Problem set
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1 Systems of linear equations, Matrices, and vector spaces, rank of matrices, Linear maps, and affine spaces

Problem 1.

We consider $(\mathbb{R} \setminus \{-1\}, \star)$ where:

$$a \star b = a + ab + b$$

for $a, b \in \mathbb{R} \setminus \{-1\}$. Show that this is an abelian group and solve $3 \star x \star x = 15$.

Solution. We prove this showing that the five axioms are true:

- First, suppose that $a + ab + b = -1$ and so we would have that $a(1+b) + b = -1$, and so we have $a(1+b) = -(1+b)$ and since $b \neq 1$ we can assure that $1 + b \neq 0$ so $a = -1$ which is a contradiction to the hypothesis that $a \neq 1$, so $\star$ is closed under the set.

- Now, we want to prove associativity(We pay attention to the associativity in the real numbers)

$$\begin{aligned} a \star (b \star c) &= a \star (b + bc + c) \\ &= a + a(b + bc + c) + (b + bc + c) \\ &= a + ab + abc + ac + b + bc + c \end{aligned} \qquad \begin{aligned} (a \star b) \star c &= (a + ab + b) \star c \\ &= (a + ab + b) + c(a + ab + b) + c \\ &= a + ab + b + ac + abc + bc + c \end{aligned}$$

  and so it is clear that they are the same.

- Note that the element 0 is an identity element because:

$$\begin{aligned} a \star 0 &= a + a \cdot 0 + 0 \\ &= a \end{aligned} \qquad \begin{aligned} 0 \star a &= 0 + a \cdot 0 + a \\ &= a \end{aligned}$$

- The inverse element for x is $\frac{-x}{1+x}$ since:

$$\begin{aligned} x \star \frac{-x}{1+x} &= x - x \cdot \frac{x}{1+x} - \frac{x}{1+x} \\ &= x - \frac{x^2}{1+x} - \frac{x}{1+x} \\ &= \frac{x + x^2}{1+x} - \frac{x^2}{1+x} - \frac{x}{1+x} \\ &= 0 \end{aligned}$$

  And the commutated case is the same, so it has inverses.

- The commutativity is a consequence of these properties in $\mathbb{R}$:

$$\begin{aligned} a \star b &= a + ab + b \\ &= b + ba + a \\ &= b \star a \end{aligned}$$

And so we conclude that $(\mathbb{R} \setminus \{-1\}, \star)$ is an abelian group. For the equation, we do:

$$\begin{aligned} 3 \star x \star x &= 15 \\ 3 \star (2x + x^2) &= 15 \\ 3 + 3(2x + x^2) + (2x + x^2) &= 15 \\ 6x + 3x^2 + 2x + x^2 &= 12 \\ 4x^2 + 8x &= 12 \\ x^2 + 2x &= 3 \\ x^2 + 2x - 3 &= 0 \\ (x-1)(x+3) &= 0 \end{aligned}$$

And therefore we conclude that $x = 1$ or $x = -3$. If we put this into the equation we have:

$$\begin{aligned} 3 \star 1 \star 1 &= 3 \star 3 \\ &= 3 + 9 + 3 \\ &= 15 \end{aligned} \qquad \begin{aligned} 3 \star (-3 \star (-3)) &= 3 \star 3 \\ &= 3 + 9 + 3 \\ &= 15 \end{aligned}$$

So the solutions are $x = 1$ and $x = -3$.

Problem 2.

Consider the set $\mathcal{G}$ of 3×3 matrices defined as follows:

$$\mathcal{G} = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{3\times 3} \,\middle|\, x, y, z \in \mathbb{R} \right\}$$

And we define $\cdot$ as the standard matrix multiplication. Is $(\mathcal{G}, \cdot)$ a group? If yes, is it abelian?

Solution. We prove this showing that the four axioms are true:

- First, it is closed under $\cdot$ since:

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+x & c+bx+z \\ 0 & 1 & b+y \\ 0 & 0 & 1 \end{bmatrix}$$

  And since the operations of addition and product of real numbers are closed, by definition the matrix is also in $\mathcal{G}$ so it is a closed operation.

- We can prove the associativity by taking three matrices and show that their product don't vary.

$$\begin{aligned} \left(\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \right) \cdot \begin{bmatrix} 1 & k & m \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix} &= \begin{bmatrix} 1 & a+x & c+bx+z \\ 0 & 1 & b+y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & a+x & c+bx+z \\ 0 & 1 & b+y \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & a+k+x & m+an+ax+c+bx+z \\ 0 & 1 & b+n+y \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

  And if we make the other option:

$$\begin{aligned} \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \cdot \left(\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & k & m \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix} \right) &= \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & a+k & m+an+c \\ 0 & 1 & n+b \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & a+k+x & m+an+c+xn+bx \\ 0 & 1 & b+n+y \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

  And we can see that they are the same, so $\cdot$ is associative.

- Note that the matrix I_3 is also in the set since $0 \in \mathbb{R}$, and we know that any matrix 3×3 operated with I_3 is the same, so I_3 is the identity for $\mathcal{G}$.
- For finding the inverse matrix for an element in $\mathcal{G}$ we do the next:

$$\left[\begin{array}{ccc|ccc} 1 & x & z & 1 & 0 & 0 \\ 0 & 1 & y & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|ccc} 1 & x & 0 & 1 & 0 & -z \\ 0 & 1 & 0 & 0 & 1 & -y \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]\begin{array}{l} \\ -zR_3 \\ -yR_3 \end{array}$$

$$\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -x & xy-z \\ 0 & 1 & 0 & 0 & 1 & -y \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]\begin{array}{l} -xR_2 \\ \\ \\ \end{array}$$

  So the matrix done in the right side is the inverse of the matrix in $\mathcal{G}$. For that, note that:

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -x & xy-z \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

  And the same is true for the converse, and since $-x, -y$ and $xy-z$ are also real numbers, we have shown the existence of inverses in $\mathcal{G}$.

Note that this group is not abelian since:

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+x & c+bx+z \\ 0 & 1 & b+y \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+x & z+ay+c \\ 0 & 1 & b+y \\ 0 & 0 & 1 \end{bmatrix}$$

And if $bx \neq ay$ then they are not the same. So, the group is not abelian.

Problem 3.

If it is possible compute the next products.

Solution. The products are:

1. $$\begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

   This product is not possible since the first matrix is a 3×2 matrix and the other one is a 3×3 matrix, and hence $3 \neq 2$ we cannot operate it.

2. $$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 5 \\ 10 & 9 & 11 \\ 16 & 15 & 17 \end{bmatrix}$$

3. $$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 5 & 7 & 9 \\ 11 & 13 & 15 \\ 8 & 10 & 12 \end{bmatrix}$$

4. $$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 4 & 1 & -1 & -4 \end{bmatrix} \cdot \begin{bmatrix} 0 & 3 \\ 1 & -1 \\ 2 & 1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 14 & 5 \\ -21 & 2 \end{bmatrix}$$

5.

$$\begin{bmatrix} 0 & 3 \\ 1 & -1 \\ 2 & 1 \\ 5 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 1 & 2 \\ 4 & 1 & -1 & -4 \end{bmatrix} = \begin{bmatrix} 12 & 3 & -3 & -12 \\ -3 & 1 & 2 & 6 \\ 6 & 5 & 1 & 0 \\ 13 & 12 & 3 & 2 \end{bmatrix}$$

Problem 4.

Find all the solutions in $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$ of the equation system $Ax = 12x$ where:

$$A = \begin{bmatrix} 6 & 4 & 3 \\ 6 & 0 & 9 \\ 0 & 8 & 0 \end{bmatrix}$$

and $x_1 + x_2 + x_3 = 1$.

Solution. First, note that Ax is:

$$\begin{bmatrix} 6 & 4 & 3 \\ 6 & 0 & 9 \\ 0 & 8 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6x_1 + 4x_2 + 3x_3 \\ 6x_1 + 9x_3 \\ 8x_2 \end{bmatrix}$$

And so we want that:

$$\begin{bmatrix} 6x_1 + 4x_2 + 3x_3 \\ 6x_1 + 9x_3 \\ 8x_2 \end{bmatrix} = \begin{bmatrix} 12x_1 \\ 12x_2 \\ 12x_3 \end{bmatrix}$$

So we have the equations

$$\begin{array}{rcrcrl} 6x_1 & + & 4x_2 & + & 3x_3 & = 12x_1 \\ 6x_1 & & & + & 9x_3 & = 12x_2 \\ & & 8x_2 & & & = 12x_3 \end{array}$$

And so we end up with the system:

$$\begin{array}{rcrcrl} -6x_1 & + & 4x_2 & + & 3x_3 & = 0 \\ 6x_1 & - & 12x_2 & + & 9x_3 & = 0 \\ & & 8x_2 & - & 12x_3 & = 0 \end{array}$$

Which can be expresed in the next matrix:

$$\begin{aligned} \begin{bmatrix} -6 & 4 & 3 \\ 6 & -12 & 9 \\ 0 & 8 & -12 \end{bmatrix} &\longrightarrow \begin{bmatrix} -6 & 4 & 3 \\ 0 & -8 & 12 \\ 0 & 8 & -12 \end{bmatrix} -R_1 \\ &\longrightarrow \begin{bmatrix} -6 & 4 & 3 \\ 0 & -8 & 12 \\ 0 & 0 & 0 \end{bmatrix} -R_2 \\ &\longrightarrow \begin{bmatrix} -6 & 4 & 3 \\ 0 & -2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \tfrac{1}{4}R_2 \\ &\longrightarrow \begin{bmatrix} -6 & 6 & 0 \\ 0 & -2 & 3 \\ 0 & 0 & 0 \end{bmatrix} -R_2 \end{aligned}$$

Which lead us to the next equations:

$$-6x_1 + 6x_2 = 0$$
$$-2x_2 + 3x_3 = 0$$

From which we derive that:

$$x = \begin{bmatrix} \frac{3}{2}x_3 \\ \frac{3}{2}x_3 \\ x_3 \end{bmatrix}$$

And with the other condition, we must satisfy the equation as:

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ \frac{3}{2}x_3 + \frac{3}{2}x_3 + x_3 &= 1 \\ 4x_3 &= 1 \\ x_3 &= \frac{1}{4} \end{aligned}$$

Problem 5.

Which of the following sets are subsets of $\mathbb{R}^3$?

Solution.

1. $A = \{(\lambda, \lambda + \mu^3, \lambda - \mu^3) | \lambda, \mu \in \mathbb{R}\}$. Take two elements in A and a scalar $c \in \mathbb{R}$, then:

   $$\begin{aligned} c(\lambda_1, \lambda_1 + \mu_1^3, \lambda_1 - \mu_1^3) + (\lambda_2, \lambda_2 + \mu_2^3, \lambda_2 - \mu_2^3) &= (c\lambda_1, c\lambda_1 + c\mu_1^3, c\lambda_1 - c\mu_1^3) + (\lambda_2, \lambda_2 + \mu_2^3, \lambda_2 - \mu_2^3) \\ &= (c\lambda_1 + \lambda_2, (c\lambda_1 + \lambda_2) + (c\mu_1^3 + \mu_2^3), (c\lambda_1 + \lambda_2) - (c\mu_1^3 + \mu_2^3)) \end{aligned}$$

   And since $\sqrt[3]{c\mu_1^3 + \mu_2^3}$ will always be a real number, then the linear combination of these elements is in A and so it is a subspace of $\mathbb{R}^3$

2. $B = \{(\lambda^2, -\lambda^2, 0) | \lambda \in \mathbb{R}\}$. If you take $c = -1$ and two elements in B we have:

   $$\begin{aligned} -(\lambda_1^2, -\lambda_1^2, 0) + (\lambda_2^2, -\lambda_2^2, 0) &= (-\lambda_1^2, \lambda_1^2, 0) + (\lambda_2^2, -\lambda_2^2, 0) \\ &= (\lambda_2^2 - \lambda_1^2, \lambda_1^2 - \lambda_2^2, 0) \end{aligned}$$

   And if $\lambda_1^2 > \lambda_2^2$ then it is not defined its square and so it would not be an element of B. So B is not a subspace of $\mathbb{R}^3$

3. $C = \{(\lambda_1, \lambda_2, \lambda_3) | \lambda_1 - 2\lambda_2 + 3\lambda_3 = \gamma\}$ for a fixed γ. If we take two elements of C we would have the equations:

   $$\lambda_1 - 2\lambda_2 + 3\lambda_3 = \gamma$$
   $$\psi_1 - 2\psi_2 + 3\psi_3 = \gamma$$

   And adding them up we get:

   $$(\lambda_1 + \psi_1) - 2(\lambda_2 + \psi_2) + 3(\lambda_3 + \psi_3) = 2\gamma$$

   So, unless $\gamma = 0$ we conclude that C is not a subspace of $\mathbb{R}^3$.

4. $D = \{(\lambda_1, \lambda_2, \lambda_3) | \lambda_2 \in \mathbb{Z}\}$. If we take an element $c \in \mathbb{R}$ such that c is irrational and we multiply it by an element of D then $c\lambda_2$ would not be an integer and so that element would not be on D. Therefore, D is not a subspace of $\mathbb{R}^3$.

Problem 6.

Write the vector $\begin{bmatrix}1\\-2\\5\end{bmatrix}$ as a linear combination of the vectors $x_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$, $x_2 = \begin{bmatrix}1\\2\\3\end{bmatrix}$ and $x_3 \begin{bmatrix}2\\-1\\1\end{bmatrix}$.

Solution. For that, let's write the matrix of the system of equations as:

$$\left[\begin{array}{ccc|c}1 & 1 & 2 & 1\\1 & 2 & -1 & -2\\1 & 3 & 1 & 5\end{array}\right] \longrightarrow \left[\begin{array}{ccc|c}1 & 1 & 2 & 1\\0 & 1 & -3 & -3\\0 & 2 & -1 & 4\end{array}\right] \begin{array}{l}\\-R_1\\-R_1\end{array}$$

$$\longrightarrow \left[\begin{array}{ccc|c}1 & 1 & 2 & 1\\0 & 1 & -3 & -3\\0 & 0 & 5 & 10\end{array}\right] \begin{array}{l}\\\\-2R_2\end{array}$$

$$\longrightarrow \left[\begin{array}{ccc|c}1 & 1 & 2 & 1\\0 & 1 & -3 & -3\\0 & 0 & 1 & 2\end{array}\right] \begin{array}{l}\\\\\frac{1}{2}R_3\end{array}$$

$$\longrightarrow \left[\begin{array}{ccc|c}1 & 1 & 0 & -3\\0 & 1 & 0 & 3\\0 & 0 & 1 & 2\end{array}\right] \begin{array}{l}-2R_3\\+3R_3\\\\\end{array}$$

$$\longrightarrow \left[\begin{array}{ccc|c}1 & 0 & 0 & -6\\0 & 1 & 0 & 3\\0 & 0 & 1 & 2\end{array}\right] \begin{array}{l}-R_2\\\\\\\end{array}$$

And so we have the linear combination:

$$\begin{aligned}-6\begin{bmatrix}1\\1\\1\end{bmatrix} + 3\begin{bmatrix}1\\2\\3\end{bmatrix} + 2\begin{bmatrix}2\\-1\\1\end{bmatrix} &= \begin{bmatrix}-6\\-6\\-6\end{bmatrix} + \begin{bmatrix}3\\6\\9\end{bmatrix} + \begin{bmatrix}4\\-2\\2\end{bmatrix}\\ &= \begin{bmatrix}1\\-2\\5\end{bmatrix}\end{aligned}$$

Problem 7.

Consider the linear mapping defined by:

$$\begin{array}{cccc}\phi: & \mathbb{R}^3 & \to & \mathbb{R}^4\\ & (x_1, x_2, x_3) & \mapsto & (3x_1 + 2x_2 + x_3, x_1 + x_2 + x_3, x_1 - 3x_2, 2x_1 + 3x_2 + x_3)\end{array}$$

Find the matrix associated with the linear transformation, determine the rank of the matrix and compute the image, the kernel and their dimension.

Solution. First, we apply the transformation to the canonical basis in $\mathbb{R}^3$ as:

$$\begin{aligned}\phi(1,0,0) &= (3,1,1,2)\\ \phi(0,1,0) &= (2,1,-3,3)\\ \phi(0,0,1) &= (1,1,0,1)\end{aligned}$$

And so we end up with the matrix:

$$A = \begin{bmatrix}3 & 2 & 1\\1 & 1 & 1\\1 & -3 & 0\\2 & 3 & 1\end{bmatrix}$$

Now, if we reduce it to the echelon form:

$$
\begin{aligned}
\left[\begin{array}{ccc|c} 3 & 2 & 1 & a \\ 1 & 1 & 1 & b \\ 1 & -3 & 0 & c \\ 2 & 3 & 1 & d \end{array}\right]
&\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & b \\ 3 & 2 & 1 & a \\ 1 & -3 & 0 & c \\ 2 & 3 & 1 & d \end{array}\right] \begin{array}{l} R_2 \\ R_1 \\ \\ \\ \end{array} \\
&\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & b \\ 0 & -1 & -2 & a-3b \\ 0 & -4 & -1 & c-b \\ 0 & 1 & -1 & d-2b \end{array}\right] \begin{array}{l} \\ -3R_1 \\ -R_1 \\ -2R_1 \end{array} \\
&\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & b \\ 0 & 1 & 2 & 3b-a \\ 0 & 4 & 1 & b-c \\ 0 & -1 & 1 & 2b-d \end{array}\right] \begin{array}{l} \\ -R_2 \\ -R_3 \\ -R_4 \end{array} \\
&\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & b \\ 0 & 1 & 2 & 3b-a \\ 0 & 0 & -3 & -3b-c \\ 0 & 0 & -2 & 3b-d \end{array}\right] \begin{array}{l} \\ \\ -4R_1 \\ +R_1 \end{array} \\
&\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & b \\ 0 & 1 & 2 & 3b-a \\ 0 & 0 & 1 & b+\frac{c}{3} \\ 0 & 0 & -1 & d-3b \end{array}\right] \begin{array}{l} \\ \\ -\frac{1}{3}R_3 \\ -R_4 \end{array} \\
&\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & b \\ 0 & 1 & 2 & 3b-a \\ 0 & 0 & 1 & b+\frac{c}{3} \\ 0 & 0 & 0 & d+\frac{c}{3}-2b \end{array}\right] \begin{array}{l} \\ \\ \\ +R_3 \end{array} \\
&\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & -\frac{c}{3} \\ 0 & 1 & 0 & b-a-\frac{2c}{3} \\ 0 & 0 & 1 & b+\frac{c}{3} \\ 0 & 0 & 0 & d+\frac{c}{3}-2b \end{array}\right] \begin{array}{l} -R_3 \\ -2R_3 \\ \\ \\ \end{array} \\
&\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & a-b+\frac{c}{3} \\ 0 & 1 & 0 & b-a-\frac{2c}{3} \\ 0 & 0 & 1 & b+\frac{c}{3} \\ 0 & 0 & 0 & d+\frac{c}{3}-2b \end{array}\right] \begin{array}{l} -R_2 \\ \\ \\ \\ \end{array}
\end{aligned}
$$

So the image is the set:

$$Im(\phi) = \{(a,b,c,d) \in \mathbb{R}^4 | d + \frac{c}{3} - 2b = 0\}$$

Note that when $(a,b,c,d) = (0,0,0,0)$ then the system only has the trivial solution and so we conclude that the rank of A is 3 and that the kernel is just $\{0\}$. By the rank theorem, we have:

$$
\begin{aligned}
\dim V &= \dim(\ker(\phi)) + \dim(\phi(V)) \\
\dim \mathbb{R}^3 &= \dim(\{0\}) + \dim(\phi(V)) \\
3 &= 0 + \dim(\phi(V)) \\
\dim(\phi(V)) &= 3
\end{aligned}
$$

And so we conclude that even when the linear transformation is monic, it is not epic.

Problem 8.

Let V be a vector space. Let f and g two automorphism over E such that $f \circ g = Id_E$. Show that $\ker(f) = \ker(g \circ f)$, $Im(g) = Im(g \circ f)$ and that $\ker(f) \cap Im(g) = \{0_E\}$.

Solution. First, we want to show that $\ker(g \circ f) = \{0_E\}$ since f is an automorphism. So, suppose that $v \in E$ is such that $(g \circ f)(v) = g(f(v)) = 0$. Since g is an automorphism, we conclude that $f(v) = 0$ and since f is also an automorphism, we conclude that $v = 0$. So, $\ker(f) = \ker(g \circ f)$.

Also, we want to show $Im(g \circ f) = E$, so take $v \in E$, since g is an automorphism we can find $u \in E$ such that $g(u) = v$. And also, since f is an automorphism, we can find $w \in E$ such that $f(w) = u$, so we have that:

$$\begin{aligned}(g \circ f)(w) &= g(f(w)) \\ &= g(u) \\ &= v\end{aligned}$$

And so we conclude that $Im(g) = Im(g \circ f)$. Note that also $0 \in Im(g)$ but the only element in $\ker(f)$ is 0, so we must have that $\ker(f) \cap Im(g) = \{0_E\}$.

Problem 9.

Let $F = \{(x, y, z) \in \mathbb{R}^3 | x + y - z = 0\}$ and $G = \{(a - b, a + b, a - 3b) \in \mathbb{R}^3 | a, b \in \mathbb{R}\}$. Prove that they are subspaces of $\mathbb{R}^3$, calculate $F \cap G$ and then using basis for F and G check the result.

Solution. First, we are going that both sets are subspaces of $\mathbb{R}^3$:

- Let $(x, y, z), (a, b, c) \in F$ and let $r \in \mathbb{R}$, we want to show that their linear combination is in F:

  $$\begin{aligned}r(x, y, z) + (a, b, c) &= (rx, ry, rz) + (a, b, c) \\ &= (rx + a, ry + b, rz + c)\end{aligned}$$

  And by definition, we have the equations:

  $$\begin{aligned}x + y - z &= 0 \\ a + b - c &= 0\end{aligned}$$

  If we multiply the first by r we get:

  $$\begin{aligned}rx + ry - rz &= 0 \\ a + b - c &= 0\end{aligned}$$

  and if we add them up:

  $$(rx + a) + (ry + b) - (rz + c) = 0$$

  Which proves that $(rx + a, ry + b, rz + c) \in F$ and so it is a subspace.

- Let $(a - b, a + b, a - 3b), (x - y, x + y, x - 3y)$ and $r \in \mathbb{R}$ we want to show that their linear combination is in G:

  $$\begin{aligned}r(a - b, a + b, a - 3b) + (x - y, x + y, x - 3y) &= (ar - br, ar + br, ar - 3br) + (x - y, x + y, x - 3y) \\ &= ((ar + x) - (br + y), (ar + x) + (br + y), (ar + x) - 3(br + y))\end{aligned}$$

  and since $ar + x, br + y \in \mathbb{R}$, by definition $((ar + x) - (br + y), (ar + x) + (br + y), (ar + x) - 3(br + y))$ and so G is a subspace.

Now, if we calculate $F \cap G$, we take a vector $(x, y, z) \in F \cap G$ then it is necessary that $x + y - z = 0$ and for $a, b \in \mathbb{R}$ we got $x = a - b$, $y = a + b$ and $z = a - 3b$. If we replace it into the another equation we get:

$$\begin{aligned} x + y - z &= 0 \\ a - b + a + b - a + 3b &= 0 \\ a + 3b &= 0 \\ b &= -\frac{a}{3} \end{aligned}$$

and so we get:

$$\begin{aligned} x &= \frac{4a}{3} \\ y &= \frac{2a}{3} \\ z &= 2a \end{aligned}$$

so we got vectors of the form $\left(\frac{4a}{3}, \frac{2a}{3}, 2a\right)$ such that $a \in \mathbb{R}$. Now, we are going to find a basis for F and for G.

- For F, note that each vector is of the form (x, y, z) with $x + y - z = 0$ and so $z = x + y$, so we can express it:

$$\begin{aligned} (x, y, z) &= (x, y, x + y) \\ &= (x, 0, x) + (0, y, y) \\ &= x(1, 0, 1) + y(0, 1, 1) \end{aligned}$$

  And so a basis for F is $\{(1, 0, 1), (0, 1, 1)\}$.

- For G, we have vectors of the form $(a - b, a + b, a - 3b)$ so we can descompose it like:

$$\begin{aligned} (a - b, a + b, a - 3b) &= (a, a, a) + (-b, b, -3b) \\ &= a(1, 1, 1) + b(-1, 1, -3) \end{aligned}$$

  And so a basis for G is $\{(1, 1, 1), (-1, 1, -3)\}$.

And now we express a vector of $F \cap G$ as a linear combination with scalars $x, y, a, b \in \mathbb{R}$ as:

$$\begin{aligned} x(1, 0, 1) + y(0, 1, 1) &= a(1, 1, 1) + b(-1, 1, -3) \\ (x, 0, x) + (0, y, y) &= (a, a, a) + (-b, b, -3b) \\ (x, y, x + y) &= (a - b, a + b, a - 3b) \end{aligned}$$

and so we get $x = a - b$, $y = a + b$ and $x + y = a - 3b$, but we can replace in this last equation as:

$$\begin{aligned} x + y &= a - 3b \\ a - b + a + b &= a - 3b \\ 2a &= a - 3b \\ a &= -3b \\ b &= -\frac{a}{3} \end{aligned}$$

and so we get the same result as we did without the basis.