Analytic Model for an Helicopter 2DoF using Euler-Lagrange equations

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***Abstract*—In this paper is presented the analytic model from a two degrees of freedom(2DoF) helicopter by using the Euler-Lagrange equations. We applied several techniques as linearization, cylindrical coordinates, space-state model.**

***Keywords*—Analytic, MIMO, Lagrange, control, 2DoF, linearization, model.**

I. Introduction

A helicopter requires many systems to control the tilt, turn, balance, altitude and rotation; for this matter it have to be used a group of controllers to manage the forces applied by the helicopter rotors and systems. To successfully make those controllers, a compliant model must be obtained. This kind of model could be obtained by measuring and breaking down experimental data, or by making a mathematical analysis of the system.

In this paper it is discussed the method to obtain the model of the Heli2DoF by the analytic method. The Heli2DoF (shown in figure 1) consists of a helicopter-like system attached to ground by it's pivot. It has only two degrees of freedom (2DoF); two axis, one horizontal and transverse tho the moving beam, and one vertical, also transverse; both axis making the pivot around the beam's supports. The beam has a rotor attached to each tip, one further from the support that controls mainly vertical movement, and the other for horizontal movement.

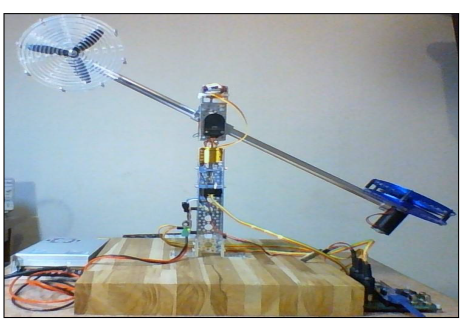

Figure 1. Helicopter 2DoF photo

II. Euler Lagrange analysis

In this section we are going to analysis the helicopter with two degrees of freedom using the Euler Lagrange method. First, we have to analyze the helicopter, considering the mass center, the pitch axis and yaw axis. In the figure 2 we can see the mentioned parts in a simplified helicopter model:

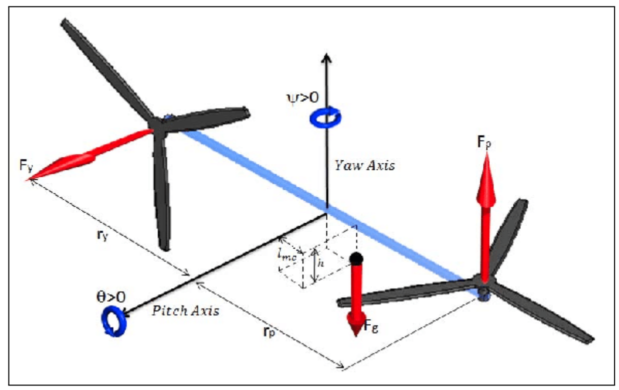

Figure 2. Helicopter 2DoF simplified model [1]

As we can see in figure 2, the mass center position is marked with a rectangle. Respect to the plane, we have a distance **lmc** and a height distance defined as **h**. In the figure 3 we can see how the mass center moves in the space:

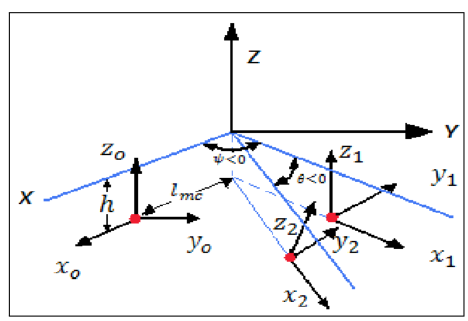

Figure 3. Mass center position analysis [1]

Analyzing the figure 3 we can get the next mass center position equations [1]:

$$\begin{cases} X_{mc} = \cos\left(\Psi\left(t\right)\right)\,\left(\mathrm{lmc}\,\cos\left(\theta\left(t\right)\right) + h\,\sin\left(\theta\left(t\right)\right)\right) \\ Y_{mc} = -\sin\left(\Psi\left(t\right)\right)\,\left(\mathrm{lmc}\,\cos\left(\theta\left(t\right)\right) + h\,\sin\left(\theta\left(t\right)\right)\right) \\ Z_{mc} = \mathrm{lmc}\,\sin\left(\theta\left(t\right)\right) - h\,\cos\left(\theta\left(t\right)\right) \end{cases} \tag{1}$$

A. Kinetic energy

From the analysis of the mass center, we continue to describe the kinetic energy as:

$$T = T_{r,p} + T_{r,y} + T_{tras} \tag{2}$$

Where $T_{r,p}$, $T_{r,y}$ and T_{tras} are presented as:

$$\begin{cases} T_{rp} = \frac{1}{2} \cdot J_{eqp} \cdot \omega_\theta^2 = \frac{1}{2} \cdot J_{eqp} \cdot \dot{\theta}^2 \\ T_{ry} = \frac{1}{2} \cdot J_{eqy} \cdot \omega_\psi^2 = \frac{1}{2} \cdot J_{eqy} \cdot \dot{\psi}^2 \\ T_{tras} = \frac{1}{2} \cdot m_{heli} \cdot V^2 = \frac{1}{2} \cdot m_{heli} \cdot (\dot{X_{mc}}^2 + \dot{Y_{mc}}^2 + \dot{Z_{mc}}^2) \end{cases} \tag{3}$$

J_{eqp} and J_{eqy} are the moments of inertia from pitch and yaw.

B. Potential energy

Given the gravity the potential energy is:

$$V = m_{heli} \cdot g \cdot Z_{mc} \tag{4}$$

C. Euler-Lagrange Equations

We begin defining the Lagrangian function as:

$$L = T - V \tag{5}$$

Using equations 3 and 4 in 5

$$L = \frac{1}{2}\left[\cdot J_{eqy} \cdot \dot{\psi}^2 + \cdot J_{eqp} \cdot \dot{\theta}^2 + \cdot m_{heli} \cdot (\dot{X_{mc}}^2 + \dot{Y_{mc}}^2 + \dot{Z_{mc}}^2)\right] - m_{heli} \cdot g \cdot Z_{mc} \tag{6}$$

Now, we have to develop the Euler-Lagrange equations:

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} = Q_1 \tag{7}$$

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \dot{\psi}} - \frac{\partial L}{\partial \psi} = Q_2 \tag{8}$$

Where:

$$Q_1 = \tau_{pp} + \tau_{py} - B_p \cdot \dot{\theta} = (K_{pp} \cdot V_{mp} + F_{cpp}) + (K_{py} \cdot V_{my} + F_{cpy}) - B_p \cdot \dot{\theta} \tag{9}$$

$$Q_2 = \tau_{yp} + \tau_{yy} - B_y \cdot \dot{\psi} = (K_{yp} \cdot V_{mp} + F_{cyp})cos\theta + (K_{yy} \cdot V_{my} + F_{cyy})cos\theta - B_y \cdot \dot{\psi} \tag{10}$$

At this point is important to explain equations 9 and 10, we have to consider that both motors, the one located in pitch and the one located in Yaw, they have also effect in the other axis, the effect can be few, but is important to taking in consideration in the analysis, that's why we consider Q_1 and Q_2, the firts is how pitch axis feel by its own motor and also the yaw motor, and second one is how yaw feels the action of both motors too.

And now, we solve the partial derivatives and equalizes to Q_1 and Q_2:

$$(J_{eqp} + M_{Heli}(l_{mc}^2 + h^2))\ddot{\theta} + M_{Heli}\left[\frac{\sin 2\theta \cdot (l_{mc}^2 - h^2)}{2} - l_{mc}h\cos 2\theta\right] \cdot \dot{\psi}^2 + M_{Heli}g(l_{mc}\sin\theta + h\cos\theta) + B_p \cdot \dot{\theta} = (K_{pp}V_{mp} + F_{cpp}) + (K_{py}V_{my} + F_{cpy}) \tag{11}$$

$$(J_{eqy} + M_{Heli}[\cos^2\theta(l_{mc}^2 - h^2) + l_{mc}h\sin(2\theta) + h^2])\ddot{\psi} + M_{Heli}\left[\sin(2\theta)(h^2 - l_{mc}^2) - 2l_{mc}h\cos(2\theta)\right]\dot{\theta}\dot{\psi} + B_y\dot{\psi} = (K_{yy}V_{mp} + F_{cyp})\cos\theta + (K_{yy}V_{my} + F_{cyy})\cos\theta \tag{12}$$

The friction is few at the axis so we can despise it:

$$F_{cpp} = F_{cpy} = F_{cyp} = F_{cyy} = 0 \tag{13}$$

Now, is important to rewrite the equations 11 and 12, substituting the constants with literal expressions as letters and linearizing the expressions around the balance point θ:

$$\alpha = J_{eqp} + M_{Heli}(l_{mc}^2 + h^2) \tag{14}$$

$$\beta = M_{Heli}\left[\frac{\sin(2\theta)(l_{mc}^2 - h^2)}{2} - l_{mc}h\cos(2\theta)\right] \tag{15}$$

$$\varphi = M_{Heli}g(-l_{mc}\sin\theta + h\cos\theta) \tag{16}$$

$$\rho = (J_{eqy} + M_{Heli}[\cos^2\theta(l_{mc}^2 - h^2) + l_{mc}h\sin(2\theta) + h^2]) \tag{17}$$

So, now we have the expressions:

$$\alpha\ddot{\theta} + \beta\dot{\psi}^2 + \varphi\theta + B_p\dot{\theta} = K_{pp}V_{mp} + K_{py}V_{my} \tag{18}$$

$$\rho\ddot{\psi} + \beta\dot{\theta}\dot{\psi} + B_y\dot{\psi} = K_{yy}V_{mp}\cos\theta + K_{yy}V_{my}\cos\theta \tag{19}$$

1) State Space Model: Using the equation that describes a space state model:

$$\dot{x} = Ax + Bu \tag{20}$$

$$y = Cx + Du \tag{21}$$

And considering the following states:

$$x_1 = \theta \qquad x_2 = \psi \qquad x_3 = \dot{\theta} \qquad x_4 = \dot{\psi} \tag{22}$$

$$x = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^T \tag{23}$$

There is a linearization around the balance point leaded by the constant $\bar{x}_1 = \bar{\theta}$, obtainnig the linear model parameterized by $\bar{\theta}$ as shown:

$$\begin{cases} \dot{x_1} = x_3 \\ \dot{x_2} = x_4 \\ \dot{x_3} = -\frac{\varphi}{\alpha}x_1 - \frac{B_p}{\alpha}x_3 - \frac{\beta}{\alpha}{x_4}^2 + \frac{K_{pp}}{\alpha}V_{mp} + \frac{K_{py}}{\alpha}V_{my} \\ \dot{x_4} = -\frac{\beta}{\rho}x_3x_4 - \frac{B_y}{\rho}x_4 + \frac{K_{yy}\cos\theta}{\rho}V_{mp} + \frac{K_{yy}\cos\theta}{\rho}V_{my} \end{cases} \tag{24}$$

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-\varphi}{\alpha} & 0 & \frac{-B_p}{\alpha} & 0 \\ 0 & 0 & 0 & \frac{-B_y}{\rho} \end{bmatrix} \tag{25}$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{K_{pp}}{\alpha} & \frac{K_{py}}{\alpha} \\ \frac{K_{yp}\cos\theta}{\rho} & \frac{K_{yy}\cos\theta}{\rho} \end{bmatrix} \tag{26}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \tag{27}$$

$$D = [0] \tag{28}$$

Table I
PARAMETERS VALUES [1]

Parameters	***Value***	***Units***
k_{PP}	0.02638	$\frac{N\cdot m}{V}$
k_{PY}	0.00189	$\frac{N\cdot m}{V}$
k_{YP}	0.002096	$\frac{N\cdot m}{V}$
k_{YY}	0.01871	$\frac{N\cdot m}{V}$
B_P	0.01325	$\frac{N}{V}$
B_Y	0.8513	$\frac{N}{V}$
J_{eqp}	0.0332	$kg\cdot m^2$
J_{eqy}	0.0371	$kg\cdot m^2$
l_{mc}	0.0122	m
h	0.00714	m
m_{heli}	1.3872	kg

III. RESULTS ANALYSIS

In order to evaluate the model and generate the step and impulse response, it's necessary to replace some variables for respective numbers. We also used the small angle approximation:

$$\begin{cases} sin\ \theta \simeq \theta \simeq 0 \\ cos\ \theta \simeq 1 \end{cases} \tag{29}$$

It is wise to evaluate the expected values from the simulations by analyzing the plant's real behavior. Since the inputs are measured by voltages and the output by degrees these will be the terms to be considered.

When there is a voltage continuously supplied to the pitch rotor (step response) it is expected that the pitch elevates or turns to a certain value of degrees from the resting position, and stays in that position after settling; this behavior can be seen in the figure 4 that also shows an underdamping, also inside the expected results. Meanwhile the yaw should start rotating or increasing it's degree value constantly but slowly since there is a small torque applied by the rotor and friction is not being considered. In this case "slow" and "small" are by comparison with the output values when applying both input voltages. As shown in the figure 5, the simulated output complies with the expected.

Similarly, when the voltage is applied as an impulse to the pitch rotor, the pitch is expected to elevate to a certain degree and then settle back at the resting position. This is the kind of behavior observable in the figure 6 adding also an underdamping. While the yaw is expected to increase it's degree until stopping still; as well this behavior can be seen in the figure 7.

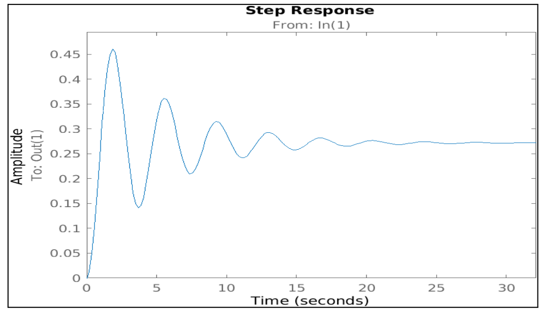

Figure 4. Pitch step response from pitch propeller

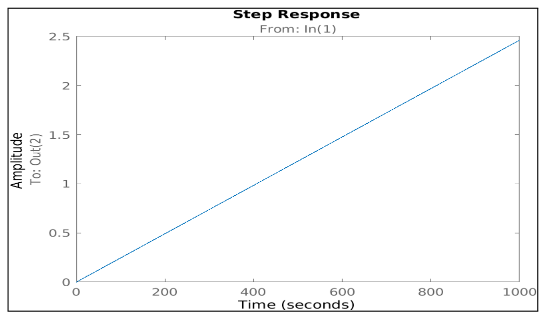

Figure 5. Yaw step response from pitch propeller

Secondly, when there is a voltage supplied to the yaw rotor, be as step or as impulse, there should be similar response as with the pitch rotor just that increasing it's effect in the degrees axis for the yaw output, and decreasing for the pitch output. As seen in the figure 8 the yaw output shows an increased slope for step response; and in figure 9 exhibit an increase in the traveled degrees before the axis stops.

As for the pitch output, the figure 10 has a step response settled at a lower value than showed before, and in the figure 11 it is shown that the overshoot occurs much lower than before.

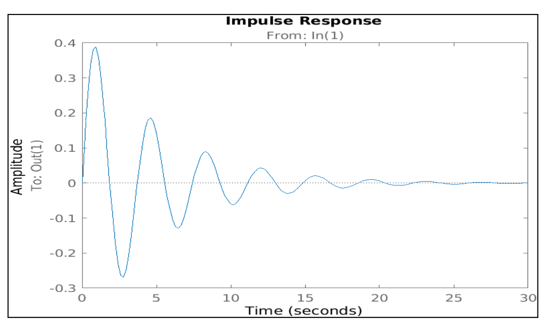

Figure 6. Pitch impulse response from pitch rotor

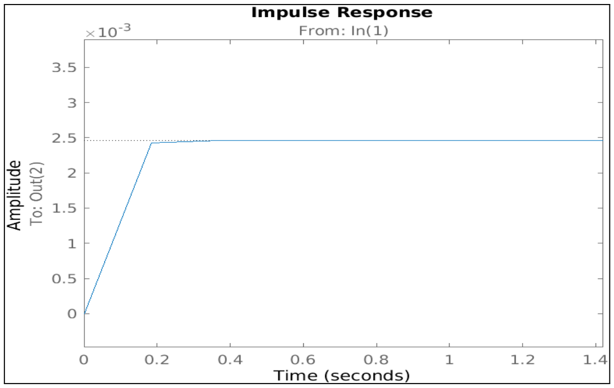

Figure 7. Yaw impulse response from pitch rotor

The overshoot exhibited in the pitch output figures are expected since the voltage is being applied with no control, and gravity acts as an opposite force that makes the output bounce around the settle point. Within future work the overshoot should be suppressed or minimized.

In the other hand the yaw axis does not have an opposite force that behaves as in the pitch, that is why it is not expected from the yaw output to have an overshoot; insted it should stop gradually at the end of the slope, as seen in the figures.

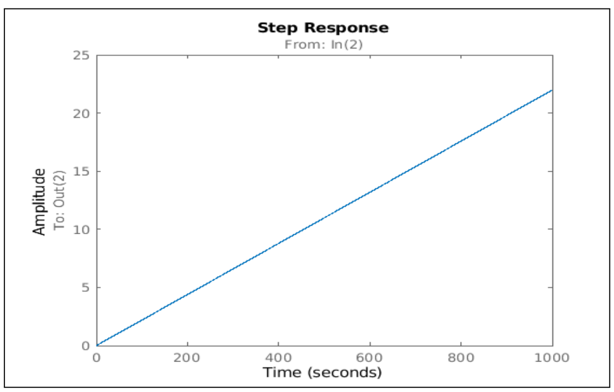

Figure 8. Yaw step response from pitch propeller

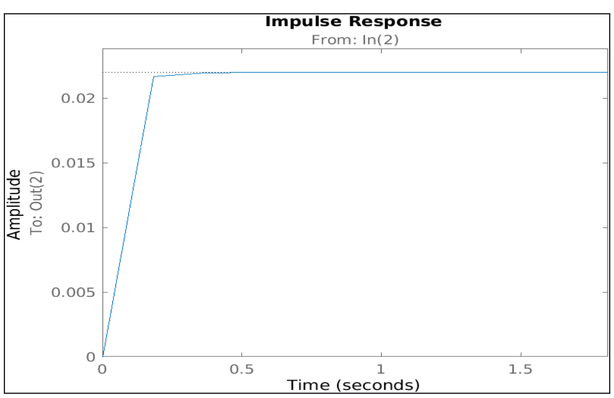

Figure 9. Yaw impulse response from yaw rotor

The controller to do in future work is expected to manage the pitch elevation angle, keeping it steady at the specified degrees making variations on the pitch rotor depending on the yaw rotor's force. Similarly, the yaw turn angle should stay steady by using the yaw rotor to make a counter-force to the torque applied by the pitch rotor.

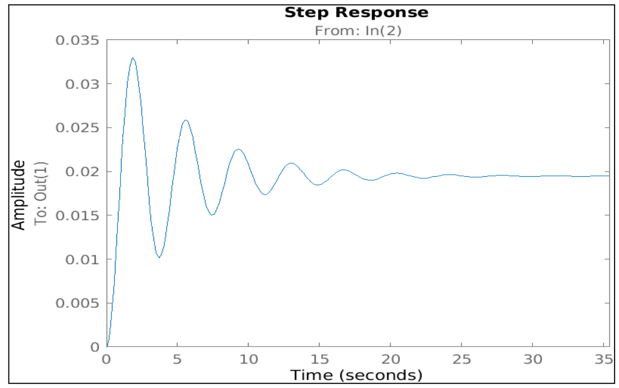

Figure 10. Pitch step response from yaw propeller

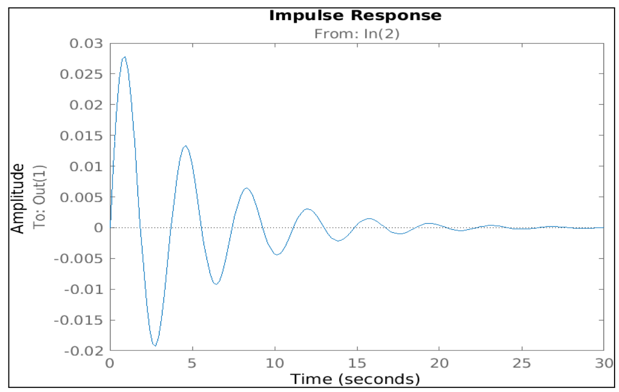

Figure 11. Pitch impulse response from yaw rotor

A. Recommended Control for the system

It is recommended a Linear Quadratic Regulator control (LQR) with an integral effect. It minimizes the function cost and give priority to certain states. The states are defined as integral pitch and yaw error.

IV. Conclusions

All the output figures comply with the expected results of the plant. Therefore it is considered that the model obtained is correct. We can consider that its a stable system because when a pulse or impulse is applied the system returns to the same position after some seconds. The Euler Lagrange method is highly efficient and simplify the analysis of forces using energy instead. Also, was necessary to applied linearization around the balance point θ, in order to generate the state-model equations.

Access to Github Matlab code: GitHub

References

[1] D. M. Rivera, "Model and Observer-Based Controller Design for a Quanser Helicopter with two DOF." Universidad Pedagogica Nacional, Colombia, 2012. [Online]. Available: https://ieeexplore.ieee.org/document/6524589