Reasoning and Logic, Tantalizing TA-check 4

Deadline: the 1st of October 23:59

Introduction

This assignment is about modules 1 through 4. Every question has an indication for how long the question should take you in an exam-like setting. In total this set of exercises should take you about 234 minutes.

Questions of Tantalizing TA-check 4 (Monday 29th August, 2022, 15:54)

0. (5 min.) **Splitting the work**

   This TA-check should be done in pairs. However, as those of you using the skill circuits probably noticed, we do not recommend doing all of the TA-check in one go. Instead we recommend you do specific questions from this TA-check after studying specific concepts. To make sure you can both work through the material at your own pace, we recommend you first divide the work between the two of you.

   We recommend you both do the first subquestion of each question and then split the remaining subquestions (one taking the odd ones, the other the even ones) and do those individually. Since the difficulty increases, make sure you alternate and do not divide first half vs second half! Then in question 6 you should discuss your answers and merge them into one set of answers to request feedback on.

1. **Methods of proof**

   For each of the following claims, determine whether it holds or not. If it holds, prove this. If it does not, give a counterexample to show this. First rewrite the statements if the structure is not sufficiently explicit.

   (a) (5 min.) **Claim.** For all integers a and b: if $a \mid b$, then $a \mid (3b^3 - 15b^2 + 6b)$.

   (b) (5 min.) Formulate the converse of the previous subquestion. Does it hold? Give a proof or a counterexample.

   (c) (15 min.) **Challenging claim.** $\sqrt{6}$ is irrational.
   Hint: In your proof, you may use (without proof) that $6 \nmid n \rightarrow (2 \nmid n \vee 3 \nmid n)$.

   Question 1: 25 min.

2. **Recursion and induction**

   (a) (20 min.) The Fibonacci numbers are named after their discoverer, the medieval mathematician Leonardo Bonacci, who investigated the way a population of rabbits would develop under certain assumptions (among others, immortality).[1] A pair of rabbits (a male and a female) is put in a green meadow with a sufficient supply of roses and bonbons. From the moment that a couple is one month old, they start mating monthly. A month after mating, a new pair of rabbits is born, always a male and a female, which also start mating after their first month. At the end of the first month, the first couple mates. At the end of the second month, the new pair that the first couple have conceived is born, and the first couple mates again. At the end of the third month, another pair is born and the second couple starts mating too. At the end of the fourth month, two new pairs are born, one from the first couple and one from the second. The number of pairs of rabbits at the end of each month forms the following sequence of numbers:

   $$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \ldots$$

   The pattern should be clear: if we index the sequence from 1, then we can define it as follows:

   $$\begin{aligned} F_n &= 1 && \text{for } n = 1, 2 \\ F_n &= F_{n-1} + F_{n-2} && \text{for } n \geqslant 3 \end{aligned}$$

[1] This sequence of numbers had been discovered centuries before in India.

The Fibonacci numbers show a variety of interesting properties. For instance, there exists a Fibonacci Heap data structure. (A data structure is a structured way of storing data in memory—more complex data than primitves such as integer, float or boolean. You will learn about these things in the algorithms and data structures course in the second quarter.) Furthermore, the ratio between consecutive Fibonacci numbers approaches closer and closer the golden ratio φ, the solution of $\varphi = 1 + \frac{1}{\varphi}$, a ratio that is found in many places in nature and the arts and that is supposed to have pleasing aesthetic qualities.

If you extend the sequence of Fibonacci numbers for a while, you can see that every fourth Fibonacci number $(3, 21, 144, \dots)$ is divisible by 3. Prove the following theorem using mathematical induction:

Theorem. *For all* $n \geqslant 1$*:* $3 \mid F_{4n}$.

(b) (20 min.) Use mathematical induction to prove the following theorem:

Theorem. *For all integers* $n \geqslant 0$*:* $5 \mid 6^n - 1$.

(c) (30 min.) Prove the following theorem using mathematical induction:

Theorem. *For all integers* $n \geqslant 1$*:*

$$\left(\sum_{i=1}^{n} i\right)^2 = \sum_{i=1}^{n} i^3.$$

Hint: Keep in mind that $(a+b)^2 = a^2 + 2ab + b^2$.

(d) (15 min.) Suppose we have a sequence of numbers defined recursively as follows:

$$d_n = \begin{cases} 0 & \text{if } n = 1; \\ 2d_{n-1} + 2 & \text{if } n \geqslant 2. \end{cases}$$

Using mathematical induction, prove that for every integer $n \geqslant 1$, $d_n = 2^n - 2$.

Question 2: 85 min.

3. **Tree-time**

(a) (5 min.) Consider the following traversal method called `stefan-order` of a binary tree:

1. Do a `stefan-order` traversal of the right-child.
2. Do a `stefan-order` traversal of the left-child.
3. List the node's value.

Give a tree containing the values 1 through 10 (inclusive) of maximum height 3 such that the `stefan-order` traversal of your tree is the sorted list 1 through 10 (inclusive).

(b) For each of the following expressions give a parse tree:

i. (4 min.) $\frac{12+144+20+3\sqrt{4}}{7} + 5 \times 11$ (Add $= 9^2 + 0$ and you've got yourself a Limerick! `https://www.youtube.com/watch?v=wGt1gz-e5-w`)

ii. (5 min.) $(\neg r \vee \neg q) \rightarrow \neg(\neg p \wedge \neg q)$. Use your parse tree, to also rewrite this in prefix notation.

iii. (5 min.) The following is a bit of LaTeX that can also be expressed as a tree!

```
\begin{question}
    \begin{parts}
        \begin{part}
            \begin{itemize}
                \item
                \item
            \end{itemize}
            \begin{subparts}
                \begin{subpart}
                    \begin{solution}
                    \end{solution}
                \end{subpart}
                \begin{subpart}
                    \begin{solution}
                        \begin{itemize}
                            \item
```

```
                                                                \item
                                                            \end{itemize}
                                                        \end{solution}
                                                    \end{subpart}
                                                \end{subparts}
                                            \end{part}
                                            \begin{part}
                                                \begin{equation}
                                                \end{equation}
                                                \begin{solution}
                                                    \begin{equation}

                                                    \end{equation}
                                                \end{solution}
                                            \end{part}
                                    \end{parts}
                                \end{question}
```

4. **Repetition**

(a) (20 min.) Suppose we have the following predicates and functions with their corresponding meanings:

- $D(x, y)$: x is the defense attorney of case y;
- $P(x, y)$: x is the prosecutor of case y;
- $J(x, y)$: x judges case y;
- $W(x, y)$: x is a witness in case y;
- $d(x)$: returns the defendant of case x.
- $w(x)$: returns the star-witness of case x;
- $c(x)$: returns the last case person x was the star-witness in;

Give translations to first-order logic of the following sentences using only the predicates and functions given (and possibly the $=$-sign). You may introduce constants to denote proper names where necessary.

i. Phoenix is the defense attorney of the case Fey.
ii. Lotta is not a witness in case Skye.
iii. The Fey case and the Engarde case featured the same star witness.
iv. Nobody is the defense attorney for a case that (s)he prosecutes.
v. There is someone who is now the defendant of a case, after being the star-witness in another case.
vi. Godot has prosecuted exactly one case.
vii. People who have been a defendant in at least one case always end up as a defense attorney, prosecutor or judge in another case.

(b) (15 min.) Give the negation of each of the following formulae.

i. $\exists x \forall y (P(x, y) \leftrightarrow Q(x, y))$.
ii. $\forall y \exists x (P(x, y) \rightarrow \neg \exists x (Q(x, y) \vee (x = y)))$.
iii. $\exists x (R(x) \vee \forall y (P(x, y) \wedge (f(x, y) = y)))$.
iv. $\exists x \forall y (\neg P(x, y) \leftrightarrow \neg Q(x, y))$.
v. $\forall x \forall y \exists z (S(x, y, z) \rightarrow (R(z) \vee \neg P(z)))$.

(c) (20 min.) Suppose we have a structure $\mathcal{A}$, with the domain $D = \{a, b, c, d, e\}$ and predicates $P = \{a, d\}$, $Q = \{(a, b), (d, e), (a, c), (b, c)\}$ and $R = \{(a, d), (b, e), (c, a), (d, b), (e, c)\}$.

For each of the following formulas, indicate whether they are true in $\mathcal{A}$. If a formula is true, explain why, and if it is not, give a suitable counterexample.

i. $\forall x (P(x) \rightarrow \exists y (Q(x, y)))$.
ii. $\forall x \forall y (Q(x, y) \rightarrow \neg R(x, y))$.
iii. $\forall x (P(x) \rightarrow \forall y (Q(x, y) \vee \neg R(x, y)))$.

iv. $(\neg P(a) \lor \neg\exists x \forall y Q(x, y)) \leftrightarrow \exists x(Q(a, x) \land R(c, x))$.

v. $(P(d) \land \exists y R(y, d)) \to \forall x(R(x, c))$.

Question 4: 55 min.

5. **Essay questions**

   (a) (5 min.) In your own words, describe the difference between weak and strong induction. Also provide two examples of theorems, which can be proven by weak and by strong induction.

   (b) (10 min.) **Do this question together. Reflection:**

   - How did you use the feedback from the teaching assistants on the previous assignment in this weeks' helpful homework?
   - What questions do you have for a teaching assistant when you go and discuss your work with them?
   - What was the hardest question for you to answer and what will you do to improve your skills in answering that type of question on an exam?

Question 5: 15 min.

6. (30 min.) **Combining the work**

   Having each done half of the homework, you should now briefly discuss the work you did. For each question pick at least one subquestion each to discuss with your partner. If there are other answers you are not sure about, discuss those too. Make changes, and update the answers.

   Now scan your answers and submit them on Brightspace as a group. Next week you can then book a time slot with a TA to get your feedback!