Problem set 2023
Sebastián Caballero 30 Days of ML

1 Systems of linear equations, Matrices, and vector spaces, rank of matrices, Linear maps, and affine spaces

Problem 1.

We consider $(\mathbb{R} \setminus \{-1\}, \star)$ where:

$$a \star b = a + ab + b$$

for $a, b \in \mathbb{R} \setminus \{-1\}$. Show that this is an abelian group and solve $3 \star x \star x = 15$.

Solution. We prove this showing that the five axioms are true:

- First, suppose that $a + ab + b = -1$ and so we would have that $a(1 + b) + b = -1$, and so we have $a(1 + b) = -(1 + b)$ and since $b \neq 1$ we can assure that $1 + b \neq 0$ so $a = -1$ which is a contradiction to the hypothesis that $a \neq 1$, so $\star$ is closed under the set.

- Now, we want to prove associativity(We pay attention to the associativity in the real numbers)

$$\begin{aligned} a \star (b \star c) &= a \star (b + bc + c) \\ &= a + a(b + bc + c) + (b + bc + c) \\ &= a + ab + abc + ac + b + bc + c \end{aligned} \qquad \begin{aligned} (a \star b) \star c &= (a + ab + b) \star c \\ &= (a + ab + b) + c(a + ab + b) + c \\ &= a + ab + b + ac + abc + bc + c \end{aligned}$$

  and so it is clear that they are the same.

- Note that the element 0 is an identity element because:

$$\begin{aligned} a \star 0 &= a + a \cdot 0 + 0 \\ &= a \end{aligned} \qquad \begin{aligned} 0 \star a &= 0 + a \cdot 0 + a \\ &= a \end{aligned}$$

- The inverse element for x is $\frac{-x}{1+x}$ since:

$$\begin{aligned} x \star \frac{-x}{1+x} &= x - x \cdot \frac{x}{1+x} - \frac{x}{1+x} \\ &= x - \frac{x^2}{1+x} - \frac{x}{1+x} \\ &= \frac{x + x^2}{1+x} - \frac{x^2}{1+x} - \frac{x}{1+x} \\ &= 0 \end{aligned}$$

  And the commutated case is the same, so it has inverses.

- The commutativity is a consequence of these properties in $\mathbb{R}$:

$$\begin{aligned} a \star b &= a + ab + b \\ &= b + ba + a \\ &= b \star a \end{aligned}$$

And so we conclude that $(\mathbb{R}\setminus\{-1\},\star)$ is an abelian group. For the equation, we do:

$$\begin{aligned}
3\star x\star x &= 15\\
3\star(2x+x^2) &= 15\\
3+3(2x+x^2)+(2x+x^2) &= 15\\
6x+3x^2+2x+x^2 &= 12\\
4x^2+8x &= 12\\
x^2+2x &= 3\\
x^2+2x-3 &= 0\\
(x-1)(x+3) &= 0
\end{aligned}$$

And therefore we conclude that $x=1$ or $x=-3$. If we put this into the equation we have:

$$\begin{aligned}
3\star 1\star 1 &= 3\star 3\\
&= 3+9+3\\
&= 15
\end{aligned}
\qquad\qquad
\begin{aligned}
3\star(-3\star(-3)) &= 3\star 3\\
&= 3+9+3\\
&= 15
\end{aligned}$$

So the solutions are $x=1$ and $x=-3$.

Problem 2.

Consider the set $\mathcal{G}$ of 3×3 matrices defined as follows:

$$\mathcal{G}=\left\{\begin{bmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{bmatrix}\in\mathbb{R}^{3\times 3}\,\middle|\, x,y,z\in\mathbb{R}\right\}$$

And we define $\cdot$ as the standard matrix multiplication. Is $(\mathcal{G},\cdot)$ a group? If yes, is it abelian?

Solution. We prove this showing that the four axioms are true:

- First, it is closed under $\cdot$ since:

  $$\begin{bmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{bmatrix}\cdot\begin{bmatrix}1 & a & c\\ 0 & 1 & b\\ 0 & 0 & 1\end{bmatrix}=\begin{bmatrix}1 & a+x & c+bx+z\\ 0 & 1 & b+y\\ 0 & 0 & 1\end{bmatrix}$$

  And since the operations of addition and product of real numbers are closed, by definition the matrix is also in $\mathcal{G}$ so it is a closed operation.

- We can prove the associativity by taking three matrices and show that their product don't vary.

  $$\begin{aligned}
  \left(\begin{bmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{bmatrix}\cdot\begin{bmatrix}1 & a & c\\ 0 & 1 & b\\ 0 & 0 & 1\end{bmatrix}\right)\cdot\begin{bmatrix}1 & k & m\\ 0 & 1 & n\\ 0 & 0 & 1\end{bmatrix} &= \begin{bmatrix}1 & a+x & c+bx+z\\ 0 & 1 & b+y\\ 0 & 0 & 1\end{bmatrix}\cdot\begin{bmatrix}1 & a+x & c+bx+z\\ 0 & 1 & b+y\\ 0 & 0 & 1\end{bmatrix}\\
  &= \begin{bmatrix}1 & a+k+x & m+an+ax+c+bx+z\\ 0 & 1 & b+n+y\\ 0 & 0 & 1\end{bmatrix}
  \end{aligned}$$

  And if we make the other option:

  $$\begin{aligned}
  \begin{bmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{bmatrix}\cdot\left(\begin{bmatrix}1 & a & c\\ 0 & 1 & b\\ 0 & 0 & 1\end{bmatrix}\cdot\begin{bmatrix}1 & k & m\\ 0 & 1 & n\\ 0 & 0 & 1\end{bmatrix}\right) &= \begin{bmatrix}1 & x & z\\ 0 & 1 & y\\ 0 & 0 & 1\end{bmatrix}\cdot\begin{bmatrix}1 & a+k & m+an+c\\ 0 & 1 & n+b\\ 0 & 0 & 1\end{bmatrix}\\
  &= \begin{bmatrix}1 & a+k+x & m+an+c+xn+bx\\ 0 & 1 & b+n+y\\ 0 & 0 & 1\end{bmatrix}
  \end{aligned}$$

  And we can see that they are the same, so $\cdot$ is associative.

- Note that the matrix I_3 is also in the set since $0 \in \mathbb{R}$, and we know that any matrix 3×3 operated with I_3 is the same, so I_3 is the identity for $\mathcal{G}$.
- For finding the inverse matrix for an element in $\mathcal{G}$ we do the next:

$$\left[\begin{array}{ccc|ccc} 1 & x & z & 1 & 0 & 0 \\ 0 & 1 & y & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & x & 0 & 1 & 0 & -z \\ 0 & 1 & 0 & 0 & 1 & -y \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ -zR_3 \\ -yR_3 \end{array}$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -x & xy-z \\ 0 & 1 & 0 & 0 & 1 & -y \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} -xR_2 \\ \\ \\ \end{array}$$

So the matrix done in the right side is the inverse of the matrix in $\mathcal{G}$. For that, note that:

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -x & xy-z \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And the same is true for the converse, and since $-x, -y$ and $xy - z$ are also real numbers, we have shown the existence of inverses in $\mathcal{G}$.

Note that this group is not abelian since:

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+x & c+bx+z \\ 0 & 1 & b+y \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+x & z+ay+c \\ 0 & 1 & b+y \\ 0 & 0 & 1 \end{bmatrix}$$

And if $bx \neq ay$ then they are not the same. So, the group is not abelian.

Problem 3.

If it is possible compute the next products.

Solution. The products are:

1.

$$\begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

This product is not possible since the first matrix is a 3×2 matrix and the other one is a 3×3 matrix, and hence $3 \neq 2$ we cannot operate it.

2.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 5 \\ 10 & 9 & 11 \\ 16 & 15 & 17 \end{bmatrix}$$

3.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 5 & 7 & 9 \\ 11 & 13 & 15 \\ 8 & 10 & 12 \end{bmatrix}$$

4.

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 4 & 1 & -1 & -4 \end{bmatrix} \cdot \begin{bmatrix} 0 & 3 \\ 1 & -1 \\ 2 & 1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 14 & 5 \\ -21 & 2 \end{bmatrix}$$

5.

$$\begin{bmatrix} 0 & 3 \\ 1 & -1 \\ 2 & 1 \\ 5 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 1 & 2 \\ 4 & 1 & -1 & -4 \end{bmatrix} = \begin{bmatrix} 12 & 3 & -3 & -12 \\ -3 & 1 & 2 & 6 \\ 6 & 5 & 1 & 0 \\ 13 & 12 & 3 & 2 \end{bmatrix}$$

Problem 4.

Find all the solutions in $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$ of the equation system $Ax = 12x$ where:

$$A = \begin{bmatrix} 6 & 4 & 3 \\ 6 & 0 & 9 \\ 0 & 8 & 0 \end{bmatrix}$$

and $x_1 + x_2 + x_3 = 1$.

Solution. First, note that Ax is:

$$\begin{bmatrix} 6 & 4 & 3 \\ 6 & 0 & 9 \\ 0 & 8 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6x_1 + 4x_2 + 3x_3 \\ 6x_1 + 9x_3 \\ 8x_2 \end{bmatrix}$$

And so we want that:

$$\begin{bmatrix} 6x_1 + 4x_2 + 3x_3 \\ 6x_1 + 9x_3 \\ 8x_2 \end{bmatrix} = \begin{bmatrix} 12x_1 \\ 12x_2 \\ 12x_3 \end{bmatrix}$$

So we have the equations

$$\begin{array}{ccccccl} 6x_1 & + & 4x_2 & + & 3x_3 & = 12x_1 \\ 6x_1 & & & + & 9x_3 & = 12x_2 \\ & & 8x_2 & & & = 12x_3 \end{array}$$

And so we end up with the system:

$$\begin{array}{ccccccl} -6x_1 & + & 4x_2 & + & 3x_3 & = 0 \\ 6x_1 & - & 12x_2 & + & 9x_3 & = 0 \\ & & 8x_2 & - & 12x_3 & = 0 \end{array}$$

Which can be expresed in the next matrix:

$$\begin{aligned} \begin{bmatrix} -6 & 4 & 3 \\ 6 & -12 & 9 \\ 0 & 8 & -12 \end{bmatrix} &\longrightarrow \begin{bmatrix} -6 & 4 & 3 \\ 0 & -8 & 12 \\ 0 & 8 & -12 \end{bmatrix} -R_1 \\ &\longrightarrow \begin{bmatrix} -6 & 4 & 3 \\ 0 & -8 & 12 \\ 0 & 0 & 0 \end{bmatrix} -R_2 \\ &\longrightarrow \begin{bmatrix} -6 & 4 & 3 \\ 0 & -2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \tfrac{1}{4}R_2 \\ &\longrightarrow \begin{bmatrix} -6 & 6 & 0 \\ 0 & -2 & 3 \\ 0 & 0 & 0 \end{bmatrix} -R_2 \end{aligned}$$

Which lead us to the next equations:

$$-6x_1 + 6x_2 = 0$$
$$-2x_2 + 3x_3 = 0$$

From which we derive that:

$$x = \begin{bmatrix} \frac{3}{2}x_3 \\ \frac{3}{2}x_3 \\ x_3 \end{bmatrix}$$

And with the other condition, we must satisfy the equation as:

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ \frac{3}{2}x_3 + \frac{3}{2}x_3 + x_3 &= 1 \\ 4x_3 &= 1 \\ x_3 &= \frac{1}{4} \end{aligned}$$

Problem 5.

Which of the following sets are subsets of $\mathbb{R}^3$?

Solution.

1. $A = \{(\lambda, \lambda + \mu^3, \lambda - \mu^3) | \lambda, \mu \in \mathbb{R}\}$. Take two elements in A and a scalar $c \in \mathbb{R}$, then:

   $$\begin{aligned} c(\lambda_1, \lambda_1 + \mu_1^3, \lambda_1 - \mu_1^3) + (\lambda_2, \lambda_2 + \mu_2^3, \lambda_2 - \mu_2^3) &= (c\lambda_1, c\lambda_1 + c\mu_1^3, c\lambda_1 - c\mu_1^3) + (\lambda_2, \lambda_2 + \mu_2^3, \lambda_2 - \mu_2^3) \\ &= (c\lambda_1 + \lambda_2, (c\lambda_1 + \lambda_2) + (c\mu_1^3 + \mu_2^3), (c\lambda_1 + \lambda_2) - (c\mu_1^3 + \mu_2^3)) \end{aligned}$$

   And since $\sqrt[3]{c\mu_1^3 + \mu_2^3}$ will always be a real number, then the linear combination of these elements is in A and so it is a subspace of $\mathbb{R}^3$

2. $B = \{(\lambda^2, -\lambda^2, 0) | \lambda \in \mathbb{R}\}$. If you take $c = -1$ and two elements in B we have:

   $$\begin{aligned} -(\lambda_1^2, -\lambda_1^2, 0) + (\lambda_2^2, -\lambda_2^2, 0) &= (-\lambda_1^2, \lambda_1^2, 0) + (\lambda_2^2, -\lambda_2^2, 0) \\ &= (\lambda_2^2 - \lambda_1^2, \lambda_1^2 - \lambda_2^2, 0) \end{aligned}$$

   And if $\lambda_1^2 > \lambda_2^2$ then it is not defined its square and so it would not be an element of B. So B is not a subspace of $\mathbb{R}^3$

3. $C = \{(\lambda_1, \lambda_2, \lambda_3) | \lambda_1 - 2\lambda_2 + 3\lambda_3 = \gamma\}$ for a fixed γ. If we take two elements of C we would have the equations:

   $$\lambda_1 - 2\lambda_2 + 3\lambda_3 = \gamma$$
   $$\psi_1 - 2\psi_2 + 3\psi_3 = \gamma$$

   And adding them up we get:

   $$(\lambda_1 + \psi_1) - 2(\lambda_2 + \psi_2) + 3(\lambda_3 + \psi_3) = 2\gamma$$

   So, unless $\gamma = 0$ we conclude that C is not a subspace of $\mathbb{R}^3$.

4. $D = \{(\lambda_1, \lambda_2, \lambda_3) | \lambda_2 \in \mathbb{Z}\}$. If we take an element $c \in \mathbb{R}$ such that c is irrational and we multiply it by an element of D then $c\lambda_2$ would not be an integer and so that element would not be on D. Therefore, D is not a subspace of $\mathbb{R}^3$.

Problem 6.

Write the vector $\begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$ as a linear combination of the vectors $x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $x_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$.

Solution. For that, let's write the matrix of the system of equations as:

$$\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 2 & -1 & -2 \\ 1 & 3 & 1 & 5 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 2 & -1 & 4 \end{array}\right] \begin{array}{l} \\ -R_1 \\ -R_1 \end{array}$$
$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 5 & 10 \end{array}\right] \begin{array}{l} \\ \\ -2R_2 \end{array}$$
$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} \\ \\ \frac{1}{2}R_3 \end{array}$$
$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & -3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} -2R_3 \\ +3R_3 \\ \\ \end{array}$$
$$\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} -R_2 \\ \\ \\ \end{array}$$

And so we have the linear combination:

$$\begin{aligned} -6 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} &= \begin{bmatrix} -6 \\ -6 \\ -6 \end{bmatrix} + \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} + \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix} \end{aligned}$$

Problem 7.

Consider the linear mapping defined by:

$$\begin{array}{rccc} \phi : & \mathbb{R}^3 & \to & \mathbb{R}^4 \\ & (x_1, x_2, x_3) & \mapsto & (3x_1 + 2x_2 + x_3, x_1 + x_2 + x_3, x_1 - 3x_2, 2x_1 + 3x_2 + x_3) \end{array}$$

Find the matrix associated with the linear transformation, determine the rank of the matrix and compute the image, the kernel and their dimension.

Solution. First, we apply the transformation to the canonical basis in $\mathbb{R}^3$ as:

$$\begin{aligned} \phi(1,0,0) &= (3,1,1,2) \\ \phi(0,1,0) &= (2,1,-3,3) \\ \phi(0,0,1) &= (1,1,0,1) \end{aligned}$$

And so we end up with the matrix:

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & -3 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

Now, if we reduce it to the echelon form:

$$
\begin{aligned}
\left[\begin{array}{ccc|c} 3 & 2 & 1 & a \\ 1 & 1 & 1 & b \\ 1 & -3 & 0 & c \\ 2 & 3 & 1 & d \end{array}\right]
&\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & b \\ 3 & 2 & 1 & a \\ 1 & -3 & 0 & c \\ 2 & 3 & 1 & d \end{array}\right] \begin{array}{l} R_2 \\ R_1 \\ \\ \\ \end{array} \\
&\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & b \\ 0 & -1 & -2 & a-3b \\ 0 & -4 & -1 & c-b \\ 0 & 1 & -1 & d-2b \end{array}\right] \begin{array}{l} \\ -3R_1 \\ -R_1 \\ -2R_1 \end{array} \\
&\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & b \\ 0 & 1 & 2 & 3b-a \\ 0 & 4 & 1 & b-c \\ 0 & -1 & 1 & 2b-d \end{array}\right] \begin{array}{l} \\ -R_2 \\ -R_3 \\ -R_4 \end{array} \\
&\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & b \\ 0 & 1 & 2 & 3b-a \\ 0 & 0 & -3 & -3b-c \\ 0 & 0 & -2 & 3b-d \end{array}\right] \begin{array}{l} \\ \\ -4R_1 \\ +R_1 \end{array} \\
&\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & b \\ 0 & 1 & 2 & 3b-a \\ 0 & 0 & 1 & b+\frac{c}{3} \\ 0 & 0 & -1 & d-3b \end{array}\right] \begin{array}{l} \\ \\ -\frac{1}{3}R_3 \\ -R_4 \end{array} \\
&\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & b \\ 0 & 1 & 2 & 3b-a \\ 0 & 0 & 1 & b+\frac{c}{3} \\ 0 & 0 & 0 & d+\frac{c}{3}-2b \end{array}\right] \begin{array}{l} \\ \\ \\ +R_3 \end{array} \\
&\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & -\frac{c}{3} \\ 0 & 1 & 0 & b-a-\frac{2c}{3} \\ 0 & 0 & 1 & b+\frac{c}{3} \\ 0 & 0 & 0 & d+\frac{c}{3}-2b \end{array}\right] \begin{array}{l} -R_3 \\ -2R_3 \\ \\ \\ \end{array} \\
&\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & a-b+\frac{c}{3} \\ 0 & 1 & 0 & b-a-\frac{2c}{3} \\ 0 & 0 & 1 & b+\frac{c}{3} \\ 0 & 0 & 0 & d+\frac{c}{3}-2b \end{array}\right] \begin{array}{l} -R_2 \\ \\ \\ \\ \end{array}
\end{aligned}
$$

So the image is the set:

$$Im(\phi) = \{(a,b,c,d) \in \mathbb{R}^4 | d + \frac{c}{3} - 2b = 0\}$$

Note that when $(a,b,c,d) = (0,0,0,0)$ then the system only has the trivial solution and so we conclude that the rank of A is 3 and that the kernel is just $\{0\}$. By the rank theorem, we have:

$$
\begin{aligned}
\dim V &= \dim(\ker(\phi)) + \dim(\phi(V)) \\
\dim \mathbb{R}^3 &= \dim(\{0\}) + \dim(\phi(V)) \\
3 &= 0 + \dim(\phi(V)) \\
\dim(\phi(V)) &= 3
\end{aligned}
$$

And so we conclude that even when the linear transformation is monic, it is not epic.

Problem 8.

Let V be a vector space. Let f and g two automorphism over E such that $f \circ g = Id_E$. Show that $\ker(f) = \ker(g \circ f)$, $Im(g) = Im(g \circ f)$ and that $\ker(f) \cap Im(g) = \{0_E\}$.

Solution. First, we want to show that $\ker(g \circ f) = \{0_E\}$ since f is an automorphism. So, suppose that $v \in E$ is such that $(g \circ f)(v) = g(f(v)) = 0$. Since g is an automorphism, we conclude that $f(v) = 0$ and since f is also an automorphism, we conclude that $v = 0$. So, $\ker(f) = \ker(g \circ f)$.

Also, we want to show $Im(g \circ f) = E$, so take $v \in E$, since g is an automorphism we can find $u \in E$ such that $g(u) = v$. And also, since f is an automorphism, we can find $w \in E$ such that $f(w) = u$, so we have that:

$$\begin{aligned}(g \circ f)(w) &= g(f(w)) \\ &= g(u) \\ &= v\end{aligned}$$

And so we conclude that $Im(g) = Im(g \circ f)$. Note that also $0 \in Im(g)$ but the only element in $\ker(f)$ is 0, so we must have that $\ker(f) \cap Im(g) = \{0_E\}$.

Problem 9.

Let $F = \{(x, y, z) \in \mathbb{R}^3 | x + y - z = 0\}$ and $G = \{(a - b, a + b, a - 3b) \in \mathbb{R}^3 | a, b \in \mathbb{R}\}$. Prove that they are subspaces of $\mathbb{R}^3$, calculate $F \cap G$ and then using basis for F and G check the result.

Solution. First, we are going that both sets are subspaces of $\mathbb{R}^3$:

- Let $(x, y, z), (a, b, c) \in F$ and let $r \in \mathbb{R}$, we want to show that their linear combination is in F:

  $$\begin{aligned}r(x, y, z) + (a, b, c) &= (rx, ry, rz) + (a, b, c) \\ &= (rx + a, ry + b, rz + c)\end{aligned}$$

  And by definition, we have the equations:

  $$\begin{aligned}x + y - z &= 0 \\ a + b - c &= 0\end{aligned}$$

  If we multiply the first by r we get:

  $$\begin{aligned}rx + ry - rz &= 0 \\ a + b - c &= 0\end{aligned}$$

  and if we add them up:

  $$(rx + a) + (ry + b) - (rz + c) = 0$$

  Which proves that $(rx + a, ry + b, rz + c) \in F$ and so it is a subspace.

- Let $(a - b, a + b, a - 3b), (x - y, x + y, x - 3y)$ and $r \in \mathbb{R}$ we want to show that their linear combination is in G:

  $$\begin{aligned}r(a - b, a + b, a - 3b) + (x - y, x + y, x - 3y) &= (ar - br, ar + br, ar - 3br) + (x - y, x + y, x - 3y) \\ &= ((ar + x) - (br + y), (ar + x) + (br + y), (ar + x) - 3(br + y))\end{aligned}$$

  and since $ar + x, br + y \in \mathbb{R}$, by definition $((ar + x) - (br + y), (ar + x) + (br + y), (ar + x) - 3(br + y))$ and so G is a subspace.

Now, if we calculate $F \cap G$, we take a vector $(x, y, z) \in F \cap G$ then it is necessary that $x + y - z = 0$ and for $a, b \in \mathbb{R}$ we got $x = a - b$, $y = a + b$ and $z = a - 3b$. If we replace it into the another equation we get:

$$\begin{aligned} x + y - z &= 0 \\ a - b + a + b - a + 3b &= 0 \\ a + 3b &= 0 \\ b &= -\frac{a}{3} \end{aligned}$$

and so we get:

$$\begin{aligned} x &= \frac{4a}{3} \\ y &= \frac{2a}{3} \\ z &= 2a \end{aligned}$$

so we got vectors of the form $\left(\frac{4a}{3}, \frac{2a}{3}, 2a\right)$ such that $a \in \mathbb{R}$. Now, we are going to find a basis for F and for G.

- For F, note that each vector is of the form (x, y, z) with $x + y - z = 0$ and so $z = x + y$, so we can express it:

  $$\begin{aligned} (x, y, z) &= (x, y, x + y) \\ &= (x, 0, x) + (0, y, y) \\ &= x(1, 0, 1) + y(0, 1, 1) \end{aligned}$$

  And so a basis for F is $\{(1, 0, 1), (0, 1, 1)\}$.

- For G, we have vectors of the form $(a - b, a + b, a - 3b)$ so we can descompose it like:

  $$\begin{aligned} (a - b, a + b, a - 3b) &= (a, a, a) + (-b, b, -3b) \\ &= a(1, 1, 1) + b(-1, 1, -3) \end{aligned}$$

  And so a basis for G is $\{(1, 1, 1), (-1, 1, -3)\}$.

And now we express a vector of $F \cap G$ as a linear combination with scalars $x, y, a, b \in \mathbb{R}$ as:

$$\begin{aligned} x(1, 0, 1) + y(0, 1, 1) &= a(1, 1, 1) + b(-1, 1, -3) \\ (x, 0, x) + (0, y, y) &= (a, a, a) + (-b, b, -3b) \\ (x, y, x + y) &= (a - b, a + b, a - 3b) \end{aligned}$$

and so we get $x = a - b$, $y = a + b$ and $x + y = a - 3b$, but we can replace in this last equation as:

$$\begin{aligned} x + y &= a - 3b \\ a - b + a + b &= a - 3b \\ 2a &= a - 3b \\ a &= -3b \\ b &= -\frac{a}{3} \end{aligned}$$

and so we get the same result as we did without the basis.

2 Analytic Geometry

Problem 10.

Show that $\langle \cdot, \cdot \rangle$ defined for all $x = (x_1, x_2) \in \mathbb{R}^2$ and $y = (y_1, y_2) \in \mathbb{R}^2$ by:

$$\langle x, y \rangle = x_1 y_1 - (x_1 y_2 + x_2 y_1) + 2(x_2 y_2)$$

is an inner product.

Solution. We need to prove three things. That $\langle \cdot, \cdot \rangle$ is a bilinear map, that it is symmetric and that is positive definite.

- First, to prove that it is a bilinear transformation, take $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in \mathbb{R}^2$ and $a, b \in \mathbb{R}$. Then:

$$\begin{aligned} \langle (x_1, x_2), (ay_1 + bz_1, ay_2 + bz_2) \rangle &= x_1(ay_1 + bz_1) - (x_1 \cdot (ay_2 + bz_2) + x_2(ay_1 + bz_1)) + 2x_2(ay_2 + bz_2) \\ &= ax_1y_1 + bx_1z_1 - (ax_1y_2 + bx_1z_2 + ax_2y_1 + bx_2z_1) + 2ax_2y_2 + 2bx_2z_2 \\ &= ax_1y_1 - ax_1y_2 - ax_2y_1 + 2a(x_2y_2) + bx_1z_1 - bx_1z_2 - bx_2z_1 + 2b(x_2z_2) \\ &= a(x_1y_1 - (x_1y_2 + x_2y_1) + 2(x_2y_2)) + b(x_1z_1 - (x_1z_2 + x_2z_1) + 2(x_2z_2)) \\ &= a\langle (x_1, x_2), (y_1, y_2) \rangle + b\langle (x_1, x_2), (z_1, z_2) \rangle \end{aligned}$$

  In a similar way we prove that $\langle (ax_1 + by_1, ax_2 + by_2), (z_1, z_2) \rangle = a\langle (x_1, x_2), (z_1, z_2) \rangle + b\langle (y_1, y_2), (z_1, z_2) \rangle$, so we conclude that it is a bilinear transformation.

- We want to show that this transformation is symmetric. So:

$$\begin{aligned} \langle x, y \rangle &= x_1y_1 - (x_1y_2 + x_2y_1) + 2(x_2y_2) \\ &= y_1x_1 - (y_1x_2 + y_2x_1) + 2(y_2x_2) \\ &= \langle y, x \rangle \end{aligned}$$

- And for the last part, we want to show that it is positive define. That is, suppose that $(x_1, x_2) \neq (0, 0)$ so we want to show that its inner product is positive:

$$\begin{aligned} \langle (x_1, x_2), (x_1, x_2) \rangle &= x_1 \cdot x_1 - (x_1x_2 + x_1x_2) + 2(x_2 \cdot x_2) \\ &= x_1^2 - 2x_1x_2 + 2x_2^2 \\ &= x_1^2 - 2x_1x_2 + x_2^2 + x_2^2 \\ &= (x_1 - x_2)^2 + x_2^2 \end{aligned}$$

  And no matter the chooses of x_1, x_2 we always end up with a positive sum of squares, so $\langle x, x \rangle > 0$ and it is easy to see that $\langle 0, 0 \rangle = 0$.

Problem 11.

Consider $\mathbb{R}^2$ with $\langle \cdot, \cdot \rangle$ defined for all x and y in $\mathbb{R}^2$ as:

$$\langle x, y \rangle := x^\top \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} y$$

Is it an inner product?

Solution. First, we could rearrange the expression taking $x = (x_1, x_2)$ and $y = (y_1, y_2)$ as:

$$\begin{aligned} \langle x, y \rangle &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2y_1 \\ y_1 + 2y_2 \end{bmatrix} \\ &= 2x_1y_1 + x_2y_1 + 2x_2y_2 \end{aligned}$$

And now we must to test the three conditions:

- First, we want to prove that it is a bilinear transformation:

$$\begin{aligned}\langle (x_1,x_2),(ay_1+bz_1,ay_2+bz_2)\rangle &= 2x_1(ay_1+bz_1)+x_2(ay_1+bz_1)+2x_2(ay_2+bz_2)\\ &= 2ax_1y_1+2bx_1z_1+ax_2y_1+bx_2z_1+2ax_2y_2+2bx_2z_2\\ &= 2ax_1y_1+ax_2y_1+2ax_2y_2+2bx_1z_1+bx_2z_1+2bx_2z_2\\ &= a(2x_1y_1+x_2y_1+2x_2y_2)+b(x_1z_1+x_2z_1+x_2z_2)\\ &= a\langle (x_1,x_2),(y_1,y_2)\rangle + b\langle (x_1,x_2),(z_1,z_2)\rangle\end{aligned}$$

  And we can prove in a similar way that $\langle (ax_1+by_1,ax_2+by_2),(z_1,z_2)\rangle = a\langle (x_1,x_2),(y_1,y_2)\rangle + b\langle (x_1,x_2),(z_1,z_2)\rangle$. So we conclude that $\langle \cdot,\cdot\rangle$ is a bilinear transformation.

- Now, we need to show that it is symmetric:

$$\begin{aligned}\langle x,y\rangle &= 2x_1y_1+x_2y_1+2x_2y_2\\ &= 2x_2y_2+y_1x_2+2x_1y_1\end{aligned}$$

  But this does not implies that $y_1x_2 = y_2x_1$ so it is not symmetric and therefore it is not an inner product.

Problem 12.

Compute the distance between $x=(1,2,3)$ and $y=(-1,-1,0)$ using:

1. $\langle x,y\rangle := x^\top y$
2. $\langle x,y\rangle := x^\top yA$ with $A := \begin{bmatrix}2 & 1 & 0\\ 1 & 3 & -1\\ 0 & -1 & 2\end{bmatrix}$

Solution. Remmber that the distance induced by a product is the norm induced by that inner product. We can first calculate $x-y$ as:

$$\begin{aligned}x-y &= (1,2,3)-(-1,-1,0)\\ &= (2,3,3)\end{aligned}$$

And so we do the norm of that vector under each of the inner products defined:

1. For $\langle x,y\rangle := x^\top y$, we have:

$$\begin{aligned}d(x,y) &= \|x-y\|\\ &= \sqrt{\langle x-y,x-y\rangle}\\ &= \sqrt{\langle (2,3,3),(2,3,3)\rangle}\\ &= \sqrt{2^2+3^2+3^2}\\ &= \sqrt{4+9+9}\\ &= \sqrt{22}\end{aligned}$$

2. For $\langle x,y\rangle := x^\top yA$ with $A := \begin{bmatrix}2 & 1 & 0\\ 1 & 3 & -1\\ 0 & -1 & 2\end{bmatrix}$, so if we make $\langle x-y,x-y\rangle$ we got:

$$\begin{aligned}\begin{bmatrix}2 & 3 & 3\end{bmatrix}\begin{bmatrix}2 & 1 & 0\\ 1 & 3 & -1\\ 0 & -1 & 2\end{bmatrix}\begin{bmatrix}2\\3\\3\end{bmatrix} &= \begin{bmatrix}2 & 3 & 3\end{bmatrix}\begin{bmatrix}7\\8\\3\end{bmatrix}\\ &= 2\cdot 7+3\cdot 8+3\cdot 3\\ &= 14+24+9\\ &= 47\end{aligned}$$

So we have:

$$\begin{aligned} d(x,y) &= \|x-y\| \\ &= \sqrt{\langle x-y, x-y\rangle} \\ &= \sqrt{\langle (2,3,3),(2,3,3)\rangle} \\ &= \sqrt{47} \end{aligned}$$

Problem 13.

Compute the angle between:

$$x = (1,2) \qquad\qquad y = (-1,-1)$$

using the inner products:

1. $\langle x, y\rangle = x^\top y$
2. $\langle x, y\rangle = x^\top B y$ with $B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$

Solution.

1. With the usual product and the norm induced by that, we consider that:

$$\begin{aligned} \cos\omega &= \frac{|\langle x,y\rangle|}{\|x\|\|y\|} \\ &= \frac{|1\cdot(-1)+2\cdot(-1)|}{\sqrt{1^2+2^2}\sqrt{(-1)^2+(-1)^2}} \\ &= \frac{|-1-2|}{\sqrt{5}\sqrt{2}} \\ &= \frac{3}{\sqrt{10}} \end{aligned}$$

   And so we get that $\omega = \arccos\frac{3}{\sqrt{10}} \approx 18\,\mathrm{deg}$

2. It is convenient to compute the inner product with arbitrary (x_1,x_2) and (y_1,y_2):

$$\begin{aligned} \langle (x_1,x_2),(y_1,y_2)\rangle &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2y_1 + y_2 \\ y_1 + 3y_2 \end{bmatrix} \\ &= 2x_1y_1 + x_1y_2 + y_1x_2 + 3x_2y_2 \end{aligned}$$

Now, with that we are ready to compute the angle given by this inner product.

$$\begin{aligned}
\cos\omega &= \frac{|\langle x, y\rangle|}{\|x\|\|y\|} \\
&= \frac{|2(1)(-1)+(1)(-1)+(2)(-1)+3(2)(-1)|}{\sqrt{2(1)(1)+(1)(2)+(1)(2)+3(2)(2)}\sqrt{(2)(-1)(-1)+(-1)(-1)+(-1)(-1)+(3)(-1)(-1)}} \\
&= \frac{|-2-1-2-6|}{\sqrt{2+2+2+12}\sqrt{2+1+1+3}} \\
&= \frac{|-11|}{\sqrt{18}\sqrt{7}} \\
&= \frac{|11|}{\sqrt{126}} \\
&= \frac{11}{\sqrt{126}}
\end{aligned}$$

And so we end up that $\omega = \arccos\frac{11}{\sqrt{126}} \approx 11.5\,\text{deg}$.

Problem 14.

Consider $\mathbb{R}^3$ with the inner product:

$$\langle x, y\rangle = x^\top \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} y$$

and let e_1, e_2, e_3 be the canonical basis of $\mathbb{R}^3$.

- Define the orthogonal projection $\pi_U(e_2)$ onto $U = span[e_1, e_3]$
- Compute the distance $d(e_2, U)$
- Draw the scenario.

Solution. Since we want to project e_2 into the set $span[e_1, e_3]$ for the projction $\pi_U(e_2)$ there would be λ_1, λ_2 such that:

$$\pi_U(e_2) = \lambda_1 \cdot e_1 + \lambda_2 \cdot e_3$$

so we must have that:

$$\begin{aligned}
\langle e_1, e_2 - \pi_U(e_2)\rangle = 0 \\
\langle e_3, e_2 - \pi_U(e_2)\rangle = 0
\end{aligned}$$

Now, we end up with the system:

$$e_1^\top B(e_2 - \pi_U(e_2)) = 0 e_3^\top B(e_2 - \pi_U(e_2)) \qquad = 0$$

And this can be showed as a matricial product like:

$$\begin{bmatrix} e_1^\top \\ e_2^\top \end{bmatrix} B(e_2 - \pi_U(e_2)) = 0$$

And if we call $A = \begin{bmatrix} e_1^\top \\ e_2^\top \end{bmatrix}$ and knowing that π_U shall be expressed as a linear combination of $P = [e_1, e_3]$ then it becomes:

$$\begin{aligned}
AB(e_2 - \pi_U(e_2)) &= 0 \\
ABe_2 - ABP\lambda &= 0 \\
ABP\lambda &= ABe_2
\end{aligned}$$

If we compute the product ABP we got:

$$
\begin{aligned}
AB &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix}
\end{aligned}
$$

And if we multiply by P then:

$$
\begin{aligned}
ABP &= \begin{bmatrix} 2 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \\
&= 2I
\end{aligned}
$$

And so we get:

$$
\begin{aligned}
ABP\lambda &= ABe_2 \\
2I\lambda &= ABe_2 \\
2\lambda &= ABe_2
\end{aligned}
$$

And so we only need to compute ABe_2:

$$
\begin{aligned}
ABe_2 &= \begin{bmatrix} 2 & 1 & 0 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\end{aligned}
$$

And if we divide by 2 we get:

$$
\lambda = \begin{bmatrix} \frac{1}{2} \\ \frac{-1}{2} \end{bmatrix}
$$

And therefore we get that:

$$
\begin{aligned}
\pi_U(e_2) &= e_1 \cdot \lambda_1 + e_3 \cdot \lambda_3 \\
&= \begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ -\frac{1}{2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix}
\end{aligned}
$$

And that is the projection of e_2 into U.

Now, for compute the distance between e_2 and U we compute what is $e_2 - \pi_U(e_2)$ and then we apply it to the norm induced by the inner product:

$$
\begin{aligned}
e_2 - \pi_U(e_2) &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} \\ 0 \\ -\frac{1}{2} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{1}{2} \\ 1 \\ \frac{1}{2} \end{bmatrix}
\end{aligned}
$$

And now we apply to itself the inner product:

$$\begin{aligned}
\langle e_2 - \pi_U(e_2), e_2 - \pi_U(e_2)\rangle &= \begin{bmatrix} -\frac{1}{2} & 1 & \frac{1}{2}\end{bmatrix} \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ 1 \\ \frac{1}{2} \end{bmatrix} \\
&= \begin{bmatrix} -\frac{1}{2} & 1 & \frac{1}{2}\end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\
&= 1
\end{aligned}$$

And so we see that the distance between that two vectors is 1.

Problem 15.

Let V be a vector space and π an endomorphism of V:

- Prove that π is a projection if and only if $Id_V - \pi$ is a projection
- Assume that π is a projection. Calculate $Im(Id_V - \pi)$ and $\ker(id_V - \pi)$ as a function of $Im(\pi)$ and $\ker(\pi)$.

Solution. First, let's prove the equivalence:

$\Rightarrow$) Suppose that π is a projection. Then $\pi \circ \pi = \pi$. And if we compute $Id_V - \pi$ composed with itself we get:

$$\begin{aligned}
(Id_V - \pi)((Id_v - \pi)(x)) &= (Id_V - \pi)(Id_v(x) - \pi(x)) \\
&= (Id_V - \pi)(x - \pi(x)) \\
&= Id_V(x - \pi(x)) - \pi(x - \pi(x)) \\
&= x - \pi(x) + \pi(\pi(x) - x) \\
&= x - \pi(x) + \pi(\pi(x)) - \pi(x) \\
&= x - \pi(x) + \pi(x) - \pi(x) \\
&= x - \pi(x)
\end{aligned}$$

And so we get that $(Id_V - \pi) \circ (Id_V - \pi) = Id_V - \pi$ and so it is a projection.

$\Leftarrow$) Suppose that $Id_V - \pi$ is a projection. Then $(Id_V - \pi) \circ (Id_V - \pi) = Id_V - \pi$. If we compute $\pi \circ \pi$ we get:

$$\begin{aligned}
\pi(\pi(x)) &= \pi(-(Id - \pi - Id)(x)) \\
&= \pi(Id(x) - (Id - \pi)(x)) \\
&= \pi(x - (Id - \pi(x))) \\
&= -((Id - \pi) - Id)(x - (Id - \pi)(x)) \\
&= -((Id - \pi)(x - (Id - \pi)(x))) - Id(x - (Id - \pi)(x)) \\
&= x - (Id - \pi)(x) - (Id - \pi)(x - (Id - \pi)(x)) \\
&= x - (Id - \pi)(x) - (Id - \pi)(x) + (Id - \pi)((Id - \pi)(x)) \\
&= x - (Id - \pi)(x) - (Id - \pi)(x) + (Id - \pi)(x) \\
&= x - (Id - \pi)(x) \\
&= x - Id(x) + \pi(x) \\
&= x - x + \pi(x) = \pi(x)
\end{aligned}$$

And since $\pi \circ \pi = \pi$, then π is a projection.

Now, if we assume that π is a projection, then we are going to give a description for each one.

- $y \in Im(Id_V - \pi)$ if and only if $y \in \ker(\pi)$. Let's prove this. If $y \in Im(Id_V - \pi)$ then there is $x \in V$ such that $(Id_V - \pi)(x) = y$. And so:

$$\begin{aligned}(Id_V - \pi)(x) &= y \\ Id_V(x) - \pi(x) &= y \\ \pi(x) + y - x &= 0 \\ \pi(\pi(x) + y - x) &= \pi(0) \\ \pi(\pi(x)) + \pi(y) - \pi(x) &= 0 \\ \pi(x) + \pi(y) - \pi(x) &= 0 \\ \pi(y) &= 0\end{aligned}$$

  And so we get that $y \in \ker(\pi)$. If $y \in \ker(\pi)$ then $\pi(y) = 0$. If we add and substract y we get:

$$\begin{aligned}\pi(y) &= 0 \\ \pi(y) + y - y &= 0 \\ \pi(y) - y &= -y \\ y - \pi(y) &= y \\ Id_V(y) - \pi(y) &= y \\ (Id_V - \pi)(y) &= y\end{aligned}$$

  And by definition $y \in Im(Id_V - \pi)$. So $Im(Id_V - \pi) = \ker(\pi)$.

- In a similar way, we get that $x \in \ker(Id_V - \pi)$ if and only if $x \in Im(\pi)$. Suppose that $x \in \ker(Id_V - pi)$, that means that:

$$\begin{aligned}(Id_V - \pi)(x) &= 0 \\ Id_V(x) - \pi(x) &= 0 \\ x - \pi(x) &= 0 \\ \pi(x) &= x\end{aligned}$$

  And so we get that $x \in \Im(\pi)$. If $x \in Im(\pi)$ we know that there is $y \in V$ such that $\pi(y) = x$. If we apply π again we get that $\pi(y) = \pi(x)$. Now, we get:

$$\begin{aligned}(Id_V - \pi)(x) &= Id_V(x) - \pi(x) \\ &= x - \pi(x) \\ &= x - \pi(y) \\ &= x - x \\ &= 0\end{aligned}$$

  So $x \in \ker(\pi)$. Therefore, $\ker(Id_V - \pi) = Im(\pi)$.

Problem 16.

Using the Gram-Schmidt method, turn the basis $B = (b_1, b_2)$ of a two dimensional space $U \subseteq \mathbb{R}^3$ into an ONB $C = (c_1, c_2)$ where:

$$b_1 := \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad\qquad b_2 := \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}$$

Solution. So what we need is to project the vector b_2 into the subspace generated by b_1. For that, suppose that $\pi_V(b_2)$ is the projection of b_2 into $V = Span(b_1)$. We need that it is orthogonal, so we get:

$$\begin{aligned}\langle b_1, b_2 - \pi_V(b_2)\rangle &= 0\\ b_1^T(b_2 - \pi_V(b_2)) &= 0\\ b_1^T b_2 - b_1^T \pi_V(b_2) &= 0\\ b_1^T \pi_V(b_2) &= b_1^T b_2\end{aligned}$$

And so we compute:

$$\begin{aligned}b_1^T b_2 &= \begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}-1\\2\\0\end{bmatrix}\\ &= -1 + 2 + 0\\ &= 1\end{aligned}$$

And if we compute the other side:

$$\begin{aligned}b_1^T \pi_V(b_2) &= \begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}\lambda\\\lambda\\\lambda\end{bmatrix}\\ &= 3\lambda\end{aligned}$$

So we get:

$$\begin{aligned}3\lambda &= 1\\ \lambda &= \frac{1}{3}\end{aligned}$$

And so we get that $\pi_V(b_2) = \begin{bmatrix}\frac{1}{3}\\\frac{1}{3}\\\frac{1}{3}\end{bmatrix}$ And so we normalize these vectors:

$$\begin{aligned}c_1 &:= \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}\\ c_2 &:= \frac{3}{\sqrt{42}}\begin{bmatrix}-\frac{4}{3}\\\frac{5}{3}\\-\frac{1}{3}\end{bmatrix}\end{aligned}$$

We can see that these vectors are orthogonal to each other since:

$$\begin{aligned}b_1^\top b_2 &= \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}\frac{3}{\sqrt{42}}\begin{bmatrix}-\frac{4}{3}\\\frac{5}{3}\\-\frac{1}{3}\end{bmatrix}\\ &= \frac{3}{\sqrt{42}}(-\frac{4}{3} + \frac{5}{3} - \frac{1}{3})\\ &= 0\end{aligned}$$

And they are normal vectors. We only need to prove that they are a basis for U:

- Suppose that a_1, a_2 are scalars such that:

$$a_1 c_1 + a_2 c_2 = 0$$

But let $x_1 = a_1 \frac{1}{\sqrt{3}}$ and $x_2 = a_2 \frac{3}{\sqrt{42}}$, so we can try to find values for x_1 and x_2 with the matrix:

$$\begin{aligned} \begin{bmatrix} 1 & -\frac{4}{3} \\ 1 & \frac{5}{3} \\ 1 & -\frac{1}{3} \end{bmatrix} &\Rightarrow \begin{bmatrix} 3 & -4 \\ 3 & 5 \\ 3 & -1 \end{bmatrix} \\ &\Rightarrow \begin{bmatrix} 3 & -5 \\ 3 & 5 \\ 3 & -1 \end{bmatrix} \\ &\Rightarrow \begin{bmatrix} 6 & 0 \\ 3 & 5 \\ 3 & -1 \end{bmatrix} \\ &\Rightarrow \begin{bmatrix} 6 & 0 \\ 0 & 6 \\ 3 & -1 \end{bmatrix} \end{aligned}$$

From which we can see that $x_1 = x_2 = 0$, which necessarily implies that $a_1 = a_2 = 0$ and so they are linearly independent.

- Let $v = a_1 b_1 + a_2 b_2$, we can use this to show that is also a linear combination of c_1 and c_2. Since $c_2 = b_2 - \pi_V(b_2)$ ten $b_2 = c_2 + \pi_V(b_2)$ and then we get:

$$\begin{aligned} v &= a_1 b_1 + a_2 b_2 \\ &= a_1 \sqrt{3} c_1 + a_2 (c_2 + \pi_V(b_2)) \\ &= a_1 \sqrt{3} c_1 + a_2 c_2 + a_2 \pi_V(b_2) \\ &= a_2 \sqrt{3} c_1 + a_2 c_2 + a_2 \frac{1}{3} b_1 \\ &= a_2 \sqrt{3} c_1 + a_2 c_2 + a_2 \frac{\sqrt{3}}{3} c_1 \\ &= (a_2 \sqrt{3} + \frac{\sqrt{3}}{3}) c_1 + a_2 c_2 \end{aligned}$$

So we conclude that they span all U.

Therefore, we have found an ONB for U.

Problem 17.

Rotate the vectors:

$$x_1 := \begin{bmatrix} 2 \\ 3 \end{bmatrix} \qquad\qquad x_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

by 30.

Solution. So, we can get that the matrix of rotations is:

$$\begin{aligned} R &= \begin{bmatrix} \cos 30 & -\sin 30 \\ \sin 30 & \cos 30 \end{bmatrix} \\ &= \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix} \end{aligned}$$

And so we multiply it by each vectors for the rotation:

$$\frac{1}{2} \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 2\sqrt{3} - 3 \\ 2 + 3\sqrt{3} \end{bmatrix} \qquad\qquad \frac{1}{2} \begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -\sqrt{3} \end{bmatrix}$$