Problem 1.

Show that for all $n, m \in \mathbb{N}$, $m < n$ if and only if $m \subset n$

Solution.

- $\Rightarrow$) Suppose that $m < n$, so $m \in n$, and now suppose that $p \in m$. We have seen that in $\mathbb{N}$ the relation $\in$ is transitive, so $p \in n$, but this implies that $m \subset n$.
- $\Leftarrow$) Suppose that $m \subset n$, it is impossible that $m = n$ because $(\exists x)(x \in n \land x \notin m)$ and if $n < m$ then $n \subset m$ but this would mean that $m \subset m$ which is absurd. So, we must conclude by the trichotomy that $m < n$.

$\blacksquare$

Problem 2.

Show that for all $n \in \mathbb{N}$, if $x \in n$, then $x \in \mathbb{N}$.

Solution. For $n = 0$, it is absurdly true. If $n = 1$, then $n = \{0\}$ and $0 \in \mathbb{N}$. Suppose this is true in general for n, and suppose that $x \in n^+$. Then, $x \in n$ or $x = n$. If $x \in n$, by the hypothesis $x \in \mathbb{N}$ and if $x = n$ it is obvious that $x \in \mathbb{N}$.

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Problem 3.

Show that for all $m, n \in \mathbb{N}$, $\min\{n, m\} = n \cap m$.

Solution. Without loss of generality, we can suppose that $n \leq m$ and so $\min\{n, m\} = n$. This would implies that $n \subseteq m$ and therefore $n \cap m = n$, so $\min\{n, m\} = n \cap m$.

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Problem 4.

Show that for all $m, n, a \in \mathbb{N}$:

- If $a + m = a + n$ then $m = n$
- If $a > 0$ and $a \cdot m = a \cdot n$ then $m = n$
- If $a > 1$ and $a^m = a^n$ then $m = n$

Solution. Suppose that $m \neq n$, then $m < n$ or $n < m$. Suppose with no loss of generality that $m < n$, so by the monotony laws we have:

- $a + m < a + n$
- If $a > 0$, $a \cdot m < a \cdot n$
- If $a > 1$, $a^m < a^n$

And since the relation $<$ is irreflexive(In other words, that $n \not< n$) then it is not possible that they are equal. So:

- $a + m \neq a + n$
- If $a > 0$, $a \cdot m \neq a \cdot n$
- If $a > 1$, $a^m \neq a^n$

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Problem 5.

Let $n, m \in \mathbb{N}$:

- Show that $m + n = 0$ if and only if $m = n = 0$
- Show that $m \cdot n$ if and only if $m = 0$ or $n = 0$

Solution. It is obvious the left side of both propositions. Now, for prove the other sides:

- Suppose $n \neq 0$ and m could be or not 0. Since $n \neq 0$ then $(\exists x)(x \in \mathbb{N} \wedge x^+ = n)$. This implies that:

  $$\begin{aligned} m + n &= m + x^+ \\ &= (m + x)^+ \end{aligned}$$

  And since $a^+ \neq 0$ for any $a \in \mathbb{N}$, we have that $m + n \neq 0$.

- Suppose $m \cdot n = 0$ and that $n \neq 0$. Then, we can assure that $(\exists x)(x \in \mathbb{N} \wedge x^+ = n)$ and therefore:

  $$\begin{aligned} m \cdot n &= m \cdot x^+ \\ &= (m \cdot x) + m = 0 \end{aligned}$$

  So by the previous proposition we can conclude that $m \cdot x = 0$, and especially $m = 0$.

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Problem 6.

Prove that for all $n, m \in \mathbb{N}$, $m \leq n$ if and only if exists $k \in \mathbb{N}$ such that $m + k = n$.

Solution.

$\Rightarrow$) For $n = 0$ this is true easily true. Suppose it is true for n and suppose that $m \leq n^+$. If $m = n^+$ it is obvious so if $m < n^+$ then $m < n$ or $m = n$. For the first case, there is k such that $m + k = n$ and then $m + (k + 1) = n^+$. If $m = n$ then $m + 1 = n^+$.

$\Leftarrow$) The case for when $n = 0$ is trivial. Suppose this is true n, so that for n^+ if there is k such that $m + k = n^+$ there are two possibilities. If $k = 0$ then $m = n^+$. Else, $\exists x \in \mathbb{N}$ such that $x^+ = k$ and therefore:

$$\begin{aligned} m + k &= n^+ \\ m + x^+ &= n^+ \\ (m + x)^+ &= n^+ \\ m + x &= n \end{aligned}$$

And by the induction hypothesis, we have that $m \leq n$ and since $n < n^+$ then $m \leq n^+$.

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Problem 7.

Prove that for all $a, b \in \mathbb{N}$, if $b > 0$ then there exists unique q, r such that $r < b$ and $a = bq + r$

Solution. For a fixed $b > 0$, it is easy to see that the proposition is true for $a = 0$, since $0 = b \cdot 0 + 0$. Suppose $a = b \cdot q + r$ for $q, r \in \mathbb{N}$ and $r < b$. There are two cases, if $r^+ = b$ then:

$$\begin{aligned} a &= b \cdot q + r \\ a^+ &= b \cdot q + r^+ \\ a^+ &= b \cdot q + b \\ a^+ &= b \cdot (q+1) + 0 \end{aligned}$$

But if $r^+ < b$ then

$$\begin{aligned} a &= b \cdot q + r \\ a^+ &= b \cdot q + r^+ \end{aligned}$$

And in any case, the proposition is true. So, it is true for all a. To prove that this is unique, then take $a = b \cdot q + r$ and $a = b \cdot p + s$ such that $p, q, r, s \in \mathbb{N}$ and $r, s < b$. Assume that $r \leq s$ so we would have that

$$\begin{aligned} b \cdot p + s &= b \cdot q + r \\ &\leq b \cdot q + s \end{aligned}$$

And we conclude that $p \leq q$. In the other hand, we have:

$$\begin{aligned} b \cdot q &\leq b \cdot q + r \\ &= b \cdot p + s \\ &< b \cdot p + b \\ &= b \cdot (p+1) \end{aligned}$$

So $b \cdot q < b \cdot (p+1)$ and we would have that $q < p + 1$, so we have in summary that $p \leq q < p + 1$ which is only possible if $p = q$. And now, it follows easily that $r = s$.

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