

## Vector Spaces I

**Problem 1:** Let  $V$  and  $W$  be vector spaces over a field  $K$ . Let  $\mathcal{B} = \{v_1, v_2, \dots, v_n\}$  be a basis for  $V$  and let  $\{w_1, w_2, \dots, w_n\}$  be any vectors in  $W$ . There is a unique linear map

$$\phi: V \rightarrow W$$

Such that  $\phi(v_i) = w_i$  for all  $1 \leq i \leq n$

**Solution.** Since  $\mathcal{B}$  is a basis for  $V$ , for any element  $v \in V$  there are  $a_1, a_2, \dots, a_n \in K$  such that:

$$v = a_1 v_1 + a_2 v_2 + \dots + a_n v_n$$

so if we define  $\phi$  such that  $\phi(v_i) = w_i$  then for any vector  $v$  we would have:

$$\begin{aligned} \phi(v) &= a_1 \phi(v_1) + a_2 \phi(v_2) + \dots + a_n \phi(v_n) \\ &= a_1 w_1 + a_2 w_2 + \dots + a_n w_n \end{aligned}$$

**Problem 2:** Suppose that  $V$  is a finite dimensional vector space. Let  $\mathcal{B} = \{v_1, v_2, \dots, v_n\}$  be a basis for  $V$  then:

- Any set of  $w_1, w_2, \dots, w_n, w_{n+1}$  vectors is linearly dependent
- Any set of  $w_1, w_2, \dots, w_{n-1}$  vectors can't generate  $V$

**Solution.** For this, we are going to use the facts needed for a basis.

- Let  $w_1, w_2, \dots, w_n, w_{n+1}$  be vectors in  $V$ , we can write them in the next way:

$$\begin{aligned} w_1 &= a_{1,1} v_1 + a_{1,2} v_2 + \dots + a_{1,n} v_n \\ w_2 &= a_{2,1} v_1 + a_{2,2} v_2 + \dots + a_{2,n} v_n \\ &\dots\dots\dots \\ w_n &= a_{n,1} v_1 + a_{n,2} v_2 + \dots + a_{n,n} v_n \\ w_{n+1} &= a_{n+1,1} v_1 + a_{n+1,2} v_2 + \dots + a_{n+1,n} v_n \end{aligned}$$

If there is a  $w_i$  such that  $w_i = 0$  we are done. Suppose then that this is not true, so for each  $1 \leq i \leq n+1$  exists  $j$  such that  $a_{i,j} \neq 0$ . But since there are  $w_{n+1}$  there must be  $i_1, i_2$  such that for the same  $j$ , we have that  $a_{i_1,j} \neq 0 \neq a_{i_2,j}$ . So, we can express the vector  $v_j$  as:

$$\begin{aligned} v_j &= \frac{w_{i_1}}{a_{i_1,j}} - \frac{a_{i_1,1} v_1 + a_{i_1,2} v_2 + \dots + a_{i_1,n} v_n}{a_{i_1,j}} \\ v_j &= \frac{w_{i_2}}{a_{i_2,j}} - \frac{a_{i_2,1} v_1 + a_{i_2,2} v_2 + \dots + a_{i_2,n} v_n}{a_{i_2,j}} \end{aligned}$$

And so the set is not linearly independent.

- Let  $w_1, w_2, \dots, w_{n-1}$  be vectors of  $V$ . Suppose that indeed we can generate  $V$  with them, so in particular, we can write:

$$\begin{aligned} v_1 &= a_{1,1}w_1 + a_{1,2}w_2 + \dots + a_{1,n-1}w_{n-1} \\ v_2 &= a_{2,1}w_1 + a_{2,2}w_2 + \dots + a_{2,n-1}w_{n-1} \\ &\dots\dots\dots \\ v_n &= a_{n,1}w_1 + a_{n,2}w_2 + \dots + a_{n,n-1}w_{n-1} \end{aligned}$$

And since none of them is zero, we can be sure that for each  $1 \leq i \leq n$  exists  $j$  such that  $a_{i,j} \neq 0$ . But since there are  $n$  vectors in  $\mathcal{B}$  and just  $n-1$  vectors  $w_i$ , there must be  $i_1, i_2$  such that for the same  $j$ , we have that  $a_{i_1,j} \neq 0 \neq a_{i_2,j}$ . So, we can express the vector  $v_j$  as:

$$\begin{aligned} w_j &= \frac{v_{i_1}}{a_{i_1,j}} - \frac{a_{i_1,1}w_1 + a_{i_1,2}w_2 + \dots + a_{i_1,n}w_n}{a_{i_1,j}} \\ w_j &= \frac{v_{i_2}}{a_{i_2,j}} - \frac{a_{i_2,1}w_1 + a_{i_2,2}w_2 + \dots + a_{i_2,n}w_n}{a_{i_2,j}} \end{aligned}$$

But then this let us generate two different linear combinations within  $\mathcal{B}$  that give us the same result, contradicting the linear independency of  $\mathcal{B}$ .

**Problem 3:** Let  $V$  be a finite vector space. If  $A = \{v_1, v_2, \dots, v_n\}$  generates  $V$  then some subset of  $A$  is a basis for  $V$ .

**Solution.** For that, let declare the next set:

$$S = \{W \in \mathcal{P}(A) | W \text{ is linearly independent}\}$$

We can assure that at least there is a maximal element  $\{v_1, v_2, \dots, v_m\}$  in  $S$  since we can assure the existence of  $\{v_1\}$  and at most it can be  $A$ . Suppose then that it is not  $A$ , so  $m < n$ , and we can assure that any set  $\{v_1, \dots, v_m, v_i\}$  is linearly dependent, with  $m < i \leq n$ . Therefore we have:

$$a_1v_1 + \dots + a_nv_n + a_iv_i = 0$$

has more than the trivial solution, so we can suppose that

**Problem 4:** Let  $A = \{v_1, v_2, \dots, v_n\}$  be a subset of a vector space  $V$ . Prove that  $A$  is linearly independent if and only if the equation  $a_1v_1 + a_2v_2 + \dots + a_nv_n = 0$  has the trivial solution.

**Solution.** We prove a double implication:

- $\Rightarrow$ ) If  $A$  is linearly independent then by definition the equation  $a_1v_1 + a_2v_2 + \dots + a_nv_n = 0$  has only one solution, the trivial one.
- $\Leftarrow$ ) Suppose that  $A$  is not linearly independent, so that there are two combinations of scalars  $a_1, a_2, \dots, a_n$  and  $b_1, b_2, \dots, b_n$  such that for a  $v$  in  $V$ :

$$\begin{aligned} a_1v_1 + a_2v_2 + \dots + a_nv_n &= v \\ b_1v_1 + b_2v_2 + \dots + b_nv_n &= v \end{aligned}$$

And if we use the transitivity we have:

$$\begin{aligned} a_1v_1 + a_2v_2 + \dots + a_nv_n &= b_1v_1 + b_2v_2 + \dots + b_nv_n \\ (a_1 - b_1)v_1 + (a_2 - b_2)v_2 + \dots + (a_n - b_n)v_n &= 0 \end{aligned}$$

But note that  $a_1 \neq b_1$ ,  $a_2 \neq b_2$  and so on, so  $a_1 - b_1 \neq 0$ ,  $a_2 - b_2 \neq 0$  and so on, so the equation has another solution apart to the trivial one.

**Problem 5:** Prove the Rank theorem

**Solution.** Remember that the rank theorem says that if  $V$  and  $W$  are finite dimensional vector spaces over  $K$ , and  $\phi : V \rightarrow W$  is a linear map then:

$$\dim V = \dim \ker(\phi) + \dim \phi(V)$$

Let  $\mathcal{A} = \{v_1, v_2, \dots, v_n\}$  be a basis for  $\ker(\phi)$  and let  $\mathcal{B} = \{w_1, w_2, \dots, w_m\}$  be a basis for  $\phi(V)$ . Since  $\mathcal{B} \subseteq \phi(V)$  there are  $u_1, u_2, \dots, u_m$  such that  $\phi(u_1) = w_1, \phi(u_2) = w_2, \dots, \phi(u_m) = w_m$ . So, we can create the set:

$$\mathcal{C} = \{v_1, v_2, \dots, v_n, u_1, u_2, \dots, u_m\}$$

And we claim that this is a basis for  $V$ . For that, let's prove the two properties for that:

- Suppose that there are scalars  $a_1, a_2, \dots, a_n, b_1, \dots, b_m$  such that:

$$\begin{aligned} a_1 v_1 + a_2 v_2 + \dots + a_n v_n + b_1 u_1 + b_2 u_2 + \dots + b_m u_m &= 0 \\ a_1 v_1 + a_2 v_2 + \dots + a_n v_n &= -b_1 u_1 - b_2 u_2 - \dots - b_m u_m \\ \phi(a_1 v_1) + \phi(a_2 v_2) + \dots + \phi(a_n v_n) &= \phi(-b_1 u_1) + \phi(-b_2 u_2) + \dots + \phi(-b_m u_m) \\ a_1 \phi(v_1) + a_2 \phi(v_2) + \dots + a_n \phi(v_n) &= -b_1 \phi(u_1) - b_2 \phi(u_2) - \dots - b_m \phi(u_m) \\ a_1 0 + a_2 0 + \dots + a_n 0 &= -b_1 w_1 - b_2 w_2 - \dots - b_m w_m \\ 0 &= -b_1 w_1 - b_2 w_2 - \dots - b_m w_m \end{aligned}$$

And since  $\mathcal{B}$  is a basis then  $b_1 = b_2 = \dots = b_m = 0$ . And therefore we have that:

$$\begin{aligned} a_1 v_1 + a_2 v_2 + \dots + a_n v_n + b_1 u_1 + b_2 u_2 + \dots + b_m u_m &= 0 \\ a_1 v_1 + a_2 v_2 + \dots + a_n v_n &= 0 \end{aligned}$$

And since  $\mathcal{A}$  is a basis, then  $a_1 = a_2 = \dots = a_n = 0$ , and so  $\mathcal{C}$  is linearly independent.

- Take  $v \in V$ , we want to prove it is a linear combination of elements of  $\mathcal{C}$ . So for that, we know that  $\phi(v)$  is a linear combination of elements of  $\mathcal{B}$ :

$$\begin{aligned} b_1 w_1 + b_2 w_2 + \dots + b_m w_m &= \phi(v) \\ b_1 \phi(u_1) + b_2 \phi(u_2) + \dots + b_m \phi(u_m) &= \phi(v) \\ \phi(b_1 u_1 + b_2 u_2 + \dots + b_m u_m) &= \phi(v) \\ \phi(b_1 u_1 + b_2 u_2 + \dots + b_m u_m) - \phi(v) &= 0 \\ \phi(b_1 u_1 + b_2 u_2 + \dots + b_m u_m - v) &= 0 \end{aligned}$$

And since  $b_1 u_1 + b_2 u_2 + \dots + b_m u_m - v \in \ker(\phi)$  we can derive a linear combination of the form:

$$\begin{aligned} a_1 v_1 + a_2 v_2 + \dots + a_n v_n &= b_1 u_1 + b_2 u_2 + \dots + b_m u_m - v \\ a_1 v_1 + a_2 v_2 + \dots + a_n v_n - b_1 u_1 - b_2 u_2 - \dots - b_m u_m &= -v \\ b_1 u_1 + b_2 u_2 + \dots + b_m u_m - a_1 v_1 - a_2 v_2 - \dots - a_n v_n &= v \end{aligned}$$

And so we have that  $v$  is a linear combination of  $\mathcal{C}$ , so  $\text{Span}(\mathcal{C}) = V$ .

And that way we conclude that  $\mathcal{C}$  is a basis for  $V$  and note that  $|\mathcal{C}| = |\mathcal{A}| + |\mathcal{B}|$ , so  $\dim V = \dim \ker(\phi) + \dim \phi(V)$ .

**Problem 6:** Determine whether or not  $\{(1, 1, 0), (2, 0, -1), (-3, 1, 1)\}$  is basis for  $\mathbb{R}^3$

**Solution.** First, let's determine whenever it is linearly independent or not.

- Suppose that  $a_1(1, 1, 0) + a_2(2, 0, -1) + a_3(-3, 1, 1) = 0$ . So, if we add those vectors we would have:

$$\begin{aligned} a_1(1, 1, 0) + a_2(2, 0, -1) + a_3(-3, 1, 1) &= (a_1, a_1, 0) + (2a_2, 0, -a_2) + (-3a_3, a_3, a_3) \\ &= (a_1 + 2a_2 - 3a_3, a_1 + a_3, -a_2 + a_3) = (0, 0, 0) \end{aligned}$$

So we would need that:

$$\begin{aligned} a_1 + 2a_2 - 3a_3 &= 0 \\ a_1 + a_3 &= 0 \\ a_3 - a_2 &= 0 \end{aligned}$$

If we solve the last two equations for  $a_1$  and  $a_2$  we would have:

$$\begin{aligned} a_1 &= -a_3 \\ a_2 &= a_3 \end{aligned}$$

And replacing in the first equation we would have:

$$\begin{aligned} a_1 + 2a_2 - 3a_3 &= 0 \\ -a_3 + 2a_3 - 3a_3 &= 0 \\ -2a_3 &= 0 \\ a_3 &= 0 \end{aligned}$$

And so we conclude that  $a_1 = a_2 = a_3 = 0$ , so this set is linearly independent.

- Take now any vector  $(x, y, z) \in \mathbb{R}^3$ , we want to prove that we can always find a linear combination of the vectors that give us  $(x, y, z)$ . For that, suppose that there are such combinations, so:

$$\begin{aligned} a_1(1, 1, 0) + a_2(2, 0, -1) + a_3(-3, 1, 1) &= (x, y, z) \\ (a_1, a_1, 0) + (2a_2, 0, -a_2) + (-3a_3, a_3, a_3) &= (x, y, z) \\ (a_1 + 2a_2 - 3a_3, a_1 + a_3, a_3 - a_2) &= (x, y, z) \end{aligned}$$

And so we have:

$$\begin{aligned} a_1 + 2a_2 - 3a_3 &= x \\ a_1 + a_3 &= y \\ a_3 - a_2 &= z \end{aligned}$$

Then we have:

$$\begin{aligned} a_1 &= y - a_3 \\ a_2 &= a_3 - z \end{aligned}$$

And plugging into the first equation we have:

$$\begin{aligned} a_1 + 2a_2 - 3a_3 &= x \\ y - a_3 + 2(a_3 - z) - 3a_3 &= x \\ y - a_3 + 2a_3 - 2z - 3a_3 &= x \\ y - 2z - 2a_3 &= x \\ a_3 &= \frac{2z - x - y}{2} \end{aligned}$$

And plugging into the next equation:

$$\begin{aligned} a_1 &= y - a_3 \\ a_1 &= y - \frac{x + y - 2z}{2} \\ a_1 &= y + z - \frac{x}{2} + \frac{y}{2} \\ a_1 &= \frac{3}{2}y + z - \frac{x}{2} \end{aligned}$$

And plugging into the last equation:

$$\begin{aligned} a_2 &= a_3 - z \\ a_2 &= z - \frac{x}{2} - \frac{y}{2} - z \\ a_2 &= \frac{-x - y}{2} \end{aligned}$$

And if you try this combination, you would get  $(x, y, z)$  so we can see  $\text{Span}(\{(1, 1, 0), (2, 0, -1), (-3, 1, 1)\}) = \mathbb{R}^3$ .

And so we have proved that  $\{(1, 1, 0), (2, 0, -1), (-3, 1, 1)\}$  is a basis for  $\mathbb{R}^3$ .

**Problem 7:** Let  $\phi : V \rightarrow W$  be linear. Suppose that  $v_1, \dots, v_n \in V$  are such that  $\phi(v_1), \dots, \phi(v_n)$  are linearly independent in  $W$ . Show that  $v_1, \dots, v_n$  are linearly independent.

**Solution.** For that, since  $\phi(v_1), \dots, \phi(v_n)$  are linearly independent, we can assure that the equation:

$$a_1\phi(v_1) + a_2\phi(v_2) + \dots + a_n\phi(v_n) = 0$$

has only the trivial solution. Suppose that the equation:

$$b_1v_1 + b_2v_2 + \dots + b_nv_n = 0$$

has a solution that is not trivial. That this, we can assure that at least  $b_1$  is not 0. And if we apply to both sides the linear map  $\phi$  we get:

$$\begin{aligned} \phi(b_1v_1 + b_2v_2 + \dots + b_nv_n) &= \phi(0) \\ \phi(b_1v_1) + \phi(b_2v_2) + \dots + \phi(b_nv_n) &= 0 \\ b_1\phi(v_1) + b_2\phi(v_2) + \dots + b_n\phi(v_n) &= 0 \end{aligned}$$

But this is a contradiction since this equation can only have the trivial solution. So we can conclude that  $v_1, \dots, v_n$ .

**Problem 8:** If  $\{v_1, \dots, v_n\}$  is a basis for  $V$  and  $\{w_1, \dots, w_m\}$  is a basis for  $W$  then:

$$\{(v_1, 0), \dots, (v_n, 0), (0, w_1), \dots, (0, w_m)\}$$

is a basis for  $V \oplus W$

**Solution.** We need to prove two things:

- First, to prove that this set is linearly independent, we need to show that the homogeneous equation has only the trivial solution. So we have:

$$\begin{aligned} a_1(v_1, 0) + a_2(v_2, 0) + \dots + a_n(v_n, 0) + b_1(0, w_1) + b_2(0, w_2) + \dots + b_m(0, w_m) &= (0, 0) \\ (a_1v_1, 0) + (a_2v_2, 0) + \dots + (a_nv_n, 0) + (0, b_1w_1) + (0, b_2w_2) + \dots + (0, b_mw_m) &= (0, 0) \\ (a_1v_1 + a_2v_2 + \dots + a_nv_n, b_1w_1 + b_2w_2 + \dots + b_mw_m) &= (0, 0) \end{aligned}$$

And this means that:

$$\begin{aligned} a_1v_1 + a_2v_2 + \dots + a_nv_n &= 0 \\ b_1w_1 + b_2w_2 + \dots + b_mw_m &= 0 \end{aligned}$$

And since those vectors are basis for each vector space  $a_1 = a_2 = \dots = a_n = b_1 = b_2 = \dots = b_m$ .

- For an element  $(v, w) \in V \oplus W$ , we know that  $v$  can be expressed as a linear combination  $a_1v_1 + a_2v_2 + \dots + a_nv_n = v$ , and also  $w$  can be expressed as  $b_1w_1 + b_2w_2 + \dots + b_mw_m = w$ , so the combination of the vectors in our set will rise:

$$a_1(v_1, 0) + a_2(v_2, 0) + \dots + a_n(v_n, 0) + b_1(0, w_1) + b_2(0, w_2) + \dots + b_m(0, w_m) = (v, w)$$

**Problem 9:** Let  $W$  be a subspace of the finite-dimensional vector space  $V$ . Show that there is a subspace  $U$  of  $V$  such that  $V \cong U \oplus W$ .

**Solution.** For this, define  $U$  as follows:

$$U := V \setminus W \cup \{0\}$$

First, we need to prove that this is a subspace of  $V$ :

Note that for any  $v \in U$  different from 0 and any  $c \in K$ , if  $cv \in W$  then  $c^{-1}cv = v \in W$  which contradicts the definition of  $U$ . If  $u, w \in U$  are not both 0, and if  $u + w \in W$  then that means that  $u, w \in W$  since  $W$  is closed over the operations, which again, contradicts the definition for  $U$ , so  $u + w \in U$ .

Now, we want to prove that this is an internal sum of  $V$ , so we have:

- If  $w \in W$  and  $u \in U$  are such that  $w + u = 0$ , then we would have  $w = -u$ , which means that  $w \in U$  and also that  $u = -w \in W$ , which means that since its only common element is 0,  $u = w = 0$ .
- For any element  $v \in V$ , there are two alternatives. If  $v \in W$  then we can express  $v$  as  $v + 0$  and  $0 \in U$ . If  $v \notin W$  then  $v \in U$  by definition and so  $v = 0 + v$  with  $0 \in W$ .

And so we conclude that  $U \oplus W$  is an internal sum of  $V$ .

**Problem 10:** A linear map  $\rho : V \rightarrow V$  is idempotent if  $\rho\rho = \rho$ . Show that  $\rho$  acts as an identity over  $\rho(V)$  if  $\rho$  is idempotent.

**Solution.** For that, we want to prove that  $\rho^2 = Id_{\rho(V)}$ . For that, let  $v \in \rho(V)$ , we know that there is  $w \in V$  such that  $\rho(w) = v$ . Now, if we apply again the function we would have:

$$\begin{aligned}\rho(\rho(w)) &= \rho(v) \\ \rho(w) &= \rho(v) \\ v &= \rho(v)\end{aligned}$$

So we conclude that  $\rho^2 = Id_{\rho(V)}$ .

**Problem 11:** Decide if  $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  given by  $\phi(x, y) = (x + y, 2x - y)$  is an isomorphism. If it is, find a formula for  $\phi^{-1}(x, y)$  and prove they are inverses.

**Solution.** Suppose that for a vector  $(a, b) \in \mathbb{R}^2$ , exists  $(x, y) \in \mathbb{R}^2$  whose image under  $\phi$  is  $(a, b)$ . We would have:

$$\phi(x, y) = (x + y, 2x - y) = (a, b)$$

And so we can write the next equations:

$$\begin{aligned}x + y &= a \\ 2x - y &= b\end{aligned}$$

If we solve for  $x$  in the first equation we would have:

$$x = a - y$$

And replacing in the second equation we would have:

$$\begin{aligned}2x - y &= 2(a - y) - y = b \\ 2a - 2y - y &= b \\ 2a - 3y &= b \\ -3y &= b - 2a \\ y &= \frac{2a - b}{3}\end{aligned}$$

And so if we plug in into the second equation we would have:

$$\begin{aligned}
 x &= a - y \\
 &= a - \frac{2a - b}{3} \\
 &= a - \frac{2a}{3} + \frac{b}{3} \\
 &= \frac{a}{3} + \frac{b}{3} \\
 &= \frac{a + b}{3}
 \end{aligned}$$

And so we would have:

$$\phi^{-1}(x, y) = \left( \frac{x + y}{3}, \frac{2x - y}{3} \right)$$

We can prove also that this indeed the inverse isomorphism by composing them:

- First, if we compose  $\phi$  and  $\phi^{-1}$  we would have:

$$\begin{aligned}
 \phi(\phi^{-1}(x, y)) &= \phi\left(\frac{x + y}{3}, \frac{2x - y}{3}\right) \\
 &= \left(\frac{x + y}{3} + \frac{2x - y}{3}, 2 \cdot \frac{x + y}{3} - \frac{2x - y}{3}\right) \\
 &= \left(\frac{3x}{3}, \frac{2x + 2y}{3} + \frac{y - 2x}{3}\right) \\
 &= \left(x, \frac{3y}{3}\right) \\
 &= (x, y)
 \end{aligned}$$

- And now, if we compose  $\phi^{-1}$  and  $\phi$  we get:

$$\begin{aligned}
 \phi^{-1}(\phi(x, y)) &= \phi^{-1}(x + y, 2x - y) \\
 &= \left(\frac{x + y + 2x - y}{3}, \frac{2(x + y) - (2x - y)}{3}\right) \\
 &= \left(\frac{3x}{3}, \frac{2x + 2y - 2x + y}{3}\right) \\
 &= \left(x, \frac{3y}{3}\right) \\
 &= (x, y)
 \end{aligned}$$

So we conclude that  $\phi$  and  $\phi^{-1}$  are inverses and so they are isomorphisms.

**Problem 12:** Let  $V$  be a vector space over a field  $k$  and let  $U, W$  be finite dimensional subspaces of  $V$ . Prove that both  $U + W$  and  $U \cap W$  are finite-dimensional subspaces of  $V$  and that

$$\dim(U + W) + \dim(U \cap W) = \dim U + \dim W$$

**Solution.** First, note that if  $U \cap W$  is the empty set, then its dimension is 0 and so it is finite-dimensional. Suppose it is not empty, so there is at least one  $v \in U \cap W$ . If we suppose that  $\mathcal{B}$  is an infinite basis for  $U \cap W$  then  $v$  is a linear combination of the elements in  $\mathcal{B}$ . But also  $v \in U$  but this would be a contradiction because this implies that  $\mathcal{B}$  is linearly dependent and so it cannot be a basis for  $U \cap W$ .

Now, we can find basis for each vector spaces as follows:

$$\mathcal{B} = \{v_1, \dots, v_k\} \text{ (Basis for } U \cap W)$$

$$\mathcal{B}_1 = \{v_1, \dots, v_k, u_1, \dots, u_n\} \text{ (Basis for } U, \text{ since we can extend any basis)}$$

$$\mathcal{B}_2 = \{v_1, \dots, v_k, w_1, \dots, w_m\} \text{ (Basis for } W, \text{ since we can extend any basis)}$$

We are going to prove that  $\mathcal{A} = \{v_1, \dots, v_k, u_1, \dots, u_n, w_1, \dots, w_m\}$  is a basis for  $U + W$ .

- First, suppose that  $a_1, \dots, a_k, b_1, \dots, b_n, c_1, \dots, c_m$  are scalars in  $k$  such that:

$$a_1v_1 + \dots + a_kv_k + b_1u_1 + \dots + b_nu_n + c_1w_1 + \dots + c_mw_m = 0$$

Suppose with no lose of generality that  $a_1 \neq 0$ , so we can express  $v_1$  in the next way:

$$v_1 = \frac{-a_2v_2 - \dots - a_kv_k - b_1u_1 - \dots - b_nu_n - c_1w_1 - \dots - c_mw_m}{a_1}$$

But note that  $v_1 \in U$  and  $v_1 \in W$ , so we can assure that

- For any element  $v \in U + W$ , we can express it as  $u + w$  with  $u \in U$  and  $w \in W$ . Now, for that we can express  $u$  and  $w$  as:

$$\begin{aligned} u &= a_1v_1 + a_2v_2 + \dots + a_kv_k + x_1u_1 + x_2u_2 + \dots + x_nu_n \\ w &= b_1v_1 + b_2v_2 + \dots + b_kv_k + y_1w_1 + y_2w_2 + \dots + y_mw_m \end{aligned}$$

And if we add them up we get:

$$\begin{aligned} u + w &= a_1v_1 + a_2v_2 + \dots + a_kv_k + x_1u_1 + x_2u_2 + \dots + x_nu_n + b_1v_1 + b_2v_2 + \dots + b_kv_k + y_1w_1 + y_2w_2 + \dots + y_mw_m \\ v &= (a_1 + b_1)v_1 + (a_2 + b_2)v_2 + \dots + (a_k + b_k)v_k + x_1u_1 + x_2u_2 + \dots + x_nu_n + y_1w_1 + y_2w_2 + \dots + y_mw_m \end{aligned}$$

And so we have proved that  $\text{Span}(\mathcal{A}) = U + W$ .

Therefore, we conclude that  $\mathcal{A}$  is a basis for  $U + W$ . But since the basis for  $U$  and the basis for  $W$  includes both the basis for  $U \cap W$ , we need to extract it, so:

$$\begin{aligned} \dim(U + W) &= \dim U + \dim W - \dim(U \cap W) \\ \dim(U + W) + \dim(U \cap W) &= \dim U + \dim W \end{aligned}$$

**Problem 13:** Let  $\phi \in \text{End}(V)$  for a finite dimensional vector space  $V$ . Prove that  $\phi$  is monic if and only if it is epic if and only if it is an isomorphism

**Solution.** Since  $V$  is a finite dimensional vector space we can use the rank theorem to find the dimensions of the kernel, images and  $V$ .

- Suppose that  $\phi$  is monic, so that  $\ker(\phi) = \{0\}$ . We would have then that  $\dim \ker(\phi) = 0$  and so  $\dim V = \dim \phi(V)$ , and since  $\phi(V) \subseteq V$  we conclude that  $\phi(V) = V$  so that  $\phi$  is epic.
- Suppose that  $\phi$  is epic, so that  $\phi(V) = V$ . We would have then that  $\dim \ker(\phi) = 0$ , and since  $\ker(\phi)$  is a subspace of  $V$ , if there would be a vector different from 0 into the set, it would make a basis and so  $\dim \ker(\phi) > 0$ , so we would only have that  $\ker(\phi) = \{0\}$  and so  $\phi$  is monic.
- If we suppose that  $\phi$  is monic or epic, we get the other one and so it is an isomorphism. If it is an isomorphism we are granted that it is monic and epic.



**Problem 14:** If  $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$  is defined as  $\phi(x, y) = (x + y, 2x - y)$  then determine what is  $p(\phi)$  when  $p(x) = x^2 - 2x + 1$

**Solution.** Note that  $\phi^2 = \phi \circ \phi$  and so  $\phi^0 = Id_{\mathbb{R}^2}$ , and we can write the polynomial as:

$$p(x) = x^2 - 2x + 1x^0$$

So if we apply it to  $\phi$  we would have:

$$\begin{aligned} p(\phi(x, y)) &= \phi^2(x, y) - 2\phi(x, y) + 1\phi^0(x, y) \\ &= \phi(x + y, 2x - y) - 2(x + y, 2x - y) + 1(x, y) \\ &= (3x, 3y) + (-2x - 2y, 2y - 4x) + (x, y) \\ &= (3x - 2x - 2y + x, 3y + 2y - 4x + y) \\ &= (2x - 2y, 6y - 4x) \end{aligned}$$