Problem set
Sebastián Caballero

2023
Self Study

Vector Spaces I

<u>Problem 1:</u> Let V and W be vector spaces over a field K. Let $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ be a basis for V and let $\{w_1, w_2, \ldots, w_n\}$ be any vectors in W. There is a unique linear map

$$\phi: \quad V \quad \to \quad W$$

Such that $\phi(v_i) = w_i$ for all $1 \leq i \leq n$

Solution. Since $\mathcal{B}$ is a basis for V, for any element $v \in V$ there are $a_1, a_2, \ldots, a_n \in K$ such that:

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$$

so if we define ϕ such that $\phi(v_i) = w_i$ then for any vector v we would have:

$$\begin{aligned}\phi(v) &= a_1\phi(v_1) + a_2\phi(v_2) + \cdots + a_n\phi(a_n) \\ &= a_1 w_1 + a_2 w_2 + \cdots + a_n w_n\end{aligned}$$

<u>Problem 2:</u> Suppose that V is a finite dimensional vector space. Let $\mathcal{B} = \{v_1, v_2, \ldots, v_n\}$ be a basis for V then:

- Any set of $w_1, w_2, \ldots, w_n, w_{n+1}$ vectors is linearly dependent
- Any set of $w_1, w_2, \ldots, w_{n-1}$ vectors can't generate V

Solution. For this, we are going to use the facts needed for a basis.

- Let $w_1, w_2, \ldots, w_n, w_{n+1}$ be vectors in V, we can write them in the next way:

$$\begin{aligned}w_1 &= a_{1,1}v_1 + a_{1,2}v_2 + \cdots + a_{1,n}v_n \\ w_2 &= a_{2,1}v_1 + a_{2,2}v_2 + \cdots + a_{2,n}v_n \\ &\cdots\cdots \\ w_n &= a_{n,1}v_1 + a_{n,2}v_2 + \cdots + a_{n,n}v_n \\ w_{n+1} &= a_{n+1,1}v_1 + a_{n+1,2}v_2 + \cdots + a_{n+1,n}v_n\end{aligned}$$

  If there is a w_i such that $w_i = 0$ we are done. Suppose then that this is not true, so for each $1 \leq i \leq n+1$ exists j such that $a_{i,j} \neq 0$. But since there are w_{n+1} there must be i_1, i_2 such that for the same j, we have that $a_{i_1,j} \neq 0 \neq a_{i_2,j}$. So, we can express the vector v_j as:

$$\begin{aligned}v_j &= \frac{w_{i_1}}{a_{i_1,j}} - \frac{a_{i_1,1}v_1 + a_{i_1,2}v_2 + \cdots + a_{i_1,n}v_n}{a_{i_1,j}} \\ v_j &= \frac{w_{i_2}}{a_{i_2,j}} - \frac{a_{i_2,1}v_1 + a_{i_2,2}v_2 + \cdots + a_{i_2,n}v_n}{a_{i_2,j}}\end{aligned}$$

  And so the set is not linearly independent.

- Let $w_1, w_2, \ldots, w_{n-1}$ be vectors of V. Suppose that indeed we can generate V with them, so in particular, we can write:

$$
\begin{aligned}
v_1 &= a_{1,1}w_1 + a_{1,2}w_2 + \cdots + a_{1,n-1}w_{n-1}\\
v_2 &= a_{2,1}w_1 + a_{2,2}w_2 + \cdots + a_{2,n-1}w_{n-1}\\
&\ldots\ldots\\
v_n &= a_{n,1}w_1 + a_{n,2}w_2 + \cdots + a_{n,n-1}w_{n-1}
\end{aligned}
$$

  And since none of them is zero, we can be fure that for each $1 \leq i \leq n$ exists j such that $a_{i,j} \neq 0$. But since there are n vectors in $\mathcal{B}$ and just $n-1$ vectors w_i, there must be i_1, i_2 such that for the same j, we have that $a_{i_1,j} \neq 0 \neq a_{i_2,j}$. So, we can express the vector v_j as:

$$
\begin{aligned}
w_j &= \frac{v_{i_1}}{a_{i_1,j}} - \frac{a_{i_1,1}w_1 + a_{i_1,2}w_2 + \cdots + a_{i_1,n}w_n}{a_{i_1,j}}\\
w_j &= \frac{v_{i_2}}{a_{i_2,j}} - \frac{a_{i_2,1}w_1 + a_{i_2,2}w_2 + \cdots + a_{i_2,n}w_n}{a_{i_2,j}}
\end{aligned}
$$

  But then this let us generate two different linear combinations within $\mathcal{B}$ that give us the same result, contradicting the linear independency of $\mathcal{B}$.

<u>Problem 3:</u> Let V be a finite vector space. If $A = \{v_1, v_2, \ldots, v_n\}$ generates V then some subset of A is a basis for V.

Solution. For that, let declare the next set:

$$S = \{W \in \mathcal{P}(A) | W \text{ is linearly independent}\}$$

We can assure that at least there is a maximal element $\{v_1, v_2, \ldots, v_m\}$ in S since we can assure the existence of $\{v_1\}$ and at most it can be A. Suppose then that it is not A, so $m < n$, and we can assure that any set $\{v_1, \ldots, v_m, v_i\}$ is linearly dependent, with $m < i \leq n$. Therefore we have:

$$a_1v_1 + \cdots + a_nv_n + a_iv_i = 0$$

has more than the trivial solution, so we can suppose that

<u>Problem 4:</u> Let $A = \{v_1, v_2, \ldots, v_n\}$ be a subset of a vector space V. Prove that A is linearly independent if and only if the equation $a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0$ has the trivial solution.

Solution. We prove a double implication:

$\Rightarrow$) If A is linearly independent then by definition the equation $a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0$ has only one solution, the trivial one.

$\Leftarrow$) Suppose that A is not linearly independent, so that there are two combinations of scalars $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ such that for a v in V:

$$
\begin{aligned}
a_1v_1 + a_2v_2 + \cdots + a_nv_n &= v\\
b_1v_1 + b_2v_2 + \cdots + b_nv_n &= v
\end{aligned}
$$

And if we use the transitivity we have:

$$
\begin{aligned}
a_1v_1 + a_2v_2 + \cdots + a_nv_n &= b_1v_1 + b_2v_2 + \cdots + b_nv_n\\
(a_1 - b_1)v_1 + (a_2 - b_2)v_2 + \cdots + (a_n - b_n)v_n &= 0
\end{aligned}
$$

But note that $a_1 \neq b_1$, $a_2 \neq b_2$ and so on, so $a_1 - b_1 \neq 0$, $a_2 - b_2 \neq 0$ and so on, so the equation has another solution apart to the trivial one.

Problem 5: Prove the Rank theorem

Solution. Remember that the rank theorem says that if V and W are finite dimensional vector spaces over K, and $\phi : V \to W$ is a linear map then:

$$\dim V = \dim \ker(\phi) + \dim \phi(V)$$

Let $\mathcal{A} = \{v_1, v_2, \dots, v_n\}$ be a basis for $ker(\phi)$ and let $\mathcal{B} = \{w_1, w_2, \dots, w_m\}$ be a basis for $\phi(V)$. Since $\mathcal{B} \subseteq \phi(V)$ there are $u_1, u_2, \dots, u_m$ such that $\phi(u_1) = w_1, \phi(u_2) = w_2, \dots, \phi(u_m) = w_m$. So, we can create the set:

$$\mathcal{C} = \{v_1, v_2, \dots, v_n, u_1, u_2, \dots, u_m\}$$

And we claim that this is a basis for V. For that, let's prove the two properties for that:

- Suppose that there are scalars $a_1, a_2, \dots, a_n, b_1, \dots, b_m$ such that:

$$\begin{aligned} a_1v_1 + a_2v_2 + \cdots + a_nv_n + b_1u_1 + b_2u_2 + \cdots + b_mu_m &= 0 \\ a_1v_1 + a_2v_2 + \cdots + a_nv_n &= -b_1u_1 - b_2u_2 - \cdots - b_mu_m \\ \phi(a_1v_1) + \phi(a_2v_2) + \cdots + \phi(a_nv_n) &= \phi(-b_1u_1) + \phi(-b_2u_2) + \cdots + \phi(-b_mu_m) \\ a_1\phi(v_1) + a_2\phi(v_2) + \cdots + a_n\phi(v_n) &= -b_1\phi(u_1) - b_2\phi(u_2) - \cdots - b_n\phi(u_m) \\ a_10 + a_20 + \cdots + a_n0 &= -b_1w_1 - b_2w_2 - \cdots - b_mw_m \\ 0 &= -b_1w_1 - b_2w_2 - \cdots - b_mw_m \end{aligned}$$

  And since $\mathcal{B}$ is a basis then $b_1 = b_2 = \cdots = b_m = 0$. And therefore we have that:

$$\begin{aligned} a_1v_1 + a_2v_2 + \cdots + a_nv_n + b_1u_2 + b_2u_2 + \cdots + b_mu_m &= 0 \\ a_1v_1 + a_2v_2 + \cdots + a_nv_n &= 0 \end{aligned}$$

  And since $\mathcal{A}$ is a basis, then $a_1 = a_2 = \cdots = a_n = 0$, and so $\mathcal{C}$ is linearly independent.

- Take $v \in V$, we want to prove it is a linear combination of elements of $\mathcal{C}$. So for that, we know that $\phi(v)$ is a linear combination of elements of $\mathcal{B}$:

$$\begin{aligned} b_1w_1 + b_2w_2 + \cdots + b_mw_m &= \phi(v) \\ b_1\phi(u_1) + b_2\phi(u_2) + \cdots + b_m\phi(u_m) &= \phi(v) \\ \phi(b_1u_1 + b_2u_2 + \cdots + b_mu_m) &= \phi(v) \\ \phi(b_1u_1 + b_2u_2 + \cdots + b_mu_m) - \phi(v) &= 0 \\ \phi(b_1u_1 + b_2u_2 + \cdots + b_mu_m - v) &= 0 \end{aligned}$$

  And since $b_1u_1 + b_2u_2 + \cdots + b_mu_m - v \in \ker(\phi)$ we can derive a linear combination of the form:

$$\begin{aligned} a_1v_1 + a_2v_2 + \cdots + a_nv_n &= b_1u_1 + b_2u_2 + \cdots + b_mu_m - v \\ a_1v_1 + a_2v_2 + \cdots + a_nv_n - b_1u_1 - b_2u_2 - \cdots - b_mu_m &= -v \\ b_1u_1 + b_2u_2 + \cdots + b_mu_m - a_1v_1 - a_2v_2 - \cdots - a_nv_n &= v \end{aligned}$$

  And so we have that v is a linear combination of $\mathcal{C}$, so $Span(\mathcal{C}) = V$.

And that way we conclude that $\mathcal{C}$ is a basis for V and note that $|\mathcal{C}| = |\mathcal{A}| + |\mathcal{B}|$, so $\dim V = \dim \ker(\phi) + \dim \phi(V)$.

Problem 6: Determine whether or not $\{(1, 1, 0), (2, 0, -1), (-3, 1, 1)\}$ is basis for $\mathbb{R}^3$

Solution. First, let's determine whenever it is linearly independent or not.

- Suppose that $a_1(1,1,0)+a_2(2,0,-1)+a_3(-3,1,1)=0$. So, if we add those vectors we would have:
$$\begin{aligned} a_1(1,1,0)+a_2(2,0,-1)+a_3(-3,1,1) &= (a_1,a_1,0)+(2a_2,0,-a_2)+(-3a_3,a_3,a_3) \\ &= (a_1+2a_2-3a_3, a_1+a_3, -a_2+a_3) = (0,0,0) \end{aligned}$$
So we would need that:
$$\begin{aligned} a_1+2a_2-3a_3 &= 0 \\ a_1+a_3 &= 0 \\ a_3-a_2 &= 0 \end{aligned}$$
If we solve the last two equations for a_1 and a_2 we would have:
$$\begin{aligned} a_1 &= -a_3 \\ a_2 &= a_3 \end{aligned}$$
And replacing in the first equation we would have:
$$\begin{aligned} a_1+2a_2-3a_3 &= 0 \\ -a_3+2a_3-3a_3 &= 0 \\ -2a_3 &= 0 \\ a_3 &= 0 \end{aligned}$$
And so we conclude that $a_1=a_2=a_3=0$, so this set is linearly independent.
- Take now any vector $(x,y,z)\in\mathbb{R}^3$, we want to prove that we can always find a linear combination of the vectors that give us (x,y,z). For that, suppose that there are such combinations, so:
$$\begin{aligned} a_1(1,1,0)+a_2(2,0,-1)+a_3(-3,1,1) &= (x,y,z) \\ (a_1,a_1,0)+(2a_2,0,-a_2)+(-3a_3,a_3,a_3) &= (x,y,z) \\ (a_1+2a_2-3a_3, a_1+a_3, a_3-a_2) &= (x,y,z) \end{aligned}$$
And so we have:
$$\begin{aligned} a_1+2a_2-3a_3 &= x \\ a_1+a_3 &= y \\ a_3-a_2 &= z \end{aligned}$$
Then we have:
$$\begin{aligned} a_1 &= y-a_3 \\ a_2 &= a_3-z \end{aligned}$$
And plugging into the first equation we have:
$$\begin{aligned} a_1+2a_2-3a_3 &= x \\ y-a_3+2(a_3-z)-3a_3 &= x \\ y-a_3+2a_3-2z-3a_3 &= x \\ y-2z-2a_3 &= x \\ a_3 &= \frac{2z-x-y}{2} \end{aligned}$$
And plugging into the next equation:
$$\begin{aligned} a_1 &= y-a_3 \\ a_1 &= y-\frac{x+y-2z}{2} \\ a_1 &= y+z-\frac{x}{2}+\frac{y}{2} \\ a_1 &= \frac{3}{2}y+z-\frac{x}{2} \end{aligned}$$

And plugging into the last equation:

$$a_2 = a_3 - z$$
$$a_2 = z - \frac{x}{2} - \frac{y}{2} - z$$
$$a_2 = \frac{-x-y}{2}$$

And if you try this combination, you would get (x, y, z) so we can see $Span(\{(1,1,0),(2,0,-1),(-3,1,1)\}) = \mathbb{R}^3$.

And so we have proved that $\{(1,1,0),(2,0,-1),(-3,1,1)\}$ is a basis for $\mathbb{R}^3$.

Problem 7: Let $\phi : V \to W$ be linear. Suppose that $v_1, \ldots, v_n \in V$ are such that $\phi(v_1), \ldots, \phi(v_n)$ are linearly independent in W. Show that $v_1, \ldots, v_n$ are linearly independent.

Solution. For that, since $\phi(v_1), \ldots, \phi(v_n)$ are linearly independent, we can assure that the equation:

$$a_1\phi(v_1) + a_2\phi(v_2) + \cdots + a_n\phi(v_n) = 0$$

has only the trivial solution. Suppose that the equation:

$$b_1v_1 + b_2v_2 + \cdots + b_nv_n = 0$$

has a solution that is not trivial. That this, we can assure that at least b_1 is not 0. And if we apply to both sides the linear map ϕ we get:

$$\phi(b_1v_1 + b_2v_2 + \cdots + b_nv_n) = \phi(0)$$
$$\phi(b_1v_1) + \phi(b_2v_2) + \cdots + \phi(b_nv_n) = 0$$
$$b_1\phi(v_1) + b_2\phi(v_2) + \cdots + b_n\phi(v_n) = 0$$

But this is a contradiction since this equation can only have the trivial solution. So we can conclude that $v_1, \ldots v_n$.

Problem 8: If $\{v_1, \ldots, v_n\}$ is a basis for V and $\{w_1, \ldots, w_m\}$ is a basis for W then:

$$\{(v_1, 0), \ldots, (v_n, 0), (0, w_1), \ldots, (0, w_n)\}$$

is a basis for $V \oplus W$

Solution. We need to prove two things:

- First, to prove that this set is linearly independent, we need to show that the homogeneous equation has only the trivial solution. So we have:

$$a_1(v_1,0) + a_2(v_2,0) + \cdots + a_n(v_n,0) + b_1(0,w_1) + b_2(0,w_2) + \cdots + b_n(0,w_n) = (0,0)$$
$$(a_1v_1,0) + (a_2v_2,0) + \cdots + (a_nv_n,0) + (0,b_1w_1) + (0,b_2w_2) + \cdots + (0,b_nw_n) = (0,0)$$
$$(a_1v_1 + a_2v_2 + \cdots + a_nv_n, b_1w_1 + b_2w_2 + \cdots + b_nw_n) = (0,0)$$

  And this means that:

$$a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0$$
$$b_1w_1 + b_2w_2 + \cdots + b_nw_n = 0$$

  And since those vectors are basis for each vector space $a_1 = a_2 = \cdots = a_n = b_1 = b_2 = \cdots = b_n$.

- For an element $(v, w) \in V \oplus W$, we know that v can be expressed as a linear combination $a_1v_1 + a_2v_2 + \cdots + a_nv_n = v$, and also w can be expressed as $b_1w_1 + b_2w_2 + \cdots + b_nv_n = w$, so the combination of the vectors in our set will rise:

$$a_1(v_1,0) + a_2(v_2,0) + \cdots + a_n(v_n,0) + b_1(0,w_1) + b_2(0,w_2) + \cdots + b_n(0,w_n) = (v,w)$$

Problem 9: Let W be a subspace of the finite-dimensional vector space V. Show that there is a subspace U of V such that $V \cong U \oplus W$.

Solution. For this, define U as follows:

$$U := V \setminus W \cup \{0\}$$

First, we need to prove that this is a subspace of V:

> Note that for any $v \in U$ different from 0 and any $c \in K$, if $cv \in W$ then $c^{-1}cv = v \in W$ which contradicts the definition of U. If $u, w \in U$ are not both 0, and if $u + w \in W$ then that means that $u, w \in W$ since W is closed over the operations, which again, contradicts the definition for U, so $u + w \in U$.

Now, we want to prove that this is an internal sum of V, so we have:

- If $w \in W$ and $u \in U$ are such that $w + u = 0$, then we would have $w = -u$, which means that $w \in U$ and also that $u = -w \in V$, which means that since its only common element is 0, $u = w = 0$.
- For any element $v \in V$, there are two alternatives. If $v \in W$ then we can express v as $v + 0$ and $0 \in U$. If $v \notin W$ then $v \in U$ by definition and so $v = 0 + v$ with $0 \in W$.

And so we conclude that $U \oplus W$ is an internal sum of V.

Problem 10: A linear map $\rho : V \to V$ is idempotent if $\rho\rho = \rho$. Show that ρ acts as an identity over $\rho(V)$ if ρ is idempotent.

Solution. For that, we want to prove that $\rho^2 = Id_{\rho(V)}$. For that, let $v \in \rho(V)$, we know that there is $w \in V$ such that $\rho(w) = v$. Now, if we apply again the function we would have:

$$\begin{aligned} \rho(\rho(w)) &= \rho(v) \\ \rho(w) &= \rho(v) \\ v &= \rho(v) \end{aligned}$$

So we conclude that $\rho^2 = Id_{\rho(V)}$.

Problem 11: Decide if $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ given by $\phi(x, y) = (x + y, 2x - y)$ is an isomorphism. If it is, find a formula for $\phi^{-1}(x, y)$ and prove they are inverses.

Solution. Suppose that for a vector $(a, b) \in \mathbb{R}^2$, exists $(x, y) \in \mathbb{R}^2$ whose image under ϕ is (a, b). We would have:

$$\phi(x, y) = (x + y, 2x - y) = (a, b)$$

And so we can write the next equations:

$$\begin{aligned} x + y &= a \\ 2x - y &= b \end{aligned}$$

If we solve for x in the first equation we would have:

$$x = a - y$$

And replacing in the second equation we would have:

$$\begin{aligned} 2x - y = 2(a - y) - y &= b \\ 2a - 2y - y &= b \\ 2a - 3y &= b \\ -3y &= b - 2a \\ y &= \frac{2a - b}{3} \end{aligned}$$

And so if we plug in into the second equation we would have:

$$\begin{aligned} x &= a - y \\ &= a - \frac{2a-b}{3} \\ &= a - \frac{2a}{3} - \frac{b}{3} \\ &= \frac{a}{3} - \frac{b}{3} \\ &= \frac{a-b}{3} \end{aligned}$$

And so we would have:

$$\phi^{-1}(x, y) = \left(\frac{x-y}{3}, \frac{2x-y}{3} \right)$$

We can prove also that this indeed the inverse isomorphism by composing them:

- First, if we compose ϕ and ϕ^{-1} we would have:

$$\begin{aligned} \phi(\phi^{-1}(x, y)) &= \phi\left(\frac{x-y}{3}, \frac{2x-y}{3} \right) \\ &= \left(\frac{x-y}{3} + \frac{2x-y}{3}, 2 \cdot \frac{x-y}{3} - \frac{2x-y}{3} \right) \\ &= \left(\frac{3x}{3}, \frac{2x-2y}{3} + \frac{y-2x}{3} \right) \end{aligned}$$