Reasoning and Logic, Tantalizing TA-check 2

Deadline: the 17th of September 23:59

Introduction

This assignment is about Modules 1 and 2 and the start of module 3. Every question has an indication for how long the question should take you in an exam-like setting. In total this set of exercises should take you about 177 minutes.

Questions of Tantalizing TA-check 2 (Monday 29th August, 2022, 15:54)

0. (5 min.) **Splitting the work**

   This TA-check should be done in pairs. However, as those of you using the skill circuits probably noticed, we do not recommend doing all of the TA-check in one go. Instead we recommend you do specific questions from this TA-check after studying specific concepts. To make sure you can both work through the material at your own pace, we recommend you first divide the work between the two of you.

   We recommend you both do the first subquestion of each question and then split the remaining subquestions (one taking the odd ones, the other the even ones) and do those individually. Since the difficulty increases, make sure you alternate and do not divide first half vs second half! Then in question 6 you should discuss your answers and merge them into one set of answers to request feedback on.

1. **Sufficient and Necessary conditions**

   (a) The definition of divisibility is as follows:

   > An integer n is divisible by an integer m if there exists an integer k such that $n = km$. We write $m \mid n$, which is pronounced as "m divides n" or "n is divisible by m."

   As an example: $3 \mid 15$, but $4 \nmid 15$ (notice the difference between $\mid$ and $\nmid$).

   The definition of a prime factor is as follows:

   > An integer p is a prime factor of an integer n iff $p \mid n$ and p is prime.

   For each of the following statements, indicate whether they are *necessary* for $8 \mid n$, with n an arbitrary integer. Give a convincing argument for your answer.

   i. (3 min.) $n > 3$.
   ii. (3 min.) $512 \mid n^3$.
   iii. (3 min.) $7 \nmid n$.

   (b) For each of the following statements, indicate whether they are *sufficient* for $5 \mid n$.

   i. (3 min.) $5 \mid \sqrt{n}$.
   ii. (3 min.) $3 \nmid n$.
   iii. (3 min.) $n = 25^k$ for some integer $k \geqslant 3$.

Question 1: 18 min.

2. **Tarski's world**

   (a) In *Tarski's world*, it is possible to describe situations using formulas whose truth can be evaluated, which are expressed in a first-order language that uses predicates such as Rightof(x, y), which means that x is situated—somewhere, not necessarily directly—to the right of y, or Blue(x), which means that x is blue. In the world in Figure 2.9 in Delftse Foundations of Computation (p. 30), for instance, the formula $\forall x(\text{Triangle}(x) \rightarrow \text{Blue}(x))$ holds, since all triangles are blue, but the converse of this formula, $\forall x(\text{Blue}(x) \rightarrow \text{Triangle}(x))$, does not hold, since object c is blue but not a triangle.

   Alfred Tarski contributed much to the semantics of first-order languages and the systematic development of the concept of truth. In the program *Tarski's World*, described in the book *Language, Proof and Logic* by Barwise and Etchemendy, a 'Tarski World' is a visualization of a 'mathematical structure,' a core concept in that theory: a structure is a description of a 'situation' wherein one can

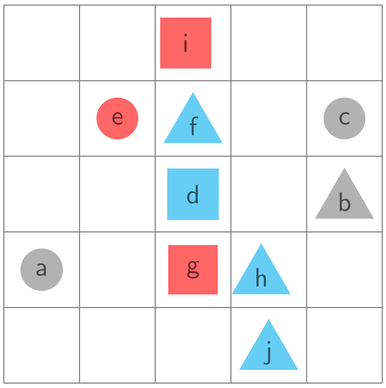

Figure 1: An instance of a Tarski World.

evaluate whether a statement in a given first-order language (say, T) is true or false. A structure contains a domain with objects, plus an indication of all (combinations of) objects with the properties denoted by the predicate symbols in the language T: Blue(a) is true in a structure (e.g. named $\mathcal{A}$) in which the object with the name 'a' belongs to the set of objects with the property of being blue. That set is called $B^{\mathcal{A}}$: the set in structure $\mathcal{A}$ that indicates which objects have the property indicated by predicate symbol B. In *Tarski's World*, color indicates to which set each object belongs, which could be notated more generally by describing the set B as $B^{\mathcal{A}} = \{a, c, e, g\}$. The relation that corresponds to the predicate RightOf is indicated by the positioning between the objects, but this can be denoted more generally by describing the relation as the set (see chapter 1) of all pairs $(x, y) \in D \times D$ for which x is situated right of y in the picture: RightOf$^{\mathcal{A}} = \{(b, a), (c, a), (d, a), (c, f), \ldots\}$.

i. (15 min.) Give the structure that is visualized in Figure 1 (of this document) in the abstract way as described above. Take another good look at section 2.4.4 in the book, where you can find explanations about what this should entail. You may leave out LeftOf, AboveOf, and BelowOf.

ii. (15 min.) Give a formal structure in which the following four formulas all hold. For each formula, give an argument showing how you can see that it holds.
- $\exists x$(Triangle$(x) \vee$ Blue(x))
- $\exists x$(Square$(x) \wedge$ Gray(x))
- $\forall y$(Circle$(y) \rightarrow$ Red(y))
- $\forall z$(Triangle$(z) \rightarrow \neg$Gray(z))

iii. (5 min.) Give a counterexample to the following claim. Do so by drawing a Tarski World in which this claim is false and explain how your counterexample shows this falsity.
$\forall x \forall y (\mathit{RightOf}(x, y) \leftrightarrow \neg \mathit{LeftOf}(x, y))$

(b) For the following invalid arguments, give suitable counterexamples. A counterexample is a structure as in question a, in which all premises (if any) hold, while the conclusion does not hold. You also need to specify which sets in your structure correspond to which predicate symbols in the formulas. Note that the domain of a structure cannot be empty. Each time, explain why your structure forms a counterexample.

As an example consider the following argument:
$\exists x(P(x) \wedge Q(x)), \forall x(Q(x) \rightarrow R(x)) \therefore \forall x(P(x) \rightarrow R(x))$

What's being claimed is that if there is an object with property P that is also has property Q, and all objects that have Q also have R, then *all* all objects with P have R as well. This is clearly not the case, as is shown by the following structure $\mathcal{C}$ with domain $D = \{4, 5\}$. Let $P^{\mathcal{C}} = \{4, 5\}$ and $Q^{\mathcal{C}} = R^{\mathcal{C}} = \{4\}$. Now the first premise holds (take $x = 4$), the second one holds as $Q^{\mathcal{C}} = R^{\mathcal{C}}$, but the conclusion does not hold: take $x = 5$.

i. (4 min.) $\exists x(P(x)) \therefore \forall x(P(x))$

ii. (4 min.) $\therefore \forall x Q(x)$[1]

iii. (4 min.) $\forall x P(x, x) \therefore \forall x \forall y P(x, y)$

[1]Remember that an argument without premises has certain conditions under which it can be valid or invalid.

iv. (6 min.) $\neg\forall x(P(x) \wedge \exists y(\neg Q(x,y))), \forall x(\neg P(x) \rightarrow \exists y(Q(x,y))) \therefore \forall x \exists y(Q(x,y))$

v. (10 min.) $\neg\forall x(P(x) \rightarrow \exists y Q(x,y))$, $\forall x \forall y(R(x,y) \rightarrow (P(y) \vee Q(y,x)))$ $\therefore$ $\exists z(\neg P(z) \vee \forall y(Q(z,y) \wedge \neg R(y,z)))$

Question 2: 63 min.

3. **Proof Outlines. Do not split this question between you.** For each of the following abstractly formulated claims, outline a proof for the claim. Highlight what proof techniques you would use in your proof. For example for : $\forall x(P(x) \rightarrow R(x))$, we would answer: Take an arbitrary x such that $P(x)$ holds. Now we show that $R(x)$ holds. Since x was arbitrarily chosen, it will hold for all x.
   (a) (3 min.) $\exists x(P(x))$
   (b) (5 min.) $\forall x(P(x) \leftrightarrow Q(x))$
   (c) (5 min.) $\forall x(\neg P(x) \rightarrow \neg Q(x))$
   (d) (5 min.) $\forall x(P(x) \rightarrow \exists y(R(x,y))$

Question 3: 18 min.

4. **Conjuctive Normal Form**

   In the previous homework assignment we saw the disjunctive normal form that every formula can be rewritten to. Every formula from propositional calculus can also be written as a conjunction of disjunctions, in conjunctive normal form (CNF). So the formula $(p \vee q)$ is already in conjunctive normal form as the conjunction of one disjunction, but $((p \wedge q) \vee r)$ is not in CNF because it is not a conjunction of disjunctions; rewriting it to $((p \vee r) \wedge (q \vee r))$ gives a conjunctive normal form of this formula.

   Now rewrite each of the following formulas to an equivalent form in conjunctive normal form. You need to be able to justify every step you take, so write down every step explicitly. Also make truth tables for the formulas labeled with '$\star$' to show that the formula in CNF that you have found is equivalent to the formula in the assignment that you started with.
   (a) (5 min.) $((a \wedge b) \rightarrow c)$ ($\star$)
   (b) (5 min.) $\neg(a \rightarrow (b \wedge c))$ ($\star$)
   (c) (4 min.) $((a \vee b) \rightarrow (c \vee d))$
   (d) (4 min.) $\neg(p \rightarrow \neg(q \vee (\neg r \wedge s)))$

Question 4: 18 min.

5. **Do this question together. Essay Questions**
   (a) (5 min.) In your own words, explain the difference between a function and a predicate.
   (b) (5 min.) Give an example of a vacuously true statement about the real-world.
   (c) (5 min.) Old exam question: If the following claim is true, explain why it is true, or if it is false, give a counterexample. Start your answer with either the word "True" or "False" indicating which of the two options applies.
   We can translate the statement $p \wedge \neg p$ for some atomic propositions p to $\forall x \in D(P(x) \wedge \neg P(x))$ for some predicate P and some domain D. Now both statements are contradictions.
   (d) (10 min.) **Reflection:**
   - How did you use the feedback from the teaching assistants on the previous assignment in this weeks' helpful homework?
   - What questions do you have for a teaching assistant when you go and discuss your work with them?
   - What was the hardest question for you to answer and what will you do to improve your skills in answering that type of question on an exam?

Question 5: 25 min.

6. (30 min.) **Combining the work**

   Having each done half of the homework, you should now briefly discuss the work you did. For each question pick at least one subquestion each to discuss with your partner. If there are other answers you are not sure about, discuss those too. Make changes, and update the answers.

   Now scan your answers and submit them on Brightspace as a group. Next week you can then book a time slot with a TA to get your feedback!