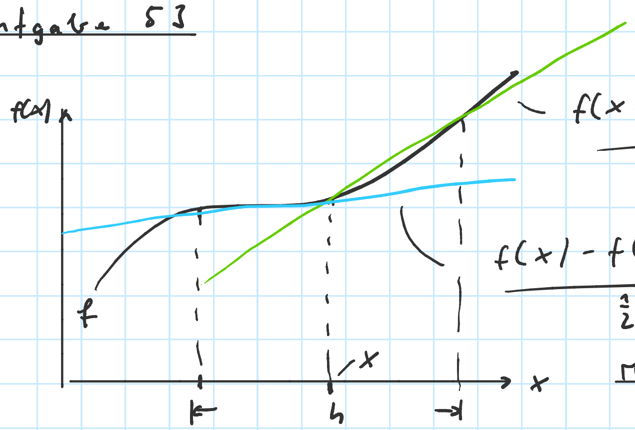
Blatt 12 Sebastian Maschke

Montag, 13. Juli 2020 17:42

Aufgabe 53



$$\frac{f(x+\frac{1}{2}h) - f(x)}{\frac{1}{2}h}$$ Vorwärtsdifferenz

$$\frac{f(x) - f(x-\frac{1}{2}h)}{\frac{1}{2}h}$$ Rückwärtsdifferenz

Mittelwertbildung:

$$f'(x) = \frac{f(x+\frac{1}{2}h) - f(x-\frac{1}{2}h)}{h} - \frac{h^2}{24} f'''(\xi)$$

Taylor-Entwicklung:

$$T_n(x) = f(x_0) + f'(x_0)\cdot(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \frac{f'''(x_0)}{6}(x-x_0)^3 + \dots + \frac{f^{(k-1)}(x_0)}{(k-1)!}(x-x_0)^{k-1} + \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\cdot(x-x_0)^{n+1}$$

$$f(x) = T_n(x) + R_n(x) \quad \text{in Umgebung } x_0$$

$x_0 = x + \frac{1}{2}h$:

$$T_2(x) = f(x+\tfrac{1}{2}h) + f'(x+\tfrac{1}{2}h)\cdot(x-(x+\tfrac{1}{2}h)) + \frac{f''(x+\frac{1}{2}h)}{2}(x-(x+\tfrac{1}{2}h))^2$$

$$= f(x+\tfrac{1}{2}h) - f'(x+\tfrac{1}{2}h)\cdot\tfrac{1}{2}h + \tfrac{1}{8}h^2 f''(x+\tfrac{1}{2}h)$$

$$R_2(x) = \frac{f'''(\xi)}{6}(x-(x+\tfrac{1}{2}h))^3 = \frac{f'''(\xi)}{6}(-\tfrac{1}{2}h)^3 = -\frac{f'''(\xi)}{48}h^3$$

$$f(x) = T_2(x) + R_2(x) = f(x+\tfrac{1}{2}h) - f'(x+\tfrac{1}{2}h)\cdot\tfrac{1}{2}h + \tfrac{1}{8}h^2 f''(x+\tfrac{1}{2}h) - \frac{f'''(\xi)}{48}h^3 \quad (1)$$

$x_0 = x - \frac{1}{2}h$:

$$T_2(x) = f(x-\tfrac{1}{2}h) + f'(x-\tfrac{1}{2}h)\cdot\tfrac{1}{2}h + \tfrac{1}{8}h^2\cdot f''(x-\tfrac{1}{2}h)$$

$$R_2(x) = \frac{f'''(\xi)}{6}(\tfrac{1}{2}h)^3 = \frac{f'''(\xi)}{48}h^3$$

$$f(x) = f(x-\tfrac{1}{2}h) + f'(x-\tfrac{1}{2}h)\cdot\tfrac{1}{2}h + \tfrac{1}{8}h^2\cdot f''(x-\tfrac{1}{2}h) + \frac{f'''(\xi)}{48}h^3 \quad (2)$$

zentrale Differenzenquotient:

$$\boxed{\frac{\Delta y}{\Delta x} := \frac{f(x+\Delta x) - f(x-\Delta x)}{2\,\Delta x}}$$

$$f'(x) = \frac{f(x+\frac{1}{2}h) - f(x-\frac{1}{2}h)}{h}$$

$$= \frac{\left[f(x+\tfrac{1}{2}h) - f'(x+\tfrac{1}{2}h)\cdot\tfrac{1}{2}h + \tfrac{1}{8}h^2 f''(x+\tfrac{1}{2}h) - \tfrac{f'''(\xi)}{48}h^3\right] - \left[f(x-\tfrac{1}{2}h) + f'(x-\tfrac{1}{2}h)\cdot\tfrac{1}{2}h + \tfrac{1}{8}h^2\cdot f''(x-\tfrac{1}{2}h) + \tfrac{f'''(\xi)}{48}h^3\right]}{h}$$

$$= \frac{f(x+\frac{1}{2}h) - f(x-\frac{1}{2}h) + \frac{1}{8}h^2\left[f''(x+\frac{1}{2}h) + f''(x-\frac{1}{2}h)\right]}{h} - \frac{2h^3\,\frac{f'''(\xi)}{48}}{h}$$

$$= \underline{\underline{\frac{f(x+\frac{1}{2}h) - f(x-\frac{1}{2}h)}{h} + \frac{1}{8}h\left[f''(x+\frac{1}{2}h) - f''(x-\frac{1}{2}h)\right] - h^2\,\frac{f'''(\xi)}{24}}}$$

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