Exercise 2

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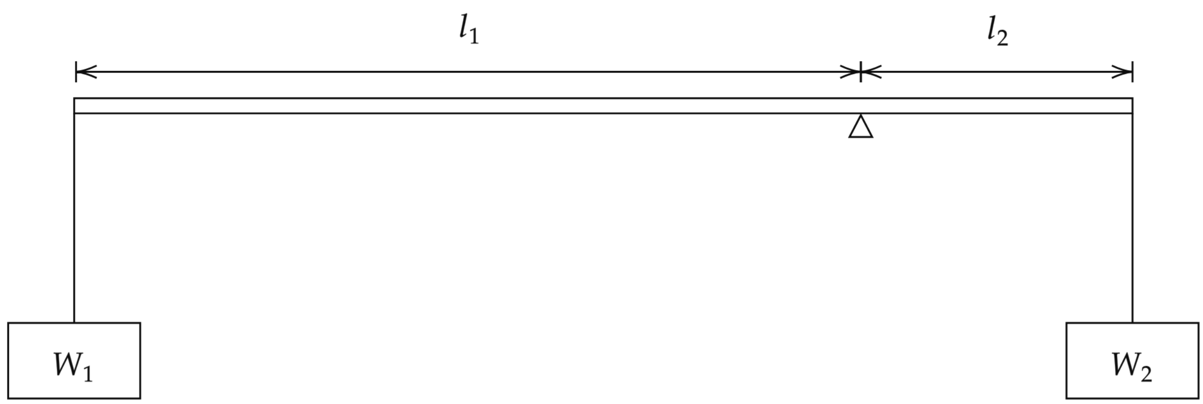
All the figures in this document is redrawn using Mathcha or Inkscape.

2.1

Use the principle of virtual work to establish the formula for an unequal- arm balance, as shown in Fig. 2-1, $W_1 l_1 = W_2 l_2$. (Neglect the weight of the cross-beam.)



Ans:
Let's say, W_1 moves down d unit, then W_2 will be lifted up $\frac{d}{l_1} l_2$.
By the conservation of energy, $W_1 d = W_2 d \frac{l_2}{l_1}$, that is, $W_1 l_1 = W_2 l_2$.

2.2

Extend the formula obtained in Ex. 2.1 to include a number of weights hung at various distances from the pivot point,

$$\sum_i W_i l_i = 0$$

(Distances on one side of the fulcrum are considered positive and on the other side, negative.)

Ans:
When one of the W_i, say W_1 is moved down in a distance d, then W_i moves down in distance $\frac{d}{l_1} l_i$ for all W_i on the same side as W_1. Similarly, W_i is lifted up in distance $\frac{d}{l_1} l_2$ for all W_i on the opposite side of W_1. By the conservation of energy,
$W_1 d + \sum_{i=2}^{k} W_i \frac{d}{l_1} l_i = \sum_{k+1}^{n} W_i \frac{d}{l_1}(-l_i)$, and thus, $\sum_i W_i l_i = 0$.

2.3

A body is acted upon by n forces and is in static equilibrium. Use the principle of virtual work to prove that:

1. **If $n = 1$, the magnitude of the force must be zero. (A trivial case.)**
2. **f $n = 2$, the two forces must be equal in magnitude, opposite in direction, and collinear.**
3. **If $n = 3$, the forces must be coplanar and their lines of action must pass through a single point.**
4. **For any n, the sum of the products of the magnitude of a force F_i times the cosine of the angle Δ_i between the force and any fixed line, is zero:**

$$\sum_{i=1}^{n} F_i \cos \Delta_i = 0$$

Ans:

1. Assume that the object moves toward the same direction as F_1 in a non-zero distance d, then, by the conservation of energy, $F_1 d = 0$. This implies that $F_1 = 0$.
2. Assume that there are two forces $\vec{F}_1$ and $\vec{F}_2$. If the object moves $\vec{r}$, by the principle of virtual work, $\vec{F}_1 \cdot \vec{r} + \vec{F}_2 \cdot \vec{r} = 0$. This equation could be developed to $\vec{F}_1 |\vec{r}|^2 + \vec{F}_2 |\vec{r}|^2 = 0$, and hence, $\vec{F}_1 = -\vec{F}_2$.
3. Suppose that there are three forces $\vec{F}_1$, $\vec{F}_2$ and $\vec{F}_3$ acting on the object. If the object moves $\vec{r}$, by the principle of virtual work, $\vec{F}_1 \cdot |\vec{r}|^2 + \vec{F}_2 \cdot |\vec{r}|^2 + \vec{F}_3 \cdot |\vec{r}|^2 = 0$, that is $\vec{F}_1 + \vec{F}_2 + \vec{F}_3 = 0$. This implies that $\vec{F}_1$, $\vec{F}_2$ and $\vec{F}_3$ are collinear or form a triangle, and therefore, the forces are coplanar. By problem 2.12, the net torque about every arbitrary points would be the same. If their lines of action don't pass through a single point, without lost of generality, say, F_1 don't pass through the intersection of the acting lines of F_2 and F_3, then set the fulcrum to be at this intersection, the torque $\tau = \vec{F}_1 \times \vec{r}_1 \neq 0$ where $\vec{r}_1 \neq \vec{0}$ is the vector pointing from the intersection of F_2 and F_3 to the acting point of F_1. This contradict the hypothesis that the body is in static equilibrium, and hence, their lines of action should pass through a single point.
4. Let the object move $\vec{r}$, by the principle of virtual work,

$$\sum_{i=1}^{n} \vec{F}_i \cdot \vec{r} = 0$$

   That is,

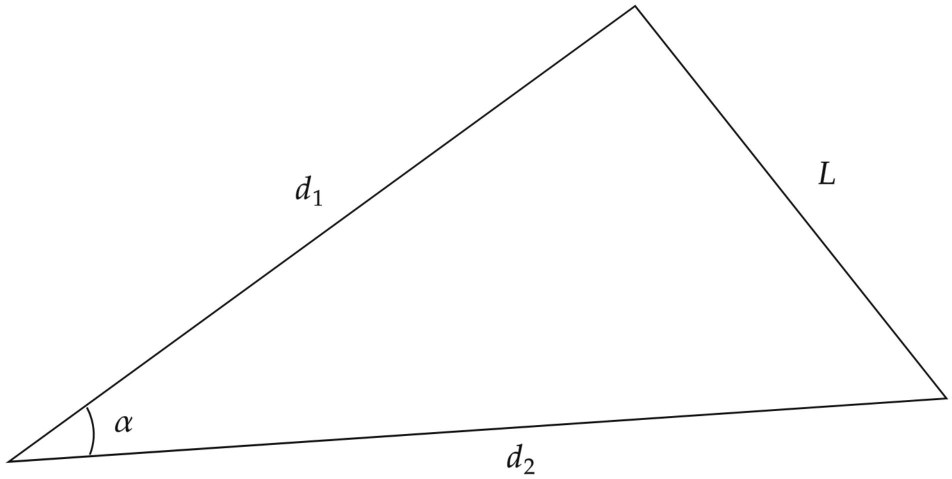
$$\sum_{i=1}^{n} |\vec{F}_i||\vec{r}| \cos \Delta_i = 0$$

   Hence,

$$\sum_{i=1}^{n} F_i \cos \Delta_i = 0$$

2.4

Problems involving static equilibrium in the absence of friction may be reduced, using the *Principle of Virtual Work*, to problems of mere geometry: Where does one point move when another moves a given small distance? In many cases this question is easily answered if the following properties of a triangle are used (referring to Figure below):



1. **1. If the sides d_1 and d_2 remain fixed in length, but the angle α changes by a small amount $\Delta\alpha$, the opposite side L changes by an amount**

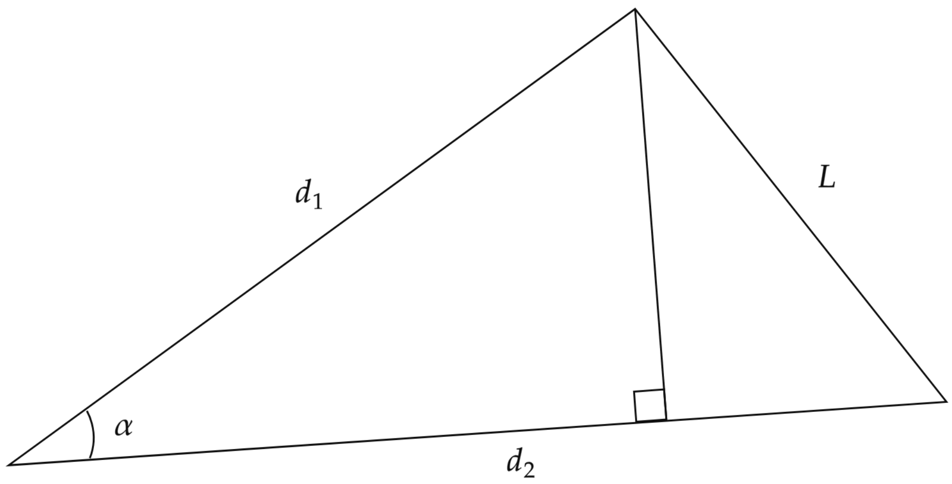
$$\Delta L = \frac{d_1 d_2}{L} \sin\alpha \Delta\alpha$$

2. **If the three sides a, b, c of a right triangle change in length by small amounts Δa, Δb and Δc, then**

$$a\Delta a + b\Delta b = c\Delta c \quad \text{(where c is the hypotenuse)}.$$

Prove these formulas.

Ans:



1. Consider the geometry,

$$\begin{aligned} & (d_1 \sin\alpha)^2 + (d_2 - d_1\cos\alpha)^2 = L^2 \\ \Rightarrow \quad & d_1^2\sin^2\alpha + d_2^2 - 2d_2d_1\cos\alpha + d_1^2\cos^2\alpha = L^2 \\ \Rightarrow \quad & d_1^2 + d_2^2 - 2d_2d_1\cos\alpha = L^2 \end{aligned}$$

Differentiate both side of the equation, we get

$$2L\frac{\mathrm{d}L}{\mathrm{d}\alpha} = 2d_2d_1\sin\alpha$$

and

$$\mathrm{d}L = \frac{d_1d_2}{L}\sin\alpha\,\mathrm{d}\alpha$$

Change d to Δ, we consequently have

$$\Delta L = \frac{d_1d_2}{L}\sin\alpha\Delta\alpha$$

2. If the right triangle have sides a, b and hypotenuse c, By the Pythagorean Theorem, $c^2 = a^2 + b^2$. Differentiate both sides of the equation with respect to a, we have

$$2c\frac{\mathrm{d}c}{\mathrm{d}a} = 2a + 2b\frac{\mathrm{d}b}{\mathrm{d}a}$$

That is,

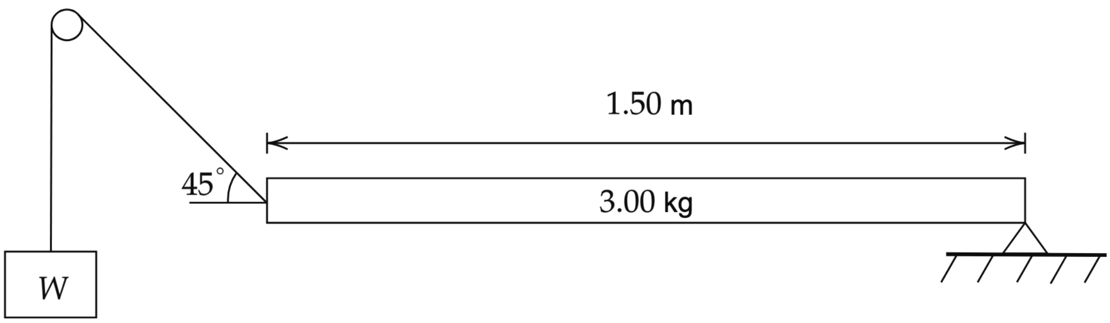
$$c\,\mathrm{d}c = a\,\mathrm{d}a + b\,\mathrm{d}b$$

d is the differential, which could be consider to be a small amount of change. Thus, we have

$$c\,\Delta c = a\,\Delta a + b\,\Delta b$$

2.5

A uniform plank $1.5\,\mathrm{m}$ **long and weighing** $3.00\,\mathrm{kg}$ **is pivoted at one end. The plank is held in equilibrium in a horizontal position by a weight and pulley arrangement, as shown in Figure. Find the weight** W **needed to balance the plank. Neglect friction.**



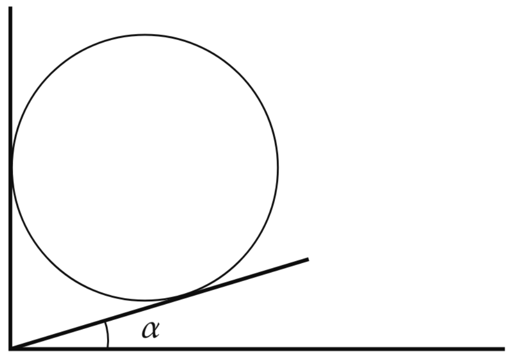
Ans:

For a uniform plank, its center of mass is at the middle of the plank. If the center of mass is moved down in a small distance d, the end of the plank is moved down $2d$, and W is lifted up $\frac{2d}{\sqrt{2}}$. By the principle of virtual work, $W \times \frac{2d}{\sqrt{2}} - 3.00 \times d = 0$, and $W = \frac{3}{\sqrt{2}}$ kg-wt.

2.6

A ball of radius $3.0\,\mathrm{cm}$ **and weight** $1.00\,\mathrm{kg}$ **rests on a plane tilted at an angle** α **with the horizontal and also touches a vertical wall, as**

shown in Figure. Both surfaces have negligible friction. Find the force with which the ball presses on the wall F_W and on the plane F_P.



Ans:

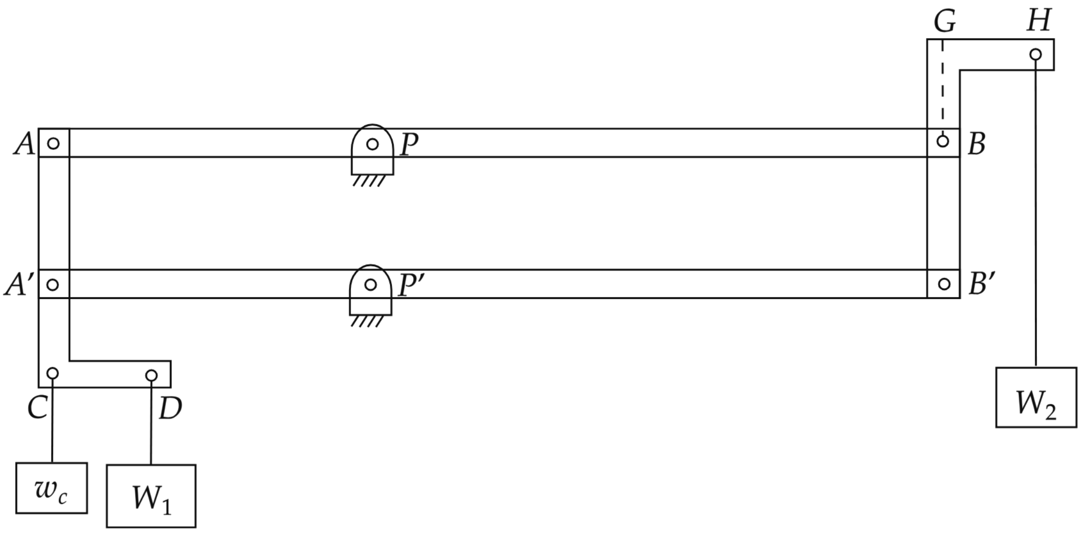
Assume that the ball moves $\vec{r} = \langle s, s\tan\alpha \rangle$. By the principle of virtual work,

$F_W s + F_P(\cos\alpha \times s\tan\alpha - \sin\alpha \times s) - 1.00g \times s\tan\alpha = 0$. And $F_P \cos\alpha = 1.00g$. Hence,

$$\begin{aligned} F_W &= g\tan\alpha - g\tan\alpha + \tan\alpha = g\tan\alpha \text{ kg-wt} \\ F_P &= g\sec\alpha \text{ kg-wt} \end{aligned}$$

2.7

The jointed parallelogram frame $AA'BB'$ is pivoted (in a vertical plane) on the pivots P and P', as shown in Figure. There is negligible friction in the pins at A, A', B, B', P and P'. The members $AA'CD$ and $B'BGH$ are rigid and identical in size. $AP = A'P' = \frac{1}{2}PB = \frac{1}{2}P'B'$. Because of the counterweight w_c, the frame is in balance without the loads W_1 and W_2. If a $0.50\,\mathrm{kg}$ weight W_1 is hung from D, what weight W_2, hung from H, is needed to produce equilibrium?

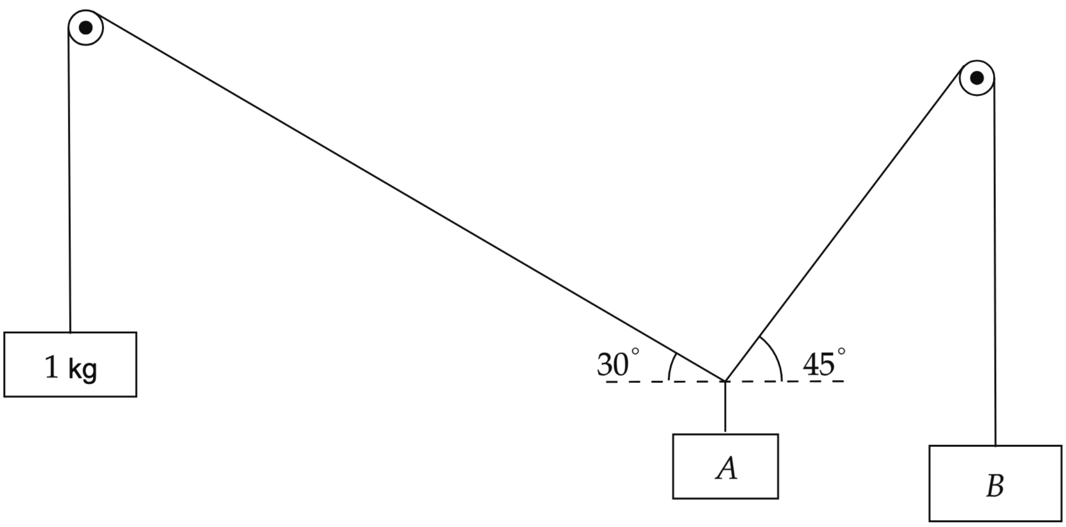


Ans:

If W_1 moves down in distance d, W_2 is lifted up $\frac{d}{AP}\overline{PB} = 2d$. Thus, by the principle of virtual work, $W_1(-d) + W_2 2d = 0$, and $W_2 = \frac{1}{2}W_1 = 0.25$ kg-wt.

2.8

The system shown in Figure is in static equilibrium. Use the principle of virtual work to find the weights A and B. Neglect the weight of the strings and the friction in the pulleys.



Ans:

`Solution I: using forces`

By analyzing the free-body diagram, $1g\cos 30^\circ = B\cos 45^\circ$. Hence, $B = \sqrt{\frac{3}{2}}$ kg-wt. And $A = 1g\sin 30^\circ + B\sin 45^\circ = \frac{1}{2}(1+\sqrt{3})$ kg-wt.

`Solution II: using virtual work`

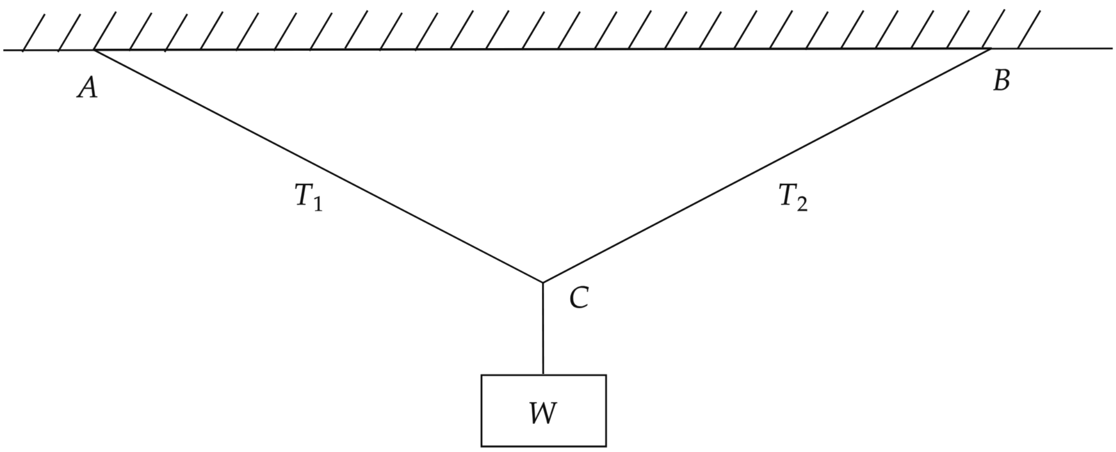
Let's say the tension of the lines connected to the 1 kg block and B is T_1 and T_B, respectively. T_1 is equal to $1g$ and T_2 is equal to B. Assume that A moves down in small distance d, the 1 kg block and B move up d_1 and d_B, respectively. By principle of virtual work,

$$
\begin{aligned}
& 1gd_1 - T_1d_1 + Bd_2 - T_2d_2 + T_1\sin 30^\circ d + T_2\sin 45^\circ d - Ad = 0 \\
\Rightarrow \quad & (1gd_1 - T_1d_1 + Bd_2 - T_2d_2) + T_1\sin 30^\circ d + T_2\sin 45^\circ d - Ad = 0 \\
\Rightarrow \quad & T_1\sin 30^\circ d + T_2\sin 45^\circ d - Ad = 0 \\
\Rightarrow \quad & 0 + 1g \times \frac{d}{2} + B \times \frac{d}{\sqrt{2}} - Ad = 0
\end{aligned}
$$

Therefore, $A = \frac{1}{2} + \frac{\sqrt{3}}{2} = \frac{1}{2}(1+\sqrt{3})$ kg-wt.

2.9

A weight $W = 50\,\text{lb}$ is suspended from the midpoint of a wire ACB as shown in Figure. $AC = CB = 5$ ft. $AB = 5\sqrt{2}$ ft. Find the tension T_1 and T_2 in the wire.

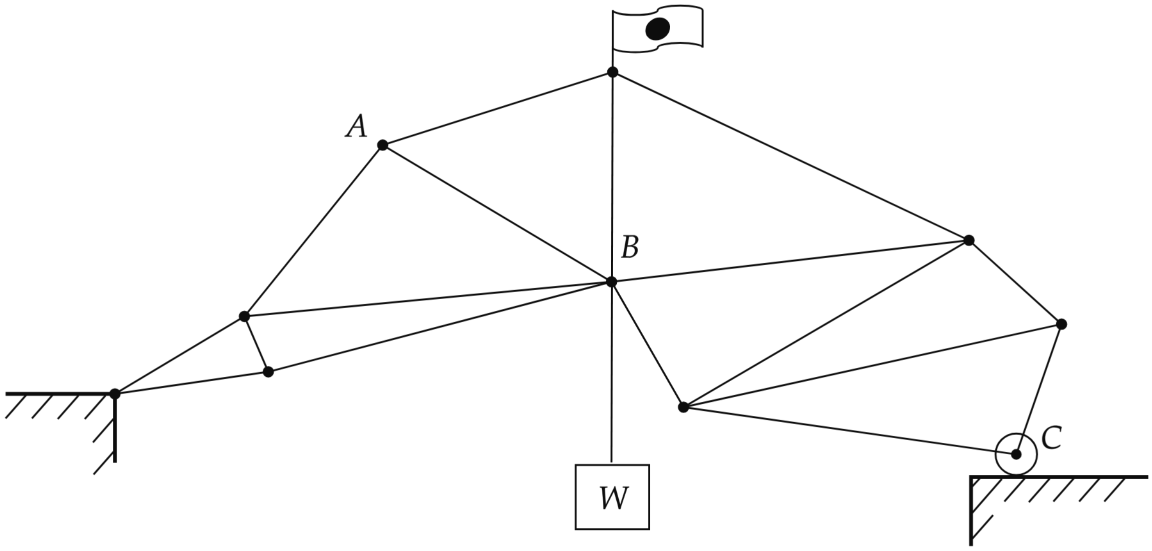


Ans:

By the principle of virtual work, $Wd = T_1 \times \frac{d}{\sqrt{2}} + T_2 \times \frac{d}{\sqrt{2}}$. In addition, $T_1 = T_2$. Hence $T_1 = T_2 = \frac{W}{\sqrt{2}} = \frac{50}{\sqrt{2}}$ lb.

2.10

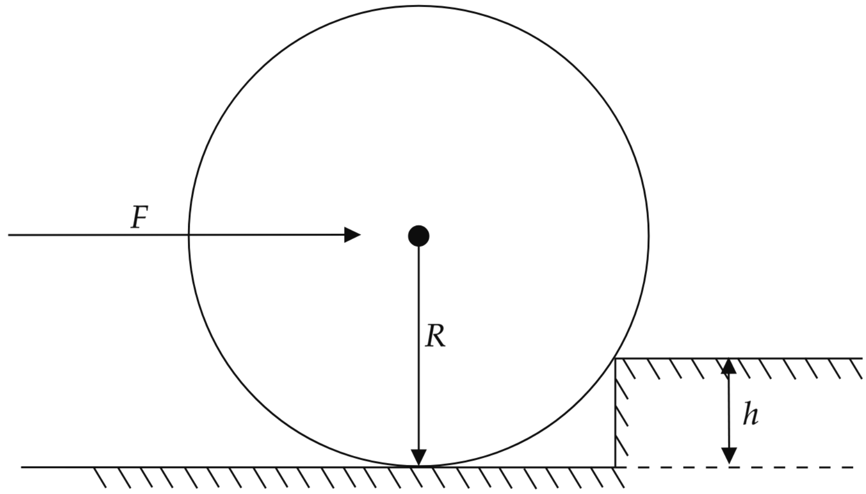
The truss shown in Figure is made of light aluminum struts pivoted at each end. At C is a roller which rolls on a smooth plate. When a workman heats up member AB with a welding torch, it is observed to increase in length by an amount x, and the load W is thereby moved vertically an amount y.



1. Is the motion of W upward or downward?
2. What is the force F in the member AB (including the sense, i.e., tension or compression)?
   Ans:
3. Downward.
4. By the principle of virtual work, $F \times x = W \times y$, that is, $F = \frac{y}{x} W$.

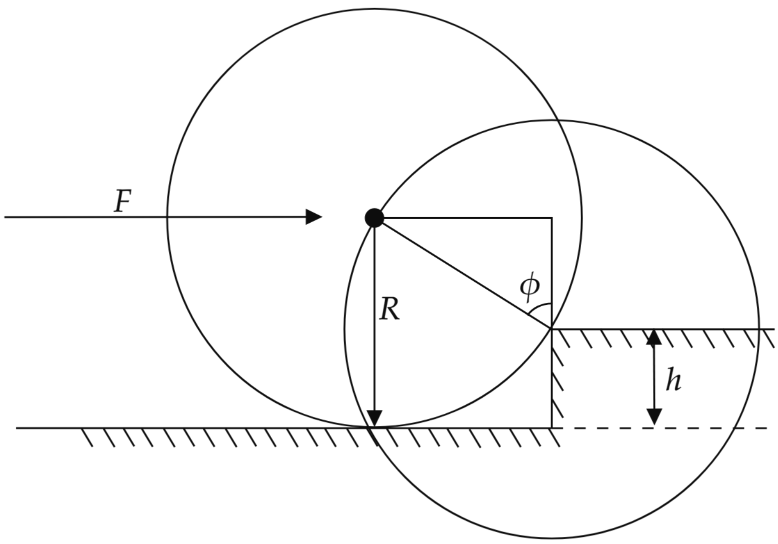
2.11

What horizontal force F (applied at the axle) is required to push a wheel of weight W and radius R over a block of height h, as shown in Figure?



Ans:

The magnitude of the horizontal force F must at least be as strong as when the object is in static equilibrium with only force of gravity and the horizontal force F acting on the wheel.



When the wheel moves in a small path s, $s_x = s\cos\phi$ and $s_y = s\sin\phi$. Hence by the principle of virtual work,

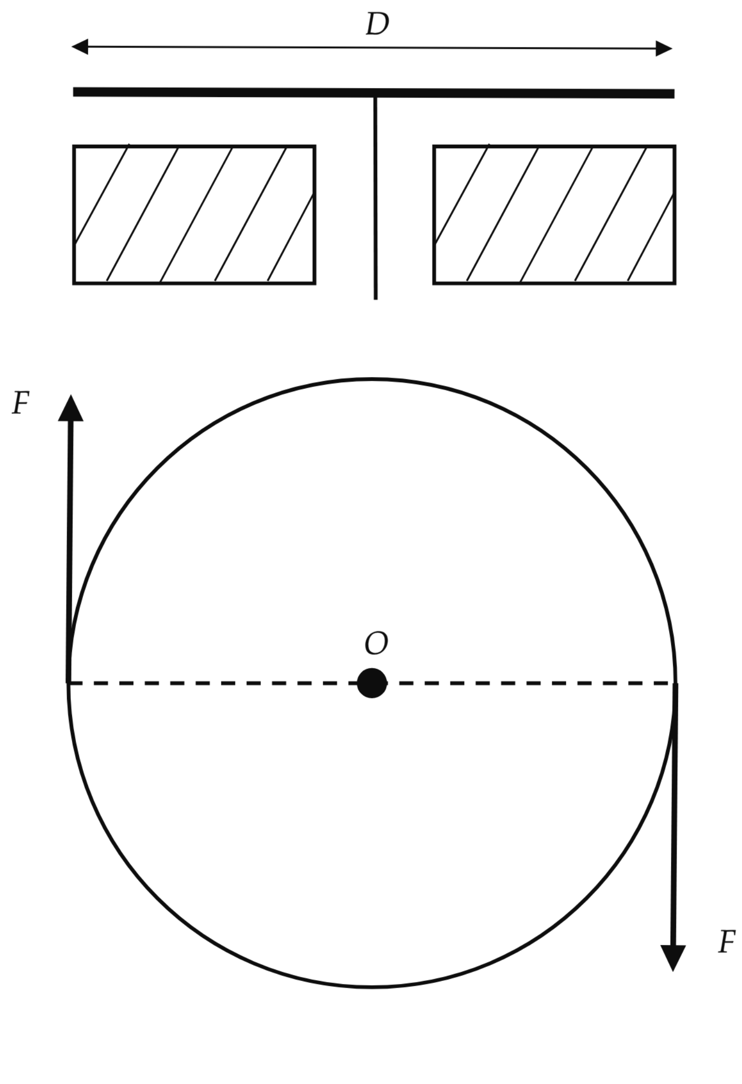
$$
\begin{aligned}
Fs_x - Ws_y &= 0 \\
Fs\cos\phi - Ws\sin\phi &= 0
\end{aligned}
$$

And therefore,

$$
\begin{aligned}
F &= W \tan \phi \\
&= \frac{\sqrt{R^2 - (R-h)^2}}{R-h} W \\
&= \frac{\sqrt{2Rh - h^2}}{R-h} W \\
&= \frac{\sqrt{h(2R-h)}}{R-h} W
\end{aligned}
$$

2.12

A horizontal turntable of diameter D is mounted on bearings with negligible friction. Two horizontal forces in the plane of the turntable of equal magnitude F, parallel to each other but pointing in opposite directions, act on the rim of the turntable on opposite ends of the diameter, as shown in Figure.



1. **What force F_B acts on the bearing?**
2. **What is the torque (= moment of the force couple) τ_O about a vertical axis through the center O?**
3. **What would be the moment τ_P about a vertical axis through an arbitrary point P in the same plane?**
4. **Is the following statement correct or false? Explain. "*Any two forces acting on a body can be combined into a single resultant force that would have the same effect.*" In framing your answer, consider the case where the two forces are opposite in direction but not quite equal in magnitude.**

Ans:

1. No force except gravity acts on the bearing. So $F_B = 0$.
2. The torque is $\tau_O = \frac{D}{2}F + \frac{D}{2}F = DF$.
3. Assume that O is the origin $\langle 0, 0\rangle$ for a two dimensional plane, then the points where the two Force is acting on are at $\left\langle \frac{D}{2}, 0\right\rangle$ and $\left\langle -\frac{D}{2}, 0\right\rangle$, and the vector representation of the two forces are $\langle 0, -F\rangle$ and $\langle 0, F\rangle$. For an arbitrary point $P = (x, y)$, the torque is

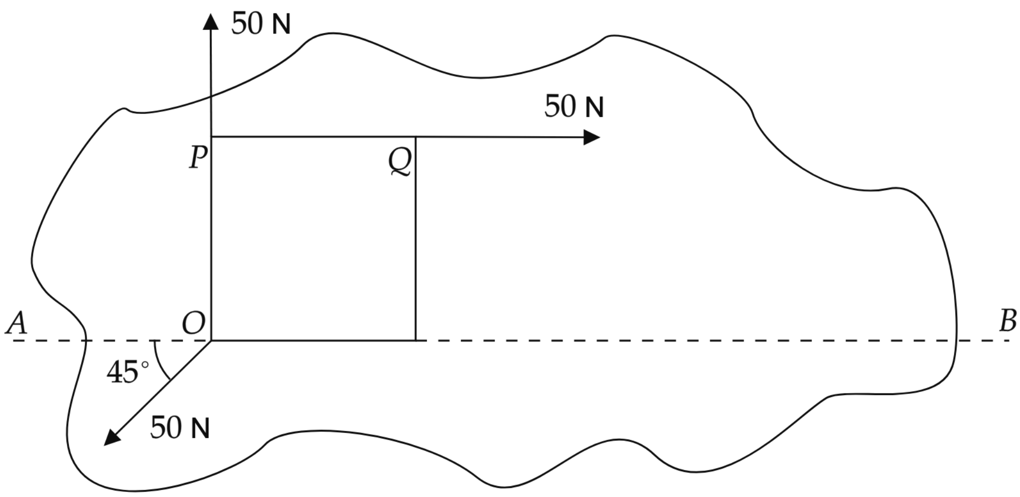
$$
\begin{aligned}
\tau_P &= \left\langle \frac{D}{2} - x, 0 - y\right\rangle \times \langle 0, -F\rangle + \left\langle -\frac{D}{2} - x, 0 - y\right\rangle \times \langle 0, F\rangle \\
&= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{D}{2} - x & -y & 0 \\ 0 & -F & 0 \end{vmatrix} + \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\frac{D}{2} - x & -y & 0 \\ 0 & F & 0 \end{vmatrix} \\
&= \left(\frac{D}{2} - x\right)(-F)\hat{k} + \left(-\frac{D}{2} - x\right)F\hat{k} \\
&= -DF\hat{k}
\end{aligned}
$$

Therefore, the magnitude of τ_P is DF as well.
4. No, as the figure shows above, if we combine the two forces into a single resultant force, that would produce a force that has the magnitude zero, which can not describe the torque

despite the fact that we already know that there is torque $\tau = DF$

2.13

A flat steel plate floating on mercury is acted upon by three forces at three corners of a square of side $0.100\,\mathrm{m}$, as shown in Figure. Find a *single* fourth force F which will hold the plate in equilibrium. Give the magnitude, direction, and point of application of F along the line AB.



Ans:
Set O to be the origin $\langle 0, 0\rangle$. Assume that the plate moves toward a direction in a distance $\vec{r} = \langle x, y\rangle$. Suppose that the forth force is $F = \langle F_x, F_y\rangle$. By the principle of virtual work,

$$\left(50 - \frac{50}{\sqrt{2}} + F_x\right)x + \left(50 - \frac{50}{\sqrt{2}} + F_y\right)y = 0$$

If the plate moves only along the x-axis or y-axis, the two equations should hold as well.
Hence,

$$\left(50-\frac{50}{\sqrt{2}}+F_x\right)x=0$$
$$\left(50-\frac{50}{\sqrt{2}}+F_y\right)y=0$$

$F_x=50\left(\frac{1}{\sqrt{2}}-1\right)$ and $F_y=50\left(\frac{1}{\sqrt{2}}-1\right)$, and the magnitude of F is

$$\begin{aligned}
F&=\sqrt{F_x^2+F_y^2}\\
&=\sqrt{\left(50\left(\frac{1}{\sqrt{2}}-1\right)\right)^2+\left(50\left(\frac{1}{\sqrt{2}}-1\right)\right)^2}\\
&=\sqrt{2}\times|50\left(\frac{1}{\sqrt{2}}-1\right)|\\
&\approx 20.7\,\mathrm{N}
\end{aligned}$$

And the direction is as same as $\langle -1,-1\rangle$.

`Point of application solution I: using torques`

Set P to be the fulcrum. and F is acting at $\langle x_F, y_F\rangle$. The torque should be zero, that is,

$$\begin{aligned}
\tau&=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & -\overline{OP} & 0\\ -\frac{50}{\sqrt{2}} & -\frac{50}{\sqrt{2}} & 0\end{vmatrix}+\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ x_F & y_F-\overline{OP} & 0\\ F_x & F_y & 0\end{vmatrix}\\
&=-\frac{50}{\sqrt{2}}\overline{OP}\hat{k}+x_F F_y\hat{k}-F_x(y_F-\overline{OP})\hat{k}=0
\end{aligned}$$

Plug in the values of F_x and F_y into the equation above, we have

$$\begin{aligned}
&-\frac{50}{\sqrt{2}}\overline{OP}\hat{k}+x_F 50\left(\frac{1}{\sqrt{2}}-1\right)\hat{k}-50\left(\frac{1}{\sqrt{2}}-1\right)(y_F-\overline{OP})\hat{k}=0\\
\Rightarrow\quad&-\frac{50}{\sqrt{2}}\overline{OP}+\frac{50}{\sqrt{2}}x_F-50x_F-\frac{50}{\sqrt{2}}y_F+\frac{50}{\sqrt{2}}\overline{OP}+50y_F-50\overline{OP}=0\\
\Rightarrow\quad&50\left(\frac{1}{\sqrt{2}}-1\right)x_F-50\left(\frac{1}{\sqrt{2}}-1\right)y_F-50\overline{OP}=0
\end{aligned}$$

Since we are looking at F that is applied along $\overline{AB}$, $y_F = 0$. So

$$50\left(\frac{1}{\sqrt{2}} - 1\right)x_F - 50\overline{OP} = 0$$

and

$$\begin{aligned}
x_F &= \frac{\overline{OP}}{\left(\frac{1}{\sqrt{2}} - 1\right)} \\
&= \frac{0.100}{\left(\frac{1}{\sqrt{2}} - 1\right)} \\
&\approx -0.341\ \mathrm{m}
\end{aligned}$$

Therefore, the force F is applied $0.34\ \mathrm{m}$ left of the point O.

`Point of application solution II: using virtual work`

Set O to be the fulcrum, if the plate rotates in a small angle ϕ, then, by the principle of virtual work,

$$\begin{aligned}
\overline{OP}\sqrt{2}\phi\cos 45^\circ \times 50\ \mathrm{N} + \frac{F_x - \frac{x_F}{y_F}F_y}{\left(\sqrt{1^2 + \frac{x_F^2}{y_F^2}}\right)^2}\sqrt{\langle 1, \frac{-x_F}{y_F}\rangle} \times \sqrt{x_F^2 + y_F^2}\phi &= \\
\Rightarrow \qquad \overline{OP}\sqrt{2}\phi\cos 45^\circ \times 50\ \mathrm{N} + (F_x y_F - F_y x_F)\phi &= \\
\Rightarrow \qquad 50\left(\frac{1}{\sqrt{2}} - 1\right)x_F - 50\left(\frac{1}{\sqrt{2}} - 1\right)y_F - 50\overline{OP} &=
\end{aligned}$$

Since we are looking at F that is applied along $\overline{AB}$, $y_F = 0$. So

$$50\left(\frac{1}{\sqrt{2}} - 1\right)x_F - 50\overline{OP} = 0$$

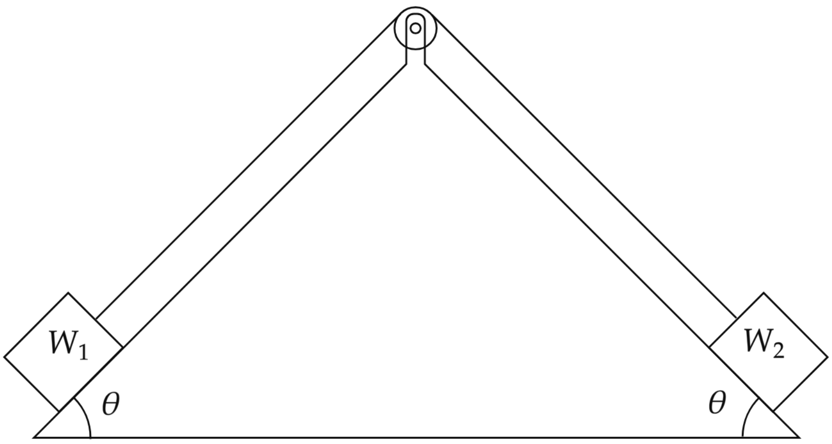
and

$$
\begin{aligned}
x_F &= \frac{\overline{OP}}{\left(\frac{1}{\sqrt{2}} - 1\right)} \\
&= \frac{0.100}{\left(\frac{1}{\sqrt{2}} - 1\right)} \\
&\approx -0.341\,\mathrm{m}
\end{aligned}
$$

Therefore, the force F is applied $0.34\,\mathrm{m}$ left of the point O.

2.14

In the absence of friction, at what speed v will the weights W_1 and W_2 in Figure be moving when they have traveled a distance D, starting from rest? ($W_1 > W_2$).



Ans:
By the conservation of energy, the change of energy is equal to zero. Hence,

$$
W_1(-D)\sin\theta + W_2 D\sin\theta + \frac{W_1 v^2}{2g} + \frac{W_2 v^2}{2g} = 0
$$

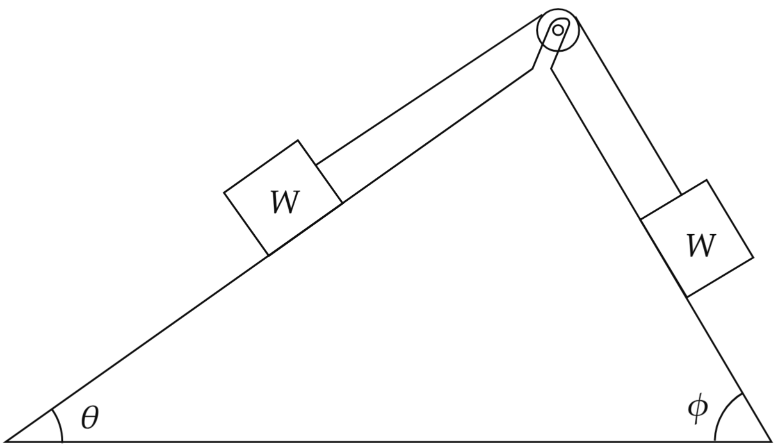
and

$$v^2 = 2\frac{(W_1 - W_2)D\sin\theta}{W_2 + W_1}g$$

$$\Rightarrow \qquad v = \sqrt{2gD\frac{(W_1 - W_2)}{W_1 + W_2}\sin\theta}$$

2.15

In Figure, the weights are equal, and there is negligible friction. If the system is released from rest, at what speed v will the weights be moving when they have traveled a distance D?



Ans:
By the conservation of energy,

$$WD\sin\theta - WD\sin\phi + \frac{2Wv^2}{2g} = 0$$

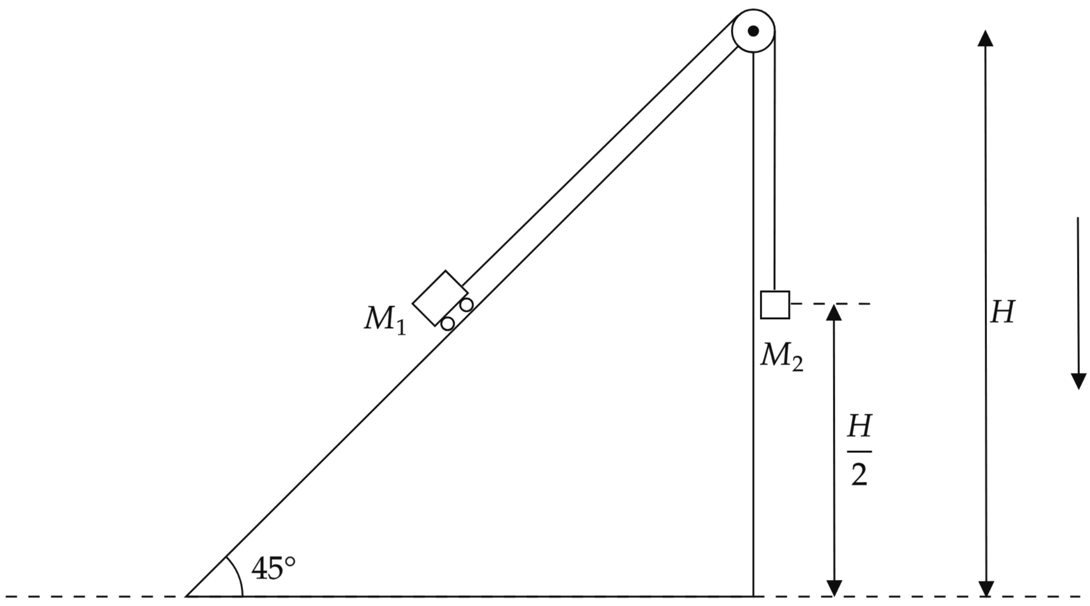
Therefore,

$$v^2 = \frac{D(\sin\phi - \sin\theta)}{1/g}$$

$$v = \sqrt{Dg(\sin\phi - \sin\theta)}$$

2.16

A mass M_1 slides on a 45° inclined plane of height H as shown in Figure. It is connected by a flexible cord of negligible mass over a small pulley (neglect its mass) to an equal mass M_2 hanging vertically as shown. The length of the cord is such that the masses can be held at rest both at height $\frac{H}{2}$. The dimensions of the masses and the pulley are negligible compared to H. At time $t = 0$ the two masses are released.



1. For $t > 0$ calculate the vertical acceleration a of M_2.
2. Which mass will move down?
3. At what time t_1 will the mass identified in part 2. strike the ground
4. If the mass identified in part 2. stops when it hits the ground, but the other mass keeps moving, will it strike the pulley?

Ans:

1. The cord has a tension force which acted both on M_1 and M_2. M_1 and M_2 have the same magnitude of acceleration. We can

simply draw the free body diagram and calculate

$$\begin{cases} -M_1 g\cos 45^\circ + T = M_1 a \\ -M_2 g + T = -M_2 a \end{cases}$$
$$\begin{cases} -M g\cos 45^\circ + T = M a \\ -M g + T = -M a \end{cases}$$

$$\Rightarrow Mg(1-\cos 45^\circ) = 2Ma$$
$$\Rightarrow a = \frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right)g$$

2. M_2 will move down.
3. We know that $v = \int a\,\mathrm{dt} = at + C$. Plug in $t = 0$ we have C is the initial velocity for v. So $v = v_0 + at$. Similarly, $s = \int v\,\mathrm{dt} = \int v_0 + at\,\mathrm{dt} = v_0 t + \frac{1}{2}at^2 + D$. Plug in $t = 0$, we have D equals the initial position. So $s = s_0 + v_0 t + \frac{1}{2}at^2$. Hence,

$$\begin{aligned} & -\frac{H}{2} = 0 + 0t_1 - \frac{1}{2}at_1^2 \\ \Rightarrow \quad & t_1 = \sqrt{\frac{H}{a}} \end{aligned}$$

4. By the formula above, when M_2 strikes, it has the speed $v = 0 + at_1 = \sqrt{aH}$, and so does M_1. By the conservation of energy, the total energy of M_1 remains constant. And it could move up until all the kinetic energy of M_1 changes into the gravitational potential energy, that is,

$$\frac{1}{2}M_1 v^2 = M_1 gh$$

So we get

$$\begin{aligned} h &= \frac{aH}{2g} \\ &= \frac{\frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right)}{g} \times \frac{H}{2} \end{aligned}$$

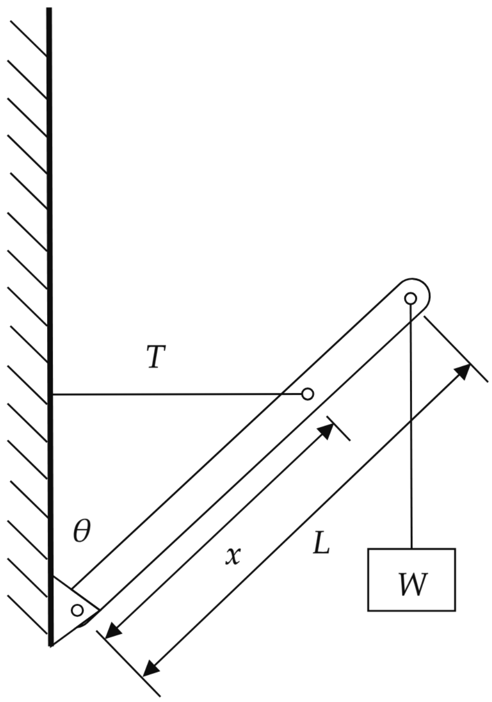
M_1 will move up to

$$
\begin{aligned}
& \frac{H}{2}\sin 45^\circ + \frac{\frac{1}{2}\left(1-\frac{1}{\sqrt{2}}\right)}{g}\frac{H}{2} \\
= & \left(\frac{1}{\sqrt{2}} + \frac{1-\frac{1}{\sqrt{2}}}{2g}\right)\frac{H}{2} \\
\approx & \; 0.72 \times \frac{H}{2} < \frac{H}{2}
\end{aligned}
$$

Therefore, M_1 will not strike the pully.

2.17

A derrick is made of a uniform boom of length L and weight w, pivoted at its lower end, as shown in Figure. It is supported at an angle θ with the vertical by a horizontal cable attached at a point a distance x from the pivot, and a weight W is slung from its upper end. Find the tension T in the horizontal cable.



Ans:

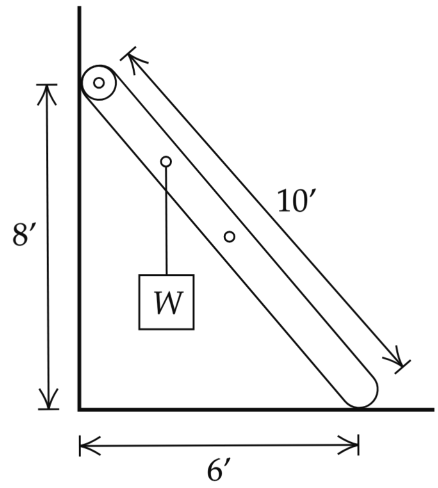
When θ changes in a small angle $\Delta\theta$, the change in distance of the cable is $\Delta S = \frac{x \times x\cos\theta}{x\sin\theta}\sin\theta\Delta\theta$ (See problem 2.4.1). The center of mass of the derrick moves $\frac{2}{L}\Delta\theta\sin\theta$ in the vertical direction and W moves $L\Delta\theta\sin\theta$. Hence, by the principle of virtual work,

$$-T\times\frac{x\times x\cos\theta}{x\sin\theta}\sin\theta\Delta\theta+w\times\frac{L}{2}\Delta\theta\sin\theta+W\times L\Delta\theta\sin\theta=0$$

$$\Rightarrow\qquad T=\frac{L\left(\frac{w}{2}+W\right)}{x}\tan\theta=\frac{L}{x}\left(\frac{w}{2}+W\right)\tan\theta$$

2.18

A uniform ladder $10\,\mathrm{ft}$ long with rollers at the top end leans against a smooth vertical wall, as shown in Figure. The ladder weighs $30\,\mathrm{lb}$. A weight $W=60\,\mathrm{lb}$ is hung from a rung $2.5\,\mathrm{ft}$ from the top end. Find



1. **the force F_R with which the rollers push on the wall.**
2. **the horizontal and vertical forces F_h and F_v with which the ladder pushes on the ground.**

Ans:

1. We set the bottom end of the ladder be the pivot. If the ladder rotates in a small angle $\Delta\phi$, then the rollers move $10\Delta\phi\times\frac{8}{10}$ horizontally, W and the center of mass of the ladder move $7.5\Delta\phi\times\frac{6}{10}$ and $5\Delta\phi\times\frac{6}{10}$,vertically, respectively. By the

principle of virtual work,

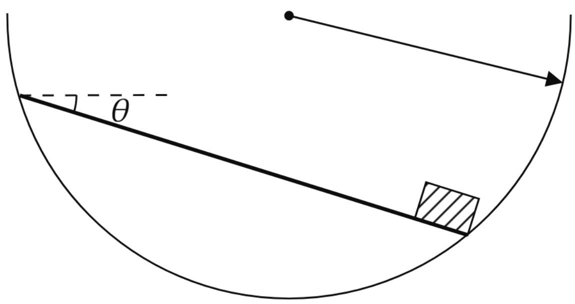
$$F_R \times 10\Delta\phi \times \frac{8}{10} - W \times 7.5\Delta\phi \times \frac{6}{10} - w \times 5\Delta\phi \times \frac{6}{10} = 0$$

where w is the weight of the ladder. Plug in the values of each variable, we have $F_R = 45\,\text{lb-wt}$.

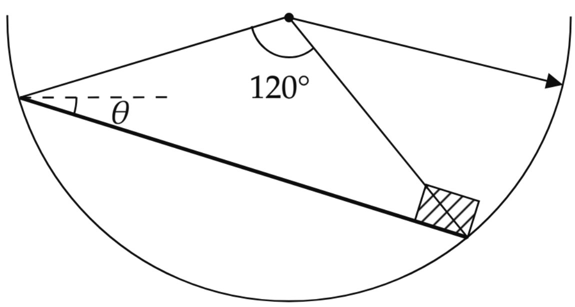
2. Since the system is static equilibrium, it is easy to see that $F_h = F_R = 45\,\text{lb-wt}$ and $F_v = W + w = 90\,\text{lb-wt}$.

2.19

A plank of weight W and length $\sqrt{3}R$ lies in a smooth circular trough of radius R, as shown in Figure. At one end of the plank is a weight $\frac{W}{2}$. Calculate the angle θ which the plank lies when it is in equilibrium.



Ans:
Analyzing the figure, we have

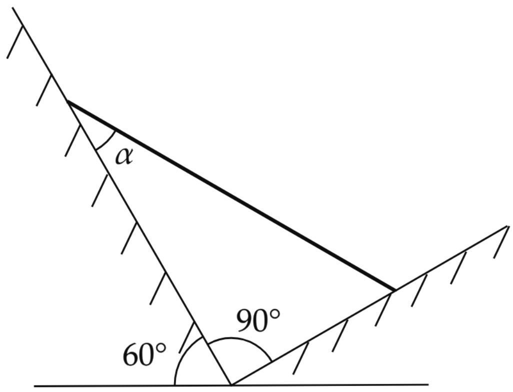


If the plank rotates along the smooth surface so that the one end walks in distance s, the angle of 120° changes in $\frac{s}{R}$, and the center of mass of the plank walks in distance $\frac{R}{2} \times \frac{s}{R} = \frac{s}{2}$. The center of mass of the plank and the one end move $\frac{s}{2} \times \sin\theta$ and $s \times \cos(30^\circ + \theta)$ vertically, respectively. Therefore, by the principle of virtual work,

$$
\begin{aligned}
& \frac{W}{2} \times s\cos(30^\circ + \theta) - W \times \frac{s}{2}\sin\theta = 0 \\
\Rightarrow \quad & \cos 30^\circ \cos\theta - \sin 30^\circ \sin\theta - \sin\theta = 0 \\
\Rightarrow \quad & \frac{\sqrt{3}}{2}\cos\theta - \left(\frac{1}{2} + 1\right)\sin\theta = 0 \\
\Rightarrow \quad & \tan\theta = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2} + 1} = \frac{1}{\sqrt{3}} \\
\Rightarrow \quad & \theta = 30^\circ
\end{aligned}
$$

2.20

A uniform bar of length l and weight W is supported at its ends by two inclined planes as shown in Figure. From the principle of virtual work find the angle α at which the bar is in equilibrium. (Neglect friction.)



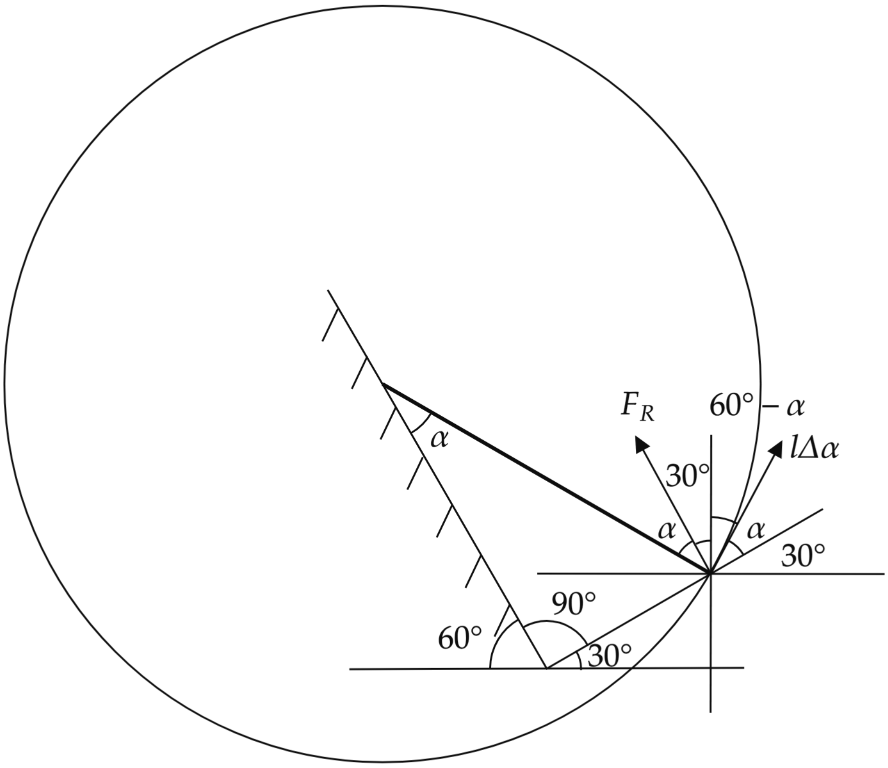
Ans:

Let the forces that the two inclined planes give the bar be F_R and F_L correspond to the left plane and right plane. If the bar slides up along the left inclined plane in a small distance s, by the principle of virtual work,

$$
\begin{aligned}
& -Ws\sin 60^\circ + F_R s = 0 \\
\Rightarrow \qquad & F_R = \frac{\sqrt{3}}{2}W
\end{aligned}
$$

Similarly, $F_L = \frac{1}{2}W$.

If α changes in a small angle $\Delta\alpha$, the bottom end of the bar walks along the circle with center at the top of bar and radius l in distance $l\Delta\alpha$, and the center of mass of the bar walks along the circle with the same center but radius $\frac{l}{2}$ in distance $\frac{l}{2}\Delta\alpha$.



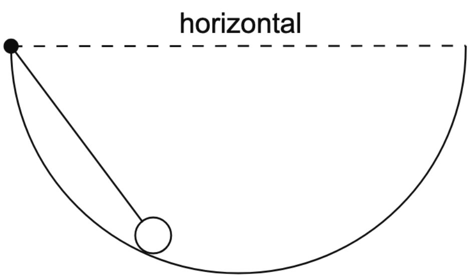
By the principle of virtual work,

$$
\begin{aligned}
& -W\frac{l}{2}\Delta\alpha\cos(60^\circ-\alpha) + F_R\cos 30^\circ l\Delta\alpha\cos(60^\circ-\alpha) \\
& \qquad - F_R\sin 30^\circ l\Delta\alpha\sin(60^\circ-\alpha) = 0 \\
\Rightarrow \qquad & -\frac{1}{2}+\frac{3}{4}-\frac{\sqrt{3}}{4}\tan(60^\circ-\alpha) = 0
\end{aligned}
$$

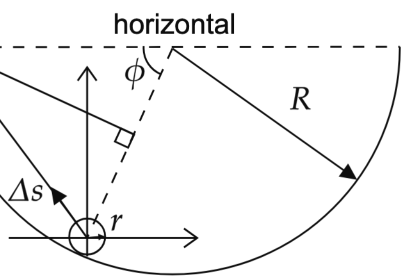
Therefore, $\tan(60^\circ-\alpha) = \frac{1}{\sqrt{3}}$. This implies that $\alpha = 30^\circ$.

2.21

A small solid sphere of radius $4.5\,\mathrm{cm}$ and weight W, is to be suspended by a string from the ends of a smooth hemispherical bowl of radius $49\,\mathrm{cm}$, as shown in Figure. It is found that if the string is any shorter than $40\,\mathrm{cm}$, it breaks. Use the principle of virtual work to find the breaking strength F of the string.



Ans:
It is easy to see that the triangle drawn below is an isosceles triangle.
Let r and R be the radii of the solid sphere and of the bowl, respectively, l be the length of the string when $l = 40\,\mathrm{cm}$, N be the force that the bowl gives to the ball. And let ϕ be the angle in the following figure. We can calculate $\cos\phi = \frac{R^2+(R-r)^2-(l+r)^2}{2R(R-r)}$.



`solution I: force`

N could be compute via analyzing the free-body diagram.

$$\begin{aligned} N\cos\phi &= F\cos\phi \\ N\sin\phi + F\sin\phi &= W \end{aligned}$$

So $N = F$ and $F = W\frac{\csc\phi}{2} \approx 0.6W$

`solution II: virtual work`

Assume that the ball moves along the bowl in a small distance Δs, then the ball will move upward in distance $\Delta s\cos\phi$.

The component of F along the path that the ball moves is $F\cos(2\phi - 90^\circ) = F\sin 2\phi$.

By the principle of virtual work,

$$\begin{aligned} & F(\sin 2\phi)\Delta s - W\Delta s\cos\phi = 0 \\ \Rightarrow \qquad & -F(2\sin\phi\cos\phi)\Delta s + -W\cos\phi\Delta s = 0 \end{aligned}$$

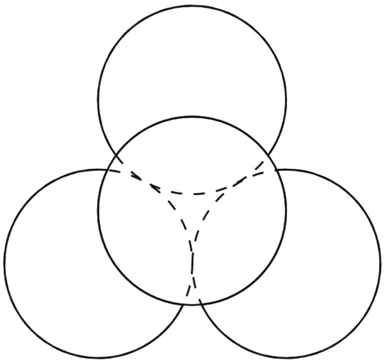
We have

$$\begin{aligned} F &= \frac{\cos\phi}{2\sin\phi\cos\phi}W \\ &= \frac{\csc\phi}{2}W \\ &\approx 0.6W \end{aligned}$$

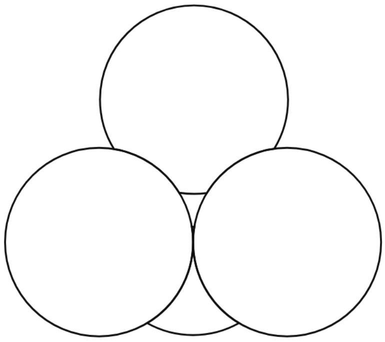
2.22

An ornament for a courtyard at a World's Fair is to be made up of four identical, frictionless metal spheres, each weighing $2\sqrt{6}$ ton-wt. The spheres are to be arranged as shown in Figure, with three resting on a horizontal surface and touching each other; the fourth is to rest freely on the other three. The bottom three are kept from separating by spot welds at the points of contact with each other.

Allowing for a factor of safety of 3, how much tension T must the spot welds withstand?

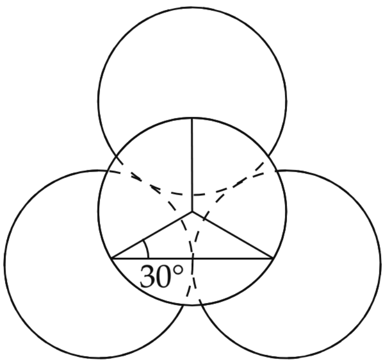


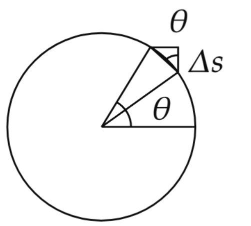
Top View



Horizontal View

Ans:
Assume that the top ball moves down in a small distance Δs, the ball will move apart from each other in a distance $\Delta s \tan\theta \cos 30°$ where $\cos\theta = \frac{1}{3}$ (regular tetrahedron).





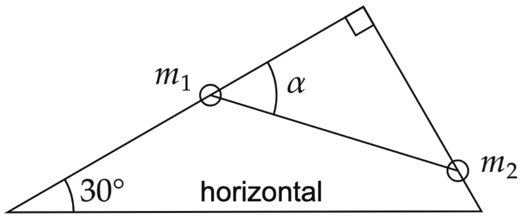
Hence, by the principle of virtual work,

$$
\begin{aligned}
& & W\Delta s - 3T\Delta s \tan\theta \cos 30° &= 0 \\
&\Rightarrow & W - 3 \times 3T \times 2\sqrt{2} \times \frac{\sqrt{3}}{2} &= 0 \\
&\Rightarrow & W - 3\sqrt{6}T &= 0
\end{aligned}
$$

and $T = \frac{W}{3\sqrt{6}} = \frac{2\sqrt{6}}{3\sqrt{6}} = \frac{2}{3}$ ton-wt.
Allowing a factor of three, the welds withstand 2 ton-wt.

2.23

A rigid wire frame is formed in a right triangle, and set in a vertical plane as shown in Figure. Two beads of masses $m_1 = 100\,\mathrm{g}$, $m_2 = 300\,\mathrm{g}$ slide without friction on the wires, and are connected by a cord. When the system is in static equilibrium, what is the tension T in the cord, and what angle α does it make with the first wire?



Ans:
Set the length of the triangle with the angle α to be a, b, and c, where a is the adjacent for α, b the opposite and c the hypotenuse. By problem 2.4.2, $c\Delta c = a\Delta a + b\Delta b$. Assume that m_1 along the slide down in a small distance Δa, and that the length of the wire is a constant$(\Delta b = -\frac{a}{b}\Delta a)$. Applying virtual work, we have

$$
\begin{aligned}
& m_1 g\Delta a \sin 30^\circ - m_2 g\Delta b \sin 60^\circ - T\cos\alpha\Delta a + T\sin\alpha\Delta b = 0 \\
\Rightarrow\quad & m_1 g\frac{1}{2} - m_2 g\left(-\frac{a}{b}\right)\frac{\sqrt{3}}{2} - T\left(\cos\alpha - \sin\alpha\left(-\frac{a}{b}\right)\right) = 0 \\
\Rightarrow\quad & \frac{1}{2}m_1 g + \frac{\sqrt{3}}{2}m_2 g\cot\alpha - T(\cos\alpha + \cos\alpha) = 0 \\
\Rightarrow\quad & \frac{1}{2}m_1 g + \frac{\sqrt{3}}{2}m_2 g\cot\alpha - 2T\cos\alpha = 0\ldots(1)
\end{aligned}
$$

If m_2 slides up in a small distance Δb. By the principle of virtual work,

$$T\sin\alpha\Delta b = m_2 g\Delta b\sin 60^\circ$$

So $T\sin\alpha = \frac{\sqrt{3}}{2}m_2g\ldots(2)$.
By (1) and (2),

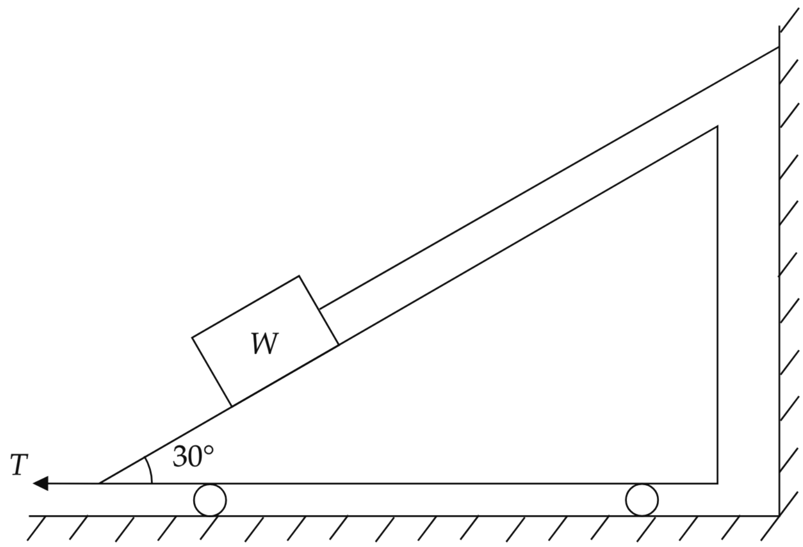
$$\begin{cases} 4T\cos\alpha = m_1g + \sqrt{3}m_2g\cot\alpha \\ T\sin\alpha = \dfrac{\sqrt{3}}{2}m_2g \end{cases}$$
$$\Rightarrow \qquad 4\cot\alpha = \frac{2}{\sqrt{3}}\frac{m_1}{m_2} + 2\cot\alpha$$
$$\Rightarrow \qquad \cot\alpha = \frac{\frac{2}{\sqrt{3}}\frac{m_1}{m_2}}{2} = \frac{1}{\sqrt{3}}\frac{m_1}{m_2}$$

So we have $\cot\alpha = \frac{1}{3\sqrt{3}}$, and $\alpha \approx 79.1^\circ$.
Plug in the value of α in (2), we have
$T = \sqrt{1+\cot^2\alpha}\frac{\sqrt{3}}{2}m_2g \approx 265$ g-wt.

2.24

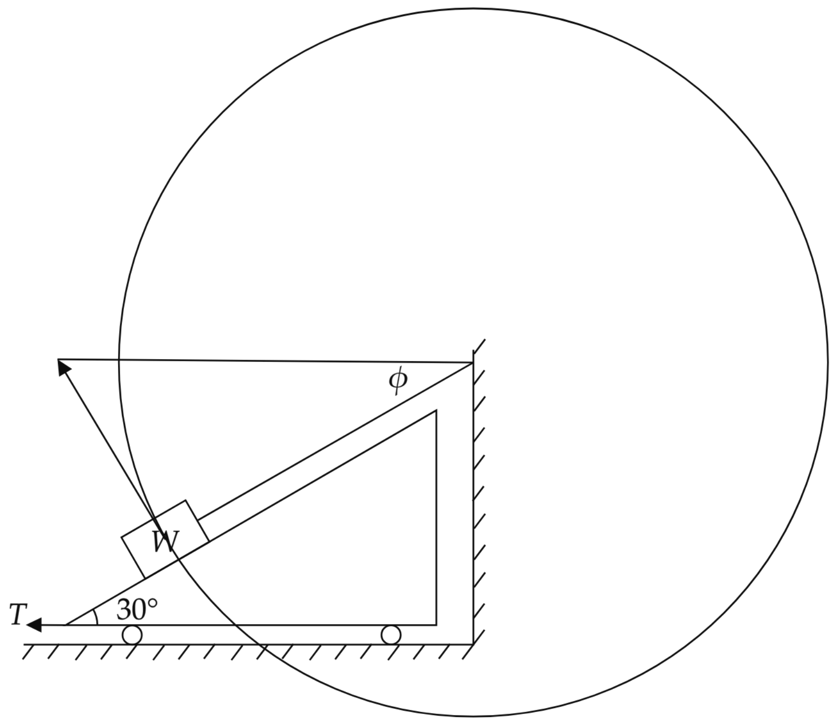
Find the tension T needed to hold the cart shown in Figure in equilibrium, if there is no friction.

1. **Using the principle of virtual work.**
2. **Using force components.**



Ans:

1. Let N denote the force that the cart gives to W. Assume that W moves along the circle with circle at the end of the cord which holds W and radius equals the length of the cord in a small distance Δs.



Then W will move up $\Delta s \cos\phi$. Here $\phi = 30°$. By the principle of virtual work,

$$-W\Delta s \cos\phi + N\Delta s = 0$$

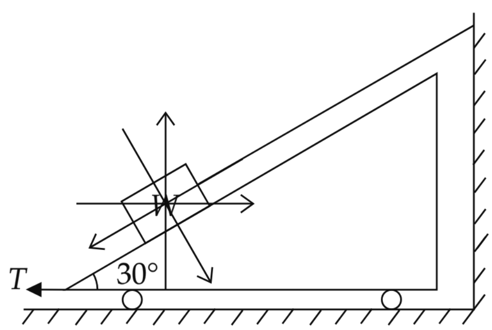
So $N = W\cos\phi$.

If the cart move forward in a small distance Δx, then, by the principle of virtual,

$$T\Delta x - N\sin 30°\Delta s = 0$$

Therefore, $T = W\cos 30° \sin 30° = \frac{\sqrt{3}}{4}W$.

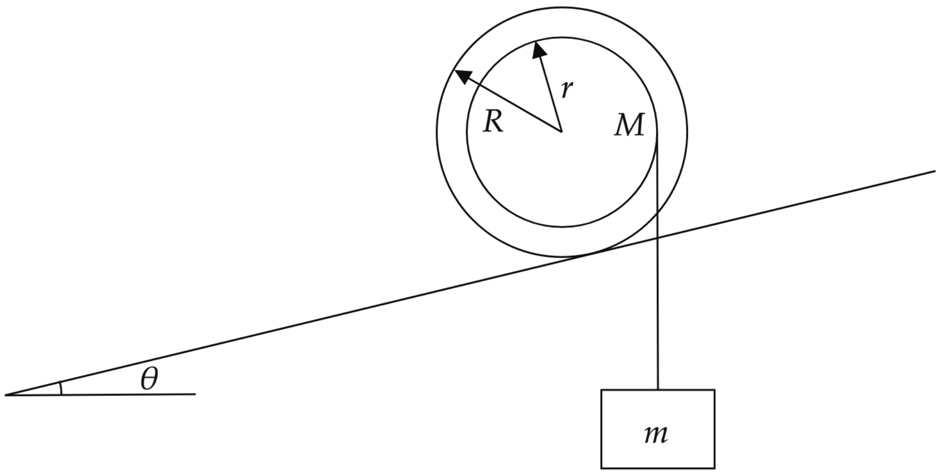
2. By analyzing the free-body diagram,



$W \cos 30° \sin 30° = T$. And therefore, $T = \frac{\sqrt{3}}{4} W$.

2.25

A bobbin of mass $M = 3\,\mathrm{kg}$ consists of a central cylinder of radius $r = 5\,\mathrm{cm}$ and two end plates of radius $R = 6\,\mathrm{cm}$. It is placed on a slotted incline on which it will roll but not slip, and a mass $m = 4.5\,\mathrm{kg}$ is suspended from a cord wound around the bobbin, as shown in Figure. It is observed that the system is in static equilibrium. What is the angle of tilt of tilt θ the incline?



Ans:
Assume that the string that holds m is lengthened in a small distance Δs vertically, then the bobbin will rotate in an angle $\frac{\Delta s}{r}$. And the bobbin will move along the incline in distance $-R\frac{\Delta s}{r}$. So m will move in distance $\Delta s - R\frac{\Delta s}{r}\sin\theta$ and M will move $-R\frac{\Delta s}{r}\sin\theta$. By the principle of virtual work,

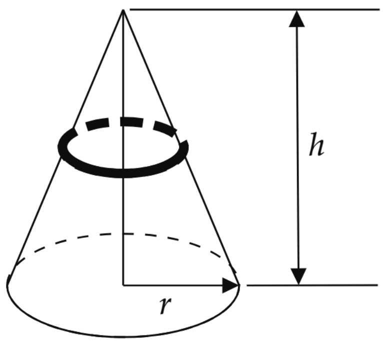
$$mg\left(\Delta s - R\frac{\Delta s}{r}\sin\theta\right) + Mg\left(-R\frac{\Delta s}{r}\sin\theta\right) = 0$$

$$\Rightarrow \qquad mr - (m+M)R\sin\theta = 0$$

Hence, $\sin\theta = \frac{mr}{R(m+M)} = \frac{4.5\times 5}{6\times(4.5+3)} = 0.5$, and $\theta = 30°$

2.26

A loop of flexible chain, of total weight W, rests on a smooth right circular cone of base radius r and height h, as shown in Figure. The chain rests in a horizontal circle on the cone, whose axis is vertical. Find the tension T in the chain. Neglect friction.



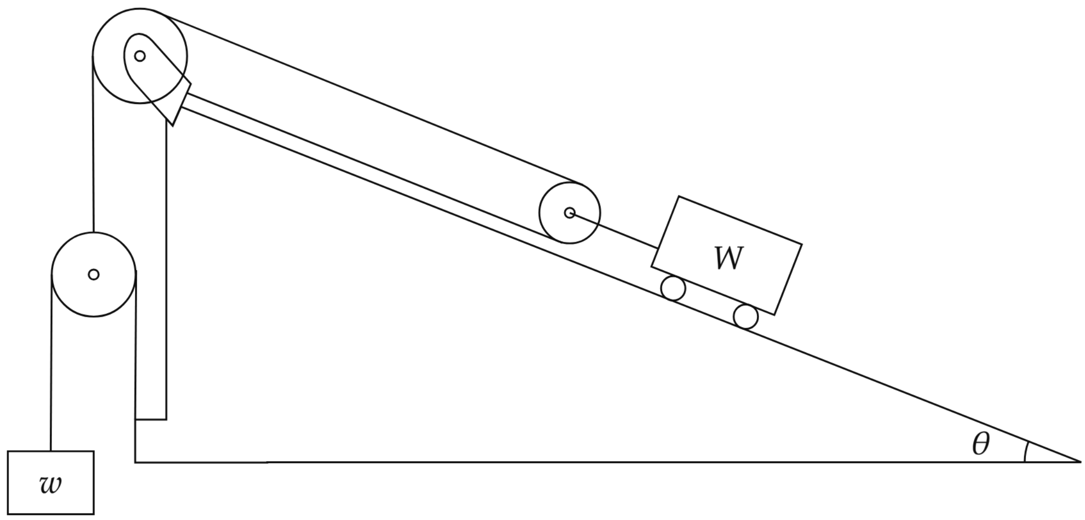
Ans:
Assume that the chain moves down in a small distance Δx, then the chain would be stretched $2\pi\Delta x \tan\theta$ where $\tan\theta = \frac{r}{h}$. By the principle of virtual work,

$$\begin{aligned} & W\Delta x - T2\pi\Delta x \tan\theta = 0 \\ \Rightarrow \quad & W - 2\pi\frac{r}{h}T = 0 \end{aligned}$$

Therefore, $T = \frac{1}{2\pi}\frac{h}{r}W$.

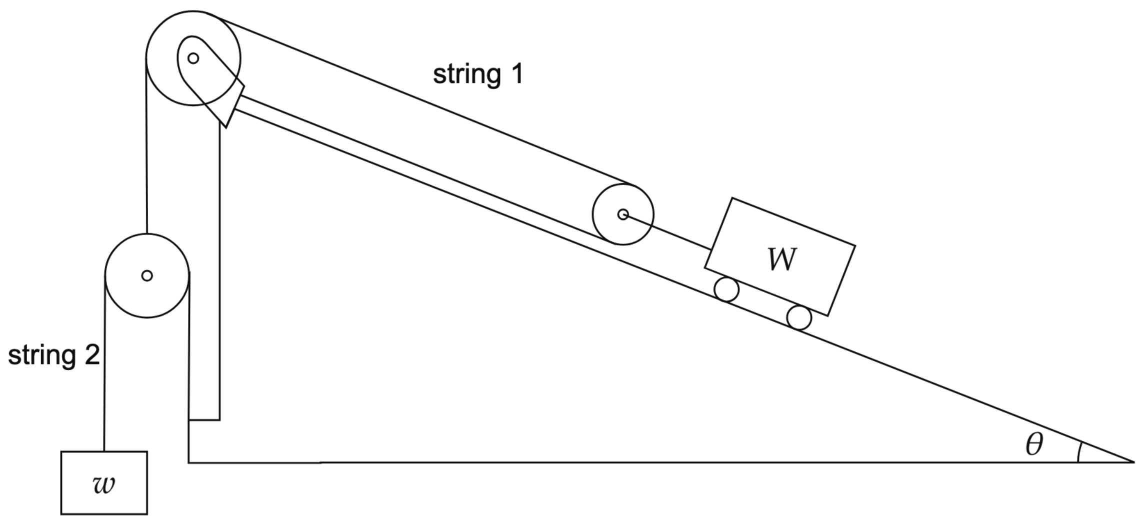
2.27

A cart on an inclined plane is balanced by the weight w as shown in Figure. All parts have negligible friction. Find the weight W of the cart.



Ans:

When W moves along the inclined plane in a distance Δs, *string 1* will change in $2\Delta s$. *string 2* will change in twice *string 1*.



By the principle of virtual work,

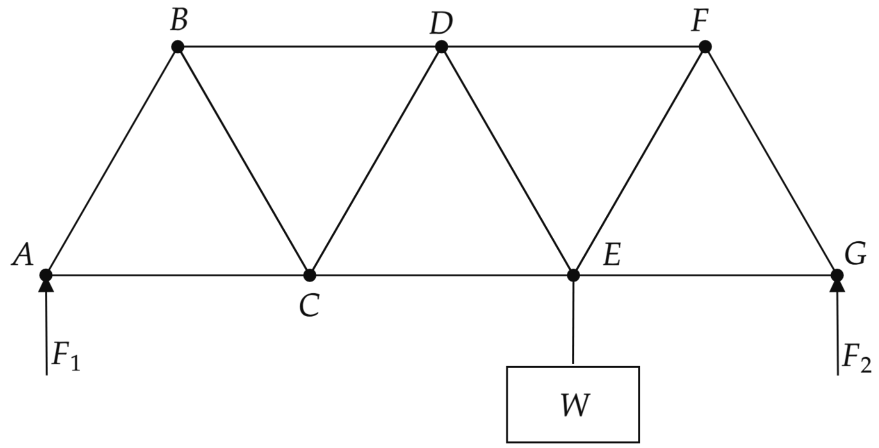
$$W\Delta s \sin\theta - w4\Delta s = 0$$

Therefore, $W = 4\csc\theta w$.

2.28

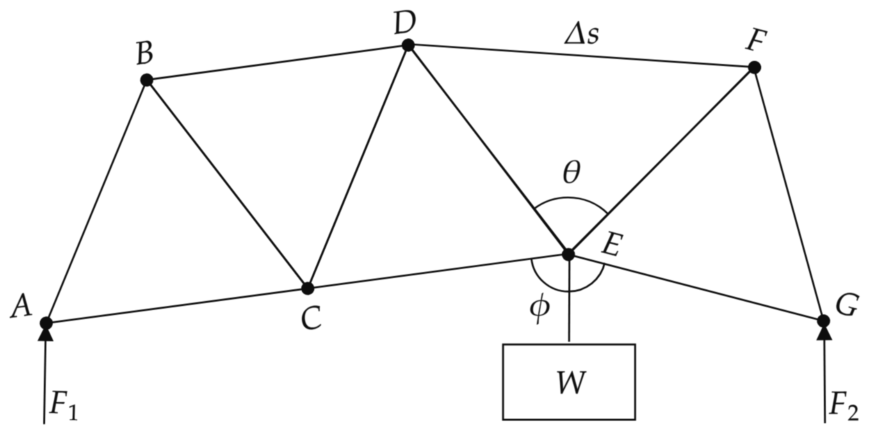
A bridge truss is constructed as shown in Figure. All joints may be considered frictionless pivots and all members rigid, weightless, and of equal length. Find the reaction forces F_1 and F_2 and the force

F_{DF} **in the member** DF.



Ans:

Set E to be a fulcrum. Since $\sum \vec{\tau} = 0$, $2F_1 = F_2$. In addition, $F_1 + F_2 = W$. Hence, $F_1 = \frac{1}{3}W$ and $F_2 = \frac{2}{3}W$.



Assume that $\overline{DF}$ is stretched in a distance Δs, by problem 2.4.1, $\Delta s = \frac{l^2}{l} \sin\theta\Delta\theta = l\sin\theta\Delta\theta$ where l is the length of the triangles. $\phi = 360° - 2 \times 60° - \theta = 240° - \theta$, and $\Delta\phi = -\Delta\theta$. Here, $\theta = 60°$. If W is lifted up Δy, then

$$\frac{1}{2}2l \times l\sin\phi = \frac{1}{2}3l \times y$$

(Left hand side of the equation is the formula for finding the area of a triangle using sine, and right hand side is the area formula for

a triangle).
So

$$l^2 \cos\phi \Delta\phi = \frac{3}{2} l \Delta y$$

Thus, $\Delta y = \frac{2}{3} l \cos\phi \Delta\phi$

By the principle of virtual work,

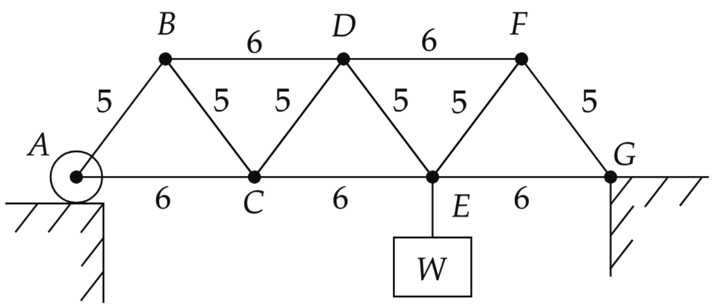
$$\begin{aligned}
& & -F_{DF}\Delta s + W\Delta y = 0 \\
\Rightarrow & & -F_{DF}\Delta s + W \times \frac{2}{3} l \cos\phi \Delta\phi = 0 \\
\Rightarrow & & -F_{DF}\Delta s - \frac{2}{3} W l \cos(240^\circ - \theta)\Delta\theta = 0 \\
\Rightarrow & & -F_{DF}\Delta s - \frac{2}{3} W l \cos(240^\circ - \theta) \times \frac{1}{l} \csc\theta \Delta s = 0 \\
\Rightarrow & & -F_{DF} - \frac{2}{3} W \cos 180^\circ \csc 60^\circ = 0
\end{aligned}$$

Consequently, we have

$$\begin{aligned}
F_{DF} &= \frac{2}{3} W \times 1 \times \frac{2}{\sqrt{3}} \\
&= \frac{4}{3\sqrt{3}} W
\end{aligned}$$

2.29

In the truss shown in Figure, all diagonal struts are of length 5 units and all horizontal ones are of length 6 units. All joints are freely hinged, and the weight of the truss is negligible.



1. **Which of the members could be replaced with flexible cables, for the load position shown?**
2. **Find the forces in struts BD and DE.**

Ans:

1. I don't know.
2. Similar as problem 2.28. $\Delta BD = \frac{5\times 5}{6}\sin\theta\Delta\theta$ where $\sin\frac{\theta}{2} = \frac{3}{5}$. So $\Delta BD = \frac{25}{6} \times 2 \times \frac{3}{5} \times \frac{4}{5}\Delta\theta = 4\Delta\theta$.
   $\Delta y_C = -\frac{2}{3} \times 6\cos(2\pi - 2\angle ACB - \theta)\Delta\theta$ (from problem 2.30).

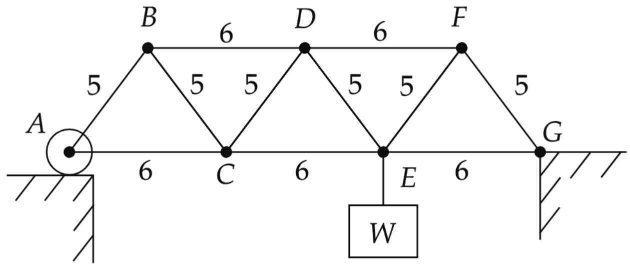
$$\begin{aligned}\Delta y_C &= -4\cos(2\angle ACB + \theta)\Delta\theta \\ &= -4\cos(\pi)\Delta\theta \\ &= \Delta BD\end{aligned}$$

And$\Delta y_E = \frac{1}{2}\Delta y_C$. Hence, by the principle of virtual work,

$$-F_{BD}\Delta BD + W\frac{1}{2}\Delta BD = 0$$

This implies that $F_{BD} = \frac{1}{2}W$. For F_{DE}, by the problem 2.4.1,

$$\Delta DE = \frac{6\times 5}{5} \times \frac{4}{5}\Delta(\angle DFE) = \frac{24}{5}\Delta(\angle DFE)$$



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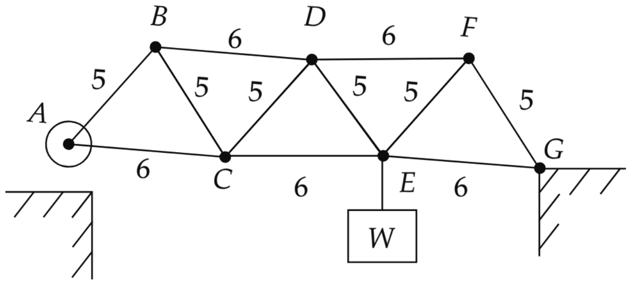

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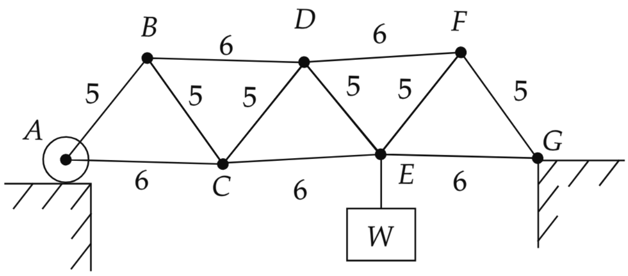
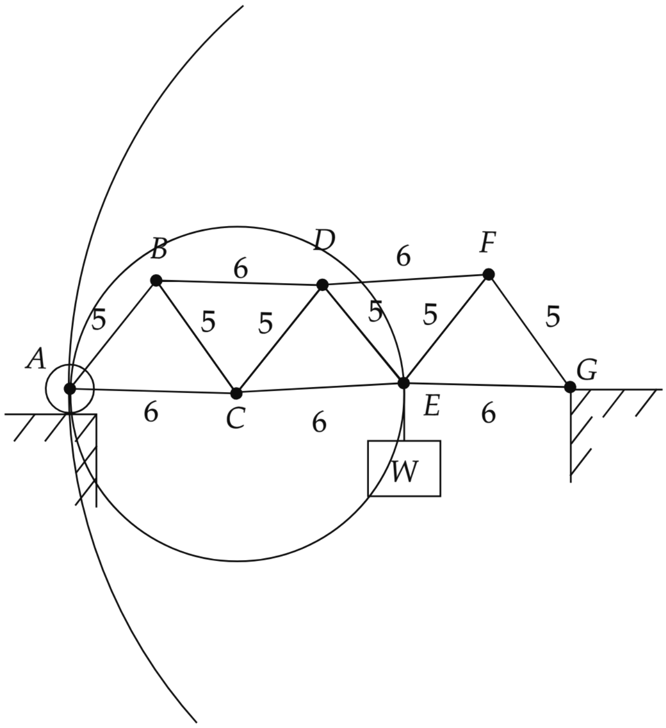


If we want $\overline{DF}$ remain horizontal, then $\triangle FEG$ need to be rotated in an angle $-\Delta(\angle DFE)$ with point G fixed. Similarly, $\triangle BDC$ is rotated $-\Delta(\angle DFE)$ with point C fixed. Therefore, A will be lifted up in distance $18(-\Delta(\angle DFE)) + 6\Delta(-\Delta(\angle DFE))$

.



But A should be on the ground. So we move the whole truss down $18(-\Delta(\angle DFE)) + 6\Delta(-\Delta(\angle DFE))$. From W's perspective, W is lifted up $6(-\Delta(\angle DFE))$ and dropped down $\frac{1}{3}(18(-\Delta(\angle DFE)) + 6(-\Delta(\angle DFE)))$. Thus,

$$\begin{aligned}\Delta y_W &= \frac{1}{3}(18\Delta(\angle DFE) + 6\Delta(\angle DFE)) - 6\Delta(\angle DFE) \\ &= 2\Delta(\angle DFE) = \frac{5}{12}\Delta DE\end{aligned}$$

Hence, by the principle of virtual work

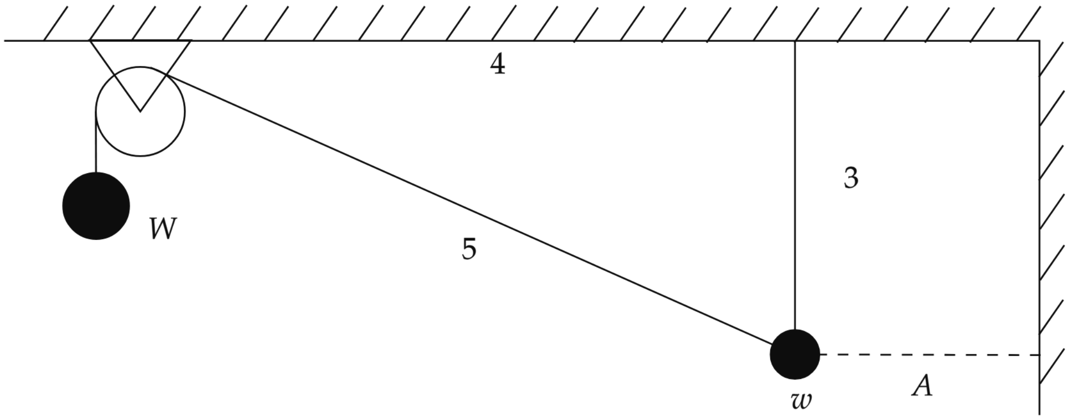
$$\begin{aligned} & -F_{DE}\Delta DE + W\Delta y_W = 0 \\ \Rightarrow \quad & -F_{DE}\Delta DE + W\frac{5}{12}\Delta DE = 0\end{aligned}$$

This implies that $F_{DE} = \frac{5}{12}W$.

2.30

In the system shown in Figure, a pendulum bob of weight w is initially held in the vertical position by a thread A. When this thread is burned, releasing the pendulum, it swings to the left and barely reaches the ceiling at its maximum swing. Find the weight

W. (Neglect friction, the radius of the pulley, and the finite sizes of the weights.)



Ans:

When w is swung to the ceiling, W would move down $5-(4-3)=4\,\text{unit}$. By the conservation of energy,

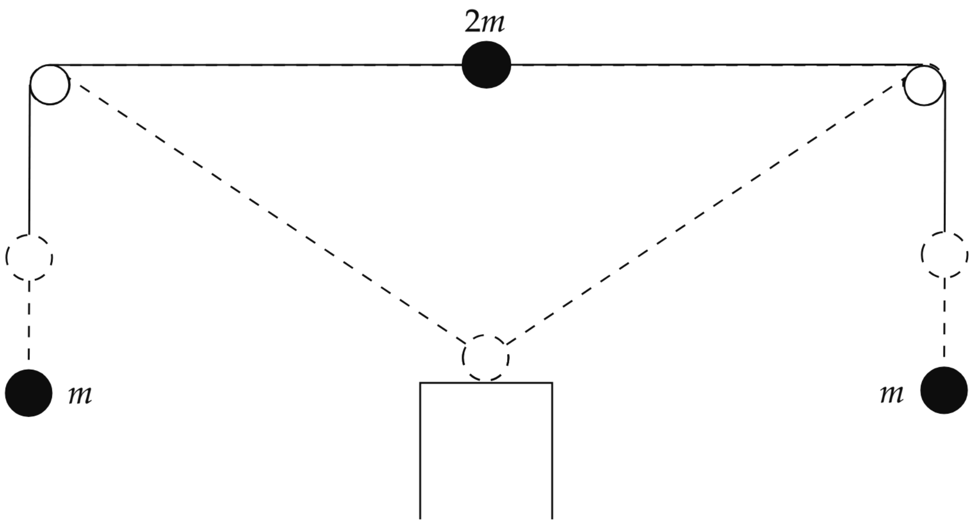
$$-3w+4W=0$$

and $W=\frac{3}{4}w$.

2.31

Two equal masses m are attached to a third mass $2m$ by equal lengths of fine thread and the thread is passed over two small pulleys with negligible friction situated $100\,\text{cm}$ apart, as shown in Figure. The mass $2m$ is initially held level with the pulleys midway between them, and is then released from rest. When it has descended a distance of $50\,\text{cm}$ it strikes a table top. What is it's

speed v when it reaches the table top?



Ans:

$L^2 + x^2 = y^2 \Rightarrow 2x\Delta x = 2y\Delta y \Rightarrow x\Delta x = y\Delta$ where Δx and Δy are the change in distance of $2m$ and m, respectively. So

$50v_{2m} = 50\sqrt{2}v_m$, and$v_m = \frac{1}{\sqrt{2}}v_{2m}$.

By the conservation of energy,

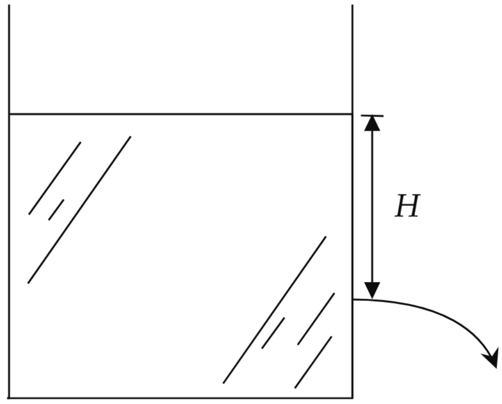
$$2mg \times 50 - 2 \times mg \times (50\sqrt{2} - 50) = \frac{1}{2}2mv^2 + 2 \times \frac{1}{2}m\left(\frac{1}{\sqrt{2}}v\right)^2$$

$$\Rightarrow \qquad (100 - 100\sqrt{2} + 100)g = \frac{3}{2}v^2$$

Therefore, $v = \sqrt{\frac{2}{3}(200 - 100\sqrt{2})g} \approx 196\ \mathrm{cm\ s^{-1}}$.

2.32

A tank of cross-sectional area A contains a liquid having density ρ. The liquid squirts freely from a small hole of area a distance H below the free surface of the liquid, as shown in Figure. If the liquid has no internal friction (viscosity), with what speed v does it emerge?



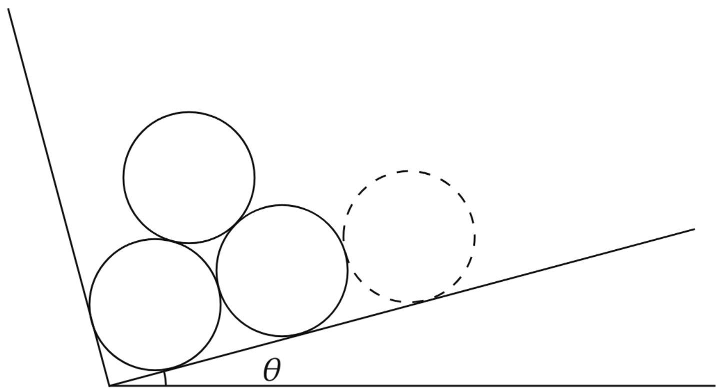
Ans:
The volume of the liquid that is above the level of the hole is equal to the volume with bottom area equals a and its height $s = \frac{HA}{a}$. So $\Delta s = \frac{A}{a}\Delta H$.
By the conservation of energy,

$$AH\rho g\Delta H = \frac{1}{2}a\Delta s\rho v^2$$
$$AH\rho g\Delta H = \frac{1}{2}A\Delta H\rho v^2$$

Therefore, $v = \sqrt{2gH}$.

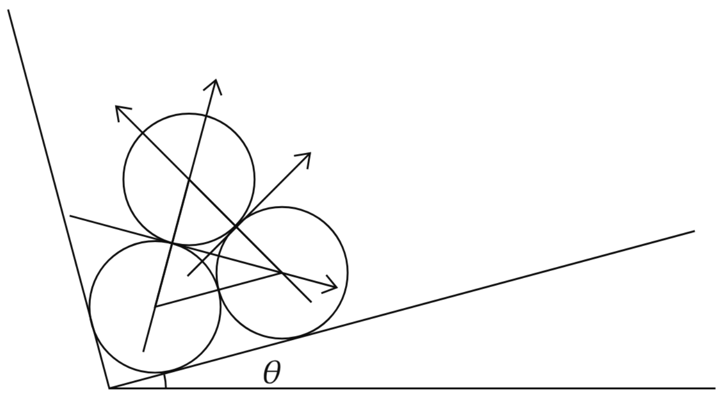
2.33

Smooth, identical logs are piled in a stake truck. The truck is forced off the highway and comes to rest on an even keel lengthwise but with the bed at an angle θ with the horizontal, as shown in Figure. As the truck is unloaded, the removal of the log shown dotted leaves the remaining three in a condition where they are just ready to slide, that is, if θ were any smaller, the logs would fall down. Find θ.



Ans:
Let W and R be the weight and radius of each log, and let N_l and N_u be the normal forces the lower ball and the upper give to the top ball.

By the principle of virtual work, if the top log moves perpendicular to N_l in a small distance Δs,



then by the principle of virtual work,

$$W\Delta s \sin(30^\circ - \theta) - N_u \Delta s \cos 30^\circ = 0$$

$$\Rightarrow \qquad W\left(\frac{1}{2}\cos\theta - \frac{\sqrt{3}}{2}\sin\theta\right) - N_u\frac{\sqrt{3}}{2} = 0$$

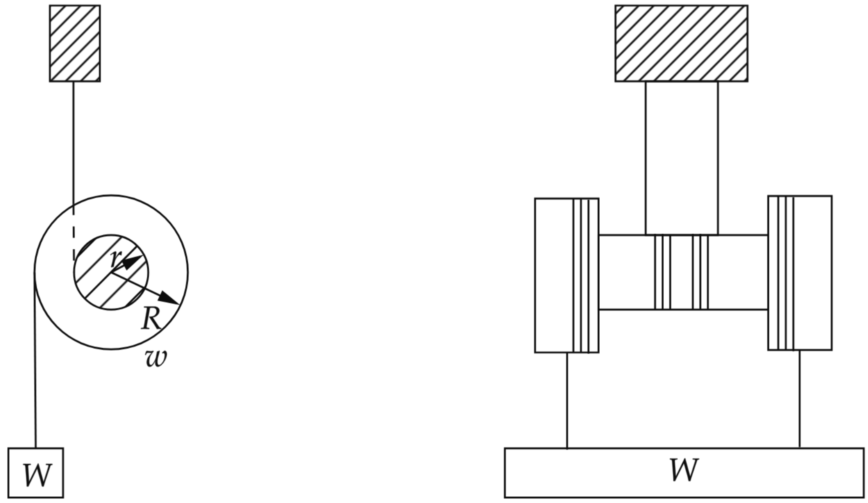
That is, $N_u = \left(\frac{1}{\sqrt{3}}\cos\theta - \sin\theta\right)W$.
If the right bottom log moves along the incline in a small distance Δs, by the principle of virtual work,

$$
\begin{aligned}
& -W\Delta s\sin\theta + N_u\sin 30^\circ\Delta s = 0 \\
\Rightarrow \qquad & -\sin\theta + \frac{1}{2\sqrt{3}}\cos\theta - \frac{1}{2}\sin\theta = 0
\end{aligned}
$$

Consequently, $\frac{3}{2}\sin\theta = \frac{1}{2\sqrt{3}}\cos\theta$ and $\tan\theta = \frac{1}{3\sqrt{3}}$, which implies that $\theta \approx 10.9^\circ$.

2.34

A spool of weight w and radii r and R is wound with cord, and suspended from a fixed support by two cords wound on the smaller radius; a weight W is then suspended from two cords wound on the larger radius, as shown in Figure. W is chosen so that the spool is just balanced. Find W.



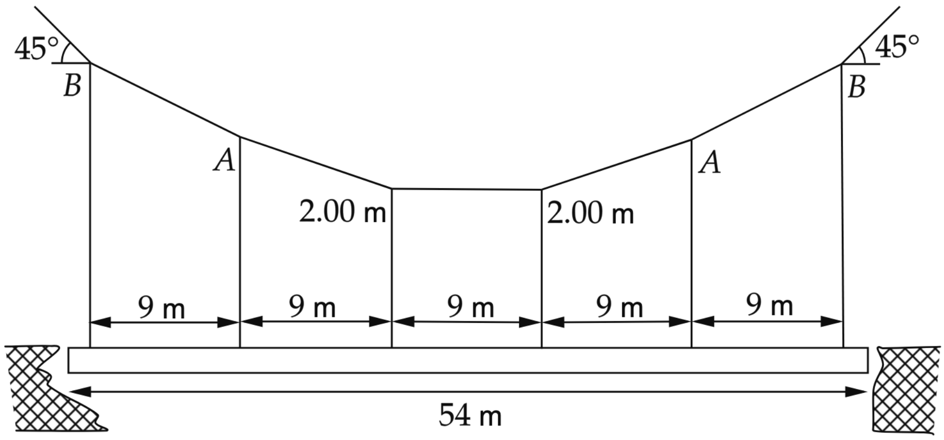
Ans:
Assume that the spool rotates in a small angle $\Delta\theta$, then the spool will move $-r\Delta\theta$ vertically, and the cord will move $R\Delta\theta - r\Delta\theta$ vertically. By the principle of virtual work,

$$w(-r\Delta\theta) + W(R\Delta\theta - r\Delta\theta) = 0$$

Which implies that $W = \frac{r}{R-r}w$.

2.35

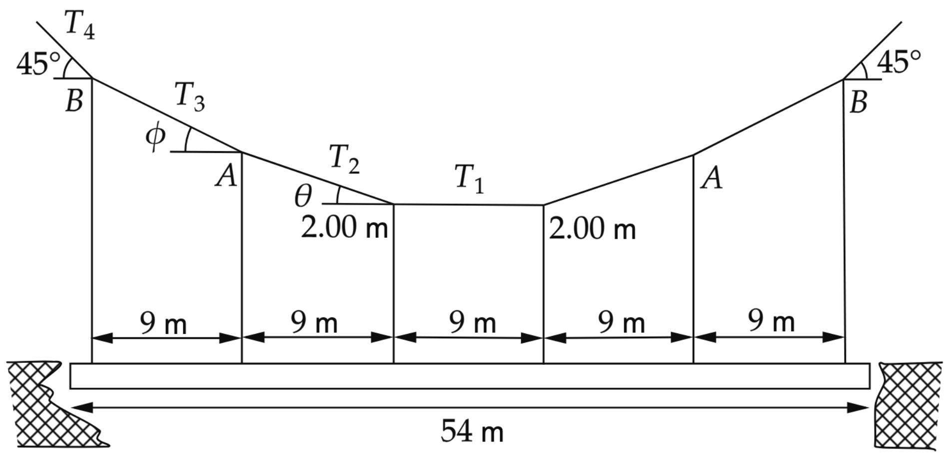
A suspension bridge is to span a deep gorge $54\,\mathrm{m}$ wide. The roadway will consist of a steel truss supported by six pairs of vertical cables spaced $9.0\,\mathrm{m}$ apart, as shown in Figure. Each cable is to carry an equal share of the $4.80 \times 10^4\,\mathrm{kg}$ weight. The two pairs of cables nearest the center are to be $2.00\,\mathrm{m}$ long. Find the proper lengths for the remaining vertical cables A and B and the maximum tension T_{max} in the two longitudinal cables, if the latter are to be at a 45° angle with the horizontal at their ends.



Ans:

Let w be the weight each of the six cable carries. Analyzing the free-body diagram, we have

$$T_1 = T_2 \cos\theta = T_3 \cos\phi = T_4 \cos 45^\circ \ldots (1)$$
$$w = T_2 \sin\theta = T_3 \sin\phi - T_2 \sin\theta = T_4 \sin 45^\circ - T_3 \sin\phi \ldots (2)$$



From (2),

$$\begin{array}{r@{}l}
w &= T_2 \sin\theta \\
w &= T_3 \sin\phi - T_2 \sin\theta \\
w &= T_4 \sin 45^\circ - T_3 \sin\phi \\
\hline
3w &= T_4 \sin 45^\circ
\end{array}$$

Now we have

$$\begin{aligned}
T_1 &= T_4 \cos 45^\circ \\
T_1 &= T_2 \cos\theta \\
w &= T_2 \sin\theta \\
3w &= T_4 \sin 45^\circ
\end{aligned}$$

So

$$\begin{aligned}
& & T_2 \cos\theta &= T_4 \cos 45^\circ \\
&\Rightarrow & w \csc\theta \cos\theta &= 3w \csc 45^\circ \cos 45^\circ \\
&\Rightarrow & \cot\theta &= 3 \cot 45^\circ
\end{aligned}$$

Thus, $\tan\theta = \frac{1}{3}$ and the length of the vertical cable A is $2.00 + 9.0 \tan\theta = 5.0\,\mathrm{m}$.

Similarly,

$$\begin{aligned}
T_1 &= T_3 \cos\phi \\
T_1 &= T_4 \cos 45^\circ \\
w &= T_4 \sin 45^\circ - T_3 \sin\phi \\
3w &= T_4 \sin 45^\circ
\end{aligned}$$

So

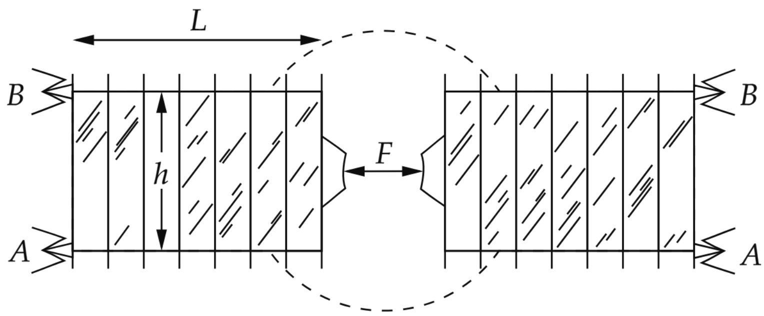
$$\begin{aligned}
& & T_4 \cos 45^\circ &= T_3 \cos\phi \\
&\Rightarrow & 3w \csc 45^\circ \cos 45^\circ &= 2w \csc\phi \cos\phi \\
&\Rightarrow & \frac{3}{2} \cot 45^\circ &= \cot\phi
\end{aligned}$$

Thus, $\tan\phi = \frac{2}{3}$ and the length of the vertical cable B is $2.00 + 9.0\tan\theta + 9.0\tan\phi = 11$ m

$w = \frac{4.8\times10^4}{6} = 8\times10^3$ kg-wt. T_4 is the maximum tension, which is $T_4 = 3w\csc 45^\circ \approx 34\times10^3$ kg-wt.

2.36

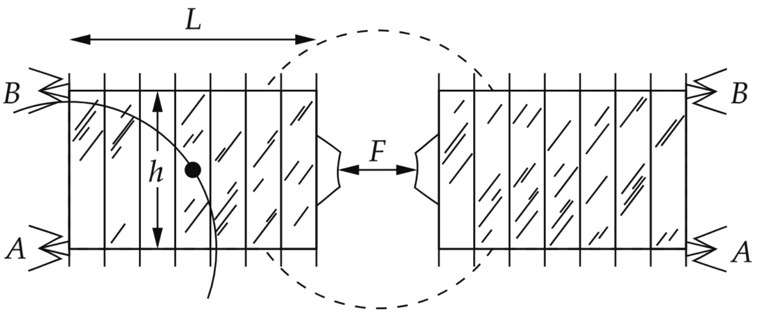
The insulating support structure of a Tandem Van de Graaff may be represented, as shown in Figure: two blocks of about uniform density, length L , height h and weight W , supported from vertical bulkheads by pivot joints (A and B) and forced apart by a screw jack (F) at the center. Since the material of the blocks cannot support tension, the jack must be adjusted to give zero force on the upper pivot.



1. **What force F is required?**
2. **What is the total force (magnitude and direction) F_a on one of the lower pivots A**

Ans:

1. `solution I: using virtual work`



If the left one block rotates around A in an angle $\Delta\theta$, then by the center of mass of the block moves
$\sqrt{\left(\frac{h}{2}\right)^2+\left(\frac{L}{2}\right)^2}\Delta\theta\frac{\frac{L}{2}}{\sqrt{\left(\frac{h}{2}\right)^2+\left(\frac{L}{2}\right)^2}}=\frac{L}{2}\Delta\theta$ downward and
$\sqrt{\left(\frac{h}{2}\right)^2+\left(\frac{L}{2}\right)^2}\Delta\theta\frac{\frac{h}{2}}{\sqrt{\left(\frac{h}{2}\right)^2+\left(\frac{L}{2}\right)^2}}=\frac{h}{2}\Delta\theta$ toward right hand side. By the principle of virtual work

$$W\frac{L}{2}\Delta\theta-F\frac{h}{2}\Delta\theta=0$$

Which implies that $F=\frac{L}{h}W$

`solution II: using force`

Since the system is in static equilibrium, the net torque is equal to zero. Hence, if we set A to be a fulcrum,

$$\frac{h}{2}F=\frac{L}{2}W$$

Which implies that $F=\frac{L}{h}W$.

2. The block acted upon by 3 forces and is in static equilibrium, by problem $2.3.3$, the lines of action of the forces must pass through a single point, which in this case is the center of mass of the block. Thus, F_A points toward the center of the mass of the block, which is $\arctan\left(\frac{h}{L}\right)$ with horizontal. The horizontal component of F_A must equal F and the vertical component of F_A must equal W to remain equilibrium. Therefore,
$F_A\frac{\frac{h}{2}}{\sqrt{\left(\frac{h}{2}\right)^2+\left(\frac{L}{2}\right)^2}}=W$ and $F_A=\sqrt{1+\left(\frac{L}{h}\right)^2}W$.