

**Problem 1.**

Determine the fibers of the projections  $p_j$

**Solution.** Let  $A = \{X_1, \dots, X_n\}$  be an indexed collection of sets over  $n$ , the projection  $p_j$  is a function from  $\prod_{i=1}^n X_i$  to  $X_j$  and assigns to a tuple  $(a_1, a_2, \dots, a_n)$  the element  $a_j$ . Now, the fibers for an element  $x \in X_j$  is the set of all tuples which  $j$ -th element is  $x$ . That means that:

$$f^{-1}(x) = \{(a_1, a_2, \dots, a_n) : a_j = x\}$$

Or in a simple way, the product  $X_1 \times X_2 \times \dots \times \{x\} \times \dots X_n$ .

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**Problem 2.**

Prove that, for each nonempty set  $X$  the function

$$\begin{array}{ccc} f : \mathcal{P}(X) & \rightarrow & \{0,1\}^X \\ A & \mapsto & \chi_A \end{array}$$

is a bijection.

**Solution.** We need to prove two things, that  $f$  is injective and surjective.

- **Injective:** Suppose  $A, B \in \mathcal{P}(X)$  are sets, such that  $f(A) = f(B)$ . That means that  $\chi_A = \chi_B$ . So if  $x \in A$ ,  $\chi_A(x) = 1$ , but it implies that  $\chi_B(x) = 1$  so  $x \in B$ . It proves that  $A \subseteq B$  and in a similar way you can prove that  $B \subseteq A$ , therefore  $A = B$ .
- **Surjective:** Let  $g : X \rightarrow \{0,1\}$  be a function. Define the set  $A$  as:

$$A := \{x \in X : g(x) = 1\}$$

By definition,  $A \subseteq X$  so  $A \in \mathcal{P}(X)$ . Now, if you do  $f(A)$  which is  $\chi_A$  by definition it is the same function  $g$ .

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**Problem 3.**

Let  $f : X \rightarrow Y$  be a function and  $i : A \rightarrow X$  the inclusion function of a subset  $A$  in  $X$ . Show that:

1.  $f|_A = f \circ i$
2.  $(f|_A)^{-1}(B) = A \cap f^{-1}(B)$ ,  $B \subseteq Y$

**Solution.** Let  $f : X \rightarrow Y$  be a function and  $i : A \rightarrow X$  the inclusion function of a subset  $A$  in  $X$ .

1. First, remember that  $f|_A$  is defined from  $A$  to  $Y$ , and by definition of composition, the function  $f \circ i$  is defined also from  $A$  to  $Y$ . Now, if you take  $x \in A$ , then  $(f \circ i)(x) = f(i(x))$ , but we know that  $i(x) = x$  so it is  $f(x)$ , which is  $f|_A(x)$  since  $x \in A$ . Therefore, both functions are the same.
2. By definition,  $(f|_A)^{-1}(B)$  is the set

$$\{x \in A : f(x) \in B\}$$

But if  $f(x) \in B$ , then  $x \in f^{-1}(B)$ , so  $x \in A \cap f^{-1}(B)$ . If  $x \in A \cap f^{-1}(B)$  then  $x \in A$  and  $x \in f^{-1}(B)$ , which means that  $f(x) \in B$ . So, by definition,  $x \in (f|_A)^{-1}(B)$ , so they are the same.

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**Problem 4.**

Let  $f : X \rightarrow Y$  be a function. Show that the following are equivalent:

1.  $f$  is injective
2.  $f^{-1}(f(A)) = A$ ,  $A \subseteq X$
3.  $f(A \cap B) = f(A) \cap f(B)$  for all  $A, B \subseteq X$

**Solution.** First, suppose that  $f$  is injective, so for any  $x, y \in X$ ,  $f(x) = f(y)$  implies that  $x = y$ . Take  $x \in A$ , then  $f(x) \in f(A)$  and by definition,  $x \in f^{-1}(f(A))$ . If  $x \in f^{-1}(f(A))$  then  $f(x) \in f(A)$ . It implies then that  $x \in A$ , thanks to the properties of  $f$ , because there is not other element in  $X$  such that its image is  $f(x)$ . Now, suppose that  $f$  is not injective, then  $f(x) = f(y)$  but  $x \neq y$  for some  $x, y \in X$ . So,  $f(x) \in f(\{x\})$  and  $x \in f^{-1}(f(\{x\}))$  but also  $y \in f^{-1}(f(\{x\}))$  but it is evident that  $y \notin \{x\}$ , so  $f^{-1}(f(A)) \neq A$  for at least one  $A \subseteq X$ .

Finally, suppose that  $f$  is not injective. Then there are two values  $x, y \in X$  such that  $f(x) = f(y)$  but  $x \neq y$ . Now, the sets  $\{x\}$  and  $\{y\}$  are disjoint, so

$$\begin{aligned} f(\{x\} \cap \{y\}) &= f(\emptyset) \\ &= \emptyset \end{aligned}$$

But  $f(x) \in f(\{x\})$  and also  $f(x) \in f(\{y\})$ , so their intersection is not empty and hence  $f(\{x\} \cap \{y\}) \neq f(\{x\}) \cap f(\{y\})$ . Suppose also that  $f$  is injective. If  $f(x) \in f(A \cap B)$  then  $x \in A \cap B$  since  $x$  is the unique value in  $X$  such that its image is  $f(x)$ . So,  $x \in A$  and  $x \in B$ , therefore  $f(x) \in f(A)$  and  $f(x) \in f(B)$  and we conclude that  $f(x) \in f(A) \cap f(B)$ . If  $f(x) \in f(A) \cap f(B)$  then  $f(x) \in f(A)$  and  $f(x) \in f(B)$ , and we conclude that  $x \in A$  and  $x \in B$ , so  $x \in A \cap B$  and  $f(x) \in f(A \cap B)$ , so  $f(A \cap B) = f(A) \cap f(B)$ . ■

**Problem 5.**

An operation  $\odot$  on a set  $X$  is called *anticommutative* if it satisfies the following:

1. There is a right identity element  $r := r_X$ , that is,  $\exists r \in X : x \odot r = x$  for all  $x \in X$ .
2.  $x \odot y = r \Leftrightarrow (x \odot y) \odot (y \odot x) = r \Leftrightarrow x = y$  for all  $x, y \in X$ .

Show that, whenever  $X$  has more than one element, an anticommutative operation  $\odot$  on  $X$  is not commutative and has no identity element.

**Solution.** Suppose that  $X$  has at least two element or more and  $\odot$  has a right element  $r$ . Suppose that  $x \odot y = y \odot x$  for some  $x, y$ . Then we have:

$$\begin{aligned}x \odot y &= y \odot x \\(x \odot y) \odot (y \odot x) &= (y \odot x) \odot (y \odot x)\end{aligned}$$

And by the property 2, we conclude that:

$$(x \odot y) \odot (y \odot x) = r$$

But this also implies that  $x = y$ . So, if they are different,  $x \odot y \neq y \odot x$  and therefore the operation is not commutative. Now, suppose it has an identity element  $e$ , it is easy to see that  $e = r$ . Now, we know that at least we can pick a different element of  $e$ , name it  $x$ . But by definition,  $e \odot x = x \odot e = x$ , which implies that  $e = x$  but we have picked them different. So, it cannot have an Identity element.

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**Problem 6.**

Let  $\odot$  and  $\otimes$  anticommutative operations on  $X$  and  $Y$ . Further, let  $f : X \rightarrow Y$  satisfy:

$$f(r_X) = r_Y, \quad f(x \odot y) = f(x) \otimes f(y), \quad x, y \in X$$

Prove that:

1.  $x \sim y$  if and only if  $f(x \odot y) = r_Y$  defines an equivalence relation on  $X$ .
2. The function

$$\begin{array}{ccc} \bar{f} : & X/\sim & \rightarrow & Y \\ & [x] & \mapsto & f(x) \end{array}$$

is well defined and injective. If, in addition,  $f$  is surjective, then  $\bar{f}$  is bijective.

**Solution.**

1. To prove that, we need to prove that the relation is reflexive, symmetric and transitive.

- **Reflexive:** Since  $x \odot x = r_X$  for all  $x \in X$  and  $f(r_X) = r_Y$  it is easy to see that  $x \sim x$ .
- **Symmetry:** Suppose that  $x \sim y$ . That means that  $f(x \odot y) = r_Y$ . We know that  $f(x \odot y) = f(x) \otimes f(y) = r_Y$ , so we conclude that  $f(x) = f(y)$  and therefore  $f(y) \otimes f(x) = f(y \odot x) = r_Y$ , so  $y \sim x$ .
- **Transitivity:** Suppose that  $f(x \odot y) = f(y \odot z) = r_Y$ . Since  $f(x) \otimes f(y) = r_Y$  and  $f(y) \otimes f(z) = r_Y$  then  $f(x) = f(y) = f(z)$ . So,  $f(x) \otimes f(z) = f(x \odot z) = r_Y$  and we conclude that  $x \sim y$ .

so we have proved that it defines an equivalence relation.

2. Since we have proved this is an equivalence relation and since  $f$  is a function,  $\bar{f}$  is well defined. Suppose that we have two classes such that  $\bar{f}([x]) = \bar{f}([y])$ . By definition,  $f(x) = f(y)$ , so we have that  $f(x) \otimes f(y) = r_Y$  which is that  $f(x \odot y) = r_Y$ , and therefore  $x \sim y$ , so  $[x] = [y]$ . We have concluded that the function is injective.

Suppose that  $f$  is surjective. That means, that for any element  $y \in Y$ , there is  $x \in X$  such that  $f(x) = y$ . Now, we can assure then the existence of  $[x]$  and therefore we know that  $\bar{f}([x]) = f(x) = y$ , so we know that  $\bar{f}$  is Surjective and then bijective.

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**Problem 7.**

Let  $R$  be a relation on  $X$  and  $S$  a relation on  $Y$ . Define a relation  $R \times S$  on  $X \times Y$  by

$$(x, y)(R \times S)(u, v) \iff (xRu) \wedge (ySv)$$

for  $(x, y), (u, v) \in X \times Y$ . Prove that if  $R$  and  $S$  are equivalence relations, then so is  $R \times S$ .

**Solution.** First, the order pair  $(x, y)$  is related to itself since  $R$  and  $S$  are equivalence relations and  $xRx$  and  $ySy$ . Now, if  $(x, y)(R \times S)(u, v)$  then  $xRu$  and  $ySv$ , but then  $uRx$  and  $vRy$  so  $(u, v)(R \times S)(x, y)$ . At last, if  $(x, y)(R \times S)(u, v)$  and  $(u, v)(R \times S)(a, b)$  then  $xRu$ ,  $ySv$ ,  $uRa$  and  $vSb$ , and by transitivity of both relations  $xRa$  and  $ySb$ , so  $(x, y)(R \times S)(a, b)$ . ■