Problem set
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2023
30 Days of ML

1 Systems of linear equations, Matrices, and vector spaces

Problem 1.

We consider $(\mathbb{R} \setminus \{-1\}, \star)$ where:

$$a \star b = a + ab + b$$

for $a, b \in \mathbb{R} \setminus \{-1\}$. Show that this is an abelian group and solve $3 \star x \star x = 15$.

Solution. We prove this showing that the five axioms are true:

- First, suppose that $a + ab + b = -1$ and so we would have that $a(1+b) + b = -1$, and so we have $a(1+b) = -(1+b)$ and since $b \neq 1$ we can assure that $1 + b \neq 0$ so $a = -1$ which is a contradiction to the hypothesis that $a \neq 1$, so $\star$ is closed under the set.
- Now, we want to prove associativity(We pay attention to the associativity in the real numbers)

$$\begin{aligned} a \star (b \star c) &= a \star (b + bc + c) \\ &= a + a(b + bc + c) + (b + bc + c) \\ &= a + ab + abc + ac + b + bc + c \end{aligned} \qquad \begin{aligned} (a \star b) \star c &= (a + ab + b) \star c \\ &= (a + ab + b) + c(a + ab + b) + c \\ &= a + ab + b + ac + abc + bc + c \end{aligned}$$

  and so it is clear that they are the same.
- Note that the element 0 is an identity element because:

$$\begin{aligned} a \star 0 &= a + a \cdot 0 + 0 \\ &= a \end{aligned} \qquad \begin{aligned} 0 \star a &= 0 + a \cdot 0 + a \\ &= a \end{aligned}$$

- The inverse element for x is $\frac{-x}{1+x}$ since:

$$\begin{aligned} x \star \frac{-x}{1+x} &= x - x \cdot \frac{x}{1+x} - \frac{x}{1+x} \\ &= x - \frac{x^2}{1+x} - \frac{x}{1+x} \\ &= \frac{x + x^2}{1+x} - \frac{x^2}{1+x} - \frac{x}{1+x} \\ &= 0 \end{aligned}$$

  And the commutated case is the same, so it has inverses.
- The commutativity is a consequence of these properties in $\mathbb{R}$:

$$\begin{aligned} a \star b &= a + ab + b \\ &= b + ba + a \\ &= b \star a \end{aligned}$$

And so we conclude that $(\mathbb{R} \setminus \{-1\}, \star)$ is an abelian group. For the equation, we do:

$$\begin{aligned}
3 \star x \star x &= 15 \\
3 \star (2x + x^2) &= 15 \\
3 + 3(2x + x^2) + (2x + x^2) &= 15 \\
6x + 3x^2 + 2x + x^2 &= 12 \\
4x^2 + 8x &= 12 \\
x^2 + 2x &= 3 \\
x^2 + 2x - 3 &= 0 \\
(x-1)(x+3) &= 0
\end{aligned}$$

And therefore we conclude that $x = 1$ or $x = -3$. If we put this into the equation we have:

$$\begin{aligned}
3 \star 1 \star 1 &= 3 \star 3 \\
&= 3 + 9 + 3 \\
&= 15
\end{aligned}
\qquad\qquad
\begin{aligned}
3 \star (-3 \star (-3)) &= 3 \star 3 \\
&= 3 + 9 + 3 \\
&= 15
\end{aligned}$$

So the solutions are $x = 1$ and $x = -3$.

Problem 2.

Consider the set $\mathcal{G}$ of 3×3 matrices defined as follows:

$$\mathcal{G} = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 3} \,\middle|\, x, y, z \in \mathbb{R} \right\}$$

And we define $\cdot$ as the standard matrix multiplication. Is $(\mathcal{G}, \cdot)$ a group? If yes, is it abelian?

Solution. We prove this showing that the four axioms are true:

- First, it is closed under $\cdot$ since:

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+x & c+bx+z \\ 0 & 1 & b+y \\ 0 & 0 & 1 \end{bmatrix}$$

  And since the operations of addition and product of real numbers are closed, by definition the matrix is also in $\mathcal{G}$ so it is a closed operation.

- We can prove the associativity by taking three matrices and show that their product don't vary.

$$\begin{aligned}
\left(\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \right) \cdot \begin{bmatrix} 1 & k & m \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix} &= \begin{bmatrix} 1 & a+x & c+bx+z \\ 0 & 1 & b+y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & a+x & c+bx+z \\ 0 & 1 & b+y \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & a+k+x & m+an+ax+c+bx+z \\ 0 & 1 & b+n+y \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

  And if we make the other option:

$$\begin{aligned}
\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \cdot \left(\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & k & m \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix} \right) &= \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & a+k & m+an+c \\ 0 & 1 & n+b \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & a+k+x & m+an+c+xn+bx \\ 0 & 1 & b+n+y \\ 0 & 0 & 1 \end{bmatrix}
\end{aligned}$$

  And we can see that they are the same, so $\cdot$ is associative.

- Note that the matrix I_3 is also in the set since $0 \in \mathbb{R}$, and we know that any matrix 3×3 operated with I_3 is the same, so I_3 is the identity for $\mathcal{G}$.
- For finding the inverse matrix for an element in $\mathcal{G}$ we do the next:

$$\left[\begin{array}{ccc|ccc} 1 & x & z & 1 & 0 & 0 \\ 0 & 1 & y & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & x & 0 & 1 & 0 & -z \\ 0 & 1 & 0 & 0 & 1 & -y \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} \\ -zR_3 \\ -yR_3 \end{array}$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -x & xy-z \\ 0 & 1 & 0 & 0 & 1 & -y \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \begin{array}{l} -xR_2 \\ \\ \\ \end{array}$$

  So the matrix done in the right side is the inverse of the matrix in $\mathcal{G}$. For that, note that:

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -x & xy-z \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

  And the same is true for the converse, and since $-x, -y$ and $xy - z$ are also real numbers, we have shown the existence of inverses in $\mathcal{G}$.

Note that this group is not abelian since:

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+x & c+bx+z \\ 0 & 1 & b+y \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+x & z+ay+c \\ 0 & 1 & b+y \\ 0 & 0 & 1 \end{bmatrix}$$

And if $bx \neq ay$ then they are not the same. So, the group is not abelian.

Problem 3.

If it is possible compute the next products.

Solution. The products are:

1. $$\begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

   This product is not possible since the first matrix is a 3×2 matrix and the other one is a 3×3 matrix, and hence $3 \neq 2$ we cannot operate it.

2. $$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 5 \\ 10 & 9 & 11 \\ 16 & 15 & 17 \end{bmatrix}$$

3. $$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 5 & 7 & 9 \\ 11 & 13 & 15 \\ 8 & 10 & 12 \end{bmatrix}$$

4. $$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 4 & 1 & -1 & -4 \end{bmatrix} \cdot \begin{bmatrix} 0 & 3 \\ 1 & -1 \\ 2 & 1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 14 & 5 \\ -21 & 2 \end{bmatrix}$$

5.

$$\begin{bmatrix} 0 & 3 \\ 1 & -1 \\ 2 & 1 \\ 5 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 1 & 2 \\ 4 & 1 & -1 & -4 \end{bmatrix} = \begin{bmatrix} 12 & 3 & -3 & -12 \\ -3 & 1 & 2 & 6 \\ 6 & 5 & 1 & 0 \\ 13 & 12 & 3 & 2 \end{bmatrix}$$

Problem 4.

Find all the solutions in $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$ of the equation system $Ax = 12x$ where:

$$A = \begin{bmatrix} 6 & 4 & 3 \\ 6 & 0 & 9 \\ 0 & 8 & 0 \end{bmatrix}$$

and $x_1 + x_2 + x_3 = 1$.

Solution. First, note that Ax is:

$$\begin{bmatrix} 6 & 4 & 3 \\ 6 & 0 & 9 \\ 0 & 8 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6x_1 + 4x_2 + 3x_3 \\ 6x_1 + 9x_3 \\ 8x_2 \end{bmatrix}$$

And so we want that:

$$\begin{bmatrix} 6x_1 + 4x_2 + 3x_3 \\ 6x_1 + 9x_3 \\ 8x_2 \end{bmatrix} = \begin{bmatrix} 12x_1 \\ 12x_2 \\ 12x_3 \end{bmatrix}$$

So we have the equations

$$\begin{array}{rcrcrl} 6x_1 & + & 4x_2 & + & 3x_3 & = 12x_1 \\ 6x_1 & & & + & 9x_3 & = 12x_2 \\ & & 8x_2 & & & = 12x_3 \end{array}$$

And so we end up with the system:

$$\begin{array}{rcrcrl} -6x_1 & + & 4x_2 & + & 3x_3 & = 0 \\ 6x_1 & - & 12x_2 & + & 9x_3 & = 0 \\ & & 8x_2 & - & 12x_3 & = 0 \end{array}$$

Which can be expresed in the next matrix:

$$\begin{aligned} \begin{bmatrix} -6 & 4 & 3 \\ 6 & -12 & 9 \\ 0 & 8 & -12 \end{bmatrix} &\longrightarrow \begin{bmatrix} -6 & 4 & 3 \\ 0 & -8 & 12 \\ 0 & 8 & -12 \end{bmatrix} -R_1 \\ &\longrightarrow \begin{bmatrix} -6 & 4 & 3 \\ 0 & -8 & 12 \\ 0 & 0 & 0 \end{bmatrix} -R_2 \\ &\longrightarrow \begin{bmatrix} -6 & 4 & 3 \\ 0 & -2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \tfrac{1}{4}R_2 \\ &\longrightarrow \begin{bmatrix} -6 & 6 & 0 \\ 0 & -2 & 3 \\ 0 & 0 & 0 \end{bmatrix} -R_2 \end{aligned}$$

Which lead us to the next equations:

$$-6x_1 + 6x_2 = 0$$
$$-2x_2 + 3x_3 = 0$$

From which we derive that:

$$x = \begin{bmatrix} \frac{3}{2}x_3 \\ \frac{3}{2}x_3 \\ x_3 \end{bmatrix}$$

And with the other condition, we must satisfy the equation as:

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ \frac{3}{2}x_3 + \frac{3}{2}x_3 + x_3 &= 1 \\ 4x_3 &= 1 \\ x_3 &= \frac{1}{4} \end{aligned}$$

Problem 5.

Which of the following sets are subsets of $\mathbb{R}^3$? **Solution.**

1. $A = \{(\lambda, \lambda + \mu^3, \lambda - \mu^3) | \lambda, \mu \in \mathbb{R}\}$. Take two elements in A and a scalar $c \in \mathbb{R}$, then:

$$\begin{aligned} c(\lambda_1, \lambda_1 + \mu_1^3, \lambda_1 - \mu_1^3) + (\lambda_2, \lambda_2 + \mu_2^3, \lambda_2 - \mu_2^3) &= (c\lambda_1, c\lambda_1 + c\mu_1^3, c\lambda_1 - c\mu_1^3) + (\lambda_2, \lambda_2 + \mu_2^3, \lambda_2 - \mu_2^3) \\ &= (c\lambda_1 + \lambda_2, (c\lambda_1 + \lambda_2) + (c\mu_1^3 + \mu_2^3), (c\lambda_1 + \lambda_2) - (c\mu_1^3 + \mu_2^3)) \end{aligned}$$

   And since $\sqrt[3]{c\mu_1^3 + \mu_2^3}$ will always be a real number, then the linear combination of these elements is in A and so it is a subspace of $\mathbb{R}^3$

2. $B = \{(\lambda^2, -\lambda^2, 0) | \lambda \in \mathbb{R}\}$. If you take $c = -1$ and two elements in B we have:

$$\begin{aligned} -(\lambda_1^2, -\lambda_1^2, 0) + (\lambda_2^2, -\lambda_2^2, 0) &= (-\lambda_1^2, \lambda_1^2, 0) + (\lambda_2^2, -\lambda_2^2, 0) \\ &= (\lambda_2^2 - \lambda_1^2, \lambda_1^2 - \lambda_2^2, 0) \end{aligned}$$

   And if $\lambda_1^2 > \lambda_2^2$ then it is not defined its square and so it would not be an element of B. So B is not a subspace of $\mathbb{R}^3$

3. $C = \{(\lambda_1, \lambda_2, \lambda_3) | \lambda_1 - 2\lambda_2 + 3\lambda_3 = \gamma\}$ for a fixed γ. If we take two elements of C we would have the equations:

$$\lambda_1 - 2\lambda_2 + 3\lambda_3 = \gamma$$
$$\psi_1 - 2\psi_2 + 3\psi_3 = \gamma$$

   And adding them up we get:

$$(\lambda_1 + \psi_1) - 2(\lambda_2 + \psi_2) + 3(\lambda_3 + \psi_3) = 2\gamma$$

   So, unless $\gamma = 0$ we conclude that C is not a subspace of $\mathbb{R}^3$.

4. $D = \{(\lambda_1, \lambda_2, \lambda_3) | \lambda_2 \in \mathbb{Z}\}$. If we take an element $c \in \mathbb{R}$ such that c is irrational and we multiply it by an element of D then $c\lambda_2$ would not be an integer and so that element would not be on D. Therefore, D is not a subspace of $\mathbb{R}^3$.

Problem 6.

Write the vector $\begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$ as a linear combination of the vectors $x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $x_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$.

Solution. For that, let's write the matrix of the system of equations as:

$$\begin{aligned}
\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 2 & -1 & -2 \\ 1 & 3 & 1 & 5 \end{array}\right] &\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 2 & -1 & 4 \end{array}\right] \begin{array}{l} \\ -R_1 \\ -R_1 \end{array} \\
&\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 5 & 10 \end{array}\right] \begin{array}{l} \\ \\ -2R_2 \end{array} \\
&\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} \\ \\ \frac{1}{2}R_3 \end{array} \\
&\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & -3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} -2R_3 \\ +3R_3 \\ \\ \end{array} \\
&\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right] \begin{array}{l} -R_2 \\ \\ \\ \end{array}
\end{aligned}$$

And so we have the linear combination:

$$\begin{aligned}
-6 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} &= \begin{bmatrix} -6 \\ -6 \\ -6 \end{bmatrix} + \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} + \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}
\end{aligned}$$