Reasoning and Logic, Tantalizing TA-check 5

Deadline: the 15th of October 23:59

Introduction

This assignment is about lectures 1 through 13, with an emphasis on lectures 11 to 13, which cover chapter 4 of the book. Every question has an indication for how long the question should take you in an exam-like setting. In total this set of exercises should take you about 214 minutes.

Questions of Helpful Homework 5 (Monday 29th August, 2022, 15:54)

0. (5 min.) **Splitting the work**

   This TA-check should be done in pairs. However, as those of you using the skill circuits probably noticed, we do not recommend doing all of the TA-check in one go. Instead we recommend you do specific questions from this TA-check after studying specific concepts. To make sure you can both work through the material at your own pace, we recommend you first divide the work between the two of you.

   We recommend you both do the first subquestion of each question and then split the remaining subquestions (one taking the odd ones, the other the even ones) and do those individually. Since the difficulty increases, make sure you alternate and do not divide first half vs second half! Then in question 4 you should discuss your answers and merge them into one set of answers to request feedback on.

1. **Set theory**

   (a) (15 min.) Suppose we have the sets $A = \{Gumshoe, Trucy, Ema\}$, $B = \{Ema, Klavier\}$ and $C = \{Klavier, Trucy, Apollo\}$ in a universe $U = (A \cup B) \cup C$. Write the following sets using set-roster notation:

   i. $A \cup B$

   Solution:

   $$\{Gumshoe, Trucy, Ema, Klavier\}$$

   ii. $A \cap (C - B)$

   Solution:

   $$\{Trucy\}$$

   iii. $B \times C$

   Solution:

   $$\{(Ema, Klavier), (Ema, Trucy), (Ema, Apollo), \ldots$$
   $$(Klavier, Klavier), (Klavier, Trucy), (Klavier, Apollo)\}$$

   iv. $\mathscr{P}((A \cap B) \cup (A \cap C))$

   Solution:

   $$\{\emptyset, \{Ema\}, \{Trucy\}, \{Ema, Trucy\}\}$$

   (b) For each of the following descriptions, draw a Venn diagram that correctly represents the relation(s) between the sets. So for the first question, ensure your Venn diagram contains two sets A and B and highlight the area that forms the set C. For instance, for Figure 1 the highlighted set is $A \cup B$.

   i. (3 min.) $C = A \cap B$ (Highlight C).

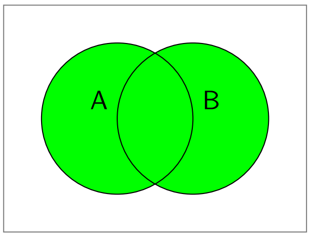
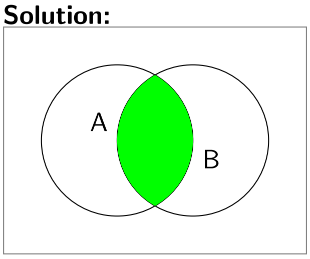


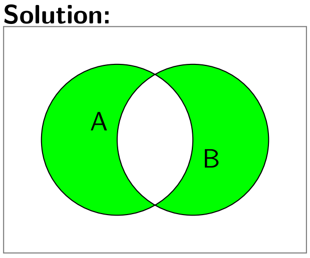
Figure 1: Example Venn diagram that represents $A \cup B$.

Solution:



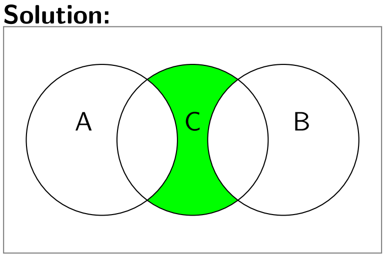
ii. (3 min.) $C = (A \cup B) - (A \cap B)$ (Highlight C).

Solution:



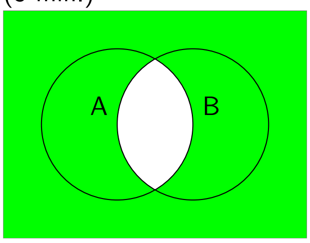
iii. (3 min.) $A \cap B = \varnothing$, $A \cap C \neq \varnothing$, $B \cap C \neq \varnothing$, $D = (C - A) - B$ (Highlight D).

Solution:



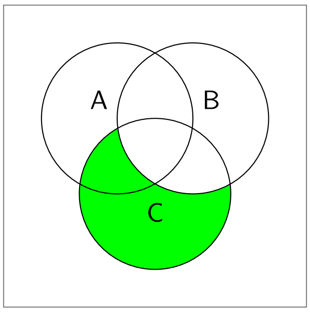
(c) For each of the following Venn diagrams denote how the highlighted set can be constructed from the other sets. For instance, for Figure 1 the highlighted set is $A \cup B$.

i. (3 min.)



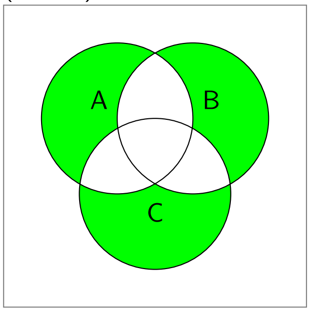
Solution: $(A \cap B)^c$

ii. (3 min.)



Solution: $C - B$

iii. (7 min.)



Solution: $(A - (B \cup C)) \cup (B - (A \cup C)) \cup (C - (B \cup A))$

(d) For each of the following claims about sets, give a proof or a suitable counterexample to show the claim holds or does not hold, respectively. The universe U is always the union of all the sets that are mentioned, A, B, C, ... as well as some elements that are not a part of any of these sets.

Hint: it can be useful sometimes to make a Venn diagram or an Euler diagram of the situations described.

i. (10 min.) **Claim.** For all sets A and B: $A^c \subseteq B \to B^c \subseteq A$

Solution: This holds.

Proof. We need to prove an implication, which means deriving the consequent after assuming the antecedent. Suppose, then, that $A^c \subseteq B$. Now we need to prove that $B^c \subseteq A$. This is another implication, namely that if some object is in B^c, then it is also in A. Take an arbitrary object $e \in B^c$.

We assumed that $A^c \subseteq B$. This means that if an object is in A^c, it is also in B. That is, if an object is not in A, then it is in B. The contrapositive of this implication also holds: if an object is not in B, then it is in A. Now we have an object $e \in B^c$, i.e., $e \notin B$, so by our assumption, $e \in A$, which is what we wanted. QED

ii. (5 min.) **Claim.** For all non-empty sets A, B, C:
$(A \cap B = \varnothing, A \cap C \neq \varnothing, B \cap C \neq \varnothing) \to \exists x(x \notin A \wedge x \notin B \wedge x \in C)$

Solution: This is clearly false. Take $A = \{1\}$, $B = \{2\}$, $C = A \cup B = \{1, 2\}$. Now all premises hold, yet there is no element in C that is not in A and not in B.

iii. (10 min.) **Claim.** For all sets A, B: $(A \cap B)^c = (A^c \cup B^c)$

Solution:

Proof. We need to prove two subset relations: $(A \cap B)^c \subseteq A^c \cup B^c$ and $(A^c \cup B^c) \subseteq (A \cap B)^c$.

$$
\begin{aligned}
x \in (A \cap B)^c & & \text{If } x \in (A \cap B)^c \text{ then } x \notin A \cap B\text{, Definition of compliment} \\
& \leftrightarrow x \notin (A \cap B) & \\
& \leftrightarrow (x \notin A) \vee (x \notin B) & \text{Definition of } \cap \\
& \leftrightarrow x \in A^c \vee x \in B^c & \\
& \leftrightarrow x \in (A^c \cup B^c) & \text{Definition of } \cup
\end{aligned}
$$

Read from bottom to top for the second subset relation. Therefore, both subset relations hold and thus the sets are equal. QED

iv. (10 min.) **Claim.** For all sets A and B: $\mathscr{P}(A) \cup \mathscr{P}(B) \subseteq \mathscr{P}(A \cup B)$

Solution: Intuition: $A \cup B$ gives all elements that occur in either A or B. The power sets of A and B will therefore be contained in the power set of $A \cup B$.

Proof. We need to prove that all elements in $\mathscr{P}(A) \cup \mathscr{P}(B)$ are also in $\mathscr{P}(A \cup B)$, so we take a random $x \in \mathscr{P}(A) \cup \mathscr{P}(B)$. Now we know that $x \subseteq A$ and/or $x \subseteq B$. Thus we use division into cases to complete our proof. For every case we will prove that $x \in \mathscr{P}(A \cup B)$, or in other words $x \subseteq A \cup B$.

Case 1: $x \subseteq A$
As we also know that $A \subseteq A \cup B$, this means that $x \subseteq A \subseteq A \cup B$, therefore $x \subseteq A \cup B$, thus $x \in \mathscr{P}(A \cup B)$.

Case 2: $x \subseteq B$
As we also know that $B \subseteq A \cup B$, this means that $x \subseteq B \subseteq A \cup B$, therefore $x \subseteq A \cup B$, thus $x \in \mathscr{P}(A \cup B)$.

Thus we have shown that $x \in \mathscr{P}(A \cup B)$ in both cases, therefore we have proven that $x \in \mathscr{P}(A) \cup \mathscr{P}(B)$ implies that $x \in \mathscr{P}(A \cup B)$. As x was randomly chosen, this means it holds for all $e \in \mathscr{P}(A) \cup \mathscr{P}(B)$. From this we can conclude that $\mathscr{P}(A) \cup \mathscr{P}(B) \subseteq \mathscr{P}(A \cup B)$ holds. QED

v. (10 min.) **Claim.** For all sets A, B, and C: $(A - B) \cup (B - C) = (A \cup B) \cap (A \cup C^c) \cap (B \cap C)^c$
Hint: this is the kind of claim that, if true, may be easiest to prove using an algebraic proof.

Solution:

Proof. Take a random element $x \in (A - B) \cup (B - C)$. We now need to prove that $x \in (A \cup B) \cap (A \cup C^c) \cap (B \cap C)^c$ holds as well, and the other way around. We do so using an algebraic proof.

$$
\begin{aligned}
x \in (A - B) \cup (B - C) &\Leftrightarrow x \in (A - B) \vee x \in (B - C) \\
&\Leftrightarrow (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin C) \\
&\Leftrightarrow (x \in A \vee (x \in B \wedge x \notin C)) \wedge (x \notin B \vee (x \in B \wedge x \notin C)) \\
&\Leftrightarrow (x \in A \vee x \in B) \wedge (x \in A \vee x \notin C) \wedge (x \notin B \vee x \in B) \wedge (x \notin B \vee x \notin C) \\
&\Leftrightarrow x \in (A \cup B) \wedge x \in (A \cup C^c) \wedge x \in (B^c \cup B) \wedge x \in (B^c \cup C^c) \\
&\Leftrightarrow x \in (A \cup B) \cap (A \cup C^c) \cap (B^c \cup C^c) \\
&\Leftrightarrow x \in (A \cup B) \cap (A \cup C^c) \cap (B \cap C)^c
\end{aligned}
$$

Since x was randomly chosen, we have that for all $x \in (A - B) \cup (B - C)$: $x \in (A \cup B) \cap (A \cup C^c) \cap (B \cap C)^c$, and since our rewriting works both ways, we have also proven that for all $y \in (A \cup B) \cap (A \cup C^c) \cap (B \cap C)^c$: $y \in (A - B) \cup (B - C)$. Thus the claim is proven. QED

vi. (10 min.) **Claim.** For all sets A, B and C: if $A \subseteq C$ *and* $B^c \subseteq C$, then $A \cap B = \varnothing$.

> **Solution:** This does not hold: take as our universe $U = \{0,1,2,3,4,5\}$, with sets $A = \{0\}$, $B = \{0,1,2,3,4\}$ and $C = \{0,5\}$. Now $A \subseteq C$. Additionally $B^c = \{5\} \subseteq C$, but $A \cap B = \{0\} \neq \varnothing$.

(e) This is an **optional** question about power sets and the empty set $\varnothing$.

i. (5 min.) Give the following sets using set-roster notation: $\mathscr{P}(\varnothing)$, $\mathscr{P}(\mathscr{P}(\varnothing))$, and $\mathscr{P}(\mathscr{P}(\mathscr{P}(\varnothing)))$.

> **Solution:** $\mathscr{P}(\varnothing) = \{\varnothing\}$, $\mathscr{P}(\mathscr{P}(\varnothing)) = \{\varnothing, \{\varnothing\}\}$, and
>
> $$\mathscr{P}(\mathscr{P}(\mathscr{P}(\varnothing))) = \{\varnothing, \{\varnothing\}, \{\{\varnothing\}\}, \{\varnothing, \{\varnothing\}\}\}.$$

ii. (5 min.) Formulate (precisely) a conjecture about the number of elements in such a set that starts with n times '$\mathscr{P}($', as a function of n.

> **Solution:** The number of elements in $\mathscr{P}^n(\varnothing)$ is $\exp_2^n(0)$ for $n \geqslant 1$, with $\mathscr{P}^b$ used to denote b applications of $\mathscr{P}(\cdot)$ and
>
> $$\exp_a^n(x) = a^{a^{\cdot^{\cdot^{\cdot^{a^x}}}}}$$
>
> with the number of as equal to n (see also Wikipedia).

iii. (10 min.) Prove your conjecture using mathematical induction on n.
Tip: Review the proof of theorem 6.3.1, which says that every set with n elements has 2^n subsets.

> **Solution:**
>
> *Proof (by mathematical induction).* We use the predicate $P(n)$ which holds when $|\mathscr{P}^n(\varnothing)| = \exp_2^n(0)$, given the definitions of as $\mathscr{P}^n(\cdot)$ and $\exp_a^n(x)$ above.
>
> **Basis (n=1)** If $n = 1$, we get $\mathscr{P}(\varnothing) = \{\varnothing\}$, with cardinality $1 = 2^0 = \exp_2^1(0)$.
>
> **Inductive step:** Suppose $P(k)$ holds for some random number $k \geqslant 1$, so $|\mathscr{P}^k(\varnothing)| = \exp_2^k(0)$ (inductive hypothesis, IH). We have $\mathscr{P}^{k+1}(\varnothing) = \mathscr{P}(\mathscr{P}^k(\varnothing))$. Since $|\mathscr{P}(A)| = 2^{|A|}$ and according to the IH, $|\mathscr{P}^k(\varnothing)| = \exp_2^k(0)$, we do get $|\mathscr{P}(\mathscr{P}^k(\varnothing))| = 2^{\exp_2^k(0)} = \exp_2^{k+1}(0)$, as desired. Since k is a random number, this holds for all $k \geqslant 1$.
>
> According to the principle of induction, we may conclude the claim holds for all $k \geqslant 1$. QED

Question 1: 112 min.

2. **Recursion and induction**

(a) (7 min.) Give a recursive definition for the set B of arbitrarily nested, properly matched brackets. For example, $[](\{\}) \in B$, but $[\{ \notin B$. (The brackets available are: $\{\ \}$, $[\ ]$, $(\ )$ and $\langle\ \rangle$.)

> **Solution:**
>
> **BASE** : $(\ ) \in B$, $[\ ] \in B$, $\{\ \} \in B$, $\langle\ \rangle \in B$.
>
> **RECURSION** : If $b \in B$, then $(b) \in B$, $[b] \in B$, $\{b\} \in B$ and $\langle b \rangle \in B$
> If $a, b \in B$, then $ab \in B$ (not a strike if they leave it out)
>
> **RESTRICTION** : Nothing is in B except what can be derived using the above rules.

(b) For the following two algorithms, prove the correctness of the algorithm using induction [1].

[1] https://www.youtube.com/watch?v=GSvqF48TVM4

i. (10 min.)

```
m = 0
n = 1
while (m <= i)
|    m := m + 4
|    n := n + 2
end while
return m + n
```

Claim : The code always returns an odd number.
Loop invariant: `m+n` is odd.

Solution:

Basis property : We have to check whether the invariant holds before the loop. Before the loop: $m = 0$, $n = 1$ so $m + n = 0 + 1 = 1$, which is odd.

Inductive property : Assume $m + n$ is odd before the loop. $m_{\text{new}} = m_{\text{old}} + 4$ and $n_{\text{new}} = n_{\text{old}} + 2$ so $m_{\text{new}} + n_{\text{new}} = m_{\text{old}} + n_{\text{old}} + 6$. We know that $m_{\text{old}} + n_{\text{old}}$ results in an odd integer, so $m_{\text{old}} + n_{\text{old}} = 2k + 1$ for some integer k. $2k + 1 + 6 = 2(k + 3) + 1 = 2c + 1$ for $c = k + 3$, so $m_{\text{new}} + n_{\text{new}}$ is odd as well.

Eventual falsity of guard : The guard is true while $m \leqslant i$, for some integer i. Since m gets incremented by 4 in every iteration and i stays constant, m will at some point be greater than i. This will terminate the loop.

Correctness : The basis property shows that $m + n$ is odd even without going into the loop, while the inductive property shows that $m + n$ will be odd after every loop. Combining these two together will show that the result from the code will always be odd. Thefore this code always returns an odd number.

ii. (10 min.)

```
i = 0
n = 1
while (i < x)
|    i := i + 1
|    n := n * i
end while
return n
```

Claim: The code computes $x!$
Loop invariant: $n = i!$

Solution:

Basis property : We have to check whether the invariant holds before the loop. Before the loop: $i = 0$ and $n = 1$ so $0! = 1 = n$.

Inductive property : Assume that $n_{\text{old}} = i_{\text{old}}!$ holds before the loop. $i_{\text{new}} = i_{\text{old}} + 1$ and $n_{\text{new}} = n_{\text{old}} * i_{\text{new}} = n_{\text{old}} * (i_{\text{old}} + 1) = i_{\text{old}}! * (i_{\text{old}} + 1) = (i_{\text{old}} + 1)! = i_{\text{new}}!$

Eventual falsity of guard : The guard is true while $i \leqslant x$, where x is some integer. Since i gets incremented by 1 in every iteration, it will at some point be equal to the constant x. This will terminate the loop.

Correctness : After the loop is done $i = x$, so from the invariant can derive that $n = x!$. Therefore this code computes $n = x!$.

(c) (10 min.) Suppose we have the following recursively defined set C:
BASE : $3 \in C$.
RECURSION : If $c \in C$, then $3c \in C$ and $-3c \in C$.

RESTRICTION : Nothing is in C except what can be derived using the above rules.

Use structural induction to prove that every integer in C is odd.

Solution:

Proof. We use the property $P(n)$ to denote that an n generated using the definition above is odd. We now use structural induction to prove that $P(n)$ holds for all possible $n \in C$.

Basis: In this case, we have to prove that 3 is an odd number. This is clearly the case, since it can be written as $2k+1$, with $k=1$.

Inductive step: We have to prove that if $x \in C$ is odd, then also $3x$ and $-3x$ are odd. Suppose, then, that we have an arbitrary element $p \in C$ that is odd. Now we need to show that $3p$ and $-3p$ are also odd.

We know p can be written as $2k+1$ for some arbitrary integer k. Filling this in, we get $3p = 3(2k+1) = 2 \cdot 3k + 3 = 2 \cdot 3k + 2 + 1 = 2(3k+1) + 1 = 2m+1$ for an integer $m = 3k+1$. This means $3p$ is also odd. The case for $-3p$ goes analogously.

Since p and k are arbitrary numbers, we can now conclude that $P(n)$ holds for any n constructed using the second rule of the recursive definition above.

According to the principle of induction, $P(n)$ now holds for all $n \in C$. QED

Question 2: 37 min.

3. **Repetition**

(a) (10 min.) Use mathematical induction to prove that for all integers $n \geqslant 1$: $133 \mid (11^{n+1} + 12^{2n-1})$.

Solution:

Proof. We use the predicate $P(n)$, which holds when $133 \mid (11^{n+1} + 12^{2n-1})$.

Basis ($n=1$): We need to prove that $P(1)$ holds, so $133 \mid (11^{1+1} + 12^{2 \cdot 1-1})$. We get:

$$11^{n+1} + 12^{2n-1} = 11^{1+1} + 12^{2 \cdot 1 - 1} = 11^2 + 12^1 = 121 + 12 = 133,$$

which is clearly divisible by 133.

Inductive step: Now suppose that for an arbitrary integer $k \geqslant 1$, $P(k)$ holds, so that $133 \mid (11^{k+1} + 12^{2k-1})$. We now need to prove that $P(k+1)$ holds too, so that $133 \mid (11^{k+1+1} + 12^{2(k+1)-1})$. Working this out gives:

$$\begin{aligned}
11^{k+1+1} + 12^{2(k+1)-1} &= 11^{k+2} + 12^{2k+2-1} \\
&= 11 \cdot 11^{k+1} + 12^2 \cdot 12^{2k-1} \\
&= 11(11^{k+1} + 12^{2k-1}) + 133 \cdot 12^{2k-1} \\
&= 11 \cdot 133m + 133 \cdot 12^{2k-1} && \text{by the IH, for some integer } m \\
&= 133(11m + 12^{2k-1}) \\
&= 133\ell,
\end{aligned}$$

for some integer $\ell = 11m + 12^{2k-1}$ (which is an integer since $k \geqslant 1$). So indeed, if $P(k)$ then also $P(k+1)$. Since k was arbitrary, $P(k) \to P(k+1)$ now holds for all $k \geqslant 1$.

According to the principle of induction, we may now conclude that $P(n)$ holds for all $n \geqslant 1$. QED

(b) (10 min.) Give a proof by contradiction that for any two positive integers m and n, $m + n \geqslant 2\sqrt{mn}$.

Solution:

Proof. We will prove the claim by contradiction, so assume the claim does not hold, i.e., there are integers x and y such that $x + y < 2\sqrt{xy}$. Squaring on both sides gives $x^2 + y^2 + 2xy < 4xy$, so $x^2 + y^2 < 2xy$. From this we can derive $x^2 + y^2 - 2xy < 0$, which would mean $(x - y)^2 < 0$. This is a contradiction, because x and y are positive integers so the difference can never result in a negative square. This tells us our assumption that $x + y < 2\sqrt{xy}$ must be false. Therefore, we conclude that $m + n > 2\sqrt{mn}$ for any m and n. QED

(c) (10 min.) **Do this question together. Reflection:**

- How did you use the feedback from the teaching assistants on the previous assignment in this weeks' helpful homework?
- What questions do you have for a teaching assistant when you go and discuss your work with them?
- What was the hardest question for you to answer and what will you do to improve your skills in answering that type of question on an exam?

Solution: It is generally good to ask yourself these questions when you are done with an(y) assignment. We will make it explicit in this course for a bit, but even if other courses do not, ask yourself these questions!

Question 3: 30 min.

4. (30 min.) **Combining the work**

Having each done half of the homework, you should now briefly discuss the work you did. For each question pick at least one subquestion each to discuss with your partner. If there are other answers you are not sure about, discuss those too. Make changes, and update the answers.

Now scan your answers and submit them on Brightspace as a group. Next week you can then book a time slot with a TA to get your feedback!