Exercise 1.13

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Exercise 1.13: Prove that Fib(n) is the closest integer to $\varphi^n/\sqrt{5}$, where $\varphi = (1+\sqrt{5})/2$. Hint: Let $\psi = (1-\sqrt{5})/2$. Use induction and the definition of the Fibonacci numbers (see Section 1.2.2) to prove that Fib(n) $= (\varphi^n - \psi^n)/\sqrt{5}$.

When $n = 0$:

$$\text{Fib}(0) = \frac{(\varphi^0 - \psi^0)}{\sqrt{5}} = \frac{1-1}{\sqrt{5}} = 0$$

When $n = 1$:

$$\text{Fib}(1) = \frac{(\varphi^1 - \psi^1)}{\sqrt{5}} = \frac{[(1+\sqrt{5}) - (1-\sqrt{5})]}{2\sqrt{5}} = \frac{2\sqrt{5}}{2\sqrt{5}} = 1$$

Assume $\text{Fib}(n) = (\varphi^n - \psi^n)/\sqrt{5}$ for $n \geq 2$

Then,

$$\text{Fib}(n+1) = (\varphi^{n+1} - \psi^{n+1})/\sqrt{5}$$

$$\text{Fib}(n+1) = \text{Fib}(n) + \text{Fib}(n-1)$$

$$= \frac{(\varphi^n - \psi^n)}{\sqrt{5}} + \frac{(\varphi^{n-1} - \psi^{n-1})}{\sqrt{5}}$$

$$= \frac{\varphi^n + \varphi^{n-1}}{\sqrt{5}} - \frac{\psi^n + \psi^{n-1}}{\sqrt{5}}$$

$$\frac{\quad}{\sqrt{5}} \qquad \frac{\quad}{\sqrt{5}}$$

$$= \frac{\varphi^{n-1}(\varphi + 1)}{\sqrt{5}} - \frac{\psi^{n-1}(\psi + 1)}{\sqrt{5}}$$

$$= \varphi^{n-1}\left(\frac{3+\sqrt{5}}{2\sqrt{5}}\right) - \psi^{n-1}\left(\frac{3-\sqrt{5}}{2\sqrt{5}}\right)$$

$$= \varphi^{n-1} \cdot \varphi^{2}/\sqrt{5} - \psi^{n-1} \cdot \psi^{2}/\sqrt{5}$$

$$= \left(\varphi^{n+1} - \psi^{n+1}\right)/\sqrt{5}$$

Thus, $\text{Fib}(n) = (\varphi^{n} - \psi^{n})/\sqrt{5}$

To prove the initial statement:

$$\text{Fib}(n) - \frac{1}{2} < \frac{\varphi^{n}}{\sqrt{5}} < \text{Fib}(n) + \frac{1}{2}$$

$$\varphi^{n} - \psi^{n} - \frac{\sqrt{5}}{2} < \varphi^{n} < \varphi^{n} - \psi^{n} + \frac{\sqrt{5}}{2}$$

Which is true as long as $-\frac{\sqrt{5}}{2} < \psi^{n} < \frac{\sqrt{5}}{2}$

But $-1 < \psi < 0$, so

$-\frac{\sqrt{5}}{2} < -1 < \psi^{n} < 1 < \frac{\sqrt{5}}{2}$ QED

$- \ \overline{2} \qquad 1 \qquad 1 \qquad 1 \qquad 2$