**Problem set** Set Theory
Sebastián Caballero Axiomatization

**Problem 1.**

Determine the fibers of the projections $p_j$

**Solution.** Let $A = \{X_1, \dots X_n\}$ be an indexed collection of sets over $n$, the projection $p_j$ is a function from $\prod_{i=1}^{n} X_i$ to $X_j$ and assigns to a tuple $(a_1, a_2, \dots, a_n)$ the element $a_j$. Now, the fibers for an element $x \in X_j$ is the set of all tuples which $j-$th element is $x$. That means that:

$$f^{-1}(x) = \{(a_1, a_2, \dots, a_n) : a_j = x\}$$

Or in a simple way, the product $X_1 \times X_2 \times \cdots \times \{x\} \times \dots X_n$.

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**Problem 2.**

Prove that, for each nonempty set $X$ the function

$$f: \begin{array}{ccc} \mathcal{P}(X) & \to & \{0,1\}^X \\ A & \mapsto & \chi_A \end{array}$$

is a bijection.

**Solution.** We need to prove two things, that $f$ is injective and surjective.

- **Injective:** Suppose $A, B \in \mathcal{P}(X)$ are sets, such that $f(A) = f(B)$. That means that $\chi_A = \chi_B$. So if $x \in A$, $\chi_A(x) = 1$, but it implies that $\chi_B(x) = 1$ so $x \in B$. It proves that $A \subseteq B$ and in a similar way you can prove that $B \subseteq A$, therefore $A = B$.
- **Surjective:** Let $g : X \to \{0,1\}$ be a function. Define the set $A$ as:

  $$A := \{x \in X : g(x) = 1\}$$

  By definition, $A \subseteq X$ so $A \in \mathcal{P}(X)$. Now, if you do $f(A)$ which is $\chi_A$ by definition it is the same function $g$.

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**Problem 3.**

Let $f : X \to Y$ be a function and $i : A \to X$ the inclusion function of a subset $A$ in $X$. Show that:

1. $f|_A = f \circ i$
2. $(f|_A)^{-1}(B) = A \cap f^{-1}(B)$, $B \subseteq Y$

**Solution.** Let $f : X \to Y$ be a function and $i : A \to X$ the inclusion function of a subset $A$ in $X$.

1. First, remember that $f|_A$ is defined from $A$ to $Y$, and by definition of composition, the function $f \circ i$ is defined also from $A$ to $Y$. Now, if you take $x \in A$, then $(f \circ i)(x) = f(i(x))$, but we know that $i(x) = x$ so it is $f(x)$, which is $f|_A(x)$ since $x \in A$. Therefore, both functions are the same.
2. By definition, $(f|_A)^{-1}(B)$ is the set

$$\{x \in A : f(x) \in B\}$$

But if $f(x) \in B$, then $x \in f^{-1}(B)$, so $x \in A \cap f^{-1}(B)$. If $x \in A \cap f^{-1}(B)$ then $x \in A$ and $x \in f^{-1}(B)$, which means that $f(x) \in B$. So, by definition, $x \in (f|_A)^{-1}(B)$, so they are the same.

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**Problem 4.**

Let $f : X \to Y$ be a function. Show that the following are equivalent:

1. $f$ is injective
2. $f^{-1}(f(A)) = A$, $A \subseteq X$
3. $f(A \cap B) = f(A) \cap f(B)$ for all $A, B \subseteq X$

**Solution.** First, suppose that $f$ is injective, so for any $x, y \in X$, $f(x) = f(y)$ implies that $x = y$. Take $x \in A$, then $f(x) \in f(A)$ and by definition, $x \in f^{-1}(f(A))$. If $x \in f^{-1}(f(A))$ then $f(x) \in f(A)$. It implies then that $x \in A$, thanks to the properties of $f$, because there is not other element in $X$ such that its image is $f(x)$. Now, suppose that $f$ is not injective, then $f(x) = f(y)$ but $x \neq y$ for some $x, y \in X$. So, $f(x) \in f(\{x\})$ and $x \in f^{-1}(f(\{x\}))$ but also $y \in f^{-1}(f(\{x\}))$ but it is evident that $y \notin \{x\}$, so $f^{-1}(f(A)) \neq A$ for at least one $A \subseteq X$.

Finally, suppose that $f$ is not injective. Then there are two values $x, y \in X$ such that $f(x) = f(y)$ but $x \neq y$. Now, the sets $\{x\}$ and $\{y\}$ are disjoint, so

$$\begin{aligned} f(\{x\} \cap \{y\}) &= f(\emptyset) \\ &= \emptyset \end{aligned}$$

But $f(x) \in f(\{x\})$ and also $f(x) \in f(\{y\})$, so their intersection is not empty and hence $f(\{x\} \cap \{y\}) \neq f(\{x\}) \cap f(\{y\})$. Suppose also that $f$ is injective. If $f(x) \in f(A \cap B)$ then $x \in A \cap B$ since $x$ is the unique value in $X$ such that its image is $f(x)$. So, $x \in A$ and $x \in B$, therefore $f(x) \in f(A)$ and $f(x) \in f(B)$ and we conclude that $f(x) \in f(A) \cap f(B)$. If $f(x) \in f(A) \cap f(B)$ then $f(x) \in f(A)$ and $f(x) \in f(B)$, and we conclude that $x \in A$ and $x \in B$, so $x \in A \cap B$ and $f(x) \in f(A \cap B)$, so $f(A \cap B) = f(A) \cap f(B)$.

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**Problem 5.**

An operation $\circledcirc$ on a set $X$ is called *anticommutative* if it satisfies the following:

1. There is a right identity element $r := r_X$, that is, $\exists r \in X : x \circledcirc r = x$ for all $x \in X$.
2. $x \circledcirc y = r \Leftrightarrow (x \circledcirc y) \circledcirc (y \circledcirc x) = r \Leftrightarrow x = y$ for all $x, y \in X$.

Show that, whenever $X$ has more than one element, an anticommutative operation $\circledcirc$ on $X$ is not commutative and has no identity element.

**Solution.** Suppose that $X$ has at least two element or more and $\circledcirc$ has a right element $r$. Suppose that $x \circledcirc y = y \circledcirc x$ for some $x, y$. Then we have:

$$x \circledcirc y = y \circledcirc x$$
$$(x \circledcirc y) \circledcirc (y \circledcirc x) = (y \circledcirc x) \circledcirc (y \circledcirc x)$$

And by the property 2, we conclude that:

$$(x \circledcirc y) \circledcirc (y \circledcirc x) = r$$

But this also implies that $x = y$. So, if they are different, $x \circledcirc y \neq y \circledcirc x$ and therefore the operation is not commutative. Now, suppose it has an identity element $e$, it is easy to see that $e = r$. Now, we know that at least we can pick a different element of $e$, name it $x$. But by definition, $e \circledcirc x = x \circledcirc e = x$, which implies that $e = x$ but we have picked them different. So, it cannot have an Identity element.

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**Problem 6.**

Let $\circledcirc$ and $\circledast$ anticommutative operations on $X$ and $Y$. Further, let $f : X \to Y$ satisfy:

$$f(r_X) = r_Y, \quad f(x \circledcirc y) = f(x) \circledast f(y), \quad x, y \in X$$

Prove that:

1. $x \sim y$ if and only if $f(x \circledcirc y) = r_Y$ defines an equivalence relation on $X$.
2. The function

$$\begin{array}{cccc} \overline{f}: & X/\sim & \to & Y \\ & [x] & \mapsto & f(x) \end{array}$$

is well defined and injective. If, in addition, $f$ is surjective, then $\overline{f}$ is bijective.

**Solution.**

1. To prove that, we need to prove that the relation is reflexive, symmetric and transitive.
   - **Reflexive:** Since $x \circledcirc x = r_X$ for all $x \in X$ and $f(r_X) = r_Y$ it is easy to see that $x \sim x$.
   - **Symmetry:** Suppose that $x \sim y$. That means that $f(x \circledcirc y) = r_Y$. We know that $f(x \circledcirc y) = f(x) \circledast f(y) = r_Y$, so we conclude that $f(x) = f(y)$ and therefore $f(y) \circledast f(x) = f(y \circledcirc x) = r_Y$, so $y \sim x$.
   - **Transitivity:** Suppose that $f(x \circledcirc y) = f(y \circledcirc z) = r_Y$. Since $f(x) \circledast f(y) = r_Y$ and $f(y) \circledast f(z) = r_Y$ then $f(x) = f(y) = f(z)$. So, $f(x) \circledast f(z) = f(x \circledcirc z) = r_Y$ and we conclude that $x \sim y$.

   so we have proved that it defines an equivalence relation.

2. Since we have proved this is an equivalence relation and since $f$ is a function, $\overline{f}$ is well defined. Suppose that we have two classes such that $\overline{f}([x]) = \overline{f}([y])$. By definition, $f(x) = f(y)$, so we have that $f(x) \circledast f(y) = r_Y$ which is that $f(x \circledcirc y) = r_Y$, and therefore $x \sim y$, so $[x] = [y]$. We have concluded that the function is injective.

   Suppose that $f$ is surjective. That means, that for any element $y \in Y$, there is $x \in X$ such that $f(x) = y$. Now, we can assure then the existence of $[x]$ and therefore we know that $\overline{f}([x]) = f(x) = y$, so we know that $\overline{f}$ is Surjective and then bijective.

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**Problem 7.**

Let $R$ be a relation on $X$ and $S$ a relation on $Y$. Define a relation $R \times S$ on $X \times Y$ by

$$(x, y)(R \times S)(u, v) \iff (xRu) \wedge (ySv)$$

for $(x, y), (u, v) \in X \times Y$. Prove that if $R$ and $S$ are equivalence relations, then so is $R \times S$.

**Solution.** First, the order pair $(x, y)$ is related to itself since $R$ and $S$ are equivalence relations and $xRx$ and $ySy$. Now, if $(x, y)(R \times S)(u, v)$ then $xRu$ and $ySv$, but then $uRx$ and $vRy$ so $(u, v)(R \times S)(x, y)$. At last, if $(x, y)(R \times S)(u, v)$ and $(u, v)(R \times S)(a, b)$ then $xRu$, $ySv$, $uRa$ and $vSb$, and by transitivity of both relations $xRa$ and $ySb$, so $(x, y)(R \times S)(a, b)$.

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