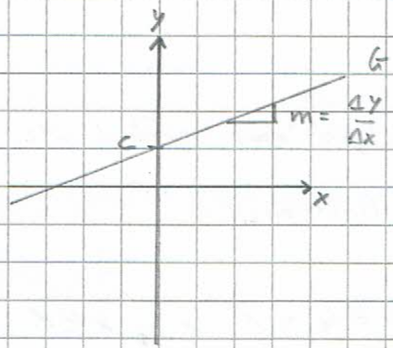
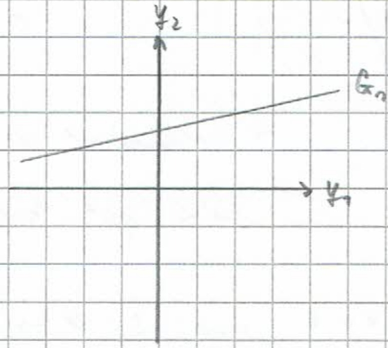
Übungsblatt 01

Aufgabe 1

a) 1) $G = \{ (x,y) \in \mathbb{R}^2 \mid y = mx + c \}$

2) $G_i = \{ (y_1, y_2) \in \mathbb{R}^2 \mid a_{i1} y_1 + a_{i2} y_2 = x_i \} \quad i = 1,2$





$y = mx + c$ $\qquad$ G_i: $y_1 = -\frac{a_{i2}}{a_{i1}} y_2 + \frac{x_i}{a_{i1}}$

$y = y_1, \quad m = -\frac{a_{i2}}{a_{i1}}, \quad c = \frac{x_i}{a_{i1}}$

$x = y_2$

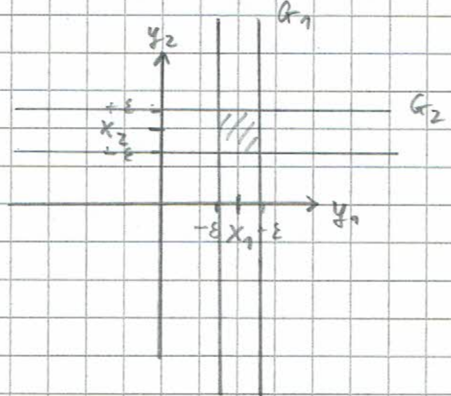
b)

$A\, y^* = x \quad \Rightarrow \quad y^* = A^{-1} \cdot x = \frac{1}{\det}\, \mathrm{adj}(A)\; x$

$= \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \cdot \begin{vmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{vmatrix} \cdot x$

c) Die Kondition untersucht die Lösung eines Problems.
Sie gibt an inwiefern ungenaue Eingabedaten sich auf die Ausgabedaten auswirken.

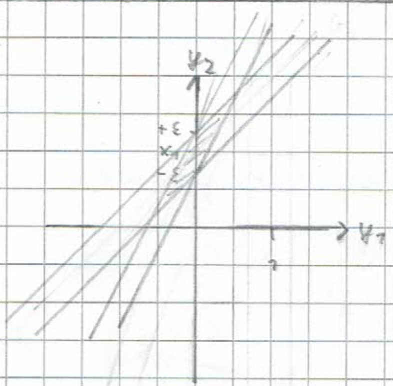
d) <u>gut konditioniertes Problem</u>



$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

schlecht konditioniertes Problem



$$A = \begin{pmatrix} -1 & 1 \\ -1{,}05 & 1 \end{pmatrix}$$

$$x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

→ Unsicherheit des Schnittpunktproblems größer als beim gut konditionierten Problem.

c) $X = Y = \mathbb{R}^2$ und $f: X \to Y,\ x \to f(x) = y = A^{-1}x$, $x \in X$, $y \in Y$, $A \in \mathbb{R}^{2\times 2}$;

$$\tilde{x} = \begin{pmatrix} x_1 + \varepsilon \\ x_2 \end{pmatrix}$$

$$\to \Delta x = \tilde{x} - x = \begin{pmatrix} \varepsilon \\ 0 \end{pmatrix}$$

$$\to \Delta y = f(\tilde{x}) - f(x) = A^{-1}(\tilde{x} - x) = A^{-1}\begin{pmatrix} \varepsilon \\ 0 \end{pmatrix}$$

$$= \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{pmatrix} a_{22} \\ -a_{21} \end{pmatrix} \cdot \varepsilon$$

$$\frac{\|\Delta y\|_\infty}{\|\Delta x\|_\infty} = \frac{\max\{\Delta y\}}{\max\{\Delta x\}} = \frac{\frac{\varepsilon}{a_{11}a_{22} - a_{12}a_{21}}}{\varepsilon} \cdot \max\{a_{22}, -a_{21}\}$$

$$= \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \max\{a_{22}, -a_{21}\}$$

→ Sensitivität

Aufgabe 2

(N1) $\|v\| > 0$, $\forall v \in V$ und $\|v\| = 0$ impliziert $v = 0$

(N2) Für alle $a \in K$, $v \in V$ gilt $\|av\| = |a|\,\|v\|$

(N3) Dreiecksungleichung: $\|v + w\| \leq \|v\| + \|w\|$

i) $\|v\|_1 := \sum_{i=1}^{n} |v_i|$

(N1): $\|v\|_1 = 0 \iff \sum_{i=1}^{n} |v_i| = 0 \rightarrow \sum_{i=1}^{n} \sqrt{v_i^2} \geq 0 \quad \forall v_i \in \mathbb{R}$

$\|v\|_1 = 0 \rightarrow v = 0$

(N2): $\|av\|_1 = \sum_{i=1}^{n} |av_i| = \sum_{i=1}^{n} |a| \cdot |v_i| = |a| \cdot \sum_{i=1}^{n} |v_i| = |a| \cdot \|v\|_1$

(N3): $\|v + w\|_1 = \sum_{i=1}^{n} |v_i + w_i| \leq \sum_{i=1}^{n} |v_i| + \sum_{i=1}^{n} |w_i|$

$$\sum_{i=1}^{n} \sqrt{(v_i + w_i)^2} \leq \sum_{i=1}^{n} \sqrt{v_i^2} + \sum_{i=1}^{n} \sqrt{w_i^2}$$

$$|v_i + w_i|^2 \leq (|v_i| + |w_i|)^2$$

$$v_i^2 + 2v_iw_i + w_i^2 \leq |v_i|^2 + 2|v_i||w_i| + |w_i|^2$$

$$2v_iw_i \leq 2|v_i||w_i|$$

ii) $\|v\|_2 := \left(\sum_{i=1}^{n} |v_i|^2\right)^{1/2}$

(N1): $\|v\|_2 = 0 \iff \sqrt{\sum_{i=1}^{n} |v_i|^2} \geq 0$ und $\sqrt{\sum_{i=1}^{n} |v_i|^2} = 0$

für $v = \underline{0}$

(N2): $\|av_i\|_2 = \sum_{i=1}^{n} \sqrt{|av_i|^2} = \sum_{i=1}^{n} \sqrt{|a|^2 \cdot |v_i|^2} = |a| \cdot \sum_{i=1}^{n} \sqrt{|v_i|^2}$

$= |a| \cdot \|v_i\|_2$

(N3): $\|v + w\|_2 = \sum_{i=1}^{n} \sqrt{|v_i + w_i|^2} = \sum_{i=1}^{n} |\langle v, w \rangle| \overset{CSU}{\leq} \sum_{i=1}^{n} \langle v_i, v_i \rangle \langle w_i, w_i \rangle$

$= \|v\|_2 + \|w\|_2$

iii) $\|v\|_\infty := \max_{i=1,\dots,n} |v_i|$

$(N1)$: $\|v\|_\infty \geq 0 \Rightarrow \max_{i=1,\dots,n} |v_i| \geq 0$ für $|v_i| \geq 0 \quad \forall i \in \{1,\dots,n\}$

$\|v\|_\infty = 0 \Rightarrow \max_{i=1,\dots,n} |v_i| = 0$ für $|v_i| = 0 \quad \forall i \in \{1 \dots n\}$

$(N2)$: $\|\alpha v\|_\infty = \max_{i=1\dots n}\{|\alpha v_i|\} = \max_{i=1\dots n}\{|\alpha| \cdot |v_i|\} = |\alpha| \cdot \max_{i=1\dots n} |v_i|$

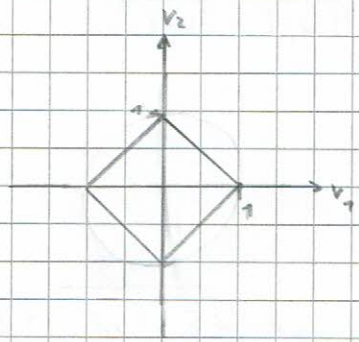
$= |\alpha| \, \|v\|_\infty$

$(N3)$: $\|v + w\|_\infty \Rightarrow \max_{1\dots n} |v_i + w_i| \leq \max_{1\dots n} |v_i| + \max_{i=1\dots n} |w_i|$

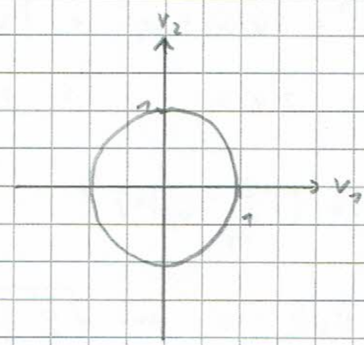
$= \|v\|_\infty + \|w\|_\infty$

b) $n = 2$

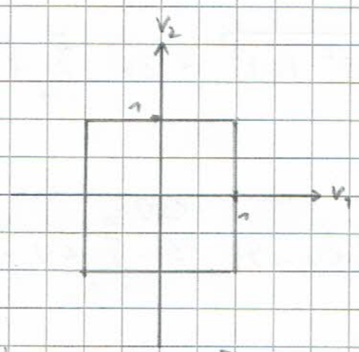
- $\{v \in \mathbb{R}^2 ; \|v\|_1 = 1\}$



- $\{v \in \mathbb{R} ; \|v\|_2 = 1\}$



- $\{v \in \mathbb{R}^2 ; \|v\|_\infty = 1\}$



c) $\|\cdot\|_\infty : C[a,b] \to \mathbb{R}, \quad g \to \max_{t \in [a,b]} |g(t)| \quad \forall g \in C[a,b]$

$(N1)$ $\|g(t)\|_\infty \geq 0 \Rightarrow \max_{t \in [a,b]} |g(t)| \geq 0 \quad \forall |g(t)| \geq 0 \quad \forall t \in [a,b]$

$\|g(t)\|_\infty = 0 \Rightarrow \max_{t \in [a,b]} |g(t)| = 0 \quad \forall g(t) = 0 \quad \forall t \in [a,b]$

$$(N2)\quad \|a \cdot g(t)\|_\infty = \max_{t\in[a,b]} |a\, g(t)| = \max_{t\in[a,b]} |a|\,|g(t)|$$

$$= |a| \cdot \max_{t\in[a,b]} |g(t)|$$

$$(N3)\quad \|g(t)+w(t)\|_\infty = \max_{t\in[a,b]} \{|g(t)+w(t)|\} \leq \max_{t\in[a,b]} \{|g(t)|+|w(t)|\}$$

$$\leq \max_{t\in[a,b]}\{|g(t)|\} + \max_{t\in[a,b]}\{|w(t)|\} = \|g(t)\|_\infty + \|w(t)\|_\infty$$

Aufgabe 3

a) $h_1 : [0,2] \to \mathbb{R}$, $x \to \sqrt{1+4x}$

$u, v \in [0,2]$

$$\|h(u) - h(v)\| \leq L \cdot \|u-v\| \quad \forall u,v \in [0,2]$$

$$\|\sqrt{1+4u} - \sqrt{1+4v}\| \leq L \cdot \|u-v\|$$

$$\left\| \frac{1+4u-(1+4v)}{\sqrt{1+4u}+\sqrt{1+4v}} \right\| \leq L \cdot \|u-v\|$$

$$\left\| \frac{4(u-v)}{\sqrt{1+4u}+\sqrt{1+4v}} \right\| \leq L \cdot \|u-v\|$$

$$4 \cdot \left\| \frac{u-v}{\sqrt{1+4u}+\sqrt{1+4v}} \right\| \leq L\, \|u-v\|$$

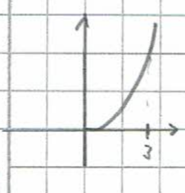
$\to \sqrt{1+4u} + \sqrt{1+4v} \geq 1 \quad \forall u,v \in [0,2]$

$\to$ L wird mit 4 abgeschätzt

b) $h_2 \quad [-4,3] \to \mathbb{R}$, $x \to h_2(x)$ mit $c > 0$

$$\|h_2'(x)\| \leq \frac{c}{2} x^2$$

$$\lim_{\Delta\to 0}\left(\frac{\|h_2(x+\Delta) - h_2(x)\|}{\|\Delta\|}\right) = \lim_{\Delta\to 0}\left(\left\|\frac{h_2(x+\Delta)-h_2(x)}{\Delta}\right\|\right) = \|h_2'(x)\|$$



$$\leq \frac{c}{2} x^2 \leq \underbrace{4{,}5\, c}_{L} \quad \forall x \in [-4,3]$$

c) Jede monoton steigende Funktion

$h_3 : \mathbb{R} \to \mathbb{R} \quad x \to h_3(x) = x^2$

$\to$ Mittelwertsatz für $\Delta \to 0$: $h_3'(x) = 2x$

$\forall L \in \mathbb{R} : \exists x \in \mathbb{R} : \| h_3'(x) \| = \| 2x \| \geq L$