

1 Systems of linear equations, Matrices, and vector spaces, rank of matrices, Linear maps, and affine spaces

Problem 1.

We consider $(\mathbb{R} \setminus \{-1\}, \star)$ where:

$$a \star b = a + ab + b$$

for $a, b \in \mathbb{R} \setminus \{-1\}$. Show that this is an abelian group and solve $3 \star x \star x = 15$.

Solution. We prove this showing that the five axioms are true:

- First, suppose that $a + ab + b = -1$ and so we would have that $a(1 + b) + b = -1$, and so we have $a(1 + b) = -(1 + b)$ and since $b \neq 1$ we can assure that $1 + b \neq 0$ so $a = -1$ which is a contradiction to the hypothesis that $a \neq -1$, so \star is closed under the set.
- Now, we want to prove associativity (We pay attention to the associativity in the real numbers)

$$\begin{aligned} a \star (b \star c) &= a \star (b + bc + c) & (a \star b) \star c &= (a + ab + b) \star c \\ &= a + a(b + bc + c) + (b + bc + c) & &= (a + ab + b) + c(a + ab + b) + c \\ &= a + ab + abc + ac + b + bc + c & &= a + ab + b + ac + abc + bc + c \end{aligned}$$

and so it is clear that they are the same.

- Note that the element 0 is an identity element because:

$$\begin{aligned} a \star 0 &= a + a \cdot 0 + 0 & 0 \star a &= 0 + a \cdot 0 + a \\ &= a & &= a \end{aligned}$$

- The inverse element for x is $\frac{-x}{1+x}$ since:

$$\begin{aligned} x \star \frac{-x}{1+x} &= x - x \cdot \frac{x}{1+x} - \frac{x}{1+x} \\ &= x - \frac{x^2}{1+x} - \frac{x}{1+x} \\ &= \frac{x + x^2}{1+x} - \frac{x^2}{1+x} - \frac{x}{1+x} \\ &= 0 \end{aligned}$$

And the commutated case is the same, so it has inverses.

- The commutativity is a consequence of these properties in \mathbb{R} :

$$\begin{aligned} a \star b &= a + ab + b \\ &= b + ba + a \\ &= b \star a \end{aligned}$$

And so we conclude that $(\mathbb{R} \setminus \{-1\}, \star)$ is an abelian group. For the equation, we do:

$$\begin{aligned}
 3 \star x \star x &= 15 \\
 3 \star (2x + x^2) &= 15 \\
 3 + 3(2x + x^2) + (2x + x^2) &= 15 \\
 6x + 3x^2 + 2x + x^2 &= 12 \\
 4x^2 + 8x &= 12 \\
 x^2 + 2x &= 3 \\
 x^2 + 2x - 3 &= 0 \\
 (x - 1)(x + 3) &= 0
 \end{aligned}$$

And therefore we conclude that $x = 1$ or $x = -3$. If we put this into the equation we have:

$$\begin{aligned}
 3 \star 1 \star 1 &= 3 \star 3 & 3 \star (-3 \star (-3)) &= 3 \star 3 \\
 &= 3 + 9 + 3 & &= 3 + 9 + 3 \\
 &= 15 & &= 15
 \end{aligned}$$

So the solutions are $x = 1$ and $x = -3$.

Problem 2.

Consider the set \mathcal{G} of 3×3 matrices defined as follows:

$$\mathcal{G} = \left\{ \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \in \mathbb{R}^{3 \times 3} \mid x, y, z \in \mathbb{R} \right\}$$

And we define \cdot as the standard matrix multiplication. Is (\mathcal{G}, \cdot) a group? If yes, is it abelian?

Solution. We prove this showing that the four axioms are true:

- First, it is closed under \cdot since:

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+x & c+bx+z \\ 0 & 1 & b+y \\ 0 & 0 & 1 \end{bmatrix}$$

And since the operations of addition and product of real numbers are closed, by definition the matrix is also in \mathcal{G} so it is a closed operation.

- We can prove the associativity by taking three matrices and show that their product don't vary.

$$\begin{aligned}
 \left(\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \right) \cdot \begin{bmatrix} 1 & k & m \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix} &= \begin{bmatrix} 1 & a+x & c+bx+z \\ 0 & 1 & b+y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & a+x & c+bx+z \\ 0 & 1 & b+y \\ 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & a+k+x & m+an+ax+c+bx+z \\ 0 & 1 & b+n+y \\ 0 & 0 & 1 \end{bmatrix}
 \end{aligned}$$

And if we make the other option:

$$\begin{aligned}
 \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \cdot \left(\begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & k & m \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix} \right) &= \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & a+k & m+an+c \\ 0 & 1 & n+b \\ 0 & 0 & 1 \end{bmatrix} \\
 &= \begin{bmatrix} 1 & a+k+x & m+an+c+xn+bx \\ 0 & 1 & b+n+y \\ 0 & 0 & 1 \end{bmatrix}
 \end{aligned}$$

And we can see that they are the same, so \cdot is associative.

- Note that the matrix I_3 is also in the set since $0 \in \mathbb{R}$, and we know that any matrix 3×3 operated with I_3 is the same, so I_3 is the identity for \mathcal{G} .
- For finding the inverse matrix for an element in \mathcal{G} we do the next:

$$\begin{aligned} \left[\begin{array}{ccc|ccc} 1 & x & z & 1 & 0 & 0 \\ 0 & 1 & y & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right] &\longrightarrow \left[\begin{array}{ccc|ccc} 1 & x & 0 & 1 & 0 & -z \\ 0 & 1 & 0 & 0 & 1 & -y \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} -zR_3 \\ -yR_3 \\ \end{array} \\ &\longrightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -x & xy-z \\ 0 & 1 & 0 & 0 & 1 & -y \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array} \right] -xR_2 \end{aligned}$$

So the matrix done in the right side is the inverse of the matrix in \mathcal{G} . For that, note that:

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -x & xy-z \\ 0 & 1 & -y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

And the same is true for the converse, and since $-x$, $-y$ and $xy - z$ are also real numbers, we have shown the existence of inverses in \mathcal{G} .

Note that this group is not abelian since:

$$\begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+x & c+bx+z \\ 0 & 1 & b+y \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & a & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+x & z+ay+c \\ 0 & 1 & b+y \\ 0 & 0 & 1 \end{bmatrix}$$

And if $bx \neq ay$ then they are not the same. So, the group is not abelian.

Problem 3.

If it is possible compute the next products.

Solution. The products are:

1.

$$\begin{bmatrix} 1 & 2 \\ 4 & 5 \\ 7 & 8 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

This product is not possible since the first matrix is a 3×2 matrix and the other one is a 3×3 matrix, and hence $3 \neq 2$ we cannot operate it.

2.

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 & 5 \\ 10 & 9 & 11 \\ 16 & 15 & 17 \end{bmatrix}$$

3.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 5 & 7 & 9 \\ 11 & 13 & 15 \\ 8 & 10 & 12 \end{bmatrix}$$

4.

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 4 & 1 & -1 & -4 \end{bmatrix} \cdot \begin{bmatrix} 0 & 3 \\ 1 & -1 \\ 2 & 1 \\ 5 & 2 \end{bmatrix} = \begin{bmatrix} 14 & 5 \\ -21 & 2 \end{bmatrix}$$

5.

$$\begin{bmatrix} 0 & 3 \\ 1 & -1 \\ 2 & 1 \\ 5 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 1 & 2 \\ 4 & 1 & -1 & -4 \end{bmatrix} = \begin{bmatrix} 12 & 3 & -3 & -12 \\ -3 & 1 & 2 & 6 \\ 6 & 5 & 1 & 0 \\ 13 & 12 & 3 & 2 \end{bmatrix}$$

Problem 4.

Find all the solutions in $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbb{R}^3$ of the equation system $Ax = 12x$ where:

$$A = \begin{bmatrix} 6 & 4 & 3 \\ 6 & 0 & 9 \\ 0 & 8 & 0 \end{bmatrix}$$

and $x_1 + x_2 + x_3 = 1$.

Solution. First, note that Ax is:

$$\begin{bmatrix} 6 & 4 & 3 \\ 6 & 0 & 9 \\ 0 & 8 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6x_1 + 4x_2 + 3x_3 \\ 6x_1 + 9x_3 \\ 8x_2 \end{bmatrix}$$

And so we want that:

$$\begin{bmatrix} 6x_1 + 4x_2 + 3x_3 \\ 6x_1 + 9x_3 \\ 8x_2 \end{bmatrix} = \begin{bmatrix} 12x_1 \\ 12x_2 \\ 12x_3 \end{bmatrix}$$

So we have the equations

$$\begin{array}{ccccccc} 6x_1 & + & 4x_2 & + & 3x_3 & = & 12x_1 \\ 6x_1 & & & + & 9x_3 & = & 12x_2 \\ & & 8x_2 & & & = & 12x_3 \end{array}$$

And so we end up with the system:

$$\begin{array}{ccccccc} -6x_1 & + & 4x_2 & + & 3x_3 & = & 0 \\ 6x_1 & - & 12x_2 & + & 9x_3 & = & 0 \\ & & 8x_2 & - & 12x_3 & = & 0 \end{array}$$

Which can be expressed in the next matrix:

$$\begin{aligned} \begin{bmatrix} -6 & 4 & 3 \\ 6 & -12 & 9 \\ 0 & 8 & -12 \end{bmatrix} &\longrightarrow \begin{bmatrix} -6 & 4 & 3 \\ 0 & -8 & 12 \\ 0 & 8 & -12 \end{bmatrix} -R_1 \\ &\longrightarrow \begin{bmatrix} -6 & 4 & 3 \\ 0 & -8 & 12 \\ 0 & 0 & 0 \end{bmatrix} -R_2 \\ &\longrightarrow \begin{bmatrix} -6 & 4 & 3 \\ 0 & -2 & 3 \\ 0 & 0 & 0 \end{bmatrix} \frac{1}{4}R_2 \\ &\longrightarrow \begin{bmatrix} -6 & 6 & 0 \\ 0 & -2 & 3 \\ 0 & 0 & 0 \end{bmatrix} -R_2 \end{aligned}$$

Which lead us to the next equations:

$$-6x_1 + 6x_2 = 0$$

$$-2x_2 + 3x_3 = 0$$

From which we derive that:

$$x = \begin{bmatrix} \frac{3}{2}x_3 \\ \frac{3}{2}x_3 \\ x_3 \end{bmatrix}$$

And with the other condition, we must satisfy the equation as:

$$x_1 + x_2 + x_3 = 1$$

$$\frac{3}{2}x_3 + \frac{3}{2}x_3 + x_3 = 1$$

$$4x_3 = 1$$

$$x_3 = \frac{1}{4}$$

Problem 5.

Which of the following sets are subsets of \mathbb{R}^3 ?

Solution.

1. $A = \{(\lambda, \lambda + \mu^3, \lambda - \mu^3) | \lambda, \mu \in \mathbb{R}\}$. Take two elements in A and a scalar $c \in \mathbb{R}$, then:

$$\begin{aligned} c(\lambda_1, \lambda_1 + \mu_1^3, \lambda_1 - \mu_1^3) + (\lambda_2, \lambda_2 + \mu_2^3, \lambda_2 - \mu_2^3) &= (c\lambda_1, c\lambda_1 + c\mu_1^3, c\lambda_1 - c\mu_1^3) + (\lambda_2, \lambda_2 + \mu_2^3, \lambda_2 - \mu_2^3) \\ &= (c\lambda_1 + \lambda_2, (c\lambda_1 + \lambda_2) + (c\mu_1^3 + \mu_2^3), (c\lambda_1 + \lambda_2) - (c\mu_1^3 + \mu_2^3)) \end{aligned}$$

And since $\sqrt[3]{c\mu_1^3 + \mu_2^3}$ will always be a real number, then the linear combination of these elements is in A and so it is a subspace of \mathbb{R}^3

2. $B = \{(\lambda^2, -\lambda^2, 0) | \lambda \in \mathbb{R}\}$. If you take $c = -1$ and two elements in B we have:

$$\begin{aligned} -(\lambda_1^2, -\lambda_1^2, 0) + (\lambda_2^2, -\lambda_2^2, 0) &= (-\lambda_1^2, \lambda_1^2, 0) + (\lambda_2^2, -\lambda_2^2, 0) \\ &= (\lambda_2^2 - \lambda_1^2, \lambda_1^2 - \lambda_2^2, 0) \end{aligned}$$

And if $\lambda_1^2 > \lambda_2^2$ then it is not defined its square and so it would not be an element of B . So B is not a subspace of \mathbb{R}^3

3. $C = \{(\lambda_1, \lambda_2, \lambda_3) | \lambda_1 - 2\lambda_2 + 3\lambda_3 = \gamma\}$ for a fixed γ . If we take two elements of C we would have the equations:

$$\begin{aligned} \lambda_1 - 2\lambda_2 + 3\lambda_3 &= \gamma \\ \psi_1 - 2\psi_2 + 3\psi_3 &= \gamma \end{aligned}$$

And adding them up we get:

$$(\lambda_1 + \psi_1) - 2(\lambda_2 + \psi_2) + 3(\lambda_3 + \psi_3) = 2\gamma$$

So, unless $\gamma = 0$ we conclude that C is not a subspace of \mathbb{R}^3 .

4. $D = \{(\lambda_1, \lambda_2, \lambda_3) | \lambda_2 \in \mathbb{Z}\}$. If we take an element $c \in \mathbb{R}$ such that c is irrational and we multiply it by an element of D then $c\lambda_2$ would not be an integer and so that element would not be on D . Therefore, D is not a subspace of \mathbb{R}^3 .

Problem 6.

Write the vector $\begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}$ as a linear combination of the vectors $x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $x_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $x_3 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$.

Solution. For that, let's write the matrix of the system of equations as:

$$\begin{aligned}
 \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 2 & -1 & -2 \\ 1 & 3 & 1 & 5 \end{array} \right] &\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 2 & -1 & 4 \end{array} \right] \begin{array}{l} \\ -R_1 \\ -R_1 \end{array} \\
 &\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 5 & 10 \end{array} \right] \begin{array}{l} \\ \\ -2R_2 \end{array} \\
 &\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 1 & 2 \end{array} \right] \begin{array}{l} \\ \\ \frac{1}{2}R_3 \end{array} \\
 &\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & -3 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array} \right] \begin{array}{l} -2R_3 \\ +3R_3 \\ \end{array} \\
 &\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -6 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array} \right] \begin{array}{l} -R_2 \\ \\ \end{array}
 \end{aligned}$$

And so we have the linear combination:

$$\begin{aligned}
 -6 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} &= \begin{bmatrix} -6 \\ -6 \\ -6 \end{bmatrix} + \begin{bmatrix} 3 \\ 6 \\ 9 \end{bmatrix} + \begin{bmatrix} 4 \\ -2 \\ 2 \end{bmatrix} \\
 &= \begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix}
 \end{aligned}$$

Problem 7.

Consider the linear mapping defined by:

$$\begin{array}{ccc}
 \phi : & \mathbb{R}^3 & \rightarrow \mathbb{R}^4 \\
 (x_1, x_2, x_3) & \mapsto & (3x_1 + 2x_2 + x_3, x_1 + x_2 + x_3, x_1 - 3x_2, 2x_1 + 3x_2 + x_3)
 \end{array}$$

Find the matrix associated with the linear transformation, determine the rank of the matrix and compute the image, the kernel and their dimension.

Solution. First, we apply the transformation to the canonical basis in \mathbb{R}^3 as:

$$\begin{aligned}
 \phi(1, 0, 0) &= (3, 1, 1, 2) \\
 \phi(0, 1, 0) &= (2, 1, -3, 3) \\
 \phi(0, 0, 1) &= (1, 1, 0, 1)
 \end{aligned}$$

And so we end up with the matrix:

$$A = \begin{bmatrix} 3 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & -3 & 0 \\ 2 & 3 & 1 \end{bmatrix}$$

Now, if we reduce it to the echelon form:

$$\begin{aligned}
\left[\begin{array}{ccc|c} 3 & 2 & 1 & a \\ 1 & 1 & 1 & b \\ 1 & -3 & 0 & c \\ 2 & 3 & 1 & d \end{array} \right] &\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & b \\ 3 & 2 & 1 & a \\ 1 & -3 & 0 & c \\ 2 & 3 & 1 & d \end{array} \right] \begin{array}{l} R_2 \\ R_1 \\ \\ \end{array} \\
&\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & b \\ 0 & -1 & -2 & a-3b \\ 0 & -4 & -1 & c-b \\ 0 & 1 & -1 & d-2b \end{array} \right] \begin{array}{l} \\ -3R_1 \\ -R_1 \\ -2R_1 \end{array} \\
&\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & b \\ 0 & 1 & 2 & 3b-a \\ 0 & 4 & 1 & b-c \\ 0 & -1 & 1 & 2b-d \end{array} \right] \begin{array}{l} \\ -R_2 \\ -R_3 \\ -R_4 \end{array} \\
&\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & b \\ 0 & 1 & 2 & 3b-a \\ 0 & 0 & -3 & -3b-c \\ 0 & 0 & -2 & 3b-d \end{array} \right] \begin{array}{l} \\ \\ -4R_1 \\ +R_1 \end{array} \\
&\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & b \\ 0 & 1 & 2 & 3b-a \\ 0 & 0 & 1 & b+\frac{c}{3} \\ 0 & 0 & -1 & d-3b \end{array} \right] \begin{array}{l} \\ \\ -\frac{1}{3}R_3 \\ -R_4 \end{array} \\
&\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 1 & b \\ 0 & 1 & 2 & 3b-a \\ 0 & 0 & 1 & b+\frac{c}{3} \\ 0 & 0 & 0 & d+\frac{c}{3}-2b \end{array} \right] \begin{array}{l} \\ \\ \\ +R_3 \end{array} \\
&\longrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & -\frac{c}{3} \\ 0 & 1 & 0 & b-a-\frac{2c}{3} \\ 0 & 0 & 1 & b+\frac{c}{3} \\ 0 & 0 & 0 & d+\frac{c}{3}-2b \end{array} \right] \begin{array}{l} -R_3 \\ -2R_3 \\ \\ \end{array} \\
&\longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & a-b+\frac{c}{3} \\ 0 & 1 & 0 & b-a-\frac{2c}{3} \\ 0 & 0 & 1 & b+\frac{c}{3} \\ 0 & 0 & 0 & d+\frac{c}{3}-2b \end{array} \right] \begin{array}{l} -R_2 \\ \\ \\ \end{array}
\end{aligned}$$

So the image is the set:

$$Im(\phi) = \{(a, b, c, d) \in \mathbb{R}^4 | d + \frac{c}{3} - 2b = 0\}$$

Note that when $(a, b, c, d) = (0, 0, 0, 0)$ then the system only has the trivial solution and so we conclude that the rank of A is 3 and that the kernel is just $\{0\}$. By the rank theorem, we have:

$$\begin{aligned}
\dim V &= \dim(\ker(\phi)) + \dim(\phi(V)) \\
\dim \mathbb{R}^3 &= \dim(\{0\}) + \dim(\phi(V)) \\
3 &= 0 + \dim(\phi(V)) \\
\dim(\phi(V)) &= 3
\end{aligned}$$

And so we conclude that even when the linear transformation is monic, it is not epic.

Problem 8.

Let V be a vector space. Let f and g two automorphism over E such that $f \circ g = Id_E$. Show that $\ker(f) = \ker(g \circ f)$, $Im(g) = Im(g \circ f)$ and that $\ker(f) \cap Im(g) = \{0_E\}$.

Solution. First, we want to show that $\ker(g \circ f) = \{0_E\}$ since f is an automorphism. So, suppose that $v \in E$ is such that $(g \circ f)(v) = g(f(v)) = 0$. Since g is an automorphism, we conclude that $f(v) = 0$ and since f is also an automorphism, we conclude that $v = 0$. So, $\ker(f) = \ker(g \circ f)$.

Also, we want to show $\text{Im}(g \circ f) = E$, so take $v \in E$, since g is an automorphism we can find $u \in E$ such that $g(u) = v$. And also, since f is an automorphism, we can find $w \in E$ such that $f(w) = u$, so we have that:

$$\begin{aligned}(g \circ f)(w) &= g(f(w)) \\ &= g(u) \\ &= v\end{aligned}$$

And so we conclude that $\text{Im}(g) = \text{Im}(g \circ f)$. Note that also $0 \in \text{Im}(g)$ but the only element in $\ker(f)$ is 0, so we must have that $\ker(f) \cap \text{Im}(g) = \{0_E\}$.

Problem 9.

Let $F = \{(x, y, z) \in \mathbb{R}^3 | x + y - z = 0\}$ and $G = \{(a - b, a + b, a - 3b) \in \mathbb{R}^3 | a, b \in \mathbb{R}\}$. Prove that they are subspaces of \mathbb{R}^3 , calculate $F \cap G$ and then using basis for F and G check the result.

Solution. First, we are going to show that both sets are subspaces of \mathbb{R}^3 :

- Let $(x, y, z), (a, b, c) \in F$ and let $r \in \mathbb{R}$, we want to show that their linear combination is in F :

$$\begin{aligned}r(x, y, z) + (a, b, c) &= (rx, ry, rz) + (a, b, c) \\ &= (rx + a, ry + b, rz + c)\end{aligned}$$

And by definition, we have the equations:

$$\begin{aligned}x + y - z &= 0 \\ a + b - c &= 0\end{aligned}$$

If we multiply the first by r we get:

$$\begin{aligned}rx + ry - rz &= 0 \\ a + b - c &= 0\end{aligned}$$

and if we add them up:

$$(rx + a) + (ry + b) - (rz + c) = 0$$

Which proves that $(rx + a, ry + b, rz + c) \in F$ and so it is a subspace.

- Let $(a - b, a + b, a - 3b), (x - y, x + y, x - 3y)$ and $r \in \mathbb{R}$ we want to show that their linear combination is in G :

$$\begin{aligned}r(a - b, a + b, a - 3b) + (x - y, x + y, x - 3y) &= (ar - br, ar + br, ar - 3br) + (x - y, x + y, x - 3y) \\ &= ((ar + x) - (br + y), (ar + x) + (br + y), (ar + x) - 3(br + y))\end{aligned}$$

and since $ar + x, br + y \in \mathbb{R}$, by definition $((ar + x) - (br + y), (ar + x) + (br + y), (ar + x) - 3(br + y))$ and so G is a subspace.

Now, if we calculate $F \cap G$, we take a vector $(x, y, z) \in F \cap G$ then it is necessary that $x + y - z = 0$ and for $a, b \in \mathbb{R}$ we got $x = a - b$, $y = a + b$ and $z = a - 3b$. If we replace it into the another equation we get:

$$\begin{aligned}x + y - z &= 0 \\a - b + a + b - a + 3b &= 0 \\a + 3b &= 0 \\b &= -\frac{a}{3}\end{aligned}$$

and so we get:

$$\begin{aligned}x &= \frac{4a}{3} \\y &= \frac{2a}{3} \\z &= 2a\end{aligned}$$

so we got vectors of the form $(\frac{4a}{3}, \frac{2a}{3}, 2a)$ such that $a \in \mathbb{R}$. Now, we are going to find a basis for F and for G .

- For F , note that each vector is of the form (x, y, z) with $x + y - z = 0$ and so $z = x + y$, so we can express it:

$$\begin{aligned}(x, y, z) &= (x, y, x + y) \\&= (x, 0, x) + (0, y, y) \\&= x(1, 0, 1) + y(0, 1, 1)\end{aligned}$$

And so a basis for F is $\{(1, 0, 1), (0, 1, 1)\}$.

- For G , we have vectors of the form $(a - b, a + b, a - 3b)$ so we can descompose it like:

$$\begin{aligned}(a - b, a + b, a - 3b) &= (a, a, a) + (-b, b, -3b) \\&= a(1, 1, 1) + b(-1, 1, -3)\end{aligned}$$

And so a basis for G is $\{(1, 1, 1), (-1, 1, -3)\}$.

And now we express a vector of $F \cap G$ as a linear combination with scalars $x, y, a, b \in \mathbb{R}$ as:

$$\begin{aligned}x(1, 0, 1) + y(0, 1, 1) &= a(1, 1, 1) + b(-1, 1, -3) \\(x, 0, x) + (0, y, y) &= (a, a, a) + (-b, b, -3b) \\(x, y, x + y) &= (a - b, a + b, a - 3b)\end{aligned}$$

and so we get $x = a - b$, $y = a + b$ and $x + y = a - 3b$, but we can replace in this last equation as:

$$\begin{aligned}x + y &= a - 3b \\a - b + a + b &= a - 3b \\2a &= a - 3b \\a &= -3b \\b &= -\frac{a}{3}\end{aligned}$$

and so we get the same result as we did without the basis.

2 Analytic Geometry

Problem 10.

Show that $\langle \cdot, \cdot \rangle$ defined for all $x = (x_1, x_2) \in \mathbb{R}^2$ and $y = (y_1, y_2) \in \mathbb{R}^2$ by:

$$\langle x, y \rangle = x_1 y_1 - (x_1 y_2 + x_2 y_1) + 2(x_2 y_2)$$

is an inner product.

Solution. We need to prove three things. That $\langle \cdot, \cdot \rangle$ is a bilinear map, that it is symmetric and that it is positive definite.

- First, to prove that it is a bilinear transformation, take $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in \mathbb{R}^2$ and $a, b \in \mathbb{R}$. Then:

$$\begin{aligned}\langle (x_1, x_2), (ay_1 + bz_1, ay_2 + bz_2) \rangle &= x_1(ay_1 + bz_1) - (x_1 \cdot (ay_2 + bz_2) + x_2(ay_1 + bz_1)) + 2x_2(ay_2 + bz_2) \\ &= ax_1 y_1 + bx_1 z_1 - (ax_1 y_2 + bx_1 z_2 + ax_2 y_1 + bx_2 z_1) + 2ax_2 y_2 + 2bx_2 z_2 \\ &= ax_1 y_1 - ax_1 y_2 - ax_2 y_1 + 2a(x_2 y_2) + bx_1 z_1 - bx_1 z_2 - bx_2 z_1 + 2b(x_2 z_2) \\ &= a(x_1 y_1 - (x_1 y_2 + x_2 y_1) + 2(x_2 y_2)) + b(x_1 z_1 - (x_1 z_2 + x_2 z_1) + 2(x_2 z_2)) \\ &= a\langle (x_1, x_2), (y_1, y_2) \rangle + b\langle (x_1, x_2), (z_1, z_2) \rangle\end{aligned}$$

In a similar way we prove that $\langle (ax_1 + by_1, ax_2 + by_2), (z_1, z_2) \rangle = a\langle (x_1, x_2), (z_1, z_2) \rangle + b\langle (y_1, y_2), (z_1, z_2) \rangle$, so we conclude that it is a bilinear transformation.

- We want to show that this transformation is symmetric. So:

$$\begin{aligned}\langle x, y \rangle &= x_1 y_1 - (x_1 y_2 + x_2 y_1) + 2(x_2 y_2) \\ &= y_1 x_1 - (y_1 x_2 + y_2 x_1) + 2(y_2 x_2) \\ &= \langle y, x \rangle\end{aligned}$$

- And for the last part, we want to show that it is positive definite. That is, suppose that $(x_1, x_2) \neq (0, 0)$ so we want to show that its inner product is positive:

$$\begin{aligned}\langle (x_1, x_2), (x_1, x_2) \rangle &= x_1 \cdot x_1 - (x_1 x_2 + x_1 x_2) + 2(x_2 \cdot x_2) \\ &= x_1^2 - 2x_1 x_2 + 2x_2^2 \\ &= x_1^2 - 2x_1 x_2 + x_2^2 + x_2^2 \\ &= (x_1 - x_2)^2 + x_2^2\end{aligned}$$

And no matter the chooses of x_1, x_2 we always end up with a positive sum of squares, so $\langle x, x \rangle > 0$ and it is easy to see that $\langle 0, 0 \rangle = 0$.

Problem 11.

Consider \mathbb{R}^2 with $\langle \cdot, \cdot \rangle$ defined for all x and y in \mathbb{R}^2 as:

$$\langle x, y \rangle := x^\top \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} y$$

Is it an inner product?

Solution. First, we could rearrange the expression taking $x = (x_1, x_2)$ and $y = (y_1, y_2)$ as:

$$\begin{aligned}\langle x, y \rangle &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\ &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2y_1 \\ y_1 + 2y_2 \end{bmatrix} \\ &= 2x_1 y_1 + x_2 y_1 + 2x_2 y_2\end{aligned}$$

And now we must to test the three conditions:

- First, we want to prove that it is a bilinear transformation:

$$\begin{aligned}
\langle (x_1, x_2), (ay_1 + bz_1, ay_2 + bz_2) \rangle &= 2x_1(ay_1 + bz_1) + x_2(ay_1 + bz_1) + 2x_2(ay_2 + bz_2) \\
&= 2ax_1y_1 + 2bx_1z_1 + ax_2y_1 + bx_2z_1 + 2ax_2y_2 + 2bx_2z_2 \\
&= 2ax_1y_1 + ax_2y_1 + 2ax_2y_2 + 2bx_1z_1 + bx_2z_1 + 2bx_2z_2 \\
&= a(2x_1y_1 + x_2y_1 + 2x_2y_2) + b(x_1z_1 + x_2z_1 + x_2z_2) \\
&= a\langle (x_1, x_2), (y_1, y_2) \rangle + b\langle (x_1, x_2), (z_1, z_2) \rangle
\end{aligned}$$

And we can prove in a similar way that $\langle (ax_1 + by_1, ax_2 + by_2), (z_1, z_2) \rangle = a\langle (x_1, x_2), (y_1, y_2) \rangle + b\langle (x_1, x_2), (z_1, z_2) \rangle$. So we conclude that $\langle \cdot, \cdot \rangle$ is a bilinear transformation.

- Now, we need to show that it is symmetric:

$$\begin{aligned}
\langle x, y \rangle &= 2x_1y_1 + x_2y_1 + 2x_2y_2 \\
&= 2x_2y_2 + y_1x_2 + 2x_1y_1
\end{aligned}$$

But this does not implies that $y_1x_2 = y_2x_1$ so it is not symmetric and therefore it is not an inner product.

Problem 12.

Compute the distance between $x = (1, 2, 3)$ and $y = (-1, -1, 0)$ using:

1. $\langle x, y \rangle := x^\top y$
2. $\langle x, y \rangle := x^\top yA$ with $A := \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

Solution. Remmber that the distance induced by a product is the norm induced by that inner product. We can first calculate $x - y$ as:

$$\begin{aligned}
x - y &= (1, 2, 3) - (-1, -1, 0) \\
&= (2, 3, 3)
\end{aligned}$$

And so we do the norm of that vector under each of the inner products defined:

1. For $\langle x, y \rangle := x^\top y$, we have:

$$\begin{aligned}
d(x, y) &= \|x - y\| \\
&= \sqrt{\langle x - y, x - y \rangle} \\
&= \sqrt{\langle (2, 3, 3), (2, 3, 3) \rangle} \\
&= \sqrt{2^2 + 3^2 + 3^2} \\
&= \sqrt{4 + 9 + 9} \\
&= \sqrt{22}
\end{aligned}$$

2. For $\langle x, y \rangle := x^\top yA$ with $A := \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}$, so if we make $\langle x - y, x - y \rangle$ we got:

$$\begin{aligned}
[2 \quad 3 \quad 3] \begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix} &= [2 \quad 3 \quad 3] \begin{bmatrix} 7 \\ 8 \\ 3 \end{bmatrix} \\
&= 2 \cdot 7 + 3 \cdot 8 + 3 \cdot 3 \\
&= 14 + 24 + 9 \\
&= 47
\end{aligned}$$

So we have:

$$\begin{aligned}d(x, y) &= \|x - y\| \\&= \sqrt{\langle x - y, x - y \rangle} \\&= \sqrt{\langle (2, 3, 3), (2, 3, 3) \rangle} \\&= \sqrt{47}\end{aligned}$$

Problem 13.

Compute the angle between:

$$x = (1, 2)$$

$$y = (-1, -1)$$

using the inner products:

1. $\langle x, y \rangle = x^\top y$

2. $\langle x, y \rangle = x^\top B y$ with $B = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$

Solution.

1. With the usual product and the norm induced by that, we consider that:

$$\begin{aligned}\cos \omega &= \frac{|\langle x, y \rangle|}{\|x\| \|y\|} \\&= \frac{|1 \cdot (-1) + 2 \cdot (-1)|}{\sqrt{1^2 + 2^2} \sqrt{(-1)^2 + (-1)^2}} \\&= \frac{|-1 - 2|}{\sqrt{5} \sqrt{2}} \\&= \frac{3}{\sqrt{10}}\end{aligned}$$

And so we get that $\omega = \arccos \frac{3}{\sqrt{10}} \approx 18 \text{ deg}$

2. It is convenient to compute the inner product with arbitrary (x_1, x_2) and (y_1, y_2) :

$$\begin{aligned}\langle (x_1, x_2), (y_1, y_2) \rangle &= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \\&= \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2y_1 + y_2 \\ y_1 + 3y_2 \end{bmatrix} \\&= 2x_1y_1 + x_1y_2 + y_1x_2 + 3x_2y_2\end{aligned}$$

Now, with that we are ready to compute the angle given by this inner product.

$$\begin{aligned}
 \cos \omega &= \frac{|\langle x, y \rangle|}{\|x\| \|y\|} \\
 &= \frac{|2(1)(-1) + (1)(-1) + (2)(-1) + 3(2)(-1)|}{\sqrt{2(1)(1) + (1)(2) + (1)(2) + 3(2)(2)} \sqrt{(2)(-1)(-1) + (-1)(-1) + (-1)(-1) + (3)(-1)(-1)}} \\
 &= \frac{|-2 - 1 - 2 - 6|}{\sqrt{2 + 2 + 2 + 12} \sqrt{2 + 1 + 1 + 3}} \\
 &= \frac{|-11|}{\sqrt{18} \sqrt{7}} \\
 &= \frac{|11|}{\sqrt{126}} \\
 &= \frac{11}{\sqrt{126}}
 \end{aligned}$$

And so we end up that $\omega = \arccos \frac{11}{\sqrt{126}} \approx 11.5 \text{ deg.}$

Problem 14.

Consider \mathbb{R}^3 with the inner product:

$$\langle x, y \rangle = x^\top \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} y$$

and let e_1, e_2, e_3 be the canonical basis of \mathbb{R}^3 .

- Define the orthogonal projection $\pi_U(e_2)$ onto $U = \text{span}[e_1, e_3]$
- Compute the distance $d(e_2, U)$
- Draw the scenario.

Solution. Since we want to project e_2 into the set $\text{span}[e_1, e_3]$ for the projection $\pi_U(e_2)$ there would be λ_1, λ_2 such that:

$$\pi_U(e_2) = \lambda_1 \cdot e_1 + \lambda_2 \cdot e_3$$

so we must have that:

$$\begin{aligned}
 \langle e_1, e_2 - \pi_U(e_2) \rangle &= 0 \\
 \langle e_3, e_2 - \pi_U(e_2) \rangle &= 0
 \end{aligned}$$

So let us first compute what this product must be:

$$\begin{aligned}
 \langle (x_1, x_2, x_3), (y_1, y_2, y_3) \rangle &= [x_1 \quad x_2 \quad x_3] \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \\
 &= [x_1 \quad x_2 \quad x_3] \begin{bmatrix} 2y_1 + y_2 \\ y_1 + 2y_2 - y_3 \\ 2y_3 - y_2 \end{bmatrix} \\
 &= 2x_1y_1 + x_1y_2 + y_1x_2 + 2x_2y_2 - x_2y_3 + 2x_3y_3 - x_3y_2
 \end{aligned}$$