Problem 1.

Let V and W be vector spaces over a field K. Let $\mathcal{B} = \{v_1, v_2, \dots, v_n\}$ be a basis for V and let $\{w_1, w_2, \dots, w_n\}$ be any vectors in W. There is a unique linear map

$$\phi : \quad V \quad \to \quad W$$

Such that $\phi(v_i) = w_i$ for all $1 \le i \le n$

Solution. Since $\mathcal{B}$ is a basis for V, for any element $v \in V$ there are $a_1, a_2, \dots, a_n \in K$ such that:

$$v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n$$

so if we define ϕ such that $\phi(v_i) = w_i$ then for any vector v we would have:

$$\begin{aligned}\phi(v) &= a_1\phi(v_1) + a_2\phi(v_2) + \cdots + a_n\phi(a_n)\\ &= a_1 w_1 + a_2 w_2 + \cdots + a_n w_n\end{aligned}$$

Problem 2.

Suppose that V is a finite dimensional vector space. Let $\mathcal{B} = \{v_1, v_2, \dots, v_n\}$ be a basis for V then:

- Any set of $w_1, w_2, \dots, w_n, w_{n+1}$ vectors is linearly dependent
- Any set of $w_1, w_2, \dots, w_{n-1}$ vectors can't generate V

Solution. For this, we are going to use the facts needed for a basis.

- Let $w_1, w_2, \dots, w_n, w_{n+1}$ be vectors in V, we can write them in the next way:

$$\begin{aligned}w_1 &= a_{1,1}v_1 + a_{1,2}v_2 + \cdots + a_{1,n}v_n\\ w_2 &= a_{2,1}v_1 + a_{2,2}v_2 + \cdots + a_{2,n}v_n\\ &\cdots\cdots\\ w_n &= a_{n,1}v_1 + a_{n,2}v_2 + \cdots + a_{n,n}v_n\\ w_{n+1} &= a_{n+1,1}v_1 + a_{n+1,2}v_2 + \cdots + a_{n+1,n}v_n\end{aligned}$$

  If there is a w_i such that $w_i = 0$ we are done. Suppose then that this is not true, so for each $1 \le i \le n+1$ exists j such that $a_{i,j} \ne 0$. But since there are w_{n+1} there must be i_1, i_2 such that for the same j, we have that $a_{i_1,j} \ne 0 \ne a_{i_2,j}$. So, we can express the vector v_j as:

$$\begin{aligned}v_j &= \frac{w_{i_1}}{a_{i_1,j}} - \frac{a_{i_1,1}v_1 + a_{i_1,2}v_2 + \cdots + a_{i_1,n}v_n}{a_{i_1,j}}\\ v_j &= \frac{w_{i_2}}{a_{i_2,j}} - \frac{a_{i_2,1}v_1 + a_{i_2,2}v_2 + \cdots + a_{i_2,n}v_n}{a_{i_2,j}}\end{aligned}$$

  And so the set is not linearly independent.

- Let $w_1, w_2, \dots, w_{n-1}$ be vectors of V. Suppose that indeed we can generate V with them, so in particular, we can write:

$$\begin{aligned}v_1 &= a_{1,1}w_1 + a_{1,2}w_2 + \cdots + a_{1,n-1}w_{n-1}\\ v_2 &= a_{2,1}w_1 + a_{2,2}w_2 + \cdots + a_{2,n-1}w_{n-1}\\ &\cdots\cdots\\ v_n &= a_{n,1}w_1 + a_{n,2}w_2 + \cdots + a_{n,n-1}w_{n-1}\end{aligned}$$

And since none of them is zero, we can be fure that for each $1 \leq i \leq n$ exists j such that $a_{i,j} \neq 0$. But since there are n vectors in $\mathcal{B}$ and just $n-1$ vectors w_i, there must be i_1, i_2 such that for the same j, we have that $a_{i_1,j} \neq 0 \neq a_{i_2,j}$. So, we can express the vector v_j as:

$$w_j = \frac{v_{i_1}}{a_{i_1,j}} - \frac{a_{i_1,1}w_1 + a_{i_1,2}w_2 + \cdots + a_{i_1,n}w_n}{a_{i_1,j}}$$
$$w_j = \frac{v_{i_2}}{a_{i_2,j}} - \frac{a_{i_2,1}w_1 + a_{i_2,2}w_2 + \cdots + a_{i_2,n}w_n}{a_{i_2,j}}$$

But then this let us generate two different linear combinations within $\mathcal{B}$ that give us the same result, contradicting the linear independency of $\mathcal{B}$.

Problem 3.

Let V be a finite vector space. If $A = \{v_1, v_2, \ldots, v_n\}$ generates V then some subset of A is a basis for V.

Solution. For that, let declare the next set:

$$S = \{W \in \mathcal{P}(A) | W \text{ is linearly independent}\}$$

We can assure that at least there is a maximal element $\{v_1, v_2, \ldots, v_m\}$ in S since we can assure the existence of $\{v_1\}$ and at most it can be A. Suppose then that it is not A, so $m < n$, and we can assure that any set $\{v_1, \ldots, v_m, v_i\}$ is linearly dependent, with $m < i \leq n$. Therefore we have:

$$a_1v_1 + \cdots + a_nv_n + a_iv_i = 0$$

has more than the trivial solution, so we can suppose that

Problem 4.

Let $A = \{v_1, v_2, \ldots, v_n\}$ be a subset of a vector space V. Prove that A is linearly independent if and only if the equation $a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0$ has the trivial solution.

Solution. We prove a double implication:

$\Rightarrow$) If A is linearly independent then by definition the equation $a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0$ has only one solution, the trivial one.

$\Leftarrow$) Suppose that A is not linearly independent, so that there are two combinations of scalars $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ such that for a v in V:

$$a_1v_1 + a_2v_2 + \cdots + a_nv_n = v$$
$$b_1v_1 + b_2v_2 + \cdots + b_nv_n = v$$

And if we use the transitivity we have:

$$a_1v_1 + a_2v_2 + \cdots + a_nv_n = b_1v_1 + b_2v_2 + \cdots + b_nv_n$$
$$(a_1 - b_1)v_1 + (a_2 - b_2)v_2 + \cdots + (a_n - b_n)v_n = 0$$

But note that $a_1 \neq b_1$, $a_2 \neq b_2$ and so on, so $a_1 - b_1 \neq 0$, $a_2 - b_2 \neq 0$ and so on, so the equation has another solution apart to the trivial one.

Problem 5.

Let W be a subspace of a finite dimensional vector space V. Any basis for W can be extended to a basis for V.

Problem 6.

Prove the Rank theorem

Problem 7.

Determine whether or not $\{(1,1,0),(2,0,-1),(-3,1,1)\}$ is basis for $\mathbb{R}^3$

Problem 8.

Let $\phi : V \to W$ be linear. Suppose that $v_1, \ldots, v_n \in V$ are such that $\phi(v_1), \ldots, \phi(v_n)$ are linearly independent in W. Show that $v_1, \ldots, v_n$ are linearly independent.

Solution. For that, since $\phi(v_1), \ldots, \phi(v_n)$ are linearly independent, we can assure that the equation:

$$a_1\phi(v_1) + a_2\phi(v_2) + \cdots + a_n\phi(v_n) = 0$$

has only the trivial solution. Suppose that the equation:

$$b_1v_1 + b_2v_2 + \cdots + b_nv_n = 0$$

has a solution that is not trivial. That this, we can assure that at least b_1 is not 0. And if we apply to both sides the linear map ϕ we get:

$$\begin{aligned}
\phi(b_1v_1 + b_2v_2 + \cdots + b_nv_n) &= \phi(0) \\
\phi(b_1v_1) + \phi(b_2v_2) + \cdots + \phi(b_nv_n) &= 0 \\
b_1\phi(v_1) + b_2\phi(v_2) + \cdots + b_n\phi(v_n) &= 0
\end{aligned}$$

But this is a contradiction since this equation can only have the trivial solution. So we can conclude that $v_1, \ldots v_n$.

Problem 9.

Let V be a vector space over a field k and let U, W be finite dimensional subspaces of V. Prove that $U + W$ and $U \cap V$ are finite dimensional subspaces of V and:

$$\dim(U + W) + \dim(U \cap W) = \dim U + \dim W$$

Problem 10.

Show that the set of real numbers $\mathbb{R}$ is a vector space over the rational numbers $\mathbb{Q}$. Show that this is not a finite-dimensional vector space over $\mathbb{Q}$.

Problem 11.

If $\{v_1, \ldots, v_n\}$ is a basis for V and $\{w_1, \ldots, w_m\}$ is a basis for W then:

$$\{(v_1, 0), \ldots, (v_n, 0), (0, w_1), \ldots, (0, w_n)\}$$

is a basis for $V \oplus W$

Solution. We need to prove two things:

- First, to prove that this set is linearly independent, we need to show that the homogeneous equation has only the trivial solution. So we have:

$$\begin{aligned}
a_1(v_1, 0) + a_2(v_2, 0) + \cdots + a_n(v_n, 0) + b_1(0, w_1) + b_2(0, w_2) + \cdots + b_n(0, w_n) &= (0, 0) \\
(a_1v_1, 0) + (a_2v_2, 0) + \cdots + (a_nv_n, 0) + (0, b_1w_1) + (0, b_2w_2) + \cdots + (0, b_nw_n) &= (0, 0) \\
(a_1v_1 + a_2v_2 + \cdots + a_nv_n, b_1w_1 + b_2w_2 + \cdots + b_nw_n) &= (0, 0)
\end{aligned}$$

  And this means that:

$$\begin{aligned}
a_1v_1 + a_2v_2 + \cdots + a_nv_n &= 0 \\
b_1w_1 + b_2w_2 + \cdots + b_nw_n &= 0
\end{aligned}$$

  And since those vectors are basis for each vector space $a_1 = a_2 = \cdots = a_n = b_1 = b_2 = \cdots = b_n$.

- For an element $(v, w) \in V \oplus W$, we know that v can be expressed as a linear combination $a_1v_1 + a_2v_2 + \cdots + a_nv_n = v$, and also w can be expressed as $b_1w_1 + b_2w_2 + \cdots + b_nv_n = w$, so the combination of the vectors in our set will rise:

$$a_1(v_1, 0) + a_2(v_2, 0) + \cdots + a_n(v_n, 0) + b_1(0, w_1) + b_2(0, w_2) + \cdots + b_n(0, w_n) = (v, w)$$

Problem 12.

Let V be a vector space over K and let W a subspace of V. With the operations induced over V/W, this becomes a vector space.

Problem 13.

Generalize the notions of external and internal direct sums to three or more summands.

Solution. For a set $V_1, V_2, \ldots, V_n$ of vector spaces defined over the same field K, we define the external direct sum as:

$$\bigoplus_{i=1}^{n} V_i = \prod_{i=1}^{n} V_I$$

With the operations defined as:

$$(v_1, v_2, \ldots, v_n) + (w_1, w_2, \ldots w_n) = (v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n)$$
$$c(v_1, v_2, \ldots, v_n) = (cv_1, cv_2, \ldots, cv_n)$$

which also produce a vector space. And let V be a vector space, $\mathcal{V} = \{V_1, V_2, \ldots, V_n\}$ a collection a subspaces of V. We define that V is the internal sum of the elements of $\mathcal{V}$ if and only if there is a function $\eta : \bigoplus_{i=1}^{n} V_i \to V$ such that:

- $\eta(v_1, v_2, \ldots, v_n) = v_1 + v_2 + \cdots + v_n$
- η is monic
- η is epic

Problem 14.

Let W be a subspace of the finite-dimensional vector space V. Show that there is a subspace U of V such that $V \cong U \oplus W$.

Solution. For this, define U as follows:

$$U := V \setminus W \cup \{0\}$$

First, we need to prove that this is a subspace of V:

> Note that for any $v \in U$ different from 0 and any $c \in K$, if $cv \in W$ then $c^{-1}cv = v \in W$ which contradicts the definition of U. If $u, w \in U$ are not both 0, and if $u + w \in W$ then that means that $u, w \in W$ since W is closed over the operations, which again, contradicts the definition for U, so $u + w \in U$.

Now, we want to prove that this is an internal sum of V, so we have:

- If $w \in W$ and $u \in U$ are such that $w + u = 0$, then we would have $w = -u$, which means that $w \in U$ and also that $u = -w \in V$, which means that since its only common element is 0, $u = w = 0$.

- For any element $v \in V$, there are two alternatives. If $v \in W$ then we can express v as $v + 0$ and $0 \in U$. If $v \notin W$ then $v \in U$ by definition and so $v = 0 + v$ with $0 \in W$.

And so we conclude that $U \oplus W$ is an internal sum of V.

Problem 15.

Prove that every vector space has a basis.

Problem 16.

Prove that the set of all infinite sequences of $0's$ and $1's$ with component-wise addition and scalar multiplication modulo 2.

Problem 17.

Can $\mathbb{C}$ be isomorphic to a subspace of $\mathbb{R}$?

Problem 18.

Prove the binomial and the multinomial theorem for rings