrt{5}-\sqrt{3})}{4}.$$

□

Theorem 3.4.6. *Let $G(q)$ be defined by* (3.1.6), *and let p be given by* (3.4.15). *Then*

$$G(e^{-\pi\sqrt{n}}) = \sqrt{\frac{\sqrt{p^2+1}-p}{\sqrt{p^2+4}+p}}$$
$$\times \left(\sqrt{\frac{p^2+4+\sqrt{(p^2+1)(p^2+4)}}{2}} - \sqrt{\frac{p^2+2+\sqrt{(p^2+1)(p^2+4)}}{2}}\right).$$

Proof. Arguing as in the proof of Theorem 3.4.5, we deduce from (3.4.17) and (3.4.12) that

$$G(e^{-\pi\sqrt{n}}) = \frac{1}{\sqrt{2}}\frac{g_n}{g_{9n}^3} = \frac{1}{\sqrt{2}}g_n^{-2}\left(\frac{g_n}{g_{9n}}\right)^3. \tag{3.4.21}$$

If

$$v(p) := \frac{p^2+2+\sqrt{(p^2+1)(p^2+4)}}{2},$$

then by (3.4.21) and (3.4.16),

$$G(e^{-\pi\sqrt{n}}) = \frac{1}{\sqrt{2}}g_n^{-2}\sqrt{\sqrt{p^2+1}-p}\left(\sqrt{v(p)+1} - \sqrt{v(p)}\right). \tag{3.4.22}$$

However, from (3.4.15), since $p > 0$,

$$g_n^{-2} = \left(\frac{\sqrt{p^2+4}+p}{2} \right)^{-1/2}. \tag{3.4.23}$$

Putting (3.4.23) in (3.4.22), we complete the proof. □

Baruah [53] has established general formulas for $G(e^{-3\pi\sqrt{n}})$ and $G(-e^{-3\pi\sqrt{n}})$ in terms of Ramanujan's theta function ψ.

We now establish the last claim in the fragment.

Entry 3.4.1 (p. 366).

$$G(e^{-\pi\sqrt{10}}) = \frac{\sqrt{9+3\sqrt{6}} - \sqrt{7+3\sqrt{6}}}{(1+\sqrt{5})\sqrt{\sqrt{6}+\sqrt{5}}}.$$

Proof. Let $n = 10$. Then, from the table in Berndt's book [63, p. 200],

$$g_{10} = \sqrt{\frac{1+\sqrt{5}}{2}}.$$

It easily follows that $p = \sqrt{5}$. Thus, Theorem 3.4.6 gives

$$G(e^{-\pi\sqrt{10}}) = \sqrt{\frac{\sqrt{6}-\sqrt{5}}{3+\sqrt{5}}} \left(\sqrt{\frac{9+3\sqrt{6}}{2}} - \sqrt{\frac{7+3\sqrt{6}}{2}} \right).$$

Upon simplification, the desired evaluation follows. □

K.G. Ramanathan [215] has also given a proof of Entry 3.4.1.

In her thesis [297], Yi systematically exploited modular equations, in particular eta-function identities, to find 22 new values for $G(e^{-\pi\sqrt{n}})$ and $G(-e^{-\pi\sqrt{n}})$. For example, she proved that

$$G(e^{-\pi/\sqrt{3}}) = \frac{\sqrt{3}-1}{2^{2/3}}, \qquad G(e^{-2\sqrt{3}\pi}) = \frac{2^{1/3}-1}{2^{1/3}(1-\sqrt{3}+2^{2/3}\sqrt{3})},$$

$$G(-e^{-\pi/3}) = -\left(\frac{1+\sqrt{3}}{4} \right)^{1/3}, \qquad G(-e^{-2\pi}) = \frac{1+\sqrt{3}-\sqrt{2}\,3^{3/4}}{2-3\sqrt{2}+3^{5/4}+3^{3/4}}.$$

Her methods can clearly produce several further evaluations.

Baruah and N. Saikia [55] and Adiga, Vasuki, and Mahadeva Naika [7], [8] have also established some further evaluations of $G(e^{-\pi\sqrt{n}})$ and $G(-e^{-\pi\sqrt{n}})$.

4

The Rogers–Ramanujan Continued Fraction and Its Connections with Partitions and Lambert Series

4.1 Introduction

Recall that the Rogers–Ramanujan continued fraction is defined by

$$R(q) := \frac{q^{1/5}}{1} \, + \, \frac{q}{1} \, + \, \frac{q^2}{1} \, + \, \frac{q^3}{1} \, + \cdots, \qquad |q| < 1, \tag{4.1.1}$$

and that it has the representation

$$R(q) = q^{1/5} \frac{(q;q^5)_\infty (q^4;q^5)_\infty}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \tag{4.1.2}$$

In this chapter, the modular properties of $R(q)$ are not so important, and so it is not really necessary to carry the appendage $q^{1/5}$. Thus, following Andrews [26], we define

$$C(q) := \frac{1}{q^{-1/5} R(q)}. \tag{4.1.3}$$

Our goal in this chapter is to prove several results in the lost notebook on $C(q)$ that are connected with either partitions or Lambert series. Most of this chapter is taken from Andrews's paper [26], with a simplification given for one of the proofs. Results in Section 4.4 are proved in Andrews's paper [22].

Define the power series coefficients $v_n, n \geq 0$, by

$$C(q) = \sum_{n=0}^{\infty} v_n q^n, \qquad |q| < 1. \tag{4.1.4}$$

In Section 4.2, we establish Ramanujan's representations for

$$\sum_{n=0}^{\infty} v_{5n+j} q^n, \qquad |q| < 1. \tag{4.1.5}$$

Although not mentioned by Ramanujan, the coefficients v_{5n+j}, $0 \leq j \leq 4$, can be represented in terms of certain partition functions. From these representations, we can readily show that

$$v_{5n} > 0, \quad v_{5n+1} > 0, \quad v_{5n+2} < 0, \quad v_{5n+3} < 0, \quad v_{5n+4} < 0. \tag{4.1.6}$$

The periodicity of the sign of v_n was first observed by M.D. Hirschhorn and G. Szekeres and was subsequently proved to hold for n sufficiently large by B. Richmond and Szekeres [232]. Their proof of (4.1.6) for sufficiently large n is a consequence of their asymptotic formula

$$v_n = \frac{\sqrt{2}}{(5n)^{3/4}} \exp\left(\frac{4\pi}{25}\sqrt{5n}\right)\left\{\cos\left(\frac{2\pi}{5}\left(n - \frac{2}{5}\right)\right) + O(n^{-1/2})\right\}.$$

Ramanujan also examined the coefficients u_n defined by

$$\frac{1}{C(q)} = \sum_{n=0}^{\infty} u_n q^n, \qquad |q| < 1, \tag{4.1.7}$$

and derived analogous formulas for

$$\sum_{n=0}^{\infty} u_{5n+j} q^n, \qquad 0 \leq j \leq 4. \tag{4.1.8}$$

We conclude Section 4.2 by deriving results for u_n analogous to (4.1.6).

Ramanujan also considered the coefficients v_n and u_n modulo 2. As we shall see in Section 4.3, these formulas involve the famous Rogers–Ramanujan functions $G(q)$ and $H(q)$, which we define in Section 4.3.

One of the most fascinating entries in the lost notebook on the Rogers–Ramanujan continued fraction gives a representation for $1/C^3(q)$ as a quotient of Lambert series, which was first proved by Andrews [22]. Page 47 in the lost notebook contains several further representations for $C(q)$, as well as for $G(q)$ and $H(q)$, in terms of Lambert series, and all of these are proved in Section 4.4.

Section 4.5 provides further q-series representations for $C(q)$, found on page 36 of the lost notebook and first proved by Andrews [26].

4.2 Connections with Partitions

We begin by stating an entry from the lost notebook that is the key to proving five identities for the coefficients v_{5n+j} in (4.1.5).

Entry 4.2.1 (p. 50). *We have*

$$C(q) = \frac{1}{(q^5;q^5)_\infty}\left(\sum_{n=-\infty}^{\infty}(-1)^n q^{(15n^2+n)/2} + q\sum_{n=-\infty}^{\infty}(-1)^n q^{(15n^2+11n)/2}\right).$$

This identity is actually the same as (1.2.21) of Chapter 1.

Entry 4.2.2 (p. 50). *If the coefficients* v_n *are defined by* (4.1.4), *then*

$$\sum_{n=0}^{\infty} v_{5n} q^n = \frac{1}{(q)_\infty} \left(\sum_{n=-\infty}^{\infty} (-1)^n q^{(75n^2+n)/2} + q^4 \sum_{n=-\infty}^{\infty} (-1)^n q^{(75n^2+49n)/2} \right), \tag{4.2.1}$$

$$\sum_{n=0}^{\infty} v_{5n+1} q^n = \frac{1}{(q)_\infty} \left(\sum_{n=-\infty}^{\infty} (-1)^n q^{(75n^2+11n)/2} + q^6 \sum_{n=-\infty}^{\infty} (-1)^n q^{(75n^2+61n)/2} \right), \tag{4.2.2}$$

$$\sum_{n=0}^{\infty} v_{5n+2} q^n = -\frac{q}{(q)_\infty} \left(\sum_{n=-\infty}^{\infty} (-1)^n q^{(75n^2+29n)/2} - q^7 \sum_{n=-\infty}^{\infty} (-1)^n q^{(75n^2+71n)/2} \right), \tag{4.2.3}$$

$$\sum_{n=0}^{\infty} v_{5n+3} q^n = -\frac{1}{(q)_\infty} \left(\sum_{n=-\infty}^{\infty} (-1)^n q^{(75n^2+19n)/2} + q \sum_{n=-\infty}^{\infty} (-1)^n q^{(75n^2+31n)/2} \right), \tag{4.2.4}$$

$$\sum_{n=0}^{\infty} v_{5n+4} q^n = -\frac{q^2}{(q)_\infty} \left(\sum_{n=-\infty}^{\infty} (-1)^n q^{(75n^2+41n)/2} - q^3 \sum_{n=-\infty}^{\infty} (-1)^n q^{(75n^2+59n)/2} \right). \tag{4.2.5}$$

Proof. Recall that the operator U_5 operating on a power series $f(q) = \sum_{n=0}^{\infty} a_n q^n$ is defined by [21, p. 161]

$$U_5 f(q) := \sum_{n=0}^{\infty} a_{5n} q^n = \frac{1}{5} \sum_{j=0}^{4} f(\zeta^j q^{1/5}), \tag{4.2.6}$$

where $\zeta = \exp(2\pi i/5)$. Hence, for $0 \le a \le 4$, by Entry 4.2.1,

$$\begin{aligned}\sum_{n=0}^{\infty} v_{5n+a}q^n &= U_5 q^{-a}C(q)\\
&= \frac{1}{5}\sum_{j=0}^{4}\zeta^{-aj}q^{-a/5}C(\zeta^j q^{1/5})\\
&= \frac{1}{5(q)_\infty}\sum_{j=0}^{4}q^{-a/5}\left(\sum_{n=-\infty}^{\infty}(-1)^n\zeta^{j(15n^2-n-2a)/2}q^{3n^2/2-n/10}\right.\\
&\qquad \left. + q^{1/5}\sum_{n=-\infty}^{\infty}(-1)^n\zeta^{j(15n^2-11n+2-2a)/2}q^{3n^2/2-11n/10}\right).\end{aligned}$$

Now, $15n^2-n-2a \equiv 0 \pmod 5$ for $n \equiv -2a \pmod 5$, while $15n^2-11n+2-2a \equiv 0 \pmod 5$ for $n \equiv 2-2a \pmod 5$. It therefore follows from above that

$$\begin{aligned}\sum_{n=0}^{\infty} v_{5n+a}q^n &= \frac{1}{(q)_\infty}\left(\sum_{n=-\infty}^{\infty}(-1)^n q^{3(5n-2a)^2/2-(5n-2a)/10-a/5}\right.\\
&\qquad \left.+q^{1/5}\sum_{n=-\infty}^{\infty}(-1)^n q^{3(5n+2-2a)^2/2-11(5n+2-2a)/10-a/5}\right)\\
&= \frac{1}{(q)_\infty}\left(q^{6a^2}\sum_{n=-\infty}^{\infty}(-1)^n q^{(75n^2-(60a+1)n)/2}\right.\\
&\qquad \left.+q^{6(1-a)^2-2(1-a)}\sum_{n=-\infty}^{\infty}(-1)^n q^{(75n^2+(49-60a)n)/2}\right). \qquad (4.2.7)\end{aligned}$$

The identities (4.2.1)–(4.2.5) now follow by setting $a = 0, 1, 2, 3, 4$, respectively, in (4.2.7). In most cases, the index of summation needs to be changed by replacing n by $-n$, $n+1$, or $n+2$ to achieve the formulations given by Ramanujan. This completes the proof. $\square$

The next theorem, which is not given by Ramanujan, gives partition-theoretic interpretations of the identities (4.2.1)–(4.2.5).

Theorem 4.2.1. *Let $B_{k,a}(n)$ denote the number of partitions of n of the form $n = b_1 + b_2 + \cdots + b_s$, where $b_i \geq b_{i+1}, b_i - b_{i+k-1} \geq 2$ and at most $a-1$ of the b_i equal 1. Recall that the coefficients v_n are defined by (4.1.4). Then*

$$v_{5n} = B_{37,37}(n) + B_{37,13}(n-4), \qquad (4.2.8)$$
$$v_{5n+1} = B_{37,32}(n) + B_{37,7}(n-6), \qquad (4.2.9)$$
$$v_{5n+2} = -(B_{37,23}(n-1) - B_{37,2}(n-8)), \qquad (4.2.10)$$
$$v_{5n+3} = -(B_{37,28}(n) + B_{37,22}(n-1)), \qquad (4.2.11)$$
$$v_{5n+4} = -(B_{37,17}(n-2) - B_{37,8}(n-5)). \qquad (4.2.12)$$

Proof. We need to recall the generating function for $B_{k,a}(n)$, namely [21, p. 111],

$$\sum_{n=0}^{\infty} B_{k,a}(n)q^n = \frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} (-1)^n q^{((2k+1)n(n+1)-2an)/2}. \tag{4.2.13}$$

We now specialize the parameters k and a in (4.2.13) in order to obtain the appropriate terms on the right sides of (4.2.1)–(4.2.5). Having done so, we then compare coefficients of q^n on each side of the resulting identities in order to obtain (4.2.8)–(4.2.12) and thus complete the proof. □

We next demonstrate the periodicity of signs in (4.1.6).

Corollary 4.2.1. *We have $v_2 = v_4 = v_9 = 0$. The remaining coefficients v_n satisfy the inequalities*

$$v_{5n} > 0, \tag{4.2.14}$$
$$v_{5n+1} > 0, \tag{4.2.15}$$
$$v_{5n+2} < 0, \tag{4.2.16}$$
$$v_{5n+3} < 0, \tag{4.2.17}$$
$$v_{5n+4} < 0. \tag{4.2.18}$$

Proof. The assertions (4.2.14), (4.2.15), and (4.2.17) follow immediately from Theorem 4.2.1. To prove (4.2.16) and (4.2.18), we require the elementary inequality

$$B_{k,a}(r) > B_{k,b}(s), \tag{4.2.19}$$

for $r > s \geq 1$, $k \geq a > b > 0$, which we now prove.

Let $\mathcal{B}_{k,a}(n)$ denote the set of partitions described in Theorem 4.2.1. We first describe an injection from $\mathcal{B}_{k,b}(s)$ into $\mathcal{B}_{k,a}(r)$. Consider any partition from $\mathcal{B}_{k,b}(s)$ and add $r-s$ to the largest part. We easily see that we obtain a partition from $\mathcal{B}_{k,a}(r)$, and so we indeed have the desired injection. To prove the strict inequality in (4.2.19), we need to find an element of $\mathcal{B}_{k,a}(r)$ that is not an image of the mapping just described. To do this, take a partition from $\mathcal{B}_{k,b}(s)$ and add $r-s-1$ to the largest part and 1 to the second-largest part. Note that since $a > b$, the restriction on the number of 1's is not violated. If there is only one part to the partition of $\mathcal{B}_{k,b}(s)$, then the map creates a second part, namely, 1. In either case, the partition obtained is not counted by the first injection. This then proves (4.2.19). □

The argument above by D. Eichhorn is shorter and more elementary than the one given by Andrews in [26]. Hirschhorn [157] makes the observation that these results can be transformed via the quintuple product identity; following this, he deduces Corollaries 4.2.1 and 4.2.2 directly.

We now establish a series of results for the coefficients u_n, defined by (4.1.7), which are completely analogous to the string of results above.

Entry 4.2.3 (p. 50). *We have*

$$\frac{1}{C(q)} = \frac{1}{(q^5;q^5)_\infty}\left(\sum_{n=-\infty}^{\infty}(-1)^n q^{(15n^2-7n)/2} - q\sum_{n=-\infty}^{\infty}(-1)^n q^{(15n^2+13n)/2}\right).$$

This identity is actually the same as (1.2.22) of Chapter 1.

Entry 4.2.4 (p. 50). *If the coefficients u_n are defined by* (4.1.7), *then*

$$\sum_{n=0}^{\infty} u_{5n}q^n = \frac{1}{(q)_\infty}\left(\sum_{n=-\infty}^{\infty}(-1)^n q^{(75n^2-7n)/2} + q^3\sum_{n=-\infty}^{\infty}(-1)^n q^{(75n^2-43n)/2}\right), \tag{4.2.20}$$

$$\sum_{n=0}^{\infty} u_{5n+1}q^n = \frac{1}{(q)_\infty}\left(-\sum_{n=-\infty}^{\infty}(-1)^n q^{(75n^2-13n)/2} - q^2\sum_{n=-\infty}^{\infty}(-1)^n q^{(75n^2-37n)/2}\right), \tag{4.2.21}$$

$$\sum_{n=0}^{\infty} u_{5n+2}q^n = \frac{1}{(q)_\infty}\left(\sum_{n=-\infty}^{\infty}(-1)^n q^{(75n^2-17n)/2} + q^7\sum_{n=-\infty}^{\infty}(-1)^n q^{(75n^2-67n)/2}\right), \tag{4.2.22}$$

$$\sum_{n=0}^{\infty} u_{5n+3}q^n = \frac{1}{(q)_\infty}\left(-q^3\sum_{n=-\infty}^{\infty}(-1)^n q^{(75n^2-47n)/2} + q^4\sum_{n=-\infty}^{\infty}(-1)^n q^{(75n^2-53n)/2}\right), \tag{4.2.23}$$

$$\sum_{n=0}^{\infty} u_{5n+4}q^n = \frac{1}{(q)_\infty}\left(-\sum_{n=-\infty}^{\infty}(-1)^n q^{(75n^2-23n)/2} - q^8\sum_{n=-\infty}^{\infty}(-1)^n q^{(75n^2-73n)/2}\right). \tag{4.2.24}$$

Proof. Recalling that the operator U_5 is defined in (4.2.6), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} u_{5n+a}q^n &= U_5 q^{-a}C^{-1}(q)\\ &= \frac{1}{5}\sum_{j=0}^{4}\zeta^{-aj}q^{-a/5}C^{-1}(\zeta^j q^{1/5})\end{aligned}$$

$$= \frac{1}{5(q)_\infty} \sum_{j=0}^{4} q^{-a/5} \left(\sum_{n=-\infty}^{\infty} (-1)^n \zeta^{j(15n^2-7n-2a)/2} q^{3n^2/2-7n/10} \right.$$
$$\left. - q^{1/5} \sum_{n=-\infty}^{\infty} (-1)^n \zeta^{j(15n^2+13n+2-2a)/2} q^{3n^2/2+13n/10} \right).$$

Now, $15n^2 - 7n - 2a \equiv 0 \pmod 5$ for $n \equiv -a \pmod 5$, while $15n^2 + 13n + 2 - 2a \equiv 0 \pmod 5$ for $n \equiv 1 - a \pmod 5$. Hence, it follows from above that

$$\sum_{n=0}^{\infty} u_{5n+a} q^n = \frac{(-1)^a}{(q)_\infty} \left(q^{a(3a+1)/2} \sum_{n=-\infty}^{\infty} (-1)^n q^{(75n^2-(30a+7)n)/2} \right.$$
$$\left. + q^{3(1-a)(2-a)/2} \sum_{n=-\infty}^{\infty} (-1)^n q^{(75n^2+(30(1-a)+13)n)/2} \right). \tag{4.2.25}$$

The identities (4.2.20)–(4.2.24) now follow by setting $a = 0, 1, 2, 3, 4$ in (4.2.25). As with the proofs of (4.2.1)–(4.2.5), to obtain the final forms of (4.2.20)–(4.2.24), changes in the index of summation need to be made. This completes the proof of Entry 4.2.4. □

Theorem 4.2.2. *Recall that $B_{k,a}(n)$ is defined in Theorem* 4.2.1 *and that the coefficients u_n are defined in* (4.1.7). *Then*

$$u_{5n} = B_{37,34}(n) + B_{37,16}(n-3), \tag{4.2.26}$$
$$u_{5n+1} = -B_{37,31}(n) - B_{37,19}(n-2), \tag{4.2.27}$$
$$u_{5n+2} = B_{37,29}(n) + B_{37,4}(n-7), \tag{4.2.28}$$
$$u_{5n+3} = -B_{37,14}(n-3) + B_{37,11}(n-4), \tag{4.2.29}$$
$$u_{5n+4} = -B_{37,26}(n) - B_{37,1}(n-8). \tag{4.2.30}$$

Proof. As in the proof of Theorem 4.2.1, we employ (4.2.13). We specialize the parameters k and a in (4.2.13) in order to obtain the appropriate terms on the right sides of (4.2.20)–(4.2.24). Having done so, we then compare coefficients of q^n on each side of the resulting identities in order to obtain (4.2.26)–(4.2.30) and thus complete the proof. □

Corollary 4.2.2. *We have $u_3 = u_8 = u_{13} = u_{23} = 0$. The remaining coefficients u_n satisfy the inequalities*

$$u_{5n} > 0, \tag{4.2.31}$$
$$u_{5n+1} < 0, \tag{4.2.32}$$
$$u_{5n+2} > 0, \tag{4.2.33}$$
$$u_{5n+3} < 0, \tag{4.2.34}$$
$$u_{5n+4} < 0. \tag{4.2.35}$$

Proof. The inequalities (4.2.31)–(4.2.33) and (4.2.35) are obvious from Theorem 4.2.2. To prove (4.2.34), use (4.2.29) and (4.2.19). □

The results in Entries 4.2.2 and 4.2.4 have been generalized by R.Y. Denis [129] and K.G. Ramanathan [219]. Hirschhorn [160] has established theorems analogous to Corollaries 4.2.1 and 4.2.2 for the Ramanujan–Göllnitz–Gordon continued fraction. Hirschhorn's result has in turn been generalized by S.H. Chan and H. Yesilyurt [122] using an entirely different method.

4.3 Further Identities Involving the Power Series Coefficients of $C(q)$ and $1/C(q)$

On page 50 in his lost notebook, Ramanujan also states analogues of (4.2.1)–(4.2.5) and (4.2.20)–(4.2.24) for v_{2n+j} and u_{2n+j}, $j = 0, 1$. Recall that $\varphi(q)$ is defined by (1.1.6) in Chapter 1.

Entry 4.3.1 (p. 50). *We have*

$$\sum_{n=0}^{\infty} v_{2n} q^n = \frac{(q^2;q^2)_\infty(-q^2;q^5)_\infty(-q^3;q^5)_\infty}{\varphi(-q^5)}, \tag{4.3.1}$$

$$\sum_{n=0}^{\infty} v_{2n+1} q^n = \frac{(q^{10};q^{10})_\infty}{(-q;q^5)_\infty(-q^4;q^5)_\infty\varphi(-q^5)}. \tag{4.3.2}$$

Proof. Recall the famous Rogers–Ramanujan identities [30], [61, p. 77, Entries 38(i), (ii)],

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty} \tag{4.3.3}$$

and

$$H(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_n} = \frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty}. \tag{4.3.4}$$

(These identities are also given in Entry 3.2.2 of Chapter 3.) Also recall that the definition of $\psi(q)$ is given in (1.1.7) of Chapter 1. Then

$$G(q)H(-q) + G(-q)H(q) = \frac{2\psi(q^2)}{(q^2;q^2)_\infty} \tag{4.3.5}$$

and

$$G(q)H(-q) - G(-q)H(q) = \frac{2q\psi(q^{10})}{(q^2;q^2)_\infty}. \tag{4.3.6}$$

The identities (4.3.5) and (4.3.6) were stated without proofs by Ramanujan among a list of forty identities of this sort, first brought before the mathematical public by B.J. Birch [98] in 1975. These two identities were first proved by G.N. Watson [288] in 1933. The entire manuscript containing these forty identities will be discussed by the authors in [38]. In the lost notebook, there is a surprising two-variable extension of (4.3.6), which has been proved by Andrews [23, Chapter 2] and which will also be proved in [38]. Observe, by (4.1.2), that

$$C(q) = \frac{G(q)}{H(q)}. \tag{4.3.7}$$

We return to the proof of (4.3.1). Using (4.1.4), (4.3.7), (4.3.5), (4.3.4), (1.1.7) in Chapter 1, Euler's identity

$$(-q;q)_\infty = \frac{1}{(q;q^2)_\infty}, \tag{4.3.8}$$

and lastly (1.1.6) in Chapter 1, we find that

$$\begin{aligned}
&\sum_{n=0}^{\infty} v_{2n} q^{2n}\\
&= \frac{1}{2}\left(C(q)+C(-q)\right)\\
&= \frac{1}{2}\left(\frac{G(q)}{H(q)}+\frac{G(-q)}{H(-q)}\right)\\
&= \frac{1}{2}\left(\frac{G(q)H(-q)+H(q)G(-q)}{H(q)H(-q)}\right)\\
&= \frac{\psi(q^2)(q^2;q^5)_\infty(q^3;q^5)_\infty(q^2;-q^5)_\infty(-q^3;-q^5)_\infty}{(q^2;q^2)_\infty}\\
&= \frac{(q^4;q^4)_\infty(q^2;q^5)_\infty(q^3;q^5)_\infty(q^2;q^{10})_\infty(-q^7;q^{10})_\infty(-q^3;q^{10})_\infty(q^8;q^{10})_\infty}{(q^2;q^4)_\infty(q^2;q^2)_\infty}\\
&= (-q^2;q^2)_\infty^2(q^2;q^{10})_\infty^2(q^6;q^{20})_\infty(q^{14};q^{20})_\infty(q^8;q^{10})_\infty^2\\
&= \frac{(-q^2;q^2)_\infty^2(q^2;q^2)_\infty(q^2;q^{10})_\infty(q^8;q^{10})_\infty}{(q^4;q^{20})_\infty(q^{10};q^{10})_\infty(q^{16};q^{20})_\infty}\\
&= \frac{(q^4;q^4)_\infty(-q^2;q^2)_\infty}{(-q^2;q^{10})_\infty(-q^8;q^{10})_\infty(q^{10};q^{10})_\infty}\\
&= \frac{(q^4;q^4)_\infty(-q^4;q^{10})_\infty(-q^6;q^{10})_\infty}{(q^{10};q^{10})_\infty/(-q^{10};q^{10})_\infty}\\
&= \frac{(q^4;q^4)_\infty(-q^4;q^{10})_\infty(-q^6;q^{10})_\infty}{\varphi(-q^{10})}.
\end{aligned} \tag{4.3.9}$$

If we replace q^2 by q in (4.3.9), we obtain (4.3.1).

The proof of (4.3.2) proceeds along similar lines. Using (4.3.7), (4.3.6), (4.3.4), (1.1.7) in Chapter 1, Euler's identity (4.3.8), and lastly (1.1.6) in Chapter 1, we find that

$$\begin{aligned}
&\sum_{n=0}^{\infty} v_{2n+1} q^{2n+1} \\
&= \frac{1}{2}\left(C(q) - C(-q)\right) \\
&= \frac{1}{2}\left(\frac{G(q)}{H(q)} - \frac{G(-q)}{H(-q)}\right) \\
&= \frac{1}{2}\left(\frac{G(q)H(-q) - H(q)G(-q)}{H(q)H(-q)}\right) \\
&= \frac{q\psi(q^{10})(q^2;q^5)_\infty(q^3;q^5)_\infty(q^2;-q^5)_\infty(-q^3;-q^5)_\infty}{(q^2;q^2)_\infty} \\
&= \frac{q(q^{20};q^{20})_\infty(q^2;q^5)_\infty(q^3;q^5)_\infty(q^2;q^{10})_\infty(-q^3;q^{10})_\infty(-q^7;q^{10})_\infty(q^8;q^{10})_\infty}{(q^{10};q^{20})_\infty(q^2;q^2)_\infty} \\
&= \frac{q(q^{20};q^{20})_\infty(q^2;q^{10})_\infty^2(q^8;q^{10})_\infty^2(q^6;q^{20})_\infty(q^{14};q^{20})_\infty}{(q^{10};q^{20})_\infty(q^2;q^2)_\infty} \\
&= \frac{q(q^{20};q^{20})_\infty(q^2;q^{10})_\infty(q^8;q^{10})_\infty}{(q^{10};q^{20})_\infty(q^4;q^{20})_\infty(q^{10};q^{10})_\infty(q^{16};q^{20})_\infty} \\
&= \frac{q(q^{20};q^{20})_\infty}{(q^{10};q^{20})_\infty(-q^2;q^{10})_\infty(-q^8;q^{10})_\infty(q^{10};q^{10})_\infty} \\
&= \frac{q(q^{20};q^{20})_\infty}{(-q^2;q^{10})_\infty(-q^8;q^{10})_\infty\varphi(-q^{10})}. \qquad (4.3.10)
\end{aligned}$$

If we replace q^2 by q in (4.3.10), we deduce (4.3.2). □

Entry 4.3.2 (p. 50). *We have*

$$\sum_{n=0}^{\infty} u_{2n} q^n = \frac{(q^2;q^2)_\infty(-q;q^5)_\infty(-q^4;q^5)_\infty}{\varphi(-q^5)}, \qquad (4.3.11)$$

$$\sum_{n=0}^{\infty} u_{2n+1} q^n = \frac{(q^{10};q^{10})_\infty}{(-q^2;q^5)_\infty(-q^3;q^5)_\infty\varphi(-q^5)}. \qquad (4.3.12)$$

Proof. The proofs of (4.3.11) and (4.3.12) follow precisely along the same lines as those for (4.3.1) and (4.3.2), respectively. □

Hirschhorn [157] conjectured refinements of Entries 4.3.1 and 4.3.2, which were later proved by R.P. Lewis and Z.–G. Liu [177].

4.4 Generalized Lambert Series

In this section we prove several representations for the Rogers–Ramanujan continued fraction involving Lambert series found on page 47 in the lost notebook. The first result is remarkable; we wonder how Ramanujan ever thought of it. A generalization has been given by Denis [131].

Entry 4.4.1 (p. 47). *We have*

$$\frac{1}{C^3(q)} = \frac{\displaystyle\sum_{n=-\infty}^{\infty} \frac{q^{2n}}{1-q^{5n+2}}}{\displaystyle\sum_{n=-\infty}^{\infty} \frac{q^{n}}{1-q^{5n+1}}} \tag{4.4.1}$$

$$= \frac{\displaystyle\sum_{n=0}^{\infty} q^{5n^2+4n}\frac{1+q^{5n+2}}{1-q^{5n+2}} - \sum_{n=0}^{\infty} q^{5n^2+6n+1}\frac{1+q^{5n+3}}{1-q^{5n+3}}}{\displaystyle\sum_{n=0}^{\infty} q^{5n^2+2n}\frac{1+q^{5n+1}}{1-q^{5n+1}} - \sum_{n=0}^{\infty} q^{5n^2+8n+3}\frac{1+q^{5n+4}}{1-q^{5n+4}}}. \tag{4.4.2}$$

The proof of Entry 4.4.1 depends on two lemmas.

Lemma 4.4.1. *For each nonnegative integer j,*

$$\sum_{n=0}^{\infty} q^{5n^2+2jn}\frac{1+q^{5n+j}}{1-q^{5n+j}} = \sum_{n=0}^{\infty} \frac{q^{jn}}{1-q^{5n+j}}. \tag{4.4.3}$$

Proof. In the following, we first expand the summands in geometric series, secondly invert the order of summation in the first series in the second equality and make the change of index $k = m-n-1$ in the second series, thirdly make the change of index $k = n-m$ in the first series, and lastly sum the geometric series. Accordingly, we find that

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{q^{jn}}{1-q^{5n+j}} &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} q^{jn+5nm+jm} \\
&= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} + \sum_{m=n+1}^{\infty}\right) q^{jn+5nm+jm} \\
&= \sum_{m=0}^{\infty}\sum_{n=m}^{\infty} q^{jn+5nm+jm} + \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} q^{jn+5n(k+n+1)+j(k+n+1)} \\
&= \sum_{m=0}^{\infty}\sum_{k=0}^{\infty} q^{j(m+k)+5m(m+k)+jm} \\
&\quad + \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} q^{jn+5n(k+n+1)+j(k+n+1)} \\
&= \sum_{m=0}^{\infty} \frac{q^{5m^2+2jm}}{1-q^{5m+j}} + \sum_{n=0}^{\infty} \frac{q^{5n^2+2nj+5n+j}}{1-q^{5n+j}}.
\end{aligned}$$

The lemma now follows by replacing m by n in the first sum on the far right side above and then combining the two series together. ∎

Lemma 4.4.2. *For every pair of nonnegative integers i, j,*

$$\sum_{n=-\infty}^{\infty} \frac{q^{in}}{1-q^{5n+j}} = \frac{(q^5;q^5)_\infty^2 (q^{i+j};q^5)_\infty (q^{5-i-j};q^5)_\infty}{(q^j;q^5)_\infty (q^{5-j};q^5)_\infty (q^i;q^5)_\infty (q^{5-i};q^5)_\infty}. \tag{4.4.4}$$

Proof. We extend the definition of $(a;q)_n$, given in (1.1.3) of Chapter 1, by defining, for *all* integers n,

$$(a)_n := (a;q)_n := \frac{(a;q)_\infty}{(aq^n;q)_\infty}. \tag{4.4.5}$$

We shall utilize Ramanujan's famous ${}_1\psi_1$ summation [61, pp. 32, 34]. For any complex numbers a, b, z with $|z| < 1$ and $|b/a| < 1$,

$$\sum_{n=-\infty}^{\infty} \frac{(a)_n}{(b)_n} z^n = \frac{(az)_\infty (q/(az))_\infty (q)_\infty (b/a)_\infty}{(z)_\infty (b/(az))_\infty (b)_\infty (q/a)_\infty}, \tag{4.4.6}$$

where we employ the notation (4.4.5). Now replace q by q^5 and set $a = q^j$, $b = q^{5+j}$, and $z = q^i$ in (4.4.6). Multiplying both sides by $1/(1-q^j)$ and simplifying, we complete the proof of Lemma 4.4.2. □

Proof of Entry 4.4.1. First applying Lemma 4.4.1 four times and then invoking Lemma 4.4.2 twice, we find that

$$\begin{aligned}
&\frac{\displaystyle\sum_{n=0}^{\infty} q^{5n^2+4n}\frac{1+q^{5n+2}}{1-q^{5n+2}} - \sum_{n=0}^{\infty} q^{5n^2+6n+1}\frac{1+q^{5n+3}}{1-q^{5n+3}}}{\displaystyle\sum_{n=0}^{\infty} q^{5n^2+2n}\frac{1+q^{5n+1}}{1-q^{5n+1}} - \sum_{n=0}^{\infty} q^{5n^2+8n+3}\frac{1+q^{5n+4}}{1-q^{5n+4}}} \\
&= \frac{\displaystyle\sum_{n=0}^{\infty} \frac{q^{2n}}{1-q^{5n+2}} - \sum_{n=0}^{\infty} \frac{q^{3n+1}}{1-q^{5n+3}}}{\displaystyle\sum_{n=0}^{\infty} \frac{q^{n}}{1-q^{5n+1}} - \sum_{n=0}^{\infty} \frac{q^{4n+3}}{1-q^{5n+4}}} \\
&= \frac{\displaystyle\sum_{n=-\infty}^{\infty} \frac{q^{2n}}{1-q^{5n+2}}}{\displaystyle\sum_{n=-\infty}^{\infty} \frac{q^{n}}{1-q^{5n+1}}} \\
&= \frac{(q^5;q^5)_\infty^2 (q^4;q^5)_\infty (q;q^5)_\infty}{(q^2;q^5)_\infty^2 (q^3;q^5)_\infty^2} \frac{(q;q^5)_\infty^2 (q^4;q^5)_\infty^2}{(q^5;q^5)_\infty^2 (q^2;q^5)_\infty (q^3;q^5)_\infty} \\
&= \left(\frac{(q;q^5)_\infty (q^4;q^5)_\infty}{(q^2;q^5)_\infty (q^3;q^5)_\infty} \right)^3 .
\end{aligned}$$

Appealing to (4.1.2) and (4.1.3), we complete the proof. □

There are 13 further identities of this type given by Ramanujan for $C(q)$, $G(q)$, and $H(q)$. We offer them in the next two entries.

Entry 4.4.2 (p. 47). *Recall that $G(q)$ and $H(q)$ are defined, respectively, in* (4.3.3) *and* (4.3.4). *Then*

$$(q^5;q^5)_\infty^2 G(q) = \sum_{n=-\infty}^{\infty} \frac{q^n}{1-q^{5n+2}}, \tag{4.4.7}$$

$$(q^5;q^5)_\infty^2 H(q) = \sum_{n=-\infty}^{\infty} \frac{q^{3n}}{1-q^{5n+1}}, \tag{4.4.8}$$

$$(q^5;q^5)_\infty^2 \frac{G^2(q)}{H(q)} = \sum_{n=-\infty}^{\infty} \frac{q^n}{1-q^{5n+1}}, \tag{4.4.9}$$

$$(q^5;q^5)_\infty^2 \frac{H^2(q)}{G(q)} = \sum_{n=-\infty}^{\infty} \frac{q^{2n}}{1-q^{5n+2}}, \tag{4.4.10}$$

$$(q^5;q^5)_\infty^2 G(q) = \sum_{n=-\infty}^{\infty} \frac{q^{2n}}{1-q^{5n+1}}, \tag{4.4.11}$$

$$(q^5;q^5)_\infty^2 H(q) = \sum_{n=-\infty}^{\infty} \frac{q^n}{1-q^{5n+3}}, \tag{4.4.12}$$

$$(q^5;q^5)_\infty^2 \frac{G^2(q)}{H(q)} = \sum_{n=-\infty}^{\infty} q^{5n^2+2n} \frac{1+q^{5n+1}}{1-q^{5n+1}}, \tag{4.4.13}$$

$$(q^5;q^5)_\infty^2 \frac{H^2(q)}{G(q)} = \sum_{n=-\infty}^{\infty} q^{5n^2+4} \frac{1+q^{5n+2}}{1-q^{5n+2}}, \tag{4.4.14}$$

$$(q^5;q^5)_\infty^2 G(q) = \sum_{n=-\infty}^{\infty} \frac{q^{4n}}{1-q^{10n+1}}, \tag{4.4.15}$$

$$(q^5;q^5)_\infty^2 H(q) = \sum_{n=-\infty}^{\infty} \frac{q^{2n}}{1-q^{10n+3}}. \tag{4.4.16}$$

Proof. The proofs below make frequent use of (4.1.2), (4.3.3), and (4.3.4).

To prove (4.4.7), use (4.4.4) with $i=1$ and $j=2$.

To prove (4.4.8), use (4.4.4) with $i=3$ and $j=1$.

To prove (4.4.9), use (4.4.4) with $i=1$ and $j=1$.

To prove (4.4.10), use (4.4.4) with $i=2$ and $j=2$.

To prove (4.4.11), use (4.4.4) with $i=2$ and $j=1$.

To prove (4.4.12), use (4.4.4) with $i=1$ and $j=3$.

We next prove (4.4.13). We first appeal to (4.4.9). For $n<0$ below, replace n by $-n-1$. Then employing (4.4.3) twice, with $j=1$ and $j=4$, we find that

$$(q^5;q^5)_\infty^2 \frac{G^2(q)}{H(q)} = \sum_{n=-\infty}^{\infty} \frac{q^n}{1-q^{5n+1}}$$

$$
\begin{aligned}
&= \sum_{n=0}^{\infty} \frac{q^n}{1-q^{5n+1}} - q^3 \sum_{n=0}^{\infty} \frac{q^{4n}}{1-q^{5n+4}} \\
&= \sum_{n=0}^{\infty} q^{5n^2+2n} \frac{1+q^{5n+1}}{1-q^{5n+1}} - q^3 \sum_{n=0}^{\infty} q^{5n^2+8n} \frac{1+q^{5n+4}}{1-q^{5n+4}} \\
&= \sum_{n=0}^{\infty} q^{5n^2+2n} \frac{1+q^{5n+1}}{1-q^{5n+1}} + \sum_{n=-\infty}^{-1} q^{5n^2+2n} \frac{1+q^{5n+1}}{1-q^{5n+1}} \\
&= \sum_{n=-\infty}^{\infty} q^{5n^2+2n} \frac{1+q^{5n+1}}{1-q^{5n+1}},
\end{aligned}
$$

where in the penultimate equality we replaced n by $-n-1$ in the second sum. This completes the proof of (4.4.13).

The proof of (4.4.14) begins with (4.4.10) and follows exactly the same steps as in the previous proof, but appeals to (4.4.3) in the cases $j = 2$ and $j = 3$ instead of $j = 1$ and $j = 4$.

To prove (4.4.15), once again use (4.4.4), but now with q replaced by q^2, and with $i = 2$ and $j = \frac{1}{2}$. A mild amount of simplification is required.

To prove (4.4.16), once again use (4.4.4) with q replaced by q^2, but with $i = 1$ and $j = \frac{3}{2}$. A mild amount of simplification is required. □

The next three results are simple consequences of parts of the foregoing entry.

Entry 4.4.3 (p. 47). *We have*

$$
C(q) = \frac{\displaystyle\sum_{n=-\infty}^{\infty} \frac{q^n}{1-q^{5n+2}}}{\displaystyle\sum_{n=-\infty}^{\infty} \frac{q^{3n}}{1-q^{5n+1}}}, \tag{4.4.17}
$$

$$
C^2(q) = \frac{\displaystyle\sum_{n=-\infty}^{\infty} \frac{q^n}{1-q^{5n+1}}}{\displaystyle\sum_{n=-\infty}^{\infty} \frac{q^{3n}}{1-q^{5n+1}}}, \tag{4.4.18}
$$

$$
C^2(q) = \frac{\displaystyle\sum_{n=-\infty}^{\infty} \frac{q^n}{1-q^{5n+2}}}{\displaystyle\sum_{n=-\infty}^{\infty} \frac{q^{2n}}{1-q^{5n+2}}}. \tag{4.4.19}
$$

Proof. To prove (4.4.17), divide (4.4.7) by (4.4.8) and use (4.3.7).

To prove (4.4.18), divide (4.4.9) by (4.4.8) and use (4.3.7).

To prove (4.4.19), divide (4.4.7) by (4.4.10) and use (4.3.7). □

Several of the generalized Lambert series identities in this section were generalized by Denis [129].

4.5 Further q-Series Representations for $C(q)$

In this last section of the chapter, we establish four identities for $C(q)$ found on page 36 in the lost notebook.

Entry 4.5.1 (p. 36). *We have*

$$\frac{(q^5;q^5)_\infty}{C(q)} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{(5n^2+3n)/2}}{(q^2;q^5)_{n+1}} - \sum_{n=0}^{\infty} \frac{(-1)^n q^{(5n^2+7n+2)/2}}{(q^3;q^5)_{n+1}}, \quad (4.5.1)$$

$$(q^5;q^5)_\infty C(q) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{(5n^2-n)/2}}{(q;q^5)_{n+1}} - \sum_{n=0}^{\infty} \frac{(-1)^n q^{(5n^2+11n+6)/2}}{(q^4;q^5)_{n+1}}, \quad (4.5.2)$$

and

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-1)^n q^{(5n^2+7n+2)/2}}{(q^3;q^5)_{n+1}} &= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n q^{(5n^2+7n)/2}(1+q^{8n+4}) \\ &\quad + \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n q^{(5n^2+13n+2)/2}(1+q^{2n+1}) - \frac{(q^5;q^5)_\infty}{2C(q)},\end{aligned} \quad (4.5.3)$$

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-1)^n q^{(5n^2+3n)/2}}{(q^2;q^5)_{n+1}} &= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n q^{(5n^2+7n)/2}(1+q^{8n+4}) \\ &\quad + \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n q^{(5n^2+13n+2)/2}(1+q^{2n+1}) + \frac{(q^5;q^5)_\infty}{2C(q)}.\end{aligned} \quad (4.5.4)$$

Proof. We use the Rogers–Fine identity [137, p. 15, equation (14.10)]. If a, b, and t are complex numbers with $|t| < 1$, then

$$\sum_{n=0}^{\infty} \frac{(a)_n}{(b)_{n+1}} t^n = \sum_{n=0}^{\infty} \frac{(a)_n (at/b)_n b^n t^n q^{n^2}(1-atq^{2n})}{(b)_{n+1}(t)_{n+1}}. \quad (4.5.5)$$

This result is not stated in Ramanujan's notebooks or lost notebook. However, we have used it many times to prove Ramanujan's formulas. All of Chapter 9 in this volume is devoted to formulas from the lost notebook that arise from using the Rogers–Fine identity.

We begin with the proof of (4.5.1). We apply (4.5.5) to each sum on the right side of (4.5.1). In the first sum, we replace q by q^5, set $a = q^4/t$ and $b = q^2$, and let $t \to 0$; in the second sum, we replace q by q^5, set $a = q^6/t$ and $b = q^3$, and let $t \to 0$. We then find that

$$\begin{aligned}&\sum_{n=0}^{\infty} \frac{(-1)^n q^{(5n^2+3n)/2}}{(q^2;q^5)_{n+1}} - \sum_{n=0}^{\infty} \frac{(-1)^n q^{(5n^2+7n+2)/2}}{(q^3;q^5)_{n+1}} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n q^{(5n^2+3n)/2}(q^2;q^5)_n q^{2n} q^{5n^2}(1-q^{10n+4})}{(q^2;q^5)_{n+1}}\end{aligned}$$

$$
\begin{aligned}
&- q\sum_{n=0}^{\infty} \frac{(-1)^n q^{(5n^2+7n)/2}(q^3;q^5)_n q^{3n} q^{5n^2}(1-q^{10n+6})}{(q^3;q^5)_{n+1}} \\
&= \sum_{n=0}^{\infty}(-1)^n q^{(15n^2+7n)/2}(1+q^{5n+2}) - \sum_{n=0}^{\infty}(-1)^n q^{(15n^2+13n+2)/2}(1+q^{5n+3}) \\
&= \sum_{n=-\infty}^{\infty} (-1)^n q^{(15n^2+7n)/2}(1-q^{3n+1}) \\
&= \sum_{n=-\infty}^{\infty} (-1)^n q^{(15n^2-7n)/2} - q\sum_{n=-\infty}^{\infty} (-1)^n q^{(15n^2+13n)/2} \\
&= \frac{(q^5;q^5)_\infty}{C(q)},
\end{aligned}
$$

by Entry 4.2.3.

To prove (4.5.2), we proceed in the same fashion. We apply (4.5.5) to each of the series on the right side of (4.5.2). In the first sum, replace q by q^5, set $a = q^2/t$ and $b = q$, and let $t \to 0$; in the second sum, we replace q by q^5, set $a = q^8/t$ and $b = q^4$, and let $t \to 0$. Accordingly, we find that after performing routine simplification,

$$
\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(-1)^n q^{(5n^2-n)/2}}{(q;q^5)_{n+1}} - \sum_{n=0}^{\infty} \frac{(-1)^n q^{(5n^2+11n+6)/2}}{(q^4;q^5)_{n+1}} \\
&= \sum_{n=0}^{\infty}(-1)^n q^{(15n^2+n)/2}(1+q^{5n+1}) - \sum_{n=0}^{\infty}(-1)^n q^{(15n^2+19n+6)/2}(1+q^{5n+4}) \\
&= \sum_{n=-\infty}^{\infty} (-1)^n q^{(15n^2+n)/2} - \sum_{n=-\infty}^{\infty} (-1)^n q^{(15n^2+19n+6)/2} \\
&= \sum_{n=-\infty}^{\infty} (-1)^n q^{(15n^2+n)/2} + q\sum_{n=-\infty}^{\infty} (-1)^n q^{(15n^2+11n)/2} \\
&= (q^5;q^5)C(q),
\end{aligned}
$$

by Entry 4.2.1, where in the antepenultimate line we replaced n by $-n-1$.

We now observe that if we subtract (4.5.3) from (4.5.4), we obtain (4.5.1). Therefore, since (4.5.1) has already been established, we need only prove (4.5.3). From the proof of (4.5.1) and from another application of Entry 4.2.3, we find that

$$
\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(-1)^n q^{(5n^2+7n+2)/2}}{(q^3;q^5)_{n+1}} \\
&\qquad = -\frac{(q^5;q^5)_\infty}{C(q)} + \sum_{n=0}^{\infty}(-1)^n q^{(15n^2+7n)/2}(1+q^{5n+2})
\end{aligned}
$$

$$= -\frac{(q^5;q^5)_\infty}{2C(q)} + \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n q^{(15n^2+7n)/2}(1+q^{5n+2})$$
$$+ \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n q^{(15n^2+13n+2)/2}(1+q^{5n+3})$$
$$= -\frac{(q^5;q^5)_\infty}{2C(q)} + \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n q^{(15n^2+7n)/2}(1+q^{8n+4})$$
$$+ \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n q^{(15n^2+13n+2)/2}(1+q^{2n+1}).$$

This completes the proof of (4.5.3) and with it (4.5.4). □

N.J. Fine [137] has found many applications of (4.5.5). See also MacMahon's *Collected Papers* [186, Chapter 16, Section 16.2]. Several arithmetical applications of (4.5.5) have been made by Andrews in [20].

S. Bhargava [90] has employed Ramanujan's ${}_1\psi_1$ summation theorem to give another proof of (4.5.1) and (4.5.2).

5

Finite Rogers–Ramanujan Continued Fractions

5.1 Introduction

We begin with some basic notation. For a continued fraction of the form

$$\frac{a_1}{1}\,\underset{+}{}\,\frac{a_2}{1}\,\underset{+}{}\,\frac{a_3}{1}\,\underset{+}{}\,\underset{\cdots}{}, \tag{5.1.1}$$

let

$$\frac{P_n}{Q_n} := \frac{a_1}{1}\,\underset{+}{}\,\frac{a_2}{1}\,\underset{+}{}\,\underset{\cdots}{}\,\underset{+}{}\,\frac{a_n}{1}, \quad n \geq 1, \tag{5.1.2}$$

be the nth *convergent* (or *approximant*). Set $P_{-1} = 1$, $Q_{-1} = 0$, $P_0 = 0$, and $Q_0 = 1$. By convention, the value of (5.1.1), if it exists, is defined to be the limit of the sequence $\{P_n/Q_n\}$ as n tends to infinity. The partial numerators and denominators, P_n and Q_n, respectively, satisfy the basic relations [182, p. 9]

$$P_n = P_{n-1} + a_n P_{n-2}, \qquad Q_n = Q_{n-1} + a_n Q_{n-2}, \tag{5.1.3}$$

and

$$P_n Q_{n-1} - P_{n-1} Q_n = (-1)^{n-1} a_1 a_2 \cdots a_n, \tag{5.1.4}$$

where $n = 1, 2, 3, \ldots$.

Define, for $|q| < 1$ and complex a,

$$R(a) := \frac{a}{1}\,\underset{+}{}\,\frac{aq}{1}\,\underset{+}{}\,\frac{aq^2}{1}\,\underset{+}{}\,\underset{\cdots}{}. \tag{5.1.5}$$

(We have adhered here to Ramanujan's notation, although it conflicts with the notation for the infinite Rogers–Ramanujan continued fraction $R(q)$ employed in the previous chapters. The infinite Rogers–Ramanujan continued fraction does not appear in this chapter, and so no confusion should arise.) On page 46 of his lost notebook [228], Ramanujan evaluates $R(a)$ in terms of its $(m-1)$th convergent when q is a primitive mth root of unity. More precisely, Ramanujan claims that given any primitive mth root of unity q,

$$R(a) = \frac{P(a) + \frac{1}{2}(-1 + \sqrt{1 + 4a^m})}{Q(a)}, \tag{5.1.6}$$

where

$$\frac{P(a)}{Q(a)} := \frac{P_{m-1}(a)}{Q_{m-1}(a)} := \frac{a}{1} + \frac{aq}{1} + \cdots + \frac{aq^{m-2}}{1}$$

is the $(m-1)$th convergent of $R(a)$. We suppress the index m, because m is regarded as fixed. Although it appears that (5.1.6) can be obtained by simply solving a quadratic equation, many difficulties arise.

We first proceed with the study of the convergents of $R(a)$. In particular, Theorem 5.2.1 below is one of the main ingredients needed to prove (5.1.6). It turns out that Theorem 5.2.1 can be established by proving an interesting identity, (5.2.1). To prove (5.2.1), we need the concept of *relative decompositions* $(\bmod\, m)$, which was first introduced and studied by H. Stern [260], [261] in 1863 and then further developed and generalized by R.D. von Sterneck [262], [263], [264] in 1902–1905. Next, we restate (5.1.6) as Entry 5.2.1 in a more precise way and then prove it by using results of Worpitzky (Lemma 5.2.2) and Vitali (Lemma 5.2.3).

Next, we generalize Entry 5.2.1 by proving some results on page 57 of Ramanujan's lost notebook. We also shall explain a mistake Ramanujan made in his ordinary notebooks [227] on evaluating the Rogers–Ramanujan continued fraction at primitive roots of unity [63, p. 35].

All the results in Sections 5.2 and 5.3 were first proved by S.–S. Huang [164].

In Section 5.4, we show that certain finite Rogers–Ramanujan continued fractions have zeros that can be expressed in terms of singular moduli. It is tempting to contemplate that these curious examples might be illustrations of a more general theory, but we doubt that this is the case. The results in this section were first published in [78].

In the last section, Section 5.5, we examine an identity for certain finite generalized Rogers–Ramanujan continued fractions. Results in this last section are taken from J. Sohn's thesis [253].

5.2 Evaluations of Finite Generalized Rogers–Ramanujan Continued Fractions at Primitive Roots of Unity

To simplify notation, we define the two sets

$$\mathcal{A}_n := \{\mathbf{v} = (n_1, \ldots, n_r) \in \mathbf{N}^r | r \geq 1,\ n_1 = 1,\ n_{i+1} - n_i \geq 2,\ \text{and } n_r \leq n\}$$

and

$$\mathcal{B}_n := \{\mathbf{v} = (n_1, \ldots, n_r) \in \mathbf{N}^r | r \geq 1,\ n_1 \geq 2,\ n_{i+1} - n_i \geq 2,\ \text{and } n_r \leq n\}.$$

Lemma 5.2.1. *For each positive integer n,*

(i)
$$P_n \;=\; \sum_{\mathbf{v}\in\mathcal{A}_n} a_{n_1}\cdots a_{n_r}$$

and

(ii)
$$Q_n \;=1\;+\;\sum_{\mathbf{v}\in\mathcal{B}_n} a_{n_1}\cdots a_{n_r},$$

where P_n and Q_n are defined in (5.1.2)*.*

Proof of (i)*.* Use induction on n. Clearly, (i) is valid for $n=1$. Assume that (i) is true up to n. Then, by the first recurrence relation in (5.1.3),

$$\begin{aligned} P_{n+1} &= P_n + a_{n+1}P_{n-1} \\ &= \sum_{\mathbf{v}\in\mathcal{A}_n} a_{n_1}\cdots a_{n_r} + \sum_{\mathbf{v}\in\mathcal{A}_{n-1}} a_{n_1}\cdots a_{n_r}a_{n+1} \\ &= \sum_{\mathbf{v}\in\mathcal{A}_{n+1}} a_{n_1}\cdots a_{n_r}. \end{aligned}$$

Identity (ii) can be proved in a similar manner by using the second recurrence relation in (5.1.3). □

Lemma 5.2.1 is known as the Euler–Minding theorem [206, p. 9].

In the sequel, we denote the nth convergent of $R(a)$ by $P_n(a)/Q_n(a)$, i.e.,

$$\frac{P_n(a)}{Q_n(a)} = \frac{a}{1}\,+\,\frac{aq}{1}\,+\cdots+\,\frac{aq^{n-1}}{1}.$$

Theorem 5.2.1. *For any number a, and any primitive mth root of unity q,*

$$P_{m-1}(a) \;+\; Q_m(a) \;=\; 1.$$

Before we prove Theorem 5.2.1, let us take a closer look at the sum of $P_{m-1}(a)$ and $Q_m(a)$. First, define $\mathcal{A}_n(l)$ to be the subset of $\mathcal{A}_n$ that contains all the l-dimensional vectors. Similarly, $\mathcal{B}_n(l)$ contains all the l-dimensional vectors in $\mathcal{B}_n$. Then, by Lemma 5.2.1, we find that

$$\begin{aligned} &P_{m-1}(a) \;+\; Q_m(a) \\ &= \sum_{\mathbf{v}\in\mathcal{A}_{m-1}} aq^{n_1-1}\cdots aq^{n_r-1} \;+\;1\;+\; \sum_{\mathbf{v}\in\mathcal{B}_m} aq^{n_1-1}\cdots aq^{n_r-1} \\ &= 1\;+\;\sum_{r=1}^{[m/2]} a^rq^{-r}\sum_{\mathbf{v}\in\mathcal{A}_{m-1}(r)} q^{n_1+\cdots+n_r} \;+\; \sum_{r=1}^{[m/2]} a^rq^{-r}\sum_{\mathbf{v}\in\mathcal{B}_m(r)} q^{n_1+\cdots+n_r} \\ &= 1\;+\;\sum_{r=1}^{[m/2]} a^rq^{-r}\left(\sum_{\mathbf{v}\in\mathcal{A}_{m-1}(r)} q^{n_1+\cdots+n_r} \;+\; \sum_{\mathbf{v}\in\mathcal{B}_m(r)} q^{n_1+\cdots+n_r}\right) \end{aligned}$$

$$= 1 + \sum_{r=1}^{[m/2]} a^r q^{-r} \left(\sum_{\mathbf{v} \in \mathcal{C}_m(r)} q^{n_1+\cdots+n_r} \right),$$

where $\mathcal{C}_m(r)$ is the union of $\mathcal{A}_{m-1}(r)$ and $\mathcal{B}_m(r)$.

Therefore, to prove Theorem 5.2.1, it suffices to show that given any primitive mth root of unity q,

$$\sum_{\mathbf{v} \in \mathcal{C}_m(r)} q^{n_1+\cdots+n_r} = 0, \quad \text{for each} \quad r = 1, 2, \ldots, [m/2]. \tag{5.2.1}$$

We will prove (5.2.1) as a corollary of the next theorem.

The following definition was introduced by H. Stern [260], and the name *relative decompositions* $(\mathrm{mod}\, m)$ was given by P. Bachmann [48, Part II, Chapter 5].

Definition 5.2.1. *Let n be a positive integer. A sequence $(n_1, n_2, \ldots, n_r)$ of positive integers is called a relative decomposition* $(\mathrm{mod}\, m)$ *of n* (*with r parts*) *if*

$$0 \le n_1 < n_2 < \cdots < n_r \le m-1 \tag{5.2.2}$$

and

$$n \equiv n_1 + n_2 + \cdots + n_r \pmod{m}. \tag{5.2.3}$$

Also, we adopt von Sterneck's notation $(n)_r$ to indicate the number of all possible relative decompositions (mod m) of n with r parts. The function $(n)_r$ can be viewed as an analogue of $p(n, r)$, the number of ordinary partitions of n into r parts. It is easy to see that $p(n, r) \le (n)_r$. For work on relative decompositions, we refer readers to Bachmann's text [48] and the papers of Stern [260], [261] and von Sterneck [262]–[264].

In the sequel, instead of considering $(n)_r$, we focus on restricted relative decompositions (mod m) with r parts. More precisely, for each $n = 0, 1, \ldots, m-1$, let $\mathfrak{G}_r(n)$ denote the set of all the relative decompositions (mod m) of n with r parts subject to the conditions

$$n_{i+1} - n_i \ge 2, \quad \text{for each } i = 1, 2, \ldots, r-1, \tag{5.2.4}$$

and

$$n_r - n_1 \le m - 2. \tag{5.2.5}$$

Note that $\mathfrak{G}_r(n)$, $n = 0, 1, \ldots, m-1$, are disjoint. Also, let $\mathfrak{g}_r(n)$ be the cardinality of $\mathfrak{G}_r(n)$.

Theorem 5.2.2. *Let r and m be positive integers with greatest common divisor d and let $j_1, j_2 \in \{0, 1, \ldots, m-1\}$. If d divides $j_1 - j_2$ or $j_1 + j_2$, then*

$$\mathfrak{g}_r(j_1) = \mathfrak{g}_r(j_2).$$

In particular, if r and m are relatively prime, then

$$\mathfrak{g}_r(0) = \mathfrak{g}_r(1) = \cdots = \mathfrak{g}_r(m-1).$$

Proof. First, suppose that d divides $j_1 - j_2$. Then, we can write $j_2 = j_1 + ud$ for some integer u. To prove the result, it suffices to find a one-to-one mapping from $\mathfrak{G}_r(j_1)$ onto $\mathfrak{G}_r(j_2)$. Note that since $d = (r, m)$, there exist integers α and β such that $\alpha r + \beta m = d$.

Next, given an element $(n_1, n_2, \dots, n_r)$ in $\mathfrak{G}_r(j_1)$, define a new sequence of positive integers

$$(\overline{n_1 + u\alpha}, \overline{n_2 + u\alpha}, \dots, \overline{n_r + u\alpha}), \tag{5.2.6}$$

where $\bar{z}$ designates the smallest positive residue of z modulo m. Finally, denote the sequence (5.2.6) by $(k_1, k_2, \dots, k_r)$, after rearranging the coordinates in nondecreasing order.

Now define φ from $\mathfrak{G}_r(j_1)$ to $\mathfrak{G}_r(j_2)$ by assigning to each element $(n_1, n_2, \dots, n_r)$ in $\mathfrak{G}_r(j_1)$ the sequence $(k_1, k_2, \dots, k_r)$ obtained by the procedure described above. The mapping φ is clearly one-to-one and onto (u and α are fixed), provided that φ is well-defined. Thus, it remains to show that $(k_1, k_2, \dots, k_r)$ satisfies (5.2.3)–(5.2.5) with n replaced by j_2. By taking congruences modulo m and using the fact $\alpha r + \beta m = d$, we find that

$$\sum_{i=1}^{r} k_i \equiv \sum_{i=1}^{r} n_i + u\alpha r \equiv j_1 + ud = j_2 \pmod{m},$$

and hence (5.2.3) is justified.

Next, observe that

$$(k_1, \dots, k_r) = (\overline{n_1 + u\alpha}, \dots, \overline{n_r + u\alpha}), \quad \text{if } \overline{n_1 + u\alpha} \le \cdots \le \overline{n_r + u\alpha},$$

and otherwise,

$$(k_1, \dots, k_r) = (\overline{n_\nu + u\alpha}, \dots, \overline{n_r + u\alpha}, \overline{n_1 + u\alpha}, \dots, \overline{n_{\nu-1} + u\alpha}),$$

where ν is the smallest integer such that $\overline{n_\nu + u\alpha} < \overline{n_{\nu-1} + u\alpha}$. In any case, (5.2.4) and (5.2.5) are satisfied. Therefore, $\mathfrak{G}_r(j_1) \cong \mathfrak{G}_r(j_2)$ (as *sets*), i.e., $\mathfrak{g}_r(j_1) = \mathfrak{g}_r(j_2)$. The case in which d divides $j_1 + j_2$ is proved similarly. □

Theorem 5.2.2 yields immediately the following result.

Corollary 5.2.1. *If r and m are positive integers with $(r, m) = d$, then*

$$\mathfrak{g}_r(ld + k) = \mathfrak{g}_r(k),$$

for any $k \in \{0, 1, \dots, d-1\}$ and any $l \in \left\{0, 1, \dots, \frac{m}{d} - 1\right\}$.

Now, we are in a position to prove (5.2.1) and finish the proof of Theorem 5.2.1.

Corollary 5.2.2. *For any primitive mth root of unity q,* (5.2.1) *holds.*

Proof. Given $r \in \{1, 2, \ldots, [m/2]\}$, we denote (r, m) by d. By the definitions of $\mathcal{C}_m(r)$ and $\mathfrak{G}_r(j)$,

$$\mathcal{C}_m(r) = \mathfrak{G}_r(0) \cup \mathfrak{G}_r(1) \cup \cdots \cup \mathfrak{G}_r(m-1). \tag{5.2.7}$$

Then, by (5.2.7), the disjointness of the sets $\mathfrak{G}_r(j)$, and Corollary 5.2.1,

$$\begin{aligned}\sum_{\mathbf{v}\in\mathcal{C}_m(r)} q^{n_1+\cdots+n_r} = \sum_{j=0}^{m-1} \mathfrak{g}_r(j)\, q^j &= \sum_{k=0}^{d-1} \sum_{l=0}^{(m/d)-1} \mathfrak{g}_r(ld+k)\, q^{ld+k} \\ &= \sum_{k=0}^{d-1} \mathfrak{g}_r(k)\, q^k \sum_{l=0}^{(m/d)-1} q^{ld} = 0,\end{aligned}$$

where the last equality follows from the fact that q^d is a primitive (m/d)th root of unity. Hence, (5.2.1) is established, and so is Theorem 5.2.1. □

We still need three additional lemmas before embarking on the proof of (5.1.6).

Lemma 5.2.2 (Worpitzky's theorem). *Let* $\mathbf{K}(a_n/1)$ *be the continued fraction defined in* (5.1.1). *If* $|a_n| \leq 1/4$, *then* $\mathbf{K}(a_n/1)$ *converges. Moreover, all approximants* P_n/Q_n, *defined in* (5.1.2), *are in the disk* $|w| < 1/2$, *and the value of the continued fraction is in the disk* $|w| \leq 1/2$.

See [182, p. 35] for a proof.

Definition 5.2.2. *Let* Λ *be a set of functions, all defined on the same domain* G, *and suppose that for every compact subset* $F \subset G$, *there is a number* $M(F) > 0$ *such that*

$$|f(z)| \leq M(F)$$

for all $f \in \Lambda$ *and* $z \in F$. *Then* Λ *is said to be uniformly bounded inside* G.

Lemma 5.2.3 (Vitali's theorem). *Let* G *be a domain, and let* $\{f_n\}$ *be a sequence of analytic functions in* G. *Suppose that the sequence* $\{f_n\}$ *is uniformly bounded inside* G *and converges on a set of points* $E \subset G$ *with a limit point in* G. *Then* $\{f_n\}$ *converges uniformly inside* G.

See [193, pp. 415–417] for a proof.

Lemma 5.2.4. *Recall that* $R(a)$ *is defined by* (5.1.5). *For each fixed primitive root of unity* q, $R(a)$ *is an analytic function of* a *inside the domain* $G = \{a : |a| < 1/4\}$.

Proof. For convenience, let $P_n(a)/Q_n(a)$ be denoted by $f_n(a)$ for each $n \in \mathbf{N}$. By Lemma 5.2.2, we may deduce that for each $n \in \mathbf{N}$ and each $a \in G$,

$$|f_n(a)| < 1/2, \tag{5.2.8}$$

and $\{f_n(a)\}$ converges to $R(a)$ in the domain G. Hence, $\{f_n\}$ is uniformly bounded inside G. On the other hand, by Lemma 5.2.1, $P_n(a)$ and $Q_n(a)$ are polynomials in a with coefficients in $\mathbf{C}$. This, combined with (5.2.8), implies that $\{f_n\}$ is indeed a sequence of analytic functions in G. Therefore, by Lemma 5.2.3, $\{f_n(a)\}$ converges uniformly to $R(a)$ in G. Finally, the analyticity of $R(a)$ follows from Weierstrass's uniform convergence theorem [193, p. 333].

□

We emphasize that Ramanujan recorded (5.1.6) with no indication of any admissible range for a. However, this can be done without too much difficulty. Indeed, the domain G in Lemma 5.2.4 is, in general, the best possible circular domain for a according to Lemma 5.2.2 and the fact that the continued fraction $\mathbf{K}(a/1)$ diverges for real a with $a < -1/4$. In the following, we restate Ramanujan's assertion (5.1.6) in a more precise way.

Entry 5.2.1 (p. 46). *Let q be a primitive mth root of unity and $|a| < 1/4$. Let $R(a)$ be the continued fraction defined in* (5.1.5). *Then*

$$R(a) = \frac{P_{m-1}(a) + \frac{1}{2}\left\{-1 + \sqrt{1+4a^m}\right\}}{Q_{m-1}(a)},$$

where

$$\frac{P_{m-1}(a)}{Q_{m-1}(a)} = \frac{a}{1} + \frac{aq}{1} + \cdots + \frac{aq^{m-2}}{1}.$$

Proof. Observe that $R(a)$ becomes a periodic continued fraction when $q^m = 1$. Hence,

$$\begin{aligned} R(a) &= \frac{a}{1} + \frac{aq}{1} + \frac{aq^2}{1} + \cdots + \frac{aq^{m-1}}{1} + \frac{R(a)}{1} \\ &= \frac{P_m(a) + R(a)P_{m-1}(a)}{Q_m(a) + R(a)Q_{m-1}(a)}, \end{aligned} \tag{5.2.9}$$

by (5.1.3). The identity (5.2.9) gives a quadratic equation in $R(a)$, namely,

$$Q_{m-1}(a)R^2(a) - \{P_{m-1}(a) - Q_m(a)\}R(a) - P_m(a) = 0. \tag{5.2.10}$$

Solving (5.2.10) by the quadratic formula, we find that

$$R(a) = \frac{\{P_{m-1}(a) - Q_m(a)\} \pm \sqrt{\{P_{m-1}(a) - Q_m(a)\}^2 + 4P_m(a)Q_{m-1}(a)}}{2Q_{m-1}(a)}.$$

By Theorem 5.2.1, the last identity can be rewritten in the form

$$R(a) = \frac{P_{m-1}(a) + \frac{1}{2}\left\{-1 \pm \sqrt{1 + 4\{P_m(a)Q_{m-1}(a) - P_{m-1}(a)Q_m(a)\}}\right\}}{Q_{m-1}(a)}. \tag{5.2.11}$$

Let us write $q = \exp(2\pi i h/m)$, with $(h, m) = 1$. Then, by (5.1.4),

$$\begin{aligned} P_m(a)Q_{m-1}(a) - P_{m-1}(a)Q_m(a) &= (-1)^{m-1} a \cdot aq \cdots aq^{m-1} \\ &= (-1)^{m-1} a^m q^{m(m-1)/2} \\ &= (-1)^{m-1} a^m e^{(2\pi i h/m) \cdot m(m-1)/2} \\ &= (-1)^{(h+1)(m-1)} a^m \\ &= a^m, \end{aligned}$$

where the last equality follows from the fact that h and m are coprime. Hence, from (5.2.11), we find that either

$$R(a) = \frac{P_{m-1}(a) + \frac{1}{2}\left\{-1 + \sqrt{1+4a^m}\right\}}{Q_{m-1}(a)} \tag{5.2.12}$$

or

$$R(a) = \frac{P_{m-1}(a) + \frac{1}{2}\left\{-1 - \sqrt{1+4a^m}\right\}}{Q_{m-1}(a)}. \tag{5.2.13}$$

Now it remains to exclude (5.2.13). By Lemma 5.2.1, $P_{m-1}(a)$ and $Q_{m-1}(a)$ are both polynomials in a and approach 0 and 1, respectively, when a tends to zero. Hence, when a is inside a small neighborhood of 0, the quantity on the right side of (5.2.13) will be outside the disk $|w| \le 1/2$, which contradicts Lemma 5.2.2. This implies that (5.2.12) is valid for $|a| \le \rho$, where ρ is a small positive number depending on m only. Finally, the desired result follows by Lemma 5.2.4 and analytic continuation. □

To conclude this section, we state a result of I. Schur [238, pp. 319–321], [239, pp. 117–136] and Ramanujan [63, p. 35] (who stated it incorrectly) in the case $a = 1$ and relate it to Entry 5.2.1.

Theorem 5.2.3. *Let $F(q) := R(1)$, where q is a primitive mth root of unity. If m is a multiple of 5, $F(q)$ diverges. Otherwise, $F(q)$ converges and*

$$F(q) = \alpha F(\alpha)\; q^{(\alpha\rho m - 1)/5}, \tag{5.2.14}$$

where α denotes the Legendre symbol $\left(\frac{m}{5}\right)$ and ρ is the least positive residue of m modulo 5. Moreover, in the latter case,

$$P_{m-1}(1) = \tfrac{1}{2}(1-\alpha) \quad \textit{and} \quad Q_{m-1}(1) = \alpha\; q^{(1-\alpha\rho m)/5}. \tag{5.2.15}$$

According to the table on page 57 of his lost notebook, Ramanujan apparently tried to establish results like (5.2.15) to obtain (5.2.14). Ramanujan's table is given as follows. The caption beneath the table is given by Ramanujan.

	$P_{n-2}(1)$	$P_{n-2}(x)$	$P_{n-1}(1)$	$P_{n-3}(x)$
$n \equiv 1, 4 \pmod 5$	$\sqrt[5]{x}$	$\dfrac{1}{\sqrt[5]{x}}$	1	0
$n \equiv 2, 3 \pmod 5$	$-\sqrt[5]{x}$	$-\dfrac{1}{\sqrt[5]{x}}$	0	1
$n \equiv 0 \pmod 5$	0^*	0^*	$-(x^{-2n/5} + x^{2n/5})$	$-(x^{-n/5} + x^{n/5})$

* x need not be primitive; it is enough that $\dfrac{x^n - 1}{x - 1} = 0$.

Unfortunately, this table is not completely correct. We reproduce below a table of Schur [238, p. 319], [239, p. 134], where we have changed his notation to conform to that of Ramanujan. Let ρ denote the least positive residue of n modulo 5.

	$P_{n-2}(1)$	$P_{n-2}(x)$	$P_{n-1}(1)$	$P_{n-3}(x)$
$n \equiv 1, 4 \pmod 5$	$x^{(1-\rho n)/5}$	$x^{-(1-\rho n)/5}$	1	0
$n \equiv 2, 3 \pmod 5$	$-x^{(1+\rho n)/5}$	$-x^{-(1+\rho n)/5}$	0	1
$n \equiv 0 \pmod 5$	0	0	$-(x^{-2n/5} + x^{2n/5})$	$-(x^{-n/5} + x^{n/5})$

By letting $n \to \infty$ in these miscalculations, Ramanujan probably was led to the following (incorrect) result, which was recorded on page 383 of his second notebook [227]:

> If $u := x^{1/5}F(x)$, then $u^2 + u - 1 = 0$ when $x^n = 1$, where n is any positive integer except multiples of 5 in which case u is not definite.

To obtain the result above, Ramanujan might have used his table and applied Entry 5.2.1 with $a = 1$. If so, Ramanujan considered Entry 5.2.1 to be valid for $a = 1$. Indeed, this turns out to be the case, since when $a = 1$, Entry 5.2.1 reduces to (5.2.14) simply by using (5.2.15). Therefore, it is likely that Entry 5.2.1 holds for a larger region of a. Finally, we emphasize that the convergence and divergence of $F(q)$ on the unit circle, except at primitive roots of unity, remains unresolved. However, D. Bowman and J. McLaughlin [102] have found another set of measure 0 for which $F(q)$ diverges.

5.3 A generalization of Entry 5.2.1

On page 57 of his lost notebook, Ramanujan generalizes Entry 5.2.1 by considering the continued fraction

$$\varepsilon = \frac{1}{1} \, \frac{}{+} \, \frac{aq}{1} \, \frac{}{+} \, \frac{aq^2}{1} \, \frac{}{+} \cdots \frac{}{+} \, \frac{aq^n}{1 + \lambda\varepsilon}, \tag{5.3.1}$$

where n is a fixed positive integer. Observe that by choosing $\lambda = a$, the continued fraction (5.3.1) reduces to $a^{-1}R(a)$.

As in previous chapters, let, for $n \in \mathbf{N}$,

$$(q;q)_n := (q)_n := (1-q)(1-q^2)\cdots(1-q^n).$$

Define the Gaussian coefficients $\left[{k \atop l} \right]_q$ by

$$\begin{bmatrix} k \\ 0 \end{bmatrix}_q := \begin{bmatrix} k \\ k \end{bmatrix}_q := 1$$

and

$$\begin{bmatrix} k \\ l \end{bmatrix}_q := \frac{(q)_k}{(q)_l (q)_{k-l}}, \tag{5.3.2}$$

when $0 < l < k$. Here we consider only integral values for k and l. Note that (5.3.2) is indeed a polynomial in q.

Entry 5.3.1 (p. 57). *Let $A_0 \equiv 1, A_{-1} \equiv 1$, and $A_{-2} \equiv 0$. For $n \geq 1$, let*

$$A_n(a) = \sum_{j=0}^{[(n+1)/2]} a^j \; q^{j^2} \begin{bmatrix} n-j+1 \\ j \end{bmatrix}_q, \qquad n \in \mathbf{N}.$$

Then, for $n \geq 0$,

(i) $$A_{n-1}(a)A_{n-1}(aq) - A_n(a)A_{n-2}(aq) = (-a)^n q^{n(n+1)/2},$$

(ii) $$A_n(a) = A_{n-1}(aq) + aqA_{n-2}(aq^2),$$

(iii) $$A_n(a) = A_{n-1}(a) + aq^n A_{n-2}(a),$$

(iv) $$\frac{1}{1}\, \underset{+}{}\, \frac{aq}{1}\, \underset{+}{}\, \frac{aq^2}{1}\, \underset{+}{}\, \cdots \underset{+}{}\, \frac{aq^n}{1+\eta} = \frac{A_{n-1}(aq) + \eta A_{n-2}(aq)}{A_n(a) + \eta A_{n-1}(a)}.$$

Proof. In Chapter 16 of his second notebook [227], [61, p. 31, Entry 16], Ramanujan, in fact, determined the numerator and denominator of the nth convergent of $R(a)$, and indeed $A_n(a)$ is the numerator of the nth convergent of $a/R(a)$. In other words,

$$\frac{A_n(a)}{B_n(a)} = 1 + \frac{aq}{1}\, \underset{+}{}\, \frac{aq^2}{1}\, \underset{+}{}\, \cdots \underset{+}{}\, \frac{aq^n}{1},$$

where $B_n(a)$ denotes the corresponding nth denominator. In fact, one can easily show that $B_n(a) = a^{-1}P_{n+1}(a)$. Furthermore,

$$
\begin{aligned}
\frac{A_n(a)}{B_n(a)} &= 1 + \cfrac{aq}{1 + \cfrac{aq^2}{1} + \cfrac{aq^3}{1} + \cdots + \cfrac{aq^n}{1}} \\
&= 1 + \frac{aq}{A_{n-1}(aq)/B_{n-1}(aq)} \\
&= \frac{A_{n-1}(aq) + aqB_{n-1}(aq)}{A_{n-1}(aq)}.
\end{aligned} \tag{5.3.3}
$$

The equality (5.3.3) immediately implies that

$$B_n(a) = A_{n-1}(aq) \tag{5.3.4}$$

and

$$A_n(a) = A_{n-1}(aq) + aqA_{n-2}(aq^2),$$

which proves (ii). By (5.3.4), (i) follows from (5.1.4). Equality (iii) is simply the first recurrence relation in (5.1.3). Finally, (iv) follows from (5.1.3) and (5.3.4). ☐

In Chapter III of [184], P.A. MacMahon offers identities generalizing (i), (ii), and (iii). These have subsequently been rediscovered [45] and greatly generalized by A. Berkovich and P. Paule [58] and by K. Garrett [139].

Entry 5.3.2 (p. 57). *Let* $|a| < 1/4$, $|\lambda| < 1/4$, *and* $|q| \leq 1$. *If* ε *is the continued fraction defined in* (5.3.1), *then*

$$\varepsilon = \frac{A_{n-2}(aq) + Z}{A_{n-1}(a)},$$

where Z *is a root of*

$$\lambda Z^2 + \{A_n(a) + \lambda A_{n-2}(aq)\}\, Z = (-a)^n q^{n(n+1)/2}, \tag{5.3.5}$$

and the ambiguous sign in the solution of (5.3.5) *is always positive.*

Proof. Throughout the proof, we restrict a, λ, and q to be on the interior of the prescribed areas. First, the convergence of the continued fraction ε is guaranteed by Lemma 5.2.2. By Entry 5.3.1(iv),

$$\varepsilon = \frac{A_{n-1}(aq) + \lambda\varepsilon A_{n-2}(aq)}{A_n(a) + \lambda\varepsilon A_{n-1}(a)}. \tag{5.3.6}$$

Regarding (5.3.6) as a quadratic equation in ε, we find, upon solving it, that

$$\varepsilon = \frac{2\lambda A_{n-2}(aq) - Y \pm \sqrt{Y^2 + 4\lambda(-a)^n q^{n(n+1)/2}}}{2\lambda A_{n-1}(a)},$$

where we have used Entry 5.3.1(i) and for convenience we have defined

$$Y := \lambda A_{n-2}(aq) + A_n(a). \tag{5.3.7}$$

Hence,

$$\varepsilon = \frac{A_{n-2}(aq) + Z}{A_{n-1}(a)}, \tag{5.3.8}$$

where

$$Z = \frac{1}{2\lambda}\left[-Y \pm \sqrt{Y^2 + 4\lambda(-a)^n q^{n(n+1)/2}}\right]. \tag{5.3.9}$$

One can easily check that Z satisfies the equation (5.3.5). Therefore, it remains to verify that the ambiguous sign is always positive. By Lemma 5.2.1, both $A_n(a)$ and $A_{n-2}(aq)$ are polynomials in a and approach 1 when a tends to 0, and hence, from (5.3.7),

$$Y \longrightarrow 1 + \lambda, \quad \text{as } a \to 0. \tag{5.3.10}$$

Now let us first fix q and λ, with $0 < \lambda < 1/4$. Then, by (5.3.8)–(5.3.10), when a tends to 0, ε approaches 1 and $-1/\lambda$, respectively, according to the "+" and "−" signs in (5.3.9). However, by Lemma 5.2.2, ε converges to a value in the disk $|w| \le 2$, which excludes the value $-1/\lambda$ when a is small enough. In other words, the "+" sign is always correct when a is in a small neighborhood of the origin. Furthermore, an argument like that used in the proof of Entry 5.2.1 shows that ε is an analytic function of a. Therefore, by analytic continuation, Entry 5.3.2 is valid for $|q| \le 1$, $|a| < 1/4$, and $0 < \lambda < 1/4$. Finally, the desired domain for λ can be obtained by analytic continuation, since ε is also analytic in λ. □

In addition to Entries 5.3.1 and 5.3.2, Ramanujan recorded the following two results on page 57 of his lost notebook [228].

Entry 5.3.3 (p. 57). *Let*

$$A(a) := \lim_{n\to\infty} A_n(a).$$

If $q^n = 1$, where q is primitive, then

$$A_{n-1}(a) + aA_{n-3}(aq) = 1 \tag{5.3.11}$$

and

$$A_{n-2}(a)A(aq) - A_{n-3}(aq)A(a) = (-a)^{n-1}q^{n(n-1)/2}A(aq^n). \tag{5.3.12}$$

Proof. The equality (5.3.11) is actually a restatement of Theorem 5.2.1.

It is easily seen that by a similar argument, Entry 5.3.1(ii) remains valid if we replace $A_n(q)$ by $A(q)$. In other words,

$$A(a) = A(aq) + aqA(aq^2). \tag{5.3.13}$$

By Entry 5.3.1(ii) and (5.3.13),

$$\begin{aligned}
&A_{n-2}(a)A(aq) - A_{n-3}(aq)A(a) \\
&= \left\{A_{n-3}(aq) + aqA_{n-4}(aq^2)\right\} A(aq) - A_{n-3}(aq) \left\{A(aq) + aqA(aq^2)\right\} \\
&= -aq \left\{A_{n-3}(aq)A(aq^2) - A_{n-4}(aq^2)A(aq)\right\}. \qquad (5.3.14)
\end{aligned}$$

Note that the expression inside the parentheses on the far right side of (5.3.14) is exactly the expression on the left side with the subscripts reduced by 1 and with a replaced by aq. Hence, we can iterate the recurrence above to obtain

$$\begin{aligned}
&A_{n-2}(a)A(aq) - A_{n-3}(aq)A(a) \\
&= (-aq)(-aq^2)\cdots(-aq^{n-1}) \left\{A_{-1}(aq^{n-1})A(aq^n) - A_{-2}(aq^n)A(aq^{n-1})\right\} \\
&= (-a)^{n-1}q^{n(n-1)/2}A(aq^n),
\end{aligned}$$

since $A_{-1} \equiv 1$ and $A_{-2} \equiv 0$, which completes the proof of (5.3.12). □

Because of the appearance of (5.3.11) in the lost notebook, it is very likely that the proof of Entry 5.2.1 that we have given is essentially the one that Ramanujan had. However, we have no idea how Ramanujan found and proved Theorem 5.2.1, i.e., (5.3.11).

5.4 Finite Rogers–Ramanujan Continued Fractions and Class Invariants

At the bottom of page 47 in his lost notebook, Ramanujan claims that particular zeros of certain finite Rogers–Ramanujan continued fractions, or similar continued fractions, involve class invariants or singular moduli. The content of this section can be found in Huang's thesis [163]. For detailed accounts of Ramanujan's work on class invariants and singular moduli, see two papers by Berndt, H.H. Chan, and L.–C. Zhang [72] [74] and Berndt's book [63, Chapter 34]. We present here only the basic definitions and facts that are needed to describe and prove Ramanujan's results in this section.

Let

$$\chi(q) := (-q;q^2)_\infty, \qquad |q| < 1. \qquad (5.4.1)$$

If $q = q_n := \exp(-\pi\sqrt{n})$, for some positive rational number n, then the class invariant G_n is defined by

$$G_n := 2^{-1/4}q_n^{-1/24}\chi(q_n). \qquad (5.4.2)$$

Let $k := k(q)$, $0 < k < 1$, denote the modulus, and let $k' = \sqrt{1-k^2}$ denote the complementary modulus. In particular, if $q = q_n$, then $k(q_n) =: k_n$ is called a singular modulus. Also, put $k_n' := \sqrt{1-k_n^2}$. Let $K = K(k)$ and $K' = K(k')$ denote complete elliptic integrals of the first kind. If $q = \exp(-\pi K'/K)$, then $\chi(q) = 2^{-1/6}(kk'/q^2)^{-1/12}$ [61, p. 124]. In particular, if $K'/K = \sqrt{n}$, then

$$G_n = (2k_nk_n')^{-1/12}. \qquad (5.4.3)$$

Entry 5.4.1 (p. 47). *If* $K'/K = \sqrt{47}$ *and* $t := t_{47} := 2^{1/3}(k_{47}k'_{47})^{1/12}$, *then*

$$1 - \frac{t}{1} - \frac{t^2}{1} - \frac{t^3}{1} - \frac{t^4}{1} = 0. \tag{5.4.4}$$

Furthermore,

$$t_{47} = \sqrt{2}e^{-\pi\sqrt{47}/24}(q_{47}; -q_{47})_\infty. \tag{5.4.5}$$

Proof. First, from (5.4.3), it is easy to see that $G_{47} = 2^{1/4}t_{47}^{-1}$. Using (5.4.1), (5.4.2), and Euler's identity

$$\frac{1}{(-q; q^2)_\infty} = (q; -q)_\infty, \tag{5.4.6}$$

we readily deduce (5.4.5).

Now from either Weber's treatise [291, p. 723] or Ramanujan's first notebook [227, p. 234], if

$$\sqrt{2}x = e^{\pi\sqrt{47}/24}(-q_{47}; q_{47}^2)_\infty, \tag{5.4.7}$$

then

$$x^5 = (1 + x)(1 + x + x^2).$$

Hence, from (5.4.5)–(5.4.7), $t = 1/x$. Thus, t satisfies the equation

$$\left(\frac{1}{t}\right)^5 = \left(1 + \frac{1}{t}\right)\left(1 + \frac{1}{t} + \frac{1}{t^2}\right),$$

i.e.,

$$t^5 + 2t^4 + 2t^3 + t^2 - 1 = 0. \tag{5.4.8}$$

Multiply both sides of (5.4.8) by $(t - 1)$ to deduce that

$$t^6 + t^5 - t^3 - t^2 - t + 1 = 0. \tag{5.4.9}$$

However, a brief calculation shows that (5.4.9) is equivalent to (5.4.4), and this completes the proof. □

Entry 5.4.2 (p. 47). *Let* K, K', L, *and* L' *denote complete elliptic integrals of the first kind associated with the moduli* k, k', ℓ, *and* ℓ', *respectively. If* $K'/K = \sqrt{39}$, $L'/L = \sqrt{13/3}$, *and* $t := t_{39} := (k_{39}k'_{39}/\ell_{13/3}\ell'_{13/3})^{1/12}$, *then*

$$1 - \frac{t}{1} - \frac{t^2}{1} - \frac{t^3}{1} = 0. \tag{5.4.10}$$

Moreover,

$$t_{39} = e^{-\pi\sqrt{13/3}/12}\frac{(-q_{13/3}; q_{13/3}^2)_\infty}{(-q_{13/3}^3; q_{13/3}^6)_\infty}. \tag{5.4.11}$$

Ramanujan, observing that each factor in the denominator of (5.4.11) is canceled by a corresponding factor in the numerator, wrote (5.4.11) as a single infinite product.

Proof. By (5.4.3) and (5.4.2),

$$t_{39} = \frac{G_{13/3}}{G_{39}} = \frac{q_{13/3}^{-1/24}\chi(q_{13/3})}{q_{39}^{-1/24}\chi(q_{39})}, \tag{5.4.12}$$

from which, by (5.4.1), (5.4.11) trivially follows.

From either Weber's text [291, p. 722] or Ramanujan's notebooks [227, vol. 1, p. 305; vol. 2, p. 295],

$$G_{39} = 2^{1/4}\left(\frac{\sqrt{13}+3}{2}\right)^{1/6}\left(\sqrt{\frac{5+\sqrt{13}}{8}}+\sqrt{\frac{\sqrt{13}-3}{8}}\right). \tag{5.4.13}$$

The class invariant $G_{13/3}$ can be determined from (5.4.13) and a certain modular equation of degree 3 [63, p. 222, Lemma 4.3]. Accordingly, we find that

$$G_{13/3} = 2^{1/4}\left(\frac{\sqrt{13}+3}{2}\right)^{1/6}\left(\sqrt{\frac{5+\sqrt{13}}{8}}-\sqrt{\frac{\sqrt{13}-3}{8}}\right). \tag{5.4.14}$$

Thus, from (5.4.12)–(5.4.14),

$$t_{39} = \left(\sqrt{\frac{5+\sqrt{13}}{8}}-\sqrt{\frac{\sqrt{13}-3}{8}}\right)^2.$$

It is now easily checked that t_{39} is a root of the polynomial equation

$$t^4 - t^3 - t^2 - t + 1 = 0. \tag{5.4.15}$$

Observing that (5.4.10) and (5.4.15) are equivalent, we complete the proof. □

Entry 5.4.3 (p. 47). *If* $t := t_{23} := 2^{1/3}(k_{23}k'_{23})^{1/12}$, *then*

$$1 - \frac{t^2}{1} \, \underset{-}{} \, \frac{t^3}{1} = 0. \tag{5.4.16}$$

The value of t in this result was, in fact, not given by Ramanujan. If $F(t)$ denotes the continued fraction in (5.4.16), then $F(t)$ is not a finite Rogers–Ramanujan continued fraction. However, $1-t/F(t)$ is a finite Rogers–Ramanujan continued fraction.

Proof. As we argued in the proof of Entry 5.4.1, $G_{23} = 2^{1/4}t_{23}^{-1}$. From Weber's tables [291, p. 722] or Ramanujan's notebooks [227, vol. 1, pp. 295, 345, 351; vol. 2, p. 294], if $G_{23} = 2^{1/4}x$, then

$$x^3 - x - 1 = 0.$$

Thus, $t = t_{23} = 1/x$ and

$$t^3 + t^2 - 1 = 0. \tag{5.4.17}$$

It is easy to see that (5.4.17) and (5.4.16) are equivalent, and so this completes the proof. □

Entry 5.4.4 (p. 47). *If* $t := t_{31} := 2^{1/3}(k_{31}k'_{31})^{1/12}$, *then*

$$1 - \frac{t}{1} - \frac{t^3}{1} = 0. \tag{5.4.18}$$

As with Entry 5.4.3, Ramanujan did not provide the definition of t in Entry 5.4.4. Also, the continued fraction in (5.4.18) is not a finite Rogers–Ramanujan continued fraction.

Proof. By a now familiar argument, $G_{31} = 2^{1/4}t_{31}^{-1}$. From Weber's tables [291, p. 722] or Ramanujan's notebooks [227, vol. 1, pp. 296, 345, 351; vol. 2, p. 295], if $G_{31} = 2^{1/4}x$, then

$$x^3 - x^2 - 1 = 0.$$

Thus, $t := t_{31} = 1/x$ and

$$t^3 + t - 1 = 0. \tag{5.4.19}$$

Clearly, (5.4.19) and (5.4.18) are equivalent, and so the proof is complete. □

5.5 A Finite Generalized Rogers–Ramanujan Continued Fraction

Entry 5.5.1 (p. 54). *For each positive integer* n,

$$\begin{aligned} &1 + \frac{aq}{1} + \frac{a^2q^4}{1} + \frac{a^2q^8}{1} + \frac{a^2q^{12}}{1} + \cdots + \frac{a^2q^{4(n-1)}}{1} \\ &= \frac{1}{1} - \frac{aq}{1} + \frac{aq}{1} - \frac{aq^3}{1} + \frac{aq^3}{1} - \cdots - \frac{aq^{2n-1}}{1} + \frac{aq^{2n-1}}{1}, \end{aligned} \tag{5.5.1}$$

where for $n = 1$ *the left side of* (5.5.1) *is understood to equal* $1 + aq$.

Proof. We use induction on n. For $n = 1$, both sides of (5.5.1) are equal to $1 + aq$, and for $n = 2$, both sides of (5.5.1) equal

$$\frac{1 + aq + a^2q^4}{1 + a^2q^4}.$$

Now assume that (5.5.1) is valid with n replaced by $n-1$, and in this inductive assumption replace a by aq^2. Thus,

$$\begin{aligned}1+\frac{aq^3}{1}&+\frac{a^2q^8}{1}+\frac{a^2q^{12}}{1}+\cdots+\frac{a^2q^{4(n-1)}}{1}\\&=\frac{1}{1}-\frac{aq^3}{1}+\frac{aq^3}{1}-\frac{aq^5}{1}+\frac{aq^5}{1}-\cdots-\frac{aq^{2n-1}}{1}+\frac{aq^{2n-1}}{1}.\end{aligned}\tag{5.5.2}$$

Let

$$S:=1+\frac{a^2q^8}{1}+\frac{a^2q^{12}}{1}+\cdots+\frac{a^2q^{4(n-1)}}{1}.\tag{5.5.3}$$

Multiplying both sides of (5.5.2) by aq, we see that

$$aq\left(1+\frac{aq^3}{S}\right)=\frac{aq}{1}-\frac{aq^3}{1}+\frac{aq^3}{1}-\cdots-\frac{aq^{2n-1}}{1}+\frac{aq^{2n-1}}{1}.\tag{5.5.4}$$

Therefore, by (5.5.4),

$$\begin{aligned}\frac{1}{1}-\frac{aq}{1}+\frac{aq}{1}&-\frac{aq^3}{1}+\frac{aq^3}{1}-\cdots-\frac{aq^{2n-1}}{1}+\frac{aq^{2n-1}}{1}\\&=\frac{1}{1}-\frac{aq}{1+aq(1+aq^3/S)}\\&=\frac{S+a^2q^4+aqS}{S+a^2q^4}\\&=1+\frac{aqS}{S+a^2q^4}\\&=1+\frac{aq}{1+a^2q^4/S}\\&=1+\frac{aq}{1}+\frac{a^2q^4}{1}+\frac{a^2q^8}{1}+\cdots+\frac{a^2q^{4(n-1)}}{1},\end{aligned}$$

where we employed (5.5.3) in the last step. This completes the proof. □

Second Proof of Entry 5.5.1. Our second proof depends on the *odd part* of a continued fraction, which we give in the next theorem [182, p. 85].

Theorem 5.5.1 (Odd Part of a Continued Fraction). *Let A_n and B_n be the nth canonical numerator and denominator of the continued fraction $b_0+\boldsymbol{K}(a_n/b_n)$. The contraction of $b_0+\boldsymbol{K}(a_n/b_n)$ with $C_0=A_1/B_1$, $D_0=1$, $C_k=A_{2k+1}$, and $D_k=B_{2k+1}$, for $k=1,2,3,\ldots$, exists if and only if $b_{2k+1}\neq 0$ for $k=0,1,2,\ldots$, and is then given by*

$$\begin{aligned}\frac{b_0b_1+a_1}{b_1}-&\frac{a_1a_2b_3/b_1}{b_1(a_3+b_2b_3)+a_2b_3}-\frac{a_3a_4b_5b_1}{b_3(a_5+b_4b_5)+a_4b_5}-\\&\frac{a_5a_6b_7b_3}{b_5(a_7+b_6b_7)+a_6b_7}-\frac{a_7a_8b_9b_5}{b_7(a_9+b_8b_9)+a_8b_9}-\cdots,\end{aligned}$$

and it is called the odd part of $b_0+\boldsymbol{K}(a_n/b_n)$.

Observe that the left side of (5.5.1) is the odd part of the finite continued fraction on the right side of (5.5.1). The result now follows. □

There are three continued fractions on page 27 of Ramanujan's lost notebook that arise from the odd parts of continued fractions. Although these continued fractions are not necessarily q-continued fractions, it seems appropriate to include them here. These proofs may also be found in Sohn's thesis [253] and his paper with J. Lee [174].

Entry 5.5.2 (p. 27). *Let $\{a_i\}$ be an arbitrary sequence. Then*

$$\frac{1}{1}-\frac{a_1}{1}+\frac{a_1}{1}-\frac{a_2}{1}+\frac{a_2}{1}-\cdots=1+\frac{a_1}{1}+\frac{a_1a_2}{1}+\frac{a_2a_3}{1}+\cdots. \tag{5.5.5}$$

Proof. From Theorem 5.5.1, we easily verify that the odd part of the left-hand side of (5.5.5) is equal to the right-hand side of (5.5.5). □

Entry 5.5.3 (p. 27). *Let $\{a_i\}$ be an arbitrary sequence. Then*

$$\frac{1}{1}-\frac{1}{a_1}+\frac{1}{1}-\frac{1}{a_2}+\frac{1}{1}-\frac{1}{a_3}-\cdots=1+\frac{1}{a_1}+\frac{1}{a_2}+\frac{1}{a_3}+\cdots. \tag{5.5.6}$$

Proof. Using Theorem 5.5.1, we easily verify that the right-hand side of (5.5.6) is the odd part of the left-hand side of (5.5.6) □

Entry 5.5.4 (p. 27). *Let ω be a cube root of unity and let $\{a_i\}$ be an arbitrary sequence. Then*

$$\frac{1}{1}-\frac{\omega}{a_1}-\frac{\omega^2}{1}-\frac{\omega}{a_2}-\frac{\omega^2}{1}-\cdots=1+\frac{\omega}{1+a_1}-\frac{1}{1+a_2}-\frac{1}{1+a_3}-\cdots. \tag{5.5.7}$$

Proof. By Theorem 5.5.1, we easily check that the odd part of the left-hand side of (5.5.7) is equal to the right-hand side of (5.5.7). □

6

Other q-continued Fractions

6.1 Introduction

Scattered among the entries on pages 41–44 in Ramanujan's lost notebook [228] are several results on q-continued fractions, including three general theorems involving two or three parameters. Special cases of these general results include the Rogers–Ramanujan continued fraction, the Ramanujan–Göllnitz–Gordon continued fraction, a famous continued fraction of Eisenstein, and several continued fractions found by A. Selberg [241], [242, pp. 1–21]. Indeed, Ramanujan recorded all these special cases, and more. Among all the claims in the lost notebook, these general continued fractions seem to have attracted the attention of more authors than any other results.

Section 6.2 contains the main result, for which we give two proofs. These are followed by several corollaries. In the short Section 6.3, three representations of a continued fraction given in the previous section are considered. A different continued fraction for the same primary quotient of q-series examined in Section 6.2 is studied in Section 6.4. In our discourse, we mainly follow the presentations by Andrews [26] and S. Bhargava and C. Adiga [91] in their papers. Some of our proofs of Ramanujan's corollaries depend on results of Andrews, L.J. Rogers, and L.J. Slater not found in Ramanujan's work, and so we naturally wonder how Ramanujan might have argued.

In Section 6.5, we present a transformation of a certain q-continued fraction. A corollary, Entry 6.5.2, is particularly elegant.

We do not know what motivated Ramanujan to focus on the least positive zero q_0 of the generalized Rogers–Ramanujan continued fraction, but his asymptotic expansion for q_0 and the approximations in Section 6.6 are fascinating. Most of the content of this and the previous sections are found in a paper by Berndt, S.–S. Huang, J. Sohn, and S.H. Son [78].

In the penultimate section of this chapter, we examine an identity that has an (apparently) superficial relation to the generalized Rogers–Ramanujan continued fraction. Below this identity are two continued fractions akin to the generalized Rogers–Ramanujan continued fraction, but Ramanujan does not make any claim about them. Are they related to the identity above them? The content of this section is taken from a paper by Berndt and A.J. Yee [84].

To close this chapter, we examine in Section 6.8 an isolated, elementary continued fraction of Ramanujan, first established by Berndt and G. Choi [76].

6.2 The Main Theorem

Entry 6.2.1 (p. 41). *For any complex numbers a, b, λ, and q, but with $|q| < 1$, define*

$$G(a, b, \lambda) := G(a, \lambda; b; q) := \sum_{n=0}^{\infty} \frac{(-\lambda/a; q)_n a^n q^{n(n+1)/2}}{(q; q)_n (-bq; q)_n}. \tag{6.2.1}$$

Then

$$\frac{G(aq, b, \lambda q)}{G(a, b, \lambda)} = \frac{1}{1} + \frac{aq + \lambda q}{1} + \frac{bq + \lambda q^2}{1} + \frac{aq^2 + \lambda q^3}{1} + \frac{bq^2 + \lambda q^4}{1} + \cdots. \tag{6.2.2}$$

Set

$$P(a, b, \lambda) = (-bq; q)_\infty G(a, b, \lambda). \tag{6.2.3}$$

Observe, from (6.2.3), that the quotient on the left side of (6.2.2) can be expressed as $P(aq, b, \lambda q)/P(a, b, \lambda)$. The different orders of the parameters on the left side of (6.2.1) may appear to be unfortunate; the second ordering arises from the usual notation for the basic hypergeometric series ${}_2\phi_1$.

For our first proof of Entry 6.2.1, we need to establish some auxilliary lemmas.

Lemma 6.2.1. *If $P(a, b, \lambda)$ is defined by* (6.2.3), *then*

(i) $$P(a, b, \lambda) - P(aq, b, \lambda) = aqP(aq, bq, \lambda q),$$

(ii) $$P(a, b, \lambda) - P(a, b, \lambda q) = \lambda q P(aq, bq, \lambda q^2),$$

(iii) $$P(a, b, \lambda) - P(a, bq, \lambda) = bqP(aq, bq, \lambda q).$$

Proof. A straightforward calculation shows that

$$(-\lambda/a)_n - q^n(-\lambda/(aq))_n = \begin{cases} 0, & \text{if } n = 0, \\ (-\lambda/a)_{n-1}(1 - q^n), & \text{if } n > 0. \end{cases}$$

It follows that

$$\begin{aligned} P(a, b, \lambda) - P(aq, b, \lambda) &= (-bq)_\infty \sum_{n=0}^{\infty} \frac{a^n q^{n(n+1)/2}}{(q)_n (-bq)_n} \left\{ \left(-\frac{\lambda}{a}\right)_n - q^n \left(-\frac{\lambda}{aq}\right)_n \right\} \\ &= aq(-bq^2)_\infty \sum_{n=1}^{\infty} \frac{(aq)^{n-1} q^{n(n-1)/2} (-\lambda/a)_{n-1}}{(q)_{n-1}(-bq^2)_{n-1}} \\ &= aqP(aq, bq, \lambda q), \end{aligned}$$

where in the penultimate line we used the obvious identity

$$\frac{(-bq)_\infty}{(-bq)_n} = \frac{(-bq^2)_\infty}{(-bq^2)_{n-1}}.$$

Thus, (i) is proved.

The proofs of the q-difference equations (ii) and (iii) follow along the same lines. We use, respectively, the easily established identities

$$(-\lambda/a)_n - (-\lambda q/a)_n = \begin{cases} 0, & \text{if } n = 0, \\ \dfrac{\lambda}{a}(-\lambda q/a)_{n-1}(1-q^n), & \text{if } n > 0, \end{cases}$$

and

$$\frac{(-bq)_\infty}{(-bq)_n} - \frac{(-bq^2)_\infty}{(-bq^2)_n} = \frac{(-bq^2)_\infty}{(-bq^2)_n} bq^{n+1}.$$

□

Lemma 6.2.2. *We have*

(i) $$P(a,b,\lambda) = P(aq,b,\lambda q) + (aq+\lambda q)P(aq,bq,\lambda q^2),$$

(ii) $$P(a,b,\lambda) = P(a,bq,\lambda q) + (bq+\lambda q)P(aq,bq,\lambda q^2).$$

Proof. In Lemma 6.2.1, replace λ by λq in (i) and add the result to (ii). This then gives (i) of the present lemma. In Lemma 6.2.1, replace λ by λq in (iii) and add the result to (ii) to obtain (ii) of the present lemma. □

First Proof of Entry 6.2.1. In Lemma 6.2.2(i), replace a by aq^n, b by bq^n, and λ by λq^{2n}, and in Lemma 6.2.2(ii), replace a by aq^{n+1}, b by bq^n, and λ by λq^{2n+1}. We can then write (i) and (ii), respectively, in the forms

$$Q_n := \frac{P(aq^n, bq^n, \lambda q^{2n})}{P(aq^{n+1}, bq^n, \lambda q^{2n+1})} = 1 + \frac{aq^{n+1} + \lambda q^{2n+1}}{Q'_n},$$

$$Q'_n := \frac{P(aq^{n+1}, bq^n, \lambda q^{2n+1})}{P(aq^{n+1}, bq^{n+1}, \lambda q^{2n+2})} = 1 + \frac{bq^{n+1} + \lambda q^{2n+2}}{Q_{n+1}}.$$

Beginning with the first identity above with $n = 0$, alternately iterate these two identities with $n = 0, 1, 2, \dots$. This formally proves (6.2.2). Now the convergence of this continued fraction is an easy consequence of Worpitzky's theorem, Lemma 5.2.2 in Chapter 5. Since $Q_n = 1 + o(1)$ and $Q'_n = 1 + o(1)$, as $n \to \infty$, the continued fraction indeed does converge to the left side of (6.2.2). □

For our second proof of Entry 6.2.1, we need three well-known results from the theory of q-series. First [21, p. 36, Theorem 3.3], if $\left[\begin{smallmatrix} n \\ j \end{smallmatrix}\right]$ denotes the Gaussian polynomial, then

$$\sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix} (-1)^j z^j q^{j(j-1)/2} = (z;q)_n. \tag{6.2.4}$$

Second [21, p. 19, Corollary 2.2],

$$\sum_{n=0}^{\infty} \frac{t^n q^{n(n-1)/2}}{(q;q)_n} = (-t;q)_\infty. \tag{6.2.5}$$

Recall that the basic hypergeometric series ${}_2\phi_1$ is defined for $|q| < 1$ by

$${}_2\phi_1(a,b;c;q) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(q)_n (c)_n} q^n. \tag{6.2.6}$$

The third result that we need is the second iterate of Heine's transformation given by [21, p. 38, last line]

$${}_2\phi_1(a,b;c;t) = \frac{(c/b)_\infty (bt)_\infty}{(c)_\infty (t)_\infty} \; {}_2\phi_1(b, abt/c; bt; c/b). \tag{6.2.7}$$

Recall that $P(a,b,\lambda)$ is defined by (6.2.3) above. We first prove that $P(a,b,\lambda)$ is symmetric in a and b.

Entry 6.2.2 (p. 42). *We have*

$$P(a,b,\lambda) = P(b,a,\lambda), \tag{6.2.8}$$

or, equivalently,

$$(-bq)_\infty \sum_{n=0}^{\infty} \frac{(-\lambda/a)_n a^n q^{n(n+1)/2}}{(q)_n (-bq)_n} = (-aq)_\infty \sum_{n=0}^{\infty} \frac{(-\lambda/b)_n b^n q^{n(n+1)/2}}{(q)_n (-aq)_n}. \tag{6.2.9}$$

Proof. If we replace a, b, c, and t in (6.2.7) by $-\lambda/a, -dq, -bq$, and a/d, respectively, and let $d \to \infty$, we deduce (6.2.9).

We may also prove (6.2.8) directly. By using (6.2.4) and (6.2.5), and remembering that $\left[\begin{smallmatrix} n \\ j \end{smallmatrix}\right] = 0$ if $j > n$, we find that

$$\begin{aligned} P(a,b,\lambda) =& (-bq)_\infty \sum_{n=0}^{\infty} \frac{(-\lambda/a)_n a^n q^{n(n+1)/2}}{(q)_n (-bq)_n} \\ =& \sum_{n=0}^{\infty} \frac{(-\lambda/a)_n (-bq^{n+1})_\infty a^n q^{n(n+1)/2}}{(q)_n} \\ =& \sum_{n=0}^{\infty} \frac{a^n q^{n(n+1)/2}}{(q)_n} \sum_{j=0}^{\infty} \begin{bmatrix} n \\ j \end{bmatrix} \frac{\lambda^j}{a^j} q^{j(j-1)/2} \sum_{k=0}^{\infty} \frac{(bq^n)^k q^{k(k+1)/2}}{(q)_k} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j,k,m=0}^{\infty} \frac{q^{(m+j)(m+j+1)/2+j(j-1)/2} a^m \lambda^j q^{k(k+1)/2} (bq^{m+j})^k}{(q)_m (q)_j (q)_k} \\
&= \sum_{j,k,m=0}^{\infty} \frac{q^{m(m+1)/2+k(k+1)/2+j^2+mk+mj+jk} a^m b^k \lambda^j}{(q)_m (q)_j (q)_k}, \qquad (6.2.10)
\end{aligned}
$$

where in the penultimate line we set $n = m + j$. The symmetry in a and b is now evident, and so (6.2.8) has once again been shown. □

Second Proof of Entry 6.2.1. Using (6.2.10), we find that

$$
\begin{aligned}
P(aq, b, \lambda q) &= \sum_{j,k,m=0}^{\infty} \frac{q^{m(m+1)/2+k(k+1)/2+j^2+mk+mj+jk} a^m b^k \lambda^j}{(q)_m (q)_j (q)_k} q^{m+j}, \\
aqP(aq, bq, \lambda q^2) &= \sum_{j,k,m=0}^{\infty} \frac{q^{m(m+1)/2+k(k+1)/2+j^2+mk+mj+jk} a^m b^k \lambda^j}{(q)_m (q)_j (q)_k} q^j (1 - q^m), \\
\lambda qP(aq, bq, \lambda q^2) &= \sum_{j,k,m=0}^{\infty} \frac{q^{m(m+1)/2+k(k+1)/2+j^2+mk+mj+jk} a^m b^k \lambda^j}{(q)_m (q)_j (q)_k} (1 - q^j).
\end{aligned}
$$

Since

$$1 = q^{m+j} + q^j(1 - q^m) + (1 - q^j),$$

it follows that

$$P(a, b, \lambda) = P(aq, b, \lambda q) + (aq + \lambda q)P(aq, bq, \lambda q^2). \qquad (6.2.11)$$

We are now set to complete the proof of Entry 6.2.1. From (6.2.11) and (6.2.8), it follows that

$$\frac{P(a, b, \lambda)}{P(aq, b, \lambda q)} = 1 + \frac{aq + \lambda q}{\left(\dfrac{P(b, aq, \lambda q)}{P(bq, aq, \lambda q^2)} \right)}, \qquad (6.2.12)$$

and so from (6.2.12) and (6.2.3), we deduce that

$$\frac{G(a, b, \lambda)}{G(aq, b, \lambda q)} = 1 + \frac{aq + \lambda q}{\left(\dfrac{G(b, aq, \lambda q)}{G(bq, aq, \lambda q^2)} \right)}. \qquad (6.2.13)$$

By iterating (6.2.13), we formally obtain (6.2.2).

The convergence of the continued fraction in (6.2.2) follows as in the first proof. □

Ramanujan especially examines the case $\lambda = 0$ in another theorem on page 42 of the lost notebook.

Entry 6.2.3 (p. 42). *Let $G(a,b) := G(a,b,0)$, where $G(a,b,\lambda)$ is defined in Entry* 6.2.1*, i.e.,*

$$G(a,b) = \sum_{n=0}^{\infty} \frac{a^n q^{n(n+1)/2}}{(q;q)_n(-bq;q)_n}, \qquad |q| < 1.$$

Then

$$\text{(i)} \qquad \frac{G(a,b)}{G(aq,b)} = 1 + aq\frac{G(bq,aq)}{G(b,aq)} = 1 + \cfrac{aq}{\cfrac{G(b,aq)}{G(bq,aq)}}$$

$$\text{(ii)} \qquad = 1 + \cfrac{aq}{1 + bq\cfrac{G(aq^2,bq)}{G(aq,bq)}}$$

$$\text{(iii)} \qquad = 1 + \frac{aq}{1} \, + \, \frac{bq}{1} \, + \, \frac{aq^2}{1} \, + \, \frac{bq^2}{1} \, + \cdots.$$

Proof. Note that the recurrence relation of (i) tells us to switch the arguments in both functions and then multiply the variables "off the main diagonal" by q. Thus, (ii) follows from an iteration of (i).

Next, observe that (iii) is the special case $\lambda = 0$ of the continued fraction (6.2.2).

It remains to prove (i). Define $P(a,b) := P(a,b,0)$, where $P(a,b,\lambda)$ is defined by (6.2.3). Recall from Entry 6.2.2 that $P(a,b) = P(b,a)$. Thus, the proposed equality (i) can be rewritten as

$$\frac{P(a,b)}{P(aq,b)} = 1 + aq\frac{P(aq,bq)}{P(aq,b)},$$

or

$$P(a,b) = P(aq,b) + aqP(aq,bq). \tag{6.2.14}$$

But (6.2.14) is simply the case $\lambda = 0$ in Lemma 6.2.1(i), and so the proof is complete. □

On page 42 Ramanujan also records a very curious representation for $G(a,b) := G(a,b,0)$, which we now prove.

Entry 6.2.4 (p. 42). *Define the power series coefficients c_n, $0 \le n < \infty$, by*

$$\sum_{n=0}^{\infty} c_n x^n = \frac{1}{(ax)_\infty (bx)_\infty}. \tag{6.2.15}$$

Then

$$\sum_{n=0}^{\infty} c_n q^{n(n+1)/2} = (-bq)_\infty G(a,b). \tag{6.2.16}$$

Proof. We need a special case of the q-binomial theorem of Cauchy and Rothe, which can also found in Ramanujan's notebooks [61, p. 14, Entry 2], [21, p. 19, Corollary 2.1], namely,

$$\sum_{n=0}^{\infty} \frac{t^n}{(q)_n} = \frac{1}{(t)_\infty}. \tag{6.2.17}$$

Thus, upon two applications of (6.2.17),

$$\begin{aligned} \sum_{n=0}^{\infty} c_n x^n &= \frac{1}{(ax)_\infty (bx)_\infty} \qquad (6.2.18)\\ &= \sum_{j=0}^{\infty} \frac{(ax)^j}{(q)_j} \sum_{k=0}^{\infty} \frac{(bx)^k}{(q)_k} \\ &= \sum_{n=0}^{\infty} \frac{x^n}{(q)_n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} a^{n-k} b^k. \end{aligned}$$

Equating coefficients of x^n, $0 \le n < \infty$, in (6.2.18), we find that

$$c_n = \frac{1}{(q)_n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} a^{n-k} b^k. \tag{6.2.19}$$

Multiply both sides of (6.2.19) by $q^{n(n+1)/2}$ and sum on n, $0 \le n < \infty$, to deduce that

$$\begin{aligned} \sum_{n=0}^{\infty} c_n q^{n(n+1)/2} &= \sum_{n=0}^{\infty} \frac{q^{n(n+1)/2}}{(q)_n} \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} a^{n-k} b^k \\ &= \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \frac{q^{(j+k)(j+k+1)/2} a^j b^k}{(q)_j (q)_k}, \qquad (6.2.20) \end{aligned}$$

where we have inverted the order of summation and then set $j = n - k$. Rewrite (6.2.20) and use (6.2.5) to conclude that

$$\begin{aligned} \sum_{n=0}^{\infty} c_n q^{n(n+1)/2} &= \sum_{j=0}^{\infty} \frac{q^{j(j+1)/2} a^j}{(q)_j} \sum_{k=0}^{\infty} \frac{(bq^{j+1})^k q^{k(k-1)/2}}{(q)_k} \\ &= \sum_{j=0}^{\infty} \frac{q^{j(j+1)/2} a^j}{(q)_j} (-bq^{j+1})_\infty \\ &= (-bq)_\infty \sum_{j=0}^{\infty} \frac{q^{j(j+1)/2} a^j}{(q)_j (-bq)_j} \\ &= (-bq)_\infty G(a, b). \end{aligned}$$

The proof is therefore completed. $\square$

Observe that Entry 6.2.2 (in the case $\lambda = 0$) follows as an immediate corollary of Entry 6.2.4.

Andrews [22] gave the initial proof of Entry 6.2.1. The first proof of Entry 6.2.1 that we gave above is by Bhargava and Adiga [91], while the second proof that we have given is due to M.D. Hirschhorn [159]. Entry 6.2.1 is obviously a substantial generalization of the famous result of Rogers and Ramanujan representing the Rogers–Ramanujan continued fraction as a quotient of the Rogers–Ramanujan functions; see, for example, [61, p. 30, Corollary] or Corollary 6.2.6 below. Ramanujan indicated, but did not explicitly record, a generalization of Corollary 6.2.6 in his second letter to Hardy [226, p. xxviii], [81, p. 57], but it is probably not the one in Entry 6.2.1. The first significant published generalizations of Entry 6.2.1 are by Selberg [241], [242, pp. 16–17]. Candidates for the generalization alluded to by Ramanujan in his letter include those by Andrews [17] and Hirschhorn [153], [156]. Undoubtedly, the most complete generalizations of Entry 6.2.1 have been found by Andrews and D. Bowman [42] and by D.P. Gupta and D. Masson [146].

Entry 6.2.1 has been proved and generalized by several other authors, including Adiga, R.Y. Denis, and K.R. Vasuki [2], N.A. Bhagirathi [87]–[89], Bhargava and Adiga [92], [93], Bhargava, Adiga, and D.D. Somashekara [96], [97], Denis [128], [130], [132], Hirschhorn [154], K.G. Ramanathan [217], S.N. Singh [249], B. Srivastava [257], [258], Vasuki [279], Vasuki and H.S. Madhusudhan [281], and A. Verma, Denis, and K. Srinivasa Rao [282]. Hirschhorn's paper [154] is especially noteworthy in that he successfully examines the convergents associated with Entry 6.2.1. For another general theorem of this sort, see a paper by Bowman and Sohn [104].

On page 44 in his lost notebook, Ramanujan writes the continued fraction

$$\frac{1}{1} \, \underset{+}{} \, \frac{q^2+aq}{1} \, \underset{+}{} \, \frac{q^4+bq^2}{1} \, \underset{+}{} \, \frac{q^6+aq^3}{1} \, \underset{+}{} \, \cdots. \tag{6.2.21}$$

Clearly, (6.2.21) is a special case of the continued fraction in (6.2.2). Ramanujan devotes most of the remaining portion of the page to stating nine particular cases. In fact, there are but six different continued fractions, since three of the continued fractions can be obtained from three of the others by changing the signs of both a and q. We now employ Entry 6.2.1 to derive each of the six continued fractions on page 44 as well as several other corollaries found on this and nearby pages.

Corollary 6.2.1 (p. 44). *For* $|q| < 1$,

$$\frac{(q;q^2)_\infty}{(q^2;q^4)_\infty^2} = \frac{1}{1} \, \underset{+}{} \, \frac{q}{1} \, \underset{+}{} \, \frac{q+q^2}{1} \, \underset{+}{} \, \frac{q^3}{1} \, \underset{+}{} \, \frac{q^2+q^4}{1} \, \underset{+}{} \, \cdots. \tag{6.2.22}$$

Proof. In Entry 6.2.1, set $a = 0$, $b = 1$, and $\lambda = 1$, which yields the continued fraction in (6.2.22). From (6.2.5),

$$G(0,1,\lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n q^{n^2}}{(q^2;q^2)_n} = (-\lambda q;q^2)_\infty. \tag{6.2.23}$$

Thus, by (6.2.23) and Euler's identity,

$$\frac{G(0,1,q)}{G(0,1,1)} = \frac{(-q^2;q^2)_\infty}{(-q;q^2)_\infty} = \frac{1}{(q^2;q^4)_\infty(-q;q^2)_\infty} = \frac{(q;q^2)_\infty}{(q^2;q^4)_\infty^2}. \tag{6.2.24}$$

Hence, by (6.2.24), we complete the proof of (6.2.22). □

Corollary 6.2.2 (p. 43). *For any complex number $a \neq -q^{-2n+1}$, $n \geq 1$,*

$$\frac{(-aq^2;q^2)_\infty}{(-aq;q^2)_\infty} = \frac{1}{1} \underset{+}{} \frac{aq}{1} \underset{+}{} \frac{q+aq^2}{1} \underset{+}{} \frac{aq^3}{1} \underset{+}{} \frac{q^2+aq^4}{1} \underset{+}{} \cdots. \tag{6.2.25}$$

Proof. In Entry 6.2.1, set $a=0$, $b=1$, and replace λ by a new parameter a. We thus easily obtain the continued fraction in (6.2.25). On the left side of (6.2.2), we obtain the quotient

$$\frac{G(0,1,aq)}{G(0,1,a)} = \frac{\displaystyle\sum_{n=0}^\infty \frac{a^n q^{n^2+n}}{(q^2;q^2)_n}}{\displaystyle\sum_{n=0}^\infty \frac{a^n q^{n^2}}{(q^2;q^2)_n}} = \frac{(-aq^2;q^2)_\infty}{(-aq;q^2)_\infty},$$

upon two applications of (6.2.5) with q replaced by q^2, and $t = aq^2$ and $t = aq$, respectively. This completes the proof. □

Corollary 6.2.3 (p. 40). *We have*

$$\frac{G(0,b,\lambda q)}{G(0,b,\lambda)} = \frac{1}{1} \underset{+}{} \frac{\lambda q}{1} \underset{+}{} \frac{bq+\lambda q^2}{1} \underset{+}{} \frac{\lambda q^3}{1} \underset{+}{} \frac{bq^2+\lambda q^4}{1} \underset{+}{} \cdots. \tag{6.2.26}$$

Proof. This corollary is simply the case $a=0$ of Entry 6.2.1. □

Corollary 6.2.4 (p. 44). *We have*

$$\sum_{n=0}^\infty (-1)^n q^{n(n+1)/2} = \frac{1}{1} \underset{+}{} \frac{q}{1} \underset{+}{} \frac{q^2-q}{1} \underset{+}{} \frac{q^3}{1} \underset{+}{} \frac{q^4-q^2}{1} \underset{+}{} \cdots. \tag{6.2.27}$$

Proof. Set $a=0$, $\lambda=1$, and $b=-1$ in Entry 6.2.1, from which the desired continued fraction follows.

To obtain the left side of (6.2.27), we utilize a result from Chapter 16 of Ramanujan's second notebook [61, p. 18, Entry 9]. Since this result also appears in Ramanujan's lost notebook, we formally list it here as an entry.

Entry 6.2.5 (p. 362). *For any complex numbers a and b, and $|q|<1$,*

$$(aq)_\infty \sum_{n=0}^\infty \frac{b^n q^{n^2}}{(q)_n (aq)_n} = \sum_{n=0}^\infty \frac{(-1)^n (b/a)_n a^n q^{n(n+1)/2}}{(q)_n}. \tag{6.2.28}$$

Applying (6.2.28) twice, first with $a = 1$ and $b = q$, and second with $a = b = 1$, we find that

$$\frac{G(0,-1,q)}{G(0,-1,1)} = \frac{\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_n^2}}{\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n^2}} = \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2}.$$

The desired result now follows. □

Corollary 6.2.4 was recorded by Gauss [141] in his diary on February 16, 1797; see also J.J. Gray's [145] translation of Gauss's diary. Usually, Corollary 6.2.4 is attributed to G. Eisenstein [134], [135], who obtained the generalization found in the next result.

Corollary 6.2.5 (p. 43). *We have*

$$\sum_{n=0}^{\infty} (-a)^n q^{n(n+1)/2} = \frac{1}{1+} \frac{aq}{1+} \frac{a(q^2-q)}{1} \, + \frac{aq^3}{1+} \frac{a(q^4-q^2)}{1} \, + \cdots . \tag{6.2.29}$$

Proof. In Entry 6.2.1, set $a = 0$ and then replace both λ and $-b$ by a new parameter a. We thus easily obtain the continued fraction in (6.2.29). To evaluate the resulting quotient of q-series on the left side of (6.2.2), we need two results. The first is a result due to Cauchy and found as Entry 3 in Chapter 16 of Ramanujan's second notebook, namely [227], [61, p. 14],

$$\sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q)_n (aq)_n} = \frac{1}{(aq)_\infty}. \tag{6.2.30}$$

Hence, by (6.2.30),

$$G(0,-a,a) = \sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q)_n (aq)_n} = \frac{1}{(aq)_\infty}. \tag{6.2.31}$$

Second, by (6.2.28),

$$G(0,-a,aq) = \sum_{n=0}^{\infty} \frac{a^n q^{n^2+n}}{(q)_n (aq)_n} = \frac{1}{(aq)_\infty} \sum_{n=0}^{\infty} (-1)^n a^n q^{n(n+1)/2}. \tag{6.2.32}$$

Dividing the latter equality by the former, we complete the proof of (6.2.29). □

Ramanujan also recorded Corollary 6.2.5 as Entry 13 in Chapter 16 of his second notebook [227], [61, p. 27]. Corollary 6.2.4 was also established by Selberg [241], [242, p. 19, equation (55)].

The next corollary stated by Ramanujan gives the Rogers–Ramanujan continued fraction product representation on the first page of this book.

Corollary 6.2.6 (p. 44). *We have*

$$\frac{1}{1}\,\frac{}{+}\,\frac{q}{1}\,\frac{}{+}\,\frac{q^2}{1}\,\frac{}{+}\,\frac{q^3}{1}\,\frac{}{+}\,\cdots = \frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}.$$

Proof. Set $a = b = 0$ and $\lambda = 1$ in Entry 6.2.1. Thus, we obtain the continued fraction in Corollary 6.2.6. Observe that $G(0,0,q) = F(q,q)$ and $G(0,0,1) = F(1,q)$, where

$$F(a,q) := \sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q)_n}. \tag{6.2.33}$$

By the Rogers–Ramanujan identities [61, p. 77],

$$F(q,q) = \frac{1}{(q^2;q^5)_\infty(q^3;q^5)_\infty} \quad \text{and} \quad F(1,q) = \frac{1}{(q;q^5)_\infty(q^4;q^5)_\infty}. \tag{6.2.34}$$

The result now follows. □

On page 44 in his lost notebook [228], Ramanujan seemingly claimed another continued fraction representation for the left side of (6.2.27). This result, quoted exactly, is given as follows:

Entry 6.2.6 (p. 44).

$$1 - q + q^3 - q^6 + \cdots = \frac{1}{1}\,\frac{}{+}\,\frac{q+q^2}{1}\,\frac{}{+}\,\frac{q^3+q^4}{1}\,\frac{}{+}\,\frac{q^5+q^6}{1}\,\frac{}{+}\,\cdots. \tag{6.2.35}$$

However, the presumed implication is false, as we now demonstrate.

From Ramanujan's second notebook [227], [61, p. 30, Corollary],

$$\frac{F(aq,q)}{F(a,q)} = \frac{1}{1}\,\frac{}{+}\,\frac{aq}{1}\,\frac{}{+}\,\frac{aq^2}{1}\,\frac{}{+}\,\frac{aq^3}{1}\,\frac{}{+}\,\cdots, \tag{6.2.36}$$

where $F(a,q)$ is defined by (6.2.33). Hence, the continued fraction in (6.2.35) is simply

$$\begin{aligned}
\frac{F(q+q^2,q^2)}{F(1+q^{-1},q^2)} &= \frac{1 + \dfrac{(q+q^2)q^2}{1-q^2} + \dfrac{(q^2+2q^3+q^4)q^8}{(1-q^2)(1-q^4)} + O(q^{21})}{1 + \dfrac{(1+q^{-1})q^2}{1-q^2} + \dfrac{(1+2q^{-1}+q^{-2})q^8}{(1-q^2)(1-q^4)} + O(q^{15})} \\
&= \frac{1+q^3+q^4+q^5+q^6+q^7+q^8+q^9+2q^{10}+\cdots}{1+q+q^2+q^3+q^4+q^5+2q^6+3q^7+3q^8+3q^9+4q^{10}+\cdots} \\
&= 1 - q + q^3 - q^6 + q^8 - q^9 - q^{10} + O(q^{11}).
\end{aligned}$$

Thus, (6.2.35) is correct as far as it is written, but it stops just short of where the indicated triangular number pattern is violated.

Corollary 6.2.7 (p. 44). *We have*

$$\frac{(q;q^6)_\infty(q^5;q^6)_\infty}{(q^3;q^6)_\infty^2} = \frac{1}{1}\,\genfrac{}{}{0pt}{}{}{+}\,\frac{q+q^2}{1}\,\genfrac{}{}{0pt}{}{}{+}\,\frac{q^2+q^4}{1}\,\genfrac{}{}{0pt}{}{}{+}\,\frac{q^3+q^6}{1}\,\genfrac{}{}{0pt}{}{}{+\cdots}. \tag{6.2.37}$$

Proof. In Entry 6.2.1, replace q by q^2 and then set $a = 1/q$, $b = 1$, and $\lambda = 1$. The desired continued fraction easily follows.

Next,

$$G(q^{-1}, 1; 1; q^2) = \sum_{n=0}^\infty \frac{(-q;q^2)_n q^{n^2}}{(q^4;q^4)_n} = \frac{(-q;q^2)_\infty(q^3;q^3)_\infty(q^3;q^6)_\infty}{(q^2;q^2)_\infty},$$

where the last equality follows from a result of L.J. Slater [251, p. 154, equation (25)]. Secondly,

$$G(q, q^2; 1; q^2) = \sum_{n=0}^\infty \frac{(-q;q^2)_n q^{n^2+2n}}{(q^4;q^4)_n} = \frac{(-q;q^2)_\infty(q^6;q^6)_\infty(q;q^6)_\infty(q^5;q^6)_\infty}{(q^2;q^2)_\infty},$$

which follows from another identity of Slater [250, p. 469, E(4)] by setting $y = -\sqrt{q}$ and letting z approach ∞ in Slater's formula. Dividing the latter equality by the former equality, we obtain the quotient on the left side of (6.2.37). □

The continued fraction of Corollary 6.2.7 is known as Ramanujan's cubic continued fraction; see Chapter 3 for several of its properties. Corollary 6.2.7 is also given by Ramanujan in his third notebook [227, vol. 2, p. 373], [63, p. 45, Entry 18]. The first published proof of Corollary 6.2.7 is by Watson [287] in 1929, while the second is by Selberg [241, p. 19], [242] in 1936. The next proofs were by B. Gordon [143] in 1965 and Andrews [17] in 1968. See also Hirschhorn's paper [154, Theorem 2] and two papers by K.G. Ramanathan [215], [216]. L.–C. Zhang [302] has examined the continued fraction in (6.2.37) when q is a root of unity.

Corollary 6.2.8 (p. 44). *We have*

$$\frac{(q;q^8)_\infty(q^7;q^8)_\infty}{(q^3;q^8)_\infty(q^5;q^8)_\infty} = \frac{1}{1}\,\genfrac{}{}{0pt}{}{}{+}\,\frac{q+q^2}{1}\,\genfrac{}{}{0pt}{}{}{+}\,\frac{q^4}{1}\,\genfrac{}{}{0pt}{}{}{+}\,\frac{q^3+q^6}{1}\,\genfrac{}{}{0pt}{}{}{+\cdots}. \tag{6.2.38}$$

Proof. In Entry 6.2.1, replace q by q^2, and then set $a = 1/q$, $b = 0$, and $\lambda = 1$. We then immediately obtain the continued fraction in (6.2.38). To complete the proof, we need two more identities of Slater [251, p. 155, equations (34), (36)],

$$G(q, q^2; 0; q^2) = \sum_{n=0}^\infty \frac{(-q;q^2)_n q^{n^2+2n}}{(q^2;q^2)_n} = \frac{(q^2;q^2)_\infty}{(q^3;q^8)_\infty(q^5;q^8)_\infty(q^8;q^8)_\infty(-q;q^2)_\infty}$$

and

$$G(1/q,1;0;q^2)=\sum_{n=0}^{\infty}\frac{(-q;q^2)_nq^{n^2}}{(q^2;q^2)_n}=\frac{(q^2;q^2)_\infty}{(q;q^8)_\infty(q^7;q^8)_\infty(q^8;q^8)_\infty(-q;q^2)_\infty}.$$

The quotient of the latter two equalities now easily yields the left side of (6.2.38), and so the proof is complete. □

The continued fraction in (6.2.38) is called the Ramanujan–Göllnitz–Gordon continued fraction. The first proof of Corollary 6.2.8 is by Selberg [241, equation (53)], [242, pp. 18–19]. Gordon [143] and Andrews [17] found another continued fraction for the left side of (6.2.38). A beautiful theory for the Ramanujan–Gordon–Göllnitz continued fraction has been developed by H.H. Chan and S.–S. Huang [115].

Corollary 6.2.9 (p. 44). *We have*

$$\sum_{n=0}^{\infty}(-1)^nq^{3n^2+2n}(1+q^{2n+1})=\frac{1}{1}\,\underset{+}{}\,\frac{q^2-q}{1}\,\underset{+}{}\,\frac{q^4-q^2}{1}\,\underset{+}{}\,\frac{q^6-q^3}{1}\,\underset{+}{}\,\cdots. \tag{6.2.39}$$

Proof. In Entry 6.2.1 we replace q by q^2 and set $a=-1/q$, $b=-1$, and $\lambda=1$ to deduce that

$$\frac{G(-q,q^2;-1;q^2)}{G(-1/q,1;-1;q^2)}=\frac{1}{1}\,\underset{+}{}\,\frac{q^2-q}{1}\,\underset{+}{}\,\frac{q^4-q^2}{1}\,\underset{+}{}\,\frac{q^6-q^3}{1}\,\underset{+}{}\,\cdots. \tag{6.2.40}$$

To complete the proof, we need a variant of Heine's transformation, namely, for $|z|<1$,

$${}_2\phi_1(a,b;c;z)=\frac{(c/b)_\infty(bz)_\infty}{(c)_\infty(z)_\infty}\;{}_2\phi_1(abz/c,b;bz;c/b), \tag{6.2.41}$$

which evidently is due to Rogers [233]. Thus, using (6.2.41) with b replaced by q/c, we find that

$$\begin{aligned}\frac{G(-q,q^2;-1;q^2)}{G(-1/q,1;-1;q^2)}&=\frac{\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(q;q^2)_nq^{n^2+2n}}{(q^2;q^2)_n^2}}{\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(q;q^2)_nq^{n^2}}{(q^2;q^2)_n^2}}\\&=\lim_{c\to0}\frac{\displaystyle\sum_{n=0}^{\infty}\frac{(q/c;q^2)_n(q;q^2)_nc^nq^{2n}}{(q^2;q^2)_n^2}}{\displaystyle\sum_{n=0}^{\infty}\frac{(q/c;q^2)_n(q;q^2)_nc^n}{(q^2;q^2)_n^2}}\\&=\lim_{c\to0}\frac{\displaystyle\frac{(cq;q^2)_\infty(q^3;q^2)_\infty}{(q^2;q^2)_\infty(cq^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(q^2;q^2)_n(q/c;q^2)_n}{(q^3;q^2)_n(q^2;q^2)_n}(cq)^n}{\displaystyle\frac{(cq;q^2)_\infty(q;q^2)_\infty}{(q^2;q^2)_\infty(c;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(1;q^2)_n(q/c;q^2)_n}{(q;q^2)_n(q^2;q^2)_n}(cq)^n}\end{aligned}$$

$$
\begin{aligned}
&=\frac{(q^3;q^2)_\infty}{(q;q^2)_\infty}\sum_{n=0}^\infty \frac{(-1)^n q^{n^2+n}}{(q^3;q^2)_n}\\
&=\sum_{n=0}^\infty \frac{(-1)^n q^{n^2+n}}{(q;q^2)_{n+1}}\\
&=\sum_{n=0}^\infty (-1)^n q^{3n^2+2n}(1+q^{2n+1}),
\end{aligned}
$$

where the last equality follows from a result of Rogers [235, p. 333, eq. (4)], or from a result found on page 37 of Ramanujan's lost notebook arising from the Rogers–Fine identity; see Entry 9.5.1 of Chapter 9. This completes the proof. □

The first proof of Corollary 6.2.9 is due to Selberg [241], [242, p. 18].

Corollary 6.2.10 (p. 44). *We have*

$$
\frac{(-q^3;q^4)_\infty}{(-q;q^4)_\infty}=\frac{1}{1}\,\genfrac{}{}{0pt}{}{}{+}\,\frac{q}{1}\,\genfrac{}{}{0pt}{}{}{+}\,\frac{q^2+q^3}{1}\,\genfrac{}{}{0pt}{}{}{+}\,\frac{q^5}{1}\,\genfrac{}{}{0pt}{}{}{+}\,\frac{q^4+q^7}{1}\,\genfrac{}{}{0pt}{}{}{+}\cdots. \tag{6.2.42}
$$

Proof. In Entry 6.2.1, first replace q by q^2. Next set $a=0$, $b=1$, and $\lambda=1/q$. We then easily obtain the continued fraction on the right side of (6.2.42). On the other hand, the quotient on the left side of (6.2.2) equals

$$
\frac{G(0,q;1;q^2)}{G(0,1/q;1;q^2)}=\frac{\displaystyle\sum_{n=0}^\infty \frac{q^{2n^2+n}}{(q^4;q^4)_n}}{\displaystyle\sum_{n=0}^\infty \frac{q^{2n^2-n}}{(q^4;q^4)_n}}=\frac{(-q^3;q^4)_\infty}{(-q;q^4)_\infty},
$$

where we have made two applications of (6.2.5). This completes the proof. □

Ramanujan found another continued fraction for the left side of (6.2.42), namely,

$$
\frac{(q^3;q^4)_\infty}{(q;q^4)_\infty}=\frac{1}{1}\,\genfrac{}{}{0pt}{}{}{-}\,\frac{q}{1+q^2}\,\genfrac{}{}{0pt}{}{}{-}\,\frac{q^3}{1+q^4}\,\genfrac{}{}{0pt}{}{}{-}\,\frac{q^5}{1+q^6}\,\genfrac{}{}{0pt}{}{}{-}\cdots;
$$

see Berndt's book [63, p. 48, Entry 20] for a proof. The first proof of Corollary 6.2.10 is due to Andrews [26], and another is due to Ramanathan [217].

Corollary 6.2.11 (p. 44). *We have*

$$
1-\sum_{n=1}^\infty q^{n(3n-1)/2}(1-q^n)=\frac{2}{2}\,\genfrac{}{}{0pt}{}{}{+}\,\frac{q+q}{1}\,\genfrac{}{}{0pt}{}{}{+}\,\frac{q^2+q^3}{1}\,\genfrac{}{}{0pt}{}{}{+}\,\frac{q^3+q^5}{1}\,\genfrac{}{}{0pt}{}{}{+}\cdots. \tag{6.2.43}
$$

Proof. In Entry 6.2.1, replace q by q^2 and then put $b=1$ and $a=\lambda=1/q$. We thus find that

$$\frac{\displaystyle\sum_{n=0}^{\infty}\frac{(-1;q^2)_n q^{n^2+2n}}{(q^4;q^4)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{(-1;q^2)_n q^{n^2}}{(q^4;q^4)_n}} = \frac{1}{1}\,+\,\frac{q+q}{1}\,+\,\frac{q^2+q^3}{1}\,+\,\frac{q^3+q^5}{1}\,+\cdots. \tag{6.2.44}$$

The continued fraction on the right side of (6.2.43) is therefore equal to

$$CF(q) := \frac{2}{1+\dfrac{\displaystyle\sum_{n=0}^{\infty}\frac{(-1;q^2)_n q^{n^2}}{(q^4;q^4)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{(-1;q^2)_n q^{n^2+2n}}{(q^4;q^4)_n}}} = 2\frac{\displaystyle\sum_{n=0}^{\infty}\frac{(-1;q^2)_n q^{n^2+2n}}{(q^4;q^4)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{(-1;q^2)_{n+1} q^{n^2}}{(q^4;q^4)_n}}. \tag{6.2.45}$$

The sum in the denominator is easily evaluated by (6.2.5). The sum in the numerator is more troublesome, and we shall use a threefold iteration of Heine's theorem given by [61, p. 15, Equation (6.1)], for $|t|<1$,

$$\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(q)_n}t^n = \frac{(abt/c)_\infty}{(t)_\infty}\sum_{n=0}^{\infty}\frac{(c/a)_n(c/b)_n}{(c)_n(q)_n}\left(\frac{abt}{c}\right)^n. \tag{6.2.46}$$

Hence, by (6.2.45), (6.2.5), and (6.2.46),

$$\begin{aligned}
CF(q) &= \frac{1}{(-q;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(-1;q^2)_n q^{n^2+2n}}{(q^4;q^4)_n}\\
&= \frac{1}{(-q;q^2)_\infty}\lim_{t\to 0}\sum_{n=0}^{\infty}\frac{(-q/t;q^2)_n(-1;q^2)_n}{(q^2;q^2)_n(-q^2;q^2)_n}t^n q^{2n}\\
&= \frac{1}{(-q;q^2)_\infty}\lim_{t\to 0}\frac{(-q;q^2)_\infty}{(tq^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(tq;q^2)_n(q^2;q^2)_n}{(q^2;q^2)_n(-q^2;q^2)_n}(-q)^n\\
&= \sum_{n=0}^{\infty}\frac{(-q)^n}{(-q^2;q^2)_n}. \qquad (6.2.47)
\end{aligned}$$

We now offer two routes to completing the proof; both depend on results in the lost notebook connected with the Rogers–Fine identity.

First, by a result found on page 36 of the lost notebook, Entry 9.4.7 of Chapter 9, which is established by the Rogers–Fine identity, (4.5.5) in Chapter 4, or (9.1.1) of Chapter 9,

$$\begin{aligned}
\sum_{n=0}^{\infty}\frac{(-q)^n}{(-q^2;q^2)_n} &= \sum_{n=0}^{\infty}q^{n(3n+1)/2}(1-q^{2n+1})\\
&= \sum_{n=0}^{\infty}q^{n(3n+1)/2} - \sum_{n=0}^{\infty}q^{n(3n+5)/2+1}
\end{aligned}$$

$$= 1 + \sum_{n=1}^{\infty} q^{n(3n+1)/2} - \sum_{n=1}^{\infty} q^{n(3n-1)/2}$$
$$= 1 + \sum_{n=1}^{\infty} q^{n(3n-1)/2}(q^n - 1). \tag{6.2.48}$$

Combining (6.2.47) and (6.2.48), we complete the proof.

For the second approach, we transform the far right side of (6.2.47) by using the Rogers–Fine identity, (4.5.5) in Chapter 4, with q replaced by q^2, $a = 0$, $b = -1$, and $t = -q$. (This is established in more detail in our chapter on the Rogers–Fine identity; see (9.4.15).) We thus find that

$$CF(q) = \sum_{n=0}^{\infty} \frac{q^{2n^2+n}}{(-q)_{2n+1}}. \tag{6.2.49}$$

Now, by a result of Andrews [15, p. 38, equation (4.2)],

$$\sum_{n=0}^{\infty} \frac{q^{2n^2+n}}{(-q)_{2n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(-q)_n}, \tag{6.2.50}$$

where we have corrected two misprints on the left side in [15]. Lastly, by setting $x = -1$ in an exercise in Andrews's text [21, p. 29, Exercise 10], we find that

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(-q)_n} = 1 - \sum_{n=1}^{\infty} q^{n(3n-1)/2}(1 - q^n). \tag{6.2.51}$$

(Equality (6.2.51) also arises from the Rogers–Fine identity; see Entry 9.4.2 of Chapter 9.) Finally, by combining (6.2.50) and (6.2.51) with (6.2.49), we conclude that

$$CF(q) = 1 - \sum_{n=1}^{\infty} q^{n(3n-1)/2}(1 - q^n)$$

to complete the proof. □

6.3 A Second General Continued Fraction

Define

$$g(b;\lambda) = \sum_{n=0}^{\infty} \frac{\lambda^n q^{n^2}}{(q)_n(-bq)_n}. \tag{6.3.1}$$

On page 40 in his lost notebook, Ramanujan offers three continued fractions for $g(b;\lambda)$, which we relate in the next theorem.

Entry 6.3.1 (p. 40). *We have*

$$\text{(i)}\qquad \frac{g(b;\lambda q)}{g(b;\lambda)} = \frac{1}{1}\,{+}\,\frac{\lambda q}{1}\,{+}\,\frac{\lambda q^2+bq}{1}\,{+}\,\frac{\lambda q^3}{1}\,{+}\,\frac{\lambda q^4+bq^2}{1}\,{+}\cdots$$

$$\text{(ii)}\qquad = \frac{1}{1}\,{+}\,\frac{\lambda q}{1+bq}\,{+}\,\frac{\lambda q^2}{1+bq^2}\,{+}\,\frac{\lambda q^3}{1+bq^3}\,{+}\cdots$$

$$\text{(iii)}\qquad = \frac{1}{1-b}\,{+}\,\frac{b+\lambda q}{1-b}\,{+}\,\frac{b+\lambda q^2}{1-b}\,{+}\,\frac{b+\lambda q^3}{1-b}\,{+}\cdots,$$

$$\text{(iv)}\qquad g(b;\lambda) = (1-b)g(b;\lambda q) + (b+\lambda q)g(b;\lambda q^2).$$

Proof of (i). Since $g(b,\lambda) = G(0,b,\lambda)$, we observe that (i) is identical to Corollary 6.2.3. □

Proof of (ii). In Entry 15 of Chapter 16 [61, pp. 30–31], Ramanujan offers the beautiful continued fraction

$$\frac{g(-a;b)}{g(-a;bq)} = 1 + \frac{bq}{1-aq}\,{+}\,\frac{bq^2}{1-aq^2}\,{+}\,\frac{bq^3}{1-aq^3}\,{+}\cdots.$$

If we replace a by $-b$ and b by λ above, and then take the reciprocal of each side, we immediately obtain (ii). □

Proof of (iii) *and* (iv). A straightforward calculation shows that

$$\begin{aligned}
&(1-b)g(b;\lambda q) + (b+\lambda q)g(b;\lambda q^2)\\
&= \sum_{n=0}^{\infty} \frac{\lambda^n q^{n^2}}{(q)_n(-bq)_n}\left(q^n - bq^n + bq^{2n} + (1-q^n)(1+bq^n)\right)\\
&= g(b;\lambda),
\end{aligned}$$

which establishes (iv). It follows that

$$\frac{g(b;\lambda)}{g(b;\lambda q)} = 1 - b + \frac{b+\lambda q}{\dfrac{g(b;\lambda q)}{g(b;\lambda q^2)}}.$$

By successive iterations of the identity above, we formally derive the continued fraction in part (iii). That the continued fraction converges and that it converges to the left side of (iii) can be demonstrated by the same argument that we used to prove Entry 6.2.1. □

Part (ii) has also been proved by V. Ramamani [214] and Hirschhorn [152], [155]. A generalization of (iii) was established by Hirschhorn [153].

6.4 A Third General Continued Fraction

On page 43 in his lost notebook, Ramanujan gives another continued fraction for quotients of the function $G(a,\lambda;b;q)$. We follow the path of Bhargava and Adiga [91] in proving not only this continued fraction, but still another continued fraction for such quotients. The two continued fractions will then be combined to establish an identity between certain continued fractions found on page 42 in the lost notebook.

We begin with a lemma providing two needed q-difference equations. Recall that $P(a,b,\lambda)$ is defined in (6.2.3).

Lemma 6.4.1. *We have*

$$\text{(i)} \qquad P(a,b,\lambda) = P(aq,b,\lambda q) + (aq+\lambda q)P(aq,bq,\lambda q^2),$$
$$\text{(ii)} \quad P(aq,b,\lambda) = (1-aq+bq)P(aq,bq,\lambda q) + (aq+\lambda q)P(aq,bq^2,\lambda q^2).$$

Proof. In Lemma 6.2.1, replace λ by λq in (i) and add the result to (ii). We thus obtain (i) of Lemma 6.4.1. Next, return to Lemma 6.2.1 and replace λ by λq and b by bq in (i), replace b by bq in (ii), multiply (i) by -1, and add these three equalities to (iii). We then deduce (ii) of the present lemma. □

Theorem 6.4.1. *We have*

$$\frac{G(aq,\lambda q;b;q)}{G(a,\lambda;b;q)} = \frac{1}{1} \, {}_{+} \, \frac{aq+\lambda q}{1-aq+bq} \, {}_{+} \, \frac{aq+\lambda q^2}{1-aq+bq^2} \, {}_{+ \cdots +} \, \frac{aq+\lambda q^n}{1-aq+bq^n} \, {}_{+ \cdots} . \tag{6.4.1}$$

Proof. Lemma 6.4.1(i) can be reconstituted in the form

$$\frac{G(aq,\lambda q;b;q)}{G(a,\lambda;b;q)} = \frac{1}{1} \, {}_{+} \, \frac{aq+\lambda q}{\dfrac{G(aq,\lambda q;b;q)}{G(aq,\lambda q^2;bq;q)}}. \tag{6.4.2}$$

Next, in Lemma 6.4.1(ii), replace λ by λq^{n+1} and b by bq^n to deduce that

$$S_n := \frac{G(aq,\lambda q^{n+1};bq^n;q)}{G(aq,\lambda q^{n+2};bq^{n+1};q)} = (1-aq+bq^{n+1}) + \frac{aq+\lambda q^{n+2}}{S_{n+1}}. \tag{6.4.3}$$

Iterating (6.4.3) with $n=0,1,2,\dots$, and using (6.4.2), we deduce (6.4.1). The convergence of the continued fraction follows along the same lines as those in the proof of Entry 6.2.1, since $S_n \to 1$ as $n \to \infty$. □

To prove Ramanujan's next continued fraction, we once again need a couple of auxiliary q-difference equations.

Lemma 6.4.2. *We have*

$$\text{(i)} \qquad P(a,bq,\lambda) = (1+aq)P(aq,bq,\lambda q) + (\lambda q - abq^3)P(aq^2,bq^2,\lambda q^2),$$
$$\text{(ii)} \quad P(aq,b,\lambda q) = \{1+q(aq+b)\}\, P(aq^2,bq,\lambda q^2) + (\lambda q^2 - abq^4)P(aq^3,bq^2,\lambda q^3).$$

Proof. Return to Lemma 6.2.1(i) to replace a by aq, b by bq, and λ by λq and multiply the resulting equality by $-bq$. Then replace a by aq and b by bq in (ii). Thirdly, in (iii), replace a by aq. Fourthly, multiply (iii) by -1. Add all four equations to (i) to obtain (i) of the present lemma. Now replace a by aq and λ by λq in the just proved (i). Also replace a by aq and λ by λq in (iii) of Lemma 6.2.1. Adding the two equalities, we obtain (ii) of the present lemma. □

Entry 6.4.1 (p. 43). *We have*

$$\frac{G(aq,\lambda q;b;q)}{G(a,\lambda;b;q)} = \frac{1}{1+aq} + \frac{\lambda q - abq^2}{1+q(aq+b)} + \frac{\lambda q^2 - abq^4}{1+q^2(aq+b)} + \cdots + \frac{\lambda q^n - abq^{2n}}{1+q^n(aq+b)} + \cdots. \tag{6.4.4}$$

Proof. Replacing b by b/q in Lemma 6.4.2(i), we may rewrite the new equality in the form,

$$\frac{G(aq,\lambda q;b;q)}{G(a,\lambda;b;q)} = \frac{1}{1+aq} + \frac{\lambda q - abq^2}{\dfrac{G(aq,\lambda q;b;q)}{G(aq^2,\lambda q^2;bq;q)}}. \tag{6.4.5}$$

Replacing a, λ, and b by aq^{n-1}, λq^{n-1}, and bq^{n-1}, respectively, we can write Lemma 6.4.2(ii) in the form

$$U_n := \frac{G(aq^n,\lambda q^n;bq^{n-1};q)}{G(aq^{n+1},\lambda q^{n+1};bq^n;q)} = 1 + q^n(aq+b) + \frac{\lambda q^{n+1} - abq^{2n+2}}{U_{n+1}}. \tag{6.4.6}$$

Iterating (6.4.6) with $n = 1, 2, \ldots$ and employing (6.4.5), we deduce (6.4.4). The convergence follows as in Entry 6.2.1, since $U_n \to 1$ as $n \to \infty$. □

Andrews's [26] proof of Entry 6.4.1 is similar to the one by Bhargava and Adiga [91] that we have given above.

Entry 6.4.2 (p. 42). *If* $|q| < 1$, *then*

$$\frac{1}{a+c} - \frac{ab}{a+b+cq} - \cdots - \frac{ab}{a+b+cq^n} - \cdots = \frac{1}{c-b+a} + \frac{bc}{c-b+a/q} + \cdots + \frac{bc}{c-b+a/q^n} + \cdots. \tag{6.4.7}$$

Proof. In both Theorem 6.4.1 and Entry 6.4.1, we set $\lambda = 0$ and replace a and b by $-b/(aq)$ and c/a, respectively. We equate the two resulting continued fractions and take their reciprocals to find that

$$
\begin{aligned}
&\frac{G(-b/(aq),0;c/a;q)}{G(-b/a,0;c/a;q)} \\
&= 1 + \frac{-b/a}{1+(b+cq)/a} + \cdots + \frac{-b/a}{1+(b+cq^n)/a} + \cdots \\
&= \left(1-\frac{b}{a}\right) + \frac{bcq/a^2}{1+q(c-b)/a} + \cdots + \frac{bcq^{2n-1}/a^2}{1+q^n(c-b)/a} + \cdots .
\end{aligned}
\tag{6.4.8}
$$

Multiplying (6.4.8) by a, multiplying numerators and denominators by a to obtain an equivalent continued fraction, and adding c to both sides, we obtain the equivalent continued fractions

$$
\begin{aligned}
&c + a\frac{G(-b/(aq),0;c/a;q)}{G(-b/a,0;c/a;q)} \\
&= a + c + \frac{-ab}{a+b+cq} + \cdots + \frac{-ab}{a+b+cq^n} + \cdots \\
&= a + c - b + \frac{bcq}{a+(c-b)q} + \cdots + \frac{bcq^{2n-1}}{a+(c-b)q^n} + \cdots .
\end{aligned}
\tag{6.4.9}
$$

In the second continued fraction of (6.4.9), multiply numerators and denominators successively by $1/q, 1/q^2, 1/q^3, \ldots$ and then take the reciprocal of both sides of (6.4.9) to complete the proof. $\square$

Observe that the continued fraction on the right side in the next entry is the reciprocal of a finite version of the special case $c = 1$ of the continued fraction on the left side of Entry 6.4.2.

Entry 6.4.3 (p. 42). *For arbitrary complex numbers a and b, for any positive integer n, and for $q \neq 0$,*

$$
\begin{aligned}
1 + \frac{a}{1} + \frac{b}{q} + \frac{a}{1} + \frac{b}{q^2} + \cdots + \frac{b}{q^n} + \frac{a}{1} \\
= 1 + a - \frac{ab}{a+b+q} - \frac{ab}{a+b+q^2} - \cdots - \frac{ab}{a+b+q^n}.
\end{aligned}
$$

Proof. The continued fraction on the right side is the *odd part* of the continued fraction on the left side, and so the proof is complete.

6.5 A Transformation Formula

Entry 6.5.1 (p. 46). *Let $k \geq 0$, $\alpha = (1+\sqrt{1+4k})/2$, and $\beta = (-1+\sqrt{1+4k})/2$. Then, for $|q| < 1$ and* Re $q > 0$,

$$
\begin{aligned}
&\frac{1}{1} + \frac{k+q}{1} + \frac{k+q^2}{1} + \frac{k+q^3}{1} + \cdots \\
&= \frac{1}{\alpha} + \frac{q}{\alpha+\beta q} + \frac{q^2}{\alpha+\beta q^2} + \frac{q^3}{\alpha+\beta q^3} + \cdots .
\end{aligned}
\tag{6.5.1}
$$

This is a beautiful theorem, and we do not know how Ramanujan derived it. We shall use the Bauer–Muir transformation to establish Entry 6.5.1.

For $|q| > 1$, the continued fraction on the left side of (6.5.1) diverges. However, by Van Vleck's theorem (Lorentzen and Waadeland [182, p. 32]), the continued fraction on the right side of (6.5.1) converges for all q such that $\operatorname{Re} q > 0$.

If $q = 0$ in (6.5.1), then we find that

$$\frac{1}{1}+\frac{k}{1}+\frac{k}{1}+\frac{k}{1}+\cdots=\frac{1}{\alpha},$$

which can be established by elementary means.

If $q = 1$ in (6.5.1), we find that

$$\frac{1}{1}+\frac{k+1}{1}+\frac{k+1}{1}+\frac{k+1}{1}+\cdots \\ =\frac{1}{\alpha}+\frac{1}{\sqrt{1+4k}}+\frac{1}{\sqrt{1+4k}}+\frac{1}{\sqrt{1+4k}}+\cdots.$$

This identity can be easily verified by elementary computations; both sides are equal to

$$\frac{2}{1+\sqrt{5+4k}}.$$

If $k = 0$, then $\alpha = 1$ and $\beta = 0$. Thus, (6.5.1) reduces to a tautology.

If $k = 2$, then $\alpha = 2$ and $\beta = 1$. We thus obtain the following corollary, which Ramanujan also records, but with a slight misprint.

Entry 6.5.2 (p. 46). *For* $|q| < 1$,

$$\frac{1}{1}+\frac{2+q}{1}+\frac{2+q^2}{1}+\frac{2+q^3}{1}+\cdots=\frac{1}{2}+\frac{q}{2+q}+\frac{q^2}{2+q^2}+\frac{q^3}{2+q^3}+\cdots.$$

Proof of Entry 6.5.1. As indicated above, we shall apply the Bauer–Muir transformation [182, p. 76], which we now briefly describe. Given a continued fraction $b_0+\mathbf{K}(a_n/b_n)$ and a sequence of complex numbers $\{w_n\}$, $0 \le n < \infty$, define

$$\lambda_n = a_n - w_{n-1}(b_n + w_n), \qquad n = 1, 2, \ldots. \tag{6.5.2}$$

Assume that $\lambda_n \neq 0$ for every $n \geq 1$. Let

$$q_n = \lambda_{n+1}/\lambda_n, \qquad n \geq 1. \tag{6.5.3}$$

If for $n \geq 2$,

$$c_n = a_{n-1}q_{n-1} \qquad \text{and} \qquad d_n = b_n + w_n - w_{n-2}q_{n-1}, \tag{6.5.4}$$

then

$$b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \frac{a_3}{b_3} + \cdots = b_0 + w_0 + \frac{\lambda_1}{b_1 + w_1} + \frac{c_2}{d_2} + \frac{c_3}{d_3} + \cdots. \qquad (6.5.5)$$

If $C(q)$ denotes the reciprocal of the continued fraction on the left side of (6.5.1), and if we employ the notation on the left side of (6.5.5), then, for $n \geq 1$, $a_n = k + q^n$, and for $n \geq 0$, $b_n = 1$. Now set $w_n = \beta$, $n \geq 0$. Then, by (6.5.2), since $1+\beta = \alpha$ and $\alpha\beta = k$, it follows that $\lambda_n = q^n$. Thus, by (6.5.3), $q_n = q$, and by (6.5.4), if $n \geq 2$, $c_n = (k + q^{n-1})q$ and $d_n = \alpha - \beta q$, since $1 + \beta = \alpha$. Also, $b_0 + w_0 = \alpha = b_1 + w_1$. Thus, by (6.5.5),

$$C(q) = \alpha + \frac{q}{\alpha} + \frac{(k+q)q}{\alpha - \beta q} + \frac{(k+q^2)q}{\alpha - \beta q} + \cdots =: \alpha + \frac{q}{C_1(q)}. \qquad (6.5.6)$$

For the continued fraction $C_1(q)$, in the notation of the left side of (6.5.5), $b_0 = \alpha$, $b_n = \alpha - \beta q$, and $a_n = (k + q^n)q$, for $n \geq 1$. We apply the Bauer–Muir transformation a second time. Set $w_n = \beta q$,$n \geq 0$. A brief calculation shows that by (6.5.2), $\lambda_n = q^{n+1}$. Thus, $b_0 + w_0 = \alpha + \beta q$, $b_1 + w_1 = \alpha$, $c_n = (k + q^{n-1})q^2$, and $d_n = \alpha - \beta q^2$, where $n \geq 2$. Hence, after applying the Bauer–Muir transformationto $C_1(q)$ in (6.5.6), we find that

$$\begin{aligned} C(q) &= \alpha + \frac{q}{\alpha + \beta q} + \frac{q^2}{\alpha} + \frac{(k+q)q^2}{\alpha - \beta q^2} + \frac{(k+q^2)q^2}{\alpha - \beta q^2} + \cdots \\ &=: \alpha + \frac{q}{\alpha + \beta q} + \frac{q^2}{C_2(q)}. \end{aligned} \qquad (6.5.7)$$

Applying the Bauer–Muir transformation to $C_2(q)$ and proceeding as in the two previous applications, we find that if $w_n = \beta q^2$, then $\lambda_n = q^{n+2}$. Thus, $b_0 + w_0 = \alpha + \beta q^2$, $b_1 + w_1 = \alpha$, $c_n = (k + q^{n-1})q^3$, and $d_n = \alpha - \beta q^3$, where $n \geq 2$. Hence, from (6.5.7),

$$\begin{aligned} C(q) &= \alpha + \frac{q}{\alpha + \beta q} + \frac{q^2}{\alpha + \beta q^2} + \frac{q^3}{\alpha} + \frac{(k+q)q^3}{\alpha - \beta q^3} + \frac{(k+q^2)q^3}{\alpha - \beta q^3} + \cdots \\ &= \cdots \\ &= \alpha + \frac{q}{\alpha + \beta q} + \frac{q^2}{\alpha + \beta q^2} + \cdots + \frac{q^{n-1}}{\alpha + \beta q^{n-1}} \\ &\quad + \frac{q^n}{\alpha} + \frac{(k+q)q^n}{\alpha - \beta q^n} + \frac{(k+q^2)q^n}{\alpha - \beta q^n} + \cdots, \end{aligned} \qquad (6.5.8)$$

after an easy inductive argument on n. Letting n tend to ∞ in (6.5.8), we deduce (6.5.1). As indicated earlier, the transformed continued fraction converges for $\operatorname{Re} q > 0$. $\square$

Lorentzen and Waadeland [182, pp. 77–80] used the Bauer–Muir transformation to prove a special case of Entry 6.5.1 and to discuss the rapidity of convergence of the transformed continued fraction; we have followed along the same lines as their proof. D. Bowman has informed us that he can prove Entry 6.5.1 by using continued fractions for certain basis hypergeometric series and the second iterate of Heine's transformation.

6.6 Zeros of the Generalized Rogers–Ramanujan Continued Fraction

Entry 6.6.1 (p. 48). *The smallest real zero of*

$$F(q) := 1 - \frac{q}{1} - \frac{q^2}{1} - \frac{q^3}{1} - \cdots$$

is approximately equal to 0.576148.

Ramanujan actually gives the value 0.5762 for this zero. He also does not indicate the possibility of other real zeros.

We considered several approaches to Ramanujan's claim, including an examination of the zeros of convergents to $F(q)$. However, for only the method described below could we obtain a proper error analysis. Ramanujan possibly used an approximating polynomial of lower degree than that below, along with an iterative procedure such as Newton's method. However, in any case, the numerical calculations seem formidable, and we wonder how Ramanujan might have proceeded.

Proof. We employ the corollary to Entry 15 in Chapter 16 in Ramanujan's second notebook [61, p. 30], providing a representation for the reciprocal $F(a,q)$ of the generalized Rogers–Ramanujan continued fraction, namely,

$$\frac{\displaystyle\sum_{k=0}^{\infty} \frac{(-a)^k q^{k^2}}{(q)_k}}{\displaystyle\sum_{k=0}^{\infty} \frac{(-a)^k q^{k(k+1)}}{(q)_k}} = 1 - \frac{aq}{1} - \frac{aq^2}{1} - \frac{aq^3}{1} - \cdots =: F(a,q). \tag{6.6.1}$$

Setting $a = 1$ in (6.6.1), we shall examine the zeros of a partial sum of the numerator, namely,

$$\sum_{k=0}^{5} \frac{(-1)^k q^{k^2}}{(q)_k} = \frac{1}{(q)_5} \big(1 - 2q - q^2 + q^3 + 2q^4 + 2q^5 + q^6 - q^7 - 4q^8$$
$$-4q^9 - q^{10} + 2q^{11} + 2q^{12} + 4q^{13} + 2q^{14} - 2q^{15} - q^{18} - q^{21} - q^{25}\big).$$

Using *Mathematica*, we find that the only real zero is approximately

$$q_0 := 0.576148762259. \tag{6.6.2}$$

By the alternating series test, q_0 approximates the least real zero of $F(1,q) = F(q)$ with a (positive) error less than

$$\frac{q_0^{36}}{(q_0)_6} = 1.38201727 \times 10^{-8}.$$

This completes the proof. $\square$

The continued fraction $F(q)$ is central in the enumeration of "coins in a fountain" [201], and, along with its least positive zero $0.576148\ldots$, is important in the study of birth and death processes [204].

We briefly pointed out in Chapter 5 that the convergence of the Rogers–Ramanujan continued fraction on the unit circle is not completely understood. The convergence of the generalized Rogers–Ramaujan continued fraction, as a function of a, on the unit circle is also not fully understood. From its representation in (6.6.1), we see that the locations of the zeros of the generalized Rogers–Ramanujan function (as a function of a) in the denominator play a key role. When $a = \exp(2\pi i\tau)$, where τ is irrational, D.S. Lubinsky [183] and V.I. Buslaev [105] have established theorems on the convergence of (6.6.1).

Entry 6.6.2 (p. 48). *Let $q_0 = q_0(a)$ denote the least positive zero of $F(a,q)$, where $F(a,q)$ is defined by* (6.6.1)*. Then, as a tends to ∞,*

$$q_0 \sim \frac{1}{a} - \frac{1}{a^2} + \frac{2}{a^3} - \frac{6}{a^4} + \frac{21}{a^5} - \frac{79}{a^6} + \frac{311}{a^7} - \frac{1266}{a^8} + \frac{5289}{a^9} - \frac{22553}{a^{10}} + \frac{97763}{a^{11}} - \cdots . \tag{6.6.3}$$

Ramanujan calculated many asymptotic expansions in his notebooks, and it seems likely that in many instances, including the present one, Ramanujan employed the method of successive approximations. We also utilize this method below, but if Ramanujan also did so, he must have been able to more easily effect the calculations.

Proof. We shall calculate the first few coefficients in (6.6.3) by the method of successive approximations. We then describe how we used *Mathematica* for the remaining coefficients.

In view of (6.6.1), first set

$$1 - \frac{aq}{1} = 0.$$

Then $q = 1/a$ is a first approximation for q_0. Next, set

$$1 - \frac{aq}{1} \, {}_{-} \, \frac{aq^2}{1} = 0 \tag{6.6.4}$$

and set $q = 1/a + x/a^2$ in (6.6.4), where x is to be determined. Then

$$1 - a\left(\frac{1}{a} + \frac{x}{a^2}\right)^2 - a\left(\frac{1}{a} + \frac{x}{a^2}\right) = 0.$$

Equating coefficients of $1/a$, we deduce that $x = -1$. Thirdly, set

$$1 - \frac{aq}{1} \, {}_{-} \, \frac{aq^2}{1} \, {}_{-} \, \frac{aq^3}{1} = 0 \tag{6.6.5}$$

and let $q = 1/a - 1/a^2 + x/a^3$ in (6.6.5). Equating coefficients of $1/a^2$, we deduce that $x = 2$.

Continuing in this way, we find that the calculations become increasingly more difficult. Since at each stage we are approximating the zeros of a finite continued fraction, we use an analogue of (6.6.1) for the finite generalized Rogers–Ramanujan continued fraction found in Ramanujan's notebooks. Thus, for each positive integer n [61, p. 31, Entry 16],

$$\frac{\displaystyle\sum_{k=0}^{[(n+1)/2]} \frac{(-a)^k q^{k^2}(q)_{n-k+1}}{(q)_k(q)_{n-2k+1}}}{\displaystyle\sum_{k=0}^{[n/2]} \frac{(-a)^k q^{k(k+1)}(q)_{n-k}}{(q)_k(q)_{n-2k}}} = 1 - \frac{aq}{1} - \frac{aq^2}{1} - \cdots - \frac{aq^n}{1}. \tag{6.6.6}$$

To calculate the first eleven terms in the asymptotic expansion of q_0, we need to take $n = 11$ above. Discarding those terms that do not arise in the calculation of the first eleven coefficients, we successively approximate the zeros of

$$\begin{aligned}(1-q)(1-q^2)(1-q^3)(1-q^4) - aq(1-q^{11})(1-q^2)(1-q^3)(1-q^4)\\ + a^2q^4(1-q^9)(1-q^3)(1-q^4) - a^3q^9(1-q^4).\end{aligned} \tag{6.6.7}$$

We used *Mathematica* in (6.6.7) to successively calculate the coefficients of a^{-j}, $1 \leq j \leq 11$, and found them to be as indicated in (6.6.3).

We emphasize that these calculations indeed do yield an asymptotic expansion, for the error term made in approximating q_0 by the first n terms is easily seen to be $O(1/a^{n+1})$ in each case. □

Entry 6.6.3 (p. 48). *Let* q_0 *be as given in Entry* 6.6.2. *Then, as* a *tends to* ∞,

$$q_0 = f(a) + O(1/a^8),$$

where

$$\begin{aligned}f(a) &:= \frac{2}{a - 1 + \sqrt{(a+1)(a+5)}}\\ &= \frac{1}{a} - \frac{1}{a^2} + \frac{2}{a^3} - \frac{6}{a^4} + \frac{21}{a^5} - \frac{79}{a^6} + \frac{311}{a^7}\\ &\quad - \frac{1265}{a^8} + \frac{5275}{a^9} - \frac{22431}{a^{10}} + \frac{96900}{a^{11}} - \cdots.\end{aligned} \tag{6.6.8}$$

First Proof of Entry 6.6.3. Expanding $f(a)$ via *Mathematica*, we deduce the Taylor series in a^{-1} given in (6.6.8). Comparing (6.6.8) with (6.6.3), we find that the coefficients of a^{-j}, $1 \leq j \leq 7$, agree, while the coefficients of a^{-8} differ only by 1. Thus, Ramanujan's claim in Entry 6.6.3 is justified. □

Second Proof of Entry 6.6.3. Our second proof is more natural and was kindly provided for us by W. Van Assche [278].

The Hermite–Padé approximant to the two functions q_0 and q_0^2 is obtained by finding polynomials A_n and B_n, each of degree n, and a polynomial R_{n-1} of degree $n-1$ such that

$$A_n(a)q_0(a) + B_n(a)q_0^2(a) + R_{n-1}(a) = O(1/a^m), \tag{6.6.9}$$

where m is as large as possible. By setting coefficients of negative powers of a equal to 0, we obtain $2n+2$ homogeneous equations for the unknown coefficients of the the polynomials A_n and B_n, from which we use $2n+1$ equations to find the coefficients of A_n and B_n up to a multiplicative factor. The polynomial R_{n-1} contains the positive powers of a.

If we set $n = 1$ in (6.6.9), we find that

$$(1-a)q_0(a) - (2a-1)q_0^2(a) + 1 = O(1/a^7); \tag{6.6.10}$$

the error term is better than the error term we would expect, i.e., $O(1/a^4)$. We now neglect the right side of (6.6.10) and use the left side to find an algebraic approximation to q_0; i.e., we solve the equation

$$(1-a)f(a) - (2a-1)f^2(a) + 1 = 0.$$

Solving this equation, we obtain the function $f(a)$ defined in (6.6.8). □

Entry 6.6.4 (p. 48). *Let q_0 be as given in Entry 6.6.2. Then, as a tends to ∞,*

$$q_0 = g(a) + O(1/a^{11}),$$

where

$$\begin{aligned} g(a) &:= \frac{1}{\dfrac{a-1+\sqrt{(a+1)(a+5)}}{2} + \left(\dfrac{a+3-\sqrt{(a+1)(a+5)}}{a-1+\sqrt{(a+1)(a+5)}}\right)^3} \\ &= \frac{1}{a} - \frac{1}{a^2} + \frac{2}{a^3} - \frac{6}{a^4} + \frac{21}{a^5} - \frac{79}{a^6} + \frac{311}{a^7} \\ &\quad - \frac{1266}{a^8} + \frac{5289}{a^9} - \frac{22553}{a^{10}} + \frac{97760}{a^{11}} - \cdots . \end{aligned} \tag{6.6.11}$$

Proof. Expanding $g(a)$ in a Taylor series in a^{-1} with the help of *Mathematica*, we establish the expansion in (6.6.11). Comparing (6.6.11) with (6.6.3), we find that the coefficients of a^{-j}, $1 \le j \le 10$, agree, while the coefficients of a^{-11} differ only by 3. Thus, Entry 6.6.4 follows. □

In fact, in both the expansions (6.6.3) and (6.6.8), Ramanujan calculated just the first ten terms. Our statement of Entry 6.6.4 is stronger than that recorded by Ramanujan, who merely claimed that (in different notation) "$q_0 = g(a)$." Undoubtedly, however, he calculated the expansion (6.6.11). We calculated eleven terms in each expansion for the purpose of comparing accuracies. Hirschhorn [161] has also examined Ramanujan's approximations for the zero q_0.

Underneath his approximations to the zero q_0, Ramanujan records the following two algebraic numbers.

Entry 6.6.5 (p. 48). *We have*

$$\frac{1}{\sqrt{3}} = .57735 \qquad \textit{and} \qquad \frac{1}{5\left(\frac{7}{9}\sqrt{3}-1\right)} = .$$

The decimal expansion of $1/\sqrt{3}$ is correct as given. Ramanujan does not give the decimal expansion of the latter number. In fact,

$$\frac{1}{5\left(\frac{7}{9}\sqrt{3}-1\right)} \approx 0.5611879. \tag{6.6.12}$$

Note that (6.6.12) is a reasonably good approximation to the least positive zero (6.6.2) of the Rogers–Ramanujan continued fraction.

6.7 Two Entries on Page 200 of Ramanujan's Lost Notebook

In this section, we discuss two entries on page 200 in Ramanujan's lost notebook [228]. On this page, Ramanujan offers an identity bearing a superficial resemblance to the standard generating function (6.6.1) for $R(a,q)$, which we define by

$$R(a,q) := \frac{1}{1}\,\underset{+}{}\,\frac{aq}{1}\,\underset{+}{}\,\frac{aq^2}{1}\,\underset{+}{}\,\frac{aq^3}{1}\,\underset{+}{}\cdots. \tag{6.7.1}$$

We provide three proofs. The first two proofs derive from familiar transformations for q-series. The third proof is more interesting. We show that each side of the identity is a generating function for certain types of partitions. We then establish the identity by deriving a bijection between the two sets of partitions.

Below the identity described above, Ramanujan offers two close cousins of the Rogers–Ramanujan continued fraction, which he links together. We emphasize that no theorem about these continued fractions is claimed by Ramanujan, and there is no evidence (other than close proximity) that the identity mentioned above is related to these two continued fractions. We have been unable to relate the continued fractions with any other result of Ramanujan. Thus, it remains a mystery as to why Ramanujan recorded them here.

Entry 6.7.1 (p. 200). *For each complex number a and $|q|<1$,*

$$\sum_{n=0}^{\infty} \frac{a^n q^{n(n+1)}}{(q;q)_n^2} = (aq;q)_\infty \sum_{n=0}^{\infty} \frac{a^n q^n}{(q;q)_n^2}. \tag{6.7.2}$$

First Proof of Entry 6.7.1. Recall the third iterate of Heine's transformation given by [61, p. 15, equation (6.1)]

$$\sum_{n=0}^{\infty} \frac{(a;q)_n (b;q)_n}{(c;q)_n (q;q)_n} t^n = \frac{(abt/c;q)_\infty}{(t;q)_\infty} \sum_{n=0}^{\infty} \frac{(c/a;q)_n (c/b;q)_n}{(c;q)_n (q;q)_n} \left(\frac{abt}{c}\right)^n .$$

Let $c = q$ and then let a and b tend to 0. Lastly, let $t = aq$. The equality (6.7.2) then follows immediately. $\square$

Second Proof of Entry 6.7.1. In Entry 8 of Chapter 16 in his second notebook [227], Ramanujan recorded an identity arising from a basic hypergeometric series transformation. For $|a|, |q| < 1$,

$$\frac{(a;q)_\infty}{(b;q)_\infty} \sum_{n=0}^{\infty} \frac{(c;q)_n (b/a;q)_n}{(d;q)_n (q;q)_n} a^n = \sum_{n=0}^{\infty} \frac{(-1)^n (b/a;q)_n (d/c;q)_n}{(b;q)_n (d;q)_n (q;q)_n} a^n c^n q^{n(n-1)/2}. \tag{6.7.3}$$

A proof of (6.7.3) may be found in [61, p. 17]. In (6.7.3), let $d = q$, replace a by aq, and let both b and c tend to 0. The claim (6.7.2) readily follows. $\square$

Third Proof of Entry 6.7.1. Replacing aq by a and dividing by $(aq;q)_\infty$ on both sides of (6.7.2), we arrive at

$$\frac{1}{(a;q)_\infty} \sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q;q)_n^2} = \sum_{n=0}^{\infty} \frac{a^n}{(q;q)_n^2}. \tag{6.7.4}$$

We prove (6.7.4). Recall that a generating function for partitions $p(n)$ is [21, p. 21, equation (2.2.9)]

$$\frac{1}{(q;q)_\infty} = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n^2}. \tag{6.7.5}$$

For $a = 1$, the only difference between the right sides of (6.7.4) and (6.7.5) is the numerator q^{n^2}; the coefficient of q^N in $1/(q;q)_n^2$ counts the number of partitions of $N + n^2$ with the Durfee square of side n. Let $A(n,N)$ be the set of partitions of $N + n^2$ with the Durfee square of side n. Then

$$\sum_{n=0}^{\infty} \frac{a^n}{(q;q)_n^2} = \sum_{n=0}^{\infty} \sum_{N=0}^{\infty} |A(n,N)|\, a^n q^N.$$

On the other hand, the left side of (6.7.4) is the product of generating functions for two sets of certain partitions: one is for partitions with nonnegative parts and the other is for partitions with the Durfee square of side n. Thus we consider pairs of partitions. Let $B(n,N)$ be the set of pairs of partitions (μ,ν) such that $|\mu| + |\nu| = N$, μ has at most $n-d$ nonnegative parts, and ν has the Durfee square of side d, $d \le n$. Then we see that

$$\frac{1}{(a;q)_\infty} \sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q;q)_n^2} = \sum_{n=0}^{\infty} \sum_{N=0}^{\infty} |B(n,N)|\, a^n q^N.$$

To prove (6.7.4), we will establish a bijection between $A(n,N)$ and $B(n,N)$ by constructing a partition λ in $A(n,N)$ for a given pair (μ,ν) in $B(n,N)$. In the proof, we assume that parts are in decreasing order. We consider an $n\times n$ square, and then attach μ and ν to the right of and below the square, respectively. If the largest part of ν is less than or equal to n, then we obtain the desired partition λ with the Durfee square of side n. Otherwise, we need to apply a bijection of F. Franklin [268, pp. 18–19] to ν in order to obtain a partition with parts less than or equal to n.

To explain the bijection of Franklin, we define a map $f_{k,s}$ from a partition $\delta=(\delta_1,\delta_2,\dots,\delta_m)$ to a partition $\rho=(\rho_1,\rho_2,\dots,\rho_m)$ as follows. If $\delta_1-\delta_{k+1}>s$, define $f_{k,s}(\delta)=\rho$, where, for $1\le i\le m$,

$$\rho_i=\begin{cases}\delta_{i+1}-1, & \text{for} \quad i<k,\\ \delta_1-s-1, & \text{for} \quad i=k,\\ \delta_i, & \text{for} \quad i>k.\end{cases}$$

Otherwise, $f_{k,s}(\delta)=\delta$.

Let $\sigma=(\sigma_1,\sigma_2,\dots,\sigma_l)$ be the partition to the right of the Durfee square of side d of ν, and let μ' be the conjugate of μ. For convention, $\sigma_{l+1}=0$. Let r_1 be the smallest j such that $f^j_{1,n-d}(\sigma)=f^{j+1}_{1,n-d}(\sigma)$. Then we add $n-d+1$ nodes r_1 times to μ' as parts, and denote $f^{r_1}_{1,n-d}(\sigma)$ by σ to avoid a proliferation of notation. Next, we consider the second excess of σ. Let r_2 be the smallest j such that $f^j_{2,n-d}(\sigma)=f^{j+1}_{2,n-d}(\sigma)$. Then we add $n-d+2$ nodes r_2 times to μ' as parts, and denote $f^{r_2}_{2,n-d}(\sigma)$ by σ. We repeat this process with $f_{k,n-d}$ and σ, where $k=3,\dots,l$.

In this way, we can finally produce a partition with parts less than or equal to $n-d$, since the process terminates when $\sigma_1-\sigma_{l+1}\le n-d$. Furthermore, we add to μ' at each step the part $n-d+k$, which is less than or equal to n, since the old σ has at most d parts; i.e., $l\le d$. Thus the new pair σ and μ' are the desired partitions; σ has at most d parts with the largest parts less than or equal to $n-d$, and μ' has parts less than or equal to n, i.e.; μ has at most n parts. Therefore, we obtain a partition λ in $A(d,N)$ with the pair (μ,ν) in $B(d,N)$. Since the steps are invertible, the map is a bijection. This completes the third proof. □

Below (6.7.2) on page 200 in [228], Ramanujan wrote the following:

Entry 6.7.2.

$$a+\frac{q^4}{a}\,{+}\,\frac{q^8}{a}\,{+}\cdots \qquad \& \qquad \frac{q}{1}\,{-}\,\frac{aq}{1}\,{+}\,\frac{q^2}{1}\,{-}\,\frac{aq^3}{1}\,{+}\cdots. \tag{6.7.6}$$

We emphasize that no assertion about these two continued fractions is claimed by Ramanujan. The former continued fraction can be written as

$$a\frac{1}{R(1/a^2,q^4)},$$

but the latter continued fraction cannot be represented in terms of the generalized Rogers–Ramanujan continued fraction. The appearance of the ampersand & between the continued fractions most likely indicates that they have been linked together by Ramanujan in some theorem. Their appearance below (6.7.2) suggests that they are related to it. However, we have been unable to make such a connection. Note that there is a superficial resemblance with the series on the left side of (6.7.2) and the series in the numerator of the generating function of the generalized Rogers–Ramanujan continued fraction given by (6.6.1). In his third notebook [227], Ramanujan examined the limits of both the even-indexed and odd-indexed partial quotients of the Rogers–Ramanujan continued fraction when $q > 1$. Quite remarkably, these limits involve exactly the same continued fractions in (6.7.6), but with, of course, $a = 1$. See [63, p. 30, Entry 11] for a statement and proof of Ramanujan's result. Thus, it is natural to conjecture that Ramanujan had established a generalization of Entry 11 for the generalized Rogers–Ramanujan continued fraction. One can begin to prove a generalization of Entry 11 by using the same ideas. However, we are unable to identify the quotients of q-series that arise in place of those appearing on page 32 of [63]. Moreover, computer algebra does not reveal any connection of these q-series with the continued fractions of (6.7.6). Thus, it would seem that our conjecture about why Ramanujan recorded the continued fractions in (6.7.6) is groundless. But there is a connection with another result of Ramanujan, namely, a claim in his second notebook, recorded as Entry 13 in [63, p. 36]. The continued fractions of (6.7.6) are precisely those appearing in Entry 13, and Ramanujan claims that they are "close" to each other. We refer readers to [63, pp. 36–40] for the meaning of "closeness." Thus, maybe Ramanujan had Entry 13 in mind, but we have the nagging suspicion that Ramanujan had some other motivation for recording these two continued fractions, and that we have been unable to discern his reasoning.

6.8 An Elementary Continued Fraction

We conclude this chapter with an isolated, but beautiful, continued fraction, which does not fall under the purview of q-continued fractions.

Entry 6.8.1 (p. 341). *If*

$$\mu_n := \frac{\sqrt{a^2+4}}{\left(\dfrac{a+\sqrt{a^2+4}}{2}\right)^n - \left(\dfrac{a-\sqrt{a^2+4}}{2}\right)^n}, \tag{6.8.1}$$

then

$$\frac{1}{2}\left(-c+b\frac{\mu_{n+1}}{\mu_n}+\sqrt{\left(c+b\frac{\mu_{n+1}}{\mu_n}\right)^2+(-1)^n\mu_{n+1}^2}\right)$$
$$=\frac{1}{a+}\,\frac{1}{a+}\cdots+\frac{1}{a+}\,\frac{b}{c+}\,\frac{1}{a+}\,\frac{1}{a+}\cdots, \tag{6.8.2}$$

where in each grouping, there are n fractions $\frac{1}{a}$.

We first remark that this entry is difficult to read. In the denominator of μ_n the "4" at the left is hardly legible, and the other "4" in the denominator is more illegible. Second, we can easily see that (6.8.2) is false, in general. For example, suppose that $a = b = c = n = 1$. Then $\mu_1 = \mu_2 = 1$, and (6.8.2) yields

$$\frac{1}{2}\left(-1+1+\sqrt{(1+1)^2-1}\right)=\frac{\sqrt{3}}{2}=\frac{1}{1+}\,\frac{1}{1+}\,\frac{1}{1+}\cdots.$$

But it is well known and easy to prove that the continued fraction on the right side above has the value $(\sqrt{5}-1)/2$. It is surprising that Ramanujan would have made such a mistake.

The entry is an isolated one on page 341 of [228], and in fact, it may be that this entry is on a scrap of paper attached to a larger page for photocopying. The remainder of the page is devoted to generating a family of solutions to Euler's Diophantine equation $a^3+b^3=c^3+d^3$, and nothing on adjoining pages is related to continued fractions. Furthermore, immediately to the right of Entry 6.8.1 are two vertical lines drawn with a straightedge. It is possible that the entry has been cropped, and so the entry may be incomplete, providing an explanation for Ramanujan's "mistake."

We are therefore faced with the problem of finding the "correct" theorem that Ramanujan likely possessed. We have two choices: we could try to find a continued fraction for the left side of (6.8.2), or we could find an algebraic representation for the continued fraction on the right side of (6.8.2). Because the continued fraction is an extremely elegant continued fraction, the latter tack is desirable. In fact, we attempted both strategies. However, we were not able to find any kind of a continued fraction representation for the left side resembling anything similar to the continued fraction on the right side. On the other hand, we were indeed successful in finding an algebraic representation for Ramanujan's beautiful continued fraction. Of course, it is then tempting to convert our representation into a form resembling what Ramanujan claimed on the left side of (6.8.2). Our attempts, partially with computer algebra, to "correct" Ramanujan in this way were fruitless.

Our goal then is to determine an evaluation for the continued fraction on the right side of (6.8.2). Most likely, Ramanujan intended a, b, and c to be positive real numbers, and so we make this assumption in the statement of our theorem. After the conclusion of our proof, we discuss the values of the continued fraction for other real values of a, b, and c. Although we could easily examine the convergence and values for complex a, b, and c, even for real values of the parameters, it is very difficult to relate all the possibilities for the convergence and values of the continued fraction in an efficient manner. The sizes, signs, and possible zero values for each parameter, a, b, and c, and the parity of n present a large variety of cases that must be individually examined, yielding a variety of results. Our proof below comes from a paper by Berndt and G. Choi [76]. A similar proof has been found by J. Lee and J. Sohn [174].

Entry 6.8.2 (p. 341; Corrected Version). *Set*

$$\alpha := \frac{1}{a}+\frac{1}{a}+\cdots+\frac{1}{a}+\frac{b}{c}+\frac{1}{a}+\frac{1}{a}+\cdots \tag{6.8.3}$$

and recall that μ_n is defined by (6.8.1)*. Then, for any positive numbers a, b, and c,*

$$\alpha = \frac{1}{2}\left(-c+(1-b)\frac{\mu_{n+1}}{\mu_n}+\sqrt{\left(c+(1+b)\frac{\mu_{n+1}}{\mu_n}\right)^2+4b(-1)^n\mu_{n+1}^2}\right). \tag{6.8.4}$$

Proof. It will be convenient to define

$$\sigma := \frac{a+\sqrt{a^2+4}}{2} \qquad \text{and} \qquad \tau := \frac{a-\sqrt{a^2+4}}{2} = -\frac{1}{\sigma}.$$

Furthermore, define, for any nonnegative integer n,

$$\nu_n := \frac{1}{\mu_n} = \frac{\sigma^n-\tau^n}{\sqrt{a^2+4}}. \tag{6.8.5}$$

It will be more convenient to work with ν_n. Using (6.8.5), it is easy to verify that ν_n satisfies the recurrence relation

$$\nu_n = a\nu_{n-1}+\nu_{n-2}, \qquad n \geq 2, \qquad \nu_0 = 0, \quad \nu_1 = 1. \tag{6.8.6}$$

Then, from the elementary recurrence formulas for the numerator and denominator of a continued fraction [182, p. 9, equation (1.2.9)],

$$\frac{\nu_n}{\nu_{n+1}} = \frac{1}{a}+\frac{1}{a}+\cdots+\frac{1}{a}, \tag{6.8.7}$$

where there are n fractions $\frac{1}{a}$.

Now, by (6.8.3) and (6.8.7), write α in the form

$$\begin{aligned}\alpha &:= \frac{1}{a}+\frac{1}{a}+\cdots+\frac{1}{a}+\frac{b}{c}+\frac{1}{a}+\frac{1}{a}+\cdots \\ &= \frac{1}{a}+\frac{1}{a}+\cdots+\frac{1}{a}+\frac{b}{c+\alpha} \\ &= \frac{(c+\alpha)\nu_n+b\nu_{n-1}}{(c+\alpha)\nu_{n+1}+b\nu_n},\end{aligned} \tag{6.8.8}$$

where we have employed (6.8.7) and again used the elementary recurrence relations for a continued fraction's numerator and denominator [182, p. 9, equation (1.2.9)]. Solving (6.8.8) for α, we find that

$$\alpha^2\nu_{n+1} - (\nu_n - b\nu_n - c\nu_{n+1})\alpha - b\nu_{n-1} - c\nu_n = 0. \tag{6.8.9}$$

Solving (6.8.9) and taking the requisite positive root, we find that

$$\begin{aligned}\alpha &= \frac{(1-b)\nu_n - c\nu_{n+1} + \sqrt{((1-b)\nu_n - c\nu_{n+1})^2 + 4\nu_{n+1}(b\nu_{n-1} + c\nu_n)}}{2\nu_{n+1}} \\ &= \frac{1}{2}\left(-c + (1-b)\frac{\nu_n}{\nu_{n+1}} + \sqrt{\left(c + (b-1)\frac{\nu_n}{\nu_{n+1}}\right)^2 + 4\left(b\frac{\nu_{n-1}}{\nu_{n+1}} + c\frac{\nu_n}{\nu_{n+1}}\right)}\right).\end{aligned} \tag{6.8.10}$$

We now utilize another elementary relation for the numerators and denominators of continued fractions [182, p. 9, equation (1.2.10)] and apply it to (6.8.7) to deduce that

$$\nu_n^2 - \nu_{n+1}\nu_{n-1} = (-1)^{n-1}. \tag{6.8.11}$$

Solving (6.8.11) for ν_{n-1} and using the elementary relation $(A+B)^2 = (A-B)^2 + 4AB$ under the radical sign, we conclude that

$$\begin{aligned}\alpha = \frac{1}{2}\Bigg(&-c + (1-b)\frac{\nu_n}{\nu_{n+1}} \\ &+\sqrt{\left(c + (1-b)\frac{\nu_n}{\nu_{n+1}}\right)^2 + 4b\left(\frac{(-1)^n}{\nu_{n+1}^2} + \left(\frac{\nu_n}{\nu_{n+1}}\right)^2 + c\frac{\nu_n}{\nu_{n+1}}\right)}\Bigg).\end{aligned} \tag{6.8.12}$$

Since by (6.8.5), $\nu_n = 1/\mu_n$, we see that (6.8.12) is the same as (6.8.4), and this completes the proof. □

We conclude this chapter with a more thorough, but by no means complete, discussion of the conditions under which Ramanujan's continued fraction converges to either the right side of (6.8.4) or to its conjugate. For brevity, set

$$\alpha_1 := \frac{(1-b)\nu_n - c\nu_{n+1} + \sqrt{D}}{2\nu_{n+1}} \quad \text{and} \quad \alpha_2 := \frac{(1-b)\nu_n - c\nu_{n+1} - \sqrt{D}}{2\nu_{n+1}}, \tag{6.8.13}$$

where

$$D := (c\nu_{n+1} + (1+b)\nu_n)^2 + 4b(-1)^n. \tag{6.8.14}$$

Set

$$\|\alpha_1\| := \frac{|(1+b)\nu_n + c\nu_{n+1} + \sqrt{D}|}{2} \quad \text{and} \quad \|\alpha_2\| := \frac{|(1+b)\nu_n + c\nu_{n+1} - \sqrt{D}|}{2}. \tag{6.8.15}$$

From [182, p. 104, Theorem 6], α converges to α_i if $\|\alpha_i\| > \|\alpha_j\|$ for $i, j = 1, 2$, $i \neq j$. Observe that

$$\begin{aligned}
\|\alpha_1\| &> \|\alpha_2\|, && \text{if } D > 0 \quad \text{and} \quad (1+b)\nu_n + c\nu_{n+1} > 0,\\
\|\alpha_2\| &> \|\alpha_1\|, && \text{if } D > 0 \quad \text{and} \quad (1+b)\nu_n + c\nu_{n+1} < 0,\\
\alpha_1 &= \alpha_2, && \text{if } D = 0.
\end{aligned}$$

Define

$$\delta_n := (1+b)\nu_n + c\nu_{n+1}.$$

Thus, by (6.8.14),

$$D = \delta_n^2 + 4b(-1)^n.$$

Suppose first that $abc \neq 0$. Then, using the aforementioned theorem in [182], we conclude that α converges to α_1 in the following cases:

	$b > 0$	$b < 0$
n even	$\delta_n > 0$	$\delta_n > 2\sqrt{-b}$
n odd	$\delta_n > 2\sqrt{b}$	$\delta_n > 0$

Moreover, α converges to α_2 in the following cases:

	$b > 0$	$b < 0$
n even	$\delta_n < 0$	$\delta_n < -2\sqrt{-b}$
n odd	$\delta_n < -2\sqrt{b}$	$\delta_n < 0$

We do not give any details but provide some examples as an illustration. If n is odd, $ac > 0$, and $-1 < b < 0$, then α converges to α_1. Using (6.8.6), we can bound ν_n from above and below in terms of Fibonacci numbers in various cases and then give alternative criteria for convergence. If n is even, $b, c > 0$, and $a < -1$, then α converges to α_1 if

$$\frac{(1+b)|a|^{n-1}}{c} < \frac{F_{n+1}}{F_n},$$

where F_j, $j \geq 0$, denotes the jth Fibonacci number.

If $abc = 0$, then as above, we must consider separately several cases. We state one such result. Suppose that n is even, $a = 0$, $c \neq 0$, and $c^2 + 4b \geq 0$. Then the continued fraction α converges to

$$\frac{-c + (\operatorname{sgn} c)\sqrt{c^2+4b}}{2},$$

where

$$\operatorname{sgn} c = \begin{cases} +1, & \text{if } c > 0,\\ -1, & \text{if } c < 0. \end{cases}$$

Suppose that n is odd, $a = 0$, $|b| > 1$, and $c \neq 0$. Then the continued fraction α converges to

$$\frac{c}{b-1}.$$

Note that if $b = 0$, α trivially converges, since it terminates.

Lastly, note that there are cases in which α does not converge, e.g., when $a = c = 0$ and $b > 0$, and when $a = 0$ and $c^2 + 4b < 0$.

7

Asymptotic Formulas for Continued Fractions

7.1 Introduction

This chapter is devoted to proving three asymptotic formulas for continued fractions found in Ramanujan's lost notebook [228]. The three continued fractions are given by (7.1.1), (7.1.2), and (7.1.4) below. Our proofs are taken from papers by Berndt and J. Sohn [83] and Berndt and A.J. Yee [84]. In the next chapter, we return to the continued fraction (7.1.1) and, in fact, derive another type of asymptotic formula for it.

On page 45 of his lost notebook [228], Ramanujan recorded two asymptotic formulas for two continued fractions involving the Riemann zeta function and Dirichlet L-functions. These continued fractions, for $|q| < 1$, are equivalent to

$$\frac{(q^2;q^3)_\infty}{(q;q^3)_\infty} = \frac{1}{1} - \frac{q}{1+q} - \frac{q^3}{1+q^2} - \frac{q^5}{1+q^3} - \frac{q^7}{1+q^4} - \cdots \tag{7.1.1}$$

and

$$\frac{(q^3;q^4)_\infty}{(q;q^4)_\infty} = \frac{1}{1} - \frac{q}{1+q^2} - \frac{q^3}{1+q^4} - \frac{q^5}{1+q^6} - \frac{q^7}{1+q^8} - \cdots \tag{7.1.2}$$

after a change of variable. They are among the most interesting continued fractions discovered by Ramanujan. The continued fraction (7.1.2) also converges for $|q| > 1$, and it converges to

$$\frac{(q^{-3};q^{-4})_\infty}{(q^{-1};q^{-4})_\infty},$$

providing a beautiful example of symmetry. The continued fraction (7.1.1) is the most difficult to establish of all of Ramanujan's continued fractions and does not seem to fit in the same hierarchy as the other q-continued fractions found by Ramanujan. Other unusual properties of this continued fraction can also be found on page 45 of [228]. For a further discussion of these continued fractions, see [63, pp. 46–49].

As an illustration, we offer now the asymptotic formula for (7.1.1).

Let $\zeta(s)=\sum_{n=1}^{\infty} n^{-s}$, Re $s>1$, denote the Riemann zeta-function, and let $L(s,\chi)=\sum_{n=1}^{\infty}\chi(n)n^{-s}$, Re $s>0$, denote the Dirichlet L-function associated with the character $\chi(n)=\left(\frac{n}{3}\right)$, the Legendre symbol. For each integer $n\geq 2$, let

$$a_n=\frac{4\Gamma(n)\zeta(n)L(n+1,\chi)}{(2\pi/\sqrt{3})^{2n+1}}.$$

Then, for $x>0$,

$$\frac{(3x)^{1/3}}{1}\,-\,\frac{1}{1+e^{x}}\,-\,\frac{1}{1+e^{2x}}\,-\,\frac{1}{1+e^{3x}}\,-\cdots=\frac{\Gamma(\frac{1}{3})}{\Gamma(\frac{2}{3})}e^{G(x)}, \tag{7.1.3}$$

where as $x\to 0^{+}$,

$$G(x)\sim a_2x^2+a_4x^4+a_6x^6+\cdots.$$

In particular,

$$a_2=\frac{1}{108},\quad a_4=\frac{1}{4320},\quad a_6=\frac{1}{38880}.$$

Observe that after an equivalence transformation, the continued fraction in (7.1.1) is the same as that in (7.1.3), but with $q=e^{-x}$.

In Section 7.2, we prove a more general theorem for odd characters χ, and in Section 7.3 we derive Ramanujan's claims as corollaries of our theorem. We close this section with a general theorem for even characters χ, and give an asymptotic formula for the Rogers–Ramanujan continued fraction. B. Richmond and G. Szekeres [232] gave asymptotic formulas for the Rogers–Ramanujan continued fraction as $q\to 0^{+}$.

Section 7.4 is devoted to a proof of Ramanujan's asymptotic formula for the generalized Rogers–Ramanujan continued fraction found on page 26 of his lost notebook. Here we define the generalized Rogers–Ramanujan continued fraction for $|q|<1$ and any complex number a by

$$R(a,q):=\frac{1}{1}\,+\,\frac{aq}{1}\,+\,\frac{aq^2}{1}\,+\,\frac{aq^3}{1}\,+\cdots. \tag{7.1.4}$$

The Rogers–Ramanujan continued fraction $R(q)$ is the special case $R(1,q)=q^{-1/5}R(q)$. Then Ramanujan asserts that [228, p. 26] as $x\to 0^{+}$,

$$\begin{aligned}R(a,e^{-x})&=\frac{-1+\sqrt{1+4a}}{2a}\\&\quad\times\exp\left(\frac{ax}{1+4a}-\frac{a(1-a)x^2}{2(1+4a)^{5/2}}+\frac{a(1-a)(1-14a)x^3}{6(1+4a)^4}-\cdots\right).\end{aligned} \tag{7.1.5}$$

We notice that each term in the expansion from the first onward has a factor of a, which is to be expected, and each term from the second onward has a factor of $1-a$. We prove indeed that these factors do appear generally.

By the same sort of argument, we can also derive an asymptotic formula for the generalized cubic continued fraction.

7.2 The Main Theorem

We need a form of Stirling's formula; see [36, p. 539] or [125, p. 224].

Lemma 7.2.1. *As* $|t| \to \infty$,

$$|\Gamma(\sigma + it)| \sim \sqrt{2\pi} e^{-\pi|t|/2} |t|^{\sigma - 1/2},$$

uniformly in any fixed vertical strip $\alpha \leq \sigma \leq \beta$.

Theorem 7.2.1. *Let* k *be a positive integer greater than or equal to* 3, *and let* $L(s,\chi)$ *denote the Dirichlet L-function associated with* $\chi(n)$, *a primitive, real, nonprincipal, odd character modulo* k. *Then as* $x \to 0^+$,

$$(xk)^{-M_1(\chi)/k} \prod_{n=1}^{k-1} (e^{-nx}; e^{-kx})_\infty^{-\chi(n)} = \prod_{n=1}^{k-1} \Gamma\left(\frac{n}{k}\right)^{\chi(n)} e^{G(x)}, \tag{7.2.1}$$

where

$$M_1(\chi) = \sum_{n=1}^{k-1} \chi(n)\, n \tag{7.2.2}$$

and

$$G(x) \sim a_2 x^2 + a_4 x^4 + a_6 x^6 + \cdots,$$

with

$$a_\nu = \frac{4\Gamma(\nu)}{(2\pi/\sqrt{k})^{2\nu+1}} \zeta(\nu) L(\nu+1, \chi). \tag{7.2.3}$$

Also, as $x \to 0^+$,

$$\text{the minimum value of } a_\nu x^\nu \text{ is asymptotic to } \frac{k}{\pi}\sqrt{\frac{2x}{\pi}} e^{-4\pi^2/(kx)}. \tag{7.2.4}$$

Proof. Let

$$P(x) := \prod_{n=1}^{k-1} (e^{-nx}; e^{-kx})_\infty^{-\chi(n)}.$$

Then, for $x > 0$,

$$f(x) := \log P(x) = -\sum_{n=1}^{\infty} \chi(n) \log(1 - e^{-nx}) = \sum_{n=1}^{\infty} \chi(n) \sum_{m=1}^{\infty} \frac{e^{-nmx}}{m}.$$

Inverting the order of summation and integration by absolute convergence, we find that for $x > 0$,

$$\int_0^\infty f(x) x^{s-1}\, dx = \int_0^\infty \sum_{n=1}^{\infty} \chi(n) \sum_{m=1}^{\infty} \frac{e^{-nmx}}{m} x^{s-1}\, dx$$

$$\begin{aligned}
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\chi(n)}{m} \int_0^{\infty} e^{-nmx} x^{s-1}\,dx \\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{\chi(n)}{m} \int_0^{\infty} e^{-u} \left(\frac{u}{nm}\right)^{s-1} \frac{du}{nm} \\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \chi(n) \frac{1}{m^{s+1}} \frac{1}{n^s} \int_0^{\infty} e^{-u} u^{s-1}\,du \\
&= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \chi(n) \frac{1}{m^{s+1}} \frac{1}{n^s} \Gamma(s) \\
&= \Gamma(s)\zeta(s+1)L(s,\chi).
\end{aligned}$$

By Mellin's inversion formula [276, p. 7],

$$f(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \Gamma(s)\zeta(s+1)L(s,\chi)x^{-s}\,ds, \qquad c > 1. \tag{7.2.5}$$

Consider now

$$I_{C_{M,T}} := \frac{1}{2\pi i} \int_{C_{M,T}} \Gamma(s)\zeta(s+1)L(s,\chi)x^{-s}\,ds, \tag{7.2.6}$$

where $M = 2N + \frac{1}{2}$, N is any positive integer, and $C_{M,T}$ is the positively oriented rectangle with corners at (c, iT), $(-M, iT)$, $(-M, -iT)$, and $(c, -iT)$, where T is any positive number.

Recall that $\Gamma(s)$ has a simple pole at $s = -n$ with residue $(-1)^n/n!$, for each nonnegative integer n. Recall also that $\zeta(s)$ has a simple pole at $s = 1$ with residue 1, and that $\zeta(-2n) = 0$ for each positive integer n [275, pp. 16, 19]. Furthermore, since χ is odd, $L(-2n-1,\chi) = 0$ for each nonnegative integer n [126, p. 71]. Hence, the integrand of (7.2.6) has simple poles at $s = -2, -4, -6, \ldots, -2N$ and a double pole at $s = 0$ on the interior of $C_{M,T}$.

Using the expansions [144, p. 944], [275, p. 16],

$$\begin{aligned}
\Gamma(s) &= \frac{1}{s} - \gamma + \cdots\,, \\
\zeta(s+1) &= \frac{1}{s} + \gamma + \cdots\,, \\
x^{-s} &= e^{-s\log x} = 1 - s\log x + \cdots\,,
\end{aligned}$$

and

$$L(s,\chi) = L(0,\chi) + L'(0,\chi)s + \cdots\,,$$

where γ denotes Euler's constant, we find that

$$\begin{aligned}
\Gamma(s)\zeta(s+1)L(s,\chi)x^{-s} &= \left(\frac{1}{s} - \gamma + \cdots\right)\left(\frac{1}{s} + \gamma + \cdots\right) \\
&\quad \times (1 - s\log x + \cdots)(L(0,\chi) + L'(0,\chi)s + \cdots).
\end{aligned}$$

Hence, the residue at $s = 0$ is

$$\begin{aligned} R_0 :=& - L(0,\chi)\log x + L'(0,\chi) + \gamma L(0,\chi) - \gamma L(0,\chi) \\ =& - L(0,\chi)\log x + L'(0,\chi). \end{aligned} \tag{7.2.7}$$

The residue at $s = -2n$, $n \geq 1$, is

$$R_{-2n} := \frac{1}{(2n)!}\zeta(1-2n)L(-2n,\chi)x^{2n}. \tag{7.2.8}$$

Next, we estimate the integrals along the horizontal sides. First, from [275, p. 81], for $-M \leq \sigma \leq c$,

$$\zeta(1+\sigma \pm iT) = O(T^{M+1/2}), \tag{7.2.9}$$

as $T \to \infty$. Also from [46, pp. 270–273] and the Phragmén–Lindelöf theorem, for $-M \leq \sigma \leq c$,

$$L(\sigma \pm iT, \chi) = O(T^{M+1}), \tag{7.2.10}$$

as $T \to \infty$.

Hence from Lemma 7.2.1, (7.2.9), and (7.2.10), we deduce that

$$\begin{aligned} &\int_{-M}^{c} \Gamma(\sigma \pm iT)\zeta(1+\sigma \pm iT)L(\sigma \pm iT,\chi)x^{-\sigma \mp iT}\,d\sigma \\ &\qquad = O\left(\int_{-M}^{c} e^{-\pi T/2}T^{2M+c+1}x^{M}\,d\sigma\right) = o(1), \end{aligned} \tag{7.2.11}$$

as $T \to \infty$.

Thus, having let $T \to \infty$, there remains to examine

$$\int_{-\infty}^{\infty} \Gamma(-M+it)\zeta(1-M+it)L(-M+it,\chi)x^{M-it}\,dt.$$

Now by using the elementary identity $\sin^2(x+iy) = \sin^2 x + \sinh^2 y$, the reflection formula

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin \pi s}, \tag{7.2.12}$$

and Lemma 7.2.1, we deduce that

$$\begin{aligned} \Gamma(-M+it) &= \frac{\pi}{\sin \pi(-M+it)\Gamma(1+M-it)} \\ &= \frac{\pi}{\{\sin^2 \pi(-M) + \sinh^2 \pi t\}^{1/2}\Gamma(1+M-it)} \\ &= O\left(\frac{1}{e^{\pi|t|}e^{-\pi|t|/2}|t|^{M+1/2}}\right) \\ &= O\left(|t|^{-M-1/2}e^{-\pi|t|/2}\right), \end{aligned}$$

as $|t| \to \infty$.

Thus by (7.2.9), (7.2.10), and the calculation above,

$$\int_1^{\pm\infty} \Gamma(-M+it)\zeta(1-M+it)L(-M+it,\chi)x^{M-it}\,dt$$
$$= O\left(\int_1^{\pm\infty} e^{-\pi|t|/2}|t|^{M+1}x^M\,dt\right) = O\left(x^M\right), \quad (7.2.13)$$

as $x \to 0^+$. Hence, as $x \to 0^+$, by (7.2.6), the residue theorem, (7.2.5), (7.2.7), (7.2.8), (7.2.11), and (7.2.13),

$$f(x) = -L(0,\chi)\log x + L'(0,\chi) \quad (7.2.14)$$
$$+\sum_{n=1}^{N} \frac{1}{(2n)!}\zeta(1-2n)L(-2n,\chi)x^{2n} + O\left(x^{2N+1/2}\right).$$

Since χ is an odd character, the functional equation for $L(s,\chi)$ is given by [126, p. 71]

$$L(s,\chi) = \left(\frac{\pi}{k}\right)^{s-1/2} \frac{\Gamma\left(1-\frac{1}{2}s\right)}{\Gamma\left(\frac{1}{2}(s+1)\right)} L(1-s,\chi). \quad (7.2.15)$$

Now from (7.2.12), we have

$$\Gamma(\tfrac{1}{2}-n) = \frac{\pi}{\sin\pi(\frac{1}{2}-n)\Gamma(n+\frac{1}{2})} = \frac{\pi}{(-1)^n\Gamma(n+\frac{1}{2})} = \frac{\sqrt{\pi}(-1)^n 2^{2n} n!}{(2n)!}, \quad (7.2.16)$$

since $\Gamma(1/2) = \sqrt{\pi}$. Thus, from (7.2.15) and (7.2.16),

$$L(-2n,\chi) = \left(\frac{\pi}{k}\right)^{-2n-1/2} \frac{\Gamma(n+1)}{\Gamma\left(\frac{1}{2}-n\right)} L(2n+1,\chi)$$
$$= \left(\frac{\pi}{k}\right)^{-2n-1/2} \frac{n!(2n)!}{\sqrt{\pi}(-1)^n 2^{2n} n!} L(2n+1,\chi)$$
$$= \left(\frac{k}{\pi}\right)^{2n+1/2} \frac{(-1)^n(2n)!}{2^{2n}\sqrt{\pi}} L(2n+1,\chi). \quad (7.2.17)$$

By the functional equation for $\zeta(s)$ [275, p. 16, equation (2.1.8)],

$$\zeta(1-2n) = \frac{2(-1)^n(2n-1)!}{(2\pi)^{2n}}\zeta(2n). \quad (7.2.18)$$

Thus, using (7.2.17) and (7.2.18) in (7.2.14), we find that

$$
\begin{aligned}
f(x) = &- L(0,\chi)\log x + L'(0,\chi) \\
&+ \sum_{n=1}^{N} \frac{1}{(2n)!}\frac{2(-1)^n(2n-1)!}{(2\pi)^{2n}}\zeta(2n)\left(\frac{k}{\pi}\right)^{2n+1/2} \\
&\times \frac{(-1)^n(2n)!}{2^{2n}\sqrt{\pi}}L(2n+1,\chi)x^{2n} + O(x^{2N+1/2}) \\
= &- L(0,\chi)\log x + L'(0,\chi) \\
&+ \sum_{n=1}^{N} \frac{4\Gamma(2n)}{(2\pi/\sqrt{k})^{4n+1}}\zeta(2n)L(2n+1,\chi)x^{2n} + O(x^{2N+1/2}). \quad (7.2.19)
\end{aligned}
$$

Next from the functional equation (7.2.15),

$$L(0,\chi) = \frac{\sqrt{k}}{\pi}L(1,\chi).$$

But from [99, p. 336, Theorem 3],

$$L(1,\chi) = -\frac{\pi\sqrt{k}}{k^2}M_1(\chi), \quad (7.2.20)$$

where $M_1(\chi)$ is defined by (7.2.2). Thus,

$$L(0,\chi) = \frac{\sqrt{k}}{\pi}\left(-\frac{\pi\sqrt{k}}{k^2}\right)M_1(\chi) = -\frac{M_1(\chi)}{k}. \quad (7.2.21)$$

By the functional equation (7.2.15) and the product and chain rules, after simplifying, we find that

$$
\begin{aligned}
L'(s,\chi) = &\left(\frac{\pi}{k}\right)^{s-1/2}\frac{\Gamma(1-\frac{1}{2}s)}{\Gamma(\frac{1}{2}(1+s))}L(1-s,\chi) \\
&\times\left(\log\frac{\pi}{k} - \frac{1}{2}\psi\left(1-\frac{1}{2}s\right) - \frac{1}{2}\psi\left(\frac{1}{2}(s+1)\right) - \frac{L'(1-s,\chi)}{L(1-s,\chi)}\right),
\end{aligned}
$$

where $\psi(s) = \Gamma'(s)/\Gamma(s)$. Hence, at $s=0$,

$$L'(0,\chi) = \left(\frac{\pi}{k}\right)^{-1/2}\frac{\Gamma(1)}{\Gamma(\frac{1}{2})}L(1,\chi)\left(\log\frac{\pi}{k} - \frac{1}{2}\psi(1) - \frac{1}{2}\psi\left(\frac{1}{2}\right) - \frac{L'(1,\chi)}{L(1,\chi)}\right). \quad (7.2.22)$$

From [1, p. 258],

$$\psi(1) = -\gamma \qquad \text{and} \qquad \psi\left(\frac{1}{2}\right) = -\gamma - 2\log 2, \quad (7.2.23)$$

where γ is Euler's constant. Thus, from (7.2.22), (7.2.20), and (7.2.23),

$$\begin{aligned}L'(0,\chi) &= \left(\frac{\pi}{k}\right)^{-1/2}\frac{1}{\sqrt{\pi}}\left(-\frac{\pi\sqrt{k}}{k^2}\right)M_1(\chi)\\ &\qquad\times\left(\log\frac{\pi}{k}+\frac{1}{2}\gamma+\frac{1}{2}\gamma+\log 2-\frac{L'(1,\chi)}{L(1,\chi)}\right)\\ &= -\frac{M_1(\chi)}{k}\left(\log\frac{2\pi}{k}+\gamma\right)-\frac{\sqrt{k}}{\pi}L'(1,\chi). \qquad (7.2.24)\end{aligned}$$

By a theorem of C. Deninger [127, p. 182],

$$L'(1,\chi) = -\frac{\pi}{\sqrt{k}}\left((\gamma+\log 2\pi)\frac{M_1(\chi)}{k}+\sum_{n=1}^{k-1}\chi(n)\log\left(\Gamma\left(\frac{n}{k}\right)\right)\right). \qquad (7.2.25)$$

Thus, by (7.2.24) and (7.2.25),

$$\begin{aligned}L'(0,\chi) &= -\frac{M_1(\chi)}{k}\log 2\pi+\frac{M_1(\chi)}{k}\log k-\gamma\frac{M_1(\chi)}{k}\\ &\quad+\frac{\sqrt{k}}{\pi}\frac{\pi}{\sqrt{k}}\left(\gamma\frac{M_1(\chi)}{k}+\frac{M_1(\chi)}{k}\log 2\pi+\sum_{n=1}^{k-1}\chi(n)\log\left(\Gamma\left(\frac{n}{k}\right)\right)\right)\\ &= \frac{M_1(\chi)}{k}\log k+\sum_{n=1}^{k-1}\chi(n)\log\left(\Gamma\left(\frac{n}{k}\right)\right). \qquad (7.2.26)\end{aligned}$$

Hence, from (7.2.19), (7.2.21), and (7.2.26),

$$\begin{aligned}f(x) &= \frac{M_1(\chi)}{k}\log xk+\sum_{n=1}^{k-1}\chi(n)\log\left(\Gamma\left(\frac{n}{k}\right)\right)\\ &\quad+\sum_{n=1}^{N}\frac{4\Gamma(2n)}{(2\pi/\sqrt{k})^{4n+1}}\zeta(2n)L(2n+1,\chi)x^{2n}+O(x^{2N+1/2}),\end{aligned}$$

which, upon exponentiation, completes the proof of (7.2.1).

To prove (7.2.4), let

$$g(t) = \frac{4\Gamma(t)x^t}{(2\pi/\sqrt{k})^{2t+1}} = \frac{4\Gamma(t)x^t}{c^{2t+1}},$$

where $c = 2\pi/\sqrt{k}$. We want to minimize $g(t)$. By the product and chain rules,

$$\begin{aligned}g'(t) &= \frac{4\Gamma'(t)x^t}{c^{2t+1}}+\frac{4\Gamma(t)x^t\log x}{c^{2t+1}}-\frac{8\Gamma(t)x^t\log c}{c^{2t+1}}\\ &= \frac{4\Gamma(t)x^t}{c^{2t+1}}\left(\frac{\Gamma'(t)}{\Gamma(t)}+\log x-2\log c\right) = 0.\end{aligned}$$

So $g(t)$ has a minimum value at the point t that satisfies the equation

$$\psi(t) = \log\left(\frac{c^2}{x}\right),$$

where $\psi(t) = \Gamma'(t)/\Gamma(t)$. But from [1, p. 259], as $t \to \infty$,

$$\psi(t) \sim \log t,$$

and therefore

$$t = \frac{c^2}{x}.$$

Thus by Stirling's formula [1, p. 257] and the calculation above, the minimum value of $a_t x^t$ is, as $x \to 0^+$,

$$\begin{aligned}
\frac{4\sqrt{2\pi}t^{t-1/2}e^{-t}x^t}{c^{2t+1}} &\sim \frac{4\sqrt{2\pi}\left(c^2/x\right)^{c^2/x-1/2}e^{-c^2/x}x^{c^2/x}}{c^{2c^2/x+1}} \\
&= \frac{4\sqrt{2\pi}c^{2c^2/x-1}x^{1/2}e^{-c^2/x}}{c^{2c^2/x+1}} \\
&= \frac{4\sqrt{2\pi}\sqrt{x}e^{-c^2/x}}{c^2} \\
&= \frac{k}{\pi}\sqrt{\frac{2x}{\pi}}e^{-4\pi^2/(kx)},
\end{aligned}$$

which completes the proof of Theorem 7.2.1. □

7.3 Two Asymptotic Formulas Found on Page 45 of Ramanujan's Lost Notebook

In this section, we use Theorem 7.2.1 to prove two asymptotic formulas found on page 45 of Ramanujan's lost notebook [228]. First we prove a lemma that allows us to explicitly calculate $L(s,\chi)$, where $s = 1, 3, 5, 7$, and χ is odd.

Lemma 7.3.1. *Let χ be a primitive, real, nonprincipal, odd character modulo k. Then*

$$\begin{aligned}
L(1,\chi) &= \frac{\pi i}{k^2}G(\chi)M_1(\chi), \\
L(3,\chi) &= \frac{2\pi^3 i}{3k^4}G(\chi)\left(k^2M_1(\chi) - M_3(\chi)\right), \\
L(5,\chi) &= \frac{2\pi^5 i}{15k^6}G(\chi)\left(\frac{7}{3}k^4M_1(\chi) - \frac{10}{3}k^2M_3(\chi) + M_5(\chi)\right), \\
L(7,\chi) &= \frac{4\pi^7 i}{315k^8}G(\chi)\left(\frac{31}{3}k^6M_1(\chi) - \frac{49}{3}k^4M_3(\chi) + 7k^2M_5(\chi) - M_7(\chi)\right),
\end{aligned}$$

where $G(\chi)$ is the Gauss sum defined by

$$G(\chi)=\sum_{n=1}^{k-1}\chi(n)e^{2\pi in/k} \tag{7.3.1}$$

and

$$M_m(\chi)=\sum_{n=1}^{k-1}\chi(n)\,n^m. \tag{7.3.2}$$

Proof. From [59, p. 33, equation (6.12)], because χ is odd,

$$G(\chi)M_m(\chi)=-2ik^{m+1}\sum_{j=0}^{m-1}\frac{m!}{(m-j)!}(2\pi)^{-j-1}\cos\left(\frac{j\pi}{2}\right)L(j+1,\chi). \tag{7.3.3}$$

Using (7.3.3), we may calculate $L(2n-1,\chi)$ for any positive integer n. Letting $m=1$ in (7.3.3), we find that

$$L(1,\chi)=\frac{\pi i}{k^2}G(\chi)M_1(\chi). \tag{7.3.4}$$

If $m=3$, we find by (7.3.3) and (7.3.4) that

$$L(3,\chi)=\frac{2\pi^3 i}{3k^4}G(\chi)\left(k^2M_1(\chi)-M_3(\chi)\right). \tag{7.3.5}$$

If $m=5$ in (7.3.3),

$$G(\chi)M_5(\chi)=-2ik^6\left\{\frac{1}{2\pi}L(1,\chi)-\frac{5}{2\pi^3}L(3,\chi)+\frac{15}{4\pi^5}L(5,\chi)\right\}. \tag{7.3.6}$$

Thus, using (7.3.4) and (7.3.5) in (7.3.6), we deduce that

$$L(5,\chi)=\frac{2\pi^5 i}{15k^6}G(\chi)\left(\frac{7}{3}k^4M_1(\chi)-\frac{10}{3}k^2M_3(\chi)+M_5(\chi)\right).$$

Similarly, the result

$$L(7,\chi)=\frac{4\pi^7 i}{315k^8}G(\chi)\left(\frac{31}{3}k^6M_1(\chi)-\frac{49}{3}k^4M_3(\chi)+7k^2M_5(\chi)-M_7(\chi)\right)$$

follows by taking $m=7$ in (7.3.3). This completes the proof of Lemma 7.3.1. □

Entry 7.3.1 (p. 45). *As* $x\to0^+$,

$$\frac{(3x)^{1/3}}{1}\,\frac{}{-}\,\frac{1}{1+e^x}\,\frac{}{-}\,\frac{1}{1+e^{2x}}\,\frac{}{-}\,\frac{1}{1+e^{3x}}\,\frac{}{-}\cdots=\frac{\Gamma(\frac13)}{\Gamma(\frac23)}e^{G(x)}, \tag{7.3.7}$$

where

$$G(x)\sim a_2x^2+a_4x^4+a_6x^6+\cdots,$$

with

$$a_\nu = \frac{4\Gamma(\nu)\zeta(\nu)L(\nu+1,\chi)}{(2\pi/\sqrt{3})^{2\nu+1}},$$

where $\chi(n) = \left(\frac{n}{3}\right)$. *In particular,*

$$a_2 = \frac{1}{108}, \quad a_4 = \frac{1}{4320}, \quad \textit{and} \quad a_6 = \frac{1}{38880}. \tag{7.3.8}$$

Furthermore, as $x \to 0^+$,

$$\textit{the minimum value of } a_\nu x^\nu \textit{ is asymptotic to } \frac{3}{\pi}\sqrt{\frac{2x}{\pi}}\, e^{-4\pi^2/(3x)}. \tag{7.3.9}$$

Proof. The continued fraction on the left-hand side of (7.3.7) is equivalent to

$$\begin{aligned}
&(3x)^{1/3}\left(\frac{1}{1} - \frac{1}{e^x(1+e^{-x})} - \frac{1}{e^{2x}(1+e^{-2x})} - \frac{1}{e^{3x}(1+e^{-3x})} - \cdots\right) \\
&= (3x)^{1/3}\left(\frac{1}{1} - \frac{e^{-x}}{1+e^{-x}} - \frac{e^{-3x}}{1+e^{-2x}} - \frac{e^{-5x}}{1+e^{-3x}} - \cdots\right) \\
&= (3x)^{1/3}\frac{(e^{-2x};e^{-3x})_\infty}{(e^{-x};e^{-3x})_\infty},
\end{aligned}$$

by (7.1.1), which can be found in Ramanujan's second notebook [227] and which was first proved by Andrews, Berndt, L. Jacobsen, and R.L. Lamphere [39], [63, p. 46]. This expression is the case $k = 3$ in Theorem 7.2.1, since $M_1(\chi) = -1$. This completes the proof of (7.3.7) and (7.3.9).

To prove (7.3.8), we need the well-known values [209, pp. 776–777]

$$\zeta(2) = \frac{\pi^2}{6}, \qquad \zeta(4) = \frac{\pi^4}{90}, \qquad \text{and} \qquad \zeta(6) = \frac{\pi^6}{945}, \tag{7.3.10}$$

and the following values from Lemma 7.3.1 with $k = 3$,

$$L(3,\chi) = \frac{4\pi^3\sqrt{3}}{243}, \quad L(5,\chi) = \frac{4\pi^5\sqrt{3}}{3^7}, \quad \text{and} \quad L(7,\chi) = \frac{56\pi^7\sqrt{3}}{3^{10}\cdot 5},$$

since $G(\chi) = i\sqrt{3}, M_1(\chi) = -1, M_3(\chi) = -7, M_5(\chi) = -31$, and $M_7(\chi) = -127$. Therefore, the values in (7.3.8) now easily follow from (7.2.3). □

Entry 7.3.2 (p. 45). *As* $x \to 0^+$,

$$\frac{2\sqrt{x}}{1} - \frac{1}{e^x+e^{-x}} - \frac{1}{e^{2x}+e^{-2x}} - \frac{1}{e^{3x}+e^{-3x}} - \cdots = \frac{\Gamma(\frac{1}{4})}{\Gamma(\frac{3}{4})}e^{G(x)}, \tag{7.3.11}$$

where

$$G(x) \sim a_2x^2 + a_4x^4 + a_6x^6 + \cdots,$$

with

$$a_\nu = \frac{4\Gamma(\nu)\zeta(\nu)L(\nu+1,\chi)}{\pi^{2\nu+1}},$$

where χ *is the nonprincipal, primitive character modulo* 4. *Furthermore,*

$$a_2 = \frac{1}{48}, \quad a_4 = \frac{1}{1152}, \quad \textit{and} \quad a_6 = \frac{61}{362880}, \tag{7.3.12}$$

and, as $x \to 0^+$,

$$\textit{the minimum value of } a_\nu x^\nu \textit{ is asymptotic to } \frac{4}{\pi}\sqrt{\frac{2x}{\pi}}\, e^{-\pi^2/x}. \tag{7.3.13}$$

Proof. By using equivalence relations and (7.1.2), we can write the continued fraction on the left-hand side of (7.3.11) in the form

$$\begin{aligned}
&2\sqrt{x}\left(\frac{1}{1} - \frac{1}{e^x(1+e^{-2x})} - \frac{1}{e^{2x}(1+e^{-4x})} - \frac{1}{e^{3x}(1+e^{-6x})} - \cdots\right)\\
&= 2\sqrt{x}\left(\frac{1}{1} - \frac{e^{-x}}{1+e^{-2x}} - \frac{e^{-3x}}{1+e^{-4x}} - \frac{e^{-5x}}{1+e^{-6x}} - \cdots\right)\\
&= 2\sqrt{x}\frac{(e^{-3x};e^{-4x})_\infty}{(e^{-x};e^{-4x})_\infty}.
\end{aligned} \tag{7.3.14}$$

Equality (7.1.2) is in Ramanujan's second notebook [227], [63, p. 48]. It is also simply the case $a = 1$, $b = 0$ of Entry 12 in Chapter 16 of Ramanujan's second notebook [227], [61, p. 24]. Among others, K.G. Ramanathan [217] has given a proof of (7.1.2). Another continued fraction for the product on the left side of (7.1.2) is found in the lost notebook and has been proved by Andrews [26] as well as by Ramanathan [217]; see Corollary 6.2.10 in the previous chapter.

The expression on the right side of (7.3.14) is the case $k = 4$ in Theorem 7.2.1, since $M_1(\chi) = -2$. This completes the proof of (7.3.11) and (7.3.13).

By Lemma 7.3.1 with $k = 4$, we find that

$$L(3,\chi) = \frac{\pi^3}{32}, \quad L(5,\chi) = \frac{\pi^5}{4^5\cdot 15}, \quad \text{and} \quad L(7,\chi) = \frac{61\pi^7}{3^2\cdot 5\cdot 4^6},$$

since $G(\chi) = 2i$, $M_1(\chi) = -2$, $M_3(\chi) = -26$, $M_5(\chi) = -242$, and $M_7(\chi) = -2186$. Hence, using (7.3.10) and the values above in (7.2.3), we readily compute the values in (7.3.12). □

Ramanujan did not record the value of a_6. Two further corollaries can be found in [83].

In [83], the case for even χ was also considered, and we prove this result below, because we need the special case for the Rogers–Ramanujan continued fraction in the next section.

Theorem 7.3.1. *Let k be a positive integer greater than 3, and let $L(s,\chi)$ denote the Dirichlet L-function associated with $\chi(n)$, a primitive, real, non-principal, even character modulo k. Then as $x \to 0^+$,*

$$\prod_{n=1}^{k-1} (e^{-nx}; e^{-kx})_\infty^{-\chi(n)} \sim \left(\prod_{n=1}^{k-1} |1-\zeta_k^n|^{-\chi(n)/2} \right) e^{-M_2(\chi)x/(4k)},$$

where $\zeta_k = \exp(2\pi i/k)$ and $M_2(\chi)$ is defined by (7.3.2).

Proof. From [126, p. 71] we know that

$$L(s,\chi) = 0$$

if $s = 0, -2, -4, -6, \dots$. Hence the integrand of (7.2.6) has simple poles only at $s = 0$ and $s = -1$. Now if we follow the same steps as we did in the proof of Theorem 7.2.1, we deduce that for any integer $N > 1$, as x tends to 0^+,

$$\log \prod_{n=1}^{k-1} (e^{-nx}; e^{-kx})_\infty^{-\chi(n)} = L'(0,\chi) - \zeta(0)L(-1,\chi)x + O(x^N). \quad (7.3.15)$$

From [127, p. 181, equation (3.2)], if χ is even,

$$L'(0,\chi) = \frac{k}{2G(\chi)} L(1,\chi), \quad (7.3.16)$$

where $G(\chi)$ is defined by (7.3.1). But from [127, p. 182, equation (3.5)],

$$L(1,\chi) = -\frac{G(\chi)}{k} \sum_{n=1}^{k-1} \chi(n) \log|1-\zeta_k^n|, \quad (7.3.17)$$

where $\zeta_k = \exp(2\pi i/k)$.

Hence, by (7.3.16) and (7.3.17),

$$L'(0,\chi) = -\frac{1}{2} \sum_{n=1}^{k-1} \chi(n) \log|1-\zeta_k^n| = \log \left(\prod_{n=1}^{k-1} |1-\zeta_k^n|^{-\chi(n)/2} \right). \quad (7.3.18)$$

Since χ is an even character, the functional equation for $L(s,\chi)$ is given by [126, p. 72]

$$L(s,\chi) = \left(\frac{\pi}{k}\right)^{s-1/2} \frac{\Gamma(\frac{1}{2}(1-s))}{\Gamma(\frac{1}{2}s)} L(1-s,\chi). \quad (7.3.19)$$

By (7.3.19) and (7.2.16),

$$L(-1,\chi) = \left(\frac{\pi}{k}\right)^{-3/2} \frac{\Gamma(1)}{\Gamma(-\frac{1}{2})} L(2,\chi) = -\frac{k^{3/2}}{2\pi^2} L(2,\chi). \quad (7.3.20)$$

From [59, p. 32, equation (6.10)],

$$L(2,\chi) = \frac{\pi^2}{k^3} G(\chi) M_2(\chi), \tag{7.3.21}$$

where $G(\chi)$ and $M_2(\chi)$ are defined by (7.3.1) and (7.3.2), respectively. Also $G(\chi) = \sqrt{k}$, since χ is even. Hence, from (7.3.20) and (7.3.21),

$$L(-1,\chi) = -\frac{k^{3/2}}{2\pi^2} \frac{\pi^2}{k^3} \sqrt{k} M_2(\chi) = -\frac{1}{2k} M_2(\chi). \tag{7.3.22}$$

From [275, p. 19],

$$\zeta(0) = -\frac{1}{2}. \tag{7.3.23}$$

Therefore, by (7.3.15), (7.3.18), (7.3.23), and (7.3.22), we complete the proof of Theorem 7.3.1. □

By using Theorem 7.3.1, we may obtain asymptotic formulas for the Rogers–Ramanujan and Ramanujan–Göllnitz–Gordon continued fractions . We give only the corollary for the Rogers–Ramanujan continued fraction $R(q)$, defined in (1.1.1) of Chapter 1. An application will be made in the next section.

Corollary 7.3.1. *As* $x \to 0^+$,

$$R(e^{-x}) \sim \frac{\sqrt{5}-1}{2}.$$

Proof. Let $k = 5$ in Theorem 7.3.1. Since $\cos(2\pi/5) = (\sqrt{5}-1)/4$, we find by a straightforward calculation that

$$\prod_{n=1}^{4} |1-\zeta_5^n|^{-\chi(n)/2} = \frac{\sqrt{5}+1}{2},$$

where $\zeta_5 = \exp(2\pi i/5)$. Therefore,

$$\frac{(e^{-2x}, e^{-3x}; e^{-5x})_\infty}{(e^{-x}, e^{-4x}; e^{-5x})_\infty} \sim \frac{\sqrt{5}+1}{2} e^{-x/5}, \tag{7.3.24}$$

since $M_2(\chi) = 4$. By (1.1.2) in Chapter 1 and (7.3.24), we complete the proof. □

Corollary 7.3.1 was also proved by J. Lehner [176] by a different method. G. Meinardus [197] developed an asymptotic formula for more general products than those considered in the last two sections, but he determined only the leading term of his asymptotic formula. Thus, Theorem 7.3.1 and Corolary 7.3.1 are special cases of his theorem.

7.4 An Asymptotic Formula for $R(a, q)$

In this section we prove the beautiful asymptotic formula (7.1.5) described in the Introduction.

Entry 7.4.1 (p. 26). *As* $x \to 0^+$,

$$R(a, e^{-x}) = \frac{-1 + \sqrt{1 + 4a}}{2a} \times \exp\left(\frac{ax}{1 + 4a} - \frac{a(1 - a)x^2}{2(1 + 4a)^{5/2}} + \frac{a(1 - a)(1 - 14a)x^3}{6(1 + 4a)^4} - \cdots \right). \tag{7.4.1}$$

Moreover, each term of the asymptotic expansion beginning with the second has a factor of $a(1 - a)$.

Proof. For brevity, set $R(a, e^{-x}) = r(a, x)$. From the definition (7.1.4), we observe that $r(a, x)$ satisfies the functional equation

$$r(a, x) = \frac{1}{1 + ae^{-x}r(ae^{-x}, x)}. \tag{7.4.2}$$

We use a method of successive approximations. Accordingly, we first set $x = 0$, so that (7.4.2) takes the form

$$r(a, 0) = \frac{1}{1 + ar(a, 0)}. \tag{7.4.3}$$

Solving this quadratic equation for $r(a, 0)$, we find that

$$r(a, 0) = \frac{-1 \pm \sqrt{1 + 4a}}{2a}.$$

Since $r(a, 0) > 0$, the plus sign must be taken above. Thus, our first approximation is

$$r(a, x) \approx \frac{-1 + \sqrt{1 + 4a}}{2a} =: c_0(a) := c_0. \tag{7.4.4}$$

For our second approximation, set

$$r(a, x) = c_0(a)e^{c_1(a)x} = c_0e^{c_1x}. \tag{7.4.5}$$

Then from (7.4.2),

$$r(a, x) + ae^{-x}r(a, x)r(ae^{-x}, x) - 1 = 0. \tag{7.4.6}$$

Using (7.4.4) and (7.4.5) in (7.4.6), we find that

$$c_0(a)e^{c_1(a)x} + ae^{-x}c_0(a)e^{c_1(a)x} \left(\frac{-1 + \sqrt{1 + 4ae^{-x}}}{2ae^{-x}} \right) e^{c_1(ae^{-x})x} - 1 \approx 0. \tag{7.4.7}$$

Now,

$$\frac{1}{2}\left(-1+\sqrt{1+4ae^{-x}}\right) = \frac{1}{2}\left(-1+\sqrt{1+4a}-\frac{2ax}{\sqrt{1+4a}}+\cdots\right)$$
$$= ac_0 - \frac{ax}{\sqrt{1+4a}}+\cdots$$

and

$$e^{c_1(ae^{-x})x} = 1 + c_1(ae^{-x})x + \cdots = 1 + c_1(a)x + O(x^2),$$

as $x \to 0$. Using the two expansions above in (7.4.7) and displaying only the terms up to the first power of x, which are needed to obtain the next approximation, we set

$$c_0(1+c_1x+\cdots)+c_0(1+2c_1x+\cdots)\left(ac_0-\frac{ax}{\sqrt{1+4a}}+\cdots\right)-1=0. \quad (7.4.8)$$

If we equate constant coefficients in (7.4.8), we arrive at

$$c_0 + ac_0^2 - 1 = 0,$$

which again yields (7.4.4). If we equate coefficients of x in (7.4.8), we find that

$$c_1 + 2ac_0c_1 - \frac{a}{\sqrt{1+4a}} = 0.$$

Solving for c_1 and employing (7.4.4), we conclude that

$$c_1 = \frac{a}{1+4a}, \quad (7.4.9)$$

which is in agreement with what Ramanujan claims in (7.4.1).

For the third approximation, set

$$r(a,x) = c_0(a)e^{c_1(a)x+c_2(a)x^2}$$

and use this approximation in (7.4.6). We repeat the procedure detailed above to calculate $c_2(a)$. In fact, at this point, we turn to *Maple* to effect the calculations. After several iterations of (7.4.6), we deduce the asymptotic expansion

$$\begin{aligned}
r(a,x) &= \frac{-1+\sqrt{1+4a}}{2a} \\
&\times \exp\left(\frac{ax}{1+4a} - \frac{a(1-a)x^2}{2(1+4a)^{5/2}} + \frac{a(1-a)(1-14a)x^3}{6(1+4a)^4}\right. \\
&- \frac{a(1-a)(1-66a+378a^2-20a^3)x^4}{24(1+4a)^{11/2}} \\
&+ \frac{a(1-a)(1-230a+4860a^2-17000a^3+1984a^4)x^5}{120(1+4a)^7} \\
&- \frac{a(1-a)(1-726a+40530a^2-455740a^3+1155960a^4-211776a^5)x^6}{720(1+4a)^{17/2}} \\
&\left.+ \frac{a(1-a)976a^6x^6}{720(1+4a)^{17/2}} + O(x^7)\right).
\end{aligned} \quad (7.4.10)$$

This establishes (7.4.1), gives further evidence that the coefficient of x^n, $n \geq 2$, has $a(1-a)$ as a factor, and indicates that finding a general formula for the coefficient of x^n is a daunting task.

We now prove the claims about the factors a and $1-a$. The assertion about a is trivial to prove. Inducting on n, suppose that $c_j(0) = 0$, $1 \leq j \leq n-1$. Then from (7.4.6),

$$\exp\left(c_n(0)x^n + O(x^{n+1})\right) = 1.$$

It follows that $c_n(0) = 0$.

The assertion about the factor $1-a$ is deeper, but it follows from Theorem 7.3.1. In fact, our proof in the previous section gives a slightly stronger result, which we now state for only the product representation for the Rogers–Ramanujan continued fraction. For every positive number $N > 0$, as $x \to 0^+$,

$$R(1, e^{-x}) = \frac{(e^{-x}; e^{-5x})_\infty (e^{-4x}; e^{-5x})_\infty}{(e^{-2x}; e^{-5x})_\infty (e^{-3x}; e^{-5x})_\infty} = \frac{\sqrt{5}-1}{2} \exp\left(\frac{1}{5}x + O(x^N)\right). \tag{7.4.11}$$

Comparing (7.4.11) with (7.4.1), we conclude that $c_n(1) = 0$ for every $n \geq 2$, since $N > 0$ can be made arbitrarily large. This completes the proof of Entry 7.4.1. □

The ideas used to prove Entry 7.4.1 can be applied to the generalized cubic continued fraction

$$C(a, e^{-x}) := \frac{1}{1} \underset{+}{} \frac{ae^{-x} + a^2e^{-2x}}{1} \underset{+}{} \frac{ae^{-2x} + a^2e^{-4x}}{1} \underset{+}{} \frac{ae^{-3x} + a^2e^{-6x}}{1} \underset{+}{} \cdots, \tag{7.4.12}$$

where a is any complex number and $x > 0$. The continued fraction (7.4.12) generalizes Ramanujan's cubic continued fraction [112]

$$C(q) := \frac{1}{1} \underset{+}{} \frac{q+q^2}{1} \underset{+}{} \frac{q^2+q^4}{1} \underset{+}{} \frac{q^3+q^6}{1} \underset{+}{} \cdots = \frac{(q; q^6)_\infty (q^5; q^6)_\infty}{(q^3; q^6)_\infty^2},$$

where $|q| < 1$.

Theorem 7.4.1. *As $x \to 0^+$,*

$$C(a, e^{-x}) = \frac{1}{a+1} \times \exp\left(\frac{ax}{1+2a} - \frac{a(1-a)x^2}{2(1+2a)^3} + \frac{a(1-a)(1-12a-4a^2)x^3}{6(1+2a)^5} - \cdots\right).$$

Moreover, each term of the asymptotic expansion beginning with the second has a factor of $a(1-a)$.

See the paper [84] by Berndt and Yee for more details.

Ramanujan's Continued Fraction for $(q^2;q^3)_\infty/(q;q^3)_\infty$

8.1 Introduction

In Chapter 6, we proved some general theorems on continued fractions from the lost notebook that yielded several beautiful examples as special cases, in particular, the Rogers–Ramanujan continued fraction, the Ramanujan–Göllnitz–Gordon continued fraction, and Ramanujan's cubic continued fraction. In Chapter 7, we considered asymptotic formulas for continued fractions, but one of the examples on which we focused in that chapter does not fall under the purview of the general theorems in Chapter 6. Our goal in this chapter is to prove two remarkable theorems for this continued fraction

$$\frac{(q^2;q^3)_\infty}{(q;q^3)_\infty} = \frac{1}{1}\,-\,\frac{q}{1+q}\,-\,\frac{q^3}{1+q^2}\,-\,\frac{q^5}{1+q^3}\,-\,\cdots, \qquad |q|<1. \tag{8.1.1}$$

The continued fraction (8.1.1) is due to Ramanujan and is found in his second notebook [227, p. 290]. Of the many q-continued fractions found by Ramanujan, (8.1.1) is, by far, the most difficult to prove. Up until recently, the only known proof was found by Andrews, Berndt, L. Jacobsen, and R.L. Lamphere [39], [63, p. 46, Entry 19] in 1992, which uses a deep theorem of Andrews [17]. However, a considerably shorter and more natural proof was recently given by Andrews, Berndt, J. Sohn, A.J. Yee, and A. Zaharescu [40].

On page 45 in his lost notebook, Ramanujan claims, in an unorthodox fashion, that a certain q-continued fraction possesses three limit points. More precisely, he asserts that as n tends to ∞ in the three residue classes modulo 3, the nth partial quotients tend, respectively, to three distinct limits, which he explicitly gives. In fact, Ramanujan claims that a more general continued fraction has three distinct limits under the broader concept of "general convergence," which was not defined in the literature until about 70 years later. If $\omega = e^{2\pi i/3}$, then, except for the simplification of notation, Ramanujan [228, p. 45] claimed that for $|q|<1$,

$$\lim_{n\to\infty}\left(\frac{1}{1}-\frac{1}{1+q}-\frac{1}{1+q^2}-\cdots-\frac{1}{1+q^n+a}\right)$$
$$=-\omega^2\left(\frac{\Omega-\omega^{n+1}}{\Omega-\omega^{n-1}}\right)\cdot\frac{(q^2;q^3)_\infty}{(q;q^3)_\infty}, \quad (8.1.2)$$

where

$$\Omega:=\frac{1-a\omega^2}{1-a\omega}\,\frac{(\omega^2q;q)_\infty}{(\omega q;q)_\infty}. \quad (8.1.3)$$

After (8.1.2), Ramanujan appended the note, "Numerators and Denominators can be equated separately."

Of course, because of the appearance of the limiting variable n on the right side of (8.1.2), Ramanujan's claim is meaningless as it stands. But after a few minutes of reflection, we readily conclude that Ramanujan was affirming that there are three distinct limits depending on the congruence class modulo 3 in which $n\to\infty$. In the note after (8.1.3), Ramanujan evidently asserted that the limits can be obtained by determining separately the limits of both the partial numerators and denominators.

Ramanujan's claim is very interesting for several reasons.

First, if $a=0$, the left side of (8.1.2) is a continued fraction (in the normal sense) that diverges. We prove that the three partial quotients tend to the required limits if n is restricted to any one of the three residue classes modulo 3. This is in contrast to the classical result from the general theory of continued fractions, which asserts that if all the elements of a divergent continued fraction are positive, then the even and odd approximants approach distinct limits [182, pp. 96–97].

Second, if $a\neq 0$, we prove that the continued fraction in (8.1.2) converges "generally" in the sense that when n is confined to any one of the three residue classes modulo 3, the limit of the left side indeed exists and is equal to that claimed on the right side of (8.1.2) in each of the three cases. The concept of general convergence is due to L. Jacobsen [167] in 1986. See also her book with H. Waadeland [182, pp. 41–44]. For some results of Ramanujan of a different kind on general convergence, see Chapter 5. Thus, we have one further example of Ramanujan's having discovered a fundamental concept long ahead of his time, before anyone else ever thought of it.

Third, note that the continued fraction (8.1.1) can be written in the equivalent form

$$\frac{(q^2;q^3)_\infty}{(q;q^3)_\infty}=\frac{1}{1}-\frac{1}{q^{-1}+1}-\frac{1}{q^{-2}+1}-\frac{1}{q^{-3}+1}-\cdots. \quad (8.1.4)$$

Thus, when $a=0$, the continued fraction on the left side of (8.1.2) is the same as the continued fraction of (8.1.4), but with q replaced by $1/q$. Observe that, remarkably, $(q^2;q^3)_\infty/(q;q^3)_\infty$ also appears in the three limits on the right side of (8.1.2). In this sense, Ramanujan's result (8.1.2) is analogous to his theorem on the divergence of the Rogers–Ramanujan continued fraction found on pages 374 and 382 in his third notebook [227], which was first proved by Andrews, Berndt, Jacobsen, and Lamphere [39], [63, p. 30, Entry 11]. In the latter result, Ramanujan explicitly determines the limits of the even and odd indexed approximants of the divergent Rogers–Ramanujan continued fraction for $|q| > 1$ and shows that these limits can be expressed in terms of the Rogers–Ramanujan continued fraction itself, *but at different arguments.*

Thus, our first important goal in this chapter is to give a proof of (8.1.2), which we think is one of the most fascinating results in Ramanujan's lost notebook. Our proof is taken from the paper of Andrews, Berndt, Sohn, Yee, and Zaharescu [41], in which the authors also establish general theorems providing classes of continued fractions with three distinct limit points. However, Ramanujan's result (8.1.2) is deeper and does not come under the umbrella of the general theorems of [41]. At the top of page 45 in his lost notebook [228], Ramanujan states separately the special case of (8.1.2) when $a = \omega$. This can be proved in a more elementary fashion, and we do so in the section following our proof of (8.1.2).

The second major purpose of this chapter is to prove another asymptotic formula for (8.1.1), which has a flavor different from that proved in Chapter 7 and which is also found on page 45 of the lost notebook. In fact, the continued fraction examined by Ramanujan is slightly more general than (8.1.1). Although both (8.1.1) and its generalization do not converge for $q > 1$, Ramanujan claims that his asymptotic formula is valid as $q \to 1$ from both directions. However, the continued fraction satisfies a simple difference equation, which is given by Ramanujan immediately preceding the asymptotic formula. Thus, Ramanujan's asymptotic formula should be more properly interpreted as an asymptotic formula for solutions of this difference equation, which does not have a unique solution. Therefore, a sequence of arbitrary constants arises in Ramanujan's asymptotic formula. If $q > 1$, as discussed above, the continued fraction in (8.1.1) has three limit points, and so it would not be possible in any way to prescribe values to these arbitrary constants.

8.2 A Proof of Ramanujan's Formula (8.1.2)

We first introduce needed notation. Define

$$P_0(a) = 0, \qquad P_1(a) = 1, \qquad Q_0(a) = 1, \qquad Q_1(a) = 1, \tag{8.2.1}$$

and for $N \geq 2$, set

$$\frac{P_N(a)}{Q_N(a)} = \frac{1}{1} - \frac{1}{1+q} - \frac{1}{1+q^2} - \cdots - \frac{1}{1+q^{N-1}+a}. \tag{8.2.2}$$

From the general theory of continued fractions [182, p. 9, equation (1.2.9)], for $N \geq 2$, the partial numerators $P_N(0)$ and $Q_N(0)$ satisfy the recurrence relations

$$\begin{cases} P_N(0) &= (1+q^{N-1})P_{N-1}(0) - P_{N-2}(0), \\ Q_N(0) &= (1+q^{N-1})Q_{N-1}(0) - Q_{N-2}(0), \end{cases} \tag{8.2.3}$$

where $P_0(0)$, $P_1(0)$, $Q_0(0)$, and $Q_1(0)$ are defined by (8.2.1).

To prove (8.1.2), our first task will be to derive explicit formulas for $P_N(0)$ and $Q_N(0)$. To do so, we need to recall the definition of the Gaussian polynomials and two versions of the q-binomial theorem [21, pp. 35–36].

Lemma 8.2.1. *If* $\left[\begin{smallmatrix} n \\ m \end{smallmatrix}\right]$ *denotes the Gaussian polynomial defined by*

$$\begin{bmatrix} n \\ m \end{bmatrix} := \begin{bmatrix} n \\ m \end{bmatrix}_q := \begin{cases} \dfrac{(q;q)_n}{(q;q)_m(q;q)_{n-m}}, & \text{if } 0 \le m \le n, \\ 0, & \text{otherwise}, \end{cases}$$

then

$$(z;q)_N = \sum_{j=0}^{N} \begin{bmatrix} N \\ j \end{bmatrix} (-1)^j z^j q^{j(j-1)/2}, \tag{8.2.4}$$

$$\frac{1}{(z;q)_N} = \sum_{j=0}^{\infty} \begin{bmatrix} N+j-1 \\ j \end{bmatrix} z^j. \tag{8.2.5}$$

Lemma 8.2.2. *Let* $N-1 = 3v+\epsilon$, *where* $\epsilon = 0, \pm 1$. *Then*

$$(-1)^v P_N(0) = \sum_{\substack{n,r=0 \\ n+r\equiv\epsilon \pmod 3}}^{\infty} (-1)^{(\epsilon-n-r)/3} q^{n(n+1)/2+r(r-1)/2} \times \begin{bmatrix} n+1 \\ r \end{bmatrix} \begin{bmatrix} n+\frac{1}{3}(N-1-(n+r)) \\ n \end{bmatrix}_{q^3}. \tag{8.2.6}$$

Proof. Recall from (8.2.3) and (8.2.1) that $P_N(0)$ satisfies the recurrence relation

$$P_N(0) = (1+q^{N-1})P_{N-1}(0) - P_{N-2}(0), \qquad N \ge 2, \tag{8.2.7}$$

and the initial conditions $P_0(0) = 0$ and $P_1(0) = 1$.

Define

$$F(t) := \sum_{N=1}^{\infty} P_N(0) t^N.$$

Multiplying the recurrence relation (8.2.7) by t^N and summing over $N \ge 2$, we obtain

$$F(t) - t = tF(t) + tF(tq) - t^2F(t).$$

So,

$$F(t) = \frac{t}{1-t+t^2} + \frac{t}{1-t+t^2}F(tq).$$

Iterating and noting that $F(0)=0$, we find that by (8.2.4) and (8.2.5),

$$\begin{aligned}
F(t) &= \sum_{n=0}^{\infty} \frac{t^{n+1}q^{n(n+1)/2}}{\prod_{j=0}^{n}(1-tq^j+t^2q^{2j})} \\
&= \sum_{n=0}^{\infty} t^{n+1}q^{n(n+1)/2}\frac{(-t;q)_{n+1}}{(-t^3;q^3)_{n+1}} \\
&= \sum_{n,r,s=0}^{\infty} (-1)^s t^{n+1+r+3s}q^{n(n+1)/2+r(r-1)/2}\begin{bmatrix} n+1 \\ r \end{bmatrix}\begin{bmatrix} n+s \\ s \end{bmatrix}_{q^3}.
\end{aligned}$$

Now we choose the terms involving t^N by setting $s=(N-1-n-r)/3$. Hence, equating the coefficients of t^N on both sides, we find that

$$\begin{aligned}
P_N(0) = \sum_{\substack{n,r=0 \\ n+r\equiv N-1 \,(\mathrm{mod}\, 3)}}^{\infty} & (-1)^{(N-1-n-r)/3}q^{n(n+1)/2+r(r-1)/2} \\
& \times \begin{bmatrix} n+1 \\ r \end{bmatrix}\begin{bmatrix} n+\frac{1}{3}(N-1-(n+r)) \\ n \end{bmatrix}_{q^3},
\end{aligned}$$

or, with $N-1=3v+\epsilon$,

$$\begin{aligned}
(-1)^v P_N(0) = \sum_{\substack{n,r=0 \\ n+r\equiv \epsilon \,(\mathrm{mod}\, 3)}}^{\infty} & (-1)^{(\epsilon-n-r)/3}q^{n(n+1)/2+r(r-1)/2} \\
& \times \begin{bmatrix} n+1 \\ r \end{bmatrix}\begin{bmatrix} n+\frac{1}{3}(N-1-(n+r)) \\ n \end{bmatrix}_{q^3},
\end{aligned}$$

as required. □

Lemma 8.2.3. *Let $N-1=3v+\epsilon$, where $\epsilon=0,\pm1$. Then*

$$\begin{aligned}
(-1)^v Q_N(0) = \sum_{\substack{n,r=0 \\ n+r\equiv \epsilon \,(\mathrm{mod}\, 3)}}^{\infty} & (-1)^{(\epsilon-n-r)/3}q^{n(n+1)/2+r(r-1)/2} \\
& \times \begin{bmatrix} n+1 \\ r \end{bmatrix}\begin{bmatrix} n+\frac{1}{3}(N-1-(n+r)) \\ n \end{bmatrix}_{q^3} \\
- \sum_{\substack{n,r=0 \\ n+r\equiv \epsilon-1 \,(\mathrm{mod}\, 3)}}^{\infty} & (-1)^{(\epsilon-1-n-r)/3}q^{n(n+3)/2+r(r-1)/2} \\
& \times \begin{bmatrix} n+1 \\ r \end{bmatrix}\begin{bmatrix} n+\frac{1}{3}(N-2-(n+r)) \\ n \end{bmatrix}_{q^3}. \qquad (8.2.8)
\end{aligned}$$

Proof. Recall from (8.2.3) and (8.2.1) that $Q_N(0)$ satisfies the recurrence relation

$$Q_N(0) = (1+q^{N-1})Q_{N-1}(0) - Q_{N-2}(0), \qquad N \geq 2, \tag{8.2.9}$$

and the initial conditions $Q_0(0) = 1$ and $Q_1(0) = 1$.

Define

$$G(t) := \sum_{N=1}^{\infty} Q_N(0)t^N.$$

Multiplying the recurrence relation (8.2.9) by t^N and summing over $N \geq 2$, we obtain

$$G(t) - t = tG(t) + tG(tq) - t^2G(t) - t^2.$$

So,

$$G(t) = \frac{t-t^2}{1-t+t^2} + \frac{t}{1-t+t^2}G(tq).$$

Iterating and noting that $G(0) = 0$, we arrive at, by (8.2.4) and (8.2.5),

$$\begin{aligned}
G(t) &= \sum_{n=0}^{\infty} \frac{t^{n+1}(1-tq^n)q^{n(n+1)/2}}{\prod_{j=0}^{n}(1-tq^j+t^2q^{2j})} \\
&= \sum_{n=0}^{\infty} t^{n+1}(1-tq^n)q^{n(n+1)/2}\frac{(-t;q)_{n+1}}{(-t^3;q^3)_{n+1}} \\
&= \sum_{n,r,s=0}^{\infty} (-1)^s t^{n+1+r+3s}(1-tq^n)q^{n(n+1)/2+r(r-1)/2}\begin{bmatrix} n+1 \\ r \end{bmatrix}\begin{bmatrix} n+s \\ s \end{bmatrix}_{q^3}.
\end{aligned}$$

Separating the sum above into two parts, we set $s = (N-1-n-r)/3$ and $s = (N-2-n-r)/3$, respectively, in the two sums. Hence, equating coefficients of t^N on both sides, we find that

$$\begin{aligned}
Q_N(0) = &\sum_{\substack{n,r=0 \\ n+r\equiv N-1 \,(\mathrm{mod}\, 3)}}^{\infty} (-1)^{(N-1-n-r)/3}q^{n(n+1)/2+r(r-1)/2} \\
&\qquad \times \begin{bmatrix} n+1 \\ r \end{bmatrix}\begin{bmatrix} n+\frac{1}{3}(N-1-(n+r)) \\ n \end{bmatrix}_{q^3} \\
- &\sum_{\substack{n,r=0 \\ n+r\equiv N-2 \,(\mathrm{mod}\, 3)}}^{\infty} (-1)^{(N-2-n-r)/3}q^{n(n+1)/2+n+r(r-1)/2} \\
&\qquad \times \begin{bmatrix} n+1 \\ r \end{bmatrix}\begin{bmatrix} n+\frac{1}{3}(N-2-(n+r)) \\ n \end{bmatrix}_{q^3}.
\end{aligned}$$

If $N-1 = 3v+\epsilon$, then

$$
\begin{aligned}
(-1)^v Q_N(0) = &\sum_{\substack{n,r=0 \\ n+r\equiv\epsilon \pmod 3}}^{\infty} (-1)^{(\epsilon-n-r)/3} q^{n(n+1)/2+r(r-1)/2} \\
&\qquad \times \begin{bmatrix} n+1 \\ r \end{bmatrix} \begin{bmatrix} n+\frac{1}{3}(N-1-(n+r)) \\ n \end{bmatrix}_{q^3} \\
&- \sum_{\substack{n,r=0 \\ n+r\equiv\epsilon-1 \pmod 3}}^{\infty} (-1)^{(\epsilon-1-n-r)/3} q^{n(n+1)/2+n+r(r-1)/2} \\
&\qquad \times \begin{bmatrix} n+1 \\ r \end{bmatrix} \begin{bmatrix} n+\frac{1}{3}(N-2-(n+r)) \\ n \end{bmatrix}_{q^3},
\end{aligned}
$$

as required. □

The previous two lemmas are actually special cases of a theorem due to M.D. Hirschhorn [153], who gave a different proof.

To calculate the limits of $P_N(0)$ and $Q_N(0)$ as $N \to \infty$ in the three residue classes modulo 3, we need the following result from Ramanujan's lost notebook, which was first proved by Andrews [32].

Entry 8.2.1 (p. 43). *Let* $\omega = e^{2\pi i/3}$. *Then*

$$
\sum_{n=0}^{\infty} \frac{(-\omega)^n q^{n(n+1)/2} (\omega q; q)_n}{(q^3; q^3)_n} = (\omega q)_\infty (q^2; q^3)_\infty. \tag{8.2.10}
$$

Note that by conjugation, Entry 8.2.1 also holds if ω is replaced by ω^2.

Proof. Recall that if r is a nonnegative integer, the basic hypergeometric function ${}_{r+1}\phi_r$ is defined for $|q| < 1$ and $|t| < 1$ by

$$
{}_{r+1}\phi_r \left[\begin{matrix} a_0, a_1, \ldots, a_r \\ b_1, b_2, \ldots, b_r \end{matrix} ; q, t \right] := \sum_{n=0}^{\infty} \frac{(a_0)_n (a_1)_n \cdots (a_r)_n}{(b_1)_n (b_2)_n \cdots (b_r)_n (q)_n} t^n.
$$

We use Watson's [285] q-analogue of Whipple's theorem , namely,

$$
\begin{aligned}
&{}_8\phi_7 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, e, f, q^{-N} \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{b}, \frac{aq}{c}, \frac{aq}{e}, \frac{aq}{f}, aq^{N+1} \end{matrix} ; q, \frac{a^2 q^{N+2}}{bcef} \right] \\
&\qquad = \frac{(aq)_N \left(\frac{aq}{ef}\right)_N}{\left(\frac{aq}{e}\right)_N \left(\frac{aq}{f}\right)_N} {}_4\phi_3 \left[\begin{matrix} \frac{aq}{bc}, e, f, q^{-N} \\ \frac{aq}{b}, \frac{aq}{c}, \frac{efq^{-N}}{a} \end{matrix} ; q, q \right],
\end{aligned} \tag{8.2.11}
$$

where N is a nonnegative integer and a, b, c, e, and f are complex numbers with the provision that $bcef \neq 0$. We apply (8.2.11) by first letting $c, f, N \to$

∞, by then replacing a and b by ae and be, respectively, and lastly by letting e tend to 0. We then find that

$$\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n}b^{-n}q^{n(3n+1)/2}}{(aq/b)_n(aq)_n(q)_n}=\frac{1}{(aq)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n a^n q^{n(n+1)/2}}{(aq/b)_n(q)_n}. \tag{8.2.12}$$

We now turn to the left side of (8.2.10) and employ (8.2.12) with $a=\omega$ and $b=\omega^2$ to find that

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-\omega)^n q^{n(n+1)/2}(\omega q)_n}{(q^3;q^3)_n}&=\sum_{n=0}^{\infty}\frac{(-\omega)^n q^{n(n+1)/2}(\omega q)_n}{(q;q)_n(\omega q;q)_n(\omega^2 q;q)_n}\\&=\sum_{n=0}^{\infty}\frac{(-\omega)^n q^{n(n+1)/2}}{(q;q)_n(\omega^2 q;q)_n}\\&=(\omega q;q)_\infty\sum_{n=0}^{\infty}\frac{(-1)^n q^{3n(n+1)/2}}{(\omega^2 q;q)_n(\omega q;q)_n(q;q)_n}\\&=(\omega q;q)_\infty\sum_{n=0}^{\infty}\frac{(-1)^n q^{3n(n+1)/2}}{(q^3;q^3)_n}\\&=(\omega q)_\infty(q^2;q^3)_\infty,\end{aligned}$$

where in the last step we applied (8.2.4) with q replaced by q^3, $z=q^2$, and $N\to\infty$. This is what we wanted to prove, and so the proof is complete. □

Lemma 8.2.4. *Let $N-1=3v+\epsilon$, where $\epsilon=0,\pm1$. Then*

$$\lim_{v\to\infty}(-1)^v P_N(0)=\frac{1}{3}(-\omega)^\epsilon(1-\omega^2)\left(\frac{(\omega^2 q)_\infty}{(\omega q)_\infty}-\omega^{\epsilon+1}\right)(\omega q)_\infty(q^2;q^3)_\infty. \tag{8.2.13}$$

Proof. Let $N\to\infty$ through values such that $N-1\equiv\epsilon \pmod 3$. Then, from (8.2.6),

$$\begin{aligned}&\lim_{v\to\infty}(-1)^v P_N(0)\\&\quad=\sum_{\substack{n,r=0\\ n+r\equiv\epsilon \,(\mathrm{mod}\, 3)}}^{\infty}(-1)^{(\epsilon-n-r)/3}q^{n(n+1)/2+r(r-1)/2}\begin{bmatrix}n+1\\ r\end{bmatrix}\frac{1}{(q^3;q^3)_n}\\&\quad=\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(q^3;q^3)_n}\sum_{\substack{r=0\\ r\equiv\epsilon-n\,(\mathrm{mod}\, 3)}}^{\infty}q^{r(r-1)/2}\begin{bmatrix}n+1\\ r\end{bmatrix}\rho^{\epsilon-n-r},\end{aligned}$$

where $\rho=e^{\pi i/3}$. Recall that $\omega=\rho^2$. Using the elementary fact

$$\frac{1+\omega^a+\bar{\omega}^a}{3}=\begin{cases}1, & \text{if } a\equiv 0 \pmod 3,\\ 0, & \text{otherwise,}\end{cases} \tag{8.2.14}$$

we find, by (8.2.4), Entry 8.2.1, and Entry 8.2.1 with ω replaced by ω^2, that

$$\begin{aligned}
&\lim_{v\to\infty}(-1)^v P_N(0)\\
&=\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(q^3;q^3)_n}\sum_{r=0}^{\infty}q^{r(r-1)/2}\begin{bmatrix}n+1\\ r\end{bmatrix}\rho^{\epsilon-n-r}\frac{1+\omega^{\epsilon-n-r}+\bar{\omega}^{\epsilon-n-r}}{3}\\
&=\frac{1}{3}\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(q^3;q^3)_n}\left\{\rho^{\epsilon-n}(-\bar{\rho};q)_{n+1}+(-1)^{\epsilon-n}(1;q)_{n+1}+\rho^{n-\epsilon}(-\rho;q)_{n+1}\right\}\\
&=\frac{1}{3}(-\omega^2)^{\epsilon}\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}(-\omega)^n}{(q^3;q^3)_n}(\omega;q)_{n+1}\\
&\quad+\frac{1}{3}(-\omega)^{\epsilon}\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}(-\omega^2)^n}{(q^3;q^3)_n}(\omega^2;q)_{n+1}\\
&=\frac{1}{3}(-\omega^2)^{\epsilon}(1-\omega)\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}(-\omega)^n}{(q^3;q^3)_n}(\omega q;q)_n\\
&\quad+\frac{1}{3}(-\omega)^{\epsilon}(1-\omega^2)\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}(-\omega^2)^n}{(q^3;q^3)_n}(\omega^2 q;q)_n\\
&=\frac{1}{3}(-\omega^2)^{\epsilon}(1-\omega)(\omega q)_\infty(q^2;q^3)_\infty+\frac{1}{3}(-\omega)^{\epsilon}(1-\omega^2)(\omega^2 q)_\infty(q^2;q^3)_\infty\\
&=\frac{1}{3}(-\omega)^{\epsilon}(1-\omega^2)\left\{\frac{(\omega^2 q)_\infty}{(\omega q)_\infty}-\omega^{\epsilon+1}\right\}(\omega q)_\infty(q^2;q^3)_\infty.
\end{aligned}$$

∎

To establish the corresponding lemma for $Q_N(0)$, we need an analogue of Entry 8.2.1, which we will establish with the same tools that Andrews used to prove Entry 8.2.1, but with an additional lemma.

Lemma 8.2.5. *For any complex numbers a, b, with $b \neq 0$,*

$$\begin{aligned}
&\frac{1}{(aq)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n a^n q^{n(n+1)/2}}{(q)_n(aq/b)_n}-\frac{1}{(aq)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n a^n q^{n(n+1)/2+n}}{(q)_n(aq/b)_n}\\
&=\frac{b}{a}\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n}b^{-n}q^{n(3n-1)/2}}{(q)_n(aq/b)_n(aq)_n}-\frac{b}{a}\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n}b^{-n}q^{n(3n+1)/2}}{(q)_n(aq/b)_n(aq)_n}. \qquad (8.2.15)
\end{aligned}$$

Proof. We need the limiting case of Watson's q-analogue of Whipple's theorem given in (8.2.12). By replacing b by bq and multiplying both sides by $(1-a/b)$, we obtain

$$\begin{aligned}
&\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n}b^{-n}(1-aq^n/b)q^{n(3n-1)/2}}{(q)_n(aq/b)_n(aq)_n}\\
&=\frac{1}{(aq)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n a^n(1-aq^n/b)q^{n(n+1)/2}}{(q)_n(aq/b)_n},
\end{aligned}$$

or

$$\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n}b^{-n}q^{n(3n-1)/2}}{(q)_n(aq/b)_n(aq)_n}-\frac{a}{b}\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n}b^{-n}q^{n(3n+1)/2}}{(q)_n(aq/b)_n(aq)_n}$$
$$=\frac{1}{(aq)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n a^n q^{n(n+1)/2}}{(q)_n(aq/b)_n}-\frac{a}{b}\frac{1}{(aq)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n a^n q^{n(n+1)/2+n}}{(q)_n(aq/b)_n}.$$

Using (8.2.12) above, we deduce (8.2.15). □

If we set $a=\omega$ and $b=\omega^2$ in (8.2.15), we find that after some simplification,

$$\sum_{n=0}^{\infty}\frac{(-\omega)^n q^{n(n+1)/2}}{(q)_n(\omega^2 q)_n}-\sum_{n=0}^{\infty}\frac{(-\omega)^n q^{n(n+1)/2+n}}{(q)_n(\omega^2 q)_n}$$
$$=\omega(\omega q)_\infty\left\{\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(3n-1)/2}}{(q^3;q^3)_n}-\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(3n+1)/2}}{(q^3;q^3)_n}\right\}$$
$$=\omega(\omega q)_\infty\left\{(q;q^3)_\infty-(q^2;q^3)_\infty\right\}, \tag{8.2.16}$$

by letting $N\to\infty$ in the q-binomial theorem, (8.2.4), with q replaced by q^3 and z replaced by q and q^2, respectively.

By employing an argument similar to that used by Andrews [32] to prove Entry 8.2.1, we can utilize Lemma 8.2.5 to prove the following lemma.

Lemma 8.2.6. *Let $\omega=e^{2\pi i/3}$. Then*

$$-\frac{\omega}{(\omega q)_\infty}\left\{\sum_{n=0}^{\infty}\frac{(-\omega)^n q^{n(n+1)/2}}{(q)_n(\omega^2 q)_n}+\omega\sum_{n=0}^{\infty}\frac{(-\omega)^n q^{n(n+1)/2+n}}{(q)_n(\omega^2 q)_n}\right\}=(q;q^3)_\infty. \tag{8.2.17}$$

Proof. Letting $a=\omega$ and $b=\omega^2$ in Lemma 8.2.5, we obtain

$$\sum_{n=0}^{\infty}\frac{(-\omega)^n q^{n(n+1)/2}}{(q)_n(\omega^2 q)_n}-\sum_{n=0}^{\infty}\frac{(-\omega)^n q^{n(n+1)/2+n}}{(q)_n(\omega^2 q)_n}$$
$$=\omega(\omega q)_\infty\left\{\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(3n-1)/2}}{(q^3;q^3)_n}-\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(3n+1)/2}}{(q^3;q^3)_n}\right\}. \tag{8.2.18}$$

From (8.2.12), we see that

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(3n+1)/2}}{(q)_n(\omega q)_n(\omega^2 q)_n}=\frac{1}{(\omega q)_\infty}\sum_{n=0}^{\infty}\frac{(-1)^n\omega^n q^{n(n+1)/2}}{(q)_n(\omega^2 q)_n}. \tag{8.2.19}$$

Combining (8.2.18) and (8.2.19), we find that

$$-\frac{\omega}{(\omega q)_\infty}\left\{\sum_{n=0}^{\infty}\frac{(-\omega)^n q^{n(n+1)/2}}{(q)_n(\omega^2 q)_n}+\omega\sum_{n=0}^{\infty}\frac{(-\omega)^n q^{n(n+1)/2+n}}{(q)_n(\omega^2 q)_n}\right\}$$
$$=\sum_{n=0}^{\infty}\frac{(-1)^n q^{n(3n-1)/2}}{(q^3;q^3)_n}. \tag{8.2.20}$$

By letting $N\to\infty$ in the q-binomial theorem, (8.2.4), with q replaced by q^3 and $z=q$, we find that the right side of (8.2.20) equals $(q;q^3)_\infty$, and so (8.2.17) is established. □

Note that by conjugation, (8.2.17) is also valid with ω replaced by ω^2.

Lemma 8.2.7. *Let $N-1=3v+\epsilon$, where $\epsilon=0,\pm1$. Then*

$$\lim_{v\to\infty}(-1)^v Q_N(0)=\frac{1}{3}(-\omega)^{\epsilon+1}(1-\omega^2)\left(\frac{(\omega^2 q)_\infty}{(\omega q)_\infty}-\omega^{\epsilon-1}\right)(\omega q)_\infty(q;q^3)_\infty. \tag{8.2.21}$$

Proof. Since the details are similar to those in the proof of Lemma 8.2.4, we suppress some of them.

Let $N\to\infty$ through values such that $N-1\equiv\epsilon\pmod 3$. Then, from (8.2.8),

$$\lim_{v\to\infty}(-1)^v Q_N(0)=\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(q^3;q^3)_n}\sum_{\substack{r=0\\ r\equiv\epsilon-n \pmod 3}}^{\infty}q^{r(r-1)/2}\begin{bmatrix}n+1\\ r\end{bmatrix}\rho^{\epsilon-n-r}$$
$$-\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2+n}}{(q^3;q^3)_n}\sum_{\substack{r=0\\ r\equiv\epsilon-n-1 \pmod 3}}^{\infty}q^{r(r-1)/2}\begin{bmatrix}n+1\\ r\end{bmatrix}\rho^{\epsilon-1-n-r},$$

where $\rho=e^{\pi i/3}$. By (8.2.14), (8.2.4), (8.2.17), the remark following the proof of Lemma 8.2.6, and calculations analogous to those used in the proof of Lemma 8.2.4,

$$\lim_{v\to\infty}(-1)^v Q_N(0)$$
$$=\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}}{(q^3;q^3)_n}\sum_{r=0}^{\infty}q^{r(r-1)/2}\begin{bmatrix}n+1\\ r\end{bmatrix}\rho^{\epsilon-n-r}\frac{1+\omega^{\epsilon-n-r}+\bar{\omega}^{\epsilon-n-r}}{3}$$
$$-\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2+n}}{(q^3;q^3)_n}\sum_{r=0}^{\infty}q^{r(r-1)/2}\begin{bmatrix}n+1\\ r\end{bmatrix}\rho^{\epsilon-1-n-r}\frac{1+\omega^{\epsilon-1-n-r}+\bar{\omega}^{\epsilon-1-n-r}}{3}$$
$$=\frac{1}{3}(-\omega^2)^{\epsilon}(1-\omega)\left\{\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}(-\omega)^n}{(q^3;q^3)_n}(\omega q;q)_n\right.$$
$$\left.+\omega\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2+n}(-\omega)^n}{(q^3;q^3)_n}(\omega q;q)_n\right\}$$

$$+\frac{1}{3}(-\omega)^{\epsilon}(1-\omega^2)\left\{\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2}(-\omega^2)^n}{(q^3;q^3)_n}(\omega^2q;q)_n\right.$$
$$\left.+\omega^2\sum_{n=0}^{\infty}\frac{q^{n(n+1)/2+n}(-\omega^2)^n}{(q^3;q^3)_n}(\omega^2q;q)_n\right\}$$
$$=\frac{1}{3}(-\omega^2)^{\epsilon+1}(1-\omega)(\omega q)_\infty(q;q^3)_\infty+\frac{1}{3}(-\omega)^{\epsilon+1}(1-\omega^2)(\omega^2q)_\infty(q;q^3)_\infty$$
$$=\frac{1}{3}(-\omega)^{\epsilon+1}(1-\omega^2)\left\{\frac{(\omega^2q)_\infty}{(\omega q)_\infty}-\omega^{\epsilon-1}\right\}(\omega q)_\infty(q;q^3)_\infty.$$

□

Theorem 8.2.1. *Let $N-1=3v+\epsilon$, where $\epsilon=0$ or ±1. Then*

$$\lim_{N\to\infty}\left(\frac{1}{1}-\frac{1}{1+q}-\frac{1}{1+q^2}-\cdots-\frac{1}{1+q^{N-1}}\right)$$
$$=-\omega^2\frac{\dfrac{(\omega^2q;q)_\infty}{(\omega q;q)_\infty}-\omega^{\epsilon+1}}{\dfrac{(\omega^2q;q)_\infty}{(\omega q;q)_\infty}-\omega^{\epsilon-1}}\frac{(q^2;q^3)_\infty}{(q;q^3)_\infty}.$$

Proof. The result follows immediately from (8.2.2) and Lemmas 8.2.4 and 8.2.7. □

Entry 8.2.2 (p. 45). *Let $N-1=3v+\epsilon$, where $\epsilon=0$ or ±1. Then*

$$\lim_{N\to\infty}\left(\frac{1}{1}-\frac{1}{1+q}-\frac{1}{1+q^2}-\cdots-\frac{1}{1+q^{N-1}+a}\right)$$
$$=-\omega^2\frac{\Omega-\omega^{\epsilon+1}}{\Omega-\omega^{\epsilon-1}}\frac{(q^2;q^3)_\infty}{(q;q^3)_\infty},$$

where

$$\Omega=\frac{1-a\omega^2}{1-a\omega}\frac{(\omega^2q)_\infty}{(\omega q)_\infty}.$$

Proof. Recall that the partial numerators $P_N(a)$ and partial denominators $Q_N(a)$ are defined in (8.2.1) and (8.2.2). It is easily shown by induction that for $N\geq 2$,

$$P_N(a)=P_N(0)+aP_{N-1}(0),$$
$$Q_N(a)=Q_N(0)+aQ_{N-1}(0).$$

For example, see [182, p. 8]. Hence,

$$\lim_{N\to\infty}P_N(a)=\lim_{N\to\infty}P_N(0)+a\lim_{N\to\infty}P_{N-1}(0),$$
$$\lim_{N\to\infty}Q_N(a)=\lim_{N\to\infty}Q_N(0)+a\lim_{N\to\infty}Q_{N-1}(0).$$

Let $N = 3v + \epsilon + 1$, where $\epsilon = 0, \pm 1$; we consider two cases: $\epsilon = 0, 1$ and $\epsilon = -1$.

Suppose that $\epsilon = 0$ or 1. From Lemma 8.2.4,

$$\begin{aligned}
\lim_{N\to\infty}(-1)^v P_N(a) &= \lim_{N\to\infty}(-1)^v P_N(0) + a\lim_{N\to\infty}(-1)^v P_{N-1}(0)\\
&= \frac{1}{3}(-\omega)^{\epsilon}(1-\omega^2)\left\{\frac{(\omega^2 q)_\infty}{(\omega q)_\infty} - \omega^{\epsilon+1}\right\}(\omega q)_\infty (q^2;q^3)_\infty\\
&\quad + a\frac{1}{3}(-\omega)^{\epsilon-1}(1-\omega^2)\left\{\frac{(\omega^2 q)_\infty}{(\omega q)_\infty} - \omega^{\epsilon}\right\}(\omega q)_\infty (q^2;q^3)_\infty\\
&= \frac{1}{3}(-\omega)^{\epsilon}(1-\omega^2)\left\{\frac{(\omega^2 q)_\infty}{(\omega q)_\infty} - \omega^{\epsilon+1} - a\omega^2\frac{(\omega^2 q)_\infty}{(\omega q)_\infty} + a\omega^{\epsilon-1}\right\}\\
&\quad \times (\omega q)_\infty (q^2;q^3)_\infty\\
&= \frac{1}{3}(-\omega)^{\epsilon}(1-\omega^2)\left\{\frac{1-a\omega^2}{1-a\omega}\frac{(\omega^2 q)_\infty}{(\omega q)_\infty} - \omega^{\epsilon+1}\right\}\\
&\quad \times (1-a\omega)(\omega q)_\infty (q^2;q^3)_\infty.
\end{aligned}$$

On the other hand, if $\epsilon = -1$, then

$$\begin{aligned}
\lim_{N\to\infty}(-1)^v P_N(a) &= \lim_{N\to\infty}(-1)^v P_N(0) + a\lim_{N\to\infty}(-1)^v P_{N-1}(0)\\
&= \frac{1}{3}(-\omega)^{\epsilon}(1-\omega^2)\left\{\frac{(\omega^2 q)_\infty}{(\omega q)_\infty} - \omega^{\epsilon+1}\right\}(\omega q)_\infty (q^2;q^3)_\infty\\
&\quad - a\frac{1}{3}(-\omega)^{\epsilon+2}(1-\omega^2)\left\{\frac{(\omega^2 q)_\infty}{(\omega q)_\infty} - \omega^{\epsilon+3}\right\}(\omega q)_\infty (q^2;q^3)_\infty\\
&= \frac{1}{3}(-\omega)^{\epsilon}(1-\omega^2)\left\{\frac{(\omega^2 q)_\infty}{(\omega q)_\infty} - \omega^{\epsilon+1} - a\omega^2\frac{(\omega^2 q)_\infty}{(\omega q)_\infty} + a\omega^{\epsilon+2}\right\}\\
&\quad \times (\omega q)_\infty (q^2;q^3)_\infty\\
&= \frac{1}{3}(-\omega)^{\epsilon}(1-\omega^2)\left\{\frac{1-a\omega^2}{1-a\omega}\frac{(\omega^2 q)_\infty}{(\omega q)_\infty} - \omega^{\epsilon+1}\right\}\\
&\quad \times (1-a\omega)(\omega q)_\infty (q^2;q^3)_\infty.
\end{aligned}$$

Therefore, in both cases,

$$\begin{aligned}
\lim_{N\to\infty}(-1)^v P_N(a) &= \frac{1}{3}(-\omega)^{\epsilon}(1-\omega^2)\left\{\frac{1-a\omega^2}{1-a\omega}\frac{(\omega^2 q)_\infty}{(\omega q)_\infty} - \omega^{\epsilon+1}\right\}\\
&\quad \times (1-a\omega)(\omega q)_\infty (q^2;q^3)_\infty. \qquad (8.2.22)
\end{aligned}$$

Similarly, we can determine the limits of the denominator $Q_N(a)$. Suppose that $\epsilon = 0$ or 1. Then, from Lemma 8.2.7,

$$\begin{aligned}\lim_{N\to\infty}(-1)^v Q_N(a) &= \lim_{N\to\infty}(-1)^v Q_N(0) + a\lim_{N\to\infty}(-1)^v Q_{N-1}(0)\\ &= \frac{1}{3}(-\omega)^{\epsilon+1}(1-\omega^2)\left\{\frac{(\omega^2 q)_\infty}{(\omega q)_\infty} - \omega^{\epsilon-1} - a\omega^2\frac{(\omega^2 q)_\infty}{(\omega q)_\infty} + a\omega^\epsilon\right\}\\ &\quad\times(\omega q)_\infty(q;q^3)_\infty\\ &= \frac{1}{3}(-\omega)^{\epsilon+1}(1-\omega^2)\left\{\frac{1-a\omega^2}{1-a\omega}\frac{(\omega^2 q)_\infty}{(\omega q)_\infty} - \omega^{\epsilon-1}\right\}\\ &\quad\times(1-a\omega)(\omega q)_\infty(q;q^3)_\infty.\end{aligned}$$

On the other hand, if $\epsilon = -1$, then

$$\begin{aligned}\lim_{N\to\infty}(-1)^v Q_N(a) &= \lim_{N\to\infty}(-1)^v Q_N(0) + a\lim_{N\to\infty}(-1)^v Q_{N-1}(0)\\ &= \frac{1}{3}(-\omega)^{\epsilon+1}(1-\omega^2)\left\{\frac{(\omega^2 q)_\infty}{(\omega q)_\infty} - \omega^{\epsilon-1} - a\omega^2\frac{(\omega^2 q)_\infty}{(\omega q)_\infty} + a\omega^\epsilon\right\}\\ &\quad\times(\omega q)_\infty(q;q^3)_\infty\\ &= \frac{1}{3}(-\omega)^{\epsilon+1}(1-\omega^2)\left\{\frac{1-a\omega^2}{1-a\omega}\frac{(\omega^2 q)_\infty}{(\omega q)_\infty} - \omega^{\epsilon-1}\right\}\\ &\quad\times(1-a\omega)(\omega q)_\infty(q;q^3)_\infty.\end{aligned}$$

Therefore, in both cases,

$$\begin{aligned}\lim_{N\to\infty}(-1)^v Q_N(a) = \frac{1}{3}(-\omega)^{\epsilon+1}(1-\omega^2)&\left\{\frac{1-a\omega^2}{1-a\omega}\frac{(\omega^2 q)_\infty}{(\omega q)_\infty} - \omega^{\epsilon-1}\right\}\\ &\times(1-a\omega)(\omega q)_\infty(q;q^3)_\infty.\end{aligned}\tag{8.2.23}$$

Combining (8.2.22) and (8.2.23) with (8.2.2), we complete the proof. □

Observe that our proof of Entry 8.2.2 justifies the addendum made by Ramanujan after his statement of (8.1.2).

D. Bowman and J. McLaughlin [103] have generalized Entry 8.2.2 by replacing the continued fraction in (8.1.1) by a more general continued fraction (depending on a positive integral parameter m), which they demonstrate has m limit points.

8.3 The Special Case $a = \omega$ of (8.1.2)

It is interesting to note that if $a = \omega$, then $\Omega = 0$, and so the three limits in (8.1.2) are identical. This claim is made at the top of page 45 in the lost notebook. In this section, we provide a more elementary proof, by means of the Bauer–Muir transformation, of this special case. Repeated efforts at using the Bauer–Muir transformation to prove the more general Entry 8.2.2 failed.

A Bauer–Muir transformation [182, pp. 76–77] of a continued fraction $b_0+\mathbf{K}(a_n/b_n)$ is a (new) continued fraction whose approximants have the values

$$S_k(w_k) := b_0 + \frac{a_1}{b_1} + \frac{a_2}{b_2} + \cdots + \frac{a_k}{b_k + w_k}, \quad k = 0, 1, 2, \ldots. \tag{8.3.1}$$

Such a transformation exists if

$$\lambda_k := a_k - w_{k-1}(b_k + w_k) \neq 0, \quad k \geq 1, \tag{8.3.2}$$

and it is given by

$$b_0+w_0+\frac{\lambda_1}{b_1 + w_1} + \frac{a_1\lambda_2/\lambda_1}{b_2 + w_2 - w_0\lambda_2/\lambda_1} + \frac{a_2\lambda_3/\lambda_2}{b_3 + w_3 - w_1\lambda_3/\lambda_2} + \cdots. \tag{8.3.3}$$

Entry 8.3.1 (p. 45). *For a cube root of unity ω,*

$$\lim_{n\to\infty}\left(\frac{1}{1} - \frac{1}{1+q} - \frac{1}{1+q^2} - \cdots - \frac{1}{1+q^n+\omega}\right) = -\omega\frac{(q^2;q^3)_\infty}{(q;q^3)_\infty}. \tag{8.3.4}$$

Proof. Let L_n denote the reciprocal of the continued fraction on the left-hand side of (8.3.4). If we employ the notation of (8.3.1), then $b_0 = 1$, $a_n = -1$, and $b_n = 1 + q^n$, for $n \geq 1$, and $\omega_n = \omega$, for $n \geq 0$.

If $q = 0$, then (8.3.4) reduces to a tautology. Hence assume that $q \neq 0$. Then, from (8.3.2), for $n \geq 1$,

$$\lambda_n = -1 - \omega(1 + q^n + \omega) = -1 - \omega - \omega^2 - \omega q^n = -\omega q^n \neq 0.$$

Thus, by (8.3.3),

$$\begin{aligned} L_n &= 1 + \omega + \frac{-\omega q}{1 + q + \omega} + \frac{-q}{1 + q^2 + \omega - \omega q} + \frac{-q}{1 + q^3 + \omega - \omega q} + \cdots \\ &= -\omega^2 + \frac{\omega^2 q}{1 - \omega q} + \frac{\omega q}{1 + q^2 + \omega - \omega q} + \frac{-q}{1 + q^3 + \omega - \omega q} + \cdots \\ &=: -\omega^2 + \frac{\omega^2 q}{C_1}, \end{aligned} \tag{8.3.5}$$

after using an equivalence transformation for the continued fraction.

For the continued fraction C_1, in the notation of (8.3.1),

$$b_0 = 1 - \omega q, \quad a_1 = \omega q, \quad a_n = -q, \quad n \geq 2,$$

and

$$b_n = 1 + q^{n+1} + \omega - \omega q, \quad n \geq 1.$$

We apply the Bauer–Muir transformation a second time. Set $\omega_0 = -\omega^2 q$ and $\omega_i = \omega q$, for $i \geq 1$. A brief calculation shows that by (8.3.2),

$$\lambda_1 = q^3\omega^2 \neq 0 \quad \text{and} \quad \lambda_k = -q^{k+2}\omega \neq 0 \quad \text{for} \quad k \geq 2.$$

Hence, from (8.3.3), after applying the Bauer–Muir transformation to C_1, we have

$$\begin{aligned} C_1 &= 1+q+\frac{\omega^2q^3}{1+q^2+\omega} + \frac{-q^2}{1+q^3+\omega-\omega q^2} + \frac{-q^2}{1+q^4+\omega-\omega q^2} + \cdots \\ &= 1+q+\frac{-q^3}{1-\omega q^2} + \frac{\omega q^2}{1+q^3+\omega-\omega q^2} + \frac{-q^2}{1+q^4+\omega-\omega q^2} + \cdots, \end{aligned} \tag{8.3.6}$$

after applying an equivalence transformation. Combining (8.3.5) and (8.3.6), we have

$$\begin{aligned} L_n &= -\omega^2 + \frac{\omega^2 q}{1+q} + \frac{-q^3}{1-\omega q^2} + \frac{\omega q^2}{1+q^3+\omega-\omega q^2} + \frac{-q^2}{1+q^4+\omega-\omega q^2} + \cdots \\ &=: -\omega^2 + \frac{\omega^2 q}{1+q} + \frac{-q^3}{C_2}. \end{aligned} \tag{8.3.7}$$

Applying the Bauer–Muir transformation to C_2 and proceeding as in the two previous applications, we find that if $\omega_0 = -\omega^2q^2$ and $\omega_i = \omega q^2$, for $i \geq 1$, then

$$\lambda_1 = \omega^2q^5 \neq 0 \quad \text{and} \quad \lambda_k = -q^{k+4}\omega \neq 0, \quad k \geq 2.$$

Thus, $b_0+\omega_0 = 1+q^2$, $b_1+\omega_1 = 1+q^3+\omega$, $b_n+\omega_n-\omega_0\lambda_n/\lambda_{n-1} = 1+q^{n+2}+\omega-q^3\omega$, and $a_n\lambda_{n+1}/\lambda_n = -q^3$, for $n \geq 1$. Hence, from (8.3.7), after using an equivalence transformation, we find that

$$\begin{aligned} L_n &= -\omega^2 + \frac{\omega^2 q}{1+q} + \frac{-q^3}{1+q^2} + \frac{\omega^2q^5}{1+q^3+\omega} + \frac{-q^3}{1+q^4+\omega-\omega q^3} \\ &\quad + \frac{-q^3}{1+q^5+\omega-\omega q^3} + \cdots \\ &= -\omega^2 + \frac{\omega^2 q}{1+q} + \frac{-q^3}{1+q^2} + \frac{-q^5}{1-\omega q^3} + \frac{\omega q^3}{1+q^4+\omega-\omega q^3} \\ &\quad + \frac{-q^3}{1+q^5+\omega-\omega q^3} + \cdots \\ &= \cdots \\ &= -\omega^2 + \frac{\omega^2 q}{1+q} + \frac{-q^3}{1+q^2} + \frac{-q^5}{1+q^3} + \cdots + \frac{-q^{2n-1}}{C_n}, \end{aligned} \tag{8.3.8}$$

where

$$\begin{aligned} C_n &= 1-\omega q^n + \frac{\omega q^n}{1+q^{n+1}+\omega-\omega q^n} + \frac{-q^n}{1+q^{n+2}+\omega-\omega q^n} \\ &\quad + \frac{-q^n}{1+q^{n+3}+\omega-\omega q^n} + \cdots \end{aligned}$$

after an easy inductive argument on n with $\omega_0 = -\omega^2 q^n$ and $\omega_i = \omega q^n$, for $i \geq 1$, after the nth step. Upon taking the reciprocal in (8.3.8), letting n tend to ∞, and using (8.1.1), we deduce (8.3.4). □

8.4 Two Continued Fractions Related to $(q^2;q^3)_\infty/(q;q^3)_\infty$

Two further continued fractions for $(q^2;q^3)_\infty/(q;q^3)_\infty$ can be found on page 27 of Ramanujan's lost notebook.

Entry 8.4.1 (p. 27). *Let ω be a cube root of unity. Then*

$$-\omega^2 - \omega\frac{(q;q^3)_\infty}{(q^2;q^3)_\infty} \tag{8.4.1}$$

$$= \frac{1}{1} - \frac{\omega q}{1} - \frac{\omega^2 q}{1} - \frac{\omega q^2}{1} - \frac{\omega^2 q^2}{1} - \frac{\omega q^3}{1} - \frac{\omega^2 q^3}{1} - \cdots \tag{8.4.2}$$

$$= 1 + \frac{\omega}{1+q^{-1}} - \frac{1}{1+q^{-2}} - \frac{1}{1+q^{-3}} - \cdots. \tag{8.4.3}$$

Proof. By (8.1.1),

$$\frac{(q^2;q^3)_\infty}{(q;q^3)_\infty} = \frac{1}{1} - \frac{q}{1+q} - \frac{q^3}{1+q^2} - \frac{q^5}{1+q^3} - \frac{q^7}{1+q^4} - \cdots,$$

which is equivalent to the continued fraction

$$\frac{1}{1} - \frac{1}{1+q^{-1}} - \frac{1}{1+q^{-2}} - \frac{1}{1+q^{-3}} - \cdots.$$

Taking the reciprocal, we find that

$$\frac{(q;q^3)_\infty}{(q^2;q^3)_\infty} = 1 - \frac{1}{1+q^{-1}} - \frac{1}{1+q^{-2}} - \frac{1}{1+q^{-3}} - \cdots.$$

Multiplying both sides by ω and adding ω^2 to both sides, we find that

$$\begin{aligned}\omega^2 + \omega\frac{(q;q^3)_\infty}{(q^2;q^3)_\infty} &= \omega^2 + \omega - \frac{\omega}{1+q^{-1}} - \frac{1}{1+q^{-2}} - \frac{1}{1+q^{-3}} - \cdots \\ &= -1 - \frac{\omega}{1+q^{-1}} - \frac{1}{1+q^{-2}} - \frac{1}{1+q^{-3}} - \cdots,\end{aligned}$$

which establishes the equality of (8.4.1) and (8.4.3).

Now (8.4.2) is the continued fraction with the parameters

$$b_0 = 0, \quad a_1 = 1, \quad a_{2n} = -\omega q^n, \quad a_{2n+1} = -\omega^2 q^n, \quad b_n = 1, \quad n \geq 1.$$

Hence, the odd part of (8.4.2), by Theorem 5.5.1 of Chapter 5, is equal to

$$1+\frac{\omega q}{1+q}-\frac{q^3}{1+q^2}-\frac{q^5}{1+q^3}-\cdots. \tag{8.4.4}$$

But (8.4.4) is equivalent to the continued fraction

$$1+\frac{\omega}{1+q^{-1}}-\frac{1}{1+q^{-2}}-\frac{1}{1+q^{-3}}-\cdots,$$

which is (8.4.3). □

8.5 An Asymptotic Expansion

In Entry 7.3.1 of Chapter 7, we established an asymptotic expansion, as $q \to 1^-$, for the continued fraction (8.1.1). Elsewhere on page 45, Ramanujan gives an asymptotic expansion for a continued fraction that generalizes that of (8.1.1), but as we remarked in the Introduction, Ramanujan evidently derived his result from a recurrence relation, (8.5.2) below, satisfied by the continued fraction. Since Ramanujan claims that his asymptotic formula is valid for both positive and negative values of x, where $q = e^{-x}$, his assertion must be interpreted as an asymptotic expansion for solutions of (8.5.2). Because (8.5.2) does not have a unique solution, his asymptotic series includes a sequence $\phi_0, \phi_1, \phi_2, \dots$ of arbitrary constants. In this section, we establish this unusual asymptotic series claimed by Ramanujan.

Entry 8.5.1 (p. 45). *Let*

$$u_\lambda := \frac{1}{1+e^{(\lambda+1)x}}-\frac{1}{1+e^{(\lambda+2)x}}-\cdots. \tag{8.5.1}$$

Then, as $x \to 0$,

$$u_\lambda + \frac{1}{u_{\lambda-1}} = 1 + e^{\lambda x} \tag{8.5.2}$$

and

$$\begin{aligned} u_\lambda = 1 &- \frac{\phi_0}{1-\lambda\phi_0} + x\left(\frac{\lambda+1}{2} + \frac{\phi_1 + (\lambda^2-1)(\frac12 - \frac23\lambda\phi_0 + \frac14\lambda^2\phi_0^2)}{(1-\lambda\phi_0)^2}\right) \\ &+ x^2\left(\frac{\lambda(\lambda+1)(\lambda+2)}{12}\right. \\ &- \frac{\phi_2 + \lambda(\lambda^2-1)(\lambda^2-4)\left(\frac{1}{45} - \frac{1}{36}\lambda\phi_0 + \frac{1}{112}(\lambda^2+\frac13)\phi_0^2\right)}{(1-\lambda\phi_0)^2} \\ &- \left.\frac{\lambda}{(1-\lambda\phi_0)^3}\left(\phi_1 + \frac{\lambda^2-1}{6}(1-\tfrac12\lambda\phi_0)\right)^2\right) + \cdots, \end{aligned} \tag{8.5.3}$$

where ϕ_0, ϕ_1, ϕ_2, $\dots$ *are independent of* λ.

Proof. From (8.5.1),

$$u_{\lambda-1} = \frac{1}{1+e^{\lambda x}} - \frac{1}{1+e^{(\lambda+1)x}} - \frac{1}{1+e^{(\lambda+2)x}} - \cdots.$$

Hence,

$$\frac{1}{u_{\lambda-1}} = 1+e^{\lambda x} - \frac{1}{1+e^{(\lambda+1)x}} - \frac{1}{1+e^{(\lambda+2)x}} - \cdots = 1+e^{\lambda x} - u_\lambda,$$

which proves (8.5.2).

To prove (8.5.3), we use the recurrence relation (8.5.2) and the method of successive approximations . We restrict our attention to solutions of (8.5.2) that have asymptotic expansions of the form

$$u_\lambda = c_0(\lambda) + c_1(\lambda)x + c_2(\lambda)x^2 + \cdots,$$

an assumption evidently also made by Ramanujan. We first calculate $c_0(\lambda)$. Now, from (8.5.2), the constant terms yield

$$c_0(\lambda) + \frac{1}{c_0(\lambda-1)} = 2.$$

Set

$$c_0(\lambda) = 1 + f(\lambda).$$

Then

$$1 + f(\lambda) + \frac{1}{1+f(\lambda-1)} = 2,$$

or

$$f(\lambda)(1+f(\lambda-1)) = f(\lambda-1). \tag{8.5.4}$$

Next put

$$g(\lambda) = \frac{1}{f(\lambda)}.$$

Then, from (8.5.4), we easily deduce that

$$g(\lambda) - g(\lambda-1) = 1. \tag{8.5.5}$$

This is an inhomogeneous linear recurrence relation that has the characteristic root 1. Thus, the general homogeneous solution is

$$g(\lambda) = c \cdot 1^\lambda = c.$$

Since 1 is the characteristic root, a particular inhomogeneous solution has the form $k\lambda$. Therefore, from (8.5.5),

$$k\lambda - k(\lambda-1) = 1.$$

Hence, $k=1$, and the general solution for the linear recurrence relation (8.5.5) is $g(\lambda)=c+\lambda$. Thus,

$$f(\lambda)=\frac{1}{c+\lambda} \quad \text{and} \quad c_0(\lambda)=1+\frac{1}{c+\lambda}.$$

Ramanujan sets $c=-1/\phi_0$. Thus,

$$c_0(\lambda)=1+\frac{1}{\lambda-1/\phi_0}=1-\frac{\phi_0}{1-\phi_0\lambda}. \tag{8.5.6}$$

For our second approximation, from (8.5.2),

$$c_0(\lambda)+c_1(\lambda)x+\frac{1}{c_0(\lambda-1)+c_1(\lambda-1)x}=2+\lambda x,$$

or

$$(c_0(\lambda)+c_1(\lambda)x)(c_0(\lambda-1)+c_1(\lambda-1)x)+1=(2+\lambda x)(c_0(\lambda-1)+c_1(\lambda-1)x).$$

Equate coefficients of x to obtain

$$c_0(\lambda-1)c_1(\lambda)+c_0(\lambda)c_1(\lambda-1)=2c_1(\lambda-1)+\lambda c_0(\lambda-1). \tag{8.5.7}$$

From (8.5.6) and (8.5.7), we have

$$\frac{(1-\phi_0\lambda)c_1(\lambda)}{1+\phi_0-\phi_0\lambda}+\frac{(1-\phi_0\lambda-\phi_0)c_1(\lambda-1)}{1-\phi_0\lambda}=\frac{\lambda(1-\phi_0\lambda)}{1-\phi_0\lambda+\phi_0}+2c_1(\lambda-1). \tag{8.5.8}$$

Now

$$c_1(\lambda-1)\left(\frac{1-\phi_0\lambda-\phi_0}{1-\phi_0\lambda}-2\right)=c_1(\lambda-1)\left(\frac{-1+\phi_0\lambda-\phi_0}{1-\phi_0\lambda}\right). \tag{8.5.9}$$

Thus, from (8.5.8) and (8.5.9),

$$c_1(\lambda)\left(\frac{1-\phi_0\lambda}{1+\phi_0-\phi_0\lambda}\right)+c_1(\lambda-1)\left(\frac{-1+\phi_0\lambda-\phi_0}{1-\phi_0\lambda}\right)=\frac{\lambda(1-\phi_0\lambda)}{1-\phi_0\lambda+\phi_0}. \tag{8.5.10}$$

Set

$$f_1(\lambda)=(1-\phi_0\lambda)c_1(\lambda). \tag{8.5.11}$$

Then, from (8.5.10) and (8.5.11),

$$\frac{f_1(\lambda)}{1+\phi_0-\phi_0\lambda}-\frac{f_1(\lambda-1)}{1-\phi_0\lambda}=\frac{\lambda(1-\phi_0\lambda)}{1-\phi_0\lambda+\phi_0}.$$

Multiply both sides by $(1-\phi_0\lambda)(1+\phi_0-\phi_0\lambda)$ to deduce that

$$(1-\phi_0\lambda)f_1(\lambda)-(1+\phi_0-\phi_0\lambda)f_1(\lambda-1)=\lambda(1-\phi_0\lambda)^2. \tag{8.5.12}$$

Set

$$g_1(\lambda) = (1 - \phi_0\lambda) f_1(\lambda). \tag{8.5.13}$$

Hence, from (8.5.12) and (8.5.13),

$$g_1(\lambda) - g_1(\lambda - 1) = \lambda(1 - \phi_0\lambda)^2, \tag{8.5.14}$$

which has the characteristic root 1, and so the general homogeneous solution is

$$\phi_1 \cdot 1^\lambda = \phi_1,$$

for an arbitrary constant ϕ_1. Since 1 is a homogeneous solution, a particular solution for the recurrence relation (8.5.14) has the form

$$g_1(\lambda) = f_1\lambda + f_2\lambda^2 + f_3\lambda^3 + f_4\lambda^4. \tag{8.5.15}$$

Substitute (8.5.15) into (8.5.14) to find that

$$\begin{aligned} f_1 - f_2 + f_3 - f_4 + \lambda(2f_2 - 3f_3 + 4f_4) + \lambda^2(3f_3 - 6f_4) + \lambda^3(4f_4)& \\ = \lambda(1 - 2\phi_0\lambda + \phi_0^2\lambda^2)&. \end{aligned}$$

Equate coefficients of like powers of λ to deduce that

$$f_1 = \frac{1}{2} - \frac{1}{3}\phi_0, \quad f_2 = \frac{1}{2} - \phi_0 + \frac{1}{4}\phi_0^2, \quad f_3 = -\frac{2}{3}\phi_0 + \frac{1}{2}\phi_0^2, \quad \text{and} \quad f_4 = \frac{1}{4}\phi_0^2.$$

Substitute these values into (8.5.15) to find that

$$\begin{aligned} g_1(\lambda) &= \left(\frac{1}{2} - \frac{1}{3}\phi_0\right)\lambda + \left(\frac{1}{2} - \phi_0 + \frac{1}{4}\phi_0^2\right)\lambda^2 + \left(-\frac{2}{3}\phi_0 + \frac{1}{2}\phi_0^2\right)\lambda^3 + \frac{1}{4}\phi_0^2 \\ &= \frac{1}{2}(\lambda + 1)(1 - 2\phi_0\lambda + \phi_0^2\lambda^2) + \frac{\lambda^2 - 1}{2} \\ &\quad - \frac{2}{3}\phi_0\lambda^3 + \frac{2}{3}\lambda\phi_0 + \frac{1}{4}\phi_0^2\lambda^4 - \frac{1}{4}\phi_0^2\lambda^2 \\ &= \frac{1}{2}(\lambda + 1)(1 - \phi_0\lambda)^2 + (\lambda^2 - 1)\left(\frac{1}{2} - \frac{2}{3}\lambda\phi_0 + \frac{1}{4}\phi_0^2\lambda^2\right), \end{aligned}$$

by elementary algebra. Hence, the general solution for the recurrence relation (8.5.14) is

$$\phi_1 + \frac{1}{2}(\lambda + 1)(1 - \phi_0\lambda)^2 + (\lambda^2 - 1)\left(\frac{1}{2} - \frac{2}{3}\lambda\phi_0 + \frac{1}{4}\phi_0^2\lambda^2\right). \tag{8.5.16}$$

From (8.5.11), (8.5.13), and (8.5.16),

$$c_1(\lambda) = \frac{g_1(\lambda)}{(1 - \phi_0\lambda)^2} = \frac{1}{2}(\lambda + 1) + \frac{\phi_1 + (\lambda^2 - 1)\left(\frac{1}{2} - \frac{2}{3}\lambda\phi_0 + \frac{1}{4}\phi_0^2\lambda^2\right)}{(1 - \phi_0\lambda)^2},$$

as claimed by Ramanujan.

To calculate the coefficient of x^2, write $u_\lambda = c_0(\lambda) + c_1(\lambda)x + c_2(\lambda)x^2$. Then, from (8.5.2),

$$c_0(\lambda) + c_1(\lambda)x + c_2(\lambda)x^2 + \frac{1}{c_0(\lambda-1) + c_1(\lambda-1)x + c_2(\lambda-1)x^2} = 2 + \lambda x + \frac{\lambda^2 x^2}{2},$$

or

$$\begin{aligned}(c_0(\lambda) + c_1(\lambda)x + c_2(\lambda)x^2) \cdot (c_0(\lambda-1) + c_1(\lambda-1)x + c_2(\lambda-1)x^2) + 1 \\ = \left(2 + \lambda x + \frac{\lambda^2 x^2}{2}\right)(c_0(\lambda-1) + c_1(\lambda-1)x + c_2(\lambda-1)x^2). \quad (8.5.17)\end{aligned}$$

Equate coefficients of x^2 to deduce that

$$\begin{aligned}c_0(\lambda)c_2(\lambda-1) + c_1(\lambda)c_1(\lambda-1) + c_2(\lambda)c_0(\lambda-1) \\ = 2c_2(\lambda-1) + \lambda c_1(\lambda-1) + \frac{\lambda^2}{2}c_0(\lambda-1),\end{aligned}$$

or

$$c_2(\lambda)c_0(\lambda-1) + c_2(\lambda-1)(c_0(\lambda)-2) = c_0(\lambda-1)\frac{\lambda^2}{2} + c_1(\lambda-1)\lambda - c_1(\lambda)c_1(\lambda-1). \quad (8.5.18)$$

Recall that

$$c_0(\lambda) = \frac{1-\phi_0\lambda-\phi_0}{1-\phi_0\lambda} \quad \text{and} \quad c_0(\lambda) - 2 = \frac{-1+\phi_0\lambda-\phi_0}{1-\phi_0\lambda}. \quad (8.5.19)$$

Set

$$c_2(\lambda)(1-\phi_0\lambda) = f_2(\lambda). \quad (8.5.20)$$

Now from (8.5.18), (8.5.19), and (8.5.20),

$$\begin{aligned}c_2(\lambda)\frac{1-\phi_0\lambda}{1+\phi_0-\phi_0\lambda} + c_2(\lambda-1)\frac{-1+\phi_0\lambda-\phi_0}{1-\phi_0\lambda} = \frac{f_2(\lambda)}{1+\phi_0-\phi_0\lambda} - \frac{f_2(\lambda-1)}{1-\phi_0\lambda} \\ = \frac{\lambda^2}{2}\frac{1-\phi_0\lambda}{1+\phi_0-\phi_0\lambda} + \lambda c_1(\lambda-1) - c_1(\lambda)c_1(\lambda-1). \quad (8.5.21)\end{aligned}$$

Set

$$g_2(\lambda) = f_2(\lambda)(1-\phi_0\lambda) = c_2(\lambda)(1-\phi_0\lambda)^2. \quad (8.5.22)$$

Then, after multiplying both sides of (8.5.21) by $(1-\phi_0\lambda)(1+\phi_0-\phi_0\lambda)$ and using (8.5.22), we find that

$$\begin{aligned}g_2(\lambda) - g_2(\lambda-1) = (1-\phi_0\lambda)(1+\phi_0-\phi_0\lambda) \\ \times \left(\frac{\lambda^2}{2}\frac{1-\phi_0\lambda}{1+\phi_0-\phi_0\lambda} + \lambda c_1(\lambda-1) - c_1(\lambda)c_1(\lambda-1)\right). \quad (8.5.23)\end{aligned}$$

Recall that

$$c_1(\lambda) = \frac{g_1(\lambda)}{(1-\phi_0\lambda)^2} \tag{8.5.24}$$

with

$$g_1(\lambda) = \frac{1}{2}(\lambda+1)(1-\phi_0\lambda)^2 + (\lambda^2-1)\left(\frac{1}{2} - \frac{2}{3}\lambda\phi_0 + \frac{1}{4}\phi_0^2\lambda^2\right) + \phi_1. \tag{8.5.25}$$

The right-hand side of (8.5.23) is not a polynomial in λ. However, by making a judicious change of variable, we will be able to determine the general solution of the recurrence relation (8.5.23). Now multiply both sides of (8.5.23) by $(1-\phi_0\lambda)(1+\phi_0-\phi_0\lambda)$. Then

$$\begin{aligned}
&g_2(\lambda)(1-\phi_0\lambda)(1+\phi_0-\phi_0\lambda) - g_2(\lambda-1)(1-\phi_0\lambda)(1+\phi_0-\phi_0\lambda)\\
&= (1-\phi_0\lambda)^2(1+\phi_0-\phi_0\lambda)^2\\
&\quad\times\left(\frac{\lambda^2}{2}\frac{1-\phi_0\lambda}{1+\phi_0-\phi_0\lambda} + \lambda c_1(\lambda-1) - c_1(\lambda)c_1(\lambda-1)\right)\\
&= \frac{\lambda^2}{2}(1-\phi_0\lambda)^3(1+\phi_0-\phi_0\lambda) + \lambda c_1(\lambda-1)(1-\phi_0\lambda)^2(1+\phi_0-\phi_0\lambda)^2\\
&\quad - c_1(\lambda)c_1(\lambda-1)(1-\phi_0\lambda)^2(1+\phi_0-\phi_0\lambda)^2\\
&= \frac{\lambda^2}{2}(1-\phi_0\lambda)^3(1+\phi_0-\phi_0\lambda) + \lambda(1-\phi_0\lambda)^2 g_1(\lambda-1) - g_1(\lambda)g_1(\lambda-1),
\end{aligned} \tag{8.5.26}$$

where we have used (8.5.24) in the last step. Set

$$h_2(\lambda) = g_2(\lambda)(1-\phi_0\lambda). \tag{8.5.27}$$

Then we can rewrite (8.5.26) as

$$\begin{aligned}
&h_2(\lambda)(1-\phi_0\lambda+\phi_0) - h_2(\lambda-1)(1-\phi_0\lambda)\\
&= \frac{\lambda^2}{2}(1-\phi_0\lambda)^3(1+\phi_0-\phi_0\lambda) + \lambda(1-\phi_0\lambda)^2 g_1(\lambda-1) - g_1(\lambda)g_1(\lambda-1).
\end{aligned} \tag{8.5.28}$$

We see that the general solution of

$$h_2(\lambda)(1-\phi_0\lambda+\phi_0) - h_2(\lambda-1)(1-\phi_0\lambda) = 0$$

is

$$c(1-\phi_0\lambda),$$

where c is a constant, since the characteristic root for the corresponding homogeneous recurrence relation, $g_2(\lambda) - g_2(\lambda-1) = 0$, equals 1.

Note from (8.5.25) that $g_1(\lambda)$ is a polynomial of degree 4 in λ, and so the right-hand side of (8.5.28) is a polynomial of degree 8 in λ. For a particular solution to the recurrence relation (8.5.28), let

$$h_2(\lambda) = \sum_{i=0}^{8} g_i\lambda^i. \tag{8.5.29}$$

Then the general solution is

$$\begin{aligned}
c(1-\phi_0\lambda) &+ \sum_{i=0}^{8} g_i\lambda^i \\
&= (c+g_0)(1-\phi_0\lambda) + g_0\phi_0\lambda + \sum_{i=1}^{8} g_i\lambda^i \\
&= (c+g_0)(1-\phi_0\lambda) + (g_0\phi_0+g_1)\lambda + \sum_{i=2}^{8} g_i\lambda^i \\
&= \phi_2(1-\phi_0\lambda) + g_1^*\lambda + \sum_{i=2}^{8} g_i\lambda^i,
\end{aligned} \tag{8.5.30}$$

where $\phi_2 = c+g_0$ and $g_1^* = g_0\phi_0 + g_1$. From this observation, we do not need to consider the constant term, and so we need only to find a particular solution of the form

$$h_2(\lambda) = \sum_{i=1}^{8} e_i\lambda^i. \tag{8.5.31}$$

By (8.5.31), (8.5.28), and (8.5.25), we have the system of equations

$$\phi_1^2 + e_1 - e_2 + e_3 - e_4 + e_5 - e_6 + e_7 - e_8 = 0,$$

$$\begin{aligned}
\lambda(-\phi_1 - \tfrac{2}{3}\phi_0\phi_1 + 2e_2 + \phi_0e_2 - 3e_3 - \phi_0e_3 + 4e_4 + \phi_0e_4 - 5e_5 - \phi_0e_5 \\
+ 6e_6 + \phi_0e_6 - 7e_7 - \phi_0e_7 + 8e_8 + \phi_0e_8) = 0,
\end{aligned}$$

$$\begin{aligned}
\lambda^2(-\tfrac{1}{4} - \tfrac{1}{6}\phi_0 + \tfrac{1}{9}\phi_0^2 + \phi_1 + 2\phi_0\phi_1 + \tfrac{1}{2}\phi_0^2\phi_1 - \phi_0e_2 + 3e_3 + 3\phi_0e_3 - 6e_4 - 4\phi_0e_4 \\
+ 10e_5 + 5\phi_0e_5 - 15e_6 - 6\phi_0e_6 + 21e_7 + 7\phi_0e_7 - 28e_8 - 8\phi_0e_8) = 0,
\end{aligned}$$

$$\begin{aligned}
\lambda^3(-\tfrac{1}{2} + \tfrac{2}{3}\phi_0 + \tfrac{7}{12}\phi_0^2 - \tfrac{1}{6}\phi_0^3 - \tfrac{4}{3}\phi_0\phi_1 - \phi_0^2\phi_1 - 2\phi_0e_3 + 4e_4 + 6\phi_0e_4 \\
- 10e_5 - 10\phi_0e_5 + 20e_6 + 15\phi_0e_6 - 35e_7 - 21\phi_0e_7 + 56e_8 + 28\phi_0e_8) = 0,
\end{aligned}$$

$$\begin{aligned}
\lambda^4(\tfrac{1}{4} + \tfrac{5}{3}\phi_0 - \tfrac{29}{36}\phi_0^2 - \tfrac{2}{3}\phi_0^3 + \tfrac{1}{16}\phi_0^4 + \tfrac{1}{2}\phi_0^2\phi_1 - 3\phi_0e_4 + 5e_5 + 10\phi_0e_5 \\
- 15e_6 - 20\phi_0e_6 + 35e_7 + 35\phi_0e_7 - 70e_8 - 56\phi_0e_8) = 0,
\end{aligned}$$

$$\begin{aligned}
\lambda^5(-\tfrac{2}{3}\phi_0 - \tfrac{25}{12}\phi_0^2 + \tfrac{1}{2}\phi_0^3 + \tfrac{1}{4}\phi_0^4 - 4\phi_0e_5 + 6e_6 + 15\phi_0e_6 \\
- 21e_7 - 35\phi_0e_7 + 56e_8 + 70\phi_0e_8) = 0,
\end{aligned}$$

$$\lambda^6(\tfrac{25}{36}\phi_0^2 + \tfrac{7}{6}\phi_0^3 - \tfrac{1}{8}\phi_0^4 - 5\phi_0 e_6 + 7e_7 + 21\phi_0 e_7 - 28e_8 - 56\phi_0 e_8) = 0,$$
$$\lambda^7(-\tfrac{1}{3}\phi_0^3 - \tfrac{1}{4}\phi_0^4 - 6\phi_0 e_7 + 8e_8 + 28\phi_0 e_8) = 0,$$

and

$$\lambda^8(\tfrac{1}{16}\phi_0^4 - 7\phi_0 e_8) = 0.$$

If we solve this system of equations using *Mathematica*, we find that

$$\begin{aligned}
e_1 &= \frac{1}{420}(21 - 5\phi_0^2 + 140\phi_1 - 420\phi_1^2),\\
e_2 &= \frac{1}{1260}(315 - 343\phi_0 + 15\phi_0^3 - 210\phi_0\phi_1),\\
e_3 &= \frac{1}{36}(9 - 27\phi_0 + 13\phi_0^2 - 12\phi_1),\\
e_4 &= -\frac{1}{144}\phi_0(80 - 108\phi_0 + 21\phi_0^2 - 24\phi_1),\\
e_5 &= \frac{1}{180}(-980\phi_0^2 - 45\phi_0^3), \quad e_6 = \frac{1}{360}(28\phi_0 - 45\phi_0^3),\\
e_7 &= -\frac{11\phi_0^2}{252}, \quad \text{and} \quad e_8 = \frac{\phi_0^3}{112}.
\end{aligned}$$

From the equalities above, (8.5.31), (8.5.30), (8.5.27), and (8.5.22), the coefficient of x^2, after rearrangement, is equal to

$$\frac{\lambda(\lambda+1)(\lambda+2)}{12} - \frac{\phi_2 + \lambda(\lambda^2-1)(\lambda^2-4)\left(\frac{1}{45} - \frac{1}{36}\lambda\phi_0 + \frac{1}{112}(\lambda^2+\frac{1}{3})\phi_0^2\right)}{(1-\lambda\phi_0)^2}$$
$$- \frac{\lambda}{(1-\lambda\phi_0)^3}\left(\phi_1 + \frac{\lambda^2-1}{6}(1 - \tfrac{1}{2}\lambda\phi_0)\right)^2,$$

as claimed by Ramanujan. This completes the proof. □

Ramanujan claims that if $x < 0$, the coefficients $\phi_0, \phi_1, \phi_2, \phi_3, \dots$ are arbitrary, but that if $x > 0$, then $\phi_1 = \phi_2 = \phi_3 = \cdots = 0$ and

$$\phi_0 = \frac{\Gamma(\frac{2}{3})}{\Gamma(\frac{1}{3})}(3x)^{1/3}e^{-G(x)},$$

where $G(x)$ has the asymptotic expansion, as $x \to 0^+$,

$$G(x) \sim a_2x^2 + a_4x^4 + a_6x^6 + \cdots,$$

with the coefficients a_ν given by

$$a_\nu = \frac{4\Gamma(\nu)\zeta(\nu)L(\nu+1,\chi)}{(2\pi/\sqrt{3})^{2\nu+1}}.$$

Here $L(s,\chi)$ denotes the Dirichlet L-function associated with the character $\chi(n) = \left(\frac{n}{3}\right)$, where $\left(\frac{n}{3}\right)$ denotes the Legendre symbol. By rearrangement, Ramanujan is asserting that

$$\frac{1}{1-u_\lambda} \sim \frac{1}{\phi_0}, \tag{8.5.32}$$

as $x \to 0^+$. Note that when $\lambda = 0$, the continued fraction in (8.1.1) is equal to $1/(1-u_0)$, with $q = e^{-x}$. In this case, the asymptotic formula (8.5.32) is identical to the aforementioned asymptotic formula of Entry 7.3.1 proved in Chapter 7. However, if $\lambda > 0$, the method of proof used in the previous chapter does not generalize, and so in this particular situation we cannot verify Ramanujan's claim. As remarked at the beginning of this section, the constants $\phi_0, \phi_1, \phi_2, \ldots$ are indeed arbitrary when $x < 0$, because (8.5.2) does not have a unique solution.

Another proof of Entry 8.5.1 has been given by Hirschhorn [162].

9

The Rogers–Fine Identity

9.1 Introduction

This chapter is devoted to consequences of the identity

$$\sum_{n=0}^{\infty} \frac{(\alpha;q)_n}{(\beta;q)_n}\tau^n = \sum_{n=0}^{\infty} \frac{(\alpha;q)_n(\alpha\tau q/\beta;q)_n\beta^n\tau^n q^{n^2-n}(1-\alpha\tau q^{2n})}{(\beta;q)_n(\tau;q)_{n+1}}. \tag{9.1.1}$$

This result was first proved by L.J. Rogers [235]. N.J. Fine [137, p. 15] discovered it independently in his exhaustive study of the series given by the left-hand side of (9.1.1). In [20, Section 4], (9.1.1) was proved combinatorially and was christened the Rogers–Fine identity. Subsequently, it was learned that G.W. Starcher [259, p. 803], in his doctoral dissertation at the University of Illinois in 1930, had also discovered and proved most of (9.1.1). Each of the three original proofs is essentially the same; the idea is to study a defining functional equation.

While Ramanujan appears not to have stated this result explicitly, he did consider a closely related more general result, namely, Entry 7 of Chapter 16 in his second notebook [227], [61, p. 16]. In fact, (9.1.1) follows from the last equation in [61, p. 16] by setting $a = \alpha q\tau/\beta$, $c = \tau\alpha$, and $d = \tau q\alpha$.

It is quite amazing how many results follow from (9.1.1). We shall examine several q-series corollaries in Section 9.2. The remaining sections relate to false theta series, i.e., series that would be instances of classical theta series except for an alteration of the signs of the series terms.

A.J. Yee and Berndt [85] have found combinatorial proofs for nineteen of the identities in this chapter. The difficulties of their proofs range widely, and four representative samples are presented here.

9.2 Series Transformations

We begin this section with the first three entries from page 41. Define

$$\phi(a) := \sum_{n=0}^{\infty} \frac{a^n q^{n(n+1)/2}}{(bq;q)_n}. \tag{9.2.1}$$

Note that $\phi(1)$ is the generating function for partitions into distinct parts when $b = 1$.

Entry 9.2.1 (p. 41). *If* $\phi(a)$ *is defined by* (9.2.1), *then*

$$\phi(a) = (b + aq)\phi(aq) + 1 - b. \tag{9.2.2}$$

First Proof of Entry 9.2.1. From the definition (9.2.1),

$$\begin{aligned}
\phi(a) - b\phi(aq) &= \sum_{n=0}^{\infty} \frac{a^n q^{n(n+1)/2}(1 - bq^n)}{(bq;q)_n} \\
&= 1 - b + \sum_{n=0}^{\infty} \frac{a^{n+1} q^{(n+1)(n+2)/2}}{(bq;q)_n} \\
&= 1 - b + aq\phi(aq),
\end{aligned}$$

and this is equivalent to (9.2.2). □

Second Proof of Entry 9.2.1. As we noted above, $\phi(a)$ generates partitions into distinct parts. In the definition of $\phi(a)$, the power of a denotes the number of distinct parts, and the sum of the powers of a and b denotes the largest part. We now divide the partitions into two sets; one is the set of partitions having a part 1, and the other is the set of partitions not having a part 1. Consider now

$$aq\phi(aq) := \sum_{n=0}^{\infty} \frac{a^{n+1} q^{(n+1)(n+2)/2}}{(bq;q)_n}.$$

The sum above generates partitions into distinct parts. But note that the smallest part is 1, since each summand generates partitions into exactly $n+1$ parts by the numerator, whereas the denominator $(bq;q)_n$ does not have an effect on the last part. The power of a is equal to the number of parts, and the sum of the powers of a and b is equal to the largest part.

Examine

$$b\phi(aq) := \sum_{n=0}^{\infty} \frac{ba^n q^{n(n+3)/2}}{(bq;q)_n}. \tag{9.2.3}$$

In each summand, the exponent of q in the numerator is the sum of integers 2 through $n+1$. Thus, we obtain a partition into distinct parts, but now there are no 1's. The power of a in (9.2.3) still denotes the number of parts, and the sum of the powers of a and b on the right side of (9.2.3) is equal to the largest part. But observe that the empty partition corresponding to the term 1 is absent, and so we must add it. On the other hand, the term with $n = 0$ in (9.2.3) is equal to b. Thus, we must subtract it.

We have now accounted for all partitions into distinct parts on the right side of (9.2.2), and so the proof of Entry 9.2.1 is complete. □

Entry 9.2.2 (p. 41). *If* $\phi(a)$ *is defined by* (9.2.1), *then*

$$\phi(a) = \sum_{n=0}^{\infty} \frac{(-aq/b;q)_n a^n b^n q^{n(3n+1)/2}(1+aq^{2n+1})}{(bq;q)_n}. \tag{9.2.4}$$

Proof. In (9.1.1), set $\alpha = -aq/\tau$ and $\beta = -bq$, and then let $\tau \to 0$. The desired result then follows. □

Entry 9.2.3 (p. 41). *If* $\phi(a)$ *is defined by* (9.2.1), *then*

$$\phi(a) = 1 + \sum_{n=1}^{\infty} \frac{(-aq/b;q)_{n-1} a^n b^{n-1} q^{n(3n-1)/2}(1+aq^{2n})}{(bq;q)_n}. \tag{9.2.5}$$

Proof. By Entries 9.2.1 and 9.2.2,

$$\begin{aligned}
\phi(a) &= \frac{b-1}{b+a} + \frac{1}{b+a}\phi(a/q) \\
&= \frac{b-1}{b+a} + \frac{1}{b+a}\sum_{n=0}^{\infty} \frac{(-a/b;q)_n a^n b^n q^{n(3n-1)/2}(1+aq^{2n})}{(bq;q)_n} \\
&= 1 + \sum_{n=1}^{\infty} \frac{(-aq/b;q)_{n-1} a^n b^{n-1} q^{n(3n-1)/2}(1+aq^{2n})}{(bq;q)_n},
\end{aligned}$$

as desired. □

Entry 9.2.4 (p. 36). *We have*

$$\sum_{n=0}^{\infty} \frac{(-1)^n a^{2n} q^{n^2}}{(a^2q^2;q^2)_n} = 1 - a\sum_{n=1}^{\infty} \frac{a^n q^n}{(-aq;q)_n}. \tag{9.2.6}$$

Proof. In (9.2.1), set $\tau = a$, $\beta = -aq$, and $\alpha = 0$. Thus,

$$\sum_{n=0}^{\infty} \frac{a^n}{(-aq;q)_n} = \frac{1}{1-a}\sum_{n=0}^{\infty} \frac{(-1)^n a^{2n} q^{n^2}}{(-aq;q)_n (aq;q)_n}.$$

Consequently,

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-1)^n a^{2n} q^{n^2}}{(a^2q^2;q^2)_n} &= (1-a)\sum_{n=0}^{\infty} \frac{a^n}{(-aq;q)_n} \\
&= 1 + \sum_{n=1}^{\infty} \frac{a^n}{(-aq;q)_n} - \sum_{n=0}^{\infty} \frac{a^{n+1}}{(-aq;q)_n} \\
&= 1 + \sum_{n=1}^{\infty} \frac{a^n}{(-aq;q)_n}\left(1 - (1+aq^n)\right) \\
&= 1 - a\sum_{n=1}^{\infty} \frac{a^n q^n}{(-aq;q)_n},
\end{aligned}$$

which completes the proof. □

It is curious that the terms in a and q on the right side of (9.2.6) are the same as those on the left side, but with the powers diminished.

Entry 9.2.5 (p. 32). *We have*

$$\sum_{n=0}^{\infty} \frac{(-1)^n a^n q^n}{(aq;q^2)_{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n} q^{2n^2+2n}}{(a^2q^2;q^4)_{n+1}}. \tag{9.2.7}$$

Proof. In (9.1.1), replace q by q^2, then set $\beta = aq^3$, $\tau = -aq$, and $\alpha = 0$, and multiply both sides by $1/(1-aq)$. Thus, (9.2.7) is proved. □

Entry 9.2.6 (p. 30). *We have*

$$aq(-aq;q^2)_\infty \sum_{n=0}^{\infty} \frac{(-1)^n a^n q^{n(n+1)/2}}{(-a;q)_{n+1}} = \sum_{n=0}^{\infty} \frac{a^{n+1} q^{(n+1)^2}}{(q^2;q^2)_n(1+aq^{2n})}.$$

Proof. In Heine's transformation, Entry 6 of Chapter 16 in Ramanujan's second notebook [227], [61, p. 15],

$$\sum_{n=0}^{\infty} \frac{(b/a;q)_n(c;q)_n}{(d;q)_n(q;q)_n} a^n = \frac{(b;q)_\infty(c;q)_\infty}{(a;q)_\infty(d;q)_\infty} \sum_{n=0}^{\infty} \frac{(d/c;q)_n(a;q)_n}{(b;q)_n(q;q)_n} c^n, \tag{9.2.8}$$

replace q by q^2, a by t, b by $-aq^3$, c by $-a$, and d by $-aq^2$. Then let $t \to 0$ to conclude that

$$\sum_{n=0}^{\infty} \frac{a^n q^{n^2+2n}}{(q^2;q^2)_n(1+aq^{2n})} = (-aq;q^2)_\infty \sum_{m=0}^{\infty} \frac{(-a)^m}{(-aq;q^2)_{m+1}}.$$

So, to finish our proof, we need only show that

$$\sum_{m=0}^{\infty} \frac{(-a)^m}{(-aq;q^2)_{m+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n a^n q^{n(n+1)/2}}{(-a;q)_{n+1}}. \tag{9.2.9}$$

By (9.1.1) with q replaced by q^2, $\alpha = 0$, $\tau = -a$, $b = -aq^3$, and both sides multiplied by $1/(1+aq)$, we see that

$$\begin{aligned}
\sum_{m=0}^{\infty} \frac{(-a)^m}{(-aq;q^2)_{m+1}} &= \sum_{m=0}^{\infty} \frac{a^{2m} q^{2m^2+m}}{(-aq;q^2)_{m+1}(-a;q^2)_{m+1}} \\
&= \sum_{m=0}^{\infty} \frac{a^{2m} q^{2m^2+m}}{(-a;q)_{2m+2}} \\
&= \sum_{m=0}^{\infty} \frac{a^{2m} q^{2m^2+m}\left((1+aq^{2m+1}) - aq^{2m+1}\right)}{(-a;q)_{2m+2}} \\
&= \sum_{m=0}^{\infty} \frac{a^{2m} q^{2m^2+m}}{(-a;q)_{2m+1}} - \sum_{m=0}^{\infty} \frac{a^{2m+1} q^{2m^2+3m+1}}{(-a;q)_{2m+2}}
\end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n a^n q^{n(n+1)/2}}{(-a;q)_{n+1}},$$

which completes the proof of (9.2.9) and therefore also of Entry 9.2.6. $\square$

9.3 The Series $\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}$

It is quite surprising how many changes Ramanujan was able to ring on instances of the false theta series in the title of this section. While some of these are only remotely related to the Rogers–Fine identity, they are, nonetheless, so closely related to each other that it becomes compelling to record them here. We remark that Entries 9.3.2–9.3.7 were first proved in [25, Section 6].

Entry 9.3.1 (p. 29). *We have*

$$\sum_{n=0}^{\infty} \frac{(-aq;q^2)_n(-aq)^n}{(-aq^2;q^2)_n} = \sum_{n=0}^{\infty}(-a)^n q^{n(n+1)/2}. \tag{9.3.1}$$

First Proof of Entry 9.3.1. In (9.1.1), replace q by q^2 and then set $\alpha = \tau = -aq$ and $\beta = -aq^2$. Hence,

$$\sum_{n=0}^{\infty} \frac{(-aq;q^2)_n(-aq)^n}{(-aq^2;q^2)_n} = \sum_{n=0}^{\infty} a^{2n} q^{2n^2+n}(1-aq^{2n+1}) = \sum_{n=0}^{\infty}(-a)^n q^{n(n+1)/2},$$

as desired. $\square$

Second Proof of Entry 9.3.1. This theorem is difficult to prove combinatorially. We employ the concept of modular partitions first introduced by P.A. MacMahon [185], [21, p. 13]. Let m and k be positive integers. Then there exist $h \geq 0$ and $0 < j \leq k$ such that $m = kh + j$. Using the terminology of arithmetic progressions, we call k the modulus. A modular partition is a modification of the Ferrers graph such that part m is represented by a row of h k's and one j.

On the left side of (9.3.1), $(aq;q^2)_n/(aq^2;q^2)_n$ generates modular partitions $\lambda^{(1)}$, where the parts are less than or equal to n and the parts ending with 1 are distinct, and aq^n generates a partition $\lambda^{(2)}$ of only one part n. We form a new partition λ, whose Ferrers graph has boxes of either 1 or 2, by putting the Ferrers graph of $\lambda^{(1)}$ immediately below that of $\lambda^{(2)}$. For example, when $n = 3$, let $\lambda^{(1)} = 6 + 6 + 3 + 2 + 1$ and $\lambda^{(2)} = 1 + 1 + 1$ be given. Then we obtain λ with the Ferrers graph below. It is easily seen that λ is generated by the left side of (9.3.1). Note that the exponent of a represents the sum of the size of the top row of λ and the number of rows below the top row, and λ has its sign defined by $(-1)^o$, where o is the number of boxes with 1 in the rows below the top row.

1	1	1
2	2	2
2	2	2
2	1	
2		
1		

We define a sign-reversing involution as follows. Let s_1 and s_2 be the last column and last row of the Ferrers graph of λ, respectively. We divide the proof into three cases: $s_1 < s_2$, $s_2 < s_1$, and $s_1 = s_2$. Here, for example, $s_1 < s_2$ means that the sum of the elements in the boxes of s_1 is less than the sum of the elements in the boxes of s_2.

Case 1: $s_1 < s_2$. If the box in the last square of s_1 contains a 2, then put s_1 immediately below s_2 with the entries arranged in weakly decreasing order. If both the first and last boxes of s_1 have 1, then remove the first box and change 1 in the last box to 2. Move s_1 immediately below s_2, so that boxes of s_1 are in weakly decreasing order. If s_1 has only one box of 1, then the box produces an additional negative sign after the move. If s_1 has one or two boxes of 1, then the move results in losing a negative sign. In summary, each move changes the sign.

Case 2: $s_1 > s_2$. If s_2 has no box with a 1, then add an additional box with 1 in front of the first box and change 2 in the last box to 1. Move s_2 immediately to the right of s_1, so that the first box has 1 and the other boxes are in weakly decreasing order. This move changes the sign as well.

Case 3: $s_1 = s_2$. We separate two cases: when s_1 and s_2 are even and when s_1 and s_2 are odd. If s_1 and s_2 are odd, move s_2 right next to s_1, so that the box with 1 goes to the top. If s_1 and s_2 are even, then s_1 must have two boxes of 1. Remove the first box of s_1, change 1 to 2, and move s_1 to immediately below s_2, so that the box of 1 is rightmost. The move changes the sign.

In each case, we see that the sign of a partition changes under the map. Thus the map is a sign-reversing involution, which results in cancellations among such partitions.

On the other hand, there are certain partitions for which none of the moves described in Cases 1–3 is possible. These are the partitions whose Ferrers graphs are $(j+1) \times j$ rectangles or $(j+1) \times (j+1)$ rectangles for some $j \geq 1$. Furthermore, these graphs contain boxes of 1's at the top row and boxes of 2's in the other rows and so have no image under the maps described above. These are partitions of $2r^2 + r = 2r(2r+1)/2$ or $2r^2 + 3r + 1 = (2r+1)(2r+2)/2$

elements. These are counted on the right side of (9.3.1). On both sides the power of a equals the largest part plus the number of parts minus 1. □

Entry 9.3.2 (p. 13). *We have*

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2+n}(q;q^2)_n}{(-q;q)_{2n+1}}=\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}. \tag{9.3.2}$$

Proof. We first record a transformation formula of Andrews [12, p. 67, Theorem A$_3$], namely,

$$\sum_{n=0}^{\infty}\frac{(a;q^2)_n(b;q)_{2n}}{(q^2;q^2)_n(c;q)_{2n}}t^n=\frac{(b;q)_\infty(at;q^2)_\infty}{(c;q)_\infty(t;q^2)_\infty}\sum_{n=0}^{\infty}\frac{(c/b;q)_n(t;q^2)_n}{(q;q)_n(at;q^2)_n}b^n. \tag{9.3.3}$$

In (9.3.3), replace c by $-q^2$, b by q, and t by q^2/a, multiply both sides by $1/(1+q)$, and then let $a\to\infty$ to deduce that

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2+n}(q;q^2)_n}{(-q;q)_{2n+1}}=(q;q)_\infty^2\sum_{m=0}^{\infty}\frac{q^m}{(q;q)_m^2}. \tag{9.3.4}$$

Next, recall Heine's transformation (9.2.8). Set $b=c=0$ and $d=a=q$ in (9.2.8) to find that

$$\sum_{n=0}^{\infty}\frac{q^n}{(q;q)_n^2}=\frac{1}{(q;q)_\infty^2}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}. \tag{9.3.5}$$

Combining (9.3.4) and (9.3.5), we complete the proof. □

Entry 9.3.3 (p. 13). *We have*

$$\sum_{n=0}^{\infty}\frac{(q;q^2)_n q^n}{(-q;q^2)_{n+1}}=\sum_{n=0}^{\infty}(-1)^n q^{2n(n+1)}. \tag{9.3.6}$$

Proof. In (9.1.1), replace q by q^2, next set $\alpha=\tau=q$ and $\beta=-q^3$, and lastly multiply both sides by $1/(1+q)$. After an enormous amount of simplification, we deduce (9.3.6). □

Entry 9.3.4 (p. 13). *We have*

$$\sum_{n=0}^{\infty}\frac{(q;q^2)_n^2 q^n}{(-q;q)_{2n+1}}=\sum_{n=0}^{\infty}(-1)^n q^{n^2+n}. \tag{9.3.7}$$

Proof. Replacing q by q^2 in this entry, we see that the right-hand side is identical with the right-hand side in Entry 9.3.3. Hence, we need only to prove that

$$\sum_{n=0}^{\infty} \frac{(q^2;q^4)_n^2 q^{2n}}{(-q^2;q^2)_{2n+1}} = \sum_{n=0}^{\infty} \frac{(q;q^2)_n q^n}{(-q;q^2)_{n+1}}. \tag{9.3.8}$$

To prove (9.3.8), we begin by replacing q by q^2, then setting $a = t = b = q^2$, and $c = -q^4$, and lastly multiplying both sides by $1/(1+q^2)$ in Andrews's theorem (9.3.3). It follows that

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(q^2;q^4)_n^2 q^{2n}}{(-q^2;q^2)_{2n+1}} &= (q^2;q^2)_\infty (q^4;q^4)_\infty \sum_{m=0}^{\infty} \frac{(q;q^2)_m(-q;q^2)_m q^{2m}}{(q^2;q^2)_m^2} \\ &= \sum_{n=0}^{\infty} \frac{(q;q^2)_n q^n}{(-q;q^2)_{n+1}},\end{aligned}$$

by Heine's transformation (9.2.8) with q replaced by q^2, with $a = q^2$, $d = q^2$, $c = q$, and $b = -q^3$, and lastly with both sides of the resulting equality multiplied by $1/(1+q)$. Hence, (9.3.8) follows, and the proof is complete. □

Recall the definition of the Gaussian binomial coefficient

$$\begin{bmatrix} n \\ m \end{bmatrix} = \begin{bmatrix} n \\ m \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_m (q;q)_{n-m}},$$

where $0 \le m \le n$. Recall also the following two special instances of the q-binomial theorem. First [21, p. 36, equation (3.3.6)],

$$\sum_{j=0}^{N} \begin{bmatrix} N \\ j \end{bmatrix} (-1)^j z^j q^{j(j-1)/2} = (z;q)_N; \tag{9.3.9}$$

second [21, p. 19, equation (2.2.5)],

$$\sum_{n=0}^{\infty} \frac{t^n}{(q;q)_n} = \frac{1}{(t;q)_\infty}. \tag{9.3.10}$$

Entry 9.3.5 (p. 12). *We have*

$$\sum_{n=0}^{\infty} \frac{(-aq^{n+1};q)_n q^n}{(q;q)_n} = \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} a^n q^{3n(n+1)/2}. \tag{9.3.11}$$

Proof. Employing the q-binomial theorem (9.3.9), replacing n by $j+m$, and lastly invoking (9.3.10), we find that

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-aq^{n+1};q)_n q^n}{(q;q)_n} &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} \begin{bmatrix} n \\ m \end{bmatrix} \frac{a^m q^{nm+n+m(m+1)/2}}{(q;q)_n} \\ &= \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \frac{a^m q^{nm+n+m(m+1)/2}}{(q;q)_m (q;q)_{n-m}}\end{aligned}$$

$$= \sum_{m=0}^{\infty}\sum_{j=0}^{\infty} \frac{a^m q^{(j+m)m+j+m+m(m+1)/2}}{(q;q)_m (q;q)_j}$$
$$= \sum_{m=0}^{\infty} \frac{a^m q^{3m(m+1)/2}}{(q;q)_m} \cdot \frac{1}{(q^{m+1};q)_\infty}$$
$$= \frac{1}{(q;q)_\infty} \sum_{n=0}^{\infty} a^n q^{3n(n+1)/2}.$$

This completes the proof. □

We note that (9.3.11) was proved in [25, p. 159, Lemma 2] but was inadvertently not attributed to Ramanujan, and we also note that P. Hammond [147] has independently discovered and generalized (9.3.11).

Entry 9.3.6 (p. 13). *We have*

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_n q^n}{(-q;q)_{2n+1}} = \sum_{n=0}^{\infty} (-1)^n q^{3n(n+1)/2}. \tag{9.3.12}$$

Proof. By (9.3.3) with $a = 0$, $b = t = q$, and $c = -q^2$, and with both sides multiplied by $1/(1+q)$,

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_n q^n}{(-q;q)_{2n+1}} = \frac{(q;q)_\infty}{(-q;q)_\infty (q;q^2)_\infty} \sum_{m=0}^{\infty} \frac{(-q;q)_m (q;q^2)_m q^m}{(q;q)_m}. \tag{9.3.13}$$

Now it is easily verified that for each positive integer m,

$$(-q;q)_m (q;q^2)_m = (q^{m+1};q)_m.$$

Thus, (9.3.13) can be written in the form

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_n q^n}{(-q;q)_{2n+1}} = (q;q)_\infty \sum_{m=0}^{\infty} \frac{(q^{m+1};q)_m q^m}{(q;q)_m} = \sum_{n=0}^{\infty} (-1)^n q^{3n(n+1)/2},$$

by Entry 9.3.5. □

Entry 9.3.7 (p. 13). *With q replaced by $-q$ in Ramanujan's formulation in* [228]*,*

$$\sum_{n=0}^{\infty} \frac{(-1)^n (-q;q)_n q^n}{(q;q^2)_{n+1}} = \sum_{n=0}^{\infty} (-1)^n q^{3n(n+1)}.$$

Proof. By Entry 9.3.5 with q replaced by q^2,

$$\sum_{n=0}^{\infty} (-1)^n q^{3n(n+1)} = (q^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{(q^2;q^2)_{2n} q^{2n}}{(q^2;q^2)_n^2}$$

$$= (q^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{(q;q)_{2n}(-q;q)_{2n}q^{2n}}{(q^2;q^2)_n^2}$$
$$= (q^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{(q;q^2)_n(-q;q)_{2n}q^{2n}}{(q^2;q^2)_n}$$
$$= \sum_{n=0}^{\infty} \frac{(-q;q)_n(-q)^n}{(q;q^2)_{n+1}},$$

where we have applied (9.3.3) with $a = q$, $b = -q$, $c = 0$, and $t = q^2$, and then multiplied both sides of the resulting equality by $1/(1-q)$. This concludes the proof of Entry 9.3.7. □

9.4 The Series $\sum_{n=0}^{\infty} q^{n(3n+1)/2}(1-q^{2n+1})$.

While Section 9.3 was based on a false theta series variation of a famous theta function of Gauss associated with the triangular numbers, this section is devoted to a false theta variation on Euler's pentagonal number series. We remark that Entries 9.4.1 and 9.4.2 first appeared in [235, Section 10]. Entry 9.4.1 may also be derived from two entries in S.O. Warnaar's paper [284], where analytic methods are employed. Replace a by $-aq$ in the identity at the top of page 388 in [284] and multiply it by aq. Add this resulting identity to the identity above (6.14) on page 390. After simplification, (9.4.1) follows. Entry 9.4.2 is equivalent to (6.15) in Warnaar's paper [284], with q replaced by q^2 there. Entry 9.4.3 appears in [13, Section 5].

Entry 9.4.1 (p. 37). *For any complex number a,*

$$\sum_{n=0}^{\infty} \frac{(-1)^n a^{2n} q^{n(n+1)/2}}{(-aq;q)_n} = \sum_{n=0}^{\infty} a^{3n} q^{n(3n+1)/2}(1-a^2q^{2n+1}). \tag{9.4.1}$$

First Proof of Entry 9.4.1. In (9.1.1), set $\alpha = a^2q/\tau$ and $\beta = -aq$, and then let $\tau \to 0$. The desired result follows. □

Second Proof of Entry 9.4.1. Replace $-a$ by a in (9.4.1). Then,

$$\sum_{n=0}^{\infty} \frac{(-1)^n a^{2n} q^{n(n+1)/2}}{(aq;q)_n} = \sum_{n=0}^{\infty}(-1)^n a^{3n} q^{n(3n+1)/2}(1-a^2q^{2n+1})$$
$$= \sum_{n=0}^{\infty}(-1)^n a^{3n} q^{n(3n+1)/2} + \sum_{n=1}^{\infty}(-1)^n a^{3n-1} q^{n(3n-1)/2}. \tag{9.4.2}$$

This entry then immediately follows from the Franklin involution. Note that the power of a on both sides of (9.4.2) gives the number of parts plus the largest part. This completes the proof. □

M.V. Subbarao [267] was the first to recognize the possibility of refining the Franklin involution in this way. See also [20, Section 3].

Entry 9.4.2 (p. 37). *We have*

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(-q;q)_n} = \sum_{n=0}^{\infty} q^{n(3n+1)/2}(1-q^{2n+1}). \tag{9.4.3}$$

Proof. Set $a=1$ in Entry 9.4.1. □

Entry 9.4.3 (p. 37). *We have*

$$\sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(-q;q)_{2n+1}} = \sum_{n=0}^{\infty} q^{n(3n+1)/2}(1-q^{2n+1}). \tag{9.4.4}$$

Proof. It suffices to show that the left sides of (9.4.3) and (9.4.4) are equal. To that end,

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(-q;q)_n} &= \sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(-q;q)_{2n}} - \sum_{n=0}^{\infty} \frac{q^{(n+1)(2n+1)}}{(-q;q)_{2n+1}} \\ &= \sum_{n=0}^{\infty} \frac{q^{n(2n+1)}\left((1+q^{2n+1})-q^{2n+1}\right)}{(-q;q)_{2n+1}} \\ &= \sum_{n=0}^{\infty} \frac{q^{n(2n+1)}}{(-q;q)_{2n+1}},\end{aligned}$$

and so the proof is finished. □

Entry 9.4.4 (p. 37). *We have*

$$2-\sum_{n=0}^{\infty} \frac{q^{n(2n-1)}}{(-q;q)_{2n}} = \sum_{n=0}^{\infty} q^{n(3n+1)/2}(1-q^{2n+1}). \tag{9.4.5}$$

Proof. It suffices to show that the left side of (9.4.5) is identical to the left side of (9.4.3). Thus,

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{(-q;q)_n} &= 1-\sum_{n=1}^{\infty} \frac{q^{n(2n-1)}}{(-q;q)_{2n-1}} + \sum_{n=1}^{\infty} \frac{q^{n(2n+1)}}{(-q;q)_{2n}} \\ &= 1-\sum_{n=1}^{\infty} \frac{q^{n(2n-1)}\left((1+q^{2n})-q^{2n}\right)}{(-q;q)_{2n}} \\ &= 2-\sum_{n=0}^{\infty} \frac{q^{n(2n-1)}}{(-q;q)_{2n}},\end{aligned}$$

and the proof is complete. □

We note that (9.4.1) was first given by Rogers [235], and the equivalence of (9.4.3), (9.4.4), and (9.4.5) was proved combinatorially in [15, p. 38]. Identity (9.4.5) was also proved in [13, p. 140].

Entry 9.4.5 (p. 39). *If*

$$g(a) := \sum_{n=0}^{\infty} a^{3n} q^{n(3n+1)/2}(1 - a^2 q^{2n+1}), \tag{9.4.6}$$

then

$$\sum_{n=0}^{\infty} \frac{(-1)^n a^{4n+3} q^{2(n+1)^2}}{(-a^2q^3;q^4)_{n+1}} = \frac{1}{2}\left(g(a) - g(-a)\right). \tag{9.4.7}$$

Proof. In (9.1.1), replace q by q^4, set $\alpha = a^4q^6/\tau$ and $\beta = -a^2q^7$, let $\tau \to 0$, and lastly multiply both sides by $a^3q^2/(1+a^2q^3)$. It follows that

$$\sum_{n=0}^{\infty} \frac{(-1)^n a^{4n+3} q^{2(n+1)^2}}{(-a^2q^3;q^4)_{n+1}} = \sum_{n=0}^{\infty} a^{6n+3} q^{6n^2+7n+2}(1 - a^2 q^{4n+3}). \tag{9.4.8}$$

On the other hand,

$$\begin{aligned}\frac{1}{2}\left(g(a) - g(-a)\right) &= \sum_{n=0}^{\infty} a^{3(2n+1)} q^{(2n+1)(6n+4)/2}(1 - a^2 q^{2(2n+1)+1})\\ &= \sum_{n=0}^{\infty} a^{6n+3} q^{6n^2+7n+2}(1 - a^2 q^{4n+3}). \end{aligned}\tag{9.4.9}$$

Comparing (9.4.8) and (9.4.9), we deduce (9.4.7). □

Entry 9.4.6 (p. 39). *If* $g(a)$ *is defined by* (9.4.6), *then*

$$\sum_{n=0}^{\infty} \frac{(-1)^n a^{4n} q^{2n^2}}{(-a^2q;q^4)_{n+1}} = \frac{1}{2}\left(g(a) + g(-a)\right). \tag{9.4.10}$$

First Proof of Entry 9.4.6. In (9.1.1), replace q by q^4, then set $\alpha = a^4q^2/\tau$ and $\beta = -a^2q^5$, let $\tau \to 0$, and multiply both sides by $1/(1+a^2q)$. This yields

$$\sum_{n=0}^{\infty} \frac{(-1)^n a^{4n} q^{2n^2}}{(-a^2q;q^4)_{n+1}} = \sum_{n=0}^{\infty} a^{6n} q^{6n^2+n}(1 - a^2 q^{4n+1}). \tag{9.4.11}$$

On the other hand,

$$\begin{aligned}\frac{1}{2}\left(g(a) + g(-a)\right) &= \sum_{n=0}^{\infty} a^{3(2n)} q^{2n(3\cdot(2n)+1)/2}(1 - a^2 q^{2(2n)+1})\\ &= \sum_{n=0}^{\infty} a^{6n} q^{6n^2+n}(1 - a^2 q^{4n+1}). \end{aligned}\tag{9.4.12}$$

A comparison of (9.4.11) and (9.4.12) produces the desired result. □

Second Proof of Entry 9.4.6. A.J. Yee has indicated to us a second proof. By the definition (9.4.6) of $g(a)$, we find that (9.4.10) is equivalent to

$$\begin{aligned}\frac{1}{2}(g(a)+g(-a)) &= \frac{1}{2}\sum_{n=0}^{\infty}(1+(-1)^n)\,a^{3n}q^{n(3n+1)/2}(1-a^2q^{2n+1})\\ &= \sum_{n=0}^{\infty}(a^{6n}q^{6n^2+n}-a^{6n+2}q^{6n^2+5n+1})\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n a^{4n}q^{2n^2}}{(-a^2q;q^4)_{n+1}}.\end{aligned}$$

Replacing a^2q and q^2 by a and q, respectively, we find that

$$\sum_{n=0}^{\infty}(a^{3n}q^{n(3n-1)}-a^{3n+1}q^{n(3n+1)})=\sum_{n=0}^{\infty}\frac{(-1)^n a^{2n}q^{n(n-1)}}{(-a;q^2)_{n+1}}. \tag{9.4.13}$$

However, if we take (9.4.7), divide both sides by a^3q^2, replace a^2q and q^2 by a and q, respectively, and then lastly replace aq by a, we obtain (9.4.13), and so the proof is complete. □

Entry 9.4.7 (p. 36). *We have*

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^n}{(-q^2;q^2)_n}=\sum_{n=0}^{\infty}q^{n(3n+1)/2}(1-q^{2n+1}). \tag{9.4.14}$$

Proof. In (9.1.1), replace q by q^2 and then set $\alpha=0$, $\beta=-q^2$, and $\tau=-q$. This yields

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^n}{(-q^2;q^2)_n}=\sum_{n=0}^{\infty}\frac{q^{2n^2+n}}{(-q;q)_{2n+1}}, \tag{9.4.15}$$

and the result follows from Entry 9.4.3. □

Entry 9.4.8 (p. 41). *We have*

$$1+2\sum_{n=1}^{\infty}\frac{q^{n^2+2n}}{(q^2;q^2)_{n-1}(1-q^{4n})}=(-q;q^2)_\infty\sum_{n=0}^{\infty}q^{n(3n+1)/2}(1-q^{2n+1}). \tag{9.4.16}$$

Proof. In Heine's transformation (9.2.8), set $b=\beta$, $c=\tau$, and $d=\tau q$, and then let $a\to 0$. Hence,

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(-1)^n\beta^n q^{n(n-1)/2}}{(q;q)_n(1-\tau q^n)} &= (\beta;q)_\infty\sum_{n=0}^{\infty}\frac{\tau^n}{(\beta;q)_n}\\ &= (\beta;q)_\infty\sum_{n=0}^{\infty}\frac{\beta^n\tau^n q^{n^2-n}}{(\beta;q)_n(\tau;q)_{n+1}},\end{aligned} \tag{9.4.17}$$

where we applied (9.1.1) with $\alpha = 0$.

Hence, replacing q by q^2 in (9.4.17) and then setting $\tau = -1$ and $\beta = -q^3$, we see that

$$\begin{aligned}
1 + 2\sum_{n=1}^{\infty} \frac{q^{n^2+2n}}{(q^2;q^2)_{n-1}(1-q^{4n})} &= 2\sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q^2;q^2)_n(1+q^{2n})} \\
&= 2(-q;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{2n^2+n}}{(-q;q^2)_{n+1}(-1;q^2)_{n+1}} \\
&= (-q;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{2n^2+n}}{(-q;q)_{2n+1}} \\
&= (-q;q^2)_\infty \sum_{n=0}^{\infty} q^{n(3n+1)/2}(1-q^{2n+1}),
\end{aligned}$$

by Entry 9.4.3. This concludes the derivation. □

Entry 9.4.9 (p. 29). *We have*

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_n q^n}{(-q;q)_{2n}} = \sum_{n=0}^{\infty} (-1)^n q^{n(3n+1)/2}(1+q^{2n+1}). \tag{9.4.18}$$

Proof. In (9.3.3), set $a = 0$, $b = t = q$, and $c = -q$. Consequently,

$$\sum_{n=0}^{\infty} \frac{(q;q^2)_n q^n}{(-q;q)_{2n}} = (q;q)_\infty \sum_{m=0}^{\infty} \frac{(-1;q)_m (q;q^2)_m q^m}{(q;q)_m}. \tag{9.4.19}$$

Now in (9.3.11) we first substitute $a = -1/q$ and then $a = -q$. Adding the two results after multiplying the second by q, we see that

$$\begin{aligned}
&\sum_{n=0}^{\infty} (-1)^n q^{n(3n+1)/2}(1+q^{2n+1}) \\
&\quad = (q;q)_\infty \left(1 + \sum_{n=1}^{\infty} \frac{(q;q)_{2n-1} q^n}{(q;q)_n (q;q)_{n-1}} + \sum_{n=0}^{\infty} \frac{(q;q)_{2n+1} q^{n+1}}{(q;q)_{n+1}(q;q)_n}\right) \\
&\quad = (q;q)_\infty \left(1 + 2\sum_{n=0}^{\infty} \frac{(q;q)_{2n+1} q^{n+1}}{(q;q)_n (q;q)_{n+1}}\right) \\
&\quad = (q;q)_\infty \left(1 + 2\sum_{n=0}^{\infty} \frac{(q;q^2)_{n+1}(-q;q)_n q^{n+1}}{(q;q)_{n+1}}\right) \\
&\quad = (q;q)_\infty \sum_{n=0}^{\infty} \frac{(-1;q)_n (q;q^2)_n q^n}{(q;q)_n}.
\end{aligned} \tag{9.4.20}$$

Comparing (9.4.20) with (9.4.19), we see that we have completed the proof. □

We note that (9.4.18) was proved in [31].

9.5 The Series $\sum_{n=0}^{\infty} q^{3n^2+2n}(1-q^{2n+1})$

The five entries to be considered in this section are in a sense natural companions to (9.4.18). However, given that this sequence of exponents $\{3n^2 \pm 2n\}$ is not the pentagonal number sequence $\{(3n^2 \pm n)/2\}$, it seems reasonable to consider these results separately.

Earlier in this chapter, in our second proof of Entry 9.4.1, we demonstrated that some entries in this chapter can be established combinatorially by appealing to Franklin's involution. In this section we give another example, but in a different kind of setting. In our application below, we consider a variation of Ferrers graphs by putting either 0 or 1 or 2 into boxes. We put 0 in the box at the upper left corner, 1's into the boxes either in the first row or column, and 2's in each box except those in the first row and column. Such a Ferrers graph represents a partition of n, where n equals the sum of all numbers in the boxes. For example, the figure below is the Ferrers graph of a partition of 16.

0	1	1	1	1	1
1	2	2	2		
1	2				
1					

Entry 9.5.1 below is identical to the identity at the top of page 388 in [284], if in [284] we replace q by q^2 and then replace a by $-aq$. As pointed out in [284], this identity yields an identity of Rogers [235, p. 333, equation (4)]. The methods of both Warnaar [284] and Rogers [235] are analytic.

Entry 9.5.1 (p. 37). *For any complex number a,*

$$\sum_{n=0}^{\infty} \frac{(-1)^n a^{2n} q^{n(n+1)}}{(-aq;q^2)_{n+1}} = \sum_{n=0}^{\infty} a^{3n} q^{3n^2+2n}(1-aq^{2n+1}).$$

First Proof of Entry 9.5.1. In (9.1.1), replace q by q^2, then set $\beta = -aq^3$ and $\alpha = a^2q^2/\tau$, and let $\tau \to 0$. The desired result then follows. □

Second Proof of Entry 9.5.1. We rewrite the identity as

$$\sum_{n=0}^{\infty} \frac{(-1)^n (aq)^{2n} q^{n(n-1)}}{(-aq;q^2)_{n+1}} = \sum_{n=0}^{\infty} ((aq)^{3n} q^{n(3n-1)} - (aq)^{3n+1} q^{n(3n+1)}).$$

The left side of the identity above generates partitions described above with distinct rows and weight $(-1)^{c-1}$, where c denotes the number of columns of the Ferrers graph. Moreover, a keeps track of each box with a 1 in it. By applying the Franklin involution, we obtain the right side. □

Entry 9.5.2 (p. 29). *We have*

$$\sum_{n=0}^{\infty}(q;q^2)_n q^n = \sum_{n=0}^{\infty}(-1)^n q^{3n^2+2n}(1+q^{2n+1}).$$

Proof. We apply the second iterate of Heine's transformation [61, p. 15, second line of equation (6.1)]

$$\sum_{n=0}^{\infty}\frac{(a;q)_n(b;q)_n}{(c;q)_n(q;q)_n}t^n = \frac{(c/b;q)_\infty(bt;q)_\infty}{(c;q)_\infty(t;q)_\infty}\sum_{n=0}^{\infty}\frac{(b;q)_n(abt/c;q)_n}{(bt;q)_n(q;q)_n}\left(\frac{c}{b}\right)^n. \quad (9.5.1)$$

In (9.5.1), replace q by q^2, then set $a=q$, $b=q^2$, and $t=q$, and lastly let $c\to 0$. Consequently,

$$\sum_{n=0}^{\infty}(q;q^2)_n q^n = \sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2+n}}{(-q;q^2)_{n+1}} = \sum_{n=0}^{\infty} q^{3n^2+2n}(1-q^{2n+1}),$$

where we put $a=1$ in Entry 9.5.1. □

Entry 9.5.3 (p. 37). *We have*

$$\sum_{n=0}^{\infty}\frac{q^n}{(-q;q^2)_{n+1}} = \sum_{n=0}^{\infty}(-1)^n q^{6n^2+4n}(1+q^{4n+2}). \quad (9.5.2)$$

Proof. In (9.1.1), replace q by q^2, then set $\alpha=0$, $\beta=-q^3$, and $\tau=q$, and lastly multiply both sides by $1/(1+q)$. Thus,

$$\sum_{n=0}^{\infty}\frac{q^n}{(-q;q^2)_{n+1}} = \sum_{n=0}^{\infty}\frac{(-1)^n q^{2n^2+2n}}{(q^2;q^4)_{n+1}},$$

and (9.5.2) now follows from Entry 9.5.1 with q replaced by q^2 and $a=-1$ in that entry. □

Entry 9.5.3 has also been proved by Andrews [22, equation (1.2)] and Warnaar [284, third equation, p. 380]. Following his proof of (9.5.2), Andrews [22, p. 100] remarked, "It would be nice to have a cominatorial proof of this result." Berndt and Yee [85] have found such a combinatorial proof.

Entry 9.5.4 (p. 36). *We have*

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^n}{(q;q^2)_{n+1}} = \sum_{n=0}^{\infty}(-1)^n q^{6n^2+4n}(1+q^{4n+2}). \quad (9.5.3)$$

Proof. Equation (9.5.3) is simply (9.5.2) with q replaced by $-q$. ☐

Entry 9.5.5 (p. 30). *We have*

$$\sum_{n=0}^{\infty} \frac{(-q^2)^{n(n+1)/2}}{(-q;-q^2)_n} = \sum_{n=0}^{\infty} (-1)^{n(n+1)/2} q^{3n^2+2n}(1-(-1)^n q^{2n+1}).$$

Proof. This result follows from Entry 9.5.1 if we set $a=-i$ and replace q by iq. ☐

10

An Empirical Study of the Rogers–Ramanujan Identities

10.1 Introduction

On pages 358–361 of *The Lost Notebook and Other Unpublished Papers* [228], we find fragments (possibly from a letter or letters) by Ramanujan on empirical evidence for the Rogers–Ramanujan identities and related formulas. Recall that the Rogers–Ramanujan identities are given for $|q| < 1$ by

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty} \tag{10.1.1}$$

and

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q;q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty}. \tag{10.1.2}$$

The history of these identities is now well known, and for this history, proofs, and surveys of proofs, we refer readers to the notes in Ramanujan's *Collected Papers* [226, pp. 344–346], G.H. Hardy's book [148, pp. 90–99], Andrews's book [21, Chapter 7], Andrews's survey [30], Berndt's book [61, p. 77], and his survey with Y.–S. Choi and S.–Y. Kang [77] of Ramanujan's problems in the *Journal of the Indian Mathematical Society*. In the following pages, we consider the four indirect arguments that Ramanujan provides in support of (10.1.1). All four arguments are found on page 358, with some details supporting the first argument given on pages 359–361. For each of the four assertions, we quote Ramanujan at the beginning of the corresponding section below.

10.2 The First Argument

"Mr. MacMahon has verified up to q^{55} and found the result correct up to that term."

To support this appeal to authority, Ramanujan calculates both sides of (10.1.1) through q^{36}. He does this by successively adding terms in the manner we now describe. If we define

$$\frac{\nu(N)}{(q)_N} := \sum_{j=0}^{N} \frac{q^{j^2}}{(q)_j}, \qquad N \geq 0, \tag{10.2.1}$$

then we can easily determine $\nu(N)$ from the recurrence relation

$$\nu(N) = (1-q^N)\nu(N-1) + q^{N^2}, \qquad N \geq 1, \qquad \nu(0) = 1. \tag{10.2.2}$$

He calculates congruences up to $N = 7$, with the last for $\nu(7)$ being

$$\begin{aligned}\nu(7) \equiv 1 - q^2 - q^3 + q^8 + 2q^9 - q^{12} - q^{13} - q^{14} - q^{15} + q^{17} + q^{18} + q^{19}\\ + q^{20} - q^{21} - q^{23} - q^{24} + q^{27} + q^{28} - q^{32} - q^{34} + q^{36} \pmod{q^{37}}.\end{aligned} \tag{10.2.3}$$

Using Euler's corollary of the q-binomial theorem (equality (6.2.5) of Chapter 6) and (10.2.3), he then computes

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} &\equiv \frac{\nu(7)}{(q)_7} = \frac{\nu(7)}{(q)_\infty}(q^8;q)_\infty\\ &= \frac{\nu(7)}{(q)_\infty} \sum_{j=0}^{\infty} \frac{(-1)^j q^{j(j+1)/2+7j}}{(q)_j}\\ &\equiv \frac{1}{(q)_\infty}(1 - q^2 - q^3 + q^9 + q^{11} - q^{21} - q^{24}) \pmod{q^{37}},\end{aligned} \tag{10.2.4}$$

which agrees with the product on the right side of (10.1.1) up to q^{37}. In effect, Ramanujan now observes that

$$\begin{aligned}\lim_{N\to\infty} \nu(N) &= 1 + \sum_{n=1}^{\infty} (-1)^n q^{n(5n-1)/2}(1+q^n)\\ &= f(-q^2,-q^3) = (q^2;q^5)_\infty (q^3;q^5)_\infty (q^5;q^5)_\infty,\end{aligned} \tag{10.2.5}$$

by the Jacobi triple product identity, given in Lemma 1.2.2 of Chapter 1. Upon letting $N \to \infty$ and putting (10.2.5) in (10.2.1), we immediately deduce (10.1.1). We thus want to find a representation for $\nu(N)$, defined by (10.2.1), from which (10.1.1) follows upon letting $N \to \infty$.

This quest was fully accomplished in [33], and we follow that development for the remainder of this section. Our objective will be to prove that

$$\nu(N) = \frac{1}{(q^{N+1})_N} \left\{ 1 + \sum_{j=1}^{N} (-1)^j q^{j(5j-1)/2}(1+q^j)W(j,N) \right\}, \tag{10.2.6}$$

where $W(j,N)$ is a polynomial in q of the form $1+O(q^{2N+1})$, uniformly in j. Since as $N\to\infty$, $(q^{N+1})_N\to 1$ and $W(j,N)\to 1$ uniformly in j, as we saw in the previous paragraph, (10.1.1) will then immediately follow.

To prove (10.2.6), we must study a small variation on Bailey chains, a topic extensively developed in [250], [27], [28], [29], and [33]. Bailey chains concern pairs of sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ of rational functions of the variables a and q. They are said to form a Bailey pair, provided that for all $n\ge 0$,

$$\beta_n=\sum_{r=0}^{n}\frac{\alpha_r}{(q)_{n-r}(aq)_{n+r}}. \tag{10.2.7}$$

A limiting form of Bailey's lemma asserts that if (10.2.7) holds, then [29, p. 27, equation (3.33)]

$$\sum_{n=0}^{\infty}q^{n^2}a^n\beta_n=\frac{1}{(aq)_\infty}\sum_{r=0}^{\infty}q^{r^2}a^r\alpha_r. \tag{10.2.8}$$

We now consider what happens when particular instances of (10.2.8) are truncated. We shall require the Gaussian polynomials

$$\begin{bmatrix}n\\m\end{bmatrix}:=\begin{bmatrix}n\\m\end{bmatrix}_q:=\begin{cases}\dfrac{(q;q)_n}{(q;q)_m(q;q)_{n-m}}, & \text{if } 0\le m\le n.\\ 0, & \text{otherwise.}\end{cases}$$

Theorem 10.2.1. *If in* (10.2.7) $a=1$, *then*

$$\sum_{n=0}^{M}q^{n^2}\beta_n=\frac{1}{(q)_{2M}}\sum_{r=0}^{M}q^{r^2}\alpha_rW(r,M), \tag{10.2.9}$$

where

$$W(r,M):=\sum_{j=r}^{M}q^{(M+j+1)(M-j)}\left(\begin{bmatrix}2M\\M-j\end{bmatrix}-\begin{bmatrix}2M\\M-j-1\end{bmatrix}\right). \tag{10.2.10}$$

If in (10.2.7) $a=q$, *then*

$$\sum_{n=0}^{M}q^{n^2+n}\beta_n=\frac{1-q}{(q)_{2M+1}}\sum_{r=0}^{M}q^{r^2+r}\alpha_rU(r,M), \tag{10.2.11}$$

where

$$U(r,M):=\sum_{j=r}^{M}q^{(M+j+2)(M-j)}\left(\begin{bmatrix}2M+1\\M-j\end{bmatrix}-\begin{bmatrix}2M+1\\M-j-1\end{bmatrix}\right). \tag{10.2.12}$$

Proof. We begin by truncating (10.2.8) in full generality. Thus, by (10.2.7),

$$\begin{aligned}\sum_{n=0}^{M} q^{n^2} a^n \beta_n &= \sum_{n=0}^{M} q^{n^2} a^n \sum_{r=0}^{n} \frac{\alpha_r}{(q)_{n-r}(aq)_{n+r}}\\ &= \sum_{r=0}^{M} \alpha_r \sum_{n=r}^{M} \frac{q^{n^2} a^n}{(q)_{n-r}(aq)_{n+r}}\\ &= \frac{1}{(aq)_{2M}} \sum_{r=0}^{M} \alpha_r q^{r^2} a^r \sum_{n=0}^{M-r} \frac{q^{n^2+2nr} a^n (aq)_{2M}}{(q)_n (aq)_{n+2r}}. \end{aligned} \tag{10.2.13}$$

Thus, to establish (10.2.9) and (10.2.11), we need only to show that

$$W(r, M) = \sum_{n=0}^{M-r} \frac{q^{n^2+2nr} (q)_{2M}}{(q)_n (q)_{n+2r}} \tag{10.2.14}$$

and

$$U(r, M) = \sum_{n=0}^{M-r} \frac{q^{n^2+2nr+n} (q)_{2M+1}}{(q)_n (q)_{n+2r+1}}, \tag{10.2.15}$$

respectively.

We require a transformation formula for the q-hypergeometric series

$${}_3\phi_2 \left[\begin{matrix} a, b, c \\ d, e \end{matrix} ; t \right] := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n (c)_n t^n}{(q)_n (d)_n (e)_n}, \qquad |t| < 1. \tag{10.2.16}$$

The transformation formula we need was found by D.B. Sears [240, p. 174, equation (10.1)] and is given by

$${}_3\phi_2 \left[\begin{matrix} a, b, c \\ d, f \end{matrix} ; t \right] = \frac{(f/c)_\infty (df/ab)_\infty}{(f)_\infty (t)_\infty} \, {}_3\phi_2 \left[\begin{matrix} d/a, d/b, c \\ d, df/ab \end{matrix} ; f/c \right], \tag{10.2.17}$$

where $t = df/(abc)$. We apply (10.2.17) with $a = b = 1/\tau$, $c = q^{-M+r}$, $d = eq^{-M+r}$, $f = q^{2r+s+1}$, so that $t = \tau^2 e q^{2r+s+1}$. Hence, for $r \le M$,

$$\begin{aligned}\sum_{n=0}^{M-r} \frac{(1/\tau)_n^2 (q^{-M+r})_n (\tau^2 e q^{2r+s+1})^n}{(q)_n (eq^{-M+r})_n (q^{2r+s+1})_n} &= \frac{(q^{M+r+s+1})_\infty (\tau^2 e q^{-M+3r+s+1})_\infty}{(q^{2r+s+1})_\infty (\tau^2 e q^{2r+s+1})_\infty}\\ &\quad \times \sum_{n=0}^{M-r} \frac{(\tau e q^{-M+r})_n^2 (q^{-M+r})_n q^{(M+r+s+1)n}}{(q)_n (eq^{-M+r})_n (\tau^2 e q^{-M+3r+s+1})_n}. \end{aligned} \tag{10.2.18}$$

Now let $\tau \to 0$ and $e \to 1$ in (10.2.18) to deduce that

$$\sum_{n=0}^{M-r} \frac{q^{n^2+2rn+sn}}{(q)_n (q^{2r+s+1})_n} = \frac{(q^{M+r+s+1})_\infty}{(q^{2r+s+1})_\infty} \sum_{n=0}^{M-r} \frac{q^{(M+r+s+1)n}}{(q)_n}. \tag{10.2.19}$$

On the left side of (10.2.19), multiply the numerator and denominator by $(q^{s+1})_{2r}$ and then multiply both sides of (10.2.19) by $(q^{2r+s+1})_{2M-2r}$. After simplification, we find that

$$\begin{aligned} &\sum_{n=0}^{M-r} \frac{q^{n^2+2rn+sn}(q^{s+1})_{2M}}{(q)_n (q^{s+1})_{n+2r}} \\ &\qquad = (q^{M+r+s+1})_{M-r} \sum_{n=0}^{M-r} \frac{q^{(M+r+s+1)n}}{(q)_n} =: v(r,s,M). \end{aligned} \tag{10.2.20}$$

By (10.2.20),

$$\begin{aligned} v(r,s,M) - v(r+1,s,M) &= (q^{M+r+s+1})_{M-r} \sum_{j=0}^{M-r} \frac{q^{(M+r+s+1)j}}{(q)_j} \\ &\quad - (q^{M+r+s+2})_{M-r-1} \sum_{j=0}^{M-r-1} \frac{q^{(M+r+s+2)j}}{(q)_j} \\ &= (q^{M+r+s+2})_{M-r-1} \left\{ (1-q^{M+r+s+1}) \sum_{j=0}^{M-r} \frac{q^{(M+r+s+1)j}}{(q)_j} \right. \\ &\qquad \left. - \sum_{j=0}^{M-r-1} \frac{q^{(M+r+s+2)j}}{(q)_j} \right\} \\ &= (q^{M+r+s+2})_{M-r-1} \left\{ \sum_{j=0}^{M-r} \frac{q^{(M+r+s+1)j}}{(q)_j} - \sum_{j=0}^{M-r} \frac{q^{(M+r+s+1)(j+1)}}{(q)_j} \right. \\ &\qquad \left. - \sum_{j=0}^{M-r-1} \frac{q^{(M+r+s+2)j}}{(q)_j} \right\} \\ &= (q^{M+r+s+2})_{M-r-1} \left\{ \sum_{j=1}^{M-r-1} \frac{q^{(M+r+s+1)j}}{(q)_{j-1}} + \frac{q^{(M+r+s+1)(M-r)}}{(q)_{M-r}} \right. \\ &\qquad \left. - \sum_{j=0}^{M-r} \frac{q^{(M+r+s+1)(j+1)}}{(q)_j} \right\} \\ &= (q^{M+r+s+2})_{M-r-1} \left\{ \sum_{j=1}^{M-r-1} \frac{q^{(M+r+s+1)j}}{(q)_{j-1}} + \frac{q^{(M+r+s+1)(M-r)}}{(q)_{M-r}} \right. \end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad\qquad\left. - \sum_{j=1}^{M-r+1} \frac{q^{(M+r+s+1)j}}{(q)_{j-1}} \right\} \\
&= (q^{M+r+s+2})_{M-r-1} \left\{ \frac{q^{(M+r+s+1)(M-r)}}{(q)_{M-r}} - \frac{q^{(M+r+s+1)(M-r)}}{(q)_{M-r-1}} \right. \\
&\qquad\qquad\qquad\qquad\qquad\left. - \frac{q^{(M+r+s+1)(M-r+1)}}{(q)_{M-r}} \right\} \\
&= \frac{(q^{M+r+s+2})_{M-r-1} q^{(M+r+s+2)(M-r)} (1-q^{2r+s+1})}{(q)_{M-r}} \\
&= q^{(M+r+s+1)(M-r)} \left(\begin{bmatrix} 2M+s \\ M-r \end{bmatrix} - \begin{bmatrix} 2M+s \\ M-r-1 \end{bmatrix} \right). \qquad (10.2.21)
\end{aligned}
$$

Clearly, then, by the fact that $v(M+1,s,M)=0$ and by (10.2.20),

$$
\begin{aligned}
W(r,s,M) &:= \sum_{j=r}^{M} (v(j,s,M) - v(j+1,s,M)) \\
&= v(r,s,M) - v(M+1,s,M) \\
&= v(r,s,M) \\
&= \sum_{n=0}^{M-r} \frac{q^{n^2+2rn+sn} (q^{s+1})_{2M}}{(q)_n (q^{s+1})_{n+2r}}. \qquad (10.2.22)
\end{aligned}
$$

Note that $W(r,0,M) = W(r,M)$ and $W(r,1,M) = U(r,M)$. Thus, setting $s=0$ and $s=1$ in (10.2.22), we see that both (10.2.14) and (10.2.15) and hence both (10.2.9) and (10.2.11) have been proved. □

We can now easily deduce (10.1.1). The identity follows immediately from Theorem 10.2.1 by letting $a=1$, $\beta_n = 1/(q)_n$, $\alpha_0 = 1$, and $\alpha_n = (-1)^n q^{n(3n-1)/2}(1+q^n)$ for $n>0$, where we have appealed to [29, p. 28, equations (3.34), (3.35)] to secure (10.2.7). To justify the limit as $N \to \infty$, we note from (10.2.10) that

$$
\begin{aligned}
W(r,M) = &\sum_{j=r}^{M-2} q^{(M+j+1)(M-j)} \left(\begin{bmatrix} 2M \\ M-j \end{bmatrix} - \begin{bmatrix} 2M \\ M-j-1 \end{bmatrix} \right) \\
&+ q^{2M} \left(\begin{bmatrix} 2M \\ 1 \end{bmatrix} - 1 \right) + 1 \\
= &\ 1 + O(q^{2M+1}),
\end{aligned}
$$

where, with the help of the penultimate equality in (10.2.21), we see that this approximation is uniform in r.

In exactly the same way, we can prove that if

$$\frac{\mu(N)}{(q)_N} := \sum_{j=0}^{N} \frac{q^{j^2+j}}{(q)_j}, \qquad N \geq 0, \tag{10.2.23}$$

then

$$\mu(N) = \frac{1}{(q^{N+1})_{N+1}} \sum_{j=0}^{N} (-1)^j q^{j(5j+3)/2}(1-q^{2j+1})U(j,N), \tag{10.2.24}$$

where $U(j,N)$ is a polynomial in q of the form $1+O(q^{2N+2})$, uniform in j. The proof now relies on (10.2.11) and (10.2.12) with $a=q$, $\beta_n = 1/(q)_n$, and $\alpha_n = (-1)^n q^{n(3n+1)/2}(1-q^{2n+1})/(1-q)$, $n \geq 0$. That α_n and β_n form a Bailey pair was established by L. Slater [250, p. 468, equation (B3)].

10.3 The Second Argument

"It can be shown independently that,

$$\log\{\text{L H S of (10.1.1)}\} \sim \frac{\pi^2}{15(1-q)} \qquad \text{as } q \to 1 \tag{10.3.1}$$

as well as

$$\log\{\text{R H S of (10.1.1)}\} \sim \frac{\pi^2}{15(1-q)} \qquad \text{as } q \to 1.\text{''} \tag{10.3.2}$$

These assertions are in the literature, and each has been proved independently at least a few times. The first assertion was established by G. Meinardus [198], who also proved the second assertion [197]. The first assertion can be proved using a version of the saddle point method. The second assertion probably first appeared in J. Lehner's Ph.D. thesis [176].

In his third notebook [227, p. 366] and on page 359 of his lost notebook [228], Ramanujan offers an extensive generalization of (10.3.1). This was first established independently by Berndt [62, pp. 269–273] and by R. McIntosh [195], who proved an even more general theorem by applying the Euler–Maclaurin summation formula in a skillful fashion. See also [62, pp. 273–286]. The asymptotic formula (10.3.2) has been generalized by Berndt and J. Sohn [83] in a slightly different manner, and an account of this appears in Theorems 7.2.1 and 7.3.1 of Chapter 7 in this book.

10.4 The Third Argument

"The numerical results of the cont. fraction go to prove the truth of the result."

Here we must assume that Ramanujan is referring to the representation for the Rogers–Ramanujan continued fraction in terms of the Rogers–Ramanujan functions in (10.1.1) and (10.1.2) that is given by

$$\begin{aligned} C(q) &:= 1 + \frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+}\cdots \\ &= \frac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{n^2}}{(q;q)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{n^2+n}}{(q;q)_n}} = \frac{(q^2;q^5)_\infty(q^3;q^5)_\infty}{(q;q^5)_\infty(q^4;q^5)_\infty}, \qquad |q|<1. \end{aligned} \tag{10.4.1}$$

(See (1.1.2) in Chapter 1.)

If we choose a root of unity, say $e^{2\pi ih/k}$, where $(h,k)=1$ and $5\nmid k$, then the continued fraction $C(e^{2\pi ih/k})$ can be explicitly evaluated, as we shall see in the following section. On the other hand, the infinite product is a modular form. Consequently, using the transformations given by Lehner [176], one can also show that as $q\to e^{2\pi ih/k}$ on a ray emanating from the origin, the infinite product in (10.4.1) converges to the same algebraic number in any particular instance. However, there is no evidence that Ramanujan was aware of such a theorem. Thus, maybe he examined only the cases $q=\pm 1$.

As we saw in Chapter 2, Ramanujan evaluated $C(e^{-\pi\sqrt{n}})$ in closed form for several rational numbers n. Because the Rogers–Ramanujan functions in (10.4.1) converge very rapidly, it is conceivable that Ramanujan's "numerical results" arose from some of these evaluations.

10.5 The Fourth Argument

"If

$$v = \frac{q}{1+}\,\frac{q^5}{1+}\,\frac{q^{10}}{1+}\,\frac{q^{15}}{1+}\cdots$$

then $\frac{1}{v}-v-1$ vanishes when q is of the form $e^{\pi im/n}$ where m and n are any two integers prime to each other, except when n is a multiple of 25. As a matter of fact, if v is the assumed product of the continued fraction

$$\frac{q}{1+}\,\frac{q^5}{1+}\,\frac{q^{10}}{1+}\cdots\text{''}$$

The sentence above was not completed by Ramanujan in this fragment. Clearly, then, there is at least one page missing. Moreover, page 358 begins with 2., indicating that the first section of the fragment has also been lost. Note that in the notation (1.1.1) of Chapter 1, $v=R(q^5)$. Although we cannot determine what Ramanujan was going to write, we can prove his initial assertion.

In Section 5.2 of the present volume, we discussed Ramanujan's evaluation of $C(q)$ at roots of unity on page 383 of his third notebook [227]. As we demonstrated, a calculational error on page 57 of the lost notebook [228] propagated an error in Ramanujan's evaluation. I. Schur [238, pp. 319–321], [239, pp. 117–136] independently proved a correct version, which we now record. See also [63, p. 35, Theorem 12.1].

Theorem 10.5.1. *Let $C(q)$ be the continued fraction defined in* (10.4.1), *and let q be a primitive nth root of unity. If n is a multiple of* 5, *then $C(q)$ diverges. When n is not a multiple of* 5, *let $\lambda = \left(\frac{n}{5}\right)$, the Legendre symbol. Furthermore, let ρ denote the least positive residue of n modulo* 5. *Then for $n \not\equiv 0 \pmod 5$,*

$$C(q) = \lambda q^{(1-\lambda\rho n)/5} C(\lambda). \tag{10.5.1}$$

We now verify Ramanujan's claim. Recall the elementary evaluations

$$C(\lambda) = \begin{cases} \dfrac{\sqrt{5}+1}{2}, & \text{if } n \equiv 1, 4 \pmod 5, \\ \dfrac{\sqrt{5}-1}{2}, & \text{if } n \equiv 2, 3 \pmod 5. \end{cases} \tag{10.5.2}$$

Let $t_n := 1/C\left(\left(\frac{2n}{5}\right)\right)$. From (10.5.2), we easily deduce that

$$t_n = \begin{cases} \dfrac{\sqrt{5}+1}{2}, & \text{if } n \equiv 1, 4 \pmod 5, \\ \dfrac{\sqrt{5}-1}{2}, & \text{if } n \equiv 2, 3 \pmod 5. \end{cases} \tag{10.5.3}$$

Now, following Ramanujan, let $\hat{q} = e^{\pi i m/n}$, a $2n$th root of unity, where $25 \nmid n$ and $(m, n) = 1$. By (10.5.1), if ρ is the least positive residue of $2n$ modulo 5,

$$v(\hat{q}) = \frac{\hat{q}}{C(\hat{q}^5)} = \frac{\hat{q}}{\left(\frac{2n}{5}\right) \hat{q}^{1-2\lambda\rho n} C\left(\left(\frac{2n}{5}\right)\right)} = \left(\frac{2n}{5}\right) t_n, \tag{10.5.4}$$

since $\hat{q}$ is a $2n$th root of unity. From (10.5.3) and (10.5.4), we easily check that in all cases,

$$\frac{1}{v(\hat{q})} - v(\hat{q}) - 1 = 0,$$

as claimed by Ramanujan.

11

Rogers–Ramanujan–Slater Type Identities

11.1 Introduction

The Rogers–Ramanujan identities

$$G(q) := \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_n} = \frac{1}{(q;q^5)_\infty (q^4;q^5)_\infty} \tag{11.1.1}$$

and

$$H(q) := \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_n} = \frac{1}{(q^2;q^5)_\infty (q^3;q^5)_\infty} \tag{11.1.2}$$

were examined empirically in the previous chapter. They have also appeared in other chapters, for example, in Entry 3.2.2 of Chapter 3 and in the proofs of Entry 4.3.1 and Corollary 6.2.6 in Chapters 4 and 6, respectively. In particular, they are prominent in the theory of the Rogers–Ramanujan continued fraction. Moreover, various other continued fractions of Ramanujan can be established using analogues of the Rogers–Ramanujan identities; see, for example, the proofs of Corollaries 6.2.7 and 6.2.8 in Chapter 6, where certain identities relating infinite series with infinite products due to L. Slater [251] are required. Ramanujan also derived several analogues of the Rogers–Ramanujan identities, and these are the subject of the present chapter. In light of the fact that several results in this chapter are variations on Slater's theorems, we have chosen to append her name to the more familiar appellation, Rogers–Ramanujan.

After the definitive work of Rogers, Ramanujan, and Slater, an enormous amount of research has been devoted to analogues and generalizations of the Rogers–Ramanujan identities. Because of space limitation, it is impossible to cite all relevant papers here. However, sources with extensive bibliographies can be found in the monographs of Andrews [21], [29, Chapter 7], the papers of Andrews [18], [19], [30], H.L. Alder's paper [11], and the survey by

A. Berkovich and B.M. McCoy [57]. For several further new identities and for finite forms of many of the identities found by Slater and others, see the papers [245]–[248] by A.V. Sills.

The identities considered by Ramanujan in the lost notebook have infinite products connected with the moduli 3, 5, 6, 7, or 12, or else they are instances of Rogers's false theta functions. We have organized the sections of this chapter according to this classification.

11.2 Identities Associated with Modulus 5

In this section, we consider all the Rogers–Ramanujan–Slater type identities related to $G(q)$ and $H(q)$, defined in (11.1.1) and (11.1.2), respectively.

Entry 11.2.1 (p. 54). *We have*

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q)_{2n}} = \frac{G(q^4)}{(q;q^2)_\infty}, \tag{11.2.1}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_{2n+1}} = \frac{H(-q)}{(q;q^2)_\infty}, \tag{11.2.2}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_{2n}} = \frac{G(-q)}{(q;q^2)_\infty}, \tag{11.2.3}$$

$$\sum_{n=0}^{\infty} \frac{q^{n^2+2n}}{(q)_{2n+1}} = \frac{H(q^4)}{(q;q^2)_\infty}. \tag{11.2.4}$$

These four identities appear near the bottom of page 54 of the lost notebook. Identities (11.2.1) and (11.2.4) appear as equation (3) in the paper by Rogers [235], and identities equivalent to these occur in Section 6 of Rogers's paper [234]. Equivalent identities occur in Slater's paper [251, p. 162, equations (98), (94), (92), (96), respectively].

Entry 11.2.2 (p. 24). *We have*

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{3n^2}}{(q^4;q^4)_n(-q;q^2)_n} = \frac{G(q)}{(-q;q)_\infty}, \tag{11.2.5}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{3n^2-2n}}{(q^4;q^4)_n(-q;q^2)_n} = \frac{H(q)}{(-q;q)_\infty}, \tag{11.2.6}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{3n^2+2n}}{(q^4;q^4)_n(-q;q^2)_{n+1}} = \frac{H(q)}{(-q;q)_\infty}. \tag{11.2.7}$$

These are the first three identities at the top of page 24 in the lost notebook. The first two were originally found by Rogers [235], and they appear in Slater's list [251, pp. 153–154, equations (15), (19), respectively].

Identity (11.2.7) does not appear in Rogers's or in Slater's list. However, it is easily deduced from (11.2.6) as follows. Noting that the right sides of (11.2.6) and (11.2.7) are identical, we form the difference of the left sides to find that

$$\begin{aligned}
&\sum_{n=0}^{\infty} \frac{(-1)^n q^{3n^2+2n}}{(q^4;q^4)_n(-q;q^2)_{n+1}} - \sum_{n=0}^{\infty} \frac{(-1)^n q^{3n^2-2n}}{(q^4;q^4)_n(-q;q^2)_n} \\
&\quad = \sum_{n=0}^{\infty} \frac{(-1)^n q^{3n^2-2n}}{(q^4;q^4)_n(-q;q^2)_{n+1}} \left(q^{4n} - 1 - q^{2n+1}\right) \\
&\quad = \sum_{n=0}^{\infty} \frac{(-1)^n q^{3n^2-2n}}{(q^4;q^4)_n(-q;q^2)_{n+1}} \left(-(1+q^{2n+1})(1-q^{4n}) - q^{6n+1}\right) \\
&\quad = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} q^{3n^2-2n}}{(q^4;q^4)_{n-1}(-q;q^2)_n} - \sum_{n=0}^{\infty} \frac{(-1)^n q^{3n^2+4n+1}}{(q^4;q^4)_n(-q;q^2)_{n+1}} \\
&\quad = 0,
\end{aligned}$$

because the first sum on the far right side is identical with the second, once n has been replaced by $n-1$ in the second sum.

11.3 Identities Associated with the Moduli 3, 6, and 12

Just as $G(q)$ and $H(q)$ played a central role in Section 11.2, the three infinite products

$$G_6(q) := (q^3;q^6)_\infty^2 (q^6;q^6)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2} = \varphi(-q^3), \tag{11.3.1}$$

$$H_6(q) := (q;q^6)_\infty (q^5;q^6)_\infty (q^6;q^6)_\infty = \sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2-2n} = f(-q,-q^5), \tag{11.3.2}$$

$$J_6(q) := (-q;q^3)_\infty (-q^2;q^3)_\infty (q^3;q^3)_\infty = \sum_{n=-\infty}^{\infty} q^{n(3n+1)/2} = f(q,q^2), \tag{11.3.3}$$

are pivotal in this section. Here we have employed Ramanujan's notations for theta functions given in (1.1.5) and (1.1.6) in Chapter 1. In each of (11.3.1)–(11.3.3), Jacobi's triple product identity, given in Lemma 1.2.2 of Chapter 1, has been invoked to provide the theta series representation of the product.

As for Ramanujan's identities, we begin with one that is stated twice in the lost notebook. It is the fourth identity on pages 6 and 16.

Entry 11.3.1 (pp. 6, 16). *If* $G_6(q)$ *is defined by* (11.3.1), *then*

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n^2}}{(q;q)_{2n}} = \frac{G_6(q^2)}{(q;q)_\infty}. \tag{11.3.4}$$

Entry 11.3.1 is immediate from an identity in Slater's paper [251, p. 155, equation (29)].

The next identity appears as the fifth identity on pages 6 and 16 of the lost notebook.

Entry 11.3.2 (pp. 6, 16). *If* $H_6(q)$ *is defined by* (11.3.2) *and* $\varphi(q)$ *is given by* (1.1.6) *in Chapter* 1, *then*

$$\sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n q^{n(n+1)}}{(q;q)_{2n+1}} = \frac{H_6(-q)}{\varphi(-q^2)}. \tag{11.3.5}$$

The identity (11.3.5) is again one of Slater's identities [251, p. 154, eq. (28)].

A related result appears as the seventh identity on page 6 and the third identity on page 12 of the lost notebook.

Entry 11.3.3 (pp. 6, 12). *If* $H_6(q)$ *is defined by* (11.3.2) *and* $\varphi(q)$ *is given by* (1.1.6) *in Chapter* 1, *then*

$$\sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n q^{n}}{(q;q)_{2n+1}} = \frac{H_6(q^2)}{\varphi(-q)}. \tag{11.3.6}$$

Proof. To the best of our knowledge, this result is not explicitly stated in the literature. However, it follows easily from Heine's transformation [21, p. 19, Corollary 2.3] that

$$\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(q)_n} t^n = \frac{(b)_\infty (at)_\infty}{(c)_\infty (t)_\infty} \sum_{n=0}^{\infty} \frac{(c/b)_n (t)_n}{(at)_n (q)_n} b^n. \tag{11.3.7}$$

Replace q by q^2 and then set $a = -q$, $c = q^3$, and $t = q$ in (11.3.7). Then let $b \to 0$ to find that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n q^n}{(q;q)_{2n+1}} &= \frac{(-q;q^2)_\infty}{(q;q^2)_\infty^2} \sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n q^{n^2+2n}}{(q^2;q^2)_n (-q;q^2)_{n+1}} \\ &= \frac{(-q;q^2)_\infty}{(q;q^2)_\infty^2} \frac{H_6(q^2)}{(-q;-q)_\infty} \\ &= \frac{H_6(q^2)}{\varphi(-q)}, \end{aligned}$$

where in the penultimate line we used an identity of Slater [251, p. 157, equation (50)], and where in the last line we used the familiar product representation for $\varphi(q)$ given in (1.1.6) in Chapter 1. □

The next identity, found as the third identity on page 6 of the lost notebook, but incorrectly stated there by Ramanujan, is a natural companion to (11.3.6).

Entry 11.3.4 (p. 6). *If* $H_6(q)$ *is defined by* (11.3.2) *and* $\varphi(q)$ *is given by* (1.1.6) *in Chapter* 1, *then*

$$\sum_{n=0}^{\infty} \frac{(-q;q)_{2n} q^n}{(q;q)_n(-q;q)_n} = \frac{H_6(q)}{\varphi(-q)}. \tag{11.3.8}$$

Proof. In this proof, we use the second iterate of Heine's transformation given by [21, p. 38, last line]

$$\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n(q)_n} t^n = \frac{(c/b)_\infty(bt)_\infty}{(c)_\infty(t)_\infty} \sum_{n=0}^{\infty} \frac{(b)_n(abt/c)_n}{(bt)_n(q)_n}\left(\frac{c}{b}\right)^n. \tag{11.3.9}$$

Replace q by q^2 and then set $a=-q$, $b=-q^2$, $c=0$, and $t=q$ to obtain the equality

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-q;q^2)_n(-q^2;q^2)_n q^n}{(q^2;q^2)_n} &= \frac{(-q;q^2)_\infty}{(q;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n q^{n^2+n}}{(q^2;q^2)_n(-q;q^2)_{n+1}} \\ &= \frac{(-q;q^2)_\infty}{(q;q^2)_\infty} \frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty} H_6(q) \\ &= \frac{H_6(q)}{\varphi(-q)},\end{aligned}$$

where in the penultimate equality we employed another identity of Slater [251, p. 154, equation (28)], and in the last line once again used (1.1.6) in Chapter 1. ∎

The final identity in this section occurs as the sixth equation on page 6 and the second on page 12 of the lost notebook.

Entry 11.3.5 (pp. 6, 12). *If* $J_6(q)$ *is defined by* (11.3.3) *and* $\varphi(q)$ *is given by* (1.1.6) *in Chapter* 1, *then*

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^n}{(q;q)_{2n}} = \frac{J_6(-q)}{\varphi(-q)}. \tag{11.3.10}$$

Proof. Again we require Heine's transformation (11.3.7). Replace q by q^2, then set $a=-q$ and $c=t=q$, and finally let $b\to 0$ to discover that

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^n}{(q;q)_{2n}} &= \frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty^2} \sum_{n=0}^{\infty} \frac{(-1)^n(q;q^2)_n q^{n^2}}{(q^4;q^4)_n} \\ &= \frac{(-q^2;q^2)_\infty}{(q;q^2)_\infty^2} \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}(-q^3;q^6)_\infty^2(q^6;q^6)_\infty,\end{aligned}$$

by an application of another identity of Slater [251, p. 154, equation (25)]. Now, if we replace q by $-q$ above, we find that

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(q;q^2)_n(-q)^n}{(-q;-q)_{2n}} &= \frac{(-q^2;q^2)_\infty(q^3;q^3)_\infty}{(-q;q^2)_\infty(q^2;q^2)_\infty(-q^3;q^3)_\infty} \\ &= \frac{(-q;q)_\infty(q^3;q^3)_\infty}{(-q;q^2)^2_\infty(q^2;q^2)_\infty(-q^3;q^3)_\infty} \\ &= \frac{(-q;q^3)_\infty(-q^2;q^3)_\infty(q^3;q^3)_\infty}{\varphi(q)} \\ &= \frac{J_6(q)}{\varphi(q)},\end{aligned}$$

by (1.1.6) of Chapter 1 and the definition of $J_6(q)$ given in (11.3.3). The last identity above is (11.3.10) with q replaced by $-q$, and so the proof is complete. □

11.4 Identities Associated with the Modulus 7

Ramanujan found three identities connected with the modulus 7. These appear on page 24 of the lost notebook as the fourth, fifth, and sixth identities.

Entry 11.4.1 (p. 24). *Recall that Ramanujan's general theta function $f(a,b)$ is defined in* (1.1.5) *of Chapter* 1. *Then*

$$\sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(-q;q^2)_n(q^4;q^4)_n} = \frac{f(-q^{5/2},-q)}{(q^2;q^2)_\infty}, \tag{11.4.1}$$

$$\sum_{n=0}^{\infty} \frac{q^{2n^2}}{(-q;q^2)_n(q^4;q^4)_n} = \frac{f(-q^2,-q^{3/2})}{(q^2;q^2)_\infty}, \tag{11.4.2}$$

$$\sum_{n=0}^{\infty} \frac{q^{2n(n+1)}}{(-q;q^2)_{n+1}(q^4;q^4)_n} = \frac{f(-q^3,-q^{1/2})}{(q^2;q^2)_\infty}. \tag{11.4.3}$$

These identities were first found by Rogers [235]. They were rediscovered by A. Selberg [241] and are often referred to as the Rogers–Selberg identities. They are listed as equalities (32), (33), and (34) in Slater's compendium [251].

11.5 False Theta Functions

We have already encountered identities for false theta functions in Section 6.2 of Chapter 6 and in the study of the Rogers–Fine identity in Chapter 9. In this section, we focus our attention on three identities that are most naturally proved by reference to Slater's elaborate applications of Bailey's fundamental ideas. We refer the reader to [29, Chapters 2, 3] for the relevant history.

Theorem 11.5.1. *If*

$$\beta_n = \sum_{r=0}^{n} \frac{\alpha_r}{(q;q)_{n-r}(aq;q)_{n+r}}, \tag{11.5.1}$$

then

$$\sum_{n=0}^{\infty} (y;q)_n (z;q)_n \left(\frac{aq}{yz}\right)^n \beta_n$$
$$= \frac{(aq/y;q)_\infty (aq/z;q)_\infty}{(aq;q)_\infty (aq/(yz);q)_\infty} \sum_{n=0}^{\infty} \frac{(y;q)_n (z;q)_n}{(aq/y;q)_n (aq/z;q)_n} \left(\frac{aq}{yz}\right)^n \alpha_n.$$

This theorem is given by Slater [250, p. 462, equations (1.3)], with x replaced by aq.

If we now let $y \to \infty$ and set $a = z = q$, Theorem 11.5.1 yields

$$\sum_{n=0}^{\infty} (-1)^n (q;q)_n q^{n(n+1)/2} \beta_n = (1-q) \sum_{n=0}^{\infty} (-1)^n q^{n(n+1)/2} \alpha_n. \tag{11.5.2}$$

The three identities of this section are the fifth identity on page 12, the sixth identity on page 12, and the third identity on page 34 of the lost notebook.

Entry 11.5.1 (pp. 12, 12, 34, respectively). *We have*

$$\sum_{n=0}^{\infty} \frac{(-1)^n (-q;q^2)_n q^{n(n+1)/2}}{(q^{n+1};q)_{n+1}} = \sum_{n=0}^{\infty} (-1)^n q^{4n^2+n} (1+q^{6n+3}), \tag{11.5.3}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n (-q;q^2)_n q^{n(n+3)/2}}{(q^{n+1};q)_{n+1}} = \sum_{n=0}^{\infty} (-1)^n q^{4n^2+3n} (1+q^{2n+1}), \tag{11.5.4}$$

$$\sum_{n=0}^{\infty} \frac{(-1)^n (q^{4n+6};q^4)_n q^{2n(n+1)}}{(-q;q)_{4n+2}} = \sum_{n=0}^{\infty} q^{4n^2+3n} (1-q^{2n+1}). \tag{11.5.5}$$

Proof. Identity (11.5.3) follows from (11.5.2) by appealing to Slater's table [250, p. 471, third line of table]. Namely, if

$$\alpha_n = \begin{cases} q^{8r^2}, & \text{if } n = 4r \text{ or } 4r-1, \\ -q^{8r^2+8r+2}, & \text{if } n = 4r+1 \text{ or } 4r+2, \end{cases}$$

and

$$\beta_n = \frac{(-q;q^2)_n}{(q^2;q)_{2n}},$$

then (11.5.1) is satisfied. Inserting these values of α_n and β_n into (11.5.2) and dividing by $1-q$, we deduce that

$$\sum_{n=0}^{\infty}\frac{(-1)^n(-q;q^2)_nq^{n(n+1)/2}}{(q^{n+1};q)_{n+1}}$$
$$=\sum_{n=0}^{\infty}q^{2n(4n+1)+8n^2}+\sum_{n=0}^{\infty}q^{(2n+1)(4n+1)+8n^2+8n+2}$$
$$-\sum_{n=0}^{\infty}q^{(2n+1)(4n+3)+8n^2+8n+2}-\sum_{n=0}^{\infty}q^{(2n+2)(4n+3)+8(n+1)^2}$$
$$=\sum_{n=0}^{\infty}(-1)^nq^{4n^2+n}+\sum_{n=0}^{\infty}(-1)^nq^{4n^2+7n+3},$$

where we combined the first and third sums and the second and fourth sums. This is (11.5.3), and so the proof is complete.

We next prove (11.5.4). Set

$$\alpha_n=\begin{cases}-q^{8r^2-4r}, & \text{if } n=4r-2,\\ q^{8r^2-4r}, & \text{if } n=4r-1,\\ q^{8r^2+4r}, & \text{if } n=4r,\\ -q^{8r^2+4r}, & \text{if } n=4r+1,\end{cases}$$

and

$$\beta_n=\frac{q^n(-q;q^2)_n}{(q^2;q)_{2n}}.$$

Then (11.5.1) is satisfied [250, p. 471, fourth line of table]. So, we may insert these values of α_n and β_n into (11.5.2) and divide both sides by $1-q$ to arrive at

$$\sum_{n=0}^{\infty}\frac{(-1)^n(-q;q^2)_nq^{n(n+3)/2}}{(q^{n+1};q)_{n+1}}$$
$$=\sum_{n=0}^{\infty}q^{2n(4n+1)+8n^2+4n}+\sum_{n=0}^{\infty}q^{(2n+1)(4n+1)+8n^2+4n}$$
$$-\sum_{n=0}^{\infty}q^{(2n+1)(4n+3)+8(n+1)^2-4(n+1)}-\sum_{n=0}^{\infty}q^{(2n+2)(4n+3)+8(n+1)^2-4(n+1)}$$
$$=\sum_{n=0}^{\infty}(-1)^nq^{4n^2+3n}+\sum_{n=0}^{\infty}(-1)^nq^{4n^2+5n+1},$$

which is (11.5.4).

Finally, we examine (11.5.5), the right-hand side of which is the same function as that on the right-hand side in (11.5.4) with q replaced by $-q$.

Now we must consider (11.5.1) and (11.5.2) with q, a, and z all replaced by q^4. Replacing q by q^4 in [251, p. 150, equation (M2)], we obtain the pair

$$\alpha_n = \alpha_n(q) = \frac{q^{n(2n+1)}(1+q^{2n+1})}{1-q^4}$$

and

$$\beta_n = \beta_n(q) = \frac{(-q;q^2)_{2n+1}}{(q^4;q^4)_{2n+1}},$$

which satisfy (11.5.1) with q and a both replaced by q^4. Hence,

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-1)^n(q^{4n+6};q^4)_n q^{2n(n+1)}}{(-q;q)_{4n+2}} &= \sum_{n=0}^{\infty} \frac{(-1)^n(q^2;q^4)_{2n+1} q^{2n(n+1)}}{(q^2;q^4)_{n+1}(-q;q)_{4n+2}} \\
&= \sum_{n=0}^{\infty} (-1)^n (q^4;q^4)_n q^{2n(n+1)} \beta_n(-q) \\
&= (1-q^4) \sum_{n=0}^{\infty} (-1)^n q^{2n(n+1)} \alpha_n(-q) \\
&= \sum_{n=0}^{\infty} (-1)^n q^{2n(n+1)} (-1)^n q^{2n^2+n} (1-q^{2n+1}) \\
&= \sum_{n=0}^{\infty} q^{4n^2+3n}(1-q^{2n+1}),
\end{aligned}$$

as desired. ∎

12

Partial Fractions

12.1 Introduction

G.N. Watson, in his celebrated London Mathematical Society Presidential address [289, p. 67], [82, pp. 334–335], noted that many of the identities that Ramanujan had found for the third order mock theta functions could be deduced from a theta-function expansion that was, in fact, a limiting case of a partial fraction decomposition. In his lost notebook, it is clear that Ramanujan had a complete working knowledge of this method.

In this chapter, we shall examine several identities amenable to this approach. The only identities of this nature from the lost notebook that we exclude are most of those examined by Watson in [289]. It should be noted that in [24], several results were proved by a much clumsier technique. In private notes made in preparing [24], this method is referred to as "pseudo-partial fractions." It is possible that the method of [24] may apply in some situations in which partial fractions do not apply; however, to our delight, this is not the case with the formulas in the lost notebook.

In many of the identities to be considered, we encounter the q-series

$$\sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n q^{n^2}}{(-aq^2;q^2)_n(-q^2/a;q^2)_n},$$

which provides a partially unifying thread in this work. The case $a = 1$ of this series has been called a fourth order mock theta function by R. McIntosh [196], and as such has been related to the Mordell integrals in both [24] and [196].

In Section 12.2, we present the fundamental partial fraction decompositions. Section 12.3 then features those identities that follow most easily from the fundamental identities. Section 12.4 contains further identities that are appropriate for this chapter; they are closely related to the q-series from Section 12.3. However, these identities often require some tools besides a partial fraction decomposition.

We conclude this chapter with some speculation about the role that partial fractions may have played in Ramanujan's general outlook on q-series.

Throughout the sequel, we frequently need to show that certain series, usually theta functions, vanish identically. Usually, the argument one needs is elementary and arises from a judicious change of the indices of summation, showing that the sum in question is equal to the negative of itself. Alternatively, to make the same deductions, we can employ two elementary results about Ramanujan's theta function $f(a,b)$ (defined in (1.1.5) of Chapter 1) from his second notebook [61, p. 34, Entry 18(iii), (iv)], namely,

$$f(-1,a)=0 \tag{12.1.1}$$

and

$$f(a,b)=a^{n(n+1)/2}b^{n(n-1)/2}f\left(a(ab)^n,b(ab)^{-n}\right), \tag{12.1.2}$$

where n is any integer.

In manipulating products, we frequently use Euler's famous identity

$$\frac{1}{(q;q^2)_\infty}=(-q;q)_\infty. \tag{12.1.3}$$

Throughout this chapter, we shall be taking limits as $N\to\infty$ of various special cases of (12.2.1) below. In taking such limits, we shall repeatedly use without comment

$$\lim_{N\to\infty}\frac{q^{Nn}(q^{-N};q)_n}{(aq^{N+1};q)_n}=(-1)^nq^{n(n-1)/2}.$$

12.2 The Basic Partial Fractions

The fundamental identities in this section are all specializations of Watson's q-analogue of Whipple's theorem [140, p. 242, equation (III.18)]. If a, b, c, d, and e are any complex numbers such that $bcde\neq 0$, and if N is any nonnegative integer, then

$$\begin{aligned}&{}_8\phi_7\left[\begin{matrix}a,q\sqrt{a},-q\sqrt{a},b,c,d,e,q^{-N}\\ \sqrt{a},-\sqrt{a},\dfrac{aq}{b},\dfrac{aq}{c},\dfrac{aq}{d},\dfrac{aq}{e},aq^{N+1}\end{matrix};q,\frac{a^2q^{N+2}}{bcde}\right]\\ &\qquad=\frac{(aq)_N\left(\dfrac{aq}{de}\right)_N}{\left(\dfrac{aq}{d}\right)_N\left(\dfrac{aq}{e}\right)_N}\,{}_4\phi_3\left[\begin{matrix}\dfrac{aq}{bc},d,e,q^{-N}\\ \dfrac{aq}{b},\dfrac{aq}{c},\dfrac{deq^{-N}}{a}\end{matrix};q,q\right].\end{aligned} \tag{12.2.1}$$

Although we cannot find a statement of this theorem in Ramanujan's works, he recorded many deductions from it. In particular, see [61, p. 16, Entry 7] and the pages immediately following.

If in (12.2.1) we let $a = 1$ and $b = 1/c$, let $d, e \to \infty$, and then divide both sides by $(1-c)(q)_N$, we obtain, after some algebraic simplifications,

$$\frac{1}{(1-c)(q)_N} + \sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(-1)^n (q)_n q^{n(3n+1)/2}}{(q)_{n+N}} \left(\frac{1}{1-cq^n} - \frac{1}{c-q^n} \right) = \sum_{n=0}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(q)_n q^{n^2}}{(c)_{n+1}(q/c)_n}, \quad (12.2.2)$$

where $\left[{N \atop n} \right]$ denotes the Gaussian polynomial defined in Lemma 8.2.1 of Chapter 8. Clearly, the left side of (12.2.2) is the classical partial fraction decomposition of the sum on the right side. For brevity, we have deduced (12.2.2) from (12.2.1). However, one can prove (12.2.2) *ex nihilo* by noting that the right side is a proper rational function of c with simple poles at $c = q^{-N}, q^{-N+1}, \dots, q^0, q^1, \dots, q^N$; the residue at each of the poles may be calculated using nothing more than the q-Chu–Vandermonde summation [140, p. 236, equation (II.7)].

Our main interest in (12.2.2) lies in the limiting case $N \to \infty$. We note that each side converges uniformly for $|q| \le 1 - \epsilon$, for each $\epsilon > 0$, because of the quadratic exponents on q. Hence, letting $N \to \infty$ and collapsing our two sums into a bilateral series, we conclude that

$$\frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(3n+1)/2}}{1-cq^n} = \sum_{n=0}^{\infty} \frac{q^{n^2}}{(c)_{n+1}(q/c)_n}. \quad (12.2.3)$$

Our next specialization of (12.2.1) closely resembles (12.2.2). In this case, we take $a = q$, replace c by $cq^{1/2}$, then set $b = q^{1/2}/c$, and let d and e tend to ∞. We then divide both sides by $(1-cq^{1/2})(1-q^{1/2}/c)(q^2)_N$, and after some algebraic simplification, we find that

$$\sum_{n=0}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(-1)^n (q)_n q^{3n(n+1)/2}}{(q)_{n+N+1}} \left(\frac{1}{1-cq^{n+1/2}} + \frac{q^{n+1/2}}{c-q^{n+1/2}} \right) = \sum_{n=0}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(q)_n q^{n^2+n}}{(cq^{1/2})_{n+1}(q^{1/2}/c)_{n+1}}. \quad (12.2.4)$$

Again, this is a classical partial fraction expansion, and again, it can be proved directly by residue calculations that involve nothing more than the q-Chu–Vandermonde summation. As before, we are most interested in the limiting case as $N \to \infty$, which is given by

$$\frac{1}{(q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(n+1)/2}}{1-cq^{n+1/2}} = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(cq^{1/2})_{n+1}(q^{1/2}/c)_{n+1}}. \quad (12.2.5)$$

We require two further specializations of (12.2.1). Each may be derived by first taking N fixed and then letting $N \to \infty$. However, for brevity, we include only the limiting cases. The first is actually equivalent to the last formula on page 1 of the lost notebook, and is given as identity (3.4) in [24].

Entry 12.2.1 (p. 1). *For $c \neq 0$,*

$$\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{q^{2n^2+n}}{1+cq^{2n}} = \sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n q^{n^2}}{(-c;q^2)_{n+1}(-q^2/c;q^2)_n}. \tag{12.2.6}$$

Proof. The result follows directly from (12.2.1) if we replace q by q^2 and c by $-c$, then set $a=1$, $b=-1/c$, and $d=q$, and finally let e and N tend to ∞. Multiply both sides by $(q;q^2)_\infty/\{(1+c)(q^2;q^2)_\infty\}$ to complete the proof. □

Secondly,

$$\frac{(-q;q^2)_\infty}{(q^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(-1)^n(1-q^{2n+1})q^{(n+1)(2n+1)}}{(1-cq^{2n+1})(1-q^{2n+1}/c)} = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{(n+1)^2}}{(cq;q^2)_{n+1}(q/c;q^2)_{n+1}}. \tag{12.2.7}$$

This result follows directly from (12.2.1) if we replace q by q^2, then replace c by cq, then set $a=q^2$, $b=q/c$, and $d=-q$, and finally let e and N tend to ∞. Multiplying both sides by $(-q^3;q^2)_\infty/\{(1-cq)(1-q/c)(q^4;q^2)_\infty\}$ completes the proof.

Our last partial fraction expansion arises from a well-known corollary of (12.2.1), namely [140, p. 238, equation (II.20)],

$$ {}_6\phi_5 \left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, d, e, q^{-N} \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{d}, \frac{aq}{e}, aq^{N+1} \end{matrix} ; q, \frac{aq^{N+1}}{de} \right] = \frac{(aq)_N \left(\frac{aq}{de}\right)_N}{\left(\frac{aq}{d}\right)_N \left(\frac{aq}{e}\right)_N}. \tag{12.2.8}$$

Equation (12.2.8) is merely (12.2.1) with $b = aq/c$, which trivially reduces ${}_4\phi_3$ to 1.

If we set $a=1$, $d=1/c$, and $e=c$, and divide both sides by $(1-c)(q)_N$, we find that

$$\frac{1}{(1-c)(q)_N} + \sum_{n=1}^{N} \begin{bmatrix} N \\ n \end{bmatrix} \frac{(-1)^n (q)_n q^{n(n+1)/2}}{(q)_{n+N}} \left(\frac{1}{1-cq^n} - \frac{1}{c-q^n} \right) = \frac{(q)_N}{(c)_{N+1}(q/c)_N}.$$

As before, we let $N \to \infty$, and after some rearrangement, we deduce the following theorem.

Entry 12.2.2 (p. 1). *We have*

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n+1)/2}}{1-cq^n} = \frac{(q)_\infty^2}{(c)_\infty (q/c)_\infty}. \tag{12.2.9}$$

This is, in fact, the well-known expansion for the reciprocal of a theta function and is equivalent to the next to last formula on page 1 of the lost notebook. Another formulation is found on page 59 of the lost notebook, where it is recorded as the generating function for cranks. For applications of this formula to cranks, see the papers by Berndt, H.H. Chan, S.H. Chan, and W.–C. Liaw [68, Theorem 8.1], [69]. The oldest reference we have for (12.2.9) is the book by J. Tannery and J. Molk [273, Section 486, pp. 134–136]. Entry 12.2.2 is also equivalent to a theorem discovered independently by R.J. Evans [136, eq. (3.1)], V.G. Kač and D.H. Peterson [169, equation (5.26)], and Kač and M. Wakimoto [170, middle of p. 438].

12.3 Applications of the Partial Fraction Decompositions

In this section, we analyze eight identities from the lost notebook that are fairly direct corollaries of the general identities in Section 12.2.

We begin with the third identity on page 8 of the lost notebook (also proved in [24, p. 18, equation (3.8)]).

Entry 12.3.1 (p. 8). *If $\psi(q)$ is Ramanujan's classical theta function defined in* (1.1.7) *of Chapter* 1, *then*

$$\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{(n+1)^2}}{(q;q^2)_{n+1}^2} = \frac{1}{\psi(-q)} \sum_{n=0}^{\infty} \frac{(-1)^n q^{2n^2+3n+1}}{1-q^{2n+1}}. \tag{12.3.1}$$

Proof. Apply (12.2.7) with $c=1$ and recall the familiar product expansion for $\psi(q)$ given in equation (1.1.7) of Chapter 1. The result then follows. □

Next, we examine the fifth formula on page 8 of the lost notebook (also proved in [24, pp. 18–19, equation (3.8)]).

Entry 12.3.2 (p. 8). *Recall that Ramanujan's theta function $\varphi(q)$ is defined by* (1.1.6) *in Chapter* 1. *Then*

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n q^{n^2}}{(-a;q^2)_{n+1}(-q^2/a;q^2)_n} - (1+1/a) \sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n q^{(n+1)^2}}{(-aq;q^2)_{n+1}(-q/a;q^2)_{n+1}} \\ = \frac{(q;q^2)_\infty \varphi(-q)}{(-a;q)_\infty (-q/a;q)_\infty}. \end{aligned} \tag{12.3.2}$$

Proof. Let $L(q)$ denote the left side of (12.3.2). Then, by (12.2.6), (12.2.7), and lastly (12.2.9), we find that

$$L(q) = \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty} \left\{ \sum_{n=-\infty}^{\infty} \frac{q^{2n^2+n}}{1+aq^{2n}} \right.$$

$$-(1+1/a)\sum_{n=0}^{\infty}\frac{(1+q^{2n+1})q^{(n+1)(2n+1)}}{(1+aq^{2n+1})(1+q^{2n+1}/a)}\Bigg\}$$

$$=\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}\left\{\sum_{n=-\infty}^{\infty}\frac{q^{n(2n+1)}}{1+aq^{2n}}-\sum_{n=-\infty}^{\infty}\frac{q^{(n+1)(2n+1)}}{1+aq^{2n+1}}\right\} \tag{12.3.3}$$

$$=\frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(n+1)/2}}{1+aq^n} \tag{12.3.4}$$

$$=\frac{(q;q^2)_\infty(q;q)_\infty^2}{(q^2;q^2)_\infty(-a;q)_\infty(-q/a;q)_\infty}$$

$$=\frac{(q;q^2)_\infty\varphi(-q)}{(-a;q)_\infty(-q/a;q)_\infty},$$

where we have used the product representation for $\varphi(-q)$ given in (1.1.6) of Chapter 1. $\square$

Next, we prove the second identity on page 4 of the lost notebook, which we also proved in [24, p. 20, equation (3.11)].

Entry 12.3.3 (p. 4; First Version).

$$\sum_{n=0}^{\infty}\frac{(-1)^n(q;q^2)_n q^{n^2}}{(-aq^2;q^2)_n(-q^2/a;q^2)_n}=\sum_{n=0}^{\infty}\frac{(-1)^n(q;q^2)_n}{(-aq;q)_n(-q/a;q)_n}+\frac{(q;q^2)_\infty\varphi(-q)}{2(-aq;q)_\infty(-q/a;q)_\infty}. \tag{12.3.5}$$

This identity has an obvious problem; namely, the first series on the right side of (12.3.5) is clearly a divergent series. However, as was noted in [24, p. 37],

$$\lim_{\alpha\to1^-}\sum_{n=0}^{\infty}\frac{(-1)^n(\alpha q;q)_n(q;q^2)_n}{(q;q)_n(-\alpha aq;q)_n(-\alpha q/a;q)_n}\alpha^n =\frac{(q;q^2)_\infty}{2(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(1+1/a)(1+a)q^{n(n+1)/2}}{(1+aq^n)(1+q^n/a)}, \tag{12.3.6}$$

which follows from (12.2.1) by replacing a by α, then setting $b=-1/a$, $c=-a$, $d=-e=\sqrt{q}$, and finally letting $N\to\infty$ and $\alpha\to1^-$. Thus, we replace the divergent series on the right side of (12.3.5) by the right side of (12.3.6) and restate the entry.

Entry 12.3.4 (p. 4; Second Version).

$$\sum_{n=0}^{\infty}\frac{(-1)^n(q;q^2)_nq^{n^2}}{(-aq^2;q^2)_n(-q^2/a;q^2)_n}=\frac{(q;q^2)_\infty}{2(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{(1+1/a)(1+a)q^{n(n+1)/2}}{(1+aq^n)(1+q^n/a)} + \frac{(q;q^2)_\infty\varphi(-q)}{2(-aq;q)_\infty(-q/a;q)_\infty}. \tag{12.3.7}$$

Proof. There are obvious similarities between Entries 12.3.2 and 12.3.4, which we shall utilize. To more clearly light a path from the former entry to the latter, we shall, for brevity, set

$$S_1 := \sum_{n=0}^{\infty}\frac{(-1)^n(q;q^2)_nq^{n^2}}{(-aq^2;q^2)_n(-q^2/a;q^2)_n},$$
$$S_2 := \sum_{n=-\infty}^{\infty}\frac{q^{n(2n+1)}}{1+aq^{2n}}, \qquad S_3 := \sum_{n=-\infty}^{\infty}\frac{q^{(n+1)(2n+1)}}{1+aq^{2n+1}},$$
$$S_4 := \sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n(n+1)/2}}{1+aq^n},$$
$$S_5 := \sum_{n=-\infty}^{\infty}\frac{(1+1/a)(1+a)q^{n(n+1)/2}}{(1+aq^n)(1+q^n/a)},$$
$$X := \frac{(q;q^2)_\infty\varphi(-q)}{(-a;q)_\infty(-q/a;q)_\infty}, \quad \text{and} \quad Y := \frac{(q;q^2)_\infty}{(q^2;q^2)_\infty}.$$

Now, by (12.3.2) and (12.3.3), we have shown that

$$\frac{1}{1+a}S_1 - YS_3 = X,$$

or, since by (12.3.4) $S_2 - S_3 = S_4$, we have equivalently shown that

$$\frac{1}{1+a}S_1 - Y(S_2 - S_4) = X. \tag{12.3.8}$$

Now, by (12.3.7), we want to prove that

$$\frac{1}{1+a}S_1 = \frac{1}{2(1+a)}YS_5 + \frac{1}{2}X.$$

But by (12.3.8) and the obvious equality $X = YS_4$, this is equivalent to proving that

$$Y(S_2 - S_4) + \frac{1}{2}YS_4 = \frac{1}{2(1+a)}YS_5.$$

Canceling Y and rearranging, we find that this is equivalent to showing that

$$S_2 - \frac{1}{2}S_4 = \frac{1}{2(1+a)}S_5. \tag{12.3.9}$$

To that end,

$$S_2 - \frac{1}{2(1+a)}S_5 - \frac{1}{2}S_4 = -\frac{1}{2}\left\{\sum_{n=-\infty}^{\infty}\frac{(1+1/a)q^{n(n+1)/2}}{(1+aq^n)(1+q^n/a)}\right.$$
$$\left.-\sum_{n=-\infty}^{\infty}\frac{q^{n(n+1)/2}}{1+aq^n}\right\} = -\frac{1}{2a}\sum_{n=-\infty}^{\infty}\frac{(1-q^n)q^{n(n+1)/2}}{(1+aq^n)(1+q^n/a)} = 0,$$

because the last series is equal to its negative, which can be seen by replacing n by $-n$. Thus, (12.3.9) has been demonstrated, and so the proof of Entry 12.3.4 is complete. $\square$

Next on the agenda is the massive third identity on page 39 of the lost notebook, which we divided by $1+a$.

Entry 12.3.5 (p. 39). *For $a \neq 0$,*

$$\sum_{n=0}^{\infty}\frac{q^{3n^2}}{(-a;q^3)_{n+1}(-q^3/a;q^3)_n} - \sum_{n=0}^{\infty}\frac{q^{3n^2+3n+1}}{(-aq;q^3)_{n+1}(-q^2/a;q^3)_{n+1}}$$
$$-\frac{1}{a}\sum_{n=0}^{\infty}\frac{q^{3n^2+3n+1}}{(-q/a;q^3)_{n+1}(-aq^2;q^3)_{n+1}} = \frac{(q)_\infty^2}{(q^3;q^3)_\infty(-a)_\infty(-q/a)_\infty}. \quad (12.3.10)$$

Proof. To simplify the left side of (12.3.10), we apply (12.2.3) with $c=-a$ and q replaced by q^3 to the first series, apply (12.2.5) with $c=-aq^{-1/2}$ and q replaced by q^3 to the second series, and apply (12.2.5) with $c=-aq^{1/2}$ and q replaced by q^3 to the third series. The right side of (12.3.10) may be converted into partial fractions by (12.2.9). Upon multiplying both sides by $(q^3;q^3)_\infty$, we find that (12.3.10) has been transformed into the assertion that

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{3n(3n+1)/2}}{1+aq^{3n}} - \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{9n(n+1)/2+1}}{1+aq^{3n+1}}$$
$$-\frac{1}{a}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{9n(n+1)/2+1}}{1+aq^{3n+2}} = \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(n+1)/2}}{1+aq^n}. \quad (12.3.11)$$

However, this assertion is easily verified if we subdivide the sum on the right side according to residues of n modulo 3, thereby deducing that

$$\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(n+1)/2}}{1+aq^n} = \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{3n(3n+1)/2}}{1+aq^{3n}}$$
$$-\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(3n+1)(3n+2)/2}}{1+aq^{3n+1}} + \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(3n+2)(3n+3)/2}}{1+aq^{3n+2}}$$
$$= \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{3n(3n+1)/2}}{1+aq^{3n}} - \sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{9n(n+1)/2+1}}{1+aq^{3n+1}}$$

$$-\frac{1}{a}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{9n(n+1)/2+1}}{1+aq^{3n+2}}+\frac{q}{a}\sum_{n=-\infty}^{\infty}\frac{(-1)^n(1+aq^{3n+2})q^{9n(n+1)/2}}{1+aq^{3n+2}}.$$

Now the last sum above equals 0, because replacing n by $-n-1$ changes the sum into its negative. Alternatively, we can appeal to (12.1.1). Thus, we have established (12.3.11) and in turn (12.3.10). □

Our next formula, which is the first one on page 39 of the lost notebook, is almost a direct corollary of (12.3.10). The function on the left side of (12.3.12) below is $f(q^3)$, where now f denotes one of Ramanujan's third order mock theta functions.

Entry 12.3.6 (p. 39). *We have*

$$\sum_{n=0}^{\infty}\frac{q^{3n^2}}{(-q^3;q^3)_n^2}=4\sum_{n=0}^{\infty}\frac{q^{3n^2+3n+1}}{(-q;q^3)_{n+1}(-q^2;q^3)_{n+1}}+\frac{\varphi^2(-q)}{(q^3;q^3)_\infty}. \tag{12.3.12}$$

Proof. The identity (12.3.12) immediately follows from (12.3.10) if we set $a=1$ there, multiply both sides by 2, and note that by the familiar product representation in (1.1.6) of Chapter 1,

$$\frac{(q;q)_\infty^2}{(q^3;q^3)_\infty(-q;q)_\infty^2}=\frac{\varphi^2(-q)}{(q^3;q^3)_\infty}.$$

□

The formulation of the next entry is slightly different from that given in the third formula on page 39 of [228]. In the aforementioned formula, set $x=q^2$, replace a by aq, and divide both sides by $(1+aq)$. We will then obtain (12.3.15) below, provided that we can show that

$$1-a\sum_{n=0}^{\infty}\frac{q^{6n^2+6n+1}}{(-aq;q^6)_{n+1}(-q^5/a;q^6)_{n+1}}=\sum_{n=0}^{\infty}\frac{q^{6n^2}}{(-aq;q^6)_{n+1}(-q^5/a;q^6)_n}. \tag{12.3.13}$$

If we can prove that

$$\sum_{n=0}^{N}\frac{q^{6n^2}}{(-aq;q^6)_{n+1}(-q^5/a;q^6)_n}-1+aq\sum_{n=0}^{N}\frac{q^{6n^2+6n}}{(-aq;q^6)_{n+1}(-q^5/a;q^6)_{n+1}}$$
$$=-\frac{q^{6(N+1)^2}}{(-aq;q^6)_{N+1}(-q^5/a;q^6)_{N+1}}, \tag{12.3.14}$$

for every nonnegative integer N, then letting $N\to\infty$ in (12.3.14) would yield (12.3.13). We proceed by induction on N. For $N=0$, it is easily checked that (12.3.14) holds. Assume (12.3.14) holds with N replaced by $N-1$. Then on the left-hand side of (12.3.14), the difference between the Nth and $(N-1)$st cases is

$$\frac{q^{6N^2}}{(-aq;q^6)_{N+1}(-q^5/a;q^6)_N}+\frac{aq^{6N^2+6N+1}}{(-aq;q^6)_{N+1}(-q^5/a;q^6)_{N+1}}$$
$$=\frac{q^{6N^2}(1+aq^{6N+1}+q^{6N+5}/a)}{(-aq;q^6)_{N+1}(-q^5/a;q^6)_{N+1}}.$$

On the other hand, on the right side of (12.3.14), the differences between the Nth and $(N-1)$st cases is

$$-\frac{q^{6(N+1)^2}}{(-aq;q^6)_{N+1}(-q^5/a;q^6)_{N+1}}+\frac{q^{6N^2}}{(-aq;q^6)_N(-q^5/a;q^6)_N}$$
$$=\frac{q^{6N^2}(1+aq^{6N+1}+q^{6N+5}/a)}{(-aq;q^6)_{N+1}(-q^5/a;q^6)_{N+1}}.$$

Since the right-hand sides of the two foregoing equalities are identical, by induction, (12.3.14) is valid for all $N \geq 0$, and so as we have seen (12.3.13) is also valid.

Entry 12.3.7 (p. 39). *For $a \neq 0$,*

$$1-a\sum_{n=0}^{\infty}\frac{q^{6n^2+6n+1}}{(-aq;q^6)_{n+1}(-q^5/a;q^6)_{n+1}}-\frac{1}{a}\sum_{n=0}^{\infty}\frac{q^{6n^2+6n+1}}{(-q/a;q^6)_{n+1}(-aq^5;q^6)_{n+1}}$$
$$-\sum_{n=0}^{\infty}\frac{q^{6n^2+6n+2}}{(-aq^3;q^6)_{n+1}(-q^3/a;q^6)_{n+1}}=\frac{(q^2;q^2)_\infty^2}{(q^6;q^6)_\infty(-aq;q^2)_\infty(-q/a;q^2)_\infty}. \tag{12.3.15}$$

Proof. The proof of (12.3.15) follows the same pattern as that for (12.3.10). To simplify the left side of (12.3.15), we apply (12.2.5) to each of the three series of (12.3.15), after replacing q by q^6. For the first, second, and third sums in (12.3.15), we take, respectively, $c=-a/q^2$, $c=-aq^2$, and $c=-a$. Thus, (12.3.15) is equivalent to the assertion that

$$1-\frac{1}{(q^6;q^6)_\infty}\left\{aq\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{9n(n+1)}}{1+aq^{6n+1}}+\frac{q}{a}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{9n(n+1)}}{1+aq^{6n+5}}\right.$$
$$\left.-q^2\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{9n(n+1)}}{1+aq^{6n+3}}\right\}$$
$$=\frac{1}{(q^6;q^6)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{n(n+1)}}{1+aq^{2n+1}}$$
$$=\frac{1}{(q^6;q^6)_\infty}\left\{\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{3n(3n+1)}}{1+aq^{6n+1}}-q^2\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{9n(n+1)}}{1+aq^{6n+3}}\right.$$

$$+q^6 \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(3n+5)}}{1+aq^{6n+5}} \Bigg\}, \tag{12.3.16}$$

where we have dissected the sum after the first equal sign according to the residues of the index modulo 3. Now deleting the identical sums from each side above, multiplying both sides by $(q^6;q^6)_\infty$, and invoking Euler's pentagonal number theorem, given in (1.1.8) of Chapter 1, we find that (12.3.16) is equivalent to the assertion that

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{9n^2+3n} - \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n(3n+1)}(1+aq^{6n+1})}{1+aq^{6n+1}} - \frac{q}{a} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{9n(n+1)}(1+aq^{6n+5})}{1+aq^{6n+5}} = 0. \tag{12.3.17}$$

However, this last equality is almost immediate. The second sum cancels the first, and the last sum equals zero by an application of (12.1.1). The equivalence of (12.3.17), (12.3.16), and (12.3.15) reveals that (12.3.15) is true. □

Our next formula, which is the fourth formula on page 17 in the lost notebook, is an immediate corollary of (12.3.15). The series on the right side of (12.3.18) below is $\omega(\sqrt{q})$, where $\omega(q)$ is one of Ramanujan's third order mock theta functions.

Entry 12.3.8 (p. 17). *If $\psi(q)$ is the theta function defined by* (1.1.7) *in Chapter* 1, *then*

$$\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^{1/6};q)_{n+1}(q^{5/6};q)_n} = \frac{1}{2} + \frac{q^{1/3}}{2} \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q^{1/2};q)_{n+1}^2} + \frac{\psi^2(q^{1/6})}{2(q)_\infty}. \tag{12.3.18}$$

Proof. Set $a=-1$ and replace q by $q^{1/6}$ in (12.3.15). This yields, in light of the product representation for $\psi(q)$ given in (1.1.7) of Chapter 1,

$$1 + 2q^{1/6} \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q^{1/6};q)_{n+1}(q^{5/6};q)_{n+1}} - q^{1/3} \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q^{1/2};q)_{n+1}^2} = \frac{(q^{1/3};q^{1/3})_\infty^2}{(q)_\infty (q^{1/6};q^{1/3})_\infty^2}. \tag{12.3.19}$$

So we see that (12.3.19) is equivalent to (12.3.18), because

$$\begin{aligned} -1 &= -\sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^{1/6};q)_n(q^{5/6};q)_n} + \sum_{n=0}^{\infty} \frac{q^{(n+1)^2}}{(q^{1/6};q)_{n+1}(q^{5/6};q)_{n+1}} \\ &= \sum_{n=0}^{\infty} \frac{q^{n^2+n+1/6}\left(q^{n+5/6} - (1-q^{n+5/6})(1-q^{n+1/6})q^{-n-1/6}\right)}{(q^{1/6};q)_{n+1}(q^{5/6};q)_{n+1}} \end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{q^{n^2+n+1/6}\left(1-(1-q^{n+5/6})q^{-n-1/6}\right)}{(q^{1/6};q)_{n+1}(q^{5/6};q)_{n+1}}$$

$$= q^{1/6}\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q^{1/6};q)_{n+1}(q^{5/6};q)_{n+1}} - \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q^{1/6};q)_{n+1}(q^{5/6};q)_{n}}.$$

This completes the proof. □

We conclude this section with the seventh formula on page 5 of the lost notebook, also proved in [24, p. 17, equation (3.2)].

Entry 12.3.9 (p. 5). *For* $a \neq 0$,

$$\left(1+\frac{1}{a}\right)\sum_{n=0}^{\infty} \frac{(-q;q)_{2n}q^{n+1}}{(aq;q^2)_{n+1}(q/a;q^2)_{n+1}}$$
$$= \frac{1}{\varphi(-q)}\sum_{n=0}^{\infty}(-1)^n\left(\frac{q^{(n+1)^2}}{1-aq^{2n+1}}+\frac{q^{(n+1)^2}}{a-q^{2n+1}}\right). \quad (12.3.20)$$

Proof. To prove (12.3.20), return to (12.2.1) with q replaced by q^2, $a = q^2$, $d = -q$, $e = -q^2$, and $N \to \infty$. Then set $b = q/a$ and $c = aq$. Multiply the result by $q(1+1/a)/\{(1-aq)(1-q/a)\}$, cancel $1/(1-q)$, and also multiply both sides by $(-q;q)_\infty/(q^2;q)_\infty$. After algebraic simplification, we find that

$$(1+1/a)\sum_{n=0}^{\infty} \frac{(-q;q)_{2n}q^{n+1}}{(aq;q^2)_{n+1}(q/a;q^2)_{n+1}}$$
$$= \frac{(1+1/a)(-q;q)_\infty}{(q;q)_\infty}\sum_{n=0}^{\infty} \frac{(-1)^n(1-q^{2n+1})q^{(n+1)^2}}{(1-aq^{2n+1})(1-q^{2n+1}/a)}$$
$$= \frac{1}{\varphi(-q)}\sum_{n=0}^{\infty}(-1)^n\left(\frac{1}{1-aq^{2n+1}}+\frac{1}{a-q^{2n+1}}\right)q^{(n+1)^2},$$

where we used the product representation for $\varphi(-q)$ given in (1.1.6) of Chapter 1. □

12.4 Partial Fractions Plus

The first result of this section arises from Bailey's transformation [49, p. 196, equation (2.4)], which we now describe. As usual, the bilateral basic hypergeometric series ${}_r\psi_r$ is defined by

$${}_r\psi_r\left[\begin{matrix}\alpha_1,\alpha_2,\ldots,\alpha_r\\ \beta_1,\beta_2,\ldots,\beta_r\end{matrix};q,z\right] = \sum_{n=-\infty}^{\infty} \frac{(\alpha_1)_n(\alpha_2)_n\cdots(\alpha_r)_n}{(\beta_1)_n(\beta_2)_n\cdots(\beta_r)_n}z^n, \qquad |z|<1.$$

Bailey's transformation is then given by

$$
{}_2\psi_2\left[\begin{matrix}\alpha,\beta\\ \gamma,\delta\end{matrix};q,z\right]=\frac{(\alpha z)_\infty(\beta z)_\infty(\gamma q/(\alpha\beta z))_\infty(\delta q/(\alpha\beta z))_\infty}{(q/\alpha)_\infty(q/\beta)_\infty(\gamma)_\infty(\delta)_\infty}\times{}_2\psi_2\left[\begin{matrix}\alpha\beta z/\gamma,\alpha\beta z/\delta\\ \alpha z,\beta z\end{matrix};q,\frac{\gamma\delta}{\alpha\beta z}\right]. \tag{12.4.1}
$$

Entry 12.4.1 (p. 5). *For any complex number a,*

$$
\frac{1}{\varphi(-q)}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{(n+1)^2}}{1-aq^{2n+1}}=\frac{1}{f(-aq,-q/a)}\sum_{n=-\infty}^{\infty}\frac{(-1)^n a^n q^{(n+1)^2}}{1-q^{2n+1}}, \tag{12.4.2}
$$

where $f(a,b)$ denotes Ramanujan's general theta function, defined by (1.1.5) *in Chapter* 1.

Proof. We apply Bailey's transformation (12.4.1) by replacing q with q^2, then setting $\alpha=q/\tau$, $\beta=a/q$, $\gamma=\tau/q$, $\delta=aq$, $z=\tau$, and lastly letting τ approach 0. To obtain the final form of (12.4.2), one needs to apply the Jacobi triple product identity, given in Lemma 1.2.2 of Chapter 1, and also to divide both sides by $-(1-a/q)\varphi(-q)$. □

Our next result is the sixth identity on page 5 of the lost notebook. It was proved as identity (3.1) in [24].

Entry 12.4.2 (p. 5). *For $a\neq 0$,*

$$
(-aq)_\infty(-q/a)_\infty(q)_\infty\sum_{n=0}^{\infty}\frac{(-1)^n(q;q^2)_n q^{n^2}}{(-aq^2;q^2)_n(-q^2/a;q^2)_n}=1+\sum_{n=1}^{\infty}\left(2(-1)^n+a^n+a^{-n}\right)\frac{q^{n(n+1)/2}}{1+q^n}. \tag{12.4.3}
$$

Proof. We begin by noting that the right side of (12.4.3) may be transformed in the same manner that (12.4.2) was proved. Apply (12.4.1) with $\alpha=-q/\tau$, $\beta=-1$, $\gamma=\tau$, $z=a\tau$, $\delta=-q$, and then let $\tau\to 0$. We also use a familiar representation for $\varphi^2(-q)$ as a Lambert series, due to Jacobi [166, p. 238, eq. (14)] and not surprisingly rediscovered by Ramanujan [227], [61, p. 114, Entry 8(v)], and then we convert it into its infinite product representation by (1.1.6) in Chapter 1. Alternatively, we can appeal to Entry 12.2.2 with $c=-1$. Accordingly, we find that

$$
\begin{aligned}
1+&\sum_{n=1}^{\infty}\left(2(-1)^n+a^n+a^{-n}\right)\frac{q^{n(n+1)/2}}{1+q^n}\\
&=\sum_{n=-\infty}^{\infty}\frac{a^n q^{n(n+1)/2}}{1+q^n}+\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(n+1)/2}}{1+q^n}\\
&=\frac{(-a)_\infty(-q/a)_\infty}{2(-q)_\infty^2}\sum_{n=-\infty}^{\infty}\frac{q^{n(n+1)/2}}{1+aq^n}+\frac{(q)_\infty^2}{2(-q)_\infty^2}.
\end{aligned} \tag{12.4.4}
$$

Hence, replacing the right side of (12.4.3) by the right side of (12.4.4) and the series on the left side of (12.4.3) by its representation in (12.2.6), we see that (12.4.3) is equivalent to the identity

$$\frac{(q;q^2)_\infty(-a)_\infty(-q/a)_\infty(q)_\infty}{(q^2;q^2)_\infty}\sum_{n=-\infty}^{\infty}\frac{q^{2n^2+n}}{1+aq^{2n}}$$
$$=\frac{(-a)_\infty(-q/a)_\infty}{2(-q)_\infty^2}\sum_{n=-\infty}^{\infty}\frac{q^{n(n+1)/2}}{1+aq^n}+\frac{(q)_\infty^2}{2(-q)_\infty^2},$$

which in turn is equivalent to

$$\frac{(-a)_\infty(-q/a)_\infty}{2(-q)_\infty^2}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n(n+1)/2}}{1+aq^n}=\frac{(q)_\infty^2}{2(-q)_\infty^2}, \tag{12.4.5}$$

where we have used Euler's identity (12.1.3). But (12.4.5) is simply a restatement of (12.2.9) with c replaced by $-a$. Hence, (12.4.3) has been proved. □

We now turn to a rather more problematic result, the fifth identity on page 5 of the lost notebook. The technique we use is patterned after the method of Watson expounded in [289, pp. 67–68], [82, pp. 335–336] . We have divided both sides of the entry by $1+a$ before stating it below.

Entry 12.4.3 (p. 5). *For $a \neq 0$,*

$$\sum_{n=0}^{\infty}\frac{(-1)^n(q^2;q^4)_n q^{2n^2}}{(-a;q^4)_{n+1}(-q^4/a;q^4)_n}+(1+1/a)\sum_{n=0}^{\infty}\frac{(-q;q)_{2n}q^{n+1}}{(aq;q^2)_{n+1}(q/a;q^2)_{n+1}}$$
$$=\frac{(-aq^2;q^4)_\infty(-q^2/a;q^4)_\infty\psi(q)}{(q;q^2)_\infty(-a;q^4)_\infty(-q^4/a;q^4)_\infty(aq;q^2)_\infty(q/a;q^2)_\infty}. \tag{12.4.6}$$

Proof. If we apply (12.2.6) with q replaced by q^2 and $c=a$ to the first series on the left side in (12.4.6), and (12.3.20) to the second series on the left side of (12.4.6), we find that (12.4.6) is equivalent to the assertion that

$$\frac{(q^2;q^4)_\infty}{(q^4;q^4)_\infty}\sum_{n=-\infty}^{\infty}\frac{q^{4n^2+2n}}{1+aq^{4n}}-\frac{(-q;q)_\infty}{(q;q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^n q^{n^2}}{1-aq^{2n-1}}$$
$$=\frac{(-aq^2;q^4)_\infty(-q^2/a;q^4)_\infty(q^2;q^2)_\infty}{(q;q^2)_\infty^2(-a;q^4)_\infty(-q^4/a;q^4)_\infty(aq;q^2)_\infty(q/a;q^2)_\infty}, \tag{12.4.7}$$

where we have invoked the product representations for $\varphi(-q)$ and $\psi(q^2)$ given in (1.1.6) and (1.1.7) of Chapter 1, respectively.

We now consider a partial product for the right-hand side of (12.4.7) and decompose it into partial fractions. In other words, we want to calculate the coefficients $A(N,n)$ and $B(N,n)$ defined by

$$\frac{(-aq^2;q^4)_N(-q^2/a;q^4)_N(q^2;q^2)_\infty}{(q;q^2)_\infty^2(-a;q^4)_{N+1}(-q^4/a;q^4)_N(aq;q^2)_{2N}(q/a;q^2)_{2N+1}} = \sum_{n=-N}^{N}\frac{A(N,n)}{1+aq^{4n}} + \sum_{n=-2N}^{2N}\frac{B(N,n)}{1-aq^{2n-1}}. \qquad (12.4.8)$$

Observe that it appears that there is a pole at $a=0$ on the left side of (12.4.8). However, it is a removable singularity. Also note that both the left and right sides of (12.4.8) tend to 0 as $a\to\infty$. An equality of the form (12.4.8) therefore follows from the Mittag-Leffler theorem. In light of the fact that all the poles are simple, we can compute $A(N,m)$ by multiplying each side of (12.4.8) by $1+aq^{4m}$ and then setting $a=-q^{-4m}$. Similarly, we can compute $B(N,m)$ upon multiplying both sides by $1-aq^{2m-1}$ and then setting $a=q^{1-2m}$. After algebraically simplifying each computation, we find that

$$A(N,m) = \frac{(q^2;q^4)_{N-m}(q^2;q^4)_{N+m}(q^2;q^2)_\infty q^{4m^2+2m}}{(q;q^2)_\infty^2(q^4;q^4)_{N-m}(q^4;q^4)_{N+m}(-q;q^2)_{2N-2m}} \times \frac{1}{(-q;q^2)_{2N+2m+1}}, \qquad (12.4.9)$$

$$B(N,2m) = -\frac{(-q^3;q^4)_{N-m}(-q;q^4)_{N+m}(q^2;q^2)_\infty q^{(2m)^2}}{(q;q^2)_\infty^2(-q;q^4)_{N-m+1}(-q^3;q^4)_{N+m}} \times \frac{1}{(q^2;q^2)_{2N-2m}(q^2;q^2)_{2N+2m}}, \qquad (12.4.10)$$

$$B(N,2m+1) = \frac{(-q;q^4)_{N-m}(-q^3;q^4)_{N+m}(q^2;q^2)_\infty q^{(2m+1)^2}}{(q;q^2)_\infty^2(-q^3;q^4)_{N-m}(-q;q^4)_{N+m+1}} \times \frac{1}{(q^2;q^2)_{2N-2m-1}(q^2;q^2)_{2N+2m+1}}. \qquad (12.4.11)$$

If we let $N\to\infty$, we find from (12.4.9)–(12.4.11) that

$$\lim_{N\to\infty} A(N,m) = \frac{(q^2;q^4)_\infty q^{4m^2+2m}}{(q^4;q^4)_\infty} \qquad (12.4.12)$$

and

$$\lim_{N\to\infty} B(N,m) = \frac{(-1)^m(-q)_\infty q^{m^2}}{(q)_\infty}. \qquad (12.4.13)$$

We now let $N\to\infty$ in (12.4.8) and use the calculations (12.4.12) and (12.4.13). Because by (12.4.9)–(12.4.11), the series in (12.4.8) converge uniformly for $|q|\le 1-\epsilon$, for each $\epsilon>0$, taking the limits on N under the summation signs is justified. We therefore deduce (12.4.6), and the proof is complete. $\square$

We now turn to a similar identity, which is the fifth one on page 39 of the lost notebook.

Entry 12.4.4 (p. 39). *For $a \neq 0$,*

$$\begin{aligned}\frac{1}{2}\sum_{n=0}^{\infty}\frac{q^{n^2}}{(-a)_{n+1}(-q/a)_n}-\frac{1}{2}\sum_{n=0}^{\infty}\frac{q^{n^2}}{(a)_{n+1}(q/a)_n}\\ = a\sum_{n=0}^{\infty}\frac{q^{(2n+1)^2}}{(-a^2q;q^4)_{n+1}(-q^3/a^2;q^4)_{n+1}}\\ +\frac{(q^4;q^4)_\infty\varphi(q)}{a(-a^2q;q^4)_\infty(-q^3/a^2;q^4)_\infty f(-a^2q^2,-1/a^2)},\end{aligned} \tag{12.4.14}$$

where $\varphi(q)$ and $f(a,b)$ are defined in (1.1.6) *and* (1.1.5), *respectively, in Chapter* 1.

Proof. First, transform the two series on the left side of (12.4.14) by using (12.2.3) with $c=-a$ and $c=a$, respectively, and then add the resulting two series together. Then apply (12.2.5) with q replaced by q^4 and $c=-a^2/q$. Replace $\varphi(q)$ by its product representation given in (1.1.6) of Chapter 1. Finally, apply (12.1.2) with $n=1$ and then use the product representation for $f(a^2,q^2/a^2)$ from Lemma 1.2.2 of Chapter 1. After all of this, we find that (12.4.14) is equivalent to the identity

$$\begin{aligned}\frac{1}{(q)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{3n(n+1)/2}}{1-a^2q^{2n}} = -\frac{q}{(q^4;q^4)_\infty}\sum_{n=-\infty}^{\infty}\frac{(-1)^nq^{6n(n+1)}}{1+a^2q^{4n+1}}\\ +\frac{(q^4;q^4)_\infty(-q;q^2)_\infty^2}{(-a^2q;q^4)_\infty(-q^3/a^2;q^4)_\infty(a^2;q^2)_\infty(q^2/a^2;q^2)_\infty}.\end{aligned} \tag{12.4.15}$$

As with (12.4.7), we prove (12.4.15) by considering a partial fraction decomposition for a partial product truncation of the last expression on the right side of (12.4.15). To that end, write

$$\begin{aligned}\frac{(q^4;q^4)_\infty(-q;q^2)_\infty^2}{(-a^2q;q^4)_N(-q^3/a^2;q^4)_N(a^2;q^2)_{2N+1}(q^2/a^2;q^2)_{2N}}\\ =\sum_{n=-2N}^{2N}\frac{C(N,n)}{1-a^2q^{2n}}+\sum_{n=-N}^{N-1}\frac{D(N,n)}{1+a^2q^{4n+1}}.\end{aligned} \tag{12.4.16}$$

As was the case with (12.4.8), it appears that there is a pole at $a=0$ on the left side of (12.4.16), but as before, it is a removable singularity. Also note that both the left and right sides of (12.4.16) tend to 0 as $a\to\infty$, and so an equality of the form (12.4.16) therefore follows from the Mittag-Leffler theorem. In light of the fact that all the poles of the quotient on the left side above are simple, we can compute $C(N,m)$ by multiplying each side of (12.4.16) by $1-a^2q^{2m}$ and then setting $a^2=q^{-2m}$. Similarly, we can compute $D(N,m)$ upon multiplying both sides by $1+a^2q^{4m+1}$ and then setting $a^2=-q^{-4m-1}$. Upon simplification, we find that

$$C(N,2m)=\frac{(q^4;q^4)_\infty(-q;q^2)_\infty^2 q^{6m^2+3m}}{(-q;q^4)_{N-m}(-q^3;q^4)_{N+m}(q^2;q^2)_{2N-2m}(q^2;q^2)_{2N+2m}}, \tag{12.4.17}$$

$$C(N,2m-1)=-\frac{(q^4;q^4)_\infty(-q;q^2)_\infty^2 q^{6m^2-3m}}{(-q^3;q^4)_{N-m}(-q;q^4)_{N+m}(q^2;q^2)_{2N-2m+1}(q^2;q^2)_{2N+2m-1}}, \tag{12.4.18}$$

$$D(N,m)=\frac{(-1)^m(q^4;q^4)_\infty(-q;q^2)_\infty^2 q^{6m^2+6m+1}}{(q^4;q^4)_{N-m-1}(q^4;q^4)_{N+m}(-q;q^2)_{2N-2m}(-q;q^2)_{2N+2m+1}}. \tag{12.4.19}$$

If we let $N\to\infty$ in (12.4.17)–(12.4.19), we find, after much simplification, that

$$\lim_{N\to\infty}C(N,m)=\frac{(-1)^m q^{3m(m+1)/2}}{(q)_\infty} \tag{12.4.20}$$

and

$$\lim_{N\to\infty}D(N,m)=\frac{(-1)^m q^{6m^2+6m+1}}{(q^4;q^4)_\infty}. \tag{12.4.21}$$

Furthermore, it is clear from the representations for $C(N,m)$ and $D(N,m)$ given in (12.4.17)–(12.4.19) that the series on the right side of (12.4.16) converge uniformly for $|q|\leq 1-\epsilon$, for each $\epsilon>0$. We may then take the limit on N under the summation sign in (12.4.16) and use (12.4.20) and (12.4.21) to confirm the truth of (12.4.15). □

We now turn to Ramanujan's first assertion in the lost notebook, the first identity on page 1. Perhaps it is to be expected that its proof is more intricate than any other in this chapter. The proof that we have fashioned requires several different q-series devices to accomplish the task.

Entry 12.4.5 (p. 1). *For $a\neq 0$,*

$$\left(1+\frac{1}{a}\right)\sum_{n=0}^\infty\frac{(-q)_{2n}q^n}{(aq;q^2)_{n+1}(q/a;q^2)_{n+1}}$$
$$=\frac{(-q)_\infty}{(aq;q^2)_\infty}\sum_{n=0}^\infty\frac{a^{-n-1}q^{2n(n+1)}}{(q/a;q^2)_{n+1}(q;q^2)_{n+1}}-\sum_{n=0}^\infty\frac{(-q)_nq^n}{(aq;q^2)_{n+1}}. \tag{12.4.22}$$

Proof. If we repeat the argument given at the beginning of the proof of Entry 12.3.9, we deduce that

$$\left(1+\frac{1}{a}\right)\sum_{n=0}^\infty\frac{(-q)_{2n}q^n}{(aq;q^2)_{n+1}(q/a;q^2)_{n+1}}$$
$$=(1+1/a)\frac{(-q)_\infty}{(q)_\infty}\sum_{n=0}^\infty\frac{(-1)^n(1-q^{2n+1})q^{n^2+2n}}{(1-aq^{2n+1})(1-q^{2n+1}/a)}$$

$$= \frac{(-q)_\infty}{a(q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+2n}}{1-q^{2n+1}/a}. \tag{12.4.23}$$

Next, we apply Theorem A_3 of [12], namely,

$$\sum_{n=0}^{\infty} \frac{(a;q^2)_n(b;q)_{2n}}{(q^2;q^2)_n(c;q)_{2n}} t^n = \frac{(b;q)_\infty(at;q^2)_\infty}{(c;q)_\infty(t;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(c/b;q)_n(t;q^2)_n}{(q;q)_n(at;q^2)_n} b^n,$$

with $t = q^2$, $b = q$, $c = 0$, and a replaced by aq. After simplification, this yields

$$\sum_{n=0}^{\infty} \frac{(-q)_n q^n}{(aq;q^2)_{n+1}} = \frac{(-q)_\infty}{(aq;q^2)_\infty} \sum_{n=0}^{\infty} (q;q^2)_n(aq;q^2)_n q^{2n}. \tag{12.4.24}$$

Hence, in light of (12.4.24), we may transform the right side of (12.4.22) to find that

$$\begin{aligned} R(q) &:= \frac{(-q)_\infty}{(aq;q^2)_\infty} \sum_{n=0}^{\infty} \frac{a^{-n-1}q^{2n(n+1)}}{(q/a;q^2)_{n+1}(q;q^2)_{n+1}} - \sum_{n=0}^{\infty} \frac{(-q)_n q^n}{(aq;q^2)_{n+1}} \\ &= \frac{(-q)_\infty}{(aq;q^2)_\infty} \left\{ \sum_{n=0}^{\infty} \frac{a^{-n-1}q^{2n(n+1)}}{(q/a;q^2)_{n+1}(q;q^2)_{n+1}} - \sum_{n=0}^{\infty} (q;q^2)_n(aq;q^2)_n q^{2n} \right\} \\ &= \frac{(-q)_\infty}{(aq;q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{a^{-n}q^{2n(n-1)}}{(q/a;q^2)_n(q;q^2)_n}, \end{aligned} \tag{12.4.25}$$

where to obtain the last line, we replaced n by $n+1$ in the first sum and n by $-n$ in the second sum of the previous line. Observe that in the last line we have used the calculation

$$\begin{aligned} &\frac{a^n q^{2n^2+2n}}{(q/a;q^2)_{-n}(q;q^2)_{-n}} = \frac{(q^{1-2n}/a;q^2)_\infty(q^{1-2n};q^2)_\infty a^n q^{2n^2+2n}}{(q/a;q^2)_\infty(q;q^2)_\infty} \\ &\quad = (1-q^{-2n+1}/a)\cdots(1-q^{-1}/a)(1-q^{-2n+1})\cdots(1-q^{-1})a^n q^{2n^2+2n} \\ &\quad = (-1)^n a^{-n} q^{-n^2}(aq;q^2)_n(-1)^n q^{-n^2}(q;q^2)_n a^n q^{2n^2+2n} \\ &\quad = (aq;q^2)_n(q;q^2)_n q^{2n}. \end{aligned}$$

Now let, as is customary,

$$[z^0] \sum_{n=-\infty}^{\infty} A_n z^n := A_0.$$

We use below Ramanujan's famous ${}_1\psi_1$ summation [61, p. 34, equation (17.6)], [29, p. 115, equation (C.2)]

$$\sum_{n=-\infty}^{\infty} \frac{(a;q)_n}{(b;q)_n} t^n = \frac{(b/a;q)_\infty (at;q)_\infty (q/(at);q)_\infty (q;q)_\infty}{(q/a;q)_\infty (b;q)_\infty (t;q)_\infty (b/(at);q)_\infty} \tag{12.4.26}$$

and the special case [29, p. 115, equation (C.3)]

$$\sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n(n-1)/2}}{(b;q)_n} t^n = \frac{(t;q)_\infty (q/t;q)_\infty (q;q)_\infty}{(b/t;q)_\infty (b;q)_\infty}, \tag{12.4.27}$$

which arises from (12.4.26) by replacing t with t/a and letting $a \to \infty$.

Returning to our work above in (12.4.25); employing (12.4.27) twice, first with q replaced by q^2, $b = q/a$, and $t = z$, and second with q replaced by q^2, $b = q$, and $t = 1/(az)$; utilizing the Jacobi triple product identity (Lemma 1.2.2 of Chapter 1); and lastly using (12.4.26) with q replaced by q^2, with a replaced by q/a, and then with $b = q^3/a$ and $t = azq$, and then dividing both sides by $1 - q/a$; we find that

$$\begin{aligned}
R(q) &= \frac{(-q)_\infty}{(aq;q^2)_\infty} [z^0] \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2-n} z^n}{(q/a;q^2)_n} \sum_{m=-\infty}^{\infty} \frac{(-1)^m a^{-m} q^{m^2-m} z^{-m}}{(q;q^2)_m} \\
&= \frac{(-q)_\infty}{(aq;q^2)_\infty} [z^0] \frac{(z;q^2)_\infty (q^2/z;q^2)_\infty (q^2;q^2)_\infty}{(q/(az);q^2)_\infty (q/a;q^2)_\infty} \\
&\quad \times \frac{(1/(az);q^2)_\infty (azq^2;q^2)_\infty (q^2;q^2)_\infty}{(azq;q^2)_\infty (q;q^2)_\infty} \\
&= \frac{(-q)_\infty}{(q)_\infty} [z^0] (1/(az);q^2)_\infty (azq^2;q^2)_\infty (q^2;q^2)_\infty \\
&\quad \times \frac{(-z)(1/z;q^2)_\infty (zq^2;q^2)_\infty (q^2;q^2)_\infty^2}{(aq;q^2)_\infty (q/a;q^2)_\infty (azq;q^2)_\infty (q/(az);q^2)_\infty} \\
&= \frac{(-q)_\infty}{(q)_\infty} [z^0] \sum_{n=-\infty}^{\infty} a^{-n} q^{n^2-n} z^{-n} (-z) \sum_{m=-\infty}^{\infty} \frac{(azq)^m}{1 - q^{2m+1}/a} \\
&= \frac{(-q)_\infty}{(q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n a^{-n} q^{n^2-n} (-1) a^{n-1} q^{n-1}}{1 - q^{2n-1}/a} \\
&= \frac{(-q)_\infty}{a(q)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{n^2+2n}}{1 - q^{2n+1}/a},
\end{aligned} \tag{12.4.28}$$

where we replaced n by $n+1$ in the last step.

Noting that the right sides of (12.4.23) and (12.4.28) are identical, we conclude that their corresponding left sides are also identical, and this completes the proof of (12.4.23). □

12.5 Related Identities

In this section, we prove three results. Two of these involve some of the series that have arisen in previous sections of this chapter, but they are not proved using partial fractions. The third result was effectively proved by Watson in [289], and so we relegate it also to this section.

We begin with the second identity on page 8 of the lost notebook.

Entry 12.5.1 (p. 8). *We have*

$$\sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n q^{n+1}}{(q;q^2)_{n+1}} = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{(n+1)^2}}{(q;q^2)_{n+1}^2}. \tag{12.5.1}$$

Proof. We follow the proof given in [24, pp. 28–29] for identity (3.6) of that paper. We first employ a transformation formula from Gasper and Rahman's treatise [140, p. 241, equation (III.9)], namely,

$$_3\phi_2 \left[\begin{matrix} a, b, c \\ d, e \end{matrix} ; q, \frac{de}{abc} \right] = \frac{(e/a;q)_\infty (de/(bc);q)_\infty}{(e;q)_\infty (de/(abc);q)_\infty} \, {}_3\phi_2 \left[\begin{matrix} a, d/b, d/c \\ d, de/(bc) \end{matrix} ; q, \frac{e}{a} \right].$$

Replacing q by q^2 and setting $a = q^2$, $b = -q^3/\tau$, $c = -q$, and $d = e = q^3$, we find that

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{(n+1)^2}}{(q;q^2)_{n+1}^2} &= \lim_{\tau \to 0} \frac{q}{(1-q)^2} \sum_{n=0}^{\infty} \frac{(-q^3/\tau;q^2)_n (-q;q^2)_n (q^2;q^2)_n}{(q^2;q^2)_n (q^3;q^2)_n (q^3;q^2)_n} \tau^n \\
&= \frac{q}{(1-q)^2} \frac{(q;q^2)_\infty}{(q^3;q^2)_\infty} \sum_{n=0}^{\infty} \frac{(q^2;q^2)_n (-q^2;q^2)_n q^n}{(q^3;q^2)_n (q^2;q^2)_n} \\
&= \frac{q}{1-q} \sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n q^n}{(q^3;q^2)_n} \\
&= \sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n q^{n+1}}{(q;q^2)_{n+1}},
\end{aligned}$$

as desired. □

We now turn to the first identity on page 8 of the lost notebook. This formula contains series that are specializations of series already examined in this chapter; see (12.3.11). Thus, this is a natural place to include this result, even though the methods of proof do not involve partial fractions.

Entry 12.5.2 (p. 8). *If* $\varphi(q)$ *and* $\psi(q)$ *are defined by* (1.1.6) *and* (1.1.7), *respectively, in Chapter* 1, *then*

$$\sum_{n=0}^{\infty} \frac{(-q^2;q^4)_n q^{2n^2}}{(-q^4;q^4)_n^2} + 4 \sum_{n=0}^{\infty} \frac{(-q)_{2n} q^{n+1}}{(-q^2;q^4)_{n+1}} = \frac{\varphi^2(q)}{\psi(-q^2)}. \tag{12.5.2}$$

Proof. To organize our efforts, we define

$$\lambda(q) := \sum_{n=0}^{\infty} \frac{(-q)_{2n} q^{n+1}}{(-q^2;q^4)_{n+1}} \tag{12.5.3}$$

and

$$\mu(q) := \sum_{n=0}^{\infty} \frac{(-1)^n (q;q^2)_n q^{n^2}}{(-q^2;q^2)_n^2}. \tag{12.5.4}$$

Then (12.5.2) can be written in the more succinct formulation

$$\mu(-q^2) + 4\lambda(q) = \frac{\varphi^2(q)}{\psi(-q^2)}. \tag{12.5.5}$$

In the following analysis, we shall employ the q-binomial theorem [21, p. 17, Theorem 2.1], [61, p. 14, Entry 2]

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} t^n = \frac{(at;q)_\infty}{(t;q)_\infty}. \tag{12.5.6}$$

Now, by five applications of (12.5.6), first with q replaced by q^4, $a = -q^2$, and $t = q^{4n+4}$, second with q replaced by q^2, $a = -q$, and $t = q^{4m+1}$, third with q replaced by q^2, $a = 0$, and $t = q^{4m+1}$, fourth with q replaced by q^8, $a = 0$, and $t = q^{4(n+1)}$, and fifth with q replaced by q^4, $a = -q^2$, and $t = q^2$,

$$\begin{aligned}
\lambda(q) &= \sum_{n=0}^{\infty} \frac{(-q;q^2)_n(-q^2;q^2)_n(q^2;q^2)_n q^{n+1}}{(q^2;q^2)_n(-q^2;q^4)_{n+1}} \\
&= \sum_{n=0}^{\infty} \frac{(-q;q^2)_n(q^4;q^4)_n q^{n+1}}{(q^2;q^2)_n(-q^2;q^4)_{n+1}} \\
&= \frac{(q^4;q^4)_\infty}{(-q^2;q^4)_\infty} \sum_{n=0}^{\infty} \frac{(-q;q^2)_n(-q^{4n+6};q^4)_\infty q^{n+1}}{(q^2;q^2)_n(q^{4n+4};q^4)_\infty} \\
&= \frac{(q^4;q^4)_\infty}{(-q^2;q^4)_\infty} \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n+1}}{(q^2;q^2)_n} \sum_{m=0}^{\infty} \frac{(-q^2;q^4)_m q^{4m(n+1)}}{(q^4;q^4)_m} \\
&= \frac{(q^4;q^4)_\infty}{(-q^2;q^4)_\infty} \sum_{m=0}^{\infty} \frac{(-q^2;q^4)_m q^{4m+1}}{(q^4;q^4)_m} \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{n(4m+1)}}{(q^2;q^2)_n} \\
&= \frac{(q^4;q^4)_\infty}{(-q^2;q^4)_\infty} \sum_{m=0}^{\infty} \frac{(-q^2;q^4)_m q^{4m+1}}{(q^4;q^4)_m} \frac{(-q^{4m+2};q^2)_\infty}{(q^{4m+1};q^2)_\infty} \\
&= \frac{q(q^4;q^4)_\infty(-q^2;q^2)_\infty}{(-q^2;q^4)_\infty(q;q^2)_\infty} \sum_{m=0}^{\infty} \frac{(q;q^2)_{2m}(-q^2;q^4)_m q^{4m}}{(-q^2;q^2)_{2m}(q^4;q^4)_m} \\
&= \frac{q(q^8;q^8)_\infty}{(q;q^2)_\infty} \sum_{m=0}^{\infty} \frac{(q;q^2)_{2m} q^{4m}}{(q^8;q^8)_m} \\
&= q(q^8;q^8)_\infty \sum_{m=0}^{\infty} \frac{q^{4m}}{(q^8;q^8)_m} \frac{1}{(q^{4m+1};q^2)_\infty} \\
&= q(q^8;q^8)_\infty \sum_{m=0}^{\infty} \frac{q^{4m}}{(q^8;q^8)_m} \sum_{n=0}^{\infty} \frac{q^{n(4m+1)}}{(q^2;q^2)_n}
\end{aligned}$$

$$
\begin{aligned}
&= q(q^8;q^8)_\infty \sum_{n=0}^{\infty} \frac{q^n}{(q^2;q^2)_n} \sum_{m=0}^{\infty} \frac{q^{4m(n+1)}}{(q^8;q^8)_m} \\
&= q(q^8;q^8)_\infty \sum_{n=0}^{\infty} \frac{q^n}{(q^2;q^2)_n (q^{4n+4};q^8)_\infty} \\
&= q \sum_{n=0}^{\infty} \frac{(q^8;q^8)_n q^{2n+1}}{(q^2;q^2)_{2n+1}} + q \frac{(q^8;q^8)_\infty}{(q^4;q^8)_\infty} \sum_{n=0}^{\infty} \frac{(q^4;q^8)_n q^{2n}}{(q^2;q^2)_{2n}} \\
&= q \sum_{n=0}^{\infty} \frac{(q^8;q^8)_n q^{2n+1}}{(q^2;q^2)_{2n+1}} + q \frac{(q^8;q^8)_\infty}{(q^4;q^8)_\infty} \sum_{n=0}^{\infty} \frac{(q^2;q^4)_n(-q^2;q^4)_n q^{2n}}{(q^2;q^4)_n (q^4;q^4)_n} \\
&= \sum_{n=0}^{\infty} \frac{(-q^4;q^4)_n q^{2(n+1)}}{(q^2;q^4)_{n+1}} + q \frac{(q^8;q^8)_\infty(-q^4;q^4)_\infty}{(q^4;q^8)_\infty (q^2;q^4)_\infty}. \qquad (12.5.7)
\end{aligned}
$$

If we now define

$$\alpha(q) := \sum_{n=0}^{\infty} \frac{(-q^2;q^2)_n}{(q;q^2)_{n+1}} q^{n+1}, \qquad (12.5.8)$$

then we may write (12.5.7) in the form

$$\lambda(q) = \alpha(q^2) + q \frac{(q^8;q^8)_\infty(-q^4;q^4)_\infty}{(q^4;q^8)_\infty (q^2;q^4)_\infty}. \qquad (12.5.9)$$

Now, by (12.5.1),

$$\alpha(q) = \sum_{n=0}^{\infty} \frac{(-q;q^2)_n q^{(n+1)^2}}{(q;q^2)_{n+1}^2}, \qquad (12.5.10)$$

and so, by (12.3.2) with $a = 1$ and q replaced by $-q$, we find that

$$\mu(-q) + 4\alpha(q) = \frac{\varphi(q)(-q;q^2)_\infty}{(q;q^2)_\infty^2 (-q^2;q^2)_\infty^2}. \qquad (12.5.11)$$

Hence, if we replace q by q^2 in (12.5.11), use the resulting equation to eliminate $\alpha(q^2)$ from (12.5.9), and use the product representation for $\varphi(q^2)$ from (1.1.6) of Chapter 1, we see that

$$\mu(-q^2) + 4\lambda(q) = \frac{(q^4;q^4)_\infty(-q^2;q^4)_\infty^3}{(q^2;q^4)_\infty^2(-q^4;q^4)_\infty^2} + 4q \frac{(q^8;q^8)_\infty(-q^4;q^4)_\infty}{(q^4;q^8)_\infty(q^2;q^4)_\infty}.$$

Using Euler's identity (12.1.3), the representation $f(1,q^4) = 2\psi(q^4)$ ([61, p. 34, Entry 18(ii); p. 36, Entry 22(ii)]), the product representations for $\varphi(q)$ and $\psi(q)$, given in (1.1.6) and (1.1.7), respectively, of Chapter 1, and considerable elementary product manipulations, we finally find that

$$\mu(-q^2) + 4\lambda(q) = \frac{(-q^2;q^4)_\infty}{(q^4;q^4)_\infty} \left\{ (q^4;q^4)_\infty^2(-q^2;q^4)_\infty^4 + 4q(q^4;q^4)_\infty^2(-q^4;q^4)_\infty^4 \right\}$$

$$= \frac{1}{\psi(-q^2)} \left\{ \varphi^2(q^2) + 4q\psi^2(q^4) \right\}$$
$$= \frac{\varphi^2(q)}{\psi(-q^2)}, \tag{12.5.12}$$

where we have used the identity

$$\varphi^2(q^2) + 4q\psi^2(q^4) = \varphi^2(q),$$

which follows from adding the two elementary identities

$$\varphi^2(q) - \varphi^2(-q) = 8q\psi^2(q^4) \quad \text{and} \quad \varphi^2(q) + \varphi^2(-q) = 2\varphi^2(q^2),$$

found as Entries 25(v), (vi) in Chapter 16 of Ramanujan's second notebook [227], [61, p. 40]. Since (12.5.12) is (12.5.5), which is what we wanted to prove, the proof of (12.5.2) is complete. □

We conclude with a result effectively proved by Watson in [289, p. 72], [82, pp. 339–340]. Watson [289, pp. 62–63], [82, pp. 330–331] clearly suggests that it is strange that Ramanujan was unaware of such a result as this. Indeed, it is now clear that Ramanujan knew everything Watson knew, and much more.

Entry 12.5.3 (p. 32). *For $a \neq 0$,*

$$a \sum_{n=0}^{\infty} \frac{q^{8n^2}}{(-a^2;q^8)_{n+1}(-q^8/a^2;q^8)_n} = \sqrt{q} \sum_{n=0}^{\infty} \frac{q^{(2n+1)^2/2}}{(-aq;q^2)_{n+1}(-q/a;q^2)_{n+1}}$$
$$+ q \sum_{n=0}^{\infty} \frac{q^{2(2n+1)^2}}{(-aq^4;q^8)_{n+1}(-q^4/a;q^8)_{n+1}} + \frac{\varphi^2(-q^4)(q^4;q^4)_\infty}{a\, f(aq,q/a) f(a^2q^4,1/a^2)}, \tag{12.5.13}$$

where $f(a,b)$ is Ramanujan's general theta function in (1.1.5) *of Chapter* 1.

Proof. Applying (12.2.3) with q replaced by q^8 and $c = -a^2$ to the left side of (12.5.13), applying (12.2.5) with q replaced by q^2 and $c = -a$ and then with q replaced by q^8 and $c = -a$ to the two series on the right side of (12.5.13), utilizing the Jacobi triple product identity (Lemma 1.2.2 of Chapter 1), and employing the product representation for $\varphi(-q^4)$ arising from (1.1.6) of Chapter 1, we see that (12.5.13) is equivalent to

$$\frac{a}{(q^8;q^8)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{12n^2+4n}}{1+a^2q^{8n}} = \frac{q}{(q^2;q^2)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{3n^2+3n}}{1+aq^{2n+1}}$$
$$+ \frac{q^3}{(q^8;q^8)_\infty} \sum_{n=-\infty}^{\infty} \frac{(-1)^n q^{12n^2+12n}}{1+aq^{8n+4}}$$
$$+ \frac{(-q^2;-q^2)_\infty (q^4;q^8)_\infty}{a\,(-aq;q^2)_\infty(-q/a;q^2)_\infty(-a^2q^4;q^4)_\infty(-1/a^2;q^4)_\infty}. \tag{12.5.14}$$

Because the remainder of the proof is very similar to previous proofs and is highly computational, we now defer to Watson. Identity (12.5.14) was proved by Watson [289, p. 72], [82, pp. 339–340] in the case $a = 1$ by specializing his general partial fraction expansion [289, p. 67], [82, pp. 334–335] with his variable $z = 0$. If he had left z arbitrary, he would have proved precisely (12.5.14), which is equivalent to (12.5.13). We note that Watson's proof of his general expansion is precisely analogous to our proof of (12.4.6) and (12.4.14). $\square$

12.6 Remarks on the Partial Fraction Method

In this chapter, we have considered a broad collection of results directly and indirectly related to partial fractions. Most of the identities we have chosen are related closely to what McIntosh [196] has called second order mock theta functions. The modular transformations of these functions were partially examined in [24] and completed in [196].

Watson's partial fraction decomposition [289, p. 67], [82, pp. 334–335] is presented in a form that is difficult to decode. M. Jackson [165] has stated the result in quite readable notation.

It should be stressed that the *ex nihilo* approach to (12.4.6) and (12.4.14) can be used to prove every partial fraction decomposition of the type considered here. Watson [289] used his general expansion to prove all the third order mock theta function identities including (12.5.13) in the case $a = 1$. It seems clear that this method was fully understood by Ramanujan and that it may well hold the key to many further developments in the theory of mock theta functions. S.H. Chan has employed partial fractions to effect a proof of Ramanujan's ${}_1\psi_1$ summation formula [120] and to derive many new general Lambert series identites, some connected with the theory of mock theta functions [121].

13

Hadamard Products for Two q-Series

13.1 Introduction

The third identity on page 57 of Ramanujan's lost notebook is given by

$$\sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q;q)_n} = \prod_{n=1}^{\infty} \left(1 + \frac{aq^{2n-1}}{1 - q^n y_1 - q^{2n} y_2 - q^{3n} y_3 - \cdots}\right), \tag{13.1.1}$$

where

$$y_1 = \frac{1}{(1-q)\psi^2(q)}, \tag{13.1.2}$$

$$y_2 = 0, \tag{13.1.3}$$

$$y_3 = \frac{q+q^3}{(1-q)(1-q^2)(1-q^3)\psi^2(q)} - \frac{\displaystyle\sum_{n=0}^{\infty} \frac{(2n+1)q^{2n+1}}{1-q^{2n+1}}}{(1-q)^3\psi^6(q)}, \tag{13.1.4}$$

$$y_4 = y_1 y_3, \tag{13.1.5}$$

and, as usual,

$$\psi(q) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}. \tag{13.1.6}$$

The series on the left-hand side of (13.1.1) is the series that arises in the standard proofs of the Rogers–Ramanujan identities [61, Chapter 16, pp. 77–78]. On the other hand, the infinite product bears no relation whatsoever to either of the familiar products appearing in the Rogers–Ramanujan identities. For the worker grounded in q-series, Ramanujan's assertion (13.1.1) is startling.

If this were not enough, in the middle of page 26 in the lost notebook, we find the assertion

$$\sum_{n=0}^{\infty} a^n q^{n^2} = \prod_{n=1}^{\infty} \left(1 + aq^{2n-1}\left(1 + y_1(n) + y_2(n) + \cdots\right)\right), \tag{13.1.7}$$

where

$$y_1(n) = \frac{\sum_{j=n}^{\infty}(-1)^j q^{j(j+1)}}{\sum_{j=0}^{\infty}(-1)^j(2j+1)q^{j(j+1)}} \tag{13.1.8}$$

and

$$y_2(n) = \frac{\left(\sum_{j=n}^{\infty}(-1)^j(j+1)q^{j(j+1)}\right)\left(\sum_{j=n}^{\infty}(-1)^j q^{j(j+1)}\right)}{\left(\sum_{j=0}^{\infty}(-1)^j(2j+1)q^{j(j+1)}\right)^2}. \tag{13.1.9}$$

The infinite series on the left side of (13.1.7) appears in several of Ramanujan's identities. These were first considered in [25] and will be elucidated in a subsequent volume in this series. Again, one is struck by the fact that we should not expect such an explicit infinite product expansion for this series. Yet contrary to our lack of such expectations, Ramanujan not only thought differently, but in fact, determined the product expansion for this series.

In order to obtain an overview of what is transpiring, one must step back from thinking in terms of q-series. The key lies in the fact that each of (13.1.1) and (13.1.7) are *entire functions* of the variable a. Hence, each has a Hadamard factorization [274, p. 246], and in each case Ramanujan is claiming that the products in question are, in fact, the relevant Hadamard factorizations.

The problem in finding the Hadamard factorization is to locate the zeros of each function. Those familiar with entire functions know that this is usually a nontrivial task. A small alteration in the definition of a function has a dramatic impact on the location of its zeros. For example, e^z has no zeros, while $1 + e^z$ has infinitely many zeros, all lying on the imaginary axis.

Our approach is to approximate each entire function by a convergent sequence of polynomials whose zero distributions are determinable. In Sections 13.2–13.7, we prove (13.1.1), and in Sections 13.8–13.11, we establish (13.1.7).

13.2 Stieltjes–Wigert Polynomials

In [269], G. Szegő extensively studied the polynomials [269, p. 245, equation (8)], [270, p. 33]

$$K_n(x) = \sum_{\nu=0}^{n} \begin{bmatrix} n \\ \nu \end{bmatrix} q^{\nu^2+\nu} x^{\nu}, \tag{13.2.1}$$

where $0 < q < 1$ and the Gaussian polynomials $\left[\begin{smallmatrix} n \\ \nu \end{smallmatrix}\right]$ are defined in Lemma 8.2.1 of Chapter 8. In [269, §3], he sets $q = e^{-1/2k^2}$ and quotes Wigert's proof that the polynomials

$$Q_n(x) = \frac{(-1)^n q^{n/2+1/4} K_n(-q^{-1/2}x)}{\sqrt{(1-q)(1-q^2)\cdots(1-q^n)}} \tag{13.2.2}$$

are orthogonal on $[0,\infty)$ with the weight function $p(x)$ given by

$$p(x) = \frac{k}{\sqrt{\pi}} e^{-k^2 \log^2 x}; \tag{13.2.3}$$

see [294]. Szegő then applies standard arguments from the theory of orthogonal polynomials to deduce that [269, p. 250, Property III] the zeros of each $K_n(x)$ are simple, real, and negative. We can add a little bit to Szegő's deductions from the general theory provided q is small.

Theorem 13.2.1. *For $0 < q < 1/4$ and for $i = 0, 1, 2, \ldots, n$,*

$$(-1)^i K_n(-q^{-2i-1}) > 0.$$

Proof. From the definition (13.2.1),

$$\begin{aligned}
(-1)^i K_n(-q^{-2i-1}) &= (-1)^i \sum_{\nu=0}^{n} (-1)^\nu \begin{bmatrix} n \\ \nu \end{bmatrix} q^{\nu^2 - 2i\nu} \\
&= \sum_{\nu=-\infty}^{\infty} (-1)^{\nu-i} \begin{bmatrix} n \\ \nu \end{bmatrix} q^{(\nu-i)^2 - i^2} \\
&= q^{-i^2} \sum_{\nu=-\infty}^{\infty} (-1)^\nu \begin{bmatrix} n \\ \nu+i \end{bmatrix} q^{\nu^2} \\
&= q^{-i^2} \left(\begin{bmatrix} n \\ i \end{bmatrix} + \sum_{\nu=1}^{\infty} (-1)^\nu \left(\begin{bmatrix} n \\ -\nu+i \end{bmatrix} + \begin{bmatrix} n \\ \nu+i \end{bmatrix} \right) \right) q^{\nu^2}.
\end{aligned}$$

Now for $0 < q < 1/4$, we note that

$$\begin{aligned}
0 \le \begin{bmatrix} A \\ B \end{bmatrix} &\le \frac{1}{(q;q)_\infty} = \frac{1}{1 - q - q^2 + q^5 + q^7 - \cdots} \\
&< \frac{1}{1-q-q^2} < \frac{1}{1 - 1/4 - 1/16} = \frac{16}{11},
\end{aligned}$$

and since the coefficients of $\left[\begin{smallmatrix} A \\ B \end{smallmatrix}\right]$ are always nonnegative, we see that for $0 \le B \le A$,

$$\begin{bmatrix} A \\ B \end{bmatrix} \ge 1.$$

Therefore, for $0 \le i \le n$,

$$
\begin{aligned}
(-1)^i K_n(-q^{-2i-1}) &\ge q^{-i^2}\left(1 - 2\cdot\frac{16}{11}\sum_{\nu=1}^{\infty} q^{\nu^2}\right) \\
&\ge q^{-i^2}\left(1 - 2\cdot\frac{16}{11}\cdot\frac{1}{4} - \frac{32}{11}\sum_{j=4}^{\infty}\left(\frac{1}{4}\right)^j\right) \\
&= q^{-i^2}\left(1 - \frac{8}{11} - \frac{32}{11}\cdot\frac{(1/4)^4}{1-1/4}\right) \\
&\ge 1 - \frac{8}{11} - \frac{32}{11\cdot 3\cdot 4^3} \\
&\ge 1 - \frac{8}{11} - \frac{1}{4^3} \\
&\ge 1 - \frac{9}{11} = \frac{2}{11},
\end{aligned}
$$

which completes the proof. □

Corollary 13.2.1. *For $0 < q < 1/4$, the ith zero of $K_n(x)$ lies in the interval $\left(-q^{1-2i}, -q^{-1-2i}\right)$, $i = 1, 2, \dots, n$.*

Proof. This follows immediately from Theorem 13.2.1 and the fact that $K_n(x)$ is a polynomial in x of degree n with alternating positive and negative values at $-q^{-1}, -q^{-3}, \dots, -q^{-2n-1}$. □

13.3 The Hadamard Factorization

We begin this section by recalling that if $f(z)$ is an entire function of order ρ with zeros $z_1, z_2, \dots$ and $f(0) \neq 0$, then [125, p. 174]

$$
f(z) = e^{H(z)}\prod_{n=1}^{\infty}\left(1 - \frac{z}{z_n}\right), \tag{13.3.1}
$$

where $H(z)$ is a polynomial of degree not exceeding ρ,

$$
\rho = \varlimsup_{n\to\infty}\frac{n\log n}{\log(1/|a_n|)}, \tag{13.3.2}
$$

and

$$
f(z) = \sum_{n=0}^{\infty} a_n z^n. \tag{13.3.3}
$$

Let us apply this factorization to $K_\infty(z)$. In this case,

$$
\begin{aligned}
0 \le \rho &= \varlimsup_{n\to\infty}\frac{n\log n}{\log\left(\dfrac{(1-q)(1-q^2)\cdots(1-q^n)}{q^{n^2}}\right)} \\
&\le \varlimsup_{n\to\infty} -\frac{n\log n}{|\log(q;q)_\infty| + n^2\log q} = 0.
\end{aligned} \tag{13.3.4}
$$

Hence, $e^{H(z)}$ is a constant, and since $K_\infty(0) = 1$, we see that

$$K_\infty(z) = \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right). \tag{13.3.5}$$

Furthermore, it follows from Corollary 13.2.1 and the interlacing theorem [124, p. 28] that z_n lies in the interval $\left(-q^{-1-2n}, -q^{1-2n}\right)$ for $n = 1, 2, 3, \ldots$. Hence, we have proved that for $0 < q < 1/4$,

$$\sum_{n=0}^{\infty} \frac{a^n q^{n^2}}{(q;q)_n} = \prod_{n=1}^{\infty} \left(1 - \frac{a}{q z_n}\right), \tag{13.3.6}$$

with $-q^{1-2n} > z_n > -q^{-1-2n}$.

What we need now is some way of obtaining explicit series for each z_n. If we write

$$z_n = -q^{-2n}\omega_n(q), \tag{13.3.7}$$

then for $0 < q < 1/4$, we have $q < \omega_n(q) < q^{-1}$.

13.4 Some Theta Series

In the next sections, we need information about

$$\theta_{m,k} := \theta_{m,k}(q) := \sum_{n=-\infty}^{\infty} (-1)^n n(n-1)\cdots(n-k+1) q^{n^2+mn}. \tag{13.4.1}$$

These series are closely related to the classical theta series; indeed, in the notation of [292, Chapter 21],

$$\theta_{0,0} = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = \frac{(q;q)_\infty}{(-q;q)_\infty} = \theta_4 \tag{13.4.2}$$

and

$$\theta_{1,1} = \sum_{n=-\infty}^{\infty} (-1)^n n q^{n^2+n} = (q^2;q^2)_\infty^3, \tag{13.4.3}$$

an identity of Jacobi [292, p. 472].

The following identities (most of which are scattered in the literature and can be derived from their definitions and fundamental properties [61, p. 34, Entry 18]) will be utilized in Section 13.5:

$$\theta_{2m,0} = (-1)^m q^{-m^2} \theta_4, \tag{13.4.4}$$

$$\theta_{2m+1,0} = 0, \tag{13.4.5}$$

$$\theta_{2m,1} = (-1)^{m-1} q^{-m^2} m \theta_4, \tag{13.4.6}$$

$$\theta_{2m+1,1} = (-1)^m q^{-m^2-m}\theta_{1,1}, \tag{13.4.7}$$
$$\theta_{2m,2} = (-1)^m q^{-m^2+1}\theta_4' + (-1)^m m(m+1)q^{-m^2}\theta_4, \tag{13.4.8}$$
$$\theta_{2m+1,2} = 2(-1)^{m+1}(m+1)q^{-m^2-m}\theta_{1,1}, \tag{13.4.9}$$
$$\theta_{2m,3} = 3(-1)^{m-1}(m+1)q^{-m^2+1}\theta_4' + (-1)^{m-1}m(m+1)(m+2)q^{-m^2}\theta_4, \tag{13.4.10}$$
$$\theta_{2m+1,3} = (-1)^m q^{1-m-m^2}\theta_{1,1}' + 3(-1)^m(m+1)(m+2)q^{-m^2-m}\theta_{1,1}, \tag{13.4.11}$$
$$\begin{aligned}\theta_{2m+1,4} &= 4(-1)^{m+1}(m+2)q^{1-m^2-m}\theta_{1,1}' \\ &\quad + 4(-1)^{m+1}(m+1)(m+2)(m+3)q^{-m^2-m}\theta_{1,1},\end{aligned} \tag{13.4.12}$$

where the prime $'$ indicates differentiation with respect to q. Each of these is proved in the same manner. We illustrate two proofs; the remainder are similar. We will see that four of the sums that arise are equal to 0. In each case, we can demonstrate this by taking the terms with negative index and replacing n by $-n-1$. First,

$$\begin{aligned}\theta_{2m+1,2} &= \sum_{n=-\infty}^{\infty}(-1)^n n(n-1)q^{n^2+n+2mn} \\ &= (-1)^m q^{-m^2-m}\sum_{n=-\infty}^{\infty}(-1)^n(n-m)(n-m-1)q^{n^2+n} \\ &= (-1)^m q^{-m^2-m}\sum_{n=-\infty}^{\infty}(-1)^n\left(n^2+n-(2m+2)n+m(m+1)\right)q^{n^2+n} \\ &= 0 + 2(-1)^{m+1}(m+1)q^{-m^2-m}\theta_{1,1} + 0,\end{aligned}$$

and second,

$$\begin{aligned}\theta_{2m,3} &= \sum_{n=-\infty}^{\infty}(-1)^n n(n-1)(n-2)q^{n^2+2mn} \\ &= (-1)^m q^{-m^2}\sum_{n=-\infty}^{\infty}(-1)^n(n-m)(n-m-1)(n-m-2)q^{n^2} \\ &= (-1)^m q^{-m^2}\sum_{n=-\infty}^{\infty}(-1)^n\left(n^3-3(m+1)n^2\right. \\ &\qquad \left.+(3m^2+6m+2)n-m(m+1)(m+2)\right)q^{n^2} \\ &= (-1)^m q^{-m^2}\left(0-3(m+1)q\theta_4'+0-m(m+1)(m+2)\theta_4\right).\end{aligned}$$

Comparable formulas can be found for all $\theta_{m,k}$, and elegant formulas for the coefficients can be produced using the methods of [44, Section 2].

Theorem 13.4.1. *Both* $q^{m^2}\theta_{2m,k}$ *and* $q^{m^2+m}\theta_{2m+1,k}$ *are analytic functions of* q *inside* $|q|<1$.

Proof. We have

$$\begin{aligned} q^{m^2}\theta_{2m,k} &= \sum_{n=-\infty}^{\infty}(-1)^n n(n-1)\cdots(n-k+1)q^{(n+m)^2} \\ &= (-1)^m\sum_{n=-\infty}^{\infty}(-1)^n(n-m)(n-m-1)\cdots(n-m-k+1)q^{n^2} \end{aligned}$$

and

$$\begin{aligned} q^{m^2+m}\theta_{2m+1,k} &= \sum_{n=-\infty}^{\infty}(-1)^n n(n-1)\cdots(n-k+1)q^{(n+m)^2+(n+m)} \\ &= (-1)^m\sum_{n=-\infty}^{\infty}(-1)^n(n-m)(n-m-1)\cdots(n-m-k+1)q^{n^2+n}, \end{aligned}$$

from which the desired conclusions immediately follow. □

Corollary 13.4.1. $q^{m(m+1)/2}\theta_{m+1,k}$ *is analytic in* q *for* $|q|<1$.

Proof. We have

$$q^{\mu(2\mu+1)}\theta_{2\mu+1,k} = q^{\mu^2}\left(q^{\mu^2+\mu}\theta_{2\mu+1,k}\right)$$

and

$$q^{\mu(2\mu-1)}\theta_{2\mu,k} = q^{\mu^2-\mu}\left(q^{\mu^2}\theta_{2\mu,k}\right).$$

So our assertion follows from Theorem 13.4.1. □

13.5 A Formal Power Series

Theorem 13.5.1. *Let* $y_i = y_i(q),\ i=1,2,3,\ldots,$ *satisfy, for each* $N \geq 0$,

$$\sum_{j=0}^{N}\left\{\frac{(-1)^{N-j}q^{(N-j)(N-j+1)/2}}{(q;q)_{N-j}} \sum_{k_1+2k_2+\cdots+jk_j=j}\frac{(-1)^{k_1+\cdots+k_j}y_1^{k_1}\cdots y_j^{k_j}\theta_{N-j+1,k_1+\cdots+k_j}}{k_1!\cdots k_j!}\right\} = 0, \quad (13.5.1)$$

where $k_i \geq 0,\ 1 \leq i \leq j$. *Then each* y_i *is a uniquely defined function of* q *analytic inside* $|q|<1$, *and the following identity holds as a formal power series identity in* z:

$$\sum_{n=-\infty}^{\infty}(-1)^n(1-zy_1-z^2y_2-z^3y_3-\cdots)^n(zq^{n+1};q)_\infty q^{n^2+n} = 0. \quad (13.5.2)$$

Proof. We begin with $N = 1$ in (13.5.1) which asserts that

$$0 = \frac{-q\theta_{2,0}}{1-q} - y_1\theta_{1,1}.$$

Hence, by (13.4.4),

$$y_1 = \frac{\theta_4}{(1-q)(q^2;q^2)_\infty^3} = \frac{1}{(1-q)\psi^2(q)}, \tag{13.5.3}$$

which we observe in passing is identical with the y_1 appearing in (13.1.2).

For $N = 2$, in (13.5.1), we see that

$$0 = \frac{q^3}{(q;q)_2}\theta_{3,0} + \frac{q}{(1-q)}y_1\theta_{2,1} + \frac{y_1^2\theta_{1,2}}{2} - y_2\theta_{1,1},$$

and by (13.5.1), (13.4.9), and (13.4.6),

$$\begin{aligned} y_2 &= \frac{y_1}{\theta_{1,1}}\left(\frac{y_1\theta_{1,2}}{2} + \frac{q\theta_{2,1}}{(1-q)}\right) \\ &= \frac{y_1}{\theta_{1,1}}\left(-\frac{\theta_4\theta_{1,1}}{(1-q)\theta_{1,1}} + \frac{\theta_4}{(1-q)}\right) = 0, \end{aligned}$$

which coincides with (13.1.3).

For $N = 3$, we find that

$$y_3 = \frac{1}{\theta_{1,1}}\left(-\frac{q^6\theta_{4,0}}{(q;q)_3} - \frac{q^3y_1\theta_{3,1}}{(q;q)_2} - \frac{qy_1^2\theta_{2,2}}{2(1-q)} - \frac{y_1^3\theta_{1,3}}{6}\right)$$

and, after simplification, with the use of (13.4.4), (13.4.7), (13.4.8), and (13.4.11),

$$\begin{aligned} y_3 &= \frac{q+q^3}{(q;q)_3\psi^2(q)} + \frac{q}{(1-q)^3\psi^6(q)}\left(\frac{1}{2}\frac{\theta_4'}{\theta_4} - \frac{1}{6}\frac{\theta_{1,1}'}{\theta_{1,1}}\right) \\ &= \frac{q+q^3}{(q;q)_3\psi^2(q)} - \frac{\displaystyle\sum_{n=1}^{\infty}\frac{(2n-1)q^{2n-1}}{1-q^{2n-1}}}{(1-q)^3\psi^6(q)}, \end{aligned} \tag{13.5.4}$$

where the final step requires logarithmic differentiation of the product representations of θ_4 and $\theta_{1,1}$. We therefore have a result coinciding with the formula for y_3 in (13.1.4).

Putting $N = 4$ in (13.5.1), we find that

$$\begin{aligned} y_4 = \frac{1}{\theta_{1,1}}\Bigg(&\frac{q^{10}\theta_{5,0}}{(q;q)_4} + \frac{q^6y_1\theta_{4,1}}{(q;q)_3} + \frac{q^3y_1^2\theta_{3,2}}{2(q;q)_2} \\ &+ \frac{qy_1^3\theta_{2,3}}{6(1-q)} + \frac{qy_3\theta_{2,1}}{1-q} + \frac{y_1^4\theta_{1,4}}{24} + y_1y_3\theta_{1,2}\Bigg). \end{aligned}$$

Using (13.4.5), (13.4.6), (13.4.7), (13.4.9), (13.4.10), and (13.4.12), we may simplify this latter expression to one involving $\theta_4, \theta_4', \theta_{1,1}$, and $\theta_{1,1}$. Thus, after simplification and the use of (13.5.3) and (13.5.4), we deduce that

$$y_4 = \frac{2(q+q^3)y_1^2}{(1-q^2)(1-q^3)} - y_1y_3 + y_1^4\left(\frac{q\theta_4'}{\theta_4} - \frac{q}{3}\,\frac{\theta_{1,1}'}{\theta_{1,1}}\right) = y_1y_3,$$

which is in agreement with (13.1.5).

For larger N, we see that y_N always appears uniquely in (13.5.1). Indeed, the only term containing y_N arises from $k_1 = \cdots = k_{N-1} = 0$ and $k_N = 1,\ j = N$. This term is therefore

$$-y_N\theta_{1,1}.$$

Consequently, for $N > 1$, we see that

$$y_N = \frac{1}{\theta_{1,1}} \sum_{j=0}^{N} \frac{(-1)^{N-j} q^{(N-j)(N-j+1)/2}}{(q;q)_{N-j}} \times \sum_{\substack{k_1+2k_2+\cdots+jk_j=j \\ k:=k_1+k_2+\cdots+k_j,\ k_i\geq 0 \\ \text{excluding} \\ k_1=\cdots=k_{N-1}=0, k_N=1}} \frac{(-1)^k y_1^{k_1}\cdots y_j^{k_j}\theta_{N-j+1,k}}{k_1!k_2!\cdots k_j!},$$

and proceeding by mathematical induction on N with the use of Corollary 13.4.1, we see that each y_N is analytic in q inside the unit circle.

Now we turn to (13.5.2). Clearly, the left-hand side of (13.5.2) defines a formal power series in z and q. While it first appears that $(zq^{n+1};q)_\infty q^{n^2+n}$ might contribute a negative power of q, we observe that by Euler's series [21, p. 19, equation (2.2.6)],

$$(zq^{n+1};q)_\infty = \sum_{m=0}^{\infty} \frac{(-1)^m q^{m(m+1)/2+mn} z^m}{(q;q)_m}.$$

Consequently, the exponent on q is

$$n^2 + n + \binom{m+1}{2} + mn = \binom{m+n+1}{2} + \binom{n+1}{2} \geq \binom{|n|}{2} > 0.$$

Now (13.5.2) is equivalent to

$$0 = [z^0]\frac{1}{N!}\,\frac{d^N}{dz^N} \sum_{n=-\infty}^{\infty} (-1)^n (1 - zy_1 - z^2y_2 - \cdots)^n (zq^{n+1};q)_\infty q^{n^2+n},$$

for every $N \geq 0$, where

$$[z^0]\sum_{m=0}^{\infty} a_m z^m := a_0.$$

To find the formal Nth derivative, we need several facts. First

$$\begin{aligned}[z^0]\ \frac{d^H}{dz^H}(zq^{n+1};q)_\infty &= [z^0]\sum_{i=H}^{\infty}\frac{(-1)^i i(i-1)\cdots(i-H+1)z^{i-H}q^{i(i+1)/2+in}}{(q;q)_i}\\ &= \frac{(-1)^H H!q^{H(H+1)/2+Hn}}{(q;q)_H}.\end{aligned}$$

Next, by the Faà di Bruno formula,

$$\begin{aligned}&[z^0]\frac{d^M}{dz^M}(1-y_1z-y_2z^2-\cdots)^n\\ &\qquad = \sum_{\substack{k_1+2k_2+\cdots+Mk_M=M\\ k:=k_1+\cdots+k_M, k_i\geq 0}} \frac{M!n(n-1)\cdots(n-k+1)(-1)^k y_1^{k_1}y_2^{k_2}\cdots y_M^{k_M}}{k_1!k_2!\cdots k_M!}.\end{aligned}$$

(For an excellent historical exposition of Faà di Bruno's formula, see W. Johnson's article [168].) Also, by Leibniz's rule,

$$\frac{d^N}{dz^N}f(z)g(z) = \sum_{j=0}^{N}\binom{N}{j}f^{(N-j)}(z)g^{(j)}(z).$$

Therefore,

$$\begin{aligned}&[z^0]\frac{1}{N!}\ \frac{d^N}{dz^N}\ \sum_{n=-\infty}^{\infty}(-1)^n(1-zy_1-z^2y_2-\cdots)^n(zq^{n+1};q)_\infty q^{n^2+n}\\ &= \frac{1}{N!}\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2+n}\sum_{j=0}^{N}\binom{N}{j}\\ &\quad\times \sum_{\substack{k_1+2k_2+\cdots+jk_j=j\\ k:=k_1+\cdots+k_j,\, k_i\geq 0}}\frac{j!n(n-1)\cdots(n-k+1)(-1)^k y_1^{k_1}\dots y_j^{k_j}}{k_1!k_2!\cdots k_j!}\\ &\quad\times(-1)^{N-j}\frac{(N-j)!}{(q;q)_{N-j}}q^{(N-j)(N-j+1)/2+(N-j)n}\\ &= \sum_{j=0}^{N}\frac{(-1)^{N-j}q^{(N-j)(N-j+1)/2}}{(q;q)_{N-j}}\sum_{\substack{k_1+2k_2+\cdots+jk_j=j\\ k:=k_1+\cdots+k_j,\, k_i\geq 0}}\frac{(-1)^k y_1^{k_1}\dots y_j^{k_j}\theta_{N-j+1,k}}{k_1!k_2!\cdots k_j!}\\ &= 0,\end{aligned}$$

and (13.5.2) is established. □

13.6 The Zeros of $K_\infty(zx)$

We know from Section 13.3 that $K_\infty(z)$ has real, simple, negative zeros $z_1, z_2, z_3, \dots$ with

$$-q^{1-2n} > z_n > -q^{-1-2n}, \tag{13.6.1}$$

provided that $0 < q < 1/4$. We shall sharpen this inequality.

Theorem 13.6.1. *For* $0 < q < 1/4$,

$$-q^{1-2n} > z_n > -q^{-2n}. \tag{13.6.2}$$

Proof. We know that $(-1)^i K_\infty\big(-q^{-2i-1}\big) > 0$ from the proof of Theorem 13.2.1. We also need to show that $(-1)^i K_\infty\big(-q^{-2i}\big) > 0$. We require Jacobi's triple product identity from Lemma 1.2.2 of Chapter 1, namely,

$$\sum_{n=-\infty}^{\infty} q^{n^2} z^n = \left(q^2;q^2\right)_\infty \left(-zq;q^2\right)_\infty \left(-z^{-1}q;q^2\right)_\infty . \tag{13.6.3}$$

Now,

$$\begin{aligned}
K_\infty(z) &= \sum_{n=0}^{\infty} \frac{q^{n^2+n} z^n}{(q;q)_n} \\
&= \frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} q^{n^2+n} z^n \left(q^{n+1};q\right)_\infty \\
&= \frac{1}{(q;q)_\infty} \sum_{n=-\infty}^{\infty} q^{n^2+n} z^n \sum_{m=0}^{\infty} \frac{(-1)^m q^{m(m+1)/2+mn}}{(q;q)_m} \\
&= \frac{1}{(q;q)_\infty} \left\{ \sum_{m=0}^{\infty} \frac{q^{m(2m+1)-m^2-m} z^{-m}}{(q;q)_{2m}} \sum_{n=-\infty}^{\infty} q^{(n+m)^2+n+m} z^{n+m} \right. \\
&\qquad \left. - \sum_{m=1}^{\infty} \frac{q^{m(2m-1)-m^2} z^{-m}}{(q;q)_{2m-1}} \sum_{n=-\infty}^{\infty} q^{(n+m)^2} z^{n+m} \right\} \\
&= (-q;q)_\infty (-zq^2;q^2)_\infty (-z^{-1};q^2)_\infty \sum_{m=0}^{\infty} \frac{q^{m^2} z^{-m}}{(q;q)_{2m}} \\
&\quad - (-q;q)_\infty (-zq;q^2)_\infty (-z^{-1}q;q^2)_\infty \sum_{m=1}^{\infty} \frac{q^{m^2-m} z^{-m}}{(q;q)_{2m-1}},
\end{aligned} \tag{13.6.4}$$

by (13.6.3). Hence, for any positive integer i,

$$\begin{aligned}
&(-1)^i K_\infty\big(-q^{-2i}\big) \\
&\quad = -(-1)^i (-q;q)_\infty \left(q^{1-2i};q^2\right)_\infty \left(q^{1+2i};q^2\right)_\infty \sum_{m=1}^{\infty} \frac{(-1)^m q^{m^2-m+2im}}{(q;q)_{2m-1}} \\
&\quad = q^{-i^2} (q;q^2)_\infty \left(\frac{q^{2i}}{1-q} - \frac{q^{2+4i}}{(1-q)(1-q^2)(1-q^3)} + \cdots \right).
\end{aligned} \tag{13.6.5}$$

Now observing that $i \geq 1$ and $0 < q < 1/4$, we see that the infinite series above is an alternating series and consequently has the lower bound

$$\frac{q^{2i}}{1-q}\left(1 - \frac{q^{2+2i}}{(1-q^2)(1-q^3)}\right) > 0.$$

Hence, for any positive integer i,

$$(-1)^i K_\infty\big(-q^{-2i}\big) > 0. \tag{13.6.6}$$

We already know that z_n lies in the open interval $\big(-q^{-1-2n}, -q^{1-2n}\big)$. Furthermore, we have just established that $K_\infty(-q^{1-2n})$ and $K_\infty(-q^{-2n})$ are of opposite signs. Hence, invoking Theorem 13.2.1, we find that $K_\infty(z)$ must have a zero in $\big(-q^{-2n}, -q^{1-2n}\big)$, and z_n is the only candidate for this role. Thus, Theorem 13.6.1 is proved. □

Next, we note that the zeros of each $K_n(z)$ are algebraic functions of the coefficients of $K_n(z)$, which are in turn polynomials in q. Thus, in turn, the zeros of $K_n(z)$ are analytic functions of q for $0 < q < 1/4$ (the domain specified in Theorem 13.2.1). This is an immediate corollary of the implicit function theorem [283], whose hypotheses are fulfilled here. Finally, for $n \geq N$ the zeros of $K_n(z)$, say $\zeta_{n,N}$, form a decreasing sequence in n (by the interlacing theorem [124, p. 28]). Consequently, $\zeta_{\infty,N} = z_N$ is analytic in q. So, by (13.3.7) and Theorem 13.6.1, $1 > \omega_n(q) > q$. Therefore, we conclude that for some sequence $\{a_{n,j}\}$, $j \geq 0$,

$$z_n = -q^{-2n}\sum_{j=0}^{\infty} a_{n,j}q^j.$$

Hence, the equality $K_\infty(z_n) = 0$ implies that

$$\begin{aligned}
0 &= \sum_{h=0}^{\infty} \frac{q^{h^2+h-2nh}(-1)^h}{(q;q)_h}\left(\sum_{j=0}^{\infty} a_{n,j}q^j\right)^h \\
&= \frac{(-1)^n q^{-n^2+n}}{(q;q)_\infty}\sum_{h=-\infty}^{\infty}(q^{h+1};q)_\infty q^{(h-n)^2}(-q)^{h-n}\left(\sum_{j=0}^{\infty} a_{n,j}q^j\right)^h \\
&= \frac{(-1)^n q^{-n^2+n}\left(\sum_{j=0}^{\infty} a_{n,j}q^j\right)^n}{(q;q)_\infty}\sum_{h=-\infty}^{\infty}(-1)^h(q^{h+n+1};q)_\infty q^{h^2+h}\left(\sum_{j=0}^{\infty} a_{n,j}q^j\right)^h.
\end{aligned}$$

Therefore,

$$0 = \sum_{h=-\infty}^{\infty}(-1)^h(q^{h+n+1};q)_\infty q^{h^2+h}\left(\sum_{j=0}^{\infty} a_{n,j}q^j\right)^h, \tag{13.6.7}$$

and the $a_{n,j}$ are uniquely determined from (13.6.7) in the same way that the y_i were determined in Theorem 13.5.1; however, this time we know in advance that the series $\sum_{n=0}^{\infty} a_{n,j} q^j$ converges in $0 < q < 1/4$. The change from Theorem 13.5.1 is that now (13.6.7) is valid as an analytic assertion for $0 < q < 1/4$ as well as a formal power series identity in q; recall that the derivation of (13.6.7) guarantees that there are no negative powers of q.

The reduction of (13.5.2) to a formal power series in q by the replacement of z by q^n means that we must have the formal series identity

$$\sum_{j=0}^{\infty} a_{n,j}\, q^j = 1 - \sum_{i=1}^{\infty} y_i q^{ni}. \tag{13.6.8}$$

Substituting (13.6.8) and (13.3.7) back into (13.3.6), we conclude the proof of our primary theorem.

Entry 13.6.1 (p. 57). *Identity* (13.1.1) *holds for all complex a and real q with* $0 < q < 1/4$.

13.7 Small Zeros of $K_\infty(z)$

The primary consequences of our work are the remarkable formulas (13.1.2)–(13.1.5), which provide the series expansions for the zeros z_n of $K_\infty(z)$. For example,

$$\begin{aligned}
z_1 &= -q^{-2}(1 - q + q^2 - 2q^3 + 4q^4 - \cdots) \\
z_2 &= -q^{-4}(1 - q^2 + q^3 - 2q^4 + 4q^5 - 7q^6 + 11q^7 - 18q^8 + 33q^9 - \cdots) \\
z_3 &= -q^{-6}(1 - q^3 + q^4 - 2q^5 + 4q^6 - 7q^7 + 11q^8 - 17q^9 \\
&\quad + 27q^{10} - 43q^{11} + 68q^{12} - 112q^{13} + 196q^{14} - \cdots) \\
z_4 &= -q^{-8}(1 - q^4 + q^5 - 2q^6 + 4q^7 - 7q^8 + 11q^9 - 17q^{10} \\
&\quad + 27q^{11} - 42q^{12} + 62q^{13} - 91q^{14} + 138q^{15} \\
&\quad - 213q^{16} + 334q^{17} - 549q^{18} + 957q^{19} - \cdots) \\
z_5 &= -q^{-10}(1 - q^5 + q^6 - 2q^7 + 4q^8 + 7q^9 + 11q^{10} - 17q^{11} \\
&\quad + 27q^{12} - 42q^{13} + 62q^{14} - 90q^{15} + 132q^{16} \\
&\quad - 192q^{17} + 275q^{18} - 398q^{19} + 591q^{20} \\
&\quad - 900q^{21} + 1417q^{22} - 2327q^{23} + 3971q^{24} - \cdots).
\end{aligned}$$

13.8 A New Polynomial Sequence

In order to prove (13.1.7), we must study a new sequence of polynomials

$$p_n(a) = \left(q^2;q^2\right)_\infty \left(-aq;q^2\right)_n \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_{q^2} \frac{q^{2j}}{\left(-aq;q^2\right)_j}. \tag{13.8.1}$$

Theorem 13.8.1. *If $p_n(a)$ is defined by* (13.8.1) *and $|q| < 1$, then*

$$\lim_{n\to\infty} p_n(a) = \sum_{j=0}^{\infty} a^j q^{j^2}. \tag{13.8.2}$$

Proof. From (13.8.1), we see that

$$\begin{aligned}
&\lim_{n\to\infty} p_n(a) = (q^2;q^2)_\infty \left(-aq;q^2\right)_\infty \sum_{j=0}^{\infty} \frac{q^{2j}}{\left(q^2;q^2\right)_j \left(-aq;q^2\right)_j} \\
&= \left(q^2;q^2\right)_\infty \left(-aq;q^2\right)_\infty {}_2\phi_1\left(0,0;-aq;q^2,q^2\right) \\
&= \left(q^2;q^2\right)_\infty \left(-aq;q^2\right)_\infty \lim_{b\to 0} \frac{\left(b;q^2\right)_\infty}{\left(q^2;q^2\right)_\infty \left(-aq;q^2\right)_\infty} {}_2\phi_1\left(-aq/b,q^2;0;q^2,b\right) \\
&= \sum_{j=0}^{\infty} a^j q^{j^2},
\end{aligned}$$

by Heine's transformation, equation (9.2.8) in Chapter 9. □

Theorem 13.8.2. *For $0 \le m < n$ and $0 < q < 1$,*

$$p_n(-q^{-2m-1}) > 0.$$

Proof. We know that the coefficients of $\begin{bmatrix} n \\ j \end{bmatrix}_q$ are positive. So assuming that $0 \le m < n$ and $0 < q < 1$, we see that

$$\begin{aligned}
\frac{1}{\left(q^2;q^2\right)_\infty} p_n(-q^{-2m-1}) &= \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_{q^2} \left(q^{-2m+2j};q^2\right)_{n-j} q^{2j} \\
&= \sum_{j=m+1}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_{q^2} (1-q^{2j-2m})\cdots(1-q^{2n-2m-2})q^{2j} \\
&> 0,
\end{aligned}$$

because each term of this sum is clearly positive. □

Theorem 13.8.3. *For $0 \le m < (1/2)(n-1)$ and $0 < q < 1/4$,*

$$p_n(-q^{-4m-2}) < 0.$$

Proof. We start with some auxiliary inequalities. First we recall that $\begin{bmatrix} A \\ B \end{bmatrix}_q$ is the generating function for partitions with at most B parts each not exceeding $A-B$ (see [21, p. 33]). Therefore, if $p(n)$ is the number of partitions of n, by [21, p. 4] and the pentagonal number theorem, (1.1.8) in Chapter 1,

$$0 \le \begin{bmatrix} A \\ B \end{bmatrix}_{q^2} \le \sum_{n=0}^{\infty} p(n) q^{2n} = \frac{1}{(q^2;q^2)_\infty}$$
$$= \frac{1}{1-q^2-q^4+q^{10}+q^{14}-\cdots}$$
$$< \frac{1}{1-q^2-q^4} < \frac{1}{1-\frac{1}{16}-\frac{1}{256}} < \frac{1}{1-\frac{2}{16}} = \frac{8}{7}. \tag{13.8.3}$$

Also,

$$\begin{aligned}(q;q^2)_\infty &= (1-q)(1-q^3)(1-q^5)(1-q^7)\cdots \\ &> (1-q)(1-q^2)(1-q^4)(1-q^6)\cdots \\ &> (1-q)(1-q^2-q^4) \\ &> \frac{3}{4}\left(1-\frac{1}{16}-\frac{1}{256}\right) = \frac{717}{1024} > \frac{7}{10}.\end{aligned} \tag{13.8.4}$$

To avoid confusion in our subsequent calculations, we also note that for $1 \le j \le 2m$, we have $j(4m-j) \ge 0$.

Hence, for $0 < q < 1/4$ and $0 \le m \le (1/2)(n-1)$,

$$\begin{aligned}\frac{1}{(q^2;q^2)_\infty} p_n(-q^{-4m-2}) &= \sum_{j=0}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_{q^2} \left(q^{-4m-1+2j};q^2\right)_{n-j} q^{2j} \\
&= \left(q^{-4m-1};q^2\right)_n + \sum_{j=1}^{2m} \begin{bmatrix} n \\ j \end{bmatrix}_{q^2} \left(q^{-4m-1+2j};q^2\right)_{n-j} q^{2j} \\
&\quad + \sum_{j=2m+1}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_{q^2} \left(q^{-4m-1+2j};q^2\right)_{n-j} q^{2j} \\
&= \left(q^{-4m-1};q^2\right)_{2m+1} \left(q;q^2\right)_{n-2m-1} \\
&\quad + \sum_{j=1}^{2m} \begin{bmatrix} n \\ j \end{bmatrix}_{q^2} \left(q^{-4m-1+2j};q^2\right)_{2m+1-j} \left(q;q^2\right)_{n-2m-1} q^{2j} \\
&\quad + \sum_{j=2m+1}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_{q^2} \left(q^{-4m-1+2j};q^2\right)_{n-j} q^{2j} \\
&= -q^{-(2m+1)^2} \left(q;q^2\right)_{2m+1} \left(q;q^2\right)_{n-2m-1} \\
&\quad + \sum_{j=1}^{2m} (-1)^{j-1} \begin{bmatrix} n \\ j \end{bmatrix}_{q^2} \left(q;q^2\right)_{2m+1-j} \left(q;q^2\right)_{n-2m-1} q^{2j-(2m-j+1)^2} \\
&\quad + \sum_{j=2m+1}^{n} \begin{bmatrix} n \\ j \end{bmatrix}_{q^2} \left(q^{-4m-1+2j};q^2\right)_{n-j} q^{2j} \\
&< -q^{-(2m+1)^2} \left(q;q^2\right)_\infty^2\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{2m}\begin{bmatrix} n \\ j \end{bmatrix}_{q^2} q^{-(2m+1)^2+4mj-j^2+4j}+\sum_{j=2m+1}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_{q^2} q^{2j} \\
&< -q^{-(2m+1)^2}\left(\left(q;q^2\right)_\infty^2-\sum_{j=1}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_{q^2} q^{2j}\right) \\
&< -q^{-(2m+1)^2}\left(\left(\frac{7}{10}\right)^2-\frac{8}{7}\sum_{j=1}^{\infty}\left(\frac{1}{4}\right)^{2j}\right) \\
&= -\frac{869}{2100}q^{-(2m+1)^2}<0,
\end{aligned}
$$

by (13.8.3) and (13.8.4). □

The final results in this section concern a related sequence of polynomials $\overline{p_n}(a)$ defined for $n \geq 0$ by

$$
\overline{p_n}(a)=\left(q^2;q^2\right)_\infty\sum_{j=0}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_{q^2}\left(-aq^{2j+1};q^2\right)_{n-j}. \tag{13.8.5}
$$

Theorem 13.8.4. *For $n \geq 0$,*

$$
p_{n+1}(a)-(1+aq^{2n+1})p_n(a)=q^{2n+2}\overline{p_n}(aq^2). \tag{13.8.6}
$$

Proof. From the definitions (13.8.1) and (13.8.5),

$$
\begin{aligned}
&\frac{1}{\left(q^2;q^2\right)_\infty}\left(\frac{p_n(a)}{\left(-aq;q^2\right)_n}-\frac{p_{n-1}(a)}{\left(-aq;q^2\right)_{n-1}}\right) \\
&\qquad=\sum_{j=0}^{n}\left(\begin{bmatrix} n \\ j \end{bmatrix}_{q^2}-\begin{bmatrix} n-1 \\ j \end{bmatrix}_{q^2}\right)\frac{q^{2j}}{\left(-aq;q^2\right)_j} \\
&\qquad=\sum_{j=0}^{n}q^{2n-2j}\begin{bmatrix} n-1 \\ j-1 \end{bmatrix}_{q^2}\frac{q^{2j}}{\left(-aq;q^2\right)_j} \\
&\qquad=q^{2n}\sum_{j=0}^{n-1}\begin{bmatrix} n-1 \\ j \end{bmatrix}_{q^2}\frac{1}{\left(-aq;q^2\right)_{j+1}} \\
&\qquad=\frac{q^{2n}\overline{p_{n-1}}(aq^2)}{\left(q^2;q^2\right)_\infty\left(-aq;q^2\right)_n}.
\end{aligned}
$$

Multiplying this identity by $(-aq;q^2)_n(q^2;q^2)_\infty$ and then replacing n by $n+1$, we obtain Theorem 13.8.4. □

Finally, we establish the positivity of $\overline{p_n}(a)$ in certain intervals.

Theorem 13.8.5. *For* $0 \leq i \leq (1/2)n$, $0 < q < 1/4$, *and* $-q^{-4i+1} > a > -q^{-4i-1}$,

$$\overline{p_n}(a) > 0.$$

Proof. We first examine the case $i = 0$. In this case, $-q > a > -q^{-1}$, and so $-q^2 > aq > -1$. Consequently, for $h \geq 0$,

$$1 + aq^{2h+1} > 1 + aq > 1 - 1 = 0.$$

Therefore, every term of $\overline{p_n}(a)$ is positive for a in this interval. So, $\overline{p_n}(a) > 0$ for $-q > a > -q^{-1}$.

Now we assume that $0 < i \leq (1/2)n$ and $0 < q < 1/4$, and also that $-q^{-4i+1} > a > -q^{-4i-1}$. Thus,

$$\begin{aligned}
\frac{1}{(q^2;q^2)_\infty}\overline{p_n}(a) &= \left(-aq;q^2\right)_{2i}\left(-aq^{4i+1};q^2\right)_{n-2i} \\
&\quad + \sum_{j=1}^{2i-1}\begin{bmatrix} n \\ j \end{bmatrix}_{q^2}\left(-aq^{2j+1};q^2\right)_{2i-j}\left(-aq^{4i+1};q^2\right)_{n-2i} \\
&\quad + \sum_{j=2i}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_{q^2}\left(-aq^{2j+1};q^2\right)_{n-j} \\
&= (1 + aq^{4i-1})\left(-aq^{4i+1};q^2\right)_{n-2i} \\
&\quad \times \left(\left(-aq;q^2\right)_{2i-1} + \sum_{j=1}^{2i-1}\begin{bmatrix} n \\ j \end{bmatrix}_{q^2}\left(-aq^{2j+1};q^2\right)_{2i-1-j}\right) \\
&\quad + \sum_{j=2i}^{n}\begin{bmatrix} n \\ j \end{bmatrix}_{q^2}\left(-aq^{2j+1};q^2\right)_{n-j}.
\end{aligned}$$

Every term of this last sum is positive, and every factor of $\left(-aq^{4i+1};q^2\right)_{n-2i}$ is positive, while $(1 + aq^{4i-1})$ is negative. So to prove that $\overline{p_n}(a) > 0$, we must prove that

$$-\left(-aq;q^2\right)_{2i-1} - \sum_{j=1}^{2i-1}\begin{bmatrix} n \\ j \end{bmatrix}_{q^2}\left(-aq^{2j+1};q^2\right)_{2i-1-j} > 0. \qquad (13.8.7)$$

Now,

$$\begin{aligned}
-\left(-aq;q^2\right)_{2i-1} &= (-a)^{2i-1}q^{(2i-1)^2}\left(1 + \frac{1}{aq}\right)\left(1 + \frac{1}{aq^3}\right)\cdots\left(1 + \frac{1}{aq^{4i-3}}\right) \\
&> (-a)^{2i-1}q^{(2i-1)^2}(1 - q^2)(1 - q^4)\cdots \\
&> (-a)^{2i-1}q^{(2i-1)^2}(1 - q^2 - q^4) \\
&> (-a)^{2i-1}q^{(2i-1)^2}\left(1 - \frac{1}{16} - \frac{1}{256}\right)
\end{aligned}$$

$$> (-a)^{2i-1} q^{(2i-1)^2} \frac{7}{8}. \tag{13.8.8}$$

Recalling (13.8.3) and using (13.8.8), we see that

$$\begin{aligned}
&- \left(-aq;q^2\right)_{2i-1} - \sum_{j=1}^{2i-1} \begin{bmatrix} n \\ j \end{bmatrix}_{q^2} \left(-aq^{2j+1};q^2\right)_{2i-1-j} \\
&\quad > |a|^{2i-1} q^{(2i-1)^2} \frac{7}{8} - \sum_{j=1}^{2i-1} \frac{8}{7} |a|^{2i-1-j} q^{(1/2)(2i-1-j)(2j+1+4i-3)} \\
&\quad = |a|^{2i-1} q^{(2i-1)^2} \left(\frac{7}{8} - \frac{8}{7} \sum_{j=1}^{2i-1} |a|^{-j} q^{-j^2} \right) \\
&\quad > |a|^{2i-1} q^{(2i-1)^2} \left(\frac{7}{8} - \frac{8}{7} \sum_{j=1}^{2i-1} q^{(4i-1)j-j^2} \right) \\
&\quad \geq |a|^{2i-1} q^{(2i-1)^2} \left(\frac{7}{8} - \frac{8}{7} \sum_{j=1}^{\infty} q^{2j} \right) \\
&\quad > |a|^{2i-1} q^{(2i-1)^2} \left(\frac{7}{8} - \frac{8}{7} \sum_{j=1}^{\infty} \left(\frac{1}{16}\right)^j \right) \\
&\quad = \frac{671}{840} |a|^{2i-1} q^{(2i-1)^2} > 0,
\end{aligned}$$

and with the establishment of this inequality, the inequality (13.8.7) is proved. This then completes the proof of Theorem 13.8.5. □

13.9 The Zeros of $p_n(a)$

Theorem 13.9.1. *If* $0 < q < 1/4$, *the zeros of* $p_n(a)$ *are simple, real, and negative. If we denote them by* $x_{n,i}$ $(1 \leq i \leq n)$, *then*

$$\begin{aligned}
&-q^{-1} > x_{n,1} > -q^{-2} > x_{n,2} > -q^{-3} \\
&> -q^{-5} > x_{n,3} > -q^{-6} > x_{n,4} > -q^{-7} > \cdots .
\end{aligned}$$

In general,

$$-q^{-4j-1} > x_{n,2j+1} > -q^{-4j-2} > x_{n,2j+2} > -q^{-4j-3}.$$

Proof. The assertion follows immediately once we recall from Theorems 13.8.2 and 13.8.3 that each of the values

$$p_n(-q^{-1}),\ \ p_n(-q^{-3}),\ \ \ldots,\ \ p_n(-q^{-(2n-1)})$$

is positive, while each of the values

$$p_n(-q^{-2}),\ \ p_n(-q^{-6}),\ \ \ldots,\ \ p_n(-q^{-(4s+2)})$$

is negative, where $4s+2$ is the largest number less than or equal to $2n$ that is congruent to 2 modulo 4.

If $2n-1$ is congruent to 3 modulo 4, this gives n sign changes in the appropriate intervals. If $2n-1$ is congruent to 1 modulo 4, then up to $-q^{-(2n-1)}$, there are $n-1$ sign changes, and there is one more in $(-q^{-(2n-1)}, -q^{-(2n)})$. In either case, the n zeros are necessarily simple, real, negative, and in the designated intervals. □

Theorem 13.9.2. *In the notation for the zeros of $p_n(a)$ given in Theorem* 13.9.1, *$\{x_{n,i}\}_{n\geq i}$ is a decreasing sequence in n if i is odd, and an increasing sequence if i is even.*

Proof. First consider $\{x_{n,2i-1}\}_{n\geq 2i-1}$. By Theorem 13.9.1,

$$-q^{-4i+3} > x_{n,2i-1} > -q^{-4i+2},$$

and by Theorem 13.8.4,

$$p_{n+1}(x_{n,2i-1}) = q^{2n+2}\overline{p_n}(q^2 x_{n,2i-1}).$$

Note that

$$-q^{-4i+5} > q^2 x_{n,2i-1} > -q^{-4i+4} > -q^{-4i+3},$$

and so by Theorem 13.8.5, $\overline{p_n}(q^2 x_{n,2i-1}) > 0$. Therefore,

$$p_{n+1}(x_{n,2i-1}) > 0.$$

But by Theorem 13.8.3,

$$p_{n+1}(-q^{-4i+2}) < 0,$$

and so

$$x_{n,2i-1} > x_{n+1,2i-1} > -q^{-4i+2},$$

which establishes that $\{x_{n,2i-1}\}_{n\geq 2i-1}$ is decreasing.

Now consider $\{x_{n,2i}\}_{n\geq 2i}$. By Theorem 13.9.1,

$$-q^{-4i+2} > x_{n,2i} > -q^{-4i+1},$$

and by Theorem 13.8.4,

$$p_{n+1}(x_{n,2i}) = q^{2n+2}\overline{p_n}(q^2 x_{n,2i}).$$

Note that

$$-q^{-4i+5} > -q^{-4i+4} > q^2 x_{n,2i} > -q^{-4i+3},$$

and so by Theorem 13.8.5, $\overline{p_n}(q^2 x_{n,2i}) > 0$. Therefore,

$$p_{n+1}(x_{n,2i}) > 0.$$

But by Theorem 13.8.3,

$$p_{n+1}(-q^{-4i+2}) < 0,$$

and so

$$-q^{-4i+2} > x_{n+1,2i} > x_{n,2i},$$

which establishes that $\{x_{n,2i}\}_{n\geq 2i}$ is increasing. □

Theorem 13.9.3. *For $0 < q < 1/4$, the entire function*

$$p_\infty(a) = \sum_{n=0}^{\infty} a^n q^{n^2}$$

has simple, negative, real zeros x_i that satisfy

$$-q^{-1} > x_1 > -q^{-2} > x_2 > -q^{-3} > -q^{-5} > x_3 > -q^{-6} > x_4 > -q^{-7} > \cdots .$$

Proof. Given that $p_\infty(a)$ is the uniform limit of the sequence $p_n(a)$, that the zeros $x_{n,i}$ are simple and lie in the same interval as indicated for x_i, and that the $x_{n,i}$ are monotone in n, the desired result follows. □

13.10 A Theta Function Expansion

Theorem 13.10.1. *If $|q| < 1$ and $w = 1 + q/a$, then as $w \to 0$,*

$$\left(q^2;q^2\right)_\infty \left(-aq;q^2\right)_\infty \left(-a^{-1}q;q^2\right)_\infty = w\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n^2+n} + O(w^3).$$

Proof. Using Jacobi's triple product identity (13.6.3), we find that

$$\begin{aligned}\left(q^2;q^2\right)_\infty \left(-aq;q^2\right)_\infty \left(-a^{-1}q;q^2\right)_\infty &= \sum_{n=-\infty}^{\infty} a^{-n}q^{n^2} \\ &= \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2-n}(1-w)^n \\ &= \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2-n}\left(1 - nw + \binom{n}{2}w^2 + O(w^3)\right).\end{aligned}$$

Replacing n by $1-n$ reveals that

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2-n} = -\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2-n}$$

and

$$\sum_{n=-\infty}^{\infty}(-1)^n\binom{n}{2}q^{n^2-n}=-\sum_{n=-\infty}^{\infty}(-1)^n\binom{n}{2}q^{n^2-n},$$

and so each series is identically zero. Therefore,

$$\begin{aligned}\left(q^2;q^2\right)_\infty\left(-aq;q^2\right)_\infty\left(-a^{-1}q;q^2\right)_\infty &= w\sum_{n=-\infty}^{\infty}(-1)^{n-1}nq^{n^2-n}+O(w^3)\\ &= w\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n^2+n}+O(w^3),\end{aligned}$$

as desired. □

Theorem 13.10.2. *If*

$$F(a)=\left(q^2;q^2\right)_\infty\left(-aq;q^2\right)_\infty\left(-a^{-1}q;q^2\right)_\infty,$$

then for any integer N*,*

$$F(a)=a^Nq^{N^2}F(aq^{2N}). \tag{13.10.1}$$

Proof. The identity (13.10.1) is a special case of Entry 18(iv) of Chapter 16 in Ramanujan's second notebook [61, p. 34]. □

13.11 Ramanujan's Product for $p_\infty(a)$

Entry 13.11.1 (p. 26). *The expansion* (13.1.7) *holds for* $0<q<1/4$.

Proof. We define

$$\mathcal{F}(a):=p_\infty(a)=\sum_{n=0}^{\infty}a^nq^{n^2} \tag{13.11.1}$$

and

$$\mathcal{G}(a):=\mathcal{F}(a^{-1})-1=\sum_{n=1}^{\infty}a^{-n}q^{n^2}. \tag{13.11.2}$$

Hence,

$$\left(q^2;q^2\right)_\infty\left(-aq;q^2\right)_\infty\left(-a^{-1}q;q^2\right)_\infty=\sum_{n=-\infty}^{\infty}a^{-n}q^{n^2}=F(a)=\mathcal{F}(a)+\mathcal{G}(a). \tag{13.11.3}$$

By Theorem 13.9.3, we see that the zeros, x_i, of $\mathcal{F}(a)$ satisfy the inequality

$$\sum_{i=1}^{\infty}\frac{1}{|x_i|}<\infty.$$

Consequently, by the product theorem for entire functions [125, p. 174],

$$\sum_{n=0}^{\infty} a^n q^{n^2} = \mathcal{F}(a) = \prod_{i=1}^{\infty}\left(1-\frac{a}{x_i}\right). \tag{13.11.4}$$

Furthermore, by Theorem 13.9.3, we know that

$$x_N = \frac{-q^{1-2N}}{1+Y_1(N)}, \tag{13.11.5}$$

where

$$Y_1(N) = O(q), \tag{13.11.6}$$

and $Y_1(N)$ is analytic in q by the implicit function theorem [283].

Therefore, by (13.11.3), (13.10.1), and (13.11.5),

$$\begin{aligned}\mathcal{G}(x_N) &= F(x_N) - \mathcal{F}(x_N) = F(x_N)\\ &= x_N^N q^{N^2} F(x_N q^{2N})\\ &= \frac{(-1)^N q^{N-N^2}}{\big(1+Y_1(N)\big)^N} F\left(-\frac{q}{1+Y_1(N)}\right).\end{aligned} \tag{13.11.7}$$

Consequently, rewriting (13.11.7) and using Theorem 13.10.1 and (13.11.2), we find that

$$\begin{aligned}\sum_{n=1}^{\infty}&(-1)^{n-N} q^{n^2+2Nn-n+N^2-N}\big(1+Y_1(N)\big)^{n+N} = F\left(-\frac{q}{1+Y_1(N)}\right)\\ &= -Y_1(N)\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n^2+n} + O(Y_1^3(N)),\end{aligned}$$

and so

$$\begin{aligned}\sum_{n=N}^{\infty}&(-1)^n q^{n^2+n}\big(1+Y_1(N)\big)^{n+1}\\ &= Y_1(N)\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n^2+n} + O\big(Y_1^3(N)\big).\end{aligned} \tag{13.11.8}$$

But by (13.11.8) and the analyticity of $Y_1(N)$, we see that the lowest power of q appearing in $Y_1(N)$ must be q^{N^2+N}. Hence, by (13.11.8) and the definition (13.1.8),

$$Y_1(N) \equiv \frac{\displaystyle\sum_{n=N}^{\infty}(-1)^n q^{n^2+n}}{\displaystyle\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n^2+n}} = y_1(N) \pmod{q^{2N^2+2N}}. \tag{13.11.9}$$

Now let

$$Y_2(N) = Y_1(N) - y_1(N), \tag{13.11.10}$$

and substitute for $Y_1(N)$ in (13.11.8). Hence,

$$\begin{aligned}&\sum_{n=N}^{\infty}(-1)^n q^{n^2+n}\big(1+y_1(N)+Y_2(N)\big)^{n+1}\\&\qquad=\big(y_1(N)+Y_2(N)\big)\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n^2+n} \pmod{q^{3N^2+3N}}.\end{aligned}$$

Because $Y_2(N) = O(q^{2N^2+2N})$ by (13.11.9) and (13.11.10), we find that

$$\begin{aligned}&(y_1(N)+Y_2(N))\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n^2+n}\\&\quad\equiv\sum_{n=N}^{\infty}(-1)^n q^{n^2+n}(1+y_1(N))^{n+1}\\&\quad\equiv\sum_{n=N}^{\infty}(-1)^n q^{n^2+n}(1+(n+1)y_1(N)) \pmod{q^{2N^2+2N}}.\end{aligned} \tag{13.11.11}$$

Hence, recalling from (13.1.8) that

$$\sum_{n=N}^{\infty}(-1)^n q^{n^2+n} = y_1(N)\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n^2+n}, \tag{13.11.12}$$

and substituting (13.11.12) into (13.11.11), we find that

$$\begin{aligned}&y_1(N)\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n^2+n}+Y_2(N)\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n^2+n}\\&\quad\equiv y_1(N)\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n^2+n}\\&\qquad+y_1(N)\sum_{n=N}^{\infty}(-1)^n(n+1)q^{n^2+n} \pmod{q^{3N^2+3N}}.\end{aligned}$$

Therefore,

$$Y_2(N) \equiv \frac{y_1(N)\displaystyle\sum_{n=N}^{\infty}(-1)^n(n+1)q^{n^2+n}}{\displaystyle\sum_{n=0}^{\infty}(-1)^n(2n+1)q^{n^2+n}} = y_2(N) \pmod{q^{3N^2+3N}}.$$

In conclusion, we see that

$$x_N = \frac{-q^{1-2N}}{1 + y_1(N) + y_2(N) + \cdots},$$

and the denominator of x_N is valid modulo q^{2N^2+2N}, which is quantitatively stronger than what Ramanujan intends by the "$\cdots$" in the formula above. This completes our proof. □

W. Bergweiler and W.K. Hayman [56] and Hayman [151] have established very general results for large classes of basic hypergeometric series satisfying certain general q-difference equations, in which the zeros are prescribed less precisely than those in the two theorems of Ramanujan proved in this chapter. We conclude this chapter with a statement of Hayman's theorem [151], which includes Entry 13.6.1 as a special case.

Theorem 13.11.1. *Let*

$$f(z) := \sum_{n=0}^{\infty} \frac{q^{n^2} z^n}{(q;q)_n (aq;q)_n},$$

and let the zeros z_n, $n \geq 1$, of $f(z)$ be arranged according to nondecreasing moduli. Then, if k is any positive integer, as $n \to \infty$, we have the asymptotic expansion

$$z_n = -q^{1-2n} \left\{ 1 + \sum_{\nu=1}^{k} b_\nu q^{n\nu} + O\left(q^{(k+1)n}\right) \right\},$$

where the constants b_ν depend on a and q. In particular,

$$b_1 = -\frac{1+a}{(1-q)\psi^2(q)}.$$

We have used Hayman's theorem in the case $a = 0$ to verify the values of $y_1, \ldots, y_4$ given in Entry 13.6.1; the difficulty of the calculations is the same as in the presentation we have given above.

14

Integrals of Theta Functions

14.1 Introduction

On pages 207 and 46 in his lost notebook [228], Ramanujan recorded eight evaluations of integrals of theta functions. Two of these give integral representations for the Rogers–Ramanujan continued fraction defined by

$$R(q) := \frac{q^{1/5}}{1} \, \frac{}{+} \, \frac{q}{1} \, \frac{}{+} \, \frac{q^2}{1} \, \frac{}{+} \, \frac{q^3}{1} \, \frac{}{+} \cdots, \qquad |q| < 1. \tag{14.1.1}$$

For example, on page 46, Ramanujan asserted that

$$R(q) = \frac{\sqrt{5}-1}{2} \exp\left(-\frac{1}{5}\int_q^1 \frac{(1-t)^5(1-t^2)^5\cdots}{(1-t^5)(1-t^{10})\cdots}\frac{dt}{t}\right) \tag{14.1.2}$$

$$= \frac{\sqrt{5}-1}{2} - \frac{\sqrt{5}}{1+\dfrac{3+\sqrt{5}}{2}\exp\left(\dfrac{1}{\sqrt{5}}\displaystyle\int_0^q \frac{(1-t)^5(1-t^2)^5\cdots}{(1-t^{1/5})(1-t^{2/5})\cdots}\frac{dt}{t^{4/5}}\right)}, \tag{14.1.3}$$

where $0 < q < 1$. The first of these representations was proved by Andrews [26], and the second was proved by S.H. Son [255].

However, the deepest result is the following claim, which appears on page 207. For $0 < q < 1$,

$$q^{1/9}\frac{(1-q)(1-q^4)^4(1-q^7)^7\cdots}{(1-q^2)^2(1-q^5)^5(1-q^8)^8\cdots} = \exp\left(-C - \frac{1}{9}\int_q^1 \frac{f^9(-t)}{f^3(-t^3)}\frac{dt}{t}\right), \tag{14.1.4}$$

where

$$C := \frac{3\sqrt{3}}{4\pi}\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\frac{1}{n^2}, \tag{14.1.5}$$

where $\left(\frac{n}{3}\right)$ denotes the Legendre symbol. In fact, after his formula for C, Ramanujan appends two question marks, indicating perhaps some uncertainty about the value of C. The formula (14.1.4) was first established by Son [255], but he did not determine the value of C. The first proof of (14.1.4) that was also accompanied by a proof of (14.1.5) was given by Berndt and A. Zaharescu [86], who used an argument different from that of Son. Note that the product on the left side of (14.1.4) can be regarded as a character analogue of the Dedekind eta function.

Quite remarkably, (14.1.2), (14.1.3), and (14.1.4) are special instances of one general theorem, namely a theorem on integrals of Eisenstein series motivated by (14.1.4) and proved by S. Ahlgren, Berndt, A.J. Yee, and Zaharescu [10]. This theorem will be briefly discussed at the conclusion of this chapter.

Our objective in this chapter is to prove the eight integral formulas found on pages 46 and 207, which have been proved by Andrews [26], Son [255], and Berndt and Zaharescu [86]. Representations of certain products and quotients of theta functions as Lambert series are the key ingredients in our proofs.

Before proceeding with some ancillary lemmas, we note some related work by N.J. Fine [137, pp. 88–90] and L.–C. Zhang [301]. Fine evaluated three definite integrals using formulas for the number of representations of an integer by certain diagonal quadratic forms. Zhang used the theory of modular forms to generalize one of Fine's integrals and to evaluate two similar integrals.

Furthermore, Ramanujan recorded several identities involving integrals of quotients of Dedekind eta functions on the left side and incomplete elliptic integrals of the first kind on the right side. These theorems were proved by S. Raghavan and S.S. Rangachari [213] and by Berndt, H.H. Chan, and S.–S. Huang [70]. An account of this work will be provided in Chapter 15.

14.2 Preliminary Results

We first review some notation from Chapter 1. It is assumed throughout the sequel that $|q| < 1$. Ramanujan's general theta function is defined by

$$f(a,b) := \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \tag{14.2.1}$$

where $|ab| < 1$. Furthermore, define

$$f(-q) := f(-q,-q^2) = (q;q)_\infty, \tag{14.2.2}$$

$$\varphi(q) := f(q,q) = \frac{(-q;-q)_\infty}{(q;-q)_\infty}, \tag{14.2.3}$$

$$\psi(q) := f(q,q^3) = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}. \tag{14.2.4}$$

The product representations of these theta functions can be derived from the Jacobi triple product identity, Lemma 1.2.2 in Chapter 1.

We need the following well-known transformation formula [61, p. 43, Entry 27(iii)].

Lemma 14.2.1. *If* $\alpha, \beta > 0$ *and* $\alpha\,\beta = \pi^2$, *then*

$$\alpha^{1/4}\, e^{-\alpha/12}\, f(-e^{-2\alpha}) = \beta^{1/4}\, e^{-\beta/12}\, f(-e^{-2\beta}). \tag{14.2.5}$$

Lemma 14.2.2. *Let* m *and* n *be positive numbers. Then as* q *tends to* 1^-,

$$f(-q^m, -q^n) \sim 2\sqrt{\frac{2\pi}{(m+n)|\log q|}}\exp\left(\frac{-\pi^2}{2(m+n)|\log q|}\right)\sin\left(\frac{\pi m}{m+n}\right) \tag{14.2.6}$$

and

$$f(q^m, q^n) \sim \sqrt{\frac{2\pi}{(m+n)|\log q|}}. \tag{14.2.7}$$

Proof. For a proof of (14.2.6), see [61, p. 141].

To prove (14.2.7), the argument is similar. By the definition of $f(a,b)$ in (14.2.1),

$$f(q^m, q^n) = \sum_{j=-\infty}^{\infty} (q^m)^{j(j+1)/2}(q^n)^{j(j-1)/2} = \sum_{j=-\infty}^{\infty} (q^a)^{j^2}(q^b)^j,$$

where $a = (m+n)/2$ and $b = (m-n)/2$. Observe that

$$f(q^m, q^n) = \theta_3(z,\tau) := \sum_{j=-\infty}^{\infty} (e^{\pi i\tau})^{j^2} e^{2jiz}, \qquad \operatorname{Im}\tau > 0, \tag{14.2.8}$$

where $z = -i(b\log q)/2$ and $\tau = -i(a\log q)/\pi$.

Applying the transformation formula [292, p. 475]

$$\theta_3(z,\tau) = (-i\tau)^{-1/2}\exp\left(\frac{z^2}{\pi i\tau}\right)\theta_3\left(\frac{z}{\tau}, -\frac{1}{\tau}\right)$$

in (14.2.8), we find that

$$\begin{aligned} f(q^m, q^n) &= \sqrt{\frac{\pi}{-a\log q}}\exp\left(-\frac{b^2\log q}{4a}\right)\left(1 + \sum_{\substack{j=-\infty \\ j\neq 0}}^{\infty}\exp\left(\frac{\pi^2 j^2}{a\log q} + \frac{\pi j b i}{a}\right)\right) \\ &\sim \sqrt{\frac{\pi}{-a\log q}}, \end{aligned}$$

as q tends to 1^-. □

In the sequel, six Lambert series identities are needed.

Lemma 14.2.3. *Recall that the theta functions* φ *and* ψ *are defined by* (14.2.3) *and* (14.2.4), *respectively. Then we have the Lambert series representations*

(i) $$q\psi^2(q)\psi^2(q^3) = \sum_{n=1}^{\infty} \frac{nq^n}{1-q^{2n}} - 3\sum_{n=1}^{\infty} \frac{nq^{3n}}{1-q^{6n}},$$

(ii) $$\varphi^2(q)\varphi^2(q^3) = 1 + 4\sum_{n=1}^{\infty} \frac{nq^n}{1-(-q)^n} - 12\sum_{n=1}^{\infty} \frac{nq^{3n}}{1-(-q)^{3n}},$$

(iii) $$q\psi^4(q^2) = \sum_{n=0}^{\infty} \frac{(2n+1)q^{2n+1}}{1-q^{4n+2}},$$

(iv) $$\varphi^4(q) = 1 + 8\sum_{n=1}^{\infty} \frac{nq^n}{1+(-q)^n}.$$

For proofs of (i)–(iv), see [61, Entries 3(iii), (iv), p. 223 (especially, a formula at the middle of p. 226); Example (iii), p. 139; and Entry 8 (ii), p. 114, respectively]. To derive two additional Lambert series identities, we need the following identity from Fine's book [137, p. 22, equation (18.86)].

Lemma 14.2.4. *For* $|q| < |t| < |q|^{-1}$,

$$\frac{(q;q)_\infty^6 (t^{-2}q;q)_\infty (t^2;q)_\infty}{\{(t^{-1}q;q)_\infty (t;q)_\infty\}^4} = \frac{1+t}{(1-t)^3} + t^{-1} \sum_{n,k=1}^{\infty} q^{kn} k^2 (t^k - t^{-k}). \quad (14.2.9)$$

Lemma 14.2.5. *Recall that* $f(-q)$ *is defined in* (14.2.2). *For* $|q| < 1$,

$$\frac{f^9(-q)}{f^3(-q^3)} = 1 + 9\sum_{n=1}^{\infty} \left\{ \frac{(3n-1)^2 q^{3n-1}}{1-q^{3n-1}} - \frac{(3n-2)^2 q^{3n-2}}{1-q^{3n-2}} \right\}. \quad (14.2.10)$$

Proof. If $L(q)$ and $R(q)$ denote, respectively, the left and the right sides of (14.2.9), with $t = \omega := \exp(2\pi i/3)$, then

$$\begin{aligned} L(q) &= \frac{(q;q)_\infty^6 (\omega q;q)_\infty (\omega^2;q)_\infty}{\{(\omega^2 q;q)_\infty (\omega;q)_\infty\}^4} = \frac{1-\omega^2}{(1-\omega)^4} \cdot \frac{(q;q)_\infty^6}{\{(\omega^2 q;q)_\infty (\omega;q)_\infty\}^3} \\ &= \frac{1+\omega}{(1-\omega)^3} \cdot \frac{(q;q)_\infty^9}{(q^3;q^3)_\infty^3} = \frac{1+\omega}{(1-\omega)^3} \cdot \frac{f^9(-q)}{f^3(-q^3)} \end{aligned} \quad (14.2.11)$$

and

$$\begin{aligned} R(q) &= \frac{1+\omega}{(1-\omega)^3} + \omega^{-1} \sum_{k=1}^{\infty} k^2 (\omega^k - \omega^{-k}) \sum_{n=1}^{\infty} q^{kn} \\ &= \frac{1+\omega}{(1-\omega)^3} + \omega^{-1} \sum_{k=1}^{\infty} k^2 \left(2i \sin \frac{2\pi k}{3}\right) \cdot \frac{q^k}{1-q^k}. \end{aligned} \quad (14.2.12)$$

Combining (14.2.11) and (14.2.12) and dividing both sides by $(1+\omega)/(1-\omega)^3$, we deduce that

$$\frac{f^9(-q)}{f^3(-q^3)} = 1 - \frac{i\sqrt{3}(1-\omega)^3}{\omega(1+\omega)} \sum_{n=1}^{\infty} \left\{ \frac{(3n-1)^2 q^{3n-1}}{1-q^{3n-1}} - \frac{(3n-2)^2 q^{3n-2}}{1-q^{3n-2}} \right\}.$$

After some simplification, we complete the proof. □

L. Carlitz [107] was evidently the first mathematician to prove Lemma 14.2.5.

In our proof of Entry 14.3.6, a different representation for $f^9(-q)/f^3(-q^3)$ arises, and we establish this in the next lemma.

Lemma 14.2.6. *For* $|q| < 1$,

$$\frac{f^9(-q)}{f^3(-q^3)} = 1 - 9 \sum_{n=1}^{\infty} \frac{q^n - q^{2n} - 6q^{3n} - q^{4n} + q^{5n}}{(1+q^n+q^{2n})^3}. \tag{14.2.13}$$

Proof. Multiplying numerators and denominators by $(1-q^n)^3$ and then inverting the order of summation, we find that

$$\begin{aligned}
&\sum_{n=1}^{\infty} \frac{q^n - q^{2n} - 6q^{3n} - q^{4n} + q^{5n}}{(1+q^n+q^{2n})^3} \\
&= \sum_{n=1}^{\infty} \frac{q^n - 4q^{2n} + 13q^{4n} - 13q^{5n} + 4q^{7n} - q^{8n}}{(1-q^{3n})^3} \\
&= \frac{1}{2} \sum_{n=1}^{\infty} \sum_{m=2}^{\infty} m(m-1) \left(q^n - 4q^{2n} + 13q^{4n} - 13q^{5n} + 4q^{7n} - q^{8n}\right) q^{3n(m-2)} \\
&= \frac{1}{2} \sum_{m=2}^{\infty} m(m-1) \left(\frac{q^{3m-5}}{1-q^{3m-5}} - 4\frac{q^{3m-4}}{1-q^{3m-4}} + 13\frac{q^{3m-2}}{1-q^{3m-2}} \right. \\
&\quad \left. -13\frac{q^{3m-1}}{1-q^{3m-1}} + 4\frac{q^{3m+1}}{1-q^{3m+1}} - \frac{q^{3m+2}}{1-q^{3m+2}} \right) \\
&= \frac{1}{2} \sum_{m=1}^{\infty} (m+1)m \frac{q^{3m-2}}{1-q^{3m-2}} - 2 \sum_{m=1}^{\infty} (m+1)m \frac{q^{3m-1}}{1-q^{3m-1}} \\
&\quad + \frac{13}{2} \sum_{m=1}^{\infty} m(m-1) \frac{q^{3m-2}}{1-q^{3m-2}} - \frac{13}{2} \sum_{m=1}^{\infty} m(m-1) \frac{q^{3m-1}}{1-q^{3m-1}} \\
&\quad + 2 \sum_{m=1}^{\infty} (m-1)(m-2) \frac{q^{3m-2}}{1-q^{3m-2}} - \frac{1}{2} \sum_{m=1}^{\infty} (m-1)(m-2) \frac{q^{3m-1}}{1-q^{3m-1}} \\
&= - \sum_{m=1}^{\infty} \left\{ \frac{(3m-1)^2 q^{3m-1}}{1-q^{3m-1}} - \frac{(3m-2)^2 q^{3m-2}}{1-q^{3m-2}} \right\},
\end{aligned}$$

where in the last step we merely added together the coefficients of each of the two distinct q-quotients. The result now follows from Lemma 14.2.5. □

Lemma 14.2.7. *For* $|q| < 1$,

$$\frac{qf^9(-q^3)}{f^3(-q)} = \sum_{n=1}^{\infty} \frac{n^2 q^n}{1+q^n+q^{2n}}. \tag{14.2.14}$$

Proof. After q is replaced by q^3 and t is replaced by q in (14.2.9), we let $L(q)$ and $R(q)$ denote, respectively, the left and the right sides of the identity. It transpires that

$$L(q) = \frac{(q^3;q^3)_\infty^6 (q;q^3)_\infty (q^2;q^3)_\infty}{\{(q^2;q^3)_\infty (q;q^3)_\infty\}^4} = \frac{(q^3;q^3)_\infty^9}{(q;q)_\infty^3} = \frac{f^9(-q^3)}{f^3(-q)}. \tag{14.2.15}$$

Since

$$\frac{1+q}{(1-q)^3} = \sum_{k=1}^{\infty} k^2 q^{k-1},$$

we find that

$$\begin{aligned} R(q) &= \frac{1+q}{(1-q)^3} + q^{-1} \sum_{k=1}^{\infty} k^2 q^k \sum_{n=1}^{\infty} q^{3kn} - q^{-1} \sum_{k=1}^{\infty} k^2 q^{-k} \sum_{n=1}^{\infty} q^{3kn} \\ &= \sum_{k=1}^{\infty} k^2 q^{k-1} \left(1 + \sum_{n=1}^{\infty} q^{3kn}\right) - \sum_{k=1}^{\infty} k^2 q^{-k-1} \frac{q^{3k}}{1-q^{3k}} \\ &= \sum_{k=1}^{\infty} k^2 q^{k-1} \left(\frac{1}{1-q^{3k}} - \frac{q^k}{1-q^{3k}}\right) = \sum_{k=1}^{\infty} \frac{k^2 q^{k-1}}{1+q^k+q^{2k}}. \end{aligned} \tag{14.2.16}$$

Combining (14.2.15) and (14.2.16), we complete the proof. □

With the left sides of (14.2.10) and (14.2.14) expressed as cubes of "cubic theta functions," J.M. Borwein and P.B. Borwein stated (14.2.10) and (14.2.14) without proofs in their paper [100, p. 697]. That these cubic theta functions have the representations given in terms of $f(-q)$ was proved by the Borweins and F.G. Garvan in [101, pp. 37–38]. A more general formula of Ramanujan was proved by Berndt, S. Bhargava, and Garvan in [66, p. 4212], [63, pp. 143–145].

14.3 The Identities on Page 207

We shall use the Lambert series identities featured in Section 14.2 and asymptotic properties of Ramanujan's theta functions in Lemma 14.2.2 to prove the identities on page 207.

Entry 14.3.1 (p. 207). *For* $0 < q < 1$,

$$\frac{\varphi(-q^3)}{\varphi(-q)} = \exp\left(2\int_0^q \psi^2(t)\psi^2(t^3)\,dt\right).$$

Proof. Using (14.2.3), we easily find that

$$\begin{aligned}\log\frac{\varphi(-q^3)}{\varphi(-q)}&=\log(q^3;q^3)_\infty-\log(-q^3;q^3)_\infty-\log(q;q)_\infty+\log(-q;q)_\infty\\&=\sum_{n=1}^{\infty}\left(\{\log(1-q^{3n})-\log(1+q^{3n})\}-\{\log(1-q^n)-\log(1+q^n)\}\right).\end{aligned}$$

Taking the derivative of both sides, we find that

$$\begin{aligned}&\frac{d}{dq}\left(\log\frac{\varphi(-q^3)}{\varphi(-q)}\right)\\&=\sum_{n=1}^{\infty}\left\{\left(\frac{-3nq^{3n-1}}{1-q^{3n}}-\frac{3nq^{3n-1}}{1+q^{3n}}\right)-\left(\frac{-nq^{n-1}}{1-q^n}-\frac{nq^{n-1}}{1+q^n}\right)\right\}\\&=\frac{2}{q}\left\{\sum_{n=1}^{\infty}\frac{nq^n}{1-q^{2n}}-3\sum_{n=1}^{\infty}\frac{nq^{3n}}{1-q^{6n}}\right\}=2\psi^2(q)\psi^2(q^3),\end{aligned}$$

by Lemma 14.2.3(i). Since $\varphi(0)=1$, we can integrate both sides over $[0,q]$. Thus,

$$\log\frac{\varphi(-q^3)}{\varphi(-q)}=2\int_0^q\psi^2(t)\psi^2(t^3)\,dt.$$

Exponentiating, we complete the proof. $\square$

Entry 14.3.2 (p. 207). *For* $0<q<1$,

$$q^{1/4}\frac{\psi(-q^3)}{\psi(-q)}=\exp\left(\frac{1}{4}\int\varphi^2(q)\varphi^2(q^3)\,\frac{dq}{q}\right).$$

Proof. Using (14.2.4), we see that

$$\begin{aligned}\log\frac{\psi(-q^3)}{\psi(-q)}=\sum_{n=1}^{\infty}&\left(\{\log(1-q^{6n})-\log(1+q^{6n-3})\}\right.\\&\left.-\{\log(1-q^{2n})-\log(1+q^{2n-1})\}\right).\end{aligned}$$

Taking the derivative of both sides, we find that

$$\begin{aligned}\frac{d}{dq}\left(\log\frac{\psi(-q^3)}{\psi(-q)}\right)&=\sum_{n=1}^{\infty}\left\{\left(\frac{-6nq^{6n-1}}{1-q^{6n}}-\frac{(6n-3)q^{6n-4}}{1+q^{6n-3}}\right)\right.\\&\qquad\left.-\left(\frac{-2nq^{2n-1}}{1-q^{2n}}-\frac{(2n-1)q^{2n-2}}{1+q^{2n-1}}\right)\right\}\\&=\sum_{n=1}^{\infty}\left(\frac{nq^{n-1}}{1-(-q)^n}-\frac{3nq^{3n-1}}{1-(-q)^{3n}}\right)\\&=\frac{1}{q}\left(\frac{1}{4}\cdot\varphi^2(q)\varphi^2(q^3)-\frac{1}{4}\right),\end{aligned}$$

by Lemma 14.2.3(ii). Hence, integrating both sides and exponentiating, we complete the proof. □

Ramanujan expressed Entry 14.3.2 in terms of an indefinite integral, because both sides tend to ∞ as q tends to 1^-. To see this, we apply (14.2.6) to find that

$$\frac{\psi(-q^3)}{\psi(-q)} = \frac{f(-q^3,-q^9)}{f(-q,-q^3)} \sim \sqrt{\frac{1}{3}} \exp\left(-\frac{\pi^2}{12\log q}\right) \to \infty.$$

Since

$$\frac{\varphi^2(t)\varphi^2(t^3)}{t} \sim \frac{\pi^2}{9} \cdot \frac{1}{t\log^2 t},$$

as t tends to 1^-, the integral

$$\int_q^1 \varphi^2(t)\varphi^2(t^3)\,\frac{dt}{t}$$

diverges.

Entry 14.3.3 (p. 207). *For* $0 < q < 1$,

$$\frac{\psi(-q)}{\psi(q)} = \exp\left(-2\int_0^q \psi^4(t^2)\,dt\right).$$

Proof. Using (14.2.4), we easily find that

$$\frac{\psi(-q)}{\psi(q)} = \frac{(q;q^2)_\infty}{(-q;q^2)_\infty}.$$

Thus,

$$\log\frac{\psi(-q)}{\psi(q)} = \sum_{n=0}^{\infty}\{\log(1-q^{2n+1}) - \log(1+q^{2n+1})\}.$$

Taking the derivative of both sides, we find that

$$\begin{aligned}\frac{d}{dq}\left(\log\frac{\psi(-q)}{\psi(q)}\right) &= \sum_{n=0}^{\infty}\left\{-\frac{(2n+1)q^{2n}}{1-q^{2n+1}} - \frac{(2n+1)q^{2n}}{1+q^{2n+1}}\right\}\\ &= -\frac{2}{q}\sum_{n=0}^{\infty}\frac{(2n+1)q^{2n+1}}{1-q^{4n+2}} = -2\psi^4(q^2),\end{aligned}$$

by Lemma 14.2.3(iii). Noting that $\psi(0) = 1$, integrating both sides over $[0,q]$, and exponentiating, we complete the proof. □

Entry 14.3.4 (p. 207). *For* $0 < q < 1$,

$$\frac{\psi(-q)}{\psi(q^2)} = q^{1/8}\exp\left(-\frac{1}{8}\int\frac{\varphi^4(q)}{q}\,dq\right).$$

Proof. Using (14.2.4), we easily find that

$$\frac{\psi(-q)}{\psi(q^2)} = \frac{(q;q^2)_\infty}{(-q^2;q^2)_\infty}.$$

Thus,

$$\log \frac{\psi(-q)}{\psi(q^2)} = \sum_{n=1}^{\infty} \left\{ \log(1-q^{2n-1}) - \log(1+q^{2n}) \right\}.$$

Taking the derivative of both sides, we find that

$$\begin{aligned}\frac{d}{dq}\left(\log \frac{\psi(-q)}{\psi(q^2)}\right) &= \sum_{n=1}^{\infty} \left\{ -\frac{(2n-1)q^{2n-2}}{1-q^{2n-1}} - \frac{2nq^{2n-1}}{1+q^{2n}} \right\} \\ &= -\sum_{k=1}^{\infty} \frac{kq^{k-1}}{1+(-q)^k} = \frac{1}{8q} - \frac{1}{8} \cdot \frac{\varphi^4(q)}{q},\end{aligned}$$

by Lemma 14.2.3(iv). Integrating both sides and exponentiating, we complete the proof. □

As q tends to 1^-,

$$\frac{\psi(-q)}{\psi(q^2)} = \frac{f(-q,-q^3)}{f(q^2,q^6)} \sim 2\exp\left(\frac{-\pi^2}{8|\log q|}\right) \to 0,$$

upon the use of (14.2.6) and (14.2.7). Similarly, using (14.2.7), we find that

$$\int_q^1 \frac{\varphi^4(t)}{t}\, dt$$

diverges. For these reasons, Ramanujan expressed Entry 14.3.4 in terms of indefinite integrals.

In [7], C. Adiga, K.R. Vasuki, and M.S. Mahadeva Naika obtained integral representations for the ratios

$$\frac{\varphi^a(\pm q^m)}{\varphi^b(\pm q^k)}, \qquad \frac{\psi^a(\pm q^m)}{\varphi^b(\pm q^k)}, \qquad \text{and} \qquad \frac{\psi^a(\pm q^m)}{\psi^b(\pm q^k)}.$$

Adiga, T. Kim, Mahadeva Naika, and H.S. Madhusudhan [4] have found a pair of integral representations for Ramanujan's cubic continued fraction $G(q)$, defined in (3.1.6) of Chapter 3.

We give two proofs of the next entry. The first is due to Son [255] and is expressed in terms of an indefinite integral. Ramanujan's formulation is given in terms of a definite integral and the constant C defined by (14.1.5). Our second proof, by Berndt and Zaharescu [86], establishes this more precise formulation of Ramanujan.

Entry 14.3.5 (p. 207). *If* $0<q<1$ *and* $\left(\frac{n}{3}\right)$ *denotes the Legendre symbol,*

$$q^{1/9}\prod_{n=1}^{\infty}(1-q^n)^{n\left(\frac{n}{3}\right)}=\exp\left(\frac{1}{9}\int\frac{f^9(-q)}{f^3(-q^3)}\frac{dq}{q}\right).$$

Proof. Let

$$A(q):=q^{1/9}\prod_{n=1}^{\infty}(1-q^n)^{n\left(\frac{n}{3}\right)}.$$

Taking the logarithm of both sides, we find that

$$\log A(q)=\frac{1}{9}\log q+\sum_{n=1}^{\infty}\{(3n-2)\log(1-q^{3n-2})-(3n-1)\log(1-q^{3n-1})\}.$$

Taking the derivative of both sides, we find that

$$\begin{aligned}\frac{d}{dq}\big(\log A(q)\big)&=\frac{1}{9q}+\sum_{n=1}^{\infty}\left\{-\frac{(3n-2)^2q^{3n-3}}{1-q^{3n-2}}-\frac{-(3n-1)^2q^{3n-2}}{1-q^{3n-1}}\right\}\\&=\frac{f^9(-q)}{9qf^3(-q^3)},\end{aligned}$$

by (14.2.10). Upon integration, we find that

$$\log A(q)=\frac{1}{9}\int\frac{f^9(-q)}{q^3(-q^3)}\frac{dq}{q}.$$

Exponentiating, we complete the proof. □

We now state and prove a more precise version of Entry 14.3.5.

Entry 14.3.6 (p. 207). *For* $0<q<1$,

$$q^{1/9}\prod_{n=1}^{\infty}(1-q^n)^{n\chi(n)}=\exp\left(-C-\frac{1}{9}\int_q^1\frac{f^9(-t)}{f^3(-t^3)}\frac{dt}{t}\right),\tag{14.3.1}$$

where

$$C:=\frac{3\sqrt{3}}{4\pi}L(2,\chi)=L'(-1,\chi),\tag{14.3.2}$$

where $\chi(n)$ *denotes the Legendre symbol* $\left(\frac{n}{3}\right)$, *and where* $L(s,\chi)$ *denotes the Dirichlet* L*-function associated with the character* χ.

We are grateful to D. Masser [194], who first informed us of the last equality in (14.3.2).

Our proof of Entry 14.3.6 proceeds in four steps. First, we show that Ramanujan's formula (14.3.1) implies (14.2.13), and conversely that (14.2.13) implies (14.3.1), except for the identification of the additive constant C. It then remains to prove that C has the prescribed value (14.3.2), which we do in three steps. We first show that C can be represented as the limit of a certain q-series as $q\to 1^-$. Second, we show that this limit can be represented by an integral. Lastly, we evaluate this integral to prove (14.3.2).

Proof. Assume throughout the proof that $0 < q < 1$. Taking the logarithm of both sides of (14.3.1) and using the Taylor expansion of $\log(1-z)$ about $z = 0$, we find that

$$\frac{1}{9}\log q - \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left(\frac{n}{3}\right)\frac{nq^{mn}}{m} = -C - \frac{1}{9}\int_q^1 \frac{f^9(-t)}{f^3(-t^3)}\frac{dt}{t}. \tag{14.3.3}$$

It is easy to see that

$$\sum_{\substack{n=1\\ n\equiv 1 \pmod 3}}^{\infty} q^n = \frac{q}{1-q^3} \qquad \text{and} \qquad \sum_{\substack{n=1\\ n\equiv 2 \pmod 3}}^{\infty} q^n = \frac{q^2}{1-q^3}. \tag{14.3.4}$$

Differentiating (14.3.4), we find that

$$\sum_{\substack{n=1\\ n\equiv 1 \pmod 3}}^{\infty} nq^{n-1} = \frac{1+2q^3}{(1-q^3)^2} \qquad \text{and} \qquad \sum_{\substack{n=1\\ n\equiv 2 \pmod 3}}^{\infty} nq^{n-1} = \frac{2q+q^4}{(1-q^3)^2}. \tag{14.3.5}$$

Combining the two equalities of (14.3.5), we deduce that

$$\sum_{n=1}^{\infty}\left(\frac{n}{3}\right) nq^n = \frac{q-q^3}{(1+q+q^2)^2}. \tag{14.3.6}$$

Using (14.3.6) in (14.3.3), we find that (14.3.3) is equivalent to

$$\sum_{m=1}^{\infty}\frac{q^m - q^{3m}}{m(1+q^m+q^{2m})^2} = \frac{1}{9}\log q + C + \frac{1}{9}\int_q^1 \frac{f^9(-t)}{f^3(-t^3)}\frac{dt}{t}. \tag{14.3.7}$$

For brevity, let L and R denote the left and right sides, respectively, of (14.3.7). Elementary differentiations show that

$$\begin{aligned} q\frac{dL}{dq} &= q\sum_{m=1}^{\infty}\left(\frac{(mq^{m-1}-3mq^{3m-1})(1+q^m+q^{2m})}{m(1+q^m+q^{2m})^3}\right.\\ &\qquad\qquad \left. - \frac{2(q^m-q^{3m})(mq^{m-1}+2mq^{2m-1})}{m(1+q^m+q^{2m})^3}\right)\\ &= \sum_{m=1}^{\infty}\frac{q^m - q^{2m} - 6q^{3m} - q^{4m} + q^{5m}}{(1+q^m+q^{2m})^3} \end{aligned} \tag{14.3.8}$$

and

$$q\frac{dR}{dq} = \frac{1}{9} - \frac{1}{9}\frac{f^9(-q)}{f^3(-q^3)}. \tag{14.3.9}$$

Employing (14.3.8) and (14.3.9) in (14.3.7), we conclude that Ramanujan's formula (14.3.1) implies the equality

$$1 - 9\sum_{m=1}^{\infty} \frac{q^m - q^{2m} - 6q^{3m} - q^{4m} + q^{5m}}{(1+q^m+q^{2m})^3} = \frac{f^9(-q)}{f^3(-q^3)}. \tag{14.3.10}$$

Conversely, (14.3.10) implies that (14.3.1) holds for $0 < q < 1$ and for some constant C. However, indeed (14.3.10) is valid by Lemma 14.2.6. Thus, it remains to prove that C has the value given by (14.3.2), which we now do in the three steps outlined above.

First, by (14.3.7), it is clear that

$$C = \lim_{q \to 1^-} \sum_{m=1}^{\infty} \frac{q^m - q^{3m}}{m(1+q^m+q^{2m})^2}. \tag{14.3.11}$$

Second, we prove that

$$C = \int_{-\infty}^{\infty} \frac{\sinh u}{u(1+2\cosh u)^2} du. \tag{14.3.12}$$

To prove (14.3.12), set $q = \exp(-1/N)$, where N is a large positive integer. Then (14.3.11) may be written in the form

$$\begin{aligned} C &= \lim_{N\to\infty} \sum_{m=1}^{\infty} \frac{e^{-m/N} - e^{-3m/N}}{m(1+e^{-m/N}+e^{-2m/N})^2} \\ &= \lim_{N\to\infty} \frac{1}{N} \sum_{m=1}^{\infty} \frac{e^{-m/N} - e^{-3m/N}}{(m/N)(1+e^{-m/N}+e^{-2m/N})^2}. \end{aligned} \tag{14.3.13}$$

On the far right side of (14.3.13), we have a Riemann sum. Taking the limit as $N \to \infty$, we deduce that

$$\begin{aligned} C &= \int_0^{\infty} \frac{e^{-u} - e^{-3u}}{u(1+e^{-u}+e^{-2u})^2} du = \int_0^{\infty} \frac{e^{u} - e^{-u}}{u(e^{u}+1+e^{-u})^2} du \\ &= 2\int_0^{\infty} \frac{\sinh u}{u(1+2\cosh u)^2} du = \int_{-\infty}^{\infty} \frac{\sinh u}{u(1+2\cosh u)^2} du, \end{aligned}$$

since the integrand is even. This establishes (14.3.12).

The function

$$g(z) := \frac{\sinh z}{z(1+2\cosh z)^2} \tag{14.3.14}$$

is meromorphic in the entire complex plane, and has double poles at the points $2\pi i n/3$, for each integer n that is not a multiple of 3. Let γ_{R_m}, $1 \le m < \infty$, be a sequence of positively oriented rectangles with vertices $\pm\sqrt{R_m}$ and $\pm\sqrt{R_m} + R_m^{3/2} i$, which are chosen so that the points $R_m^{3/2} i$ remain at a bounded distance from the points $2\pi i n/3$, as m tends to ∞. For brevity, let $L_1 = L_1(m)$ and $L_2 = L_2(m)$ denote, respectively, the left and right sides, and let $L_3 = L_3(m)$ denote the top side of γ_{R_m}. Then, it is not difficult to see that for $j = 1, 2$,

$$\left| \int_{L_j} g(z)dz \right| \ll R_m e^{-\sqrt{R_m}}, \tag{14.3.15}$$

as $R_m \to \infty$. It is also not difficult to see that

$$\left| \int_{L_3} g(z)dz \right| \ll \frac{1}{R_m}, \tag{14.3.16}$$

as $R_m \to \infty$. In summary, the inequalities (14.3.15) and (14.3.16) imply that if $\gamma'_{R_m} = L_1 \cup L_2 \cup L_3$, then

$$\int_{\gamma'_{R_m}} g(z)dz = o(1), \tag{14.3.17}$$

as $R_m \to \infty$.

Letting $R(a)$ denote the residue of $g(z)$ at a pole a, we find by the residue theorem that

$$\frac{1}{2\pi i} \int_{-\sqrt{R_m}}^{\sqrt{R_m}} g(z)dz + \frac{1}{2\pi i} \int_{\gamma'_{R_m}} g(z)dz = \sum_{\substack{1 \leq n < 3R_m^{3/2}/(2\pi) \\ 3 \nmid n}} R\left(\frac{2\pi i n}{3}\right). \tag{14.3.18}$$

Letting R_m tend to ∞ in (14.3.18) and using (14.3.17), we deduce from (14.3.12) that

$$C = 2\pi i \sum_{\substack{n=1 \\ 3 \nmid n}}^{\infty} R\left(\frac{2\pi i n}{3}\right). \tag{14.3.19}$$

In order to compute the residues, we introduce simpler notation. If the positive integer n is not a multiple of 3, set $a = 2\pi i n/3$ and $\omega = e^{2\pi i/3}$. Then $e^a = \omega$ if $n \equiv 1 \pmod 3$, and $e^a = \bar{\omega}$ if $n \equiv 2 \pmod 3$. We use the Taylor expansions,

$$\frac{1}{z} = \frac{1}{a} - \frac{z-a}{a^2} + \cdots, \tag{14.3.20}$$

$$\sinh z = \sinh a + (z-a)\cosh a + \cdots, \tag{14.3.21}$$

and

$$\cosh z = \cosh a + (z-a)\sinh a + \tfrac{1}{2}(z-a)^2 \cosh a + \cdots. \tag{14.3.22}$$

Since $1 + 2\cosh a = 0$, it follows from (14.3.22) that

$$1 + 2\cosh z = 2(z-a)\sinh a \left(1 + (z-a)\frac{\cosh a}{2\sinh a} + \cdots\right),$$

and so

$$\frac{1}{(1+2\cosh z)^2} = \frac{1-(z-a)\dfrac{\cosh a}{\sinh a}+\cdots}{4(z-a)^2\sinh^2 a}. \tag{14.3.23}$$

Using (14.3.20), (14.3.21), and (14.3.23) in (14.3.14), we find that

$$\begin{aligned} g(z) &\\ &= \frac{\left(1+(z-a)\dfrac{\cosh a}{\sinh a}+\cdots\right)\left(1-\dfrac{z-a}{a}+\cdots\right)\left(1-(z-a)\dfrac{\cosh a}{\sinh a}+\cdots\right)}{4a(z-a)^2\sinh a} \\ &= \frac{1-\dfrac{z-a}{a}+\cdots}{4a(z-a)^2\sinh a}, \end{aligned}$$

and so

$$R(a) = -\frac{1}{4a^2\sinh a} = \frac{1}{2a^2(e^{-a}-e^{a})}.$$

We distinguish two cases. If $n \equiv 1 \pmod 3$, then $e^{-a}-e^{a} = \bar{\omega}-\omega = -i\sqrt{3}$, and hence

$$R(a) = \frac{i}{2a^2\sqrt{3}} = -\frac{3\sqrt{3}\,i}{8\pi^2n^2}. \tag{14.3.24}$$

If $n \equiv 2 \pmod 3$, then $e^{-a}-e^{a} = \omega-\bar{\omega} = i\sqrt{3}$, and hence

$$R(a) = -\frac{i}{2a^2\sqrt{3}} = \frac{3\sqrt{3}\,i}{8\pi^2n^2}. \tag{14.3.25}$$

Using (14.3.24) and (14.3.25) in (14.3.19), we conclude that

$$C = \frac{3\sqrt{3}}{4\pi}\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\frac{1}{n^2},$$

which is (14.3.2). This then completes the proof of Entry 14.3.6. □

Entry 14.3.7 (p. 207). *Let* $\omega := \exp(2\pi i/3)$ *and* $0<q<1$. *Then*

$$\begin{aligned} &\left(\frac{1-q\omega}{1-q\omega^2}\right)\left(\frac{1-q^2\omega}{1-q^2\omega^2}\right)^2\left(\frac{1-q^3\omega}{1-q^3\omega^2}\right)^3\cdots \\ &= \exp\left(-(\omega-\omega^2)\int_0^q \frac{f^9(-t^3)}{f^3(-t)}\,dt\right). \end{aligned}$$

Proof. Let

$$B(q) := \left(\frac{1-q\omega}{1-q\omega^2}\right)\left(\frac{1-q^2\omega}{1-q^2\omega^2}\right)^2\left(\frac{1-q^3\omega}{1-q^3\omega^2}\right)^3\cdots.$$

Then

$$\log B(q) = \sum_{n=1}^{\infty} \{ n \log(1 - q^n \omega) - n \log(1 - q^n \omega^2) \}.$$

Taking the derivative of both sides, we find that

$$\frac{d}{dq}\big(\log B(q)\big) = \sum_{n=1}^{\infty} \left\{ \frac{-n^2 \omega q^{n-1}}{1 - q^n \omega} - \frac{-n^2 \omega^2 q^{n-1}}{1 - q^n \omega^2} \right\} = -(\omega - \omega^2) \cdot \frac{f^9(-q^3)}{f^3(-q)},$$

by (14.2.14). Since $B(0) = 1$, integrating both sides over $[0, q]$ and exponentiating, we complete the proof. □

14.4 Integral Representations of the Rogers–Ramanujan Continued Fraction

Recall that $f(-q)$ is defined in (14.2.2). The first entry below was first proved by Andrews [26], while the second was first established by Son [255].

Entry 14.4.1 (p. 46). *We have*

$$R(q) = \frac{\sqrt{5} - 1}{2} \exp\left(-\frac{1}{5} \int_q^1 \frac{f^5(-t)}{f(-t^5)} \frac{dt}{t} \right). \tag{14.4.1}$$

Proof. Taking the logarithmic derivative of both sides of (14.4.1), we find that

$$\frac{1}{5q} - \sum_{n=0}^{\infty} \left\{ \frac{(5n+1)q^{5n}}{1 - q^{5n+1}} + \frac{(5n+4)q^{5n+3}}{1 - q^{5n+4}} - \frac{(5n+2)q^{5n+1}}{1 - q^{5n+2}} - \frac{(5n+3)q^{5n+2}}{1 - q^{5n+3}} \right\} = \frac{f^5(-q)}{f(-q^5)}. \tag{14.4.2}$$

The equality (14.4.2) is a beautiful well-known identity of Ramanujan found in his notebooks [227], [61, p. 256, Entry 9(i)]. For several references to proofs of (14.4.2), see [61, pp. 261–262]. Hence, it follows that there exists an absolute constant A such that

$$R(q) = A \exp\left(-\frac{1}{5} \int_q^1 \frac{f^5(-t)}{f(-t^5)} \frac{dt}{t} \right). \tag{14.4.3}$$

Now let $q \to 1^-$. Recalling from Corollary 7.3.1 in Chapter 7 that

$$R(q) \sim \frac{\sqrt{5} - 1}{2},$$

as $q \to 1^-$, we conclude that $A = (\sqrt{5} - 1)/2$. This then completes the proof. □

In his famous (second) letter to Hardy [226, p. xxviii], [81, p. 57], Ramanujan communicated the following identity for the Rogers–Ramanujan continued fraction. The first proof is due to Watson [287]. The result can also be found in Ramanujan's notebooks [227], [61, p. 83, Entry 39(i)]; references to further proofs can be found in [61, p. 84]. See also equation (3.2.9) of Chapter 3.

Lemma 14.4.1. *Let* $\alpha, \beta > 0$, $\alpha\beta = \pi^2$, $q := e^{-2\alpha}$ *and* $Q := e^{-2\beta}$. *Then*

$$\left(\frac{\sqrt{5}+1}{2} + R(q)\right)\left(\frac{\sqrt{5}+1}{2} + R(Q)\right) = \frac{5+\sqrt{5}}{2}. \tag{14.4.4}$$

Lemma 14.4.2. *Let* α, β, q, *and* Q *be defined as in Lemma* 14.4.1. *Then*

$$\frac{f^5(-Q)}{f(-Q^5)} \log Q = \sqrt{5}\frac{f^5(-q)}{f(-q^{1/5})} q^{1/5} \log q. \tag{14.4.5}$$

Proof. Applying (14.2.5) twice, we find that

$$\frac{\left(\alpha^{1/4}e^{-\alpha/12}f(-e^{-2\alpha})\right)^5}{(\alpha/5)^{1/4}e^{-\alpha/60}f(e^{-2\alpha/5})} = \frac{\left(\beta^{1/4}e^{-\beta/12}f(-e^{-2\beta})\right)^5}{(5\beta)^{1/4}e^{-5\beta/12}f(e^{-10\beta})}.$$

Upon simplification, we find that

$$5^{1/4}\alpha e^{-2\alpha/5}\frac{f^5(-q)}{f(-q^{1/5})} = 5^{-1/4}\beta\frac{f^5(-Q)}{f(-Q^5)}.$$

Since $\alpha = -\frac{1}{2}\log q$ and $\beta = -\frac{1}{2}\log Q$, we complete the proof. □

Lemma 14.4.3. *Let* α, β, q, *and* Q *be defined as in Lemma* 14.4.1. *Then*

$$\int_Q^1 \frac{f^5(-t)}{f(-t^5)}\frac{dt}{t} = \sqrt{5}\int_0^q \frac{f^5(-t)}{f(-t^{1/5})}\frac{dt}{t^{4/5}}. \tag{14.4.6}$$

Proof. By the definitions of q and Q, we deduce that

$$(\log Q)(\log q) = (-2\alpha)(-2\beta) = 4\pi^2.$$

By differentiation, we find that

$$\frac{dQ}{dq} = -\frac{Q\log Q}{q\log q}.$$

Therefore, (14.4.5) becomes

$$-\frac{f^5(-Q)}{f(-Q^5)}\frac{1}{Q}\frac{dQ}{dq} = \sqrt{5}\cdot\frac{f^5(-q)}{f(-q^{1/5})}\frac{1}{q^{4/5}}.$$

Integrating, we complete the proof. □

Entry 14.4.2 (p. 46). *For* $0 \le q < 1$,

$$R(q) = \frac{\sqrt{5}-1}{2} - \frac{\sqrt{5}}{1 + \dfrac{3+\sqrt{5}}{2} \exp\left(\dfrac{1}{\sqrt{5}} \displaystyle\int_0^q \dfrac{f^5(-t)}{f(-q^{1/5})} \dfrac{dt}{t^{4/5}} \right)}.$$

Proof. Let α, β, q, and Q be defined as in Lemma 14.4.1, let $\epsilon := (\sqrt{5}+1)/2$, and let $F(q) := q^{-1/5}R(q)$. By Entry 14.4.1 for $0 < Q < 1$ and then by (14.4.6), we deduce that

$$\begin{aligned} Q^{1/5}\, F(Q) &= \frac{1}{\epsilon} \exp\left(-\frac{1}{5} \int_Q^1 \frac{f^5(-t)}{f(-t^5)} \frac{dt}{t} \right) \\ &= \frac{1}{\epsilon} \exp\left(-\frac{1}{\sqrt{5}} \int_0^q \frac{f^5(-t)}{f(-t^{1/5})} \frac{dt}{t^{4/5}} \right). \end{aligned}$$

Adding ϵ to both sides, we find that

$$\epsilon + Q^{1/5}\, F(Q) = \epsilon + \frac{1}{\epsilon} \exp\left(-\frac{1}{\sqrt{5}} \int_0^q \frac{f^5(-t)}{f(-t^{1/5})} \frac{dt}{t^{4/5}} \right).$$

Applying (14.4.4), we deduce that

$$\frac{\epsilon\sqrt{5}}{\epsilon + q^{1/5}F(q)} = \epsilon \left(1 + \frac{1}{\epsilon^2} \exp\left(-\frac{1}{\sqrt{5}} \int_0^q \frac{f^5(-t)}{f(-t^{1/5})} \frac{dt}{t^{4/5}} \right) \right). \tag{14.4.7}$$

Let

$$z := \frac{1}{\epsilon^2} \exp\left(-\frac{1}{\sqrt{5}} \int_0^q \frac{f^5(-t)}{f(-t^{1/5})} \frac{dt}{t^{4/5}} \right).$$

Using this notation in (14.4.7) and inverting both sides, we obtain

$$\epsilon + q^{1/5}F(q) = \frac{\sqrt{5}}{1+z} = \sqrt{5} \left(1 - \frac{1}{1 + (1/z)} \right).$$

Since

$$\frac{1}{z} = \frac{3+\sqrt{5}}{2} \exp\left(\frac{1}{\sqrt{5}} \int_0^q \frac{f^5(-t)}{f(-t^{1/5})} \frac{dt}{t^{4/5}} \right),$$

we complete the proof. □

Both Entry 14.3.6 and Entry 14.4.1 are, in fact, special cases of the following theorem of Ahlgren, Berndt, Yee, and Zaharescu [10]. Generalizations and simpler proofs of their theorem have been found by Y. Yang [296] and R. Takloo-Bighash [272].

Theorem 14.4.1. *Suppose that* α *is real, that* $k \ge 2$ *is an integer, and that* χ *is a nontrivial Dirichlet character that satisfies the condition* $\chi(-1) = (-1)^k$. *Then, for* $0 < q < 1$,

$$q^{\alpha} \prod_{n=1}^{\infty} (1-q^n)^{\chi(n)n^{k-2}} = \exp\left(-C - \int_q^1 \left\{\alpha - \sum_{n=1}^{\infty} \sum_{d|n} \chi(d) d^{k-1} t^n \right\} \frac{dt}{t}\right), \tag{14.4.8}$$

where

$$C = L'(2-k, \chi).$$

In special cases, such as in Entries 14.3.6 and 14.4.1, the integrand in (14.4.8) can be expressed in terms of eta functions. For a proof of Theorem 14.4.1 and several additional examples, see [10].

15

Incomplete Elliptic Integrals

15.1 Introduction

On pages 51–53 in his lost notebook [228], Ramanujan recorded several identities involving integrals of theta functions and incomplete elliptic integrals of the first kind. We offer here one typical example, proved in Entry 15.7.1 below. Let (in Ramanujan's notation) $f(-q) = (q;q)_\infty$. (Detailed notation is given in Section 15.2. The function f is essentially the Dedekind eta function; see (15.2.4).) Let

$$v := v(q) := q\frac{f^3(-q)f^3(-q^{15})}{f^3(-q^3)f^3(-q^5)}. \tag{15.1.1}$$

Then

$$\int_0^q f(-t)f(-t^3)f(-t^5)f(-t^{15})dt$$
$$= \frac{1}{5}\int_{2\tan^{-1}\left(\frac{1}{\sqrt{5}}\sqrt{\frac{1-11v-v^2}{1+v-v^2}}\right)}^{2\tan^{-1}(1/\sqrt{5})} \frac{d\varphi}{\sqrt{1-\frac{9}{25}\sin^2\varphi}}. \tag{15.1.2}$$

The reader will immediately realize that these are rather uncommon integrals. Indeed, we had never previously seen identities like (15.1.2) in the literature.

In a wonderful paper [213], all of these integral identities were proved by S. Raghavan and S.S. Rangachari. However, in almost all of their proofs, they used results with which Ramanujan would have been unfamiliar. In particular, they relied heavily on results from the theory of modular forms, evidently not known to Ramanujan. For example, for four identities, including (15.1.2), Raghavan and Rangachari appealed to differential equations satisfied by certain quotients of eta functions, such as (15.1.1), which can be found in R. Fricke's text [138].

In an effort to discern Ramanujan's methods and to better understand the origins of identities like (15.1.2), Berndt, H.H. Chan, and S.–S. Huang

[70] devised proofs independent of the theory of modular forms and other ideas with which Ramanujan would have been unfamiliar. In particular, they relied exclusively on results found in his ordinary notebooks [227] and his lost notebook [228]. It should be emphasized that at the time of the publication of Raghavan and Rangachari's paper [213], many of these results had not yet been proved. Particularly troublesome were the aforementioned four differential equations for quotients of eta functions. To prove them, identities for Eisenstein series found in Chapter 21 of Ramanujan's second notebook and several eta function identities scattered among the unorganized pages of his second notebook [62, Chapter 25] were used. These three authors also utilized several results in the lost notebook found on pages in close proximity to the elliptic integral identities. Furthermore, they owe a huge debt to Raghavan and Rangachari's paper [213]. In many cases, large portions of their proofs were incorporated, while in other instances different lines of attack were employed.

In Section 15.3, we prove two identities for integrals of theta functions of forms unlike (15.1.2). The first proof is virtually the same as that given by Raghavan and Rangachari, while the latter proof is completely different. In Sections 15.4–15.6, we prove several integral identities associated with modular equations of degree 5. Here some transformations of incomplete elliptic integrals due to J. Landen and Ramanujan play key roles. In Section 15.7, several identities of order 15 are established. Here two of the aforementioned differential equations are crucial. Differential equations are also central in Sections 15.8 and 15.9, where identities of orders 14 and 35, respectively, are proved.

15.2 Preliminary Results

Recall that Ramanujan's general theta function $f(a,b)$ is defined by

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \qquad |ab| < 1.$$

Theta functions satisfy the very important and useful Jacobi triple product identity [61, p. 35, Entry 19],

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty. \tag{15.2.1}$$

Recall also that the most important special cases are given by, for $|q| < 1$,

$$\varphi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{n^2} = \frac{(-q;q^2)_\infty(q^2;q^2)_\infty}{(-q^2;q^2)_\infty(q;q^2)_\infty}, \tag{15.2.2}$$

$$\psi(q) := f(q,q^3) = \sum_{n=0}^{\infty} q^{n(n+1)/2} = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty}, \tag{15.2.3}$$

and

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = (q;q)_\infty$$
$$= e^{-2\pi iz/24}\eta(z), \qquad q = e^{2\pi iz}, \qquad \operatorname{Im} z > 0. \tag{15.2.4}$$

The product representations in (15.2.2)–(15.2.4) are instances of the Jacobi triple product identity (15.2.1). The function $\eta(z)$, defined in (15.2.4), is the Dedekind eta function. It has the transformation formula

$$\eta(-1/z) = \sqrt{z/i}\,\eta(z). \tag{15.2.5}$$

The functions φ, ψ, and f in (15.2.2)–(15.2.4) can be expressed in terms of the modulus k and the hypergeometric function $z := {}_2F_1(\frac{1}{2},\frac{1}{2};1;k^2)$. For a catalogue of formulas of this type, see [61, pp. 122–124]. We will need two such formulas in the sequel. If $\alpha = k^2$ and

$$q = \exp\left(\frac{{}_2F_1(\frac{1}{2},\frac{1}{2};1;1-\alpha)}{{}_2F_1(\frac{1}{2},\frac{1}{2};1;\alpha)}\right),$$

then

$$\psi(-q) = \sqrt{\tfrac{1}{2}z}\,\{\alpha(1-\alpha)/q\}^{1/8} \tag{15.2.6}$$

and

$$f(-q^2) = \sqrt{z}2^{-1/3}\,\{\alpha(1-\alpha)/q\}^{1/12}. \tag{15.2.7}$$

The Eisenstein series $P(q)$, $Q(q)$, and $R(q)$ are defined for $|q| < 1$ by

$$P(q) := 1 - 24\sum_{n=1}^{\infty} \frac{nq^n}{1-q^n}, \tag{15.2.8}$$

$$Q(q) := 1 + 240\sum_{n=1}^{\infty} \frac{n^3q^n}{1-q^n}, \tag{15.2.9}$$

and

$$R(q) := 1 - 504\sum_{n=1}^{\infty} \frac{n^5q^n}{1-q^n}. \tag{15.2.10}$$

(This is the notation used by Ramanujan in his lost notebook and paper [223], [226, pp. 136–162], but in his ordinary notebooks, P, Q, and R are replaced by L, M, and N, respectively.)

The Rogers–Ramanujan continued fraction $u(q)$ is defined by

$$u := u(q) := \frac{q^{1/5}}{1}\,\underset{+}{}\,\frac{q}{1}\,\underset{+}{}\,\frac{q^2}{1}\,\underset{+}{}\,\frac{q^3}{1}\,\underset{+}{}\,\cdots, \qquad |q| < 1. \tag{15.2.11}$$

This notation is different from the notation $R(q)$ used in previous chapters; we have adhered here to the notation that Ramanujan employed in the entries of this chapter. With $f(-q)$ defined by (15.2.4), two of the most important properties of $u(q)$ are given by [61, p. 267, equations (11.5), (11.6)]

$$\frac{1}{u(q)} - 1 - u(q) = \frac{f(-q^{1/5})}{q^{1/5}f(-q^5)} \tag{15.2.12}$$

and

$$\frac{1}{u^5(q)} - 11 - u^5(q) = \frac{f^6(-q)}{qf^6(-q^5)}. \tag{15.2.13}$$

(See also (1.1.10) and (1.1.11) of Chapter 1.) Lastly, it can be shown that with the use of the Rogers–Ramanujan identities, given, for example, at the beginning of Chapter 10,

$$u(q) = q^{1/5}\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}. \tag{15.2.14}$$

(See also (1.1.2) of Chapter 1).

15.3 Two Simpler Integrals

Entry 15.3.1 (p. 51). *Let $P(q)$, $Q(q)$, and $R(q)$ be the Eisenstein series defined by* (15.2.8)–(15.2.10). *Then*

$$\int_{e^{-2\pi}}^{q} \sqrt{Q(t)}\frac{dt}{t} = \log\left(\frac{Q^{3/2}(q) - R(q)}{Q^{3/2}(q) + R(q)}\right).$$

Proof. Following Ramanujan's suggestion, let $z = R^2(t)/Q^3(t)$. Then

$$\frac{1}{z}\frac{dz}{dq} = \frac{2}{R}\frac{dR}{dq} - \frac{3}{Q}\frac{dQ}{dq}. \tag{15.3.1}$$

Using Ramanujan's differential equations [223, equation (30)], [228, p. 142], [61, p. 330]

$$q\frac{dR}{dq} = \frac{PR - Q^2}{2} \qquad \text{and} \qquad q\frac{dQ}{dq} = \frac{PQ - R}{3},$$

in (15.3.1), we find that

$$\frac{q}{z}\frac{dz}{dq} = \frac{R^2 - Q^3}{RQ}. \tag{15.3.2}$$

Hence, by (15.3.2),

$$q\frac{d}{dq}\log\left(\frac{Q^{3/2} - R}{Q^{3/2} + R}\right) = q\frac{d}{dq}\log\left(\frac{1 - \sqrt{z}}{1 + \sqrt{z}}\right)$$

$$
\begin{aligned}
&= q\frac{d}{dz}\log\left(\frac{1-\sqrt{z}}{1+\sqrt{z}}\right)\frac{dz}{dq} \\
&= \frac{1}{\sqrt{z}(z-1)}q\frac{dz}{dq} \\
&= \sqrt{Q}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\int_{e^{-2\pi}}^{q}\sqrt{Q(t)}\frac{dt}{t} &= \int_{e^{-2\pi}}^{q}\frac{d}{dt}\log\left(\frac{Q^{3/2}-R}{Q^{3/2}+R}\right)dt \\
&= \log\left(\frac{Q^{3/2}(q)-R(q)}{Q^{3/2}(q)+R(q)}\right) - \log\left(\frac{Q^{3/2}(e^{-2\pi})-R(e^{-2\pi})}{Q^{3/2}(e^{-2\pi})+R(e^{-2\pi})}\right).
\end{aligned}
$$

But it is well known that $R(e^{-2\pi}) = 0$ [123, p. 88], and so Entry 15.3.1 follows. □

See [211, p. 344] for some interesting comments by Raghavan on Entry 15.3.1.

Entry 15.3.2 (p. 53). *Let* $u(q)$ *denote the Rogers–Ramanujan continued fraction, defined by* (15.2.11), *and set* $v = u(q^2)$. *Recall that* $\psi(q)$ *is defined by* (15.2.3). *Then*

$$
\frac{8}{5}\int\frac{\psi^5(q)}{\psi(q^5)}\frac{dq}{q} = \log(u^2v^3) + \sqrt{5}\log\left(\frac{1+(\sqrt{5}-2)uv^2}{1-(\sqrt{5}+2)uv^2}\right). \tag{15.3.3}
$$

Proof. Let $k := k(q) := uv^2$. Then from page 53 of Ramanujan's lost notebook [228], or from page 326 of his second notebook [63, pp. 12–13],

$$
u^5 = k\left(\frac{1-k}{1+k}\right)^2 \qquad \text{and} \qquad v^5 = k^2\left(\frac{1+k}{1-k}\right). \tag{15.3.4}
$$

(In this book they are recorded in equations (1.8.1) of Chapter 1 and (2.6.16) in Chapter 2; see also S.–Y. Kang's paper [171].) It follows that

$$
\log(u^2v^3) = \frac{1}{5}\log\left(k^8\frac{1-k}{1+k}\right). \tag{15.3.5}
$$

If we set $\epsilon = (\sqrt{5}+1)/2$, we readily find that $\epsilon^3 = \sqrt{5}+2$ and $\epsilon^{-3} = \sqrt{5}-2$. Then, with the use of (15.3.5), we see that (15.3.3) is equivalent to the equality

$$
\frac{8}{5}\int\frac{\psi^5(q)}{\psi(q^5)}\frac{dq}{q} = \frac{1}{5}\log\left(k^8\frac{1-k}{1+k}\right) + \sqrt{5}\log\left(\frac{1+\epsilon^{-3}k}{1-\epsilon^3k}\right). \tag{15.3.6}
$$

Now from Entry 9(vi) in Chapter 19 of Ramanujan's second notebook [61, p. 258],

$$\frac{\psi^5(q)}{\psi(q^5)} = 25q^2\psi(q)\psi^3(q^5) + 1 - 5q\frac{d}{dq}\log\frac{f(q^2,q^3)}{f(q,q^4)}. \tag{15.3.7}$$

By the Jacobi triple product identity (15.2.1),

$$\begin{aligned}\frac{f(q^2,q^3)}{f(q,q^4)} &= \frac{(-q^2;q^5)_\infty(-q^3;q^5)_\infty}{(-q;q^5)_\infty(-q^4;q^5)_\infty}\\ &= \frac{(q;q^5)_\infty(q^4;q^5)_\infty(q^4;q^{10})_\infty(q^6;q^{10})_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty(q^2;q^{10})_\infty(q^8;q^{10})_\infty}\\ &= q^{1/5}\frac{u(q)}{v(q)}, \end{aligned}\tag{15.3.8}$$

by (15.2.14). Using (15.3.8) in (15.3.7), we find that

$$\begin{aligned}\frac{8}{5}\int\frac{\psi^5(q)}{\psi(q^5)}\frac{dq}{q} &= 40\int q\psi(q)\psi^3(q^5)dq + \int\frac{8}{5q}dq - 8\int\frac{d}{dq}\log\left(q^{1/5}u/v\right)dq\\ &= 40\int q\psi(q)\psi^3(q^5)dq - 8\log(u/v)\\ &= 40\int q\psi(q)\psi^3(q^5)dq + \frac{8}{5}\log k - \frac{24}{5}\log\frac{1-k}{1+k}, \end{aligned}\tag{15.3.9}$$

where (15.3.4) has been employed. Comparing (15.3.9) with (15.3.6), we now see that it suffices to prove that

$$8\int q\psi(q)\psi^3(q^5)dq = \log\frac{1-k}{1+k} + \frac{1}{\sqrt{5}}\log\left(\frac{1+\epsilon^{-3}k}{1-\epsilon^3 k}\right). \tag{15.3.10}$$

Upon differentiation of both sides of (15.3.10) and simplification, we find that (15.3.10) is equivalent to

$$q\psi(q)\psi^3(q^5) = \frac{k(q)k'(q)}{(1-k^2(q))(1-4k(q)-k^2(q))}. \tag{15.3.11}$$

We now prove (15.3.11). By (15.3.4) again,

$$\frac{v}{u^2} = \frac{1+k}{1-k}. \tag{15.3.12}$$

Taking the logarithmic derivative of both sides of (15.3.12), we find that

$$\frac{k'(q)}{1-k^2(q)} = \frac{1}{2}\frac{v'(q)}{v(q)} - \frac{u'(q)}{u(q)}. \tag{15.3.13}$$

By the logarithmic differentiation of (15.2.14),

$$\frac{u'(q)}{u(q)} = \frac{1}{5q} - \sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{nq^{n-1}}{1-q^n}$$

and

$$\frac{v'(q)}{v(q)} = 2\left(\frac{1}{5q} - \sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{nq^{2n-1}}{1-q^{2n}}\right),$$

where $\left(\frac{n}{5}\right)$ denotes the Legendre symbol. Using these derivatives in (15.3.13), we see that

$$\frac{k'(q)}{1-k^2(q)} = \sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{nq^{n-1}}{1-q^{2n}}. \tag{15.3.14}$$

However, from Entry 8(i) in Chapter 19 of Ramanujan's second notebook [61, p. 249],

$$\sum_{n=1}^{\infty}\left(\frac{n}{5}\right)\frac{nq^{n}}{1-q^{2n}} = q\psi^3(q)\psi(q^5) - 5q^2\psi(q)\psi^3(q^5),$$

so that by (15.3.14),

$$\frac{k'(q)}{1-k^2(q)} = \psi^3(q)\psi(q^5) - 5q\psi(q)\psi^3(q^5). \tag{15.3.15}$$

From page 56 in Ramanujan's lost notebook [228], which is Entry 1.8.2(ii) in Chapter 1 of this book,

$$\frac{\psi^2(q)}{q\psi^2(q^5)} = \frac{1-k^2(q)}{k(q)} + 1, \tag{15.3.16}$$

which has been proved by Kang [171, Theorem 4.2]. Putting (15.3.16) in (15.3.15), we deduce that

$$\frac{k'(q)}{1-k^2(q)} = \left(\frac{1-k^2(q)}{k(q)} - 4\right) q\psi(q)\psi^3(q^5). \tag{15.3.17}$$

It is easily seen that (15.3.17) is equivalent to (15.3.11), and so the proof of (15.3.3) is complete. □

15.4 Elliptic Integrals of Order 5 (I)

Entry 15.4.1 (p. 52). *With* $f(-q)$, $\psi(q)$, *and* $u(q)$ *defined by* (15.2.4), (15.2.3), *and* (15.2.11), *respectively, and with* $\epsilon = (\sqrt{5}+1)/2$,

$$5^{3/4}\int_0^q \frac{f^2(-t)f^2(-t^5)}{\sqrt{t}}dt = 2\int_{\cos^{-1}\left((\epsilon u)^{5/2}\right)}^{\pi/2} \frac{d\varphi}{\sqrt{1-\epsilon^{-5}5^{-3/2}\sin^2\varphi}} \tag{15.4.1}$$

$$= \int_0^{2\tan^{-1}\left(5^{3/4}\sqrt{q}f^3(-q^5)/f^3(-q)\right)} \frac{d\varphi}{\sqrt{1-\epsilon^{-5}5^{-3/2}\sin^2\varphi}} \tag{15.4.2}$$

$$= \sqrt{5}\int_0^{2\tan^{-1}\left(5^{1/4}\sqrt{q}\psi(q^5)/\psi(q)\right)} \frac{d\varphi}{\sqrt{1-\epsilon 5^{-1/2}\sin^2\varphi}}. \tag{15.4.3}$$

To prove (15.4.1), we need the following lemma.

Lemma 15.4.1. *Let $u(q)$ be defined by* (15.2.11)*. Then*

$$u'(q)=\frac{u(q)}{5q}\frac{f^5(-q)}{f(-q^5)}.$$

Proof. By (15.2.14) and the Jacobi triple product identity (15.2.1),

$$u(q)=q^{1/5}\frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}=q^{1/5}\frac{f(-q,-q^4)}{f(-q^2,-q^3)}.$$

By logarithmic differentiation and the use of Entry 9(v) in Chapter 19 of Ramanujan's second notebook [61, p. 258],

$$\begin{aligned}\frac{u'(q)}{u(q)}&=\frac{1}{5q}+\frac{d}{dq}\log\frac{f(-q,-q^4)}{f(-q^2,-q^3)}\\&=\frac{1}{5q}+\frac{1}{5q}\left(-1+\frac{f^5(-q)}{f(-q^5)}\right)=\frac{1}{5q}\frac{f^5(-q)}{f(-q^5)},\end{aligned}$$

which completes the proof. □

Proof of (15.4.1). Let

$$\cos^2\varphi=\epsilon^5u^5(t). \tag{15.4.4}$$

If $t=0$, then $\varphi=\pi/2$; if $t=q$, then $\varphi=\cos^{-1}\left((\epsilon u)^{5/2}\right)$. Upon differentiation and the use of Lemma 15.4.1,

$$\begin{aligned}2\cos\varphi(-\sin\varphi)\frac{d\varphi}{dt}&=5\epsilon^5u^4(t)u'(t)\\&=\epsilon^5\frac{u^5(t)}{t}\frac{f^5(-t)}{f(-t^5)}=\cos^2\varphi\frac{f^5(-t)}{tf(-t^5)}.\end{aligned} \tag{15.4.5}$$

Hence, by (15.4.5), (15.2.13), and (15.4.4),

$$\begin{aligned}&5^{3/4}\int_0^q\frac{f^2(-t)f^2(-t^5)}{\sqrt{t}}dt\\&=5^{3/4}\int_{\pi/2}^{\cos^{-1}\left((\epsilon u)^{5/2}\right)}\frac{f^2(-t)f^2(-t^5)}{\sqrt{t}}\frac{-2tf(-t^5)}{f^5(-t)}\frac{\sin\varphi}{\cos\varphi}d\varphi\\&=2\cdot5^{3/4}\int_{\cos^{-1}\left((\epsilon u)^{5/2}\right)}^{\pi/2}\sqrt{t}\frac{f^3(-t^5)}{f^3(-t)}\frac{\sin\varphi}{\cos\varphi}d\varphi\\&=2\cdot5^{3/4}\int_{\cos^{-1}\left((\epsilon u)^{5/2}\right)}^{\pi/2}\frac{1}{\sqrt{1/u^5(t)-11-u^5(t)}}\frac{\sin\varphi}{\cos\varphi}d\varphi\\&=2\cdot5^{3/4}\int_{\cos^{-1}\left((\epsilon u)^{5/2}\right)}^{\pi/2}\frac{\sin\varphi}{\sqrt{\epsilon^5-11\cos^2\varphi-\epsilon^{-5}\cos^4\varphi}}d\varphi.\end{aligned} \tag{15.4.6}$$

Since $\epsilon^{\pm 5} = (5\sqrt{5} \pm 11)/2$,

$$\begin{aligned}
\epsilon^5 - 11\cos^2\varphi - \epsilon^{-5}\cos^4\varphi &= \epsilon^5 - 11(1-\sin^2\varphi) - \epsilon^{-5}\cos^4\varphi \\
&= \epsilon^{-5} + 11\sin^2\varphi - \epsilon^{-5}\cos^4\varphi \\
&= \epsilon^{-5}(1-\cos^2\varphi)(1+\cos^2\varphi) + 11\sin^2\varphi \\
&= \epsilon^{-5}\sin^2\varphi(2-\sin^2\varphi) + 11\sin^2\varphi \\
&= \sin^2\varphi(2\epsilon^{-5} + 11 - \epsilon^{-5}\sin^2\varphi) \\
&= \sin^2\varphi(5\sqrt{5} - \epsilon^{-5}\sin^2\varphi) \\
&= 5\sqrt{5}\sin^2\varphi(1 - \epsilon^{-5}5^{-3/2}\sin^2\varphi).
\end{aligned}$$

Thus, from (15.4.6),

$$5^{3/4}\int_0^q \frac{f^2(-t)f^2(-t^5)}{\sqrt{t}}dt = 2\int_{\cos^{-1}\left((\epsilon u)^{5/2}\right)}^{\pi/2} \frac{d\varphi}{\sqrt{1-\epsilon^{-5}5^{-3/2}\sin^2\varphi}},$$

which is (15.4.1). □

To prove (15.4.2), we need two transformations for incomplete elliptic integrals found in Chapter 17 of Ramanujan's second notebook [61, pp. 105–106, Entries 7(ii), (vi)].

Lemma 15.4.2. *If* $\tan\gamma = \sqrt{1-x}\ \tan\alpha$, *then*

$$\int_0^\alpha \frac{d\varphi}{\sqrt{1-x\sin^2\varphi}} = \int_0^\gamma \frac{d\varphi}{\sqrt{1-x\cos^2\varphi}}. \tag{15.4.7}$$

If $\cot\alpha\ \tan(\beta/2) = \sqrt{1-x\sin^2\alpha}$, *then*

$$2\int_0^\alpha \frac{d\varphi}{\sqrt{1-x\sin^2\varphi}} = \int_0^\beta \frac{d\varphi}{\sqrt{1-x\sin^2\varphi}}. \tag{15.4.8}$$

Proof of (15.4.2). In (15.4.7), replace φ by $\pi/2-\varphi$ and combine the result with (15.4.8) to deduce that

$$\int_0^\beta \frac{d\varphi}{\sqrt{1-x\sin^2\varphi}} = 2\int_{\pi/2-\gamma}^{\pi/2} \frac{d\varphi}{\sqrt{1-x\sin^2\varphi}}, \tag{15.4.9}$$

provided that

(i) $\qquad \cot\alpha\ \tan(\beta/2) = \sqrt{1-x\sin^2\alpha},$

(ii) $\qquad \tan\gamma = \sqrt{1-x}\ \tan\alpha.$

Examining (15.4.1) and (15.4.2), we see that we want to set $x = \epsilon^{-5}5^{-3/2}$ and $\gamma = \frac{\pi}{2} - \cos^{-1}\left((\epsilon u)^{5/2}\right)$. We also see that, to prove (15.4.2), we will need to show that (i) and (ii) imply that

$$\beta = 2\tan^{-1}\left(5^{3/4}\sqrt{q}f^3(-q^5)/f^3(-q)\right). \tag{15.4.10}$$

Since $\epsilon^{\pm 5} = (5\sqrt{5} \pm 11)/2$, a short calculation gives

$$1 - \epsilon^{-5}5^{-3/2} = \epsilon^5 5^{-3/2}.$$

Thus, from (ii) and elementary trigonometry,

$$\begin{aligned}\tan\alpha &= \frac{1}{\sqrt{1-\epsilon^{-5}5^{-3/2}}}\cot\left(\cos^{-1}(\epsilon u)^{5/2}\right)\\ &= \epsilon^{-5/2}5^{3/4}\frac{(\epsilon u)^{5/2}}{\sqrt{1-(\epsilon u)^5}} = \frac{5^{3/4}u^{5/2}}{\sqrt{1-(\epsilon u)^5}}.\end{aligned} \tag{15.4.11}$$

Thus, by (i),

$$\tan(\beta/2) = \sqrt{1-\epsilon^{-5}5^{-3/2}\sin^2\alpha}\,\frac{5^{3/4}u^{5/2}}{\sqrt{1-(\epsilon u)^5}}. \tag{15.4.12}$$

From (15.4.11) and elementary trigonometry,

$$x\ \sin^2\alpha = \frac{\epsilon^{-5}u^5}{1+\epsilon^{-5}u^5}.$$

Using this in (15.4.12), we deduce that

$$\begin{aligned}\tan(\beta/2) &= \sqrt{1-\frac{\epsilon^{-5}u^5}{1+\epsilon^{-5}u^5}}\,\frac{5^{3/4}u^{5/2}}{\sqrt{1-(\epsilon u)^5}}\\ &= \frac{5^{3/4}u^{5/2}}{\sqrt{(1+\epsilon^{-5}u^5)(1-\epsilon^5u^5)}}\\ &= \frac{5^{3/4}u^{5/2}}{\sqrt{1-11u^5-u^{10}}}\\ &= \frac{5^{3/4}}{\sqrt{1/u^5-11-u^5}}\\ &= 5^{3/4}\sqrt{q}f^3(-q^5)/f^3(-q),\end{aligned}$$

by (15.2.13). Clearly, the last equality is equivalent to (15.4.10), and so the proof of (15.4.2) is complete. □

For the proof of (15.4.3), we need another transformation for incomplete elliptic integrals.

Lemma 15.4.3. *If* $0 < p < 1$ *and*

$$\tan\left(\frac{1}{2}(A-B)\right) = \frac{1-p}{1+2p}\tan B, \tag{15.4.13}$$

then

$$(1+2p)\int_0^A \frac{d\varphi}{\sqrt{1-p^3\left(\dfrac{2+p}{1+2p}\right)\sin^2\varphi}} = 3\int_0^B \frac{d\varphi}{\sqrt{1-p\left(\dfrac{2+p}{1+2p}\right)^3\sin^2\varphi}}.$$

This lemma is Entry 6(iv) in Chapter 19 in Ramanujan's second notebook and is a consequence of a theorem of Jacobi; see [61, pp. 238–241] for a proof.

Proof of (15.4.3). We apply Lemma 15.4.3 with

$$p = \frac{1}{\epsilon^2\sqrt{5}},$$

where $\epsilon = (\sqrt{5}+1)/2$. Then

$$1+2p = \frac{3}{\sqrt{5}} \qquad \text{and} \qquad 2+p = \frac{3\epsilon}{\sqrt{5}}, \tag{15.4.14}$$

and so

$$p^3\left(\frac{2+p}{1+2p}\right) = \epsilon^{-5}5^{-3/2} \qquad \text{and} \qquad p\left(\frac{2+p}{1+2p}\right)^3 = \frac{\epsilon}{\sqrt{5}}.$$

If we substitute these quantities in Lemma 15.4.3, and if we set

$$A = 2\tan^{-1}\left(5^{3/4}\sqrt{q}f^3(-q^5)/f^3(-q)\right) \tag{15.4.15}$$

and

$$B = 2\tan^{-1}\left(5^{1/4}\sqrt{q}\psi(q^5)/\psi(q)\right), \tag{15.4.16}$$

we shall be finished with the proof of (15.4.3) if we can prove (15.4.13).

Using the subtraction formula for the tangent function, (15.4.15), and (15.4.16), we deduce that

$$\tan\left(\frac{1}{2}(A-B)\right) = \frac{5^{3/4}\sqrt{q}f^3(-q^5)/f^3(-q) - 5^{1/4}\sqrt{q}\psi(q^5)/\psi(q)}{1+5q\dfrac{f^3(-q^5)\psi(q^5)}{f^3(-q)\psi(q)}}. \tag{15.4.17}$$

It will be convenient to use some results from the lost notebook proved by Kang [172]; see Entry 2.5.1 of Chapter 2. Set

$$t = q^{1/6}\frac{(-q^5;q^5)_\infty}{(-q;q)_\infty} \qquad \text{and} \qquad s = \frac{\varphi(-q)}{\varphi(-q^5)}, \tag{15.4.18}$$

where $\varphi(q)$ is defined by (15.2.2). Then

$$\frac{f(-q)}{q^{1/6}f(-q^5)} = \frac{s}{t} \qquad \text{and} \qquad \frac{\psi(q)}{\sqrt{q}\psi(q^5)} = \frac{s}{t^3}. \tag{15.4.19}$$

Employing (15.4.19) in (15.4.17), we readily deduce that

$$5^{-1/4}\tan\left(\frac{1}{2}(A-B)\right)=\frac{\sqrt{5}t^3s-t^3s^3}{s^4+5t^6}. \tag{15.4.20}$$

Next, a simple calculation shows that

$$1-p=\frac{3}{\epsilon\sqrt{5}}. \tag{15.4.21}$$

Hence, by (15.4.14), (15.4.21), (15.4.16), and the double angle formula,

$$\begin{aligned}5^{-1/4}\frac{1-p}{1+2p}\tan B&=5^{-1/4}\epsilon^{-1}\tan B\\&=5^{-1/4}\epsilon^{-1}\tan\left(2\tan^{-1}\left(5^{1/4}\sqrt{q}\psi(q^5)/\psi(q)\right)\right)\\&=\frac{2\epsilon^{-1}\sqrt{q}\psi(q^5)/\psi(q)}{1-\sqrt{5}\;q\psi^2(q^5)/\psi^2(q)}\\&=\frac{2\epsilon^{-1}t^3s}{s^2-\sqrt{5}\;t^6}.\end{aligned} \tag{15.4.22}$$

Comparing (15.4.20) and (15.4.22), in view of (15.4.13), we must prove that

$$\frac{2\epsilon^{-1}t^3s}{s^2-\sqrt{5}\;t^6}=\frac{\sqrt{5}\;t^3s-t^3s^3}{s^4+5t^6}.$$

After considerable simplification, the last equality is seen to be equivalent to

$$s^4+5t^6=s^2+s^2t^6. \tag{15.4.23}$$

Now, from (15.4.18) and (15.2.3), we find that

$$t=t(q)=\frac{q^{1/6}\psi(q^5)f(-q^2)}{\psi(q)f(-q^{10})}.$$

Replacing q by $-q$ and employing (15.2.6) and (15.2.7), we find that

$$t^6(-q)=-q\left(\frac{\psi(-q^5)f(-q^2)}{\psi(-q)f(-q^{10})}\right)^6=-\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4},$$

where β has degree 5 over α. On the other hand, from (15.4.18),

$$s(-q)=\frac{\varphi(q)}{\varphi(q^5)}=:\sqrt{m},$$

where m is the multiplier of degree 5. Hence, replacing q by $-q$ in (15.4.23), we see that this equality is equivalent to

$$m^2 - 5\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4} = m - m\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4}. \qquad (15.4.24)$$

Using formulas for m and $5/m$ given in Entry 13(xii) of Chapter 19 in Ramanujan's second notebook [61, pp. 281–282], namely,

$$m = \left(\frac{\beta}{\alpha}\right)^{1/4} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/4} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4}$$

and

$$\frac{5}{m} = \left(\frac{\alpha}{\beta}\right)^{1/4} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/4} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/4},$$

we may easily verify that (15.4.24) does hold to complete the proof. □

15.5 Elliptic Integrals of Order 5 (II)

Entry 15.5.1 (p. 52). *As before, let* $\epsilon = (\sqrt{5}+1)/2$, *and let* $u(q)$ *and* $f(-q)$ *be defined by* (15.2.11) *and* (15.2.4), *respectively. Then*

$$5^{-3/4}\int_0^q \frac{f^5(-t)}{\sqrt{f(-t^{1/5})f(-t^5)}}\frac{dt}{t^{9/10}} = 2\int_{\cos^{-1}\left(\sqrt{\epsilon u}\right)}^{\pi/2} \frac{d\varphi}{\sqrt{1-\epsilon^{-1}5^{-1/2}\sin^2\varphi}} \qquad (15.5.1)$$

$$= \int_0^{2\tan^{-1}\left(5^{1/4}q^{1/10}\sqrt{f(-q^5)/f(-q^{1/5})}\right)} \frac{d\varphi}{\sqrt{1-\epsilon^{-1}5^{-1/2}\sin^2\varphi}} \qquad (15.5.2)$$

$$= \frac{1}{\sqrt{5}}\int_0^{2\tan^{-1}\left(5^{3/4}q^{1/10}\left(\frac{f(-q^{1/5})+q^{1/5}f(-q^5)}{f(-q^{1/5})+5q^{1/5}f(-q^5)}\right)\sqrt{\frac{f(-q^5)}{f(-q^{1/5})}}\right)} \frac{d\varphi}{\sqrt{1-\epsilon^5 5^{-3/2}\sin^2\varphi}}. \qquad (15.5.3)$$

Proof of (15.5.1). Let

$$\cos^2\varphi = \epsilon u(t). \qquad (15.5.4)$$

Thus, if $t=0$, then $\varphi = \pi/2$; if $t=q$, then $\varphi = \cos^{-1}\left(\sqrt{\epsilon u}\right)$. Upon differentiation and the use of Lemma 15.4.1,

$$2\cos\varphi(-\sin\varphi)\frac{d\varphi}{dt} = \epsilon u'(t) = \epsilon\frac{u(t)}{5t}\frac{f^5(-t)}{f(-t^5)}. \qquad (15.5.5)$$

Therefore, by (15.5.5), (15.2.12), and (15.5.4),

$$5^{-3/4}\int_0^q \frac{f^5(-t)}{\sqrt{f(-t^{1/5})f(-t^5)}}\frac{dt}{t^{9/10}}$$

$$
\begin{aligned}
&= 2\cdot 5^{1/4}\int_{\cos^{-1}\left(\sqrt{\epsilon u}\right)}^{\pi/2}\sqrt{\frac{t^{1/5}f(-t^5)}{f(-t^{1/5})}\frac{\sin\varphi\ \cos\varphi}{\epsilon u(t)}}d\varphi\\
&= 2\cdot 5^{1/4}\int_{\cos^{-1}\left(\sqrt{\epsilon u}\right)}^{\pi/2}\frac{1}{\sqrt{1/u(t)-1-u(t)}}\frac{\sin\varphi}{\cos\varphi}d\varphi\\
&= 2\cdot 5^{1/4}\int_{\cos^{-1}\left(\sqrt{\epsilon u}\right)}^{\pi/2}\frac{\sin\varphi}{\sqrt{\epsilon-\cos^2\varphi-\epsilon^{-1}\cos^4\varphi}}d\varphi. \qquad (15.5.6)
\end{aligned}
$$

Now,

$$
\begin{aligned}
\epsilon-\cos^2\varphi-\epsilon^{-1}\cos^4\varphi &= \epsilon-(1-\sin^2\varphi)-\epsilon^{-1}\cos^4\varphi\\
&= \epsilon^{-1}+\sin^2\varphi-\epsilon^{-1}\cos^4\varphi\\
&= \epsilon^{-1}(1-\cos^2\varphi)(1+\cos^2\varphi)+\sin^2\varphi\\
&= \epsilon^{-1}\sin^2\varphi(2-\sin^2\varphi)+\sin^2\varphi\\
&= \sin^2\varphi(2\epsilon^{-1}-\epsilon^{-1}\sin^2\varphi+1)\\
&= \sin^2\varphi(\sqrt{5}-\epsilon^{-1}\sin^2\varphi).
\end{aligned}
$$

Using this calculation in (15.5.6), we find that

$$
5^{-3/4}\int_0^q\frac{f^5(-t)}{\sqrt{f(-t^{1/5})f(-t^5)}}\frac{dt}{t^{9/10}} = 2\cdot 5^{1/4}\int_{\cos^{-1}\left(\sqrt{\epsilon u}\right)}^{\pi/2}\frac{d\varphi}{\sqrt{\sqrt{5}-\epsilon^{-1}\sin^2\varphi}},
$$

from which (15.5.1) is immediate. □

Proof of (15.5.2). The proof is similar to that of (15.4.2). We begin with (15.4.9), set $x=\epsilon^{-1}5^{-1/2}$, and put $\gamma=\frac{\pi}{2}-\cos^{-1}\left(\sqrt{\epsilon u}\right)$. Thus,

$$
\tan\gamma=\cot\left(\cos^{-1}\left(\sqrt{\epsilon u}\right)\right)=\sqrt{\frac{\epsilon u}{1-\epsilon u}}. \qquad (15.5.7)
$$

As with the proof of (15.4.2), we want to show that conditions (i) and (ii) imply that

$$
\beta=2\tan^{-1}\left(5^{1/4}q^{1/10}\sqrt{f(-q^5)/f(-q^{1/5})}\right). \qquad (15.5.8)
$$

From condition (ii) and (15.5.7),

$$
\tan\alpha=\frac{\tan\gamma}{\sqrt{1-\epsilon^{-1}5^{-1/2}}}=\sqrt{\frac{\sqrt{5}\ u}{1-\epsilon u}} \qquad (15.5.9)
$$

and

$$
\sin^2\alpha=\frac{\tan^2\alpha}{1+\tan^2\alpha}=\frac{\sqrt{5}\ u}{1+\epsilon^{-1}u}. \qquad (15.5.10)
$$

Using (15.5.9) and (15.5.10) in conjunction with condition (ii), we arrive at

$$\tan(\beta/2) = (\tan\alpha)\sqrt{1-x\ \sin^2\alpha} = \sqrt{\frac{\sqrt{5}\ u}{1-\epsilon u}}\sqrt{1-\frac{\epsilon^{-1}u}{1+\epsilon^{-1}u}}$$
$$= 5^{1/4}\sqrt{\frac{1}{1/u-1-u}} = 5^{1/4}q^{1/10}\sqrt{\frac{f(-q^5)}{f(-q^{1/5})}}.$$

Hence, (15.5.8) follows, and so the proof of (15.5.2) is finished. □

To prove (15.5.3), we need another transformation for incomplete elliptic integrals from Chapter 19 in Ramanujan's second notebook [61, p. 238, Entry 6(iii)].

Lemma 15.5.1. *If*

$$\tan\left(\frac{1}{2}(\alpha+\beta)\right) = (1+p)\tan\alpha,$$

where $0 < p < 1$, *then*

$$(1+2p)\int_0^\alpha \frac{d\varphi}{\sqrt{1-p^3\left(\dfrac{2+p}{1+2p}\right)\sin^2\varphi}} = \int_0^\beta \frac{d\varphi}{\sqrt{1-p\left(\dfrac{2+p}{1+2p}\right)^3\sin^2\varphi}}.$$

Proof of (15.5.3). We apply Lemma 15.5.1 with $p = 1/\epsilon$. Thus, $1+2p = \sqrt{5}$ and $2+p = \epsilon^2$. Hence,

$$p^3\frac{2+p}{1+2p} = \frac{1}{\epsilon\sqrt{5}} \qquad \text{and} \qquad p\left(\frac{2+p}{1+2p}\right)^3 = \frac{\epsilon^5}{5\sqrt{5}}.$$

We abbreviate notation by setting

$$A = f(-q^{1/5}), \qquad B = f(-q^5), \qquad \text{and} \qquad C = q^{1/5}.$$

Put

$$\alpha = 2\tan^{-1}\left(5^{1/4}\sqrt{\frac{CB}{A}}\right)$$

and

$$\beta = 2\tan^{-1}\left(5^{3/4}\frac{A+CB}{A+5CB}\sqrt{\frac{CB}{A}}\right).$$

Examining (15.5.2) and (15.5.3) in relation to Lemma 15.5.1, we see that we will be finished with the proof if we can show that

$$\tan\left(\frac{1}{2}(\alpha+\beta)\right) = \epsilon\tan\alpha. \tag{15.5.11}$$

First, by the addition formula for the tangent function,

$$\begin{aligned}
&\tan\left(\frac{1}{2}(\alpha+\beta)\right) \\
&\quad = \tan\left(\tan^{-1}\left(5^{1/4}\sqrt{\frac{CB}{A}}\right) + \tan^{-1}\left(5^{3/4}\frac{A+CB}{A+5CB}\sqrt{\frac{CB}{A}}\right)\right) \\
&\quad = \frac{5^{1/4}\sqrt{\dfrac{CB}{A}} + 5^{3/4}\dfrac{A+CB}{A+5CB}\sqrt{\dfrac{CB}{A}}}{1-5\dfrac{CB}{A}\dfrac{A+CB}{A+5CB}} \\
&\quad = \frac{5^{1/4}\left(A+5CB+\sqrt{5}(A+CB)\right)\sqrt{ABC}}{A^2-5B^2C^2}. \qquad (15.5.12)
\end{aligned}$$

On the other hand, by the double angle formula for the tangent function,

$$\epsilon\tan\alpha = \epsilon\tan\left(2\tan^{-1}\left(5^{1/4}\sqrt{\frac{BC}{A}}\right)\right) = \frac{2\epsilon 5^{1/4}\sqrt{ABC}}{A-\sqrt{5}\,CB}. \qquad (15.5.13)$$

Comparing (15.5.12) and (15.5.13), we are required to prove that

$$\frac{2\epsilon}{A-\sqrt{5}\,CB} = \frac{A+5CB+\sqrt{5}(A+CB)}{A^2-5B^2C^2}.$$

This can be established by elementary algebra, and so the proof is complete. □

15.6 Elliptic Integrals of Order 5 (III)

Entry 15.6.1 (p. 52). *Recall that Ramanujan's continued fraction $u(q)$ is defined by* (15.2.11)*. Then there exists a constant C such that for $0<q<1$,*

$$u^5+u^{-5} = \frac{1}{2\sqrt{q}}\frac{f^3(-q)}{f^3(-q^5)}\left(C+\int_q^1\frac{f^8(-t)}{f^4(-t^5)}\frac{dt}{t^{3/2}}+125\int_0^q\frac{f^8(-t^5)}{f^4(-t)}\sqrt{t}\,dt\right). \qquad (15.6.1)$$

To prove Entry 15.6.1, we need to establish a differential equation for a certain quotient of eta functions.

Lemma 15.6.1. *Let*

$$\lambda := \lambda(q) := q\frac{f^6(-q^5)}{f^6(-q)}. \qquad (15.6.2)$$

Then

$$q\frac{d}{dq}(\lambda(q)) = \sqrt{q}f^2(-q)f^2(-q^5)\sqrt{125\lambda^3+22\lambda^2+\lambda}. \qquad (15.6.3)$$

Proof. By logarithmic differentiation,

$$\frac{1}{\lambda}\frac{d\lambda}{dq}=\frac{1}{q}-30\sum_{n=1}^{\infty}\frac{nq^{5n-1}}{1-q^{5n}}+6\sum_{n=1}^{\infty}\frac{nq^{n-1}}{1-q^{n}}.$$

We now apply Entry 4(i) in Chapter 21 of Ramanujan's second notebook [61, p. 463]. Accordingly,

$$\begin{aligned}\frac{q}{\lambda}\frac{d\lambda}{dq}&=\frac{\sqrt{f^{12}(-q)+22qf^{6}(-q)f^{6}(-q^5)+125q^2f^{12}(-q^5)}}{f(-q)f(-q^5)}\\&=\sqrt{q}f^2(-q)f^2(-q^5)\sqrt{\frac{1}{\lambda}+22+125\lambda},\end{aligned}$$

or

$$q\frac{d\lambda}{dq}=\sqrt{q}f^2(-q)f^2(-q^5)\sqrt{\lambda+22\lambda^2+125\lambda^3},$$

and the proof is complete. □

Proof of Entry 15.6.1. From (15.2.13), in the notation (15.6.2),

$$\frac{1}{u^5}-11-u^5=\frac{1}{\lambda}. \tag{15.6.4}$$

Considering (15.6.4) as a quadratic equation in $x:=u^{-5}$, we find upon solving it that

$$2x=2u^{-5}=\frac{1}{\lambda}\left(11\lambda+1+\sqrt{125\lambda^2+22\lambda+1}\right).$$

The other root of this quadratic equation is easily seen to be

$$-2u^5=\frac{1}{\lambda}\left(11\lambda+1-\sqrt{125\lambda^2+22\lambda+1}\right).$$

Hence,

$$u^5+u^{-5}=\frac{1}{\lambda}\sqrt{125\lambda^2+22\lambda+1}.$$

Thus,

$$G(q):=2\sqrt{\lambda}(u^5+u^{-5})=2\sqrt{125\lambda+22+\frac{1}{\lambda}}. \tag{15.6.5}$$

Thus, by (15.6.5) and (15.6.3),

$$\begin{aligned}\frac{dG}{dq}&=\frac{125-1/\lambda^2}{\sqrt{125\lambda+22+1/\lambda}}\frac{d\lambda}{dq}\\&=\frac{125-1/\lambda^2}{\sqrt{125\lambda+22+1/\lambda}}\frac{f^2(-q)f^2(-q^5)}{\sqrt{q}}\sqrt{125\lambda^3+22\lambda^2+\lambda}\\&=\frac{125\lambda-1/\lambda}{\sqrt{q}}f^2(-q)f^2(-q^5)\\&=125\sqrt{q}\frac{f^8(-q^5)}{f^4(-q)}-\frac{f^8(-q)}{q^{3/2}f^4(-q^5)},\end{aligned}$$

upon the use of (15.6.2) again.

Thus, for any q_0 such that $0 < q_0 < 1$,

$$G(q) - G(q_0) = \int_{q_0}^{q} \frac{dG}{dt} dt = 125 \int_{q_0}^{q} \frac{f^8(-t^5)}{f^4(-t)} \sqrt{t}\, dt - \int_{q_0}^{q} \frac{f^8(-t)}{f^4(-t^5)} \frac{dt}{t^{3/2}},$$

or, by (15.6.5) and (15.6.2),

$$\begin{aligned} u^5 &+ u^{-5} \\ &= \frac{f^3(-q)}{2\sqrt{q} f^3(-q^5)} \left(G(q_0) + 125 \int_{q_0}^{q} \frac{f^8(-t^5)}{f^4(-t)} \sqrt{t}\, dt - \int_{q_0}^{q} \frac{f^8(-t)}{f^4(-t^5)} \frac{dt}{t^{3/2}} \right) \\ &= \frac{f^3(-q)}{2\sqrt{q} f^3(-q^5)} \left(G(q_0) + 125 \int_{0}^{q} \frac{f^8(-t^5)}{f^4(-t)} \sqrt{t}\, dt - 125 \int_{0}^{q_0} \frac{f^8(-t^5)}{f^4(-t)} \sqrt{t}\, dt \right. \\ &\qquad\qquad \left. + \int_{q}^{1} \frac{f^8(-t)}{f^4(-t^5)} \frac{dt}{t^{3/2}} - \int_{q_0}^{1} \frac{f^8(-t)}{f^4(-t^5)} \frac{dt}{t^{3/2}} \right) \\ &= \frac{f^3(-q)}{2\sqrt{q} f^3(-q^5)} \left(C + 125 \int_{0}^{q} \frac{f^8(-t^5)}{f^4(-t)} \sqrt{t}\, dt + \int_{q}^{1} \frac{f^8(-t)}{f^4(-t^5)} \frac{dt}{t^{3/2}} \right), \end{aligned}$$

where

$$C = G(q_0) - 125 \int_{0}^{q_0} \frac{f^8(-t^5)}{f^4(-t)} \sqrt{t}\, dt - \int_{q_0}^{1} \frac{f^8(-t)}{f^4(-t^5)} \frac{dt}{t^{3/2}}. \tag{15.6.6}$$

Thus, we have completed the proof of Entry 15.6.1 and have furthermore shown that the constant C is given by (15.6.6). □

Now set $q_0 = e^{-2\pi/\theta}$. Ramanujan calculated $G(e^{-2\pi/\theta})$ for three values of θ.

Entry 15.6.2 (p. 52). *We have*

(i) $$G(e^{-2\pi/\sqrt{5}}) = 4 \left(\frac{\sqrt{5}+1}{2} \right)^{5/2},$$

(ii) $$G(e^{-2\pi}) = G(e^{-2\pi/5}) = 6 \cdot 5^{1/4}(3 + \sqrt{5}).$$

Ramanujan erroneously claimed that

$$G(e^{-2\pi}) = G(e^{-2\pi/5}) = 16 \cdot 5^{-1/4}(2 + \sqrt{5}).$$

Raghavan and Rangachari [213] used a different method to prove Entry 15.6.2.

Proof. To prove (i), we need to evaluate

$$\lambda(e^{-2\pi/\sqrt{5}}) = e^{-2\pi/\sqrt{5}} \frac{f^6(-e^{-2\pi\sqrt{5}})}{f^6(-e^{-2\pi/\sqrt{5}})}, \tag{15.6.7}$$

where $\lambda(q)$ is defined by (15.6.2). Recall the transformation formula [61, p. 31, Entry 27(iii)] for $f(-q)$. If $\alpha, \beta > 0$ and $\alpha\beta = \pi^2$, then

$$e^{-\alpha/12}\sqrt[4]{\alpha}f(-e^{-2\alpha}) = e^{-\beta/12}\sqrt[4]{\beta}f(-e^{-2\beta}). \tag{15.6.8}$$

Applying (15.6.8) with $\alpha = \pi/\sqrt{5}$ and $\beta = \pi\sqrt{5}$, we find, after simplification, that

$$f(-e^{-2\pi/\sqrt{5}}) = e^{-\pi/(3\sqrt{5})}5^{1/4}f(-e^{-2\pi\sqrt{5}}). \tag{15.6.9}$$

Hence, using (15.6.9) in (15.6.7), we find that

$$\lambda(e^{-2\pi/\sqrt{5}}) = 5^{-3/2}. \tag{15.6.10}$$

Thus, from (15.6.5),

$$\begin{aligned}
G(e^{-2\pi/\sqrt{5}}) &= 2\sqrt{125 \cdot 5^{-3/2} + 22 + 5^{3/2}} \\
&= 2\sqrt{2}(5^{3/2} + 11)^{1/2} \\
&= 4\left(\frac{\sqrt{5}+1}{2}\right)^{5/2}.
\end{aligned}$$

We now prove (ii). First, by (15.6.2), (15.2.4), and (15.2.5),

$$125\lambda(e^{-2\pi/5}) = 125\frac{\eta^6(i)}{\eta^6(i/5)} = \frac{\eta^6(i)}{\eta^6(5i)} = \frac{1}{\lambda(e^{-2\pi})}.$$

Hence, by (15.6.5) and the calculation above,

$$\begin{aligned}
G(e^{-2\pi/5}) &= 2\sqrt{125\lambda(e^{-2\pi/5}) + 22 + 1/\lambda(e^{-2\pi/5})} \\
&= 2\sqrt{1/\lambda(e^{-2\pi}) + 22 + 125\lambda(e^{-2\pi})} = G(e^{-2\pi}).
\end{aligned}$$

Thus, it suffices to evaluate $G(e^{-2\pi/5})$, and so we need to determine

$$\lambda(e^{-2\pi/5}) = e^{-2\pi/5}\frac{f^6(-e^{-2\pi})}{f^6(-e^{-2\pi/5})}. \tag{15.6.11}$$

To evaluate $\lambda(e^{-2\pi/5})$, we will use [73, Theorem 2.3(i)], which is proved again in Theorem 2.3.2 of Chapter 2 of this volume. Let G_n denote the Ramanujan–Weber class invariant. Set

$$V' = \frac{G_{25n}}{G_n} \tag{15.6.12}$$

and

$$A' = e^{2\pi\sqrt{n}/6}\frac{f(-e^{-2\pi\sqrt{n}})}{f(-e^{-10\pi\sqrt{n}})}. \tag{15.6.13}$$

Then

$$\frac{A'^2}{\sqrt{5}V'} - \frac{\sqrt{5}V'}{A'^2} = \frac{1}{\sqrt{5}}(V'^3 - V'^{-3}). \tag{15.6.14}$$

Set $n = 1/25$. Then, by (15.6.12) and the relation $G_n = G_{1/n}$,

$$V' = \frac{G_1}{G_{1/25}} = \frac{1}{G_{25}} = \frac{\sqrt{5}-1}{2}. \tag{15.6.15}$$

(See, e.g., [63, p. 190] for the value of G_{25}.) Set $\epsilon = (\sqrt{5}+1)/2$. Then from (15.6.14) and (15.6.15),

$$\frac{\epsilon A'^2}{\sqrt{5}} - \frac{\sqrt{5}}{\epsilon A'^2} = -\frac{4}{\sqrt{5}},$$

from which we easily find that $A'^2 = \epsilon^{-1}$. Hence, from (15.6.13) and (15.6.11), $\lambda(e^{-2\pi/5}) = A'^{-6} = \epsilon^3$. Lastly, we then conclude from (15.6.5) that

$$\begin{aligned} G(e^{-2\pi/5}) &= 2\sqrt{125\epsilon^3 + 22 + \epsilon^{-3}} \\ &= 2\sqrt{270 + 126\sqrt{5}} \\ &= 6 \cdot 5^{1/4}\sqrt{14 + 6\sqrt{5}} \\ &= 6 \cdot 5^{1/4}(3 + \sqrt{5}), \end{aligned}$$

which completes the proof of (ii). □

Ramanujan claimed that

$$\begin{aligned} C = 5^{3/4} \Bigg(-\pi + 4\int_0^{\pi/2} &\sqrt{1 - \epsilon^{-5}5^{-3/2}\sin^2\varphi}\,d\varphi \\ &-2\int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - \epsilon^{-5}5^{-3/2}\sin^2\varphi}} \Bigg), \end{aligned} \tag{15.6.16}$$

which is quite different from (15.6.6). Now in Entry 15.4.1, let q tend to 1. Then u tends to ϵ^{-1}, and so $\cos^{-1}((\epsilon u)^{5/2})$ tends to $\cos^{-1} 1 = 0$. Thus, (15.4.1) yields

$$5^{3/4}\int_0^1 \frac{f^2(-t)f^2(-t^5)}{\sqrt{t}}dt = 2\int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - \epsilon^{-5}5^{-3/2}\sin^2\varphi}}. \tag{15.6.17}$$

Thus, one of the integrals in (15.6.16) can be identified as an integral of eta functions. But this is the only progress we have made in identifying (15.6.16) with (15.6.6).

S. Raghavan [212] has communicated to us the following curious observation. Return to the integral

$$-\int_{q_0}^{1} \frac{f^8(-t)}{f^4(-t^5)} \frac{dt}{t^{3/2}} \tag{15.6.18}$$

from (15.6.6). Suppose that we *illegally* factor out $1/\lambda(t)$ from the integrand of (15.6.18), set $t = e^{-2\pi/\sqrt{5}}$, and then use the value (15.6.10). Then the integral (15.6.18) becomes

$$-\frac{1}{\lambda(e^{-2\pi/\sqrt{5}})} \int_{q_0}^{1} f^2(-t)f^2(-t^5)\frac{dt}{\sqrt{t}} = -5^{3/2} \int_{q_0}^{1} f^2(-t)f^2(-t^5)\frac{dt}{\sqrt{t}}. \tag{15.6.19}$$

Letting $q_0 \to 0$ in (15.6.19) and using (15.6.17), we obtain

$$-2 \cdot 5^{3/4} \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - \epsilon^{-5}5^{-3/2}\sin^2\varphi}},$$

which again is precisely one of the terms in (15.6.16).

Numerically, (15.6.6) and (15.6.16) do not agree. First, by (15.6.16),

$$C = 5^{3/4}(-\pi + 4 \cdot 1.56762\ldots - 2 \cdot 1.57398\ldots) = -0.06370\ldots. \tag{15.6.20}$$

To calculate C via (15.6.6), we set $q_0 = e^{-2\pi}$ and use Entry 15.6.2. Accordingly,

$$C = 6 \cdot 5^{1/4}(3 + \sqrt{5}) - 250\pi \int_1^{\infty} \frac{\eta^8(5ix)}{\eta^4(ix)} dx - 2\pi \int_0^1 \frac{\eta^8(ix)}{\eta^4(5ix)} dx. \tag{15.6.21}$$

We used *Mathematica* to calculate the integrals in (15.6.21) and found that

$$\begin{aligned} C &= 46.978487\ldots - 250\pi \cdot 8.60104\ldots \times 10^{-6} - 2\pi \cdot 5.81407\ldots \\ &= 10.44085\ldots. \end{aligned} \tag{15.6.22}$$

Thus, (15.6.20) and (15.6.22) show that Ramanujan's claim (15.6.16) is erroneous. Nonetheless, we are haunted by the possibility that a corrected version of (15.6.16) exists, for Ramanujan very rarely made a serious error.

Below Entry 15.6.1 and to the right of Entry 15.6.2, Ramanujan wrote "corresponding integral for $t + 1/t$." (For consistency, we have used u instead of t here.) Raghavan [212] has kindly worked out for us the theorem implied by Ramanujan's brief remark, and we give his proof here.

Entry 15.6.3 (p. 52). *Recall that Ramanujan's continued fraction $u(q)$ is defined by* (15.2.11). *Define*

$$\mu := \mu(q) := \frac{q^{1/5} f(-q^5)}{f(-q^{1/5})}. \tag{15.6.23}$$

Then there exists a constant C such that for $0 < q < 1$,

$$u+u^{-1}=\frac{1}{10\sqrt{\mu}}\left(C+5\int_0^q \mu^{3/2}(t)\frac{f^5(-t)}{tf(-t^5)}dt+\int_q^1 \mu^{-1/2}(t)\frac{f^5(-t)}{tf(-t^5)}dt\right), \tag{15.6.24}$$

where the constant C *is given by* (15.6.32) *below.*

Proof. Define

$$H(q):=10\sqrt{\mu}\left(u+\frac{1}{u}\right), \tag{15.6.25}$$

where μ is defined by (15.6.23). With the use of (15.2.12), it is a straightforward task to prove that

$$H(q)=10\sqrt{5\mu+2+1/\mu}. \tag{15.6.26}$$

It follows that

$$\frac{dH}{dq}=\frac{5(5\mu^2-1)}{\mu^2\sqrt{5\mu+2+1/\mu}}\frac{d\mu}{dq}. \tag{15.6.27}$$

From Entry 3.2.4 in Chapter 3, by a direct calculation, we find that

$$\frac{du}{dq}=\frac{u}{5q}\frac{f^5(-q)}{f(-q^5)}. \tag{15.6.28}$$

Thus, by (15.2.12) and (15.6.28),

$$-\frac{1}{\mu^2}\frac{d\mu}{dq}=-\left(\frac{1}{u^2}+1\right)\frac{du}{dq}=-\frac{1+u^2}{5qu}\frac{f^5(-q)}{f(-q^5)}. \tag{15.6.29}$$

Using (15.6.29) and the formula $\sqrt{5\mu+2+1/\mu}=\sqrt{\mu}(u+1/u)$, from (15.6.25) and (15.6.26), in (15.6.27), we deduce that

$$\frac{dH}{dq}=\frac{5\mu^2-1}{\sqrt{\mu}}\frac{f^5(-q)}{qf(-q^5)}. \tag{15.6.30}$$

It follows that for $0<q_0,\ q<1$,

$$\begin{aligned}H(q)-H(q_0)&=5\int_{q_0}^q \mu^{3/2}(t)\frac{f^5(-t)}{tf(-t^5)}dt-\int_{q_0}^q \mu^{-1/2}(t)\frac{f^5(-t)}{tf(-t^5)}dt\\&=C+5\int_0^q \mu^{3/2}(t)\frac{f^5(-t)}{tf(-t^5)}dt+\int_q^1 \mu^{-1/2}(t)\frac{f^5(-t)}{tf(-t^5)}dt,\end{aligned} \tag{15.6.31}$$

where C is given by

$$C=H(q_0)-5\int_0^{q_0} \mu^{3/2}(t)\frac{f^5(-t)}{tf(-t^5)}dt-\int_{q_0}^1 \mu^{-1/2}(t)\frac{f^5(-t)}{tf(-t^5)}dt. \tag{15.6.32}$$

With the use of (15.6.25) in (15.6.31), the proof of Entry 15.6.3 is complete. □

15.7 Elliptic Integrals of Order 15

The entries in this and the following two sections depend on remarkable differential equations satisfied by certain quotients of eta functions. For the first series of results, that quotient is defined by

$$v := v(q) := q\left(\frac{f(-q)f(-q^{15})}{f(-q^3)f(-q^5)}\right)^3. \tag{15.7.1}$$

We need three ancillary lemmas. The first and third are found in Ramanujan's notebooks [227].

Lemma 15.7.1. *Let v be defined by* (15.7.1), *and let*

$$R = \frac{1}{q}\left(\frac{f(-q)f(-q^5)}{f(-q^3)f(-q^{15})}\right)^2.$$

Then

$$R + 5 + \frac{9}{R} = \frac{1}{v} - v.$$

For a proof of Lemma 15.7.1, see Berndt's book [62, p. 221, Entry 62].

Lemma 15.7.2. *Let R be given above, and let*

$$P = \frac{1}{q}\left(\frac{f(-q)}{f(-q^5)}\right)^6 \qquad \textit{and} \qquad Q = \frac{1}{q^3}\left(\frac{f(-q^3)}{f(-q^{15})}\right)^6.$$

Then

$$P + \frac{125}{P} = R - 4 + \frac{135}{R} + \frac{486}{R^2} + \frac{729}{R^3}$$

and

$$Q + \frac{125}{Q} = R^3 + 6R^2 + 15R - 4 + \frac{9}{R}.$$

Proof. From Berndt's book [62, p. 223, Entry 63; p. 226, Entry 64], we have, respectively,

$$\sqrt{PQ} + \frac{125}{\sqrt{PQ}} = \sqrt{K^2+4}(K-9) \tag{15.7.2}$$

and

$$\sqrt{PQ} - \frac{125}{\sqrt{PQ}} = (K-4)\sqrt{(K-11)(K+1)}, \tag{15.7.3}$$

where

$$K = \frac{1}{v} - v, \tag{15.7.4}$$

where v is given by (15.7.1). (The forms of Entries 63 and 64 in [62] are slightly different from those in (15.7.2) and (15.7.3), respectively, but their equivalences are easily demonstrated by elementary algebra.) Multiplying (15.7.2) by $(\frac{1}{v}+v)$ and (15.7.3) by K in (15.7.4), we deduce that

$$P+\frac{125}{P}+Q+\frac{125}{Q}=(K^2+4)(K-9) \tag{15.7.5}$$

and

$$-P-\frac{125}{P}+Q+\frac{125}{Q}=K(K-4)\sqrt{(K-11)(K+1)}. \tag{15.7.6}$$

From Lemma 15.7.1, we know that

$$K=R+5+\frac{9}{R}. \tag{15.7.7}$$

Hence, from (15.7.5), (15.7.6), and (15.7.7), we deduce that

$$P+\frac{125}{P}+Q+\frac{125}{Q}=R^3+6R^2+16R-8+\frac{144}{R}+\frac{486}{R^2}+\frac{729}{R^3} \tag{15.7.8}$$

and

$$-P-\frac{125}{P}+Q+\frac{125}{Q}=R^3+6R^2+14R-\frac{126}{R}-\frac{486}{R^2}-\frac{729}{R^3}. \tag{15.7.9}$$

Solving (15.7.8) and (15.7.9) yields Lemma 15.7.2. □

Lemma 15.7.3. *We have*

$$1+6\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}-30\sum_{k=1}^{\infty}\frac{kq^{5k}}{1-q^{5k}}$$
$$=\sqrt{\frac{f^{12}(-q)+22qf^6(-q)f^6(-q^5)+125q^2f^{12}(-q^5)}{f^2(-q)f^2(-q^5)}}.$$

For a proof of Lemma 15.7.3, see Berndt's book [61, p. 463, Entry 4].

Lemma 15.7.4. *Let v be defined by* (15.7.1). *Then*

$$\frac{dv}{dq}=f(-q)f(-q^3)f(-q^5)f(-q^{15})\sqrt{1-10v-13v^2+10v^3+v^4}.$$

Proof. From the definition (15.7.1) of v, we find that

$$\begin{aligned}\frac{1}{v}\frac{dv}{dq}=\frac{d\log v}{dq}&=\frac{d\log\left\{q^{3/2}\frac{f^3(-q^{15})}{f^3(-q^3)}\right\}}{dq}+\frac{d\log\left\{q^{-1/2}\frac{f^3(-q)}{f^3(-q^5)}\right\}}{dq} \\ &=\frac{3}{2q}+9\sum_{n=1}^{\infty}\frac{nq^{3n-1}}{1-q^{3n}}-45\sum_{n=1}^{\infty}\frac{nq^{15n-1}}{1-q^{15n}} \\ &\quad-\frac{1}{2q}+15\sum_{n=1}^{\infty}\frac{nq^{5n-1}}{1-q^{5n}}-3\sum_{n=1}^{\infty}\frac{nq^{n-1}}{1-q^n}\end{aligned} \tag{15.7.10}$$

$$= \frac{3}{2q}\sqrt{\frac{f^{12}(-q^3)+22q^3f^6(-q^3)f^6(-q^{15})+125q^6f^{12}(-q^{15})}{f^2(-q^3)f^2(-q^{15})}}$$
$$-\frac{1}{2q}\sqrt{\frac{f^{12}(-q)+22qf^6(-q)f^6(-q^5)+125q^2f^{12}(-q^5)}{f^2(-q)f^2(-q^5)}},$$

by Lemma 15.7.3. Simplifying (15.7.10) by using the definitions of P, Q, and R from Lemmas 15.7.1 and 15.7.2, as well as Lemmas 15.7.2 and 15.7.1 themselves, we find that

$$\begin{aligned}
\frac{dv}{dq} &= vf(-q)f(-q^3)f(-q^5)f(-q^{15})\left(\frac{3}{2}q^{1/2}\frac{f(-q^3)f(-q^{15})}{f(-q)f(-q^5)}\right.\\
&\quad\times\left.\sqrt{Q+22+\frac{125}{Q}}-\frac{1}{2}q^{-1/2}\frac{f(-q)f(-q^5)}{f(-q^{15})f(-q^3)}\sqrt{P+22+\frac{125}{P}}\right)\\
&= vf(-q)f(-q^3)f(-q^5)f(-q^{15})\left(\frac{3}{2}\frac{1}{\sqrt{R}}\sqrt{Q+22+\frac{125}{Q}}\right.\\
&\quad\left.-\frac{1}{2}\sqrt{R}\sqrt{P+22+\frac{125}{P}}\right)\\
&= vf(-q)f(-q^3)f(-q^5)f(-q^{15})\left(\frac{3}{2}\frac{1}{\sqrt{R}}\sqrt{R^3+6R^2+15R+18+\frac{9}{R}}\right.\\
&\quad\left.-\frac{1}{2}\sqrt{R}\sqrt{R+18+\frac{135}{R}+\frac{486}{R^2}+\frac{729}{R^3}}\right)\\
&= vf(-q)f(-q^3)f(-q^5)f(-q^{15})\left(\frac{3}{2}\sqrt{\left(R+3+\frac{3}{R}\right)^2}\right.\\
&\quad\left.-\frac{1}{2}\sqrt{\left(R+9+\frac{27}{R}\right)^2}\right)\\
&= vf(-q)f(-q^3)f(-q^5)f(-q^{15})\left(\sqrt{R}-\frac{3}{\sqrt{R}}\right)\left(\sqrt{R}+\frac{3}{\sqrt{R}}\right)\\
&= vf(-q)f(-q^3)f(-q^5)f(-q^{15})\sqrt{\frac{1}{v}-v-11}\sqrt{\frac{1}{v}-v+1}\\
&= f(-q)f(-q^3)f(-q^5)f(-q^{15})\sqrt{1-10v-13v^2+10v^3+v^4}.
\end{aligned}$$

This completes the proof. □

Entry 15.7.1 (p. 51). *Let v be defined by* (15.7.1), *and let $\epsilon = (\sqrt{5}+1)/2$. Then*

$$\int_0^q f(-t)f(-t^3)f(-t^5)f(-t^{15})dt$$

$$= \frac{1}{5} \int_{2 \tan^{-1}\left(\frac{1}{\sqrt{5}} \sqrt{\frac{1-11v-v^2}{1+v-v^2}}\right)}^{2 \tan^{-1}(1/\sqrt{5})} \frac{d\varphi}{\sqrt{1 - \frac{9}{25} \sin^2 \varphi}} \tag{15.7.11}$$

$$= \frac{1}{9} \int_{2 \tan^{-1}\left(\frac{1-v\epsilon^{-3}}{1+v\epsilon^3} \sqrt{\frac{(1+v\epsilon)(1-v\epsilon^5)}{(1-v\epsilon^{-1})(1+v\epsilon^{-5})}}\right)}^{\pi/2} \frac{d\varphi}{\sqrt{1 - \frac{1}{81} \sin^2 \varphi}} \tag{15.7.12}$$

$$= \frac{1}{4} \int_{\tan^{-1}\left((3-\sqrt{5}) \sqrt{\frac{(1-v\epsilon^{-1})(1-v\epsilon^5)}{(1+v\epsilon)(1+v\epsilon^{-5})}}\right)}^{\tan^{-1}(3-\sqrt{5})} \frac{d\varphi}{\sqrt{1 - \frac{15}{16} \sin^2 \varphi}}. \tag{15.7.13}$$

Proof of (15.7.11). Let

$$\tan(\varphi/2) = \sqrt{\frac{1 - 11v(t) - v^2(t)}{5(1 + v(t) - v^2(t))}}. \tag{15.7.14}$$

Clearly,

$$\varphi(0) = 2 \tan^{-1}\left(\frac{1}{\sqrt{5}}\right) \qquad \text{and} \qquad \varphi(q) = 2 \tan^{-1} \sqrt{\frac{1 - 11v(q) - v^2(q)}{5(1 + v(q) - v^2(q))}}. \tag{15.7.15}$$

Differentiating both sides of (15.7.14) with respect to t, we find, after a modest calculation, that

$$\tan(\varphi/2) \sec^2(\varphi/2) \frac{d\varphi}{dt} = -\frac{12(1 + v^2(t))}{5(1 + v(t) - v^2(t))^2} \frac{dv}{dt}. \tag{15.7.16}$$

From (15.7.14) and elementary trigonometry, with the argument t deleted for brevity,

$$\tan(\varphi/2) \sec^2(\varphi/2) = \frac{6(1 - v - v^2)\sqrt{1 - 11v - v^2}}{(5(1 + v - v^2))^{3/2}}. \tag{15.7.17}$$

From (15.7.16) and (15.7.17), it follows that

$$\begin{aligned} \frac{d\varphi/dt}{dv/dt} &= -\frac{2\sqrt{5}(1 + v^2)}{(1 - v - v^2)\sqrt{(1 - 11v - v^2)(1 + v - v^2)}} \\ &= -\frac{2\sqrt{5}(1 + v^2)}{(1 - v - v^2)\sqrt{1 - 10v - 13v^2 + 10v^3 + v^4}}. \end{aligned} \tag{15.7.18}$$

From further elementary trigonometry,

$$\sin^2 \varphi = 4 \sin^2(\varphi/2) \cos^2(\varphi/2) = \frac{5(1 - 11v - v^2)(1 + v - v^2)}{9(1 - v - v^2)^2},$$

and so

$$1-\frac{9}{25}\sin^2\varphi=\frac{4(1+v^2)^2}{5(1-v-v^2)^2}. \tag{15.7.19}$$

From (15.7.18) and (15.7.19), we deduce that

$$\frac{d\varphi/dt}{dv/dt}=-\frac{5\sqrt{1-\frac{9}{25}\sin^2\varphi}}{\sqrt{1-10v-13v^2+10v^3+v^4}}. \tag{15.7.20}$$

Using (15.7.15) and (15.7.20), we find that with $v=v(q)$,

$$\begin{aligned}\int_0^q & f(-t)f(-t^3)f(-t^5)f(-t^{15})dt \\ &= \frac{1}{5}\int_{2\tan^{-1}\left(\frac{1}{\sqrt{5}}\sqrt{\frac{1-11v-v^2}{1+v-v^2}}\right)}^{2\tan^{-1}(1/\sqrt{5})} f(-t)f(-t^3)f(-t^5)f(-t^{15}) \\ &\quad\times\frac{\sqrt{1-10v-13v^2+10v^3+v^4}}{\dfrac{dv}{dt}\sqrt{1-\frac{9}{25}\sin^2\varphi}}d\varphi.\end{aligned}$$

Invoking Lemma 15.7.4, we complete the proof of (15.7.11). □

Lemma 15.7.5. (*First version of Landen's transformation*) *If* $0\le\alpha,\beta\le\pi/2$, $0<x<1$, *and* $\sin(2\beta-\alpha)=x\sin\alpha$, *then*

$$\int_0^\alpha\frac{d\varphi}{\sqrt{1-x^2\sin^2\varphi}}=\frac{2}{1+x}\int_0^\beta\frac{d\varphi}{\sqrt{1-\dfrac{4x}{(1+x)^2}\sin^2\varphi}}.$$

Lemma 15.7.5 can be found as Entry 7(xiii) in Chapter 17 in Ramanujan's second notebook [61, p. 113]. If we replace x by $(1-\sqrt{1-x^2})/(1+\sqrt{1-x^2})$ in Lemma 15.7.5 and interchange the roles of α and β, we obtain the following second version of Landen's transformation.

Lemma 15.7.6. (*Second version of Landen's transformation*) *If* $0\le\alpha,\beta\le\pi/2$, $0<x<1$, *and* $\tan(\beta-\alpha)=\sqrt{1-x^2}\tan\alpha$, *then*

$$\int_0^\alpha\frac{d\varphi}{\sqrt{1-x^2\sin^2\varphi}}=\frac{1}{1+\sqrt{1-x^2}}\int_0^\beta\frac{d\varphi}{\sqrt{1-\left(\dfrac{1-\sqrt{1-x^2}}{1+\sqrt{1-x^2}}\right)^2\sin^2\varphi}}.$$

Proof of (15.7.12). We apply Lemma 15.7.6 with $x=\frac{3}{5}$. Then $\tan(\beta-\alpha)=\frac{4}{5}\tan\alpha$. Suppose that

$$\alpha=2\tan^{-1}\sqrt{\frac{1-11v(q)-v^2(q)}{5(1+v(q)-v^2(q))}}. \tag{15.7.21}$$

If $q=0$, then $\alpha = 2\tan^{-1}(1/\sqrt{5})$. In comparing (15.7.11) and (15.7.12), we must prove that, with the agrument q deleted for brevity,

$$\tan(\beta/2) = \frac{(1-v\epsilon^{-3})}{(1+v\epsilon^{3})}\sqrt{\frac{(1+v\epsilon)(1-v\epsilon^{5})}{(1-v\epsilon^{-1})(1+v\epsilon^{-5})}}, \tag{15.7.22}$$

for if $q=0$, then $\beta=\pi/2$.

Set $t_1=\tan(\alpha/2)$ and $t_2=\tan\left((\beta-\alpha)/2\right)$. Then

$$\frac{2t_2}{1-t_2^2} = \tan(\beta-\alpha) = \frac{4}{5}\tan\alpha = \frac{8t_1}{5(1-t_1^2)}.$$

If we consider the extremal equality as a quadratic equation in t_2, a routine calculation gives

$$t_2 = -\frac{5(1-t_1^2)}{8t_1} + \frac{1}{2}\sqrt{\frac{25(1-t_1^2)^2}{16t_1^2}+4}, \tag{15.7.23}$$

since $t_2>0$. Using (15.7.21) and the definition of t_1, we find that

$$1-t_1^2 = \frac{4(1+4v-v^2)}{5(1+v-v^2)} \tag{15.7.24}$$

and

$$\frac{25(1-t_1^2)^2}{16t_1^2}+4 = \frac{9(1+v^2)^2}{(1+v-v^2)(1-11v-v^2)}, \tag{15.7.25}$$

after a lengthy calculation. Employing (15.7.21), (15.7.24), and (15.7.25) in (15.7.23), we conclude that

$$\begin{aligned} t_2 &= -\frac{\sqrt{5}(1+4v-v^2)+3(1+v^2)}{2\sqrt{(1+v-v^2)(1-11v-v^2)}} \\ &= \frac{\epsilon^2(\epsilon-v)(\epsilon^{-5}-v)}{\sqrt{(1-v\epsilon^{-1})(1+v\epsilon)(1-v\epsilon^{5})(1+v\epsilon^{-5})}} \\ &= \epsilon^{-2}\sqrt{\frac{(1-v\epsilon^{-1})(1-v\epsilon^{5})}{(1+v\epsilon)(1+v\epsilon^{-5})}}. \end{aligned} \tag{15.7.26}$$

Hence, by (15.7.21) and (15.7.26),

$$\begin{aligned} \tan(\beta/2) &= \tan\left(\alpha/2+(\beta-\alpha)/2\right) = \frac{t_1+t_2}{1-t_1t_2} \\ &= \frac{\sqrt{\dfrac{(1-v\epsilon^{5})(1+v\epsilon^{-5})}{5(1-v\epsilon^{-1})(1+v\epsilon)}} + \epsilon^{-2}\sqrt{\dfrac{(1-v\epsilon^{-1})(1-v\epsilon^{5})}{(1+v\epsilon)(1+v\epsilon^{-5})}}}{1-\dfrac{\epsilon^{-2}(1-v\epsilon^{5})}{\sqrt{5}(1+v\epsilon)}} \end{aligned}$$

$$= \sqrt{\frac{(1-v\epsilon^5)(1+v\epsilon)}{(1-v\epsilon^{-1})(1+v\epsilon^{-5})}} \left(\frac{(1+v\epsilon^{-5}) + \epsilon^{-2}\sqrt{5}(1-v\epsilon^{-1})}{\sqrt{5}(1+v\epsilon) - \epsilon^{-2}(1-v\epsilon^5)} \right)$$
$$= \sqrt{\frac{(1-v\epsilon^5)(1+v\epsilon)}{(1-v\epsilon^{-1})(1+v\epsilon^{-5})}} \frac{(\frac{3}{2}\sqrt{5}-\frac{3}{2})(1-v\epsilon^{-3})}{(\frac{3}{2}\sqrt{5}-\frac{3}{2})(1+v\epsilon^3)}.$$

Thus, (15.7.22) has been established, and the proof of (15.7.12) is complete. □

Proof of (15.7.13). We apply Lemma 15.7.5 with $x = \frac{3}{5}$, and let α be given by (15.7.21). Comparing (15.7.11) and (15.7.13), we see that it suffices to prove that, with the argument q deleted for brevity,

$$t := \tan\beta = 2\epsilon^{-2}\sqrt{\frac{(1-v\epsilon^{-1})(1-v\epsilon^5)}{(1+v\epsilon)(1+v\epsilon^{-5})}}, \tag{15.7.27}$$

for if $q = 0$, then $t = 2\epsilon^{-2} = 3 - \sqrt{5}$.

Now the hypothesis $\sin(2\beta - \alpha) = \frac{3}{5}\sin\alpha$ in Lemma 15.7.5, by the addition formula for the sine function and the double angle formulas for both the sine and cosine functions, easily translates to the condition

$$\tan\alpha = \frac{\sin(2\beta)}{\frac{3}{5} + \cos(2\beta)} = \frac{5\tan\beta}{4 - \tan^2\beta}. \tag{15.7.28}$$

Using (15.7.28), (15.7.27), and (15.7.21), we have

$$\begin{aligned} \frac{5t}{4-t^2} = \tan\alpha &= \frac{2\tan(\alpha/2)}{1-\tan^2(\alpha/2)} \\ &= \frac{\sqrt{5(1-11v-v^2)(1+v-v^2)}}{2(1+4v-v^2)}. \end{aligned} \tag{15.7.29}$$

Considering (15.7.29) as a quadratic equation in t, we solve it to deduce that

$$t = \frac{1}{2}\left(-\frac{2\sqrt{5}(1+4v-v^2)}{\sqrt{(1-11v-v^2)(1+v-v^2)}} + \sqrt{\frac{20(1+4v-v^2)^2}{(1-11v-v^2)(1+v-v^2)} + 16} \right),$$

since $t > 0$. Simplifying, we find that

$$\begin{aligned} t &= -\frac{\sqrt{5}(1+4v-v^2)}{\sqrt{(1-11v-v^2)(1+v-v^2)}} + \frac{3(1+v^2)}{\sqrt{(1-11v-v^2)(1+v-v^2)}} \\ &= 2\epsilon^{-2}\sqrt{\frac{(1-v\epsilon^{-1})(1-v\epsilon^5)}{(1+v\epsilon)(1-v\epsilon^{-5})}}, \end{aligned}$$

by identically the same calculation that we used in (15.7.26). Thus, (15.7.27) has been proved, and the proof of (15.7.13) is complete. □

For the remainder of this section, set

$$v = q\left(\frac{f(-q^3)f(-q^{15})}{f(-q)f(-q^5)}\right)^2. \tag{15.7.30}$$

Entry 15.7.2 (p. 53). *If v is defined by* (15.7.30)*, then*

$$\int_0^q f(-t)f(-t^3)f(-t^5)f(-t^{15})dt = \frac{1}{5}\int_{2\tan^{-1}\left(\frac{1-3v}{\sqrt{5}(1+3v)}\right)}^{2\tan^{-1}(1/\sqrt{5})} \frac{d\varphi}{\sqrt{1-\frac{9}{25}\sin^2\varphi}}. \tag{15.7.31}$$

Proof. Because of the conflict in notation between (15.7.1) and (15.7.30), for this proof only, we set

$$u := q\left(\frac{f(-q)f(-q^{15})}{f(-q^3)f(-q^5)}\right)^3. \tag{15.7.32}$$

In the notation (15.7.30) and (15.7.32), Lemma 15.7.1 takes the form

$$\frac{1}{v} + 5 + 9v = \frac{1}{u} - u.$$

By using this equality, we can easily verify that

$$\frac{1-3v}{1+3v} = \sqrt{\frac{1-11u-u^2}{1+u-u^2}}.$$

Thus, (15.7.31) follows immediately from (15.7.11). □

15.8 Elliptic Integrals of Order 14

As in the previous section, the primary theorem in the present section depends on a first-order differential equation satisfied by a certain quotient of eta functions and established through a series of lemmas. Let

$$v := v(q) := q\left(\frac{f(-q)f(-q^{14})}{f(-q^2)f(-q^7)}\right)^4. \tag{15.8.1}$$

Lemma 15.8.1. *If v is defined by* (15.8.1) *and*

$$R = \frac{1}{q}\left(\frac{f(-q)f(-q^7)}{f(-q^2)f(-q^{14})}\right)^3, \tag{15.8.2}$$

then

$$R + 7 + \frac{8}{R} = v + \frac{1}{v}. \tag{15.8.3}$$

Lemma 15.8.1 is a reformulation of Entry 19(ix) in Chapter 19 of Ramanujan's second notebook [61, p. 315]. To see this, replace q by $-q$ in the definitions of (15.8.1) and (15.8.2). Then use Entries 12(i), (iii) in Chapter 17 of the second notebook [61, p. 124] to convert (15.8.3) into a modular equation, which is readily seen to be the same as Entry 19(ix).

Lemma 15.8.2. *Let*

$$P = \frac{1}{q}\left(\frac{f(-q)}{f(-q^7)}\right)^4 \qquad \textit{and} \qquad Q = \frac{1}{q^2}\left(\frac{f(-q^2)}{f(-q^{14})}\right)^4. \tag{15.8.4}$$

Then

$$P + \frac{49}{P} = R - 1 + \frac{48}{R} + \frac{64}{R^2} \tag{15.8.5}$$

and

$$Q + \frac{49}{Q} = R^2 + 6R - 1 + \frac{8}{R}. \tag{15.8.6}$$

Proof. In the notation (15.8.4), Ramanujan discovered the eta-function identity [62, p. 209, Entry 55]

$$\begin{aligned}\sqrt{PQ} + \frac{49}{\sqrt{PQ}} &= v^{3/2} - 8v^{1/2} - 8v^{-1/2} + v^{-3/2}\\ &= \left(\frac{1}{\sqrt{v}} + \sqrt{v}\right)^3 - 11\left(\frac{1}{\sqrt{v}} + \sqrt{v}\right)\\ &= K(K^2 - 11),\end{aligned} \tag{15.8.7}$$

where

$$K = \frac{1}{\sqrt{v}} + \sqrt{v}. \tag{15.8.8}$$

Letting $c = K(K^2 - 11)$ and solving (15.8.7) for $\sqrt{PQ}$, we find that

$$\sqrt{PQ} = \frac{c + \sqrt{c^2 - 196}}{2},$$

where the correct root was found by an examination of $\sqrt{PQ}$ in a neighborhood of $q = 0$. A brief calculation now gives

$$\begin{aligned}\sqrt{PQ} - \frac{49}{\sqrt{PQ}} &= \sqrt{c^2 - 196}\\ &= \sqrt{K^6 - 22K^4 + 121K^2 - 196}\\ &= \sqrt{(K^2 - 4)(K^4 - 18K^2 + 49)}.\end{aligned} \tag{15.8.9}$$

Multiplying (15.8.7) by K (given by (15.8.8)) and (15.8.9) by $1/\sqrt{v} - \sqrt{v} = \sqrt{K^2 - 4}$, and using (15.8.3), we deduce that, respectively,

$$
\begin{aligned}
P+\frac{49}{P}+Q+\frac{49}{Q} =& K^2(K^2-11)\\
=& \left(R+\frac{8}{R}+9\right)\left(R+\frac{8}{R}-2\right)\\
=& \frac{(R+8)(R+1)(R^2-2R+8)}{R^2}
\end{aligned}
\tag{15.8.10}
$$

and

$$
\begin{aligned}
&-P-\frac{49}{P}+Q+\frac{49}{Q}\\
&=(K^2-4)\sqrt{K^4-18K^2+49}\\
&=\left(R+\frac{8}{R}+5\right)\sqrt{\left(R+\frac{8}{R}+9\right)^2-18\left(R+\frac{8}{R}+9\right)+49}\\
&=\frac{(R^2+5R+8)(R^2-8)}{R^2}.
\end{aligned}
\tag{15.8.11}
$$

Solving (15.8.10) and (15.8.11), we deduce (15.8.5) and (15.8.6). ☐

Lemma 15.8.3. *We have*

$$
\begin{aligned}
1+4\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}-28\sum_{k=1}^{\infty}\frac{kq^{7k}}{1-q^{7k}}&\\
=\left\{\frac{f^8(-q)+13qf^4(-q)f^4(-q^7)+49q^2f^8(-q^7)}{f(-q)f(-q^7)}\right\}^{2/3}&.
\end{aligned}
$$

Lemma 15.8.3 is part of Entry 5(i) in Chapter 21 of Ramanujan's second notebook [61, p. 467].

Lemma 15.8.4. *If v is defined by* (15.8.1)*, then*

$$
\frac{dv}{dq}=f(-q)f(-q^2)f(-q^7)f(-q^{14})\sqrt{1-14v+19v^2-14v^3+v^4}.
$$

Proof. From the definition (15.8.1) of v, Lemma 15.8.3, (15.8.4), (15.8.2), and Lemma 15.8.2,

$$
\begin{aligned}
\frac{q}{v}\frac{dv}{dq}&=q\frac{d\log v}{dq}=q\frac{d\log\left\{q^2\frac{f^4(-q^{14})}{f^4(-q^2)}\right\}}{dq}+q\frac{d\log\left\{q^{-1}\frac{f^4(-q)}{f^4(-q^7)}\right\}}{dq}\\
&=q\left\{\frac{2}{q}+8\sum_{n=1}^{\infty}\frac{nq^{2n-1}}{1-q^{2n}}-56\sum_{n=1}^{\infty}\frac{nq^{14n-1}}{1-q^{14n}}\right\}\\
&\quad+q\left\{-\frac{1}{q}+28\sum_{n=1}^{\infty}\frac{nq^{7n-1}}{1-q^{7n}}-4\sum_{n=1}^{\infty}\frac{nq^{n-1}}{1-q^{n}}\right\}
\end{aligned}
$$

$$= 2\left(\frac{f^8(-q^2)+13q^2f^4(-q^2)f^4(-q^{14})+49q^4f^8(-q^{14})}{f(-q^2)f(-q^{14})}\right)^{2/3}$$
$$-\left(\frac{f^8(-q)+13qf^4(-q)f^4(-q^7)+49q^2f^8(-q^7)}{f(-q)f(-q^7)}\right)^{2/3}$$
$$= qf(-q)f(-q^2)f(-q^7)f(-q^{14})\left\{2q^{1/3}\frac{f(-q^2)f(-q^{14})}{f(-q)f(-q^7)}\right.$$
$$\left.\times\left(Q+\frac{49}{Q}+13\right)^{2/3} - q^{-1/3}\frac{f(-q)f(-q^7)}{f(-q^2)f(-q^{14})}\left(P+\frac{49}{P}+13\right)^{2/3}\right\}$$
$$= qf(-q)f(-q^2)f(-q^7)f(-q^{14})\left\{\frac{2}{R^{1/3}}\left(R^2+6R+12+\frac{8}{R}\right)^{2/3}\right.$$
$$\left.-R^{1/3}\left(R+12+\frac{48}{R}+\frac{64}{R^2}\right)^{2/3}\right\}$$
$$= qf(-q)f(-q^2)f(-q^7)f(-q^{14})\left\{\frac{2}{R^{1/3}}\left(\frac{(R+2)^6}{R^2}\right)^{1/3}\right.$$
$$\left.-R^{1/3}\left(\frac{(R+4)^6}{R^4}\right)^{1/3}\right\}$$
$$= qf(-q)f(-q^2)f(-q^7)f(-q^{14})\left(R-\frac{8}{R}\right). \tag{15.8.12}$$

Now, by using Lemma 15.8.1, we can easily verify that

$$\left(R-\frac{8}{R}\right)^2 = \left(R+\frac{8}{R}\right)^2 + 14\left(R+\frac{8}{R}\right) + 49 - 14\left(R+7+\frac{8}{R}\right) + 17$$
$$= \left(R+7+\frac{8}{R}\right)^2 - 14\left(R+7+\frac{8}{R}\right) + 17$$
$$= \left(v+\frac{1}{v}\right)^2 - 14\left(v+\frac{1}{v}\right) + 17$$
$$= \frac{1-14v+19v^2-14v^3+v^4}{v^2}. \tag{15.8.13}$$

Taking the square roots of both sides of (15.8.13) and substituting in (15.8.12), we complete the proof. □

Entry 15.8.1 (p. 51). *If v is defined by* (15.8.1) *and if*

$$c = \frac{\sqrt{13+16\sqrt{2}}}{7},$$

then

$$\int_0^q f(-t)f(-t^2)f(-t^7)f(-t^{14})dt$$
$$= \frac{1}{\sqrt{8\sqrt{2}}} \int_{\cos^{-1}\left(c\frac{1+v}{1-v}\right)}^{\cos^{-1} c} \frac{d\varphi}{\sqrt{1 - \frac{16\sqrt{2}-13}{32\sqrt{2}} \sin^2 \varphi}}. \qquad (15.8.14)$$

Proof. Let

$$\cos \varphi = c \frac{1 + v(t)}{1 - v(t)}, \qquad (15.8.15)$$

so that at $t = 0, q$, we obtain the upper and lower limits, respectively, in the integral on the right side of (15.8.14). Differentiating (15.8.15), we find that

$$-\sin \varphi \frac{d\varphi}{dt} = \frac{2c}{(1 - v(t))^2} \frac{dv}{dt}. \qquad (15.8.16)$$

By elementary trigonometry,

$$\sin \varphi = \frac{\sqrt{(1 - v)^2 - c^2(1 + v)^2}}{1 - v}. \qquad (15.8.17)$$

Putting (15.8.17) in (15.8.16), we arrive at

$$\frac{d\varphi/dt}{dv/dt} = -\frac{2c}{(1 - v)\sqrt{(1 - v)^2 - c^2(1 + v)^2}}. \qquad (15.8.18)$$

Next, by (15.8.17),

$$\begin{aligned} 1 - \frac{16\sqrt{2} - 13}{32\sqrt{2}} \sin^2 \varphi &= \frac{32\sqrt{2}(1 - v)^2 - (16\sqrt{2} - 13)\left\{(1 - v)^2 - c^2(1 + v)^2\right\}}{32\sqrt{2}(1 - v)^2} \\ &= \frac{(16\sqrt{2} + 13)(1 - v)^2 + 7(1 + v)^2}{32\sqrt{2}(1 - v)^2}. \qquad (15.8.19) \end{aligned}$$

Thus, by (15.8.18) and (15.8.19),

$$\begin{aligned} &\frac{dv/dt}{d\varphi/dt}\sqrt{1 - \frac{16\sqrt{2} - 13}{32\sqrt{2}} \sin^2 \varphi} \\ &= -\frac{\sqrt{(1 - v)^2 - c^2(1 + v)^2}}{2c} \frac{\sqrt{(16\sqrt{2} + 13)(1 - v)^2 + 7(1 + v)^2}}{2\sqrt{8\sqrt{2}}} \\ &= -\frac{\sqrt{49(1 - v)^2 - (16\sqrt{2} + 13)(1 + v)^2}\sqrt{(16\sqrt{2} + 13)(1 - v)^2 + 7(1 + v)^2}}{4\sqrt{16\sqrt{2} + 13}\sqrt{8\sqrt{2}}} \\ &= -\frac{\sqrt{1 - 14v + 19v^2 - 14v^3 + v^4}}{\sqrt{8\sqrt{2}}}, \end{aligned}$$

after a calculation via *Mathematica*. Thus,

$$\begin{aligned}
&\int_0^q f(-t)f(-t^2)f(-t^7)f(-t^{14})dt\\
&= \frac{1}{\sqrt{8\sqrt{2}}}\int_{\cos^{-1}\left(c\frac{1+v}{1-v}\right)}^{\cos^{-1}c} f(-t)f(-t^2)f(-t^7)f(-t^{14})\\
&\quad\times \frac{\sqrt{1-14v+19v^2-14v^3+v^4}}{\frac{dv}{dt}\sqrt{1-\frac{16\sqrt{2}-13}{32\sqrt{2}}\sin^2\varphi}}d\varphi\\
&= \frac{1}{\sqrt{8\sqrt{2}}}\int_{\cos^{-1}\left(c\frac{1+v}{1-v}\right)}^{\cos^{-1}c} \frac{d\varphi}{\sqrt{1-\frac{16\sqrt{2}-13}{32\sqrt{2}}\sin^2\varphi}},
\end{aligned}$$

upon the use of Lemma 15.8.4. □

15.9 An Elliptic Integral of Order 35

To avoid square roots, we have modestly reformulated Ramanujan's integral equality (Entry 15.9.1 below). Throughout this section, set

$$v := v(q) := q\frac{f(-q)f(-q^{35})}{f(-q^5)f(-q^7)}. \tag{15.9.1}$$

(Ramanujan defined v by the square of the right side of (15.9.1).) Ramanujan's theorem depends on a differential equation for v, which we prove through a series of lemmas.

Lemma 15.9.1. *Let*

$$R = \frac{f(-q)f(-q^5)}{q^{3/2}f(-q^7)f(-q^{35})}.$$

Then

$$R^2 - 5 + \frac{49}{R^2} = \frac{1}{v^3} - 5\frac{1}{v^2} - 5v^2 - v^3.$$

Lemma 15.9.1 can be found on page 303 of Ramanujan's second notebook [227]; a proof is given in [62, pp. 236–242].

Lemma 15.9.2. *Let*

$$P = \frac{f(-q)}{q^{1/6}f(-q^5)} \quad \textit{and} \quad Q = \frac{f(-q^7)}{q^{7/6}f(-q^{35})}.$$

Then

$$(PQ)^3 + \frac{125}{(PQ)^3} = \left(\frac{1}{v^4} - v^4\right) - 7\left(\frac{1}{v^3} + v^3\right) + 7\left(\frac{1}{v^2} - v^2\right) + 14\left(\frac{1}{v} + v\right).$$

This eta-function identity is not found in Ramanujan's ordinary notebooks [227], but it is recorded in his lost notebook [228, p. 55] and is given in Entry 17.2.5 in Chapter 17 of this book.

Lemma 15.9.3. *We have*

$$1+6\sum_{k=1}^{\infty}\frac{kq^k}{1-q^k}-30\sum_{k=1}^{\infty}\frac{kq^{5k}}{1-q^{5k}}$$
$$=\sqrt{\frac{f^{12}(-q)+22qf^6(-q)f^6(-q^5)+125q^2f^{12}(-q^5)}{f^2(-q)f^2(-q^5)}}.$$

For a proof of this result from Chapter 21 of Ramanujan's second notebook, see Berndt's book [61, p. 463, Entry 4(i)].

Lemma 15.9.4. *If v is defined by* (15.9.1), *then*

$$v\frac{dv}{dq}$$
$$=qf(-q)f(-q^5)f(-q^7)f(-q^{35})\sqrt{(1+v-v^2)(1-5v-9v^3-5v^5-v^6)}.$$

Proof. Set

$$K=\frac{1}{v}-v. \tag{15.9.2}$$

Then Lemma 15.9.2 can be reformulated as

$$(PQ)^3+\frac{125}{(PQ)^3}=(K^3-7K^2+9K+7)\sqrt{K^2+4}. \tag{15.9.3}$$

Considering (15.9.3) as a quadratic equation in $(PQ)^3$, we solve it. Then after a tedious, but elementary, calculation, we find that

$$(PQ)^3-\frac{125}{(PQ)^3}=(K-1)(K-4)\sqrt{(K+1)(K^3-5K^2+3K-19)}. \tag{15.9.4}$$

Now multiply both sides of (15.9.3) by

$$\frac{1}{v^3}+v^3=(K^2+1)\sqrt{K^2+4}$$

and both sides of (15.9.4) by

$$\frac{1}{v^3}-v^3=K(K^2+3),$$

and use the observation $v=P/Q$ to deduce that, respectively,

$$P^6+\frac{125}{P^6}+Q^6+\frac{125}{Q^6}=U_1 \tag{15.9.5}$$

and

$$-P^6 - \frac{125}{P^6} + Q^6 + \frac{125}{Q^6} = U_2, \tag{15.9.6}$$

where

$$U_1 := (K^2+4)(K^2+1)(K^3-7K^2+9K+7) \tag{15.9.7}$$

and

$$U_2 := K(K-1)(K-4)(K^2+3)\sqrt{(K+1)(K^3-5K^2+3K-19)}. \tag{15.9.8}$$

Solving (15.9.5) and (15.9.6), we deduce that

$$P^6 + \frac{125}{P^6} = \frac{1}{2}\left(U_1 - U_2\right) \tag{15.9.9}$$

and

$$Q^6 + \frac{125}{Q^6} = \frac{1}{2}\left(U_1 + U_2\right). \tag{15.9.10}$$

Using the definition of K in (15.9.2), we can rewrite Lemma 15.9.1 in the form

$$R^2 + \frac{49}{R^2} = K^3 - 5K^2 + 3K - 5. \tag{15.9.11}$$

Considering (15.9.11) as a quadratic equation in R^2, we solve it and find that

$$R^2 = \frac{1}{2}\left(V_1 + V_2\right) \qquad \text{and} \qquad \frac{49}{R^2} = \frac{1}{2}\left(V_1 - V_2\right), \tag{15.9.12}$$

where

$$V_1 := K^3 - 5K^2 + 3K - 5 \tag{15.9.13}$$

and

$$V_2 := (K-3)\sqrt{(K+1)(K^3-5K^2+3K-19)}. \tag{15.9.14}$$

Now, by Lemma 15.9.3, we find that

$$\begin{aligned}
\frac{1}{v}\frac{dv}{dq} = \frac{d\log v}{dq} &= \frac{d\log\left\{q^{7/6}\frac{f(-q^{35})}{f(-q^7)}\right\}}{dq} + \frac{d\log\left\{q^{-1/6}\frac{f(-q)}{f(-q^5)}\right\}}{dq} \\
&= \frac{7}{6q} + 7\sum_{n=1}^{\infty}\frac{nq^{7n-1}}{1-q^{7n}} - 35\sum_{n=1}^{\infty}\frac{nq^{35n-1}}{1-q^{35n}} \\
&\quad - \frac{1}{6q} + 5\sum_{n=1}^{\infty}\frac{nq^{5n-1}}{1-q^{5n}} - \sum_{n=1}^{\infty}\frac{nq^{n-1}}{1-q^{n}} \\
&= \frac{7}{6q}\sqrt{\frac{f^{12}(-q^7) + 22q^7 f^6(-q^7)f^6(-q^{35}) + 125q^{14}f^{12}(-q^{35})}{f^2(-q^7)f^2(-q^{35})}} \\
&\quad - \frac{1}{6q}\sqrt{\frac{f^{12}(-q) + 22qf^6(-q)f^6(-q^5) + 125q^2 f^{12}(-q^5)}{f^2(-q)f^2(-q^5)}}
\end{aligned}$$

$$= qf(-q)f(-q^5)f(-q^7)f(-q^{35})$$
$$\times \left(-\frac{7}{6R}\sqrt{Q^6 + \frac{125}{Q^6} + 22} + \frac{R}{6}\sqrt{P^6 + \frac{125}{P^6} + 22} \right), \qquad (15.9.15)$$

where we have used the definitions of P, Q, and R in Lemmas 15.9.2 and 15.9.1.

Squaring both sides of (15.9.15) and simplifying with the use of (15.9.12), (15.9.9), (15.9.10), (15.9.7), (15.9.8), (15.9.13), and (15.9.14), we find that

$$\left(\frac{1}{qf(-q)f(-q^5)f(-q^7)f(-q^{35})} \frac{dv}{dq} \right)^2$$
$$= \frac{v^2}{36} \left\{ \frac{49}{R^2} \left(Q^6 + \frac{125}{Q^6} + 22 \right) + R^2 \left(P^6 + \frac{125}{P^6} + 22 \right) \right.$$
$$\left. - 14\sqrt{\left(Q^6 + \frac{125}{Q^6} + 22 \right) \left(P^6 + \frac{125}{P^6} + 22 \right)} \right\}$$
$$= \frac{v^2}{36} \left\{ \frac{(V_1 - V_2)}{2} \left(\frac{U_1 + U_2}{2} + 22 \right) + \frac{(V_1 + V_2)}{2} \left(\frac{U_1 - U_2}{2} + 22 \right) \right.$$
$$\left. - 14\sqrt{\left(\frac{U_1}{2} + 22 \right)^2 - \frac{U_2^2}{4}} \right\}$$
$$= \frac{v^2}{36} \left\{ \frac{1}{2}(U_1V_1 - U_2V_2) + 22V_1 - 14\sqrt{\left(\frac{U_1}{2} + 22 \right)^2 - \frac{U_2^2}{4}} \right\}$$
$$= v^2(K^4 - 4K^3 - 2K^2 - 16K - 19), \qquad (15.9.16)$$

where the last step involves a considerable amount of algebra. Lastly, by (15.9.2), we substitute $K = 1/v - v$ into (15.9.16). Upon simplification, factorization, and taking the square roots of both sides, we complete the proof. $\square$

Entry 15.9.1 (p. 53). *If* v *is defined by* (15.9.1), *then*

$$\int_0^q t\ f(-t)f(-t^5)f(-t^7)f(-t^{35})dt$$
$$= \int_0^v \frac{t\ dt}{\sqrt{(1 + t - t^2)(1 - 5t - 9t^3 - 5t^5 - t^6)}}.$$

Proof. Let $v(t)$ be defined by (15.9.1). Then the limits $t = 0, q$ are transformed into $0, v = v(q)$, respectively. Thus,

$$\int_0^q t\ f(-t)f(-t^5)f(-t^7)f(-t^{35})dt$$
$$= \int_0^{v(q)} \frac{t\ f(-t)f(-t^5)f(-t^7)f(-t^{35})}{dv/dt} dv$$

$$= \int_0^{v(q)} \frac{v\, dv}{\sqrt{(1+v-v^2)(1-5v-9v^3-5v^5-v^6)}},$$

upon the employment of Lemma 15.9.4. Thus, the proof is complete. □

15.10 Constructions of New Incomplete Elliptic Integral Identities

It is clear from the previous sections that some of Ramanujan's incomplete elliptic integrals arise from differential equations satisfied by quotients of eta functions. Berndt, Chan, and Huang [70] derived further differential equations and established several new results in the spirit of this chapter. In closing, we give an example and then briefly describe why such differential equations exist and how further integral identities can be found.

Theorem 15.10.1. *Define*

$$v := q\frac{f^6(-q^2)f^6(-q^6)}{f^6(-q)f^6(-q^3)}. \tag{15.10.1}$$

Then

$$\frac{dv}{dq} = q^{-1/2}f(-q)f(-q^2)f(-q^3)f(-q^6)\sqrt{v(4v+1)(16v+1)}.$$

We now offer an identity associated with an incomplete elliptic integral of order 6, which is derived from Theorem 15.10.1.

Theorem 15.10.2. *If v is defined by* (15.10.1), *then*

$$\begin{aligned}\int_0^q \frac{1}{\sqrt{t}}f(-t)f(-t^2)f(-t^3)f(-t^6)\,dt &= \frac{1}{2}\int_{\sin^{-1}\sqrt{\frac{1}{4v+1}}}^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1-\frac{3}{4}\sin^2\varphi}}\\ &= \frac{1}{2}\int_0^{\cot^{-1}\frac{1}{4\sqrt{v}}} \frac{d\varphi}{\sqrt{1-\frac{3}{4}\sin^2\varphi}}.\end{aligned} \tag{15.10.2}$$

The key to the proof of Theorem 15.10.2 is the substitution

$$\sin^2\varphi = \frac{1}{4v+1}. \tag{15.10.3}$$

We briefly describe here how we arrive at this substitution. Consider the equation

$$y^2 = x(4x+1)(16x+1)$$

for an elliptic curve E. By substituting $x_1 = 4x + 1$, we may rewrite the equation in the form

$$y^2 = x_1(x_1 - 1)(x_1 - \tfrac{3}{4}).$$

Next, by setting $y_2 = yx_2^3$ and $x_2^2 = 1/x_1$ [208, pp. 42–43], we obtain the Legendre form of the elliptic curve E, namely,

$$y_1^2 = (1 - x_2^2)(1 - \tfrac{3}{4}x_2^2).$$

Our substitution (15.10.3) is obtained by letting

$$\sin^2 \varphi = x_2^2 = \frac{1}{x_1} = \frac{1}{4x + 1}.$$

We now describe how one can construct differential equations analogous to that of Theorem 15.10.1. The quotients of eta products that appear in Ramanujan's integrals happen to be Hauptmoduls associated with discrete groups of genus zero of the form $\Gamma_0(N) + W_p$, where $p|N$ and W_p is an Atkin-Lehner involution of $\Gamma_0(N)$ (see [116] for more details). Suppose v is the Hauptmodul associated with a discrete group Γ of genus zero. Then the derivative of v with respect to q is a modular form of weight 2 under Γ. To construct a differential equation associated with v, we search for another modular form of weight 2 under Γ for which the quotient $w^{-1}\frac{dv}{dq}$ is invariant under Γ. Since every modular function invariant under Γ can be expressed as a rational function of v, we can easily determine the relation between the two modular forms.

16

Infinite Integrals of q-Products

16.1 Introduction

On page 201 in his lost notebook [228], Ramanujan records five integral evaluations that are related to the normal integral and that involve q-products, although this is not immediate, since Ramanujan set $q = e^{-2k^2}$, and he designated all products by just recording the first couple of terms. R. Askey [47] proved the last two of the formulas. Here we prove all five claims. All are dependent on the q-binomial theorem [61, p. 14, Entry 2]

$$\sum_{n=0}^{\infty} \frac{(a;q)_n}{(q;q)_n} x^n = \frac{(ax;q)_\infty}{(x;q)_\infty}, \qquad |q|, |x| < 1, \tag{16.1.1}$$

a limiting case

$$\sum_{n=0}^{\infty} \frac{(-1)^n q^{n(n-1)/2} x^n}{(q;q)_n} = (x;q)_\infty, \tag{16.1.2}$$

or other special cases of (16.1.1).

The five integral formulas of Ramanujan on page 201 are given next. We have taken the liberty of moderately altering Ramanujan's notation.

Entry 16.1.1 (p. 201). *If $|a| < 1$ and m and k are real numbers, then*

$$\int_{-\infty}^{\infty} e^{-x^2+2mx}(-aqe^{2kx};q)_\infty dx = \sqrt{\pi} \sum_{n=0}^{\infty} e^{(m+nk)^2} \frac{a^n q^{n(n+1)/2}}{(q;q)_n}. \tag{16.1.3}$$

Entry 16.1.2 (p. 201). *If $|a| < 1$ and m and k are real numbers, then*

$$\int_{-\infty}^{\infty} e^{-x^2+2mx}(-ae^{-k^2+2kx};e^{-2k^2})_\infty dx = \frac{\sqrt{\pi}e^{m^2}}{(ae^{2mk};e^{-2k^2})_\infty}. \tag{16.1.4}$$

Entry 16.1.3 (p. 201). *If $|a|<1$ and m and k are real numbers, then*

$$\int_{-\infty}^{\infty} e^{-x^2+2mx} \frac{dx}{(ae^{2ikx};e^{-2k^2})_\infty} = \sqrt{\pi}e^{m^2}(-ae^{-k^2+2imk};e^{-2k^2})_\infty. \quad (16.1.5)$$

Entry 16.1.4 (p. 201). *If $|a|,|b|<1$ and m and k are real numbers, then*

$$\int_{-\infty}^{\infty} e^{-x^2+2mx}(-ae^{-2k^2+2kx};e^{-2k^2})_\infty(-be^{-2k^2-2kx};e^{-2k^2})_\infty dx$$
$$= \sqrt{\pi}e^{m^2} \frac{(abe^{-2k^2};e^{-2k^2})_\infty}{(ae^{-k^2+2mk};e^{-2k^2})_\infty(be^{-k^2-2mk};e^{-2k^2})_\infty}. \quad (16.1.6)$$

Entry 16.1.5 (p. 201). *If $|a|,|b|<1$ and m and k are real numbers, then*

$$\int_{-\infty}^{\infty} e^{-x^2+2mx} \frac{dx}{(ae^{-k^2+2ikx};e^{-2k^2})_\infty(be^{-k^2-2ikx};e^{-2k^2})_\infty}$$
$$= \sqrt{\pi}e^{m^2} \frac{(-ae^{-2k^2+2imk};e^{-2k^2})_\infty(-be^{-2k^2-2imk};e^{-2k^2})_\infty}{(abe^{-2k^2};e^{-2k^2})_\infty}. \quad (16.1.7)$$

In our proofs in the next section we repeatedly use the normal integral evaluation

$$\int_{-\infty}^{\infty} e^{-x^2}dx = \sqrt{\pi}, \quad (16.1.8)$$

usually without comment.

16.2 Proofs

Proof of Entry 16.1.1. Using (16.1.2) and inverting the order of integration and summation by absolute convergence, we find that

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-x^2+2mx}(-aqe^{2kx};q)_\infty dx &= \sum_{n=0}^{\infty} \frac{a^n q^{n(n+1)/2}}{(q;q)_n} \int_{-\infty}^{\infty} e^{-x^2+2mx+2knx}dx \\ &= \sum_{n=0}^{\infty} \frac{a^n q^{n(n+1)/2}}{(q;q)_n} e^{(m+nk)^2} \int_{-\infty}^{\infty} e^{-(x-(m+nk))^2}dx \\ &= \sum_{n=0}^{\infty} \frac{a^n q^{n(n+1)/2}}{(q;q)_n} e^{(m+nk)^2}\sqrt{\pi},\end{aligned}$$

which completes the proof of (16.1.3). □

Proof of Entry 16.1.2. If we set $q = e^{-2k^2}$ and $b = ae^{2km}$ and make the change of variable $x = u + m$, we find that (16.1.4) takes the form

$$\int_{-\infty}^{\infty} e^{-x^2+2mx}(-ae^{-k^2+2kx};e^{-2k^2})_\infty dx$$
$$= e^{m^2}\int_{-\infty}^{\infty} e^{-u^2}(-b\sqrt{q}e^{2ku};q)_\infty du = \frac{e^{m^2}\sqrt{\pi}}{(b;q)_\infty}. \quad (16.2.1)$$

Inserting (16.1.2) and inverting the order of integration and summation by absolute convergence, we find that

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-u^2}(-b\sqrt{q}e^{2ku};q)_\infty du &= \sum_{n=0}^{\infty}\frac{b^n q^{n^2/2}}{(q;q)_n}\int_{-\infty}^{\infty} e^{-u^2+2nku}du\\
&= \sum_{n=0}^{\infty}\frac{b^n q^{n^2/2}}{(q;q)_n}e^{n^2k^2}\int_{-\infty}^{\infty} e^{-(u-nk)^2}du\\
&= \sqrt{\pi}\sum_{n=0}^{\infty}\frac{b^n}{(q;q)_n}\\
&= \frac{\sqrt{\pi}}{(b;q)_\infty},
\end{aligned}$$

by an application of the q-binomial theorem (16.1.1). We see that indeed, (16.2.1) has been established, and so the proof is complete. □

Proof of Entry 16.1.3. If we set $q = e^{-2k^2}$ and $b = ae^{2ikm}$ and make the change of variable $x = u + m$, we find from (16.1.5) that it suffices to show that

$$\int_{-\infty}^{\infty} e^{-u^2}\frac{du}{(be^{2iku};q)_\infty} = \sqrt{\pi}(-b\sqrt{q};q)_\infty. \quad (16.2.2)$$

Employing the q-binomial theorem (16.1.1) and inverting the order of integration and summation, we find that

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-u^2}\frac{du}{(be^{2iku};q)_\infty} &= \sum_{n=0}^{\infty}\frac{b^n}{(q;q)_n}\int_{-\infty}^{\infty} e^{-u^2+2inku}du\\
&= \sum_{n=0}^{\infty}\frac{b^n}{(q;q)_n}e^{-n^2k^2}\int_{-\infty}^{\infty} e^{-(u-ink)^2}du\\
&= \sqrt{\pi}\sum_{n=0}^{\infty}\frac{b^n q^{n^2/2}}{(q;q)_n}\\
&= \sqrt{\pi}(-b\sqrt{q};q)_\infty, \qquad (16.2.3)
\end{aligned}$$

by the q-binomial theorem (16.1.2). The last integral appearing in (16.2.3) can be evaluated by integrating e^{-u^2} around the positively oriented rectangle with horizontal sides $[-N, N]$ and $[-N - ink, N - ink]$, and vertical sides $[-N-ink, -N]$ and $[N-ink, N]$, applying Cauchy's theorem, letting $N \to \infty$, and lastly applying the normal integral evaluation (16.1.8). □

Proof of Entry 16.1.4. If we set $q = e^{-2k^2}$, $c = ae^{2mk}$, and $d = be^{-2mk}$, and make the change of variable $x = u + m$, then we find from (16.1.6) that it suffices to show that

$$I(c,d) := \int_{-\infty}^{\infty} e^{-u^2}(-ce^{2ku}q;q)_\infty(-de^{-2ku}q;q)_\infty du = \frac{\sqrt{\pi}(cdq;q)_\infty}{(c\sqrt{q};q)_\infty(d\sqrt{q};q)_\infty}. \tag{16.2.4}$$

Applying the q-binomial theorem (16.1.2) twice, inverting the order of summation and integration, and applying the q-binomial theorem (16.1.1) with $a = 0$, and then lastly invoking the q-binomial theorem (16.1.1) for a fourth time, we find that

$$\begin{aligned}
I(c,d) &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{c^n q^{n(n+1)/2}}{(q;q)_n}\frac{d^m q^{m(m+1)/2}}{(q;q)_m}\int_{-\infty}^{\infty} e^{-u^2+2k(n-m)u}du \\
&= \sqrt{\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{c^n d^m q^{n(n+1)/2+m(m+1)/2}}{(q;q)_n(q;q)_m}e^{k^2(n-m)^2} \\
&= \sqrt{\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{c^n d^m q^{n/2+m/2+mn}}{(q;q)_n(q;q)_m} \\
&= \sqrt{\pi}\sum_{n=0}^{\infty} \frac{c^n q^{n/2}}{(q;q)_n}\frac{1}{(dq^{n+1/2};q)_\infty} \\
&= \frac{\sqrt{\pi}}{(d\sqrt{q};q)_\infty}\sum_{n=0}^{\infty}\frac{(d\sqrt{q};q)_n}{(q;q)_n}(a\sqrt{q})^n \\
&= \frac{\sqrt{\pi}(cdq;q)_\infty}{(c\sqrt{q};q)_\infty(d\sqrt{q};q)_\infty}.
\end{aligned}$$

Hence, the proof of (16.2.4), and so that of Entry 16.1.4 as well, is complete. □

Proof of Entry 16.1.5. If we set $q = e^{-2k^2}$, $c = ae^{2imk}$, and $d = be^{-2imk}$, and make the change of variable $x = u + m$, then we find from (16.1.7) that it suffices to show that

$$J(c,d) := \int_{-\infty}^{\infty} \frac{e^{-u^2}du}{(c\sqrt{q}e^{2iku};q)_\infty(d\sqrt{q}e^{-2iku};q)_\infty} = \frac{\sqrt{\pi}(-cq;q)_\infty(-dq;q)_\infty}{(cdq;q)_\infty}. \tag{16.2.5}$$

Applying the q-binomial theorem (16.1.1) twice, inverting the order of summation and integration, applying the q-binomial theorem (16.1.2), and then lastly invoking the q-binomial theorem (16.1.1) for a fourth time, we find that

$$\begin{aligned}
J(c,d) &= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{c^n q^{n/2} d^m q^{m/2}}{(q;q)_n(q;q)_m}\int_{-\infty}^{\infty} e^{-u^2+2ik(n-m)u}du \\
&= \sqrt{\pi}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{c^n q^{n/2} d^m q^{m/2}}{(q;q)_n(q;q)_m}e^{-k^2(n-m)^2}
\end{aligned}$$

$$
\begin{aligned}
&= \sqrt{\pi} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{c^n d^m q^{n(n+1)/2+m(m+1)/2-mn}}{(q;q)_n (q;q)_m} \\
&= \sqrt{\pi} \sum_{n=0}^{\infty} \frac{c^n q^{n(n+1)/2}}{(q;q)_n} \sum_{m=0}^{\infty} \frac{(dq^{1-n})^m q^{m(m-1)/2}}{(q;q)_m} \\
&= \sqrt{\pi} \sum_{n=0}^{\infty} \frac{c^n q^{n(n+1)/2}}{(q;q)_n} (-dq^{1-n};q)_\infty \\
&= \sqrt{\pi}(-dq;q)_\infty \sum_{n=0}^{\infty} \frac{c^n q^{n(n+1)/2}(-dq^{1-n};q)_n}{(q;q)_n} \\
&= \sqrt{\pi}(-dq;q)_\infty \sum_{n=0}^{\infty} \frac{(-1/d;q)_n}{(q;q)_n} (cdq)^n \\
&= \frac{\sqrt{\pi}(-cq;q)_\infty (-dq;q)_\infty}{(cdq;q)_\infty}.
\end{aligned}
$$

Thus, (16.2.5) has been proved, and the proof of Entry 16.1.5 is complete. □

P.I. Pastro [205] has given different proofs of Entries 16.1.4 and 16.1.5. Moreover, he has found sets of orthogonal polynomials related to these two integrals. His orthogonal polynomials are q-analogues of the Laguerre polynomials.

17

Modular Equations in Ramanujan's Lost Notebook

17.1 Introduction

Ramanujan recorded several hundred modular equations in his three notebooks [227]; no other mathematician has ever discovered nearly so many. Complete proofs for all the modular equations in Ramanujan's three notebooks can be found in Berndt's books [61], [62], [63]. In particular, Chapters 19–21 in Ramanujan's second notebook are almost exclusively devoted to modular equations. Ramanujan used modular equations to evaluate class invariants, certain q-continued fractions including the Rogers–Ramanujan continued fraction, theta functions, and certain other quotients and products of theta functions and eta functions [63].

In his lost notebook, and in a few fragments published with the lost notebook [228], Ramanujan organized some of his modular equations by type, rather than by degree as he did in his second notebook. These lists cover the most important kinds of modular equations. Although many of these modular equations are found in his notebooks [227], some are not. The purpose of this chapter is to provide a list and discussion of all these modular equations and to give proofs for those not found elsewhere in Ramanujan's notebooks.

Each modular equation is equivalent to a certain theta-function identity, but a theta-function identity may not have an equivalent modular equation. Ramanujan's lost notebook contains many new and beautiful theta-function identities (not equivalent to modular equations), which will not be discussed in this chapter. However, many can be found in Chapter 1 of this volume and in our second volume [37].

In the next section we examine the modular equations on page 55 of the lost notebook. These have been called P–Q modular equations [62, p. 204], or eta-function identities, or modular equations of Schläfli type [221]. These are among the most elegant and beautiful modular equations found by Ramanujan, and they have been most useful in the applications mentioned above.

In Section 17.3, we examine a fragment on pages 350–352 of [228] containing six groups of modular equations. These include modular equations associated with the names of A.M. Legendre, H. Schröter, and R. Russell. The last of the six sets contains Ramanujan's beautiful formulas for multipliers.

The brief Section 17.4 is devoted to a fragment found on page 349 of [228].

Before proceeding further, we provide some definitions in preparation for defining a modular equation, as Ramanujan would have understood it.

The complete elliptic integral of the first kind associated with the *modulus* k, $0 < k < 1$, is defined by

$$K := K(k) := \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}}.$$

The complementary modulus k' is defined by $k' := \sqrt{1 - k^2}$; set $K' := K(k')$. If $q = \exp(-\pi K'/K)$, then one of the central theorems in the theory of elliptic functions asserts that [61, p. 101, Entry 6]

$$\varphi^2(q) = \frac{2}{\pi} K(k) = {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; k^2), \tag{17.1.1}$$

where φ denotes the classical theta function defined by

$$\varphi(q) = \sum_{j=-\infty}^{\infty} q^{j^2},$$

${}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; k^2)$ denotes the ordinary hypergeometric function, and where the last equality in (17.1.1) follows from expanding the integrand in a binomial series and integrating termwise. It is (17.1.1) upon which all of Ramanujan's modular equations ultimately rests.

Let K, K', L, and L' denote complete elliptic integrals of the first kind associated with the moduli k, k', ℓ, and $\ell' := \sqrt{1 - \ell^2}$, respectively, where $0 < k, \ell < 1$. Suppose that

$$n \frac{K'}{K} = \frac{L'}{L} \tag{17.1.2}$$

for some positive integer n. A relation between k and ℓ induced by (17.1.2) is called a *modular equation of degree n*. In fact, modular equations are always algebraic equations. After Ramanujan, set $\alpha = k^2$ and $\beta = \ell^2$. In the sequel, we shall frequently say that β has degree n over α. Lastly, the multiplier m is defined by

$$m = \frac{K}{L}.$$

At the end of Section 17.3, we shall state several formulas for multipliers in terms of α and β. These can be regarded as transformations of elliptic integrals, or, by (17.1.1), transformations for hypergeometric functions.

Most of the content of this chapter was originally published in [65].

17.2 Eta-Function Identities

After Ramanujan, define, for $q = \exp(2\pi i z)$,

$$f(-q) := q^{-1/24}\eta(z) := (q;q)_\infty = \sum_{k=-\infty}^{\infty} (-1)^k q^{k(3k-1)/2},$$

where $\eta(z)$ denotes the Dedekind eta function, $|q| < 1$, and the last equality is Euler's pentagonal number theorem.

There are four sets of modular equations on page 55 of [228]. For the first set, Ramanujan puts

$$u = \frac{f(-q)}{q^{1/6} f(-q^5)} \qquad \text{and} \qquad v = \frac{f(-q^n)}{q^{n/6} f(-q^{5n})}. \tag{17.2.1}$$

The first set comprises five identities.

Entry 17.2.1 (p. 55). *For* $n = 2$ *in* (17.2.1), *the functions* u *and* v *satisfy the modular equation of degree* 5

$$uv + \frac{5}{uv} = \left(\frac{u}{v}\right)^3 + \left(\frac{v}{u}\right)^3.$$

Entry 17.2.1 is identical to Entry 53 in Chapter 25 of Part IV [62, p. 206].

Entry 17.2.2 (p. 55). *If* $n = 3$ *in* (17.2.1), *then the functions* u *and* v *satisfy the modular equation of degree* 15

$$(uv)^3 + \left(\frac{5}{uv}\right)^3 = -\left\{\left(\frac{u}{v}\right)^6 - \left(\frac{v}{u}\right)^6\right\} - 9\left\{\left(\frac{u}{v}\right)^3 + \left(\frac{v}{u}\right)^3\right\}.$$

Entry 17.2.2 is the same as Entry 63 of Chapter 25 of Part IV [62, p. 223].

Entry 17.2.3 (p. 55). *If* $n = 4$ *in* (17.2.1), *then the functions* u *and* v *satisfy the modular equation of degree* 5

$$\begin{aligned}(uv)^3 + \left(\frac{5}{uv}\right)^3 = \left(\frac{u}{v}\right)^5 + \left(\frac{v}{u}\right)^5 - 8\left\{\left(\frac{u}{v}\right)^3 + \left(\frac{v}{u}\right)^3\right\}& \\ + 4\left(\frac{u}{v} + \frac{v}{u}\right) + 4\Big/\left(\frac{u}{v} + \frac{v}{u}\right)&. \end{aligned} \tag{17.2.2}$$

Proof. Let β have degree 5 over α, and let m denote the multiplier of degree 5. From Entries 12(ii) and 12(iv) in Chapter 17 of [61, p. 124],

$$u = \sqrt{m}\left(\frac{1-\alpha}{1-\beta}\right)^{1/6}\left(\frac{\alpha}{\beta}\right)^{1/24} \quad \text{and} \quad v = \sqrt{m}\left(\frac{1-\alpha}{1-\beta}\right)^{1/24}\left(\frac{\alpha}{\beta}\right)^{1/6}, \tag{17.2.3}$$

respectively. Recall the definition [61, p. 284, equation (13.3)]

$$\rho = (m^3 - 2m^2 + 5m)^{1/2} \tag{17.2.4}$$

and the representations [61, p. 286, equation (13.12)]

$$\left(\frac{\alpha}{\beta}\right)^{1/4} = \frac{2m+\rho}{m(m-1)} \qquad \text{and} \qquad \left(\frac{1-\alpha}{1-\beta}\right)^{1/4} = \frac{2m-\rho}{m(m-1)}. \tag{17.2.5}$$

Thus, from (17.2.3) and (17.2.5),

$$\frac{u}{v} = \left(\frac{1-\alpha}{1-\beta}\right)^{1/8}\left(\frac{\alpha}{\beta}\right)^{-1/8} = \sqrt{\frac{2m-\rho}{m(m-1)}}\sqrt{\frac{m(m-1)}{2m+\rho}} = \sqrt{\frac{2m-\rho}{2m+\rho}}. \tag{17.2.6}$$

Hence, after a modicum of elementary algebra,

$$\frac{u}{v} + \frac{v}{u} = \frac{4m}{\sqrt{4m^2-\rho^2}}. \tag{17.2.7}$$

Next, by (17.2.3) and (17.2.5),

$$(uv)^3 = m^3\left(\frac{1-\alpha}{1-\beta}\right)^{5/8}\left(\frac{\alpha}{\beta}\right)^{5/8} = \frac{(4m^2-\rho^2)^{5/2}}{m^2(m-1)^5},$$

and so

$$(uv)^3 + \left(\frac{5}{uv}\right)^3 = \frac{(4m^2-\rho^2)^{5/2}}{m^2(m-1)^5} + \frac{125m^2(m-1)^5}{(4m^2-\rho^2)^{5/2}}. \tag{17.2.8}$$

Hence, by (17.2.7), (17.2.8), and (17.2.4),

$$\begin{aligned}
&\left(\frac{u}{v}+\frac{v}{u}\right)\left((uv)^3 + \left(\frac{5}{uv}\right)^3\right) \\
&= 4m\left(\frac{(4m^2-\rho^2)^2}{m^2(m-1)^5} + \frac{125m^2(m-1)^5}{(4m^2-\rho^2)^3}\right) \\
&= 4m\left(\frac{(-m^3+6m^2-5m)^5 + 125m^4(m-1)^{10}}{m^2(m-1)^5(-m^3+6m^2-5m)^3}\right) \\
&= 4m\left(\frac{(m-1)^5(m-5)^5 + 125(m-1)^{10}/m}{(m-1)^8(m-5)^3}\right) \\
&= 4m\left(\frac{(m-5)^5 - 125(m-1)^5/m}{(m-1)^3(m-5)^3}\right).
\end{aligned} \tag{17.2.9}$$

Next, by (17.2.6) and (17.2.4),

$$\begin{aligned}
\left(\frac{u}{v}\right)^3 + \left(\frac{v}{u}\right)^3 &= \left(\frac{2m-\rho}{2m+\rho}\right)^{3/2} + \left(\frac{2m+\rho}{2m-\rho}\right)^{3/2} \\
&= \frac{16m^3+12m\rho^2}{(4m^2-\rho^2)^{3/2}} = \frac{4m(3m^3-2m^2+15m)}{(-m^3+6m^2-5m)^{3/2}}.
\end{aligned} \tag{17.2.10}$$

Hence, combining (17.2.7) and (17.2.10), with the use of (17.2.4), we find that

$$\left(\frac{u}{v}+\frac{v}{u}\right)\left(\left(\frac{u}{v}\right)^3+\left(\frac{v}{u}\right)^3\right)=\frac{16(3m^3-2m^2+15m)}{(-m^2+6m-5)^2}. \tag{17.2.11}$$

Next, by (17.2.6) and (17.2.4),

$$\begin{aligned}\left(\frac{u}{v}\right)^5+\left(\frac{v}{u}\right)^5&=\frac{64m^5+160m^3\rho^2+20m\rho^4}{(4m^2-\rho^2)^{5/2}}\\&=\frac{m^3(20m^4+80m^3+24m^2+400m+500)}{(-m^3+6m^2-5m)^{5/2}}.\end{aligned} \tag{17.2.12}$$

Hence, (17.2.7) and (17.2.12), with the aid of (17.2.4), yield

$$\left(\frac{u}{v}+\frac{v}{u}\right)\left(\left(\frac{u}{v}\right)^5+\left(\frac{v}{u}\right)^5\right)=\frac{4m(20m^4+80m^3+24m^2+400m+500)}{(-m^2+6m-5)^3}. \tag{17.2.13}$$

Hence, multiplying (17.2.2) by $u/v+v/u$, we find that by (17.2.13), (17.2.11), (17.2.7), and (17.2.4), the new right side of (17.2.2) can be written in the form

$$\begin{aligned}&\frac{16m(5m^4+20m^3+6m^2+100m+125)}{(-m^2+6m-5)^3}\\&\qquad-\frac{128m(3m^3-2m^2+15m)}{(-m^2+6m-5)^2}+\frac{64m}{-m^2+6m-5}+4.\end{aligned} \tag{17.2.14}$$

In view of (17.2.2), combining (17.2.9) and (17.2.14), we find that it suffices to prove that

$$\begin{aligned}m(m-5)^5-125(m-1)^5=&-4m(5m^4+20m^3+6m^2+100m+125)\\&-32m(3m^2-2m+15)(m^2-6m+5)\\&-16m(m^2-6m+5)^2+(m^2-6m+5)^3.\end{aligned}$$

This last equality is easily verified via *Mathematica*, and this completes the proof. □

Entry 17.2.4 (p. 55). *If* $n=5$ *in* (17.2.1), *then the functions* u *and* v *satisfy the modular equation of degree* 25

$$(uv)^2+\left(\frac{5}{uv}\right)^2+5\left(uv+\frac{5}{uv}\right)+15=\left(\frac{v}{u}\right)^3. \tag{17.2.15}$$

Proof. In [61, p. 268, equation (11.8)], we proved that

$$v^6:=\frac{f^6(-q^5)}{q^5f^6(-q^{25})}=(uv)^5+5(uv)^4+15(uv)^3+25(uv)^2+25uv,$$

where we have replaced q by q^5 in the cited formulation. Dividing the equality above by $(uv)^3$ and rearranging the terms, we easily deduce (17.2.15). □

Entry 17.2.5 (p. 55). *If* $n = 7$ *in* (17.2.1), *the functions* u *and* v *obey the modular equation of degree* 35

$$(uv)^3 + \left(\frac{5}{uv}\right)^3 = -\left\{\left(\frac{u}{v}\right)^4 - \left(\frac{v}{u}\right)^4\right\} - 7\left\{\left(\frac{u}{v}\right)^3 + \left(\frac{v}{u}\right)^3\right\}$$
$$- 7\left\{\left(\frac{u}{v}\right)^2 - \left(\frac{v}{u}\right)^2\right\} + 14\left(\frac{u}{v} + \frac{v}{u}\right). \qquad (17.2.16)$$

Proof. We will use the theory of modular forms and employ the theory developed by Berndt and L.–C. Zhang in [62, pp. 237–239].

Let $q = \exp(2\pi i z)$, where $\operatorname{Im} z > 0$, and recall that $f(-q) = q^{-1/24}\eta(z)$, where η denotes the Dedekind eta function. In the notation of [62, p. 237],

$$uv = R_{5,7}(z) \qquad \text{and} \qquad v/u = S_{5,7}(z).$$

By Lemmas 68.1 and 68.2 in [62, pp. 237, 238], we deduce that

$$R_{5,7}^3(z), S_{5,7}(z) \in \{\Gamma_0(35), 0, 1\},$$

where $\{\Gamma_0(n), 0, 1\}$ is the space of modular forms on $\Gamma_0(n)$ of weight 0 and multiplier system identically equal to 1.

From [62, p. 239], if r/s denotes a cusp with $(r, s) = 1$, then for any pair of positive integers m, n,

$$\operatorname{ord}\left(\eta(mnz); \frac{r}{s}\right) = \frac{(mn, s)^2}{24mn}, \qquad (17.2.17)$$

where (a, b) denotes the greatest common divisor of a and b. A complete set of inequivalent cusps for $\Gamma_0(35)$ is $\{0, \infty, \frac{1}{5}, \frac{1}{7}\}$. Using (17.2.17) repeatedly, we compose the following table summarizing the information that we need about the orders of certain functions at these cusps. We have abbreviated the left and right sides of (17.2.16) by $L(17.2.16)$ and $R(17.2.16)$, respectively.

cusp/order	u	v	uv	u/v	$L(17.2.16)$	$R(17.2.16)$
0	$\frac{1}{30}$	$\frac{1}{210}$	$\frac{4}{105}$	$\frac{1}{35}$	$-\frac{4}{35}$	$-\frac{4}{35}$
$\frac{1}{5}$	$-\frac{1}{6}$	$-\frac{1}{42}$	$-\frac{4}{21}$	$-\frac{1}{7}$	$-\frac{4}{7}$	$-\frac{4}{7}$
$\frac{1}{7}$	$\frac{1}{30}$	$\frac{7}{30}$	$\frac{4}{15}$	$-\frac{1}{5}$	$-\frac{4}{5}$	$-\frac{4}{5}$

If $F(z)$ denotes the difference of the left and right sides of (17.2.16), and if $\sum_\zeta$ denotes the sum over a complete set of inequivalent cusps for $\Gamma_0(35)$, then, by the valence formula [62, p. 239],

$$0 = \sum_\zeta \operatorname{ord}(F; \zeta) \geq \operatorname{ord}(F; \infty) - \tfrac{4}{35} - \tfrac{4}{7} - \tfrac{4}{5} = \operatorname{ord}(F; \infty) - \tfrac{52}{35}. \qquad (17.2.18)$$

Thus, if we can show that $F(z) = O(q^2)$ as $q \to 0$ ($z \to i\infty$), then we will have obtained a contradiction to (17.2.18), unless $F(z) \equiv 0$, which is what we want to prove. In fact, using *Mathematica*, we find that

$$L(17.2.16) = \frac{1}{q^4} - \frac{3}{q^3} + \frac{5}{q} + 3q - 16q^2 + \cdots = R(17.2.16).$$

This then completes the proof of (17.2.16). □

In the second set of eta-function identities, Ramanujan sets

$$u^2 = \frac{f(-q^{1/5})}{q^{1/5}f(-q^5)} \qquad \text{and} \qquad v^2 = \frac{f(-q^{n/5})}{q^{n/5}f(-q^{5n})}. \tag{17.2.19}$$

It is not clear why Ramanujan did not write u and v for u^2 and v^2, respectively. There are just two modular equations in the second set.

Entry 17.2.6 (p. 55). *For* $n = 2$ *in* (17.2.19), *the functions* u *and* v *satisfy the modular equation of degree* 25

$$uv + \frac{5}{uv} = \left(\frac{u}{v}\right)^3 + \left(\frac{v}{u}\right)^3 - 2\left(\frac{u}{v} + \frac{v}{u}\right). \tag{17.2.20}$$

Proof. We rewrite Entry 58 of Chapter 25 in [62, pp. 212–213] in Ramanujan's notation (17.2.19). Thus, since $P = u^2$ and $Q = v^2$, we find that

$$u^2v^2 + \frac{25}{u^2v^2} = \left(\frac{u}{v}\right)^6 + \left(\frac{v}{u}\right)^6 - 4\left\{\left(\frac{u}{v}\right)^4 + \left(\frac{v}{u}\right)^4\right\}. \tag{17.2.21}$$

In (17.2.20), observe that for sufficiently small and positive q, each side is positive. It thus suffices to show that the squares of both sides of (17.2.20) are equal, i.e., after slight simplification, we want to prove that

$$\begin{aligned} u^2v^2 + \frac{25}{u^2v^2} &= \left(\frac{u}{v}\right)^6 + \left(\frac{v}{u}\right)^6 + 4\left\{\left(\frac{u}{v}\right)^2 + \left(\frac{v}{u}\right)^2\right\} \\ &\quad - 4\left(\frac{u}{v} + \frac{v}{u}\right)\left\{\left(\frac{u}{v}\right)^3 + \left(\frac{v}{u}\right)^3\right\}. \end{aligned} \tag{17.2.22}$$

In comparing (17.2.21) with (17.2.22), we see that it remains to prove that

$$-\left(\frac{u}{v}\right)^4 - \left(\frac{v}{u}\right)^4 = \left(\frac{u}{v}\right)^2 + \left(\frac{v}{u}\right)^2 - \left(\frac{u}{v} + \frac{v}{u}\right)\left\{\left(\frac{u}{v}\right)^3 + \left(\frac{v}{u}\right)^3\right\}.$$

Since the last equality is trivial, the proof is complete. □

Entry 17.2.7 (p. 55). *If* $n = 3$ *in* (17.2.19), *the functions* u *and* v *satisfy the modular equation of degree* 75

$$\begin{aligned} u^2v^2 + \frac{25}{u^2v^2} &+ 3\left(\frac{u}{v} + \frac{v}{u}\right)\left(uv + \frac{5}{uv}\right) \\ &= \left(\frac{u}{v}\right)^4 + \left(\frac{v}{u}\right)^4 - 6\left(\left(\frac{u}{v}\right)^2 + \left(\frac{v}{u}\right)^2\right) - 9. \end{aligned} \tag{17.2.23}$$

Proof. From Schoeneberg's book [237, p. 102], if σ_∞ denotes the number of inequivalent cusps of $\Gamma_0(N)$, then

$$\sigma_\infty = \sum_{d|N} \varphi\left((d, N/d)\right),$$

where φ denotes Euler's φ–function, and (a,b) denotes the greatest common divisor of a and b. If $N=75$, then $\sigma_\infty = 12$, and a complete set of inequivalent cusps is given by $\{0, \infty, \frac{1}{3}, \frac{1}{5}, \frac{1}{10}, \frac{1}{15}, \frac{1}{20}, \frac{1}{25}, \frac{1}{30}, \frac{1}{45}, \frac{1}{50}, \frac{1}{60}\}$.

Set $U(q) = u(q^5)$ and $V(q) = v(q^5)$. In the notation of [62, pp. 237–238],

$$U^2V^2 = R_{25,3}(z) \qquad \text{and} \qquad V^2/U^2 = S_{25,3}(z),$$

where $q = \exp(2\pi i z)$. By Lemmas 68.1 and 68.2 in [62, pp. 237–238],

$$R_{25,3}(z), S_{25,3}(z) \in \{\Gamma_0(75,0,1)\}.$$

Letting $L(17.2.23)$ and $L(17.2.23)$ denote the left and right sides, respectively, of (17.2.23) and using (17.2.17), we compose the following table for orders of cusps:

cusp/order	u^2	v^2	uv	u/v	$L(17.2.23)$	$R(17.2.23)$
0	$\frac{1}{25}$	$\frac{1}{75}$	$\frac{2}{75}$	$\frac{1}{75}$	$-\frac{4}{75}$	$-\frac{4}{75}$
$\frac{1}{3}$	$\frac{1}{25}$	$\frac{3}{25}$	$\frac{2}{25}$	$-\frac{1}{25}$	$-\frac{4}{25}$	$-\frac{4}{25}$
$\frac{1}{5}$	0	0	0	0	0	0
$\frac{1}{10}$	0	0	0	0	0	0
$\frac{1}{15}$	0	0	0	0	0	0
$\frac{1}{20}$	0	0	0	0	0	0
$\frac{1}{25}$	-1	$-\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{4}{3}$	$-\frac{4}{3}$
$\frac{1}{30}$	0	0	0	0	0	0
$\frac{1}{45}$	0	0	0	0	0	0
$\frac{1}{50}$	-1	$-\frac{1}{3}$	$-\frac{2}{3}$	$-\frac{1}{3}$	$-\frac{4}{3}$	$-\frac{4}{3}$
$\frac{1}{60}$	0	0	0	0	0	0

If $F(z)$ denotes the difference of the left and right sides of (17.2.23), and if $\sum_\zeta$ denotes the sum over a complete set of inequivalent cusps, then, by the valence formula and the tables above,

$$0 = \sum_\zeta \operatorname{ord}(F;\zeta) \geq \operatorname{ord}(F;\infty) - \tfrac{4}{75} - \tfrac{4}{25} - \tfrac{4}{3} - \tfrac{4}{3} = -\tfrac{216}{75}. \tag{17.2.24}$$

Thus, if we can show that $F(z) = O(q^3)$ as q tends to 0, or z tends to $i\infty$, then we will have shown a contradiction to (17.2.24) unless $F(z) \equiv 0$, which is what we want to prove. In fact, using *Mathematica*, we find that

$$L(17.2.23) = \frac{1}{q^4} + \frac{2}{q^3} - \frac{1}{q^2} + \frac{2}{q} - 5 + 14q + 14q^2 + 44q^3 + \cdots = R(17.2.23).$$

This then completes the proof. □

The third set of eta-function identities comprises five modular equations. For these, Ramanujan sets

$$u = \frac{f(-q)}{q^{(n-1)/24} f(-q^n)} \qquad \text{and} \qquad v = \frac{f(-q^5)}{q^{5(n-1)/24} f(-q^{5n})}. \tag{17.2.25}$$

Entry 17.2.8 (p. 55). *For* $n = 2$ *in* (17.2.25)*, the functions* u *and* v *satisfy the modular equation of degree* 5

$$(uv)^2 + \left(\frac{2}{uv}\right)^2 = \left(\frac{v}{u}\right)^3 - \left(\frac{u}{v}\right)^3. \tag{17.2.26}$$

Proof. We prove that (17.2.26) is equivalent to Entry 13(xiv) in Chapter 19 of [61, p. 282]. To that end, first set

$$U = \frac{f(q)}{q^{1/24} f(-q^2)} \qquad \text{and} \qquad V = \frac{f(q^5)}{q^{5/24} f(-q^{10})}.$$

If we replace q by $-q$ in (17.2.26), we then find that (17.2.26) is equivalent to the identity

$$(UV)^2 - \left(\frac{2}{UV}\right)^2 = \left(\frac{V}{U}\right)^3 + \left(\frac{U}{V}\right)^3. \tag{17.2.27}$$

We now apply Entries 12(i) and 12(iii) in Chapter 17 of [61, p. 124] to deduce that

$$UV = 2^{1/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{-1/24} \qquad \text{and} \qquad \frac{U}{V} = \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/24},$$

where β has degree 5 over α. Thus, (17.2.27) is equivalent to the modular equation of degree 5

$$\frac{2^{2/3}}{\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/12}} - 2^{4/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/12} \\ = \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/8} + \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8}. \tag{17.2.28}$$

But with

$$P := \{16\alpha\beta(1-\alpha)(1-\beta)\}^{1/12} \qquad \text{and} \qquad Q := \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8},$$

(17.2.28) may be rewritten in the form

$$\frac{2}{P} - 2P = \frac{1}{Q} + Q,$$

which is Entry 13(xiv) in Chapter 19 of [61, p. 282]. □

Entry 17.2.9 (p. 55). *With $n = 3$ in (17.2.25), the functions u and v satisfy the modular equation of degree 15*

$$(uv)^2 + \left(\frac{3}{uv}\right)^2 + 5 = \left(\frac{v}{u}\right)^3 - \left(\frac{u}{v}\right)^3.$$

Entry 17.2.9 is identical to Entry 62 in Chapter 25 of [62, p. 221].

Entry 17.2.10 (p. 55). *With $n = 4$ in (17.2.25), the functions u and v satisfy the modular equation of degree 5*

$$(uv)^2 + \left(\frac{4}{uv}\right)^2 = \left(\frac{v}{u}\right)^3 + \left(\frac{u}{v}\right)^3 - 5\left(\frac{v}{u} + \frac{u}{v}\right). \tag{17.2.29}$$

Proof. By Entries 12(ii) and 12(iv) in Chapter 17 of [61, p. 124],

$$u := \frac{f(-q)}{q^{1/8}f(-q^4)} = \sqrt{2}\left(\frac{1-\alpha}{\alpha}\right)^{1/8}$$

and

$$v := \frac{f(-q^5)}{q^{5/8}f(-q^{20})} = \sqrt{2}\left(\frac{1-\beta}{\beta}\right)^{1/8},$$

where β has degree 5 over α. It follows that

$$uv = 2\left(\frac{(1-\alpha)(1-\beta)}{\alpha\beta}\right)^{1/8} \qquad \text{and} \qquad \frac{u}{v} = \left(\frac{\beta(1-\alpha)}{\alpha(1-\beta)}\right)^{1/8}. \tag{17.2.30}$$

Thus, using (17.2.30), we see that in order to prove (17.2.29) it suffices to prove the fifth degree modular equation

$$\begin{aligned}4\left(\frac{(1-\alpha)(1-\beta)}{\alpha\beta}\right)^{1/4} + 4\left(\frac{\alpha\beta}{(1-\alpha)(1-\beta)}\right)^{1/4} &= \left(\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right)^{3/8}\\ + \left(\frac{\beta(1-\alpha)}{\alpha(1-\beta)}\right)^{3/8} - 5\left(\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right)^{1/8} &- 5\left(\frac{\beta(1-\alpha)}{\alpha(1-\beta)}\right)^{1/8}.\end{aligned} \tag{17.2.31}$$

Recall that ρ is defined by (17.2.4). From [61, pp. 285–286, equations (13.10), (13.11)], we find that

$$\left(\frac{(1-\alpha)(1-\beta)}{\alpha\beta}\right)^{1/4} = \left(\frac{(\rho-3m+5)(\rho-m^2+3m)}{(\rho+3m-5)(\rho+m^2-3m)}\right)^{1/2},$$

where m is the multiplier of degree 5. Also, from [61, p. 286, eq. (13.12)],

$$\left(\frac{\alpha(1-\beta)}{\beta(1-\alpha)}\right)^{1/8} = \left(\frac{2m+\rho}{2m-\rho}\right)^{1/2}.$$

Thus, (17.2.31) may be recast in the form

$$\begin{aligned}
&4\left(\frac{(\rho-3m+5)(\rho-m^2+3m)}{(\rho+3m-5)(\rho+m^2-3m)}\right)^{1/2} + 4\left(\frac{(\rho+3m-5)(\rho+m^2-3m)}{(\rho-3m+5)(\rho-m^2+3m)}\right)^{1/2}\\
&= \left(\frac{2m+\rho}{2m-\rho}\right)^{3/2} + \left(\frac{2m-\rho}{2m+\rho}\right)^{3/2} - 5\left(\frac{2m+\rho}{2m-\rho}\right)^{1/2} - 5\left(\frac{2m-\rho}{2m+\rho}\right)^{1/2},
\end{aligned}$$

or

$$\begin{aligned}
&\frac{4\left\{(\rho-3m+5)(\rho-m^2+3m)+(\rho+3m-5)(\rho+m^2-3m)\right\}}{\{\rho^2-(3m-5)^2\}^{1/2}\{\rho^2-(m^2-3m)^2\}^{1/2}}\\
&\qquad = \frac{(2m+\rho)^3+(2m-\rho)^3}{(4m^2-\rho^2)^{3/2}} - \frac{20m}{(4m^2-\rho^2)^{1/2}}. \qquad (17.2.32)
\end{aligned}$$

Expanding all the numerators above, employing (17.2.4), putting the right side under one denominator, and omitting a goodly amount of elementary algebra, we find that (17.2.32) reduces to the equation

$$\begin{aligned}
&\frac{1}{(m^3-11m^2+35m-25)^{1/2}(-m^3+7m^2-11m+5)^{1/2}}\\
&= \frac{1}{(-m^2+6m-5)^{3/2}}. \qquad (17.2.33)
\end{aligned}$$

However, (17.2.33) is easily established by factoring all the polynomials in it, and so this completes the proof. □

Entry 17.2.11 (p. 55). *With* $n = 5$, *the functions* u *and* v *in* (17.2.25) *satisfy the modular equation of degree* 25

$$(uv)^2 + \left(\frac{5}{uv}\right)^2 + 5\left(uv + \frac{5}{uv}\right) + 15 = \left(\frac{v}{u}\right)^3.$$

Entry 17.2.11 is identical to Entry 17.2.4 above.

Entry 17.2.12 (p. 55). *With* $n = 7$, *the functions* u *and* v *in* (17.2.25) *satisfy the modular equation of degree* 35

$$(uv)^2 + \left(\frac{7}{uv}\right)^2 - 5 = \left(\frac{v}{u}\right)^3 - \left(\frac{u}{v}\right)^3 - 5\left\{\left(\frac{v}{u}\right)^2 + \left(\frac{u}{v}\right)^2\right\}.$$

Entry 17.2.12 is the same as Entry 71 in Chapter 25 of [62, p. 236].

The fourth and last set of eta-function identities contains two modular equations featuring

$$u = \frac{f(-q^n)}{q^{(5-n)/24} f(-q^5)} \qquad \text{and} \qquad v = \frac{f(-q)}{q^{(5n-1)/24} f(-q^{5n})}. \tag{17.2.34}$$

Entry 17.2.13 (p. 55). *With $n = 2$ in* (17.2.34)*, the two functions u and v satisfy the modular equation of degree* 5

$$uv - \frac{5}{uv} = \left(\frac{v}{u}\right)^2 - \left(\frac{2u}{v}\right)^2.$$

Entry 17.2.13 is identical to Entry 54 of Chapter 25 of [62, p. 207].

Entry 17.2.14 (p. 55). *With $n = 3$ in* (17.2.34)*, the functions u and v satisfy the modular equation of degree* 15

$$(uv)^3 - \left(\frac{5}{uv}\right)^3 = \left(\frac{v}{u}\right)^4 - \left(\frac{3u}{v}\right)^4 + \left(\frac{v}{u}\right)^2 - \left(\frac{3u}{v}\right)^2.$$

Entry 17.2.14 is identical to Entry 64 in Chapter 25 of [62, p. 226].

Perhaps put in other forms, the modular equations we have been considering so far in this chapter are called Schläfli modular equations, or modular equations of Schläfli type. For a completely different approach, using the Atkin–Lehner involution, to deriving modular equations of the type considered in this section, see a paper by H.H. Chan and M.L. Lang [116].

In her doctoral dissertation, J. Yi [297] derived a plethora of new eta-function identities and made several applications of them to the values of continued fractions and theta functions. Further new modular equations involving only the Dedekind eta function have been found by N.D. Baruah [50], [51], [52], [53], [54], Baruah and N. Saikia [55], M.S. Mahadeva Naika [189], [190], C. Adiga, T. Kim, and Mahadeva Naika [3], Adiga, Mahadeva Naika, and K. Shivashankara [6], H.S. Madhusudhan, Mahadeva Naika, and K.R. Vasuki [188], and S. Bhargava, Adiga, and Mahadeva Naika [94], [95].

17.3 Summary of Modular Equations of Six Kinds

We reproduce Ramanujan's summary of several of his modular equations found in a fragment on pages 350–352 of [228]. The modular equations are grouped into six types. It is interesting that in contrast to his work in the notebooks [227] and lost notebook [228], Ramanujan used the more standard notations of k and ℓ to denote the moduli. Since most of the modular equations in this section have been given elsewhere by Ramanujan, so that readers may more easily compare the results stated here with Ramanujan's other work, we have put all the modular equations in Ramanujan's original notation.

There are three modular equations in the first set.

Entry 17.3.1 (p. 350). *If β has degree 2 over α, then*

$$(1-\sqrt{1-\alpha})(1-\sqrt{\beta}) = 2\sqrt{\beta(1-\alpha)}.$$

Entry 17.3.2 (p. 350). *If β has degree 4 over α, then*

$$(1-\sqrt[4]{1-\alpha})(1-\sqrt[4]{\beta}) = 2\sqrt[4]{\beta(1-\alpha)}.$$

Entry 17.3.3 (p. 350). *If β has degree 8 over α, then*

$$(1-\sqrt[4]{1-\alpha})(1-\sqrt[4]{\beta}) = 2\sqrt[8]{2\beta(1-\alpha)}.$$

After some elementary algebraic manipulation, it is easily seen that Entry 17.3.1 is equivalent to part of equation (24.12) in Chapter 18 of [61, p. 213] and that Entry 17.3.2 is equivalent to (24.22) in Chapter 18 [61, p. 215]. Entry 17.3.3 is the equation just before Entry 24(vi) in [61, p. 217]. Unfortunately, Berndt erroneously claimed [61, pp. 216–217] that two of Ramanujan's modular equations with degrees 8 and 16 are incorrect. It was Berndt, not Ramanujan, who was incorrect, and his work was corrected in [63]. Modular equations of degree 2^n can be obtained from classical theta-function identities by iterating modular equations of degree 2^{n-1}. However, the complexity of these modular equations increases rapidly with n.

There are three sets of modular equations in the second and third groups as well.

Entry 17.3.4 (p. 350). *If m denotes the multiplier of degree 2 and β has degree 2 over α, then*

$$\tfrac{1}{2}m^2 = \frac{1+\sqrt{\beta}}{1+\sqrt{1-\alpha}} = \frac{1+\beta}{1+(1-\alpha)}.$$

Entry 17.3.5 (p. 350). *If m denotes the multiplier of degree 4 and β has degree 4 over α, then*

$$\tfrac{1}{2}m = \frac{1+\sqrt[4]{\beta}}{1+\sqrt[4]{1-\alpha}} = \frac{1+\sqrt{\beta}}{1+\sqrt{1-\alpha}}.$$

Entry 17.3.6 (p. 350). *If m denotes the multiplier of degree 16 and β has degree 16 over α, then*

$$\tfrac{1}{2}\sqrt{m} = \frac{1+\sqrt[4]{\beta}}{1+\sqrt[4]{1-\alpha}}.$$

The two equalities of Entry 17.3.4 are given in (24.17), as part of Entry 24(ii) in Chapter 18 of [61, p. 214]. The two equalities of Entry 17.3.5 are given in (24.20), as part of Entry 24(iii) in Chapter 18 in [61, p. 215]. Lastly, Entry 17.3.6 is given in the middle of page 216 of [61] and is part of Entry 24(iv) of Chapter 18 in [61, p. 216].

Entry 17.3.7 (p. 350). *If m is the multiplier of degree 2 and β has degree 2 over α, then*

(a)
$$m\sqrt{1-\alpha}+\sqrt{\beta}=1,$$

(b)
$$\frac{2}{m}\sqrt{\beta}+\sqrt{1-\alpha}=1,$$

(c)
$$m^2\sqrt{1-\alpha}+\beta=1,$$

(d)
$$\frac{4}{m^2}\sqrt{\beta}+(1-\alpha)=1.$$

Parts (a) and (c) are parts of Entry 24(ii) in Chapter 18 of [61, p. 214, eqs. (24.15), (24.16)]. The equation in part (b) is the reciprocal of that of (a), and the equation in part (d) is the reciprocal of that of (c). (For the definition of the *reciprocal* of a modular equation, see [61, p. 216, Entry 24(v)].)

Entry 17.3.8 (p. 350). *If m is the multiplier of degree 4 and β has degree 4 over α, then*

(a)
$$\sqrt{m}\sqrt[4]{1-\alpha}+\sqrt[4]{\beta}=1,$$

(b)
$$\frac{2}{\sqrt{m}}\sqrt[4]{\beta}+\sqrt[4]{1-\alpha}=1,$$

(c)
$$m\sqrt[4]{1-\alpha}+\sqrt{\beta}=1,$$

(d)
$$\frac{4}{m}\sqrt[4]{\beta}+\sqrt{1-\alpha}=1.$$

Parts (a) and (c) are parts of Entry 24(iii) in Chapter 18 of [61, pp. 214, 215, eqs. (24.18), (24.19)]. The modular equations in parts (b) and (d) are the reciprocals of those in parts (a) and (c), respectively.

Entry 17.3.9 (p. 350). *If m is the multiplier of degree 8 and β has degree 8 over α, then*

(a)
$$\sqrt{m}\sqrt[8]{1-\alpha}+\sqrt[4]{\beta}=1,$$

(b)
$$2\sqrt{\frac{2}{m}}\sqrt[8]{\beta}+\sqrt[4]{1-\alpha}=1.$$

Part (a) is the same as equation (24.24) on page 216 of [61], while the equation of part (b) is the reciprocal of part (a).

Ramanujan records four modular equations in his fourth set. The first, due to Legendre, is historically the first modular equation of a degree that is not a power of two. As was emphasized in [61, Chapters 19, 20], H. Schröter derived several modular equations of this sort. Many modular equations of this kind also were derived by R. Russell, but his methods are not completely rigorous. Russell's method has been put on a firm foundation by H.H. Chan and W.–C. Liaw [117].

Entry 17.3.10 (p. 350). *If β has degree* 3 *over α, then*

$$\{\alpha\beta\}^{1/4}+\{(1-\alpha)(1-\beta)\}^{1/4}=1.$$

Entry 17.3.10 is also Entry 5(ii) of Chapter 19 in [61, p. 230].

Entry 17.3.11 (p. 350). *If β has degree* 7 *over α, then*

$$\{\alpha\beta\}^{1/8}+\{(1-\alpha)(1-\beta)\}^{1/8}=1.$$

Entry 17.3.11 is identical to part of Entry 19(i) of Chapter 19 in [61, p. 314].

Entry 17.3.12 (p. 350). *If β has degree* 15 *over α, then*

$$\begin{aligned}&(\alpha\beta)^{1/16}\left(\{(1+\sqrt{\alpha})(1+\sqrt{\beta})\}^{1/4}+\{(1-\sqrt{\alpha})(1-\sqrt{\beta})\}^{1/4}\right)\\&+\{(1-\alpha)(1-\beta)\}^{1/16}\Big(\{(1+\sqrt{1-\alpha})(1+\sqrt{1-\beta})\}^{1/4}\\&+\{(1-\sqrt{1-\alpha})(1-\sqrt{1-\beta})\}^{1/4}\Big)=\sqrt{2}.\end{aligned}$$

Entry 17.3.12 is identical to Entry 20(vi) in Chapter 20 of [61, p. 384].

Entry 17.3.13 (p. 350). *If β has degree* 31 *over α, then*

$$\begin{aligned}&(\alpha\beta)^{1/32}\Big(\{(1+\sqrt{\alpha})(1+\sqrt{\beta})\}^{1/8}\\&\times\sqrt{1+\{\alpha\beta\}^{1/4}+\{(1-\sqrt{\alpha})(1-\sqrt{\beta})\}^{1/4}}\\&+\{(1-\sqrt{\alpha})(1-\sqrt{\beta})\}^{1/8}\sqrt{1+\{\alpha\beta\}^{1/4}+\{(1+\sqrt{\alpha})(1+\sqrt{\beta})\}^{1/4}}\Big)\\&+\{(1-\alpha)(1-\beta)\}^{1/32}\Big(\{(1+\sqrt{1-\alpha})(1+\sqrt{1-\beta})\}^{1/8}\\&\times\sqrt{1+\{(1-\alpha)(1-\beta)\}^{1/4}+\{(1-\sqrt{1-\alpha})(1-\sqrt{1-\beta})\}^{1/4}}\\&+\{(1-\sqrt{1-\alpha})(1-\sqrt{1-\beta})\}^{1/8}\\&\times\sqrt{1+\{(1-\alpha)(1-\beta)\}^{1/4}+\{(1+\sqrt{1-\alpha})(1+\sqrt{1-\beta})\}^{1/4}}\Big)=2^{3/4}.\end{aligned}$$

Entry 17.3.13 is the same as Entry 22(i) in Chapter 20 of [61, p. 439].

The fifth set in this fragment contains seven results. These results are similar to modular equations of Russell type. However, Ramanujan focuses on the algebraic expression

$$\sqrt{\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}},$$

which has not been prominent in the work of any other mathematician on modular equations. In the next section, we will see how useful this expression becomes in simplifying modular equations.

Entry 17.3.14 (p. 351). *If β has degree 7 over α, then*

$$\sqrt{\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}} \\ = \{\alpha\beta\}^{1/8}+\{(1-\alpha)(1-\beta)\}^{1/8}-\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}.$$

By combining both parts of Entry 19(i) in Chapter 19 of [61, p. 314], we easily deduce Entry 17.3.14.

Ramanujan claimed that the modular equation of Entry 17.3.14 also holds for $n = \frac{5}{3}$. At first, this is somewhat puzzling, since modular equations have been defined for only *integral n*. However, Ramanujan evidently had in mind modular equations of degree 15, where, in general, there are four moduli of degrees 1, 3, 5, and 15. Thus, Ramanujan asserted that the moduli of degrees 3 and 5 satisfy the modular equation above. In the proof below, we depend heavily on the parametrizations used in [61, pp. 385–387] to establish many modular equations of degree 15.

Entry 17.3.15 (p. 351). *If α and β have degrees 3 and 5, respectively, then*

$$\sqrt{\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}} \\ = \{\alpha\beta\}^{1/8}+\{(1-\alpha)(1-\beta)\}^{1/8}-\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}. \tag{17.3.1}$$

Proof. Using the notation in [61, p. 385], but with β and γ there replaced by α and β here, we set

$$B := \{\alpha\beta\}^{1/8} \qquad \text{and} \qquad B' := \{(1-\alpha)(1-\beta)\}^{1/8}.$$

Thus, (17.3.1) takes the form

$$\sqrt{\frac{1+B^4+B'^4}{2}} = B+B'-BB'. \tag{17.3.2}$$

Also define [61, p. 385; p. 386, equation (11.4)]

$$M := \sqrt{\frac{z_1 z_{15}}{z_3 z_5}}, \tag{17.3.3}$$

$$\rho^2 := \frac{1+M-M^2}{M}. \tag{17.3.4}$$

Then [61, p. 386, equation (11.3)]

$$B = \tfrac{1}{2}(M-\rho) \qquad \text{and} \qquad B' = \tfrac{1}{2}(M+\rho). \tag{17.3.5}$$

Also [61, middle of p. 387],

$$\sqrt{\frac{1+B^4+B'^4}{2}}=\frac{1+M+3M^2-M^3}{4M}. \tag{17.3.6}$$

On the other hand, by (17.3.5) and (17.3.4),

$$\begin{aligned} B+B'-BB' &= M-\tfrac{1}{4}(M^2-\rho^2) \\ &= M-\tfrac{1}{4}M^2+\frac{1+M-M^2}{4M} \\ &= \frac{1+M+3M^2-M^3}{4M}. \end{aligned} \tag{17.3.7}$$

Comparing (17.3.6) and (17.3.7), we complete the proof of (17.3.2) and so also of Entry 17.3.15. ☐

Entry 17.3.16 (p. 351). *If* β *has degree* 15 *over* α, *then*

$$\begin{aligned} &\sqrt{\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}} \\ &\qquad = \{\alpha\beta\}^{1/8}+\{(1-\alpha)(1-\beta)\}^{1/8}+\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}. \end{aligned} \tag{17.3.8}$$

Proof. The proof is similar to that above, and again we rely heavily on the notation and calculations from [61]. Set, as in [61, p. 385], but with δ there replaced by β here,

$$A=\{\alpha\beta\}^{1/8} \qquad \text{and} \qquad A^{'}=\{(1-\alpha)(1-\beta)\}^{1/8}.$$

Thus, (17.3.8) takes the form

$$\sqrt{\frac{1+A^4+A'^4}{2}}=A+A'+AA'. \tag{17.3.9}$$

With [61, p. 386]

$$A=\tfrac{1}{2}(M^{-1}-\rho) \qquad \text{and} \qquad A'=\tfrac{1}{2}(M^{-1}+\rho), \tag{17.3.10}$$

where M and ρ are defined by (17.3.3) and (17.3.4), respectively, we find that [61, near the bottom of p. 386]

$$\sqrt{\frac{1+A^4+A'^4}{2}}=\frac{1+3M-M^2+M^3}{4M^2}. \tag{17.3.11}$$

On the other hand, by using (17.3.10), (17.3.4), and a calculation similar to that in (17.3.7), we find that

$$A+A'+AA'=\frac{1+3M-M^2+M^3}{4M^2}. \tag{17.3.12}$$

Comparing (17.3.11) and (17.3.12), we see that we have established (17.3.9), and the proof is complete. ☐

Entry 17.3.17 (p. 351). *If β has degree* 23 *over α, then*

$$\sqrt{\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}} = \tfrac{1}{2}\left(1+\{\alpha\beta\}^{1/4}+\{(1-\alpha)(1-\beta)\}^{1/4}\right)+2^{1/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/12}. \tag{17.3.13}$$

Entry 17.3.17 is identical to Entry 15(ii) in Chapter 20 of [61, p. 411]. In fact, Ramanujan's formulation in the lost notebook is erroneous, since the last term on the right side of (17.3.13) was replaced by

$$2^{2/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/6}.$$

Entry 17.3.18 (p. 351). *If β has degree* 31 *over α, then*

$$\sqrt{\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}} = 1+\{\alpha\beta\}^{1/4}+\{(1-\alpha)(1-\beta)\}^{1/4} - \{\alpha\beta\}^{1/8}-\{(1-\alpha)(1-\beta)\}^{1/8}-\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}.$$

Entry 17.3.18 is the same as Entry 22(iii) in Chapter 20 of [61, p. 439].

Entry 17.3.19 (p. 351). *If β has degree* 47 *over α, then*

$$\sqrt{\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}} = \tfrac{1}{2}\left(1+\{\alpha\beta\}^{1/4}+\{(1-\alpha)(1-\beta)\}^{1/4}\right) + \left(\tfrac{1}{256}\alpha\beta(1-\alpha)(1-\beta)\right)^{1/24}\left(1+\{\alpha\beta\}^{1/8}+\{(1-\alpha)(1-\beta)\}^{1/8}\right).$$

Entry 17.3.19 is the same as Entry 23(i) in Chapter 20 of [61, p. 444].

Entry 17.3.20 (p. 351). *If β has degree* 71 *over α, then*

$$\sqrt{\frac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}} = 1+\{\alpha\beta\}^{1/4}+\{(1-\alpha)(1-\beta)\}^{1/4} - \{\alpha\beta\}^{1/8}-\{(1-\alpha)(1-\beta)\}^{1/8}+\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8} + 2^{2/3}\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/24}\left(\{\alpha\beta\}^{1/8}+\{(1-\alpha)(1-\beta)\}^{1/8}-1\right).$$

Entry 17.3.20 is identical to Entry 23(ii) of Chapter 20 in [61, p. 444].

The sixth and last group of modular equations in this fragment contains seven pairs of formulas for moduli. Formulas of this sort seem to have originated with Ramanujan, and Ramanujan's methods for deriving these equations are unknown. From the definition of a modular equation, a formula for a modulus yields a transformation between two hypergeometric functions. It appears likely that such formulas are, in fact, special cases of more general transformation formulas for hypergeometric functions involving one or more parameters. It would be worthwhile to investigate such possibilities.

Entry 17.3.21 (p. 351). *If* β *and the multiplier* m *have degree* 3, *then*

$$m^2 = \sqrt{\frac{\beta}{\alpha}} + \sqrt{\frac{1-\beta}{1-\alpha}} - \sqrt{\frac{\beta(1-\beta)}{\alpha(1-\alpha)}},$$
$$\frac{9}{m^2} = \sqrt{\frac{\alpha}{\beta}} + \sqrt{\frac{1-\alpha}{1-\beta}} - \sqrt{\frac{\alpha(1-\alpha)}{\beta(1-\beta)}}.$$

Entry 17.3.22 (p. 351). *If* β *and the multiplier* m *have degree* 5, *then*

$$m = \left(\frac{\beta}{\alpha}\right)^{1/4} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/4} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4},$$
$$\frac{5}{m} = \left(\frac{\alpha}{\beta}\right)^{1/4} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/4} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/4}.$$

Entry 17.3.23 (p. 351). *If* β *and the multiplier* m *have degree* 7, *then*

$$m^2 = \left(\frac{\beta}{\alpha}\right)^{1/2} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/2} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/2} - 8\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/3},$$
$$\frac{49}{m^2} = \left(\frac{\alpha}{\beta}\right)^{1/2} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/2} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/2} - 8\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/3}.$$

Entries 17.3.21, 17.3.22, and 17.3.23 are the same as Entries 5(vii), 13(xii), and 19(v), respectively, in Chapter 19 of [61, pp. 230, 281–282, 314].

Entry 17.3.24 (p. 351). *If* β *and the multiplier* m *have degree* 9, *then*

$$\sqrt{m} = \left(\frac{\beta}{\alpha}\right)^{1/8} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/8} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8},$$
$$\frac{3}{\sqrt{m}} = \left(\frac{\alpha}{\beta}\right)^{1/8} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/8} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/8}.$$

Entry 17.3.25 (p. 352). *If* β *and the multiplier* m *have degree* 13, *then*

$$m = \left(\frac{\beta}{\alpha}\right)^{1/4} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/4} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4} - 4\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/6},$$
$$\frac{13}{m} = \left(\frac{\alpha}{\beta}\right)^{1/4} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/4} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/4} - 4\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/6}.$$

Entry 17.3.26 (p. 352). *If* β *and the multiplier* m *have degree* 17, *then*

$$\begin{aligned}
m &= \left(\frac{\beta}{\alpha}\right)^{1/4} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/4} + \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/4} \\
&\quad - 2\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8} \left\{1 + \left(\frac{\beta}{\alpha}\right)^{1/8} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/8}\right\}, \\
\frac{17}{m} &= \left(\frac{\alpha}{\beta}\right)^{1/4} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/4} + \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/4} \\
&\quad - 2\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/8} \left\{1 + \left(\frac{\alpha}{\beta}\right)^{1/8} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/8}\right\}.
\end{aligned}$$

Entries 17.3.24, 17.3.25, and 17.3.26 are, respectively, Entries 3(x), (xi), Entries 8(iii), (iv), and Entries 12(iii), (iv) in Chapter 20 of [61, pp. 352, 376, 397–398].

Entry 17.3.27 (p. 352). *If β and the multiplier m have degree 25, then*

$$\begin{aligned}
\sqrt{m} &= \left(\frac{\beta}{\alpha}\right)^{1/8} + \left(\frac{1-\beta}{1-\alpha}\right)^{1/8} - \left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/8} - 2\left(\frac{\beta(1-\beta)}{\alpha(1-\alpha)}\right)^{1/12}, \\
\frac{5}{\sqrt{m}} &= \left(\frac{\alpha}{\beta}\right)^{1/8} + \left(\frac{1-\alpha}{1-\beta}\right)^{1/8} - \left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/8} - 2\left(\frac{\alpha(1-\alpha)}{\beta(1-\beta)}\right)^{1/12}.
\end{aligned}$$

Entry 17.3.27 is identical to Entries 15(i), (ii) in Chapter 19 of [61, p. 291].

17.4 A Fragment on Page 349

By introducing a new parameter, Q, Ramanujan found simpler forms for some old modular equations and found some new ones as well. Each degree n satisfies the congruence $n \equiv 7 \pmod{16}$. Set

$$\begin{cases}
P &= 1 - \{\alpha\beta\}^{1/8} - \{(1-\alpha)(1-\beta)\}^{1/8}, \\
Q &= \sqrt{\dfrac{1+\sqrt{\alpha\beta}+\sqrt{(1-\alpha)(1-\beta)}}{2}} \\
&\quad -\{\alpha\beta\}^{1/8} - \{(1-\alpha)(1-\beta)\}^{1/8} + \{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}, \\
R &= 4\{\alpha\beta(1-\alpha)(1-\beta)\}^{1/8}.
\end{cases} \tag{17.4.1}$$

Entry 17.4.1 (p. 349). *If β has degree 7 and P, Q, and R are defined by* (17.4.1), *then*

$$P^2 = Q = 0.$$

Both equations above are in Entry 19(i) of Chapter 19 of [61, p. 314].

Entry 17.4.2 (p. 349). *If β has degree 23 and P, Q, and R are defined by* (17.4.1), *then*

$$P^2 = Q = R^{2/3}.$$

The equality $P = R^{1/3}$ is Entry 15(i) in Chapter 20 of [61, p. 411], while the equality $P^2 = Q$, after some elementary manipulation, can be shown to be equivalent to Entry 15(ii) in Chapter 20 of [61, p. 411].

Entry 17.4.3 (p. 349). *If* P, Q, *and* R *are defined by* (17.4.1) *and* $n = 39$, *then*

$$Q(P^2 - Q) = PR.$$

Entry 17.4.4 (p. 349). *If* P, Q, *and* R *are defined by* (17.4.1) *and* $n = 55$, *then*

$$Q(P^2 - Q)^2 = R(P^3 - R).$$

Entry 17.4.5 (p. 349). *If* β *has degree* 71 *and* P, Q, *and* R *are defined by* (17.4.1), *then*

$$P^2 - Q = PR^{1/3}.$$

Entry 17.4.5 is identical to Entry 23(ii) in Chapter 20 of [61, p. 444].

Entry 17.4.6 (p. 349). *If* P, Q, *and* R *are defined by* (17.4.1) *and* $n = 119$, *then*

$$(P^2 - Q)^2 = QR^{1/3}(P - R^{1/3}).$$

At this moment, our only proofs of Entries 17.4.3, 17.4.4, and 17.4.6 require the theory of modular forms. We are grateful to Song Heng Chan for constructing these proofs. Since they are similar to other proofs in this chapter dependent on the theory of modular forms, we do not give them here.

Ramanujan also listed the numbers 103 and 167, but he did not give modular equations for these degrees.

At the top of page 349, Ramanujan wrote $n \equiv 15 \pmod{16}$, and then listed the numbers $15, 31, 47, 79, 95, 143, 191$, each indeed satisfying the given congruence. Perhaps, for these degrees Ramanujan had derived modular equations of a certain unknown type but did not record them.

18

Fragments on Lambert Series

18.1 Introduction

In a fragment published with his lost notebook [228, pp. 353–355], Ramanujan provided a list of twenty identities involving Lambert series and products or quotients of theta functions. These are immediately followed by another fragment on pages 356 and 357 with an almost identical list of twenty-one Lambert series identities. Most of these can be found in Ramanujan's second notebook [227], [61], but some are not. Several have arithmetical interpretations. These were not recorded by Ramanujan in his notebooks, but they are mentioned, although not explicitly stated, in the second of these fragments. Therefore, the purpose of this chapter is to discuss each of these Lambert series identities as well as to provide the arithmetical corollaries to which Ramanujan alluded. Several are related to the number of representations of an integer as a sum of squares or as a sum of triangular numbers. One of the most interesting identities yields a formula for the number of ways an integer can be represented as a sum of six triangular numbers. This formula is due to Jacobi [166], but outside of its appearance in H.J.S. Smith's *Report on the Theory of Numbers* [252, p. 306, formula (6)], we have been unable to find it elsewhere in the literature until very recently; see papers by V.G. Kač and M. Wakimoto [170] in 1994 and K. Ono, S. Robins, and P.T. Wahl [203] in 1995.

Let $r_k(n)$ denote the number of ways the positive integer n can be represented as a sum of k squares, with representations arising from different signs and from different orders being regarded as distinct. By convention, $r_k(0) = 1$. Also, let $t_k(n)$ denote the number of ways a positive integer n can be represented by a sum of k triangular numbers, with different orders regarded as distinct representations, and with $t_k(0)$ defined to be 1. Recall Ramanujan's definitions of the theta functions $\varphi(q)$ and $\psi(q)$,

$$\varphi(q) := \sum_{n=-\infty}^{\infty} q^{n^2} \qquad \text{and} \qquad \psi(q) := \sum_{n=0}^{\infty} q^{n(n+1)/2},$$

and the generating functions for $r_k(n)$ and $t_k(n)$,

$$\varphi^k(q) = \sum_{n=0}^{\infty} r_k(n) q^n \qquad \text{and} \qquad \psi^k(q) = \sum_{n=0}^{\infty} t_k(n) q^n,$$

where $|q| < 1$. We shall also need Ramanujan's definition

$$f(-q) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2}, \qquad |q| < 1.$$

The most complete bibliography of formulas for sums of an even number of squares can be found in S.C. Milne's paper [200]. In our citations, we have focused on papers establishing formulas for $r_{2k}(n)$ that have appeared since the publication of [200].

Lambert series identities may be derived by a variety of methods. We do not know how Ramanujan proceeded, but it seems likely that he used the results in Sections 33 and 34 in Chapter 16 of his second notebook [61, pp. 52–61] and his ${}_1\psi_1$ summation formula [61, pp. 32–34]. A. Cauchy [108], [109, pp. 55-64] and others have employed contour integration. Systematic derivations of large classes of Lambert series identities have been carried out by L.–C. Shen [243], [244].

The numberings below are those given by Ramanujan in the two fragments. Apparently, the first fragment is a rough draft of a section that he planned to put in a paper, while the second fragment seems to be a final draft of a section of a proposed paper.

18.2 Entries from the Two Fragments

Entry 18.2.1 (formula (3.12), p. 356).

$$\varphi^4(q) = 1 + 8 \sum_{n=1}^{\infty} \frac{nq^n}{1 + (-q)^n}.$$

Entry 18.2.1 is the same as Entry 8(ii) in Chapter 17 of Ramanujan's second notebook [61, p. 114]. It is well known and easy to prove that Entry 18.2.1 is equivalent to a formula of Jacobi [166] for $r_4(n)$, namely,

$$r_4(n) = 8 \sum_{\substack{d \mid n \\ 4 \nmid d}} d. \tag{18.2.1}$$

A very short proof of this formula can be found in [63, p. 377].

Entry 18.2.2 (formula (1.13), p. 353; formula (3.13), p. 356).

$$\varphi^6(q) = 1 - 4 \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)^2 q^{2n+1}}{1 - q^{2n+1}} + 16 \sum_{n=1}^{\infty} \frac{n^2 q^n}{1 + q^{2n}}.$$

Entry 18.2.2 is equivalent to another theorem of Jacobi [166],

$$r_6(n) = 4 \sum_{\substack{d|n \\ d \text{ odd}}} (-1)^{(d-1)/2} \left\{ \left(\frac{2n}{d} \right)^2 - d^2 \right\}.$$

An especially elegant and elementary proof of this classical formula has been given by S.H. Chan [119].

Entry 18.2.3 (formula (1.14), p. 353; formula (3.14), p. 356).

$$\varphi^8(q) = 1 + 16 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - (-q)^n}.$$

Entry 18.2.3 was given by Ramanujan as Example (i) in Section 17 of Chapter 17 in his second notebook [61, p. 139]. Entry 18.2.3 is also equivalent to Jacobi's famous formula [166]

$$r_8(n) = 16(-1)^n \sum_{d|n} (-1)^d d^3. \tag{18.2.2}$$

Elegant elementary proofs of (18.2.1) and (18.2.2) have been given by J.–F. Lin [178], and by B. Spearman and K.S. Williams [256] and Williams [295].

Many authors, including Ramanujan [224], have discovered formulas for $r_{2k}(n)$ for certain values of k. For very comprehensive lists of references to the classical literature on $r_{2k}(n)$, see the papers [199], [200] by Milne, in which he develops general methods for deriving infinite families of formulas for $r_{2k}(n)$. Kač and Wakimoto [170], Ono [202], and D. Zagier [300] have also found infinite families of formulas for $r_{2k}(n)$. H.H. Chan and K.S. Chua [114] discovered an elegant formula for $r_{32}(n)$.

Entry 18.2.4 (formula (3.21), p. 356).

$$\psi^2(q^4) = \sum_{n=0}^{\infty} \frac{(-1)^n q^{2n}}{1 - q^{4n+2}}.$$

We have replaced q by q^4 in Ramanujan's formulation.

It is easy to show that Entry 18.2.4 is equivalent to the representation

$$\psi^2(q^2) = \sum_{n=0}^{\infty} \frac{q^n}{1 + q^{2n+1}},$$

which is Example (iv) in Section 17 of Chapter 17 of Ramanujan's second notebook [61, p. 139].

Let $d_j(n)$ denote the number of positive divisors of the positive integer n that are congruent to j modulo 4. Then Entry 18.2.4 is equivalent to the arithmetical assertion

$$t_2(n) = d_1(4n+1) - d_3(4n+1). \tag{18.2.3}$$

Entry 18.2.4 can be easily derived from the fundamental identity

$$8q\psi^2(q^4) = \varphi^2(q) - \varphi^2(-q)$$

of Jacobi [166], found as Entry 25(v) of Chapter 16 in Ramanujan's second notebook [61, p. 40], and from a well-known Lambert series representation for $\varphi^2(q)$, [61, p. 114, Entry 8(v)]. Therefore, (18.2.3) also follows from Jacobi's well known theorem

$$r_2(n) = 4\left(d_1(n) - d_3(n)\right).$$

Entry 18.2.5 (formula (3.22), p. 356).

$$q\psi^4(q^2) = \sum_{n=0}^{\infty} \frac{(2n+1)q^{2n+1}}{1-q^{4n+2}}.$$

This is Example (iii) in Section 17 of Chapter 17 in Ramanujan's second notebook [61, p. 139]. Arithmetically, Entry 18.2.5 is equivalent to the beautiful theorem, due to Legendre [175, p. 133],

$$t_4(n) = \sigma(2n+1),$$

where $\sigma(n)$ denotes the sum of all positive divisors of n. Proofs of Entry 18.2.5 have also been given by Cauchy [108, p. 572], [109, p. 64] and Plana [207, p. 147]. Jacobi [166] claimed that Bouniakowsky first proved Entry 18.2.5, but he did not give a reference.

Entry 18.2.6 (formula (3.23), p. 356).

$$q^{3/2}\psi^6(q^2) = \frac{1}{16}\sum_{n=0}^{\infty} \frac{(2n+1)^2 q^{(2n+1)/2}}{1+q^{2n+1}} - \frac{1}{16}\sum_{n=0}^{\infty} \frac{(-1)^n(2n+1)^2 q^{(2n+1)/2}}{1-q^{2n+1}}.$$

Since the only known proofs of Entry 18.2.6 after Jacobi are nonclassical—in particular, the proof of Kač and Wakimoto [170] employs Lie algebras, and that of Ono, Robins, and Wahl [203] utilizes modular forms—we give here a proof in the spirit of Ramanujan. Furthermore, Entry 18.2.6 was not given by Ramanujan in his notebooks.

Recall Ramanujan's notation [61, p. 101]

$$x = k^2, \qquad z = {}_2F_1(\tfrac{1}{2}, \tfrac{1}{2}; 1; x),$$

and

$$y = \pi \frac{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; 1-x)}{{}_2F_1(\frac{1}{2}, \frac{1}{2}; 1; x)},$$

where k, $0 < k < 1$, denotes the *modulus* and ${}_2F_1$ denotes the ordinary hypergeometric function.

We need only the second formula in Lemma 18.2.1 below, but our proof yields at once a sequence of results of this sort, and so, in the spirit of Entries 13–17 of Chapter 17, we give the first five formulas of this type.

Lemma 18.2.1. *With x, y, and z defined above,*

$$\begin{aligned}
\sum_{n=0}^{\infty} \frac{(-1)^n}{\sinh\{\frac{1}{2}(2n+1)y\}} &= \frac{z\sqrt{x}}{2}, \\
\sum_{n=0}^{\infty} \frac{(-1)^n(2n+1)^2}{\sinh\{\frac{1}{2}(2n+1)y\}} &= \frac{z^3(1-x)\sqrt{x}}{2}, \\
\sum_{n=0}^{\infty} \frac{(-1)^n(2n+1)^4}{\sinh\{\frac{1}{2}(2n+1)y\}} &= \frac{z^5(1-6x+5x^2)\sqrt{x}}{2}, \\
\sum_{n=0}^{\infty} \frac{(-1)^n(2n+1)^6}{\sinh\{\frac{1}{2}(2n+1)y\}} &= \frac{z^7(1-x)(1-46x+61x^2)\sqrt{x}}{2}, \\
\sum_{n=0}^{\infty} \frac{(-1)^n(2n+1)^8}{\sinh\{\frac{1}{2}(2n+1)y\}} &= \frac{z^9(1-x)(1-411x+1731x^2-1385x^3)\sqrt{x}}{2}.
\end{aligned}$$

Proof. The Jacobian elliptic function $\operatorname{cd}(zt)$ has the Maclaurin series expansion

$$\begin{aligned}
\operatorname{cd}(zt) = 1 &+ (x-1)\frac{(zt)^2}{2!} + (1-6x+5x^2)\frac{(zt)^4}{4!} \\
&+ (x-1)(1-46x+61x^2)\frac{(zt)^6}{6!} \qquad (18.2.4) \\
&+ (x-1)(-1+411x-1731x^2+1385x^3)\frac{(zt)^8}{8!} + \cdots,
\end{aligned}$$

which we generated with *Mathematica.* On the other hand, if $q = \exp(-y)$ [292, p. 511],

$$\begin{aligned}
\operatorname{cd}(zt) &= \frac{4}{z\sqrt{x}} \sum_{n=0}^{\infty} \frac{(-1)^n q^{n+1/2}\cos(2n+1)t}{1-q^{2n+1}} \\
&= \frac{2}{z\sqrt{x}} \sum_{n=0}^{\infty} \frac{(-1)^n \cos(2n+1)t}{\sinh\{\frac{1}{2}(2n+1)y\}} \\
&= \frac{2}{z\sqrt{x}} \sum_{n=0}^{\infty} \frac{(-1)^n}{\sinh\{\frac{1}{2}(2n+1)y\}} \sum_{j=0}^{\infty} \frac{(-1)^j(2n+1)^{2j}t^{2j}}{(2j)!}
\end{aligned}$$

$$= \frac{2}{z\sqrt{x}} \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} \left(\sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)^{2j}}{\sinh\{\frac{1}{2}(2n+1)y\}} \right) t^{2j}. \tag{18.2.5}$$

Equating coefficients of t^{2m}, $0 \leq m \leq 4$, in (18.2.4) and (18.2.5), we derive the five equalities of Lemma 18.2.1. □

Proof of Entry 18.2.6. By Entry 11(iii) in Chapter 17 of Ramanujan's second notebook [61, p. 123],

$$q^{3/2}\psi^6(q^2) = \tfrac{1}{64} z^3 x^{3/2}. \tag{18.2.6}$$

On the other hand, by Entry 16(x) in Chapter 17 of Ramanujan's second notebook [61, p. 134] and by Lemma 18.2.1 above,

$$\begin{aligned}
&\frac{1}{16} \sum_{n=0}^{\infty} \frac{(2n+1)^2 q^{(2n+1)/2}}{1+q^{2n+1}} - \frac{1}{16} \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)^2 q^{(2n+1)/2}}{1-q^{2n+1}} \\
&= \frac{1}{32} \sum_{n=0}^{\infty} \frac{(2n+1)^2}{\cosh\{\frac{1}{2}(2n+1)y\}} - \frac{1}{32} \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)^2}{\sinh\{\frac{1}{2}(2n+1)y\}} \\
&= \tfrac{1}{64} z^3 \sqrt{x} - \tfrac{1}{64} z^3 (1-x) \sqrt{x} = \tfrac{1}{64} z^3 x^{3/2}.
\end{aligned} \tag{18.2.7}$$

Comparing (18.2.6) and (18.2.7), we see that we have completed the proof. □

Entry 18.2.6 has a beautiful arithmetical interpretation, which we now give.

Corollary 18.2.1.

$$t_6(n) = \frac{1}{8} \sum_{\substack{d|(4n+3) \\ d \equiv 3 \pmod 4}} d^2 - \frac{1}{8} \sum_{\substack{d|(4n+3) \\ d \equiv 1 \pmod 4}} d^2.$$

Proof. From Entry 18.2.6,

$$\begin{aligned}
q^{3/4}\psi^6(q) &= \frac{1}{16} \sum_{n=0}^{\infty} \frac{(2n+1)^2 q^{(2n+1)/4}}{1+q^{(2n+1)/2}} \\
&\quad - \frac{1}{16} \sum_{n=0}^{\infty} \frac{(-1)^n (2n+1)^2 q^{(2n+1)/4}}{1-q^{(2n+1)/2}} \\
&= \frac{1}{16} \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} (-1)^j (2n+1)^2 q^{(2n+1)(2j+1)/4} \\
&\quad - \frac{1}{16} \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} (-1)^n (2n+1)^2 q^{(2n+1)(2j+1)/4}.
\end{aligned} \tag{18.2.8}$$

We divide each double sum on the far right side of (18.2.8) into four cases according to the parities of n and j. The series with n and j of the same parity cancel. The corresponding sums in the remaining two cases are equal. We thus find that

$$q^{3/4}\psi^6(q) = \frac{1}{8}\sum_{\substack{m=0\\ m\equiv 3\,(\mathrm{mod}\,4)}}^{\infty}\Bigg(\sum_{\substack{d|m\\ d\equiv 3\,(\mathrm{mod}\,4)}} d^2\Bigg)q^{m/4} - \frac{1}{8}\sum_{\substack{m=0\\ m\equiv 3\,(\mathrm{mod}\,4)}}^{\infty}\Bigg(\sum_{\substack{d|m\\ d\equiv 1\,(\mathrm{mod}\,4)}} d^2\Bigg)q^{m/4}.$$

Equating coefficients of q^n on both sides above, we complete the proof. □

The formulations of Corollary 18.2.1 by Kač and Wakimoto [170, p. 444] and Ono, Robins, and Wahl [203, p. 81] are slightly different.

The reader can easily see that several of the formulas so far presented in this chapter can be grouped into pairs of very similar formulas. This is not accidental; each can be derived from the other. An excellent explanation of this observation has been given by H.H. Chan [113]. In his proofs, Chan utilized the Hecke correspondence between Dirichlet series and Fourier expansions of modular forms. In an unpublished manuscript, Yu Yang Liu [179] has continued along the lines of Chan and has shown that Entries 18.2.2 and 18.2.6 are equivalent. In his proof, the transformation formulas for φ and ψ are needed; the functional equations of the Riemann zeta function and the Dirichlet L-function $\sum_{n=0}^{\infty}(-1)^n(2n+1)^{-s}$, Re $s>0$, are also used.

Entry 18.2.7 (formula (3.24), p. 356).

$$q\psi^8(q) = \sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^{2n}}.$$

Entry 18.2.7 is identical to Example (ii) in Section 17 of Chapter 17 in Ramanujan's second notebook [61, p. 139]. We let the reader show as an exercise that Entry 18.2.7 is equivalent to the elegant arithmetical formulation

$$t_8(n) = \sum_{\substack{d|(n+1)\\ d\ \mathrm{odd}}}\left(\frac{n+1}{d}\right)^3.$$

Entry 18.2.7 and its arithmetical equivalent are due to Legendre [175, p. 133].

After the seven entries above, in the second fragment, Ramanujan writes, "These are of course well known formulae for the number of representations of a number as the sum of 2, 4, 6, and 8 squares or triangular numbers. There are also various other arithmetical problems in which the partition method gives the actual value. I shall quote a few examples and reserve the discussion of these to another paper." (This "another paper" was apparently never written.) Possibly the foregoing seven equalities were intended to be put in the paper [224], [226, pp. 179–199], for in this paper Ramanujan offers a general approach for deriving formulas for $r_{2k}(n)$ and $t_{2k}(n)$. However, he does not explicitly work out the details for any given case. It would seem worthwhile to more fully develop the details omitted by Ramanujan in this paper.

For systematic derivations of several formulas for $t_k(n)$, see the papers by Kač and Wakimoto [170], Ono, Robins, and Wahl [203], Milne [200], and Z.–G. Liu [180].

For the next four entries, let

$$a(q) := \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2}.$$

In fact, Ramanujan uses the notation S instead of $a(q)$, which is the notation introduced by J.M. and P.B. Borwein [100] for one of their "cubic" theta functions. The function $a(q)$ plays a central role in Ramanujan's theory of elliptic functions to the alternative cubic base [66], [63, Chapter 33].

Entry 18.2.8 (formula (1.81), p. 355; formula (3.31), p. 356). *If $\left(\frac{n}{3}\right)$ denotes the Legendre symbol, then*

$$a(q) = 1 + 6\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\frac{q^n}{1-q^n}.$$

Entry 18.2.8 is identical to equation (2.6) in Chapter 33 of [63, p. 93]. If $r(n)$ denotes the number of representations of the positive integer n by the quadratic form $j^2 + jk + k^2$ and if $d_{m,3}(n)$ $(m = 1, 2)$ denotes the number of positive divisors of n of the form $3\ell + m$, then Entry 18.2.8 implies that

$$r(n) = 6\left(d_{1,3}(n) - d_{2,3}(n)\right),$$

which is due to P.G.L. Dirichlet [133]. For further historical references to Entry 18.2.8 and this arithmetical identity, see the book by Berndt and Rankin [81, p. 199].

Entry 18.2.9 (formula (1.82), p. 355; formula (3.32), p. 356). *If χ_0 denotes the principal character modulo* 3*, then*

$$a^2(q) = 1 + 12\sum_{n=1}^{\infty}\chi_0(n)\frac{nq^n}{1-q^n}.$$

Entry 18.2.9 is contained in Entry 3(i) of Chapter 21 of Ramanujan's second notebook [61, p. 460]. See also [63, p. 100, Corollary 2.11].

Entry 18.2.10 (formula (1.83), p. 355; formula (3.33), p. 356). *If $\left(\frac{n}{3}\right)$ denotes the Legendre symbol, then*

$$a^3(q) = 1 - 9\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\frac{n^2q^n}{1-q^n} + 27\sum_{n=1}^{\infty}\frac{n^2q^n}{1+q^n+q^{2n}}. \tag{18.2.9}$$

Proof. By Theorem 8.7 in Chapter 33 of [63, p. 143],

$$27\sum_{n=1}^{\infty}\frac{n^2q^n}{1+q^n+q^{2n}}=27\frac{x}{27}a^3(q)=xa^3(q),\tag{18.2.10}$$

where x is the square root of the modulus in Ramanujan's cubic theory of elliptic functions. Furthermore, by Lemma 14.2.5 in Chapter 14,

$$b^3(q):=\frac{f^9(-q)}{f^3(-q^3)}=1-9\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\frac{n^2q^n}{1-q^n},\tag{18.2.11}$$

where $b(q)$ is another of the cubic theta functions (see [66, Section 2] or [63, p. 93]). By Corollary 3.2 of Chapter 33 in [63, p. 102],

$$b^3(q)=(1-x)a^3(q).\tag{18.2.12}$$

Thus, by (18.2.10)–(18.2.12), the right side of (18.2.9) is equal to

$$(1-x)a^3(q)+xa^3(q)=a^3(q),$$

as claimed by Ramanujan in (18.2.9). □

Entry 18.2.11 (formula (1.84), p. 355; formula (3.34), p. 356). *We have*

$$a^4(q)=1+24\sum_{n=1}^{\infty}\frac{n^3q^n}{1-q^n}+8\sum_{n=1}^{\infty}\frac{(3n)^3q^{3n}}{1-q^{3n}}.$$

Entry 18.2.11 is contained in Entry 3(i) of Chapter 21 in Ramanujan's second notebook [61, p. 460].

Note that Entries 18.2.9–18.2.11 yield formulas for the numbers of ways a positive integer n can be represented as a sum of 2, 3, and 4 numbers, respectively, of the form j^2+jk+k^2. For a comprehensive list of such formulas, see a paper by G.A. Lomadze [181].

Entry 18.2.12 (formula (1.71), p. 354). *If* $\left(\frac{n}{7}\right)$ *denotes the Legendre symbol, then*

$$\varphi(q)\varphi(q^7)=1+2\sum_{n=1}^{\infty}\left(\frac{n}{7}\right)\frac{q^n}{1-(-q)^n}.$$

Proof. Observe that

$$\begin{aligned}\sum_{n=1}^{\infty}\left(\frac{n}{7}\right)\frac{q^n}{1-(-q)^n}&=\sum_{n=1}^{\infty}\left(\frac{2n}{7}\right)\frac{q^{2n}}{1-q^{2n}}+\sum_{n=1}^{\infty}\left(\frac{2n-1}{7}\right)\frac{q^{2n-1}}{1+q^{2n-1}}\\&=\sum_{n=1}^{\infty}\left(\frac{n}{7}\right)\frac{q^{2n}}{1-q^{2n}}\qquad\qquad(18.2.13)\\&\quad+\sum_{n=1}^{\infty}\left(\frac{2n-1}{7}\right)\left\{\frac{q^{2n-1}}{1-q^{2n-1}}-\frac{2q^{4n-2}}{1-q^{4n-2}}\right\}.\end{aligned}$$

By carefully examining the coefficients of $q^n/(1-q^n)$ above modulo 28, we see that (18.2.13) is in agreement with Entry 17(ii) in Chapter 19 of the second notebook [61, p. 302]. This completes the proof. $\square$

Entry 18.2.13 (formula (1.72), p. 355). *If $\left(\frac{n}{7}\right)$ denotes the Legendre symbol, then*

$$q\psi(q)\psi(q^7) = \sum_{n=1}^{\infty} \left(\frac{2n-1}{7}\right) \frac{q^{2n-1}}{1-q^{2n-1}}.$$

Entry 18.2.13 is identical to Entry 17(i) in Chapter 19 in Ramanujan's second notebook [61, p. 302].

The next two entries involve an analogue of $a(q)$, namely,

$$T(q) := \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+2n^2}.$$

Entry 18.2.14 (formula (1.91), p. 355; formula (3.41), p. 357). *If $\left(\frac{n}{7}\right)$ denotes the Legendre symbol, then*

$$T(q) = 1 + 2\sum_{n=1}^{\infty} \left(\frac{n}{7}\right) \frac{q^n}{1-q^n}. \tag{18.2.14}$$

Proof. For each even integer n, set $n = 2j$. Then

$$m^2 + mn + 2n^2 = m^2 + mn + \tfrac{1}{4}n^2 + \tfrac{7}{4}n^2 = (m+j)^2 + 7j^2.$$

For each odd integer n, set $n = 2j+1$. Then

$$\begin{aligned} m^2 + mn + 2n^2 &= m^2 + mn + \tfrac{1}{4}n^2 + \tfrac{7}{4}n^2 \\ &= (m+j+\tfrac{1}{2})^2 + 7(j+\tfrac{1}{2})^2 \\ &= (m+j)(m+j+1) + 7j(j+1) + 2. \end{aligned}$$

Thus,

$$\begin{aligned} T(q) &= \sum_{m,j=-\infty}^{\infty} q^{(m+j)^2+7j^2} + \sum_{m,j=-\infty}^{\infty} q^{(m+j)(m+j+1)+7j(j+1)+2} \\ &= \sum_{m,j=-\infty}^{\infty} q^{m^2+7j^2} + \sum_{m,j=-\infty}^{\infty} q^{m(m+1)+7j(j+1)+2} \\ &= \varphi(q)\varphi(q^7) + 4q^2\psi(q^2)\psi(q^{14}). \end{aligned} \tag{18.2.15}$$

Using Entries 18.2.12 and 18.2.13 in (18.2.15), we deduce that

$$\begin{aligned} T(q) &= 1 + 2\sum_{n=1}^{\infty} \left(\frac{n}{7}\right) \frac{q^n}{1-(-q)^n} + 4\sum_{n=1}^{\infty} \left(\frac{2n-1}{7}\right) \frac{q^{2n-1}}{1-q^{2n-1}} \\ &= 1 + 2\sum_{n=1}^{\infty} \left(\frac{n}{7}\right) \frac{q^n}{1-q^n}, \end{aligned}$$

by the same argument that we used in (18.2.13). This completes the proof. $\square$

Entry 18.2.14 has an elegant arithmetic interpretation. Let $d_{1,2,4}(n)$ and $d_{3,5,6}(n)$ denote the numbers of divisors of the positive integer n that are congruent to either 1, 2, or 4 (mod 7) and to either 3, 5, or 6 (mod 7), respectively. Then, if $r(n)$ denotes the number of representations of n by the quadratic form $j^2 + jk + 2k^2$,

$$r(n) = 2\left(d_{1,2,4}(n) - d_{3,5,6}(n)\right),$$

which is originally due to Dirichlet [133].

Entry 18.2.15 (formula (1.92), p. 355; formula (3.42), p. 357). *If* $T(q)$ *is defined by* (18.2.14) *and if* $\chi_0(n)$ *denotes the principal character modulo* 7, *then*

$$T^2(q) = 1 + 4\sum_{n=1}^{\infty} \chi_0(n)\frac{nq^n}{1-q^n}.$$

Entry 18.2.15 is contained in Entry 5(i) of Chapter 21 in Ramanujan's second notebook [61, p. 467]. Entry 18.2.15 yields a formula for the number of representations of a positive integer n as a sum of two numbers of the form $j^2 + jk + 2k^2$.

Entry 18.2.16 (formula (1.21), p. 353; formula (3.51), p. 357). *If* $\left(\frac{n}{3}\right)$ *denotes the Legendre symbol, then*

$$\frac{\varphi^3(-q)}{\varphi(-q^3)} = 1 - 6\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\frac{q^n}{1+q^n}.$$

Entry 18.2.16 is identical to Entry 4(iv) in Chapter 19 in Ramanujan's second notebook [61, p. 227].

Entry 18.2.17 (formula (1.22), p. 353; formula (3.52), p. 357). *If* $\left(\frac{n}{3}\right)$ *denotes the Legendre symbol, then*

$$\frac{\varphi^3(q^3)}{\varphi(q)} = 1 - 2\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\frac{q^n}{1-(-q)^n}.$$

Entry 18.2.17 is due to Ramanujan in his notebooks [227]; see Entry 35 in Chapter 36 of [63, p. 375].

Entry 18.2.18 (formula (1.31), p. 353; formula (3.61), p. 357). *We have*

$$\frac{\psi^3(q)}{\psi(q^3)} = 1 + 3\sum_{n=0}^{\infty}\left(\frac{q^{6n+1}}{1-q^{6n+1}} - \frac{q^{6n+5}}{1-q^{6n+5}}\right).$$

Entry 18.2.18 is the same as Entry 4(iii) in Chapter 19 of Ramanujan's second notebook [61, p. 226].

Entry 18.2.19 (formula (1.32), p. 353; formula (3.62), p. 357). *If* $\left(\frac{n}{3}\right)$ *denotes the Legendre symbol, then*

$$\frac{q\psi^3(q^3)}{\psi(q)} = \sum_{n=1}^{\infty} \left(\frac{n}{3}\right) \frac{q^n}{1-q^{2n}}.$$

Entry 18.2.19 can also be found in Ramanujan's notebooks; see Entry 34 in Chapter 36 of [63, p. 374].

Entry 18.2.20 (formula (1.41), pp. 353–354; formula (3.71), p. 357). *If* $\left(\frac{n}{3}\right)$ *denotes the Legendre symbol, then*

$$\frac{f^3(-q)}{f(-q^3)} = 1 - 3\sum_{n=1}^{\infty} \left(\frac{n}{3}\right) \frac{q^n}{1-q^n} + 9\sum_{n=1}^{\infty} \left(\frac{n}{3}\right) \frac{q^{3n}}{1-q^{3n}}.$$

Entry 18.2.21 (formula (1.42), p. 354; formula (3.72), p. 357). *If* $\left(\frac{n}{3}\right)$ *denotes the Legendre symbol, then*

$$\frac{qf^3(-q^9)}{f(-q^3)} = \sum_{n=1}^{\infty} \left(\frac{n}{3}\right) \frac{q^n}{1-q^n} - \sum_{n=1}^{\infty} \left(\frac{n}{3}\right) \frac{q^{3n}}{1-q^{3n}}.$$

Proof of Entries 18.2.20 *and* 18.2.21. Recall from Lemma 5.1, (2.8), and (2.9) of Chapter 33 in [63, pp. 109, 93–94] that the cubic theta functions $b(q)$ and $c(q)$ have the representations

$$b(q) = \frac{f^3(-q)}{f(-q^3)} = \tfrac{1}{2}\left\{3a(q^3) - a(q)\right\} \tag{18.2.16}$$

and

$$c(q) = 3q^{1/3}\frac{f^3(-q^3)}{f(-q)} = \tfrac{1}{2}\left\{a(q^{1/3}) - a(q)\right\}, \tag{18.2.17}$$

respectively. If we now use Entry 18.2.8 in (18.2.16), we easily complete the proof of Entry 18.2.20. After replacing q by q^3 in (18.2.17) and employing Entry 18.2.8, we easily deduce Entry 18.2.21. □

Entry 18.2.22 (formula (1.51), p. 354; formula (3.81), p. 357). *If* $\left(\frac{n}{5}\right)$ *denotes the Legendre symbol, then*

$$\frac{f^5(-q)}{f(-q^5)} = 1 - 5\sum_{n=1}^{\infty} \left(\frac{n}{5}\right) \frac{nq^n}{1-q^n}.$$

Entry 18.2.22 can be found as Entry 9 in Chapter 19 of Ramanujan's second notebook [61, p. 257].

Entry 18.2.23 (formula (1.52), p. 354; formula (3.82), p. 357). *If* $\left(\frac{n}{5}\right)$ *denotes the Legendre symbol, then*

$$q\frac{f^5(-q^5)}{f(-q)} = \sum_{n=1}^{\infty} \left(\frac{n}{5}\right) \frac{q^n}{(1-q^n)^2}.$$

Entry 18.2.23 is a very famous result of Ramanujan that leads to the celebrated Ramanujan congruence $p(5n+4) \equiv 0 \pmod 5$ for the partition function $p(n)$. A very natural proof of Entry 18.2.23 has been given by H.H. Chan [111]. References to several other proofs may be found in the third printing of Ramanujan's *Collected Papers* [226].

Entry 18.2.24 (formula (1.61), p. 354). *If* $\left(\frac{n}{3}\right)$ *denotes the Legendre symbol, then*

$$\varphi(q)\varphi(q^3) = 1 + 2\sum_{n=1}^{\infty} \left(\frac{n}{3}\right) \frac{q^n}{1+(-q)^n}.$$

Entry 18.2.24 is the same as Entry 3(ii) in Chapter 19 of the second notebook [61, p. 223].

M.S. Mahadeva Naika and H.S. Madhusudhan [192] have found a common generalization for Entries 18.2.4, 18.2.8, 18.2.16–18.2.19, and 18.2.24.

Entry 18.2.25 (formula (1.62), p. 354). *If* $\chi_0(n)$ *denotes the principal character modulo* 3, *then*

$$\varphi^2(q)\varphi^2(q^3) = 1 + 4\sum_{n=1}^{\infty} \chi_0(n)\frac{nq^n}{1-(-q)^n}.$$

Proof. Using the elementary identity

$$\frac{nq^n}{1+q^n} = \frac{nq^n}{1-q^n} - \frac{2nq^{2n}}{1-q^{2n}}$$

twice in the second equality below, we find that

$$\begin{aligned}
\sum_{n=1}^{\infty} \chi_0(n)\frac{nq^n}{1-(-q)^n} &= \sum_{n=0}^{\infty} \frac{(6n+2)q^{6n+2}}{1-q^{6n+2}} + \sum_{n=0}^{\infty} \frac{(6n+4)q^{6n+6}}{1-q^{6n+4}} \\
&\quad + \sum_{n=0}^{\infty} \frac{(6n+1)q^{6n+1}}{1+q^{6n+1}} + \sum_{n=0}^{\infty} \frac{(6n+5)q^{6n+5}}{1+q^{6n+5}} \\
&= \sum_{n=0}^{\infty} \frac{(6n+2)q^{6n+2}}{1-q^{6n+2}} + \sum_{n=0}^{\infty} \frac{(6n+4)q^{6n+6}}{1-q^{6n+4}} \\
&\quad + \sum_{n=0}^{\infty} \frac{(6n+1)q^{6n+1}}{1-q^{6n+1}} - \sum_{n=0}^{\infty} \frac{(12n+2)q^{12n+2}}{1-q^{12n+2}} \\
&\quad + \sum_{n=0}^{\infty} \frac{(6n+5)q^{6n+5}}{1-q^{6n+5}} - \sum_{n=0}^{\infty} \frac{(12n+10)q^{12n+10}}{1-q^{12n+10}}.
\end{aligned}$$

It is now easy to see from the latter formula that the proposed formula in Entry 18.2.25 is equivalent to that of Entry 3(iv) of Chapter 19 in Ramanujan's second notebook [61, p. 223]. □

A systematic approach, via the theory of modular forms, for generating certain types of Lambert series identities has been given by O. Kolberg [173].

The arithmetic identities described in this chapter have been placed in the much more general context of convolutions of character sums and the values of Hecke L-series by V.A. Bykovsky [106].

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