Reasoning and Logic, Tantalizing TA-check 6

Deadline: the 22nd of October 23:59

Introduction

This assignment is about lectures 1 through 13, with an emphasis on lectures 11 to 13, which cover chapter 4 of the book. Every question has an indication for how long the question should take you in an exam-like setting. In total this set of exercises should take you about 155 minutes.

Questions of Helpful Homework 6 (Monday 29th August, 2022, 15:55)

0. (5 min.) **Splitting the work**

   This TA-check should be done in pairs. However, as those of you using the skill circuits probably noticed, we do not recommend doing all of the TA-check in one go. Instead we recommend you do specific questions from this TA-check after studying specific concepts. To make sure you can both work through the material at your own pace, we recommend you first divide the work between the two of you.

   We recommend you both do the first subquestion of each question and then split the remaining subquestions (one taking the odd ones, the other the even ones) and do those individually. Since the difficulty increases, make sure you alternate and do not divide first half vs second half! Then in question 3 you should discuss your answers and merge them into one set of answers to request feedback on.

1. **Relations and functions**

   (a) Suppose we have a function $g : \mathbb{R}^2 \to \mathbb{R}^2$, defined as:

   $$g(a,b) = \begin{cases} (\frac{-a}{b}, \frac{b}{-a}) & \text{if } a < 0. \\ (-b, b) & \text{if } a = 0; \\ (a\sqrt{b}, b\sqrt{a}) & \text{if } a > 0; \end{cases}$$

   i. (2 min.) The function g is a function, i.e., a set. What format do the elements of this set have?

   Solution: A function is a subset of the Cartesian product of its domain and its co-domain, so in this case we have $g : A^2 \to A^2$ and we get $g \subseteq (A \times A) \times (A \times A)$, which looks like $g = \{((a,b),(c,d)),((u,v),(x,y)),\ldots\}$.

   ii. (3 min.) Determine $g(2,5)$, $g(-5,2)$, $g(0,4)$ and $g(2,8)$, and explain your answers.

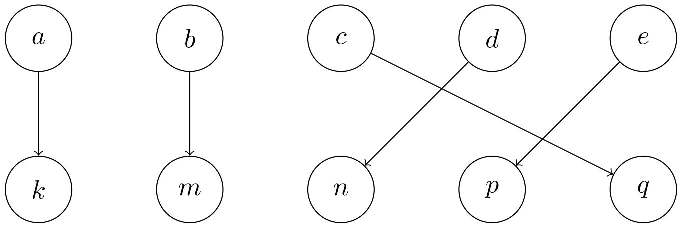
   Solution:

   $$\begin{aligned} g(2,5) &= (2\sqrt{5}, 5\sqrt{2}) \\ g(-5,2) &= (\frac{5}{2}, \frac{2}{5}) \\ g(0,4) &= (-4,4) \\ g(2,8) &= (2\sqrt{8}, 8\sqrt{2}) \end{aligned}$$

   (b) Suppose we have the sets $X = \{a,b,c,d,e\}$ and $Y = \{k,m,n,p,q\}$ and the function $h : X \to Y$ that is defined as $h(a) = k$, $h(b) = m$, $h(c) = q$, $h(d) = n$, $h(e) = p$.

   i. (2 min.) Draw a directed graph for h.

   Solution:



ii. (3 min.) Suppose we have the following sets: $A = \{k, p\}$, $C = \{a, e\}$, $D = \{n\}$, and $E = \{k, p, q\}$. Determine $h^{-1}(A)$, $h(X)$, $h(C)$, $h^{-1}(D)$, $h^{-1}(E)$, and $h^{-1}(Y)$.

Solution:

$$\begin{aligned}
h^{-1}(A) &= h^{-1}(\{k,p\}) &&= \{a,e\} \\
h(X) &= h(\{a,b,c,d,e\}) &&= \{k,m,n,p,q\} \\
h(C) &= h(\{a,e\}) &&= \{k,p\} \\
h^{-1}(D) &= h^{-1}(\{n\}) &&= \{d\} \\
h^{-1}(Y) &= h^{-1}(\{k,m,n,p,q\}) &&= \{a,b,c,d,e\}
\end{aligned}$$

(c) For each of the following functions, indicate whether it is injective and whether it is surjective. Motivate your answer.

i. (3 min.) $f_1 : \mathbb{Z} \to \mathbb{N}$, with $f_1(x) = 0$.

Solution: f_1 is not injective: $f_1(0) = f_1(1)$ yet $0 \neq 1$.
f_1 is not surjective: $1 \in \mathbb{N}$, yet there exists no $x \in \mathbb{Z}$ such that $f_1(x) = 1$.

ii. (3 min.) $f_2 : \mathbb{R} \to \mathbb{Z}$, with $f_2(x) = \lfloor x \rfloor$ [1].

Solution: f_2 is not injective: $f_2(1.1) = f_2(1.2)$, yet $1.1 \neq 1.2$.
f_2 is surjective: $\forall y \in \mathbb{Z}\ \exists x \in \mathbb{R} : f_2(x) = y$; take for instance $x = y$ in each case.

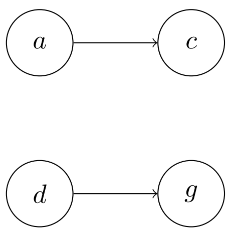
iii. (3 min.) $f_3 : \mathbb{N} \to \mathbb{R}$, with $f_3(x) = 2^x$.

Solution: f_3 is injective: every $n \in \mathbb{N}$ gives a different image.
f_3 is not surjective: there is no $n \in \mathbb{N}$ such that $f_3(n) = 2^n = -1$.

(d) Suppose we have the sets $A = \{a, b, c, d, e\}$ and $B = \{c, d, e, f, g\}$. For each of the following relations $R \subseteq A \times B$, draw a directed graph (see Epp, p. 446). Also indicate whether R is a function, and explain why each time.

i. (2 min.) $R = \{(a, c), (d, g)\}$

Solution:

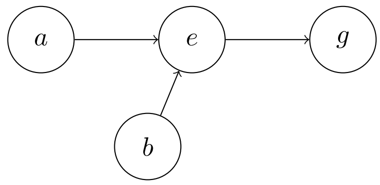


R is not a function. Not every element in the set A is mapped to an element in the set B.

ii. (3 min.) $R = \{(a, e), (b, e), (e, g)\}$

[1] $\lfloor x \rfloor$ is x rounded down to the closest integer

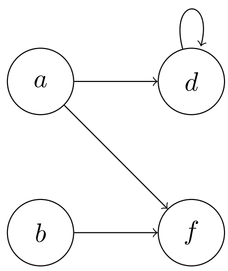
Solution:



R is not a function. Not every element in the set A is mapped to an element in the set B.

iii. (3 min.) $R = \{(a, d), (a, f), (b, f), (d, d)\}$

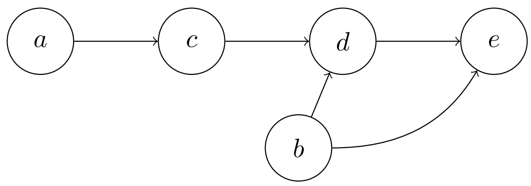
Solution:



R is not a function. Not every element in the set A is mapped to an element in the set B. Furthermore, a is mapped to both d and f.

iv. (3 min.) $R = \{(a, c), (c, d), (d, e), (b, d), (b, e)\}$

Solution:



R is not a function. Not every element in the set A is mapped to an element in the set B. Furthermore, b is mapped to both d and e.

(e) Suppose we have the set $A = \{0, 1, 2, 3, 4\}$.

i. (2 min.) Explain when a 'relation R on A' is (I) transitive, (II) reflexive and (III) symmetric. Provide examples of elements that need to be in the different relations to show the relations have these properties.

Solution: A binary (!) relation $R \subseteq A \times A$ is

- reflexive if for all $x \in A$: xRx.

  For instance $R : (=) \subseteq \mathbb{R} \times \mathbb{R}$, where x can be taken as $x = \pi$.

- symmetric if for all $x, y \in A$: $xRy \rightarrow yRx$.

  Here the same example can be used as above, $x = y = \pi$.

- transitive if for all $x, y, z \in A$: $(xRy \wedge yRz) \rightarrow xRz$.

  For instance $R : (<) \subseteq \mathbb{Z} \times \mathbb{Z}$, where for instance x, y, z can be taken as $x = 1$, $y = 2$ and $z = 3$.

ii. (3 min.) How do elements of an equivalence class for a relation R relate to each other—in terms of R?

Solution: For all elements $x, y \in [\mathrm{a}]$ for an element a, $xR\mathrm{a}$ and $yR\mathrm{a}$ hold and therefore also xRy.

iii. (15 min.) Give four relations R_1 to R_4 on A. The first three R_1 to R_3 must each have exactly one of the properties mentioned at e.i, and the last R_4 must have all three of the properties. Give the relations in set-roster notation. For R_4, also give all equivalence classes.

Solution: You might take the following relations:

$R_1 = \{(0,1),(1,2),(2,3),(0,2),(0,3),(1,3)\}$	Transitive
$R_2 = \{(0,0),(1,1),(2,2),(3,3),(4,4)\}$	Reflexive
$R_3 = \{(0,1),(1,0)\}$	Symmetric
$R_4 = \{(0,1),(1,0),(0,0),(1,1),(2,2),(3,3),(4,4)\}$	All three

The equivalence classes for R_4 are:

$$[0] = \{0,1\}$$
$$[1] = \{0,1\}$$
$$[2] = \{2\}$$
$$[3] = \{3\}$$
$$[4] = \{4\}.$$

(f) A relation R on A is symmetric iff for all $x, y \in A$: $xRy \to yRx$. A relation R on A is 'anti-symmetric' iff for all $x, y \in A$: $(xRy \land yRx) \to (x = y)$.

i. (10 min.) For each of the following two claims, give a proof or a counterexample.

I. If a relation R on a set A is transitive, then R is cannot be anti-symmetric.

II. If a relation R on a set A is reflexive, then R is symmetric.

Solution: As we can see above, neither claim holds.

(I) Take $R = \{(0,1),(1,2),(0,2)\}$. Now R is transitive, but also anti-symmetric.

(II) Take $R = \{(0,1),(0,0),(1,1)\}$. Now R is reflexive, but not symmetric.

(g) Transitivity

i. (2 min.) Give a systematic method that, given a binary relation R on a set A, constructs a graph G_R representing the relation.

Solution:

1. Make sets $V = A$ and $E = \varnothing$ for the vertices and edges of the graph.
2. For each element $(a,b) \in R$, add edge (a,b) to E.

ii. (5 min.) We will prove the following theorem: A relation R on a set A is transitive iff (If a vertex b can be reached in G_R from a, then (a,b) is also an edge in G_R).
First, show the structure of the proof as much as you can, based on the structural elements of the theorem—that is, without using the definitions of the terms used.

Solution: The theorem consists of an iff, so we prove it by proving two implications:

1. If the relation R is transitive, then (If it is possible to reach vertex b from vertex a in G_R, then (a,b) is an edge in G_R) holds.
2. (If it is possible to reach vertex b from vertex a in G_R, then (a,b) is an edge in G_R) holds, then the relation R is transitive.

Of course we prove these implications in both cases by first assuming the antecedent and then deriving the consequent. For the first implication, we then have to assume another antecedent and derive another consequent.

iii. (20 min.) Now, fill in the details of your proof, i.e., give the complete proof.

Solution:

Proof. So first we prove that if the relation R is transitive, then (If it is possible to reach vertex b from vertex a in G_R, then (a, b) is an edge in G_R) holds. To do so, suppose R is transitive. We now need to prove the consequent, that is, that the implication (If it is possible to reach vertex b from vertex a in G_R, then (a, b) is an edge in G_R) holds, so suppose there are vertices a and b such that b can be reached from a. This means that there must be a sequence of edges (a path) $(a, v_1), (v_1, v_2), \ldots, (v_n, b)$ that leads from a to b. We have assumed that R is transitive, so we can apply the definition of transitivity n times, for instance $(aRv_1 \wedge v_1Rv_2) \rightarrow aRv_2$, followed by $(aRv_2 \wedge v_2Rv_3) \rightarrow aRv_3$, and so on until we reach the desired result: $(aRv_n \wedge v_nRb) \rightarrow aRb$.
Now we prove the second implication: if (If it is possible to reach vertex b from vertex a in G_R, then (a, b) is an edge in G_R) holds, then the relation R is transitive. Suppose, then, that (If it is possible to reach vertex b from vertex a in G_R, then (a, b) is an edge in G_R) holds. Now we need to derive that R is transitive, i.e., that $\forall x, y, z \in A((xRy \wedge yRz) \rightarrow xRz)$. This turns out to be an implication as well, albeit a universally quantified one, so first take three random elements $a, b, c \in A$. Now we assume that aRb and bRc, so we know that there must be edges (a, b) and (b, c) in G_R. These edges are both incident on b, which means vertex c must be reachable from vertex a. Now our first assumption says that there must also be an edge (a, c) in the graph. This means that aRc also holds, which is the consequent that we had to derive. QED

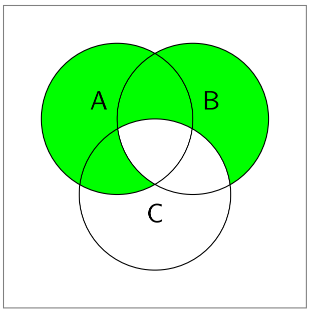
Question 1: 87 min.

2. **Repetition**

(a) *Old exam questions*

i. (3 min.) Give a Venn Diagram for the set $(A - B) \cup (B - C)$.

Solution:



ii. (5 min.) Claim: For all sets A if $|A|$ is odd and $\varnothing \notin A$ then there is **no** set B such that $\mathscr{P}(B) = A$. If the claim is true, prove it. If it is false, give a counterexample.

Solution:

Proof. Proof by contradiction: Assume there is a set A such that $|A|$ is odd, $\varnothing \notin A$ and there is also a set B such that $\mathscr{P}(B) = A$.

Since $\varnothing \subseteq B$, it must hold that $\varnothing \in \mathscr{P}(B)$. Thus $\varnothing \in A$, which contradicts the assumption that $\varnothing \notin A$. Thus by contradiction, it holds for all A. QED

Alternatively:

Proof. Proof by contrapositive: $\forall A(\exists B(\mathscr{P}(B) = A) \rightarrow (\exists k(|A| = 2k) \vee \varnothing \in A))$.
Take an arbitrary A such that there exists B with $\mathscr{P}(B) = A$.
Now consider the two exhaustive cases: $|B| = 0$ and $|B| \geqslant 1$.

- $|B| = 0$, this means that $B = \varnothing$, thus $A = \mathscr{P}(B) = \{\varnothing\}$. Thus $\varnothing \in A$, and therefore $\exists k(|A| = 2k) \vee \varnothing \in A$ is true.

- $|B| \geqslant 1$. We also know that $|A| = |\mathscr{P}(B)| = 2^{|B|}$. Since $|B| > 1$, this means $|A| = 2 \cdot 2^i$ for some $i \geqslant 0$. Thus $|A| = 2d$ for some d. Therefore $\exists k(|A| = 2k) \vee \varnothing \in A$ is true.

Since the claim holds in both cases and A was arbitrarily chosen, it holds for all A. QED

iii. (15 min.) Four young apprentices broke into a temple to steal four sacred element crystals. When the alarm went off, they panicked, and each of them swallowed the crystal they held right before they were caught. You must determine who ate which crystal. The elements compel their masters. Those who ate the earth and water crystals must speak the truth, while those who consumed fire and air must lie. The youths are too scared to confess their own transgressions. Instead, they fall to accusing each other. This is what they said:

- Coyote Caoti said something before you arrived (but you don't know what)
- Dr. Whoo says that Coyote Caoti is telling the truth about that statement
- Donna says that she didn't eat the earth crystal
- Marty says that if Coyote Caoti ate the air crystal, then Dr. Whoo ate the earth crystal

Remember that:

- Each student ate a different elemental crystal
- The earth and water crystals force their owner to tell the truth
- The fire and air crystals force their owners to lie

Who ate what crystal? Explain how you derived your answer.

Solution: Suppose Donna would be lying. Then, she ate the fire crystal or the air crystal, so she didn't eat the earth crystal. But then Donna is telling the truth, which is a contradiction. Therefore, Donna is telling the truth. Therefore, Donna did not eat the earth crystal, and also not the air or fire crystal, so Donna ate the water crystal.
Dr. Whoo says that Coyote Caoti is telling the truth. Thus, if Coyote Caoti is telling the truth, then Dr. Whoo is telling the truth, and if Coyote Caoti is lying, then Dr. Whoo is lying.
Since Donna was already telling the truth, an only 2 apprentices are telling the truth, Dr. Whoo and Coyote Caoti cannot both tell the truth, so they are both lying. Since only 2 apprentices are lying, Marty is telling the truth. Since the consequent of Martys statement is false, the antecedent of his statement must also be false. Thus, Coyote Caoti did not eat the air crystal, and since he lied, he ate the fire crystal. That leaves the air crystal for the lying Dr. Whoo and the earth crystal for Marty.

(b) (10 min.) **Do this question together. Reflection:**

- How did you use the feedback from the teaching assistants on the previous assignment in this weeks' helpful homework?
- What questions do you have for a teaching assistant when you go and discuss your work with them?
- What was the hardest question for you to answer and what will you do to improve your skills in answering that type of question on an exam?

Solution: It is generally good to ask yourself these questions when you are done with an(y) assignment. We will make it explicit in this course for a bit, but even if other courses do not, ask yourself these questions!

Question 2: 33 min.

3. (30 min.) **Combining the work**

Having each done half of the homework, you should now briefly discuss the work you did. For each question pick at least one subquestion each to discuss with your partner. If there are other answers you are not sure about, discuss those too. Make changes, and update the answers.

Now scan your answers and submit them on Brightspace as a group. Next week you can then book a time slot with a TA to get your feedback!