Reasoning and Logic, Tantalizing TA-check 3

Deadline: the 24th of September 23:59

Introduction

This assignment is about Modules 1 through 3 and the start of module 4. Every question has an indication for how long the question should take you in an exam-like setting. In total this set of exercises should take you about 241 minutes.

Questions of Tantalizing TA-check 3 (Monday 29th August, 2022, 15:54)

0. (5 min.) **Splitting the work**

   This TA-check should be done in pairs. However, as those of you using the skill circuits probably noticed, we do not recommend doing all of the TA-check in one go. Instead we recommend you do specific questions from this TA-check after studying specific concepts. To make sure you can both work through the material at your own pace, we recommend you first divide the work between the two of you.

   We recommend you both do the first subquestion of each question and then split the remaining subquestions (one taking the odd ones, the other the even ones) and do those individually. Since the difficulty increases, make sure you alternate and do not divide first half vs second half! Then in question 7 you should discuss your answers and merge them into one set of answers to request feedback on.

1. **Proving some claims!**

   For each of the following claims, give a proof or provide a counterexample with an explanation of how it makes the claim false.

   Hint: to get a feeling for the claim, it may be helpful to write down a few cases and to see if the claim holds in such cases.

   (a) (5 min.)
   Claim. *For all integers a, b and c such that $a \mid b$ and $b \mid c$, we have $a \mid c$.*

   (b) (5 min.)
   Claim. *For all integers a, b, c: if $(a+b) \mid c$ and $(a+c) \mid b$, then $(b+c) \mid a$.*

   (c) (5 min.)
   Claim. *For all integers a, b, c: if $a \mid b$ and $a \nmid c$, then $a \nmid (c-b)$.*

   (d) (15 min.)
   Claim. *There does not exist an integer n such that $n^3 = 4k+2$ for some integer k.*
   Hint: try to think of a positive claim that is equivalent to this one.

   (e) (10 min.) **Claim.** For all integers a, b, c, d: if $a \mid b$ and $c \mid d$, then $ac \mid bd$.

   (f) (10 min.) **Claim.** For all integers n: $4 \nmid (n^2 - 3)$.

   (g) (15 min.) **Claim.** For all integers $n > 0$: $\sum_{i=1}^{n} i!$ is odd.
   Hint: combine (and prove) two lemmas involving parity of numbers that are added and that are multiplied.

   (h) (10 min.) *Old exam question* For all real numbers x and y with $x \neq 0$ and $y \neq 0$, it holds that $\sqrt{x^2+y^2} \neq x+y$.

   (i) (10 min.) *Old exam question* Prove the following claim for all integers a,b,c: if $a^2+b^2=c^2$, then a or b is even.

   (j) (15 min.) **Bonus** (not mandatory, maybe consider using a computer-aided proof): Call a number n *impure* if $\sqrt{n} < \sigma(n)$, with σ the function that counts the number of integer divisors of n (including 1 and n itself).
   Claim. For all integers $n > 0$, there are more impure numbers $k \leqslant n$ than there are primes $p \leqslant n$.

Question 1: 100 min.

2. **First steps in recursive functions**
   (a) (5 min.) Consider the following recursive function $f(n)$. List the first 7 values of $f(n)$, so for $n = 0$ to $n = 6$.

$$f(n) = \begin{cases} 1 & \text{if } n \leqslant 0 \\ f(n-2) + f(3) & \text{if } n > 3 \\ f(n-1) + 3 & \text{else} \end{cases}$$

   (b) (5 min.) Consider the following recursive function $g(n)$. List the first 7 values of $g(n)$, so for $n = 0$ to $n = 6$.

$$g(n) = \begin{cases} 1 & \text{if } n \leqslant 1 \\ g(n/2) + 3 & \text{if } n > 1 \text{ and } n \text{ is even} \\ g(3n+1) - 2 & \text{else} \end{cases}$$

   (c) (10 min.) Formulate a recursive function $h(n)$ that computes the number of odd digits in a number. E.g. $h(7) = 1$, $h(12) = 1$, $h(131) = 3$, and $h(138015719) = 7$. Note that you can use $\lfloor n \rfloor$ to round down n and $n \mod x$ to compute the remainder when dividing n by x.

Question 2: 20 min.

3. (20 min.) **A first induction proof**
   Prove the following theorem using mathematical induction:

   Theorem. *For all integers $n \geqslant 1$: $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$.*

Question 3: 20 min.

4. **Satisfiability of propositions**
   (a) (3 min.) Explain when a proposition is satisfiable and how you can show that a proposition is satisfiable.
   (b) (3 min.) What property does a proposition that is not satisfiable have?
   (c) (3 min.) How can you show that a proposition is not satisfiable?
   (d) For each of the following domains and propositions, show that they are either satisfiable or not. If the proposition is satisfiable, show this using a formal structure using all elements in the domain. If the proposition is not satisfiable, explain that is it not satisfiable using the method described in your answer of c.
      i. (5 min.) $D^i = \{1, 2, 3, 4\}$

         (1) $\forall x, y(P(x, y) \rightarrow \neg P(y, x))$
         (2) $\forall x, y, z(P(x, y) \wedge P(y, z) \rightarrow P(x, z))$
         (3) $P(2, 3)$

      ii. (10 min.) $D^{ii} = \{1, 2, 3\}$

         (1) $\forall x, y(P(x, y) \rightarrow \neg P(y, x))$
         (2) $\forall x, y, z(P(x, y) \wedge P(y, z) \rightarrow P(x, z))$
         (3) $\forall x \exists y(P(x, y))$
         (4) $P(1, 2)$
         (5) $P(2, 3)$

Question 4: 24 min.

5. **Revision**
   (a) (6 min.) Consider the following argument written in propositional logic. If it is invalid provide a counterexample and explain how it shows the argument is invalid. If it is valid, prove it.

$$\begin{array}{l} p \rightarrow ((q \wedge r) \vee (r \rightarrow q)) \\ \neg q \leftrightarrow \neg(r \vee p) \\ q \\ \hline \therefore p \end{array}$$

(b) (10 min.) Consider the following argument written in predicate logic. Provide a counterexample in the form of a formal structure.

$$\begin{array}{l} \forall x(P(x) \leftrightarrow (Q(x) \land \exists y(R(x,y)))) \\ P(3) \\ \exists x(x \neq 3 \land P(x)) \\ \exists x,y(x \neq y \land R(x,y)) \\ \hline \therefore \neg\exists y(\forall x R(x,y)) \end{array}$$

Question 5: 16 min.

6. **Do this question together. Essay questions**
   (a) (8 min.) In your own words, describe what a theorem prover such as Z3 does and why this tool can be useful.
   (b) (8 min.) In your own words, explain the idea and structure of an induction proof.
   (c) (10 min.) **Reflection:**
   - How did you use the feedback from the teaching assistants on the previous assignment in this weeks' helpful homework?
   - What questions do you have for a teaching assistant when you go and discuss your work with them?
   - What was the hardest question for you to answer and what will you do to improve your skills in answering that type of question on an exam?

Question 6: 26 min.

7. (30 min.) **Combining the work**

   Having each done half of the homework, you should now briefly discuss the work you did. For each question pick at least one subquestion each to discuss with your partner. If there are other answers you are not sure about, discuss those too. Make changes, and update the answers.

   Now scan your answers and submit them on Brightspace as a group. Next week you can then book a time slot with a TA to get your feedback!