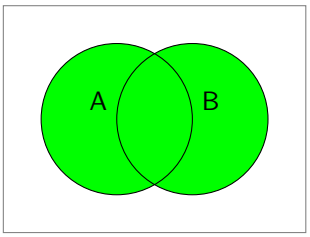


Figure 1: Example Venn diagram that represents $A \cup B$.

Reasoning and Logic, Tantalizing TA-check 5

Deadline: the 15th of October 23:59

Introduction

This assignment is about lectures 1 through 13, with an emphasis on lectures 11 to 13, which cover chapter 4 of the book. Every question has an indication for how long the question should take you in an exam-like setting. In total this set of exercises should take you about 214 minutes.

Questions of Helpful Homework 5 (Monday 29th August, 2022, 15:54)

0. (5 min.) **Splitting the work**

   This TA-check should be done in pairs. However, as those of you using the skill circuits probably noticed, we do not recommend doing all of the TA-check in one go. Instead we recommend you do specific questions from this TA-check after studying specific concepts. To make sure you can both work through the material at your own pace, we recommend you first divide the work between the two of you.

   We recommend you both do the first subquestion of each question and then split the remaining subquestions (one taking the odd ones, the other the even ones) and do those individually. Since the difficulty increases, make sure you alternate and do not divide first half vs second half! Then in question 4 you should discuss your answers and merge them into one set of answers to request feedback on.

1. **Set theory**

   (a) (15 min.) Suppose we have the sets $A = \{Gumshoe, Trucy, Ema\}$, $B = \{Ema, Klavier\}$ and $C = \{Klavier, Trucy, Apollo\}$ in a universe $U = (A \cup B) \cup C$. Write the following sets using set-roster notation:

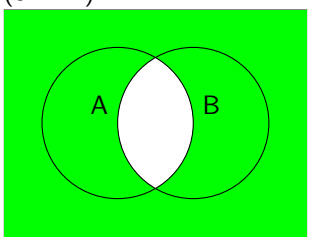
   i. $A \cup B$
   ii. $A \cap (C - B)$
   iii. $B \times C$
   iv. $\mathscr{P}((A \cap B) \cup (A \cap C))$

   (b) For each of the following descriptions, draw a Venn diagram that correctly represents the relation(s) between the sets. So for the first question, ensure your Venn diagram contains two sets A and B and highlight the area that forms the set C. For instance, for Figure 1 the highlighted set is $A \cup B$.

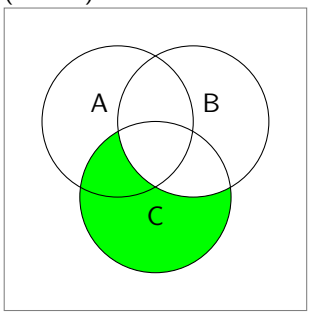
   i. (3 min.) $C = A \cap B$ (Highlight C).
   ii. (3 min.) $C = (A \cup B) - (A \cap B)$ (Highlight C).
   iii. (3 min.) $A \cap B = \varnothing$, $A \cap C \neq \varnothing$, $B \cap C \neq \varnothing$, $D = (C - A) - B$ (Highlight D).

   (c) For each of the following Venn diagrams denote how the highlighted set can be constructed from the other sets. For instance, for Figure 1 the highlighted set is $A \cup B$.

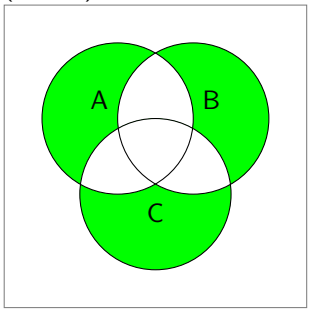
i. (3 min.)



ii. (3 min.)



iii. (7 min.)



(d) For each of the following claims about sets, give a proof or a suitable counterexample to show the claim holds or does not hold, respectively. The universe U is always the union of all the sets that are mentioned, A, B, C, ... as well as some elements that are not a part of any of these sets.

Hint: it can be useful sometimes to make a Venn diagram or an Euler diagram of the situations described.

i. (10 min.) **Claim.** For all sets A and B: $A^c \subseteq B \to B^c \subseteq A$

ii. (5 min.) **Claim.** For all non-empty sets A, B, C:
$(A \cap B = \varnothing, A \cap C \neq \varnothing, B \cap C \neq \varnothing) \to \exists x(x \notin A \wedge x \notin B \wedge x \in C)$

iii. (10 min.) **Claim.** For all sets A, B: $(A \cap B)^c = (A^c \cup B^c)$

iv. (10 min.) **Claim.** For all sets A and B: $\mathscr{P}(A) \cup \mathscr{P}(B) \subseteq \mathscr{P}(A \cup B)$

v. (10 min.) **Claim.** For all sets A, B, and C: $(A-B) \cup (B-C) = (A \cup B) \cap (A \cup C^c) \cap (B \cap C)^c$
Hint: this is the kind of claim that, if true, may be easiest to prove using an algebraic proof.

vi. (10 min.) **Claim.** For all sets A, B and C: if $A \subseteq C$ *and* $B^c \subseteq C$, then $A \cap B = \varnothing$.

(e) This is an **optional** question about power sets and the empty set $\varnothing$.

i. (5 min.) Give the following sets using set-roster notation: $\mathscr{P}(\varnothing)$, $\mathscr{P}(\mathscr{P}(\varnothing))$, and $\mathscr{P}(\mathscr{P}(\mathscr{P}(\varnothing)))$.

ii. (5 min.) Formulate (precisely) a conjecture about the number of elements in such a set that starts with n times '$\mathscr{P}$(', as a function of n.

iii. (10 min.) Prove your conjecture using mathematical induction on n.
Tip: Review the proof of theorem 6.3.1, which says that every set with n elements has 2^n subsets.

Question 1: 112 min.

2. **Recursion and induction**

(a) (7 min.) Give a recursive definition for the set B of arbitrarily nested, properly matched brackets. For example, $[](\{\}) \in B$, but $[\{ \notin B$. (The brackets available are: { }, [ ], ( ) and $\langle\ \rangle$.)

(b) For the following two algorithms, prove the correctness of the algorithm using induction [1].

i. (10 min.)

```
m = 0
n = 1
while (m <= i)
|    m := m + 4
|    n := n + 2
end while
return m + n
```

Claim : The code always returns an odd number.
Loop invariant: `m+n` is odd.

ii. (10 min.)

```
i = 0
n = 1
while (i < x)
|    i := i + 1
|    n := n * i
end while
return n
```

Claim: The code computes $x!$
Loop invariant: $n = i!$

(c) (10 min.) Suppose we have the following recursively defined set C:

BASE : $3 \in C$.
RECURSION : If $c \in C$, then $3c \in C$ and $-3c \in C$.
RESTRICTION : Nothing is in C except what can be derived using the above rules.

Use structural induction to prove that every integer in C is odd.

Question 2: 37 min.

3. **Repetition**

(a) (10 min.) Use mathematical induction to prove that for all integers $n \geqslant 1$: $133 \mid (11^{n+1} + 12^{2n-1})$.

(b) (10 min.) Give a proof by contradiction that for any two positive integers m and n, $m + n \geqslant 2\sqrt{mn}$.

(c) (10 min.) **Do this question together. Reflection:**

- How did you use the feedback from the teaching assistants on the previous assignment in this weeks' helpful homework?
- What questions do you have for a teaching assistant when you go and discuss your work with them?
- What was the hardest question for you to answer and what will you do to improve your skills in answering that type of question on an exam?

Question 3: 30 min.

4. (30 min.) **Combining the work**

Having each done half of the homework, you should now briefly discuss the work you did. For each question pick at least one subquestion each to discuss with your partner. If there are other answers you are not sure about, discuss those too. Make changes, and update the answers.

Now scan your answers and submit them on Brightspace as a group. Next week you can then book a time slot with a TA to get your feedback!

[1] https://www.youtube.com/watch?v=GSvqF48TVM4