Exercises 5

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All the figures in this document is redrawn using Mathcha or Inkscape.

5.1

Two blocks of mass $m_1 = 1\,\mathrm{kg}$, $m_2 = 2\,\mathrm{kg}$ on a horizontal surface, connected by a string, are being pulled by another string which is attached to a mass $m_3 = 2\,\mathrm{kg}$ hanging over a pulley, as shown in Fig. 5-1. Neglect friction and the masses of the pulley and strings.

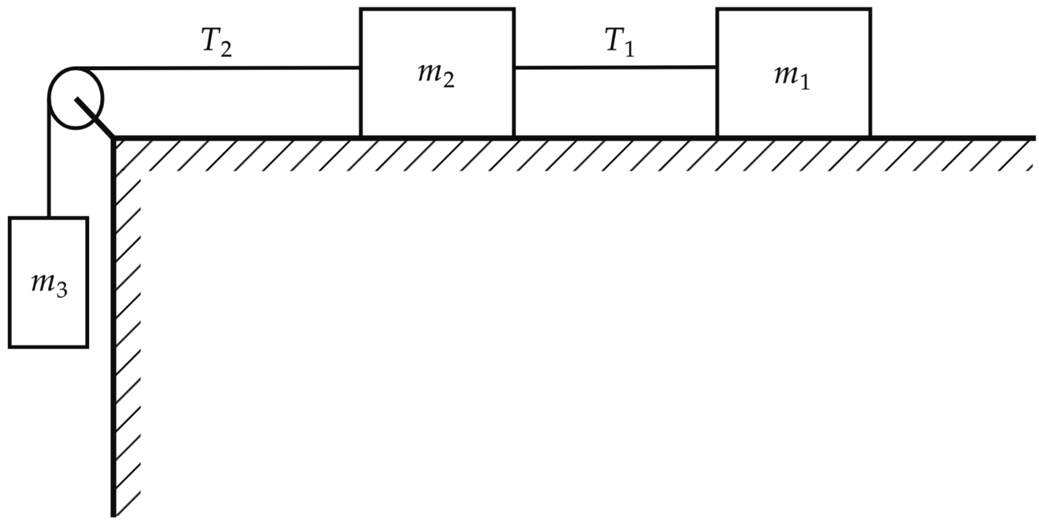
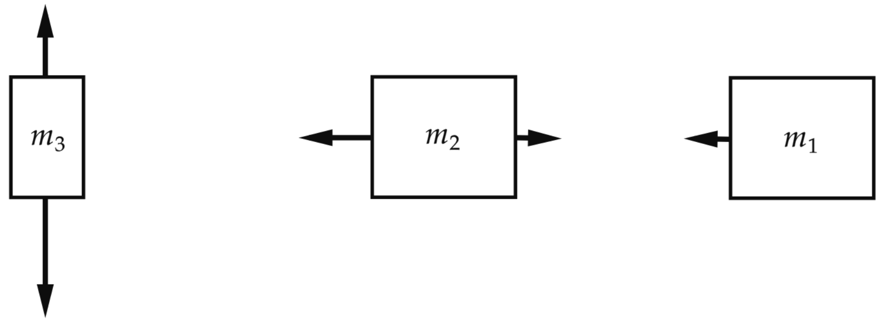


Figure 5-1

1. **Sketch free-body diagrams for all masses, showing the forces acting.**
2. **Find the acceleration a of the masses.**
3. **Find the tension T_1 and T_2 in the strings.**

Ans:

1. The free-body diagrams for m_1, m_2 and m_3 are



2. Assume that the acceleration is a, then by analyzing the free-body diagrams,

$$
\begin{aligned}
& \begin{cases} m_3 g - T_2 & = m_3 a \\ T_2 - T_1 & = m_2 a \\ T_1 & = m_1 a \end{cases} \\
\Rightarrow \quad & m_3 g = (m_1 + m_2 + m_3) a \\
\Rightarrow \quad & a = \frac{m_3}{m_1 + m_2 + m_3} g \\
\Rightarrow \quad & = \frac{2}{1+2+2} g \\
\Rightarrow \quad & = \frac{2}{5} g
\end{aligned}
$$

3. From 2. $T_1 = 1 \times \frac{2}{5} g = \frac{2}{5} g$ and $T_2 = 2 \times \frac{2}{5} g + \frac{2}{5} g = \frac{6}{5} g$.

5.2

A mass m (kg) hangs on a cord susupended from an elevator which is descending with an acceleration of $0.1g$. What is the tension T in the cord in newtons?

Ans:

By analyzing the free-body diagram, $mg - T = m(0.1g)$, which implies that $T = 0.9mg \approx 8.829m \approx 8.8m\,\mathrm{N}$.

5.3

Two objects of mass $m = 1\,\mathrm{kg}$ each, connected by a taut string of length $L = 2\,\mathrm{kg}$, move in a circular orbit with constant speed $V = 5ms^{-1}$ about their common center C in a zero-g environment, as shown in Fig. 5-2. What is the tension T in the string in

newtons?

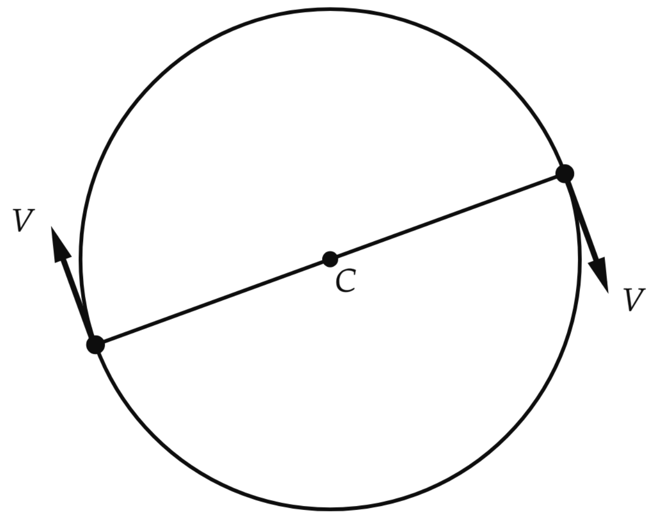
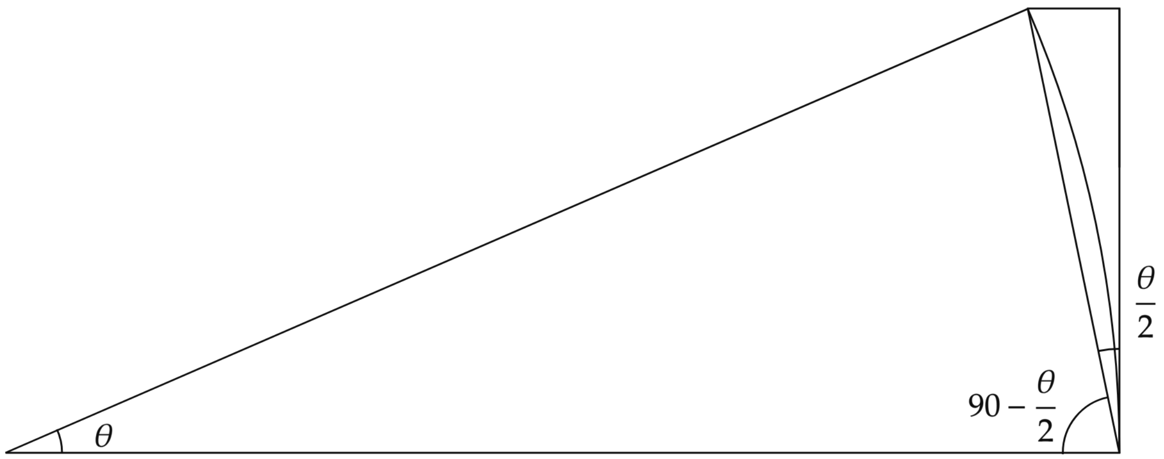


Figure 5-2

Ans:

For a motion that is circular, in a small change in time, the distance that the moving object "falls" is

$s = R\theta \times \theta/2 = \frac{R}{2}\left(\frac{V}{2\pi R}2\pi\right)^2 = \frac{V^2}{2R}$. And $a = \frac{d^2 s}{dt^2} = \frac{V^2}{R}$. Hence,



$$T = ma = m\frac{V^2}{R} = 1 \times \frac{5^2}{2/2} = 25\ \mathrm{N}$$

5.4

Referring to Fig. 5-3: What horizontal force F must be constantly spplied to M so that M_1 and M_2 do not move relative to M?

Neglect friction.

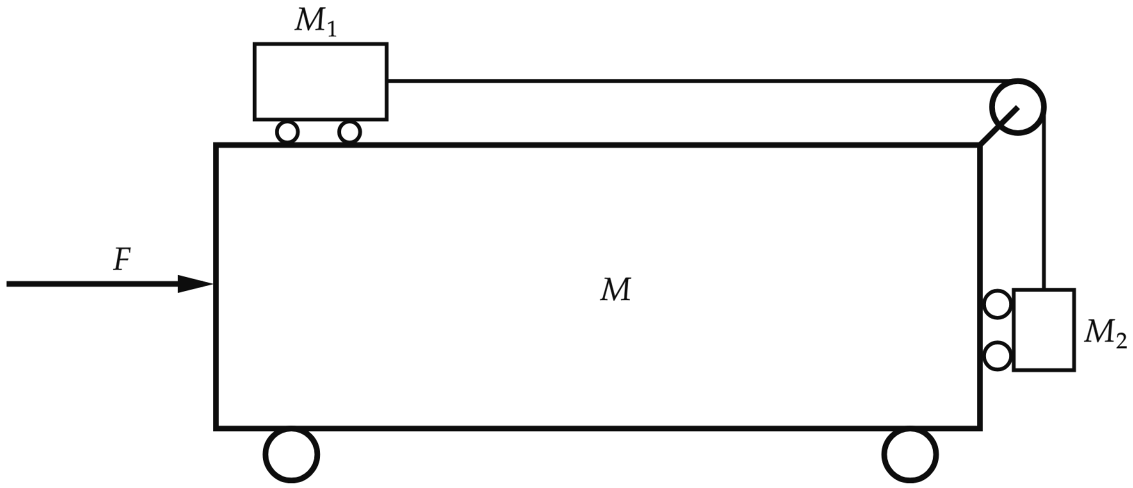


Figure 5-3

Ans:

Since M_2 is relatively static to M, M_2 is static in the vertical direction. Let's say the tension of the string is T, then $M_2 g = T$. In addition, $T = M_1 a$ where a is the acceleration of the M_1. Similarly, M_1 is relatively static to M, that is

$$F - M_2 a - T = Ma$$

which implies that

$$\begin{aligned} F &= Ma + M_2 a + T \\ &= (M + M_2)\frac{M_2 g}{M_1} + M_2 g \\ &= \frac{M_2}{M_1}(M + M_1 + M_2)g \end{aligned}$$

5.5

Referring to Fig. 5-4: what horizontal force F must be constantly applied to $M = 21\,\mathrm{kg}$ so that $m_1 = 5\,\mathrm{kg}$ does not move relative to $m_2 = 4\,\mathrm{kg}$. Neglect friction.

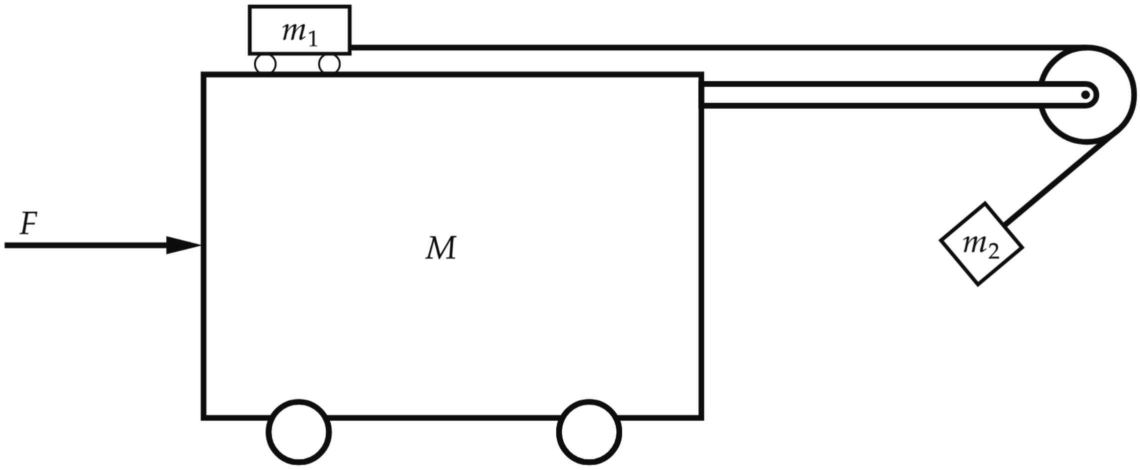


Figure 5-4

Ans:
Lets say the tension of the string is T and the angle of the string on m_2 is θ with the horizontal. m_1 does not move relative to m_2 means that they have the same acceleration, and so do M. Assume that they accelerate in acceleration a, Then

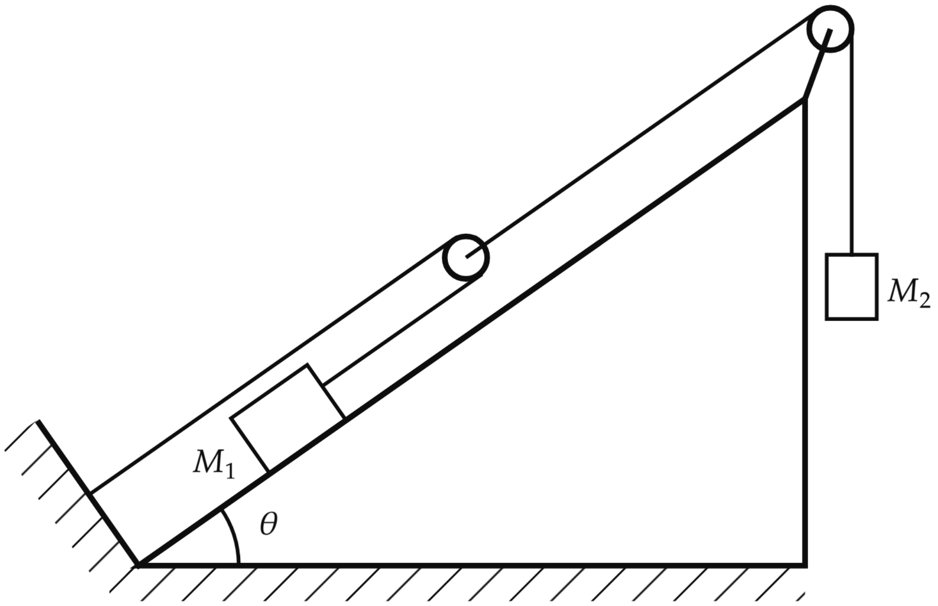
$$\begin{cases} m_2 g = T\sin\theta \\ m_2 a = T\cos\theta \end{cases}$$
$$m_1 a = T$$
$$Ma = F - T - T\cos\theta$$

Hence, $\cos\theta = m_2/m_1$ and
$T = m_2 g/\sqrt{1-(m_2/m_1)^2} = m_1 m_2 g/\sqrt{m_1^2 - m_2^2}$. Thus,

$$\begin{aligned}
\frac{m_1}{M} &= \frac{T}{F - T - Tm_2/m_1} \\
\Rightarrow \quad F &= \frac{M}{m_1}T\left(1 + \frac{m_1}{M} + \frac{m_2}{M}\right) \\
\Rightarrow \quad &= \frac{Mm_2 g(1 + (m_1+m_2)/M)}{\sqrt{m_1^2 - m_2^2}} \\
\Rightarrow \quad &= \frac{(Mm_2 + (m_1+m_2)m_2)g}{\sqrt{m_1^2 - m_2^2}} \\
\Rightarrow \quad &= \frac{(21 \times 4 \times +(5+4) \times 4) \times 9.81}{\sqrt{5^2 - 4^2}} \\
\Rightarrow \quad &= 392.4\ \mathrm{N} \\
\Rightarrow \quad &\approx 392\ \mathrm{N}
\end{aligned}$$

5.6

In the system shown in Fig. 5-5, M_1 slides without friction on the inclined plane. $\theta = 30°$, $M_1 = 400$ g, $M_2 = 200$ g. Find the acceleration a of M_2 and the tension T in the cords.



Ans:

Assume that the acceleration of M_2 is a, then

$$M_2 g - T = M_2 a$$
$$\frac{T}{2} - M_1 g \sin\theta = M_1 2a$$

Hence, $M_2 g - 2M_1 g \sin\theta = M_2 a + 4M_1 a$, that is,

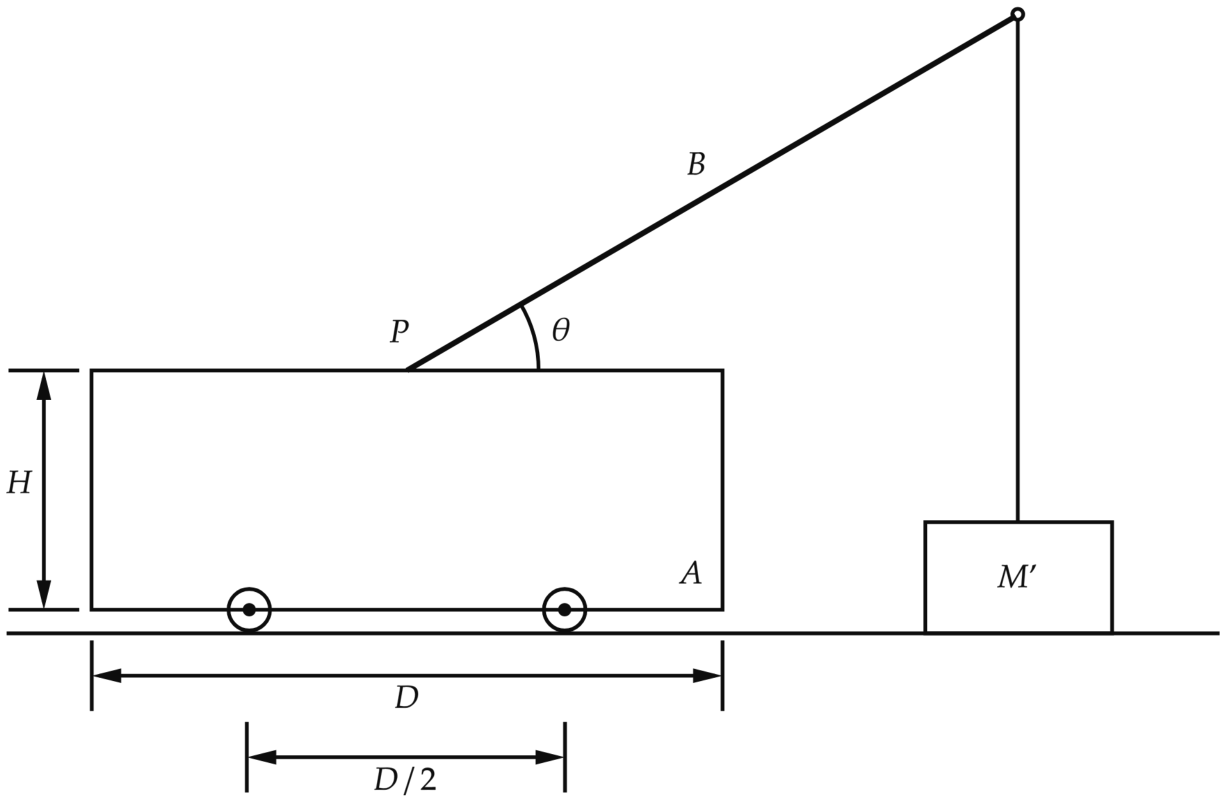
$$\begin{aligned} a &= \frac{(M_2 - 2M_1 \sin\theta) g}{M_2 + 4M_1} \\ &= \frac{200 - 2 \times 400 \times \frac{1}{2}}{200 + 4 \times 400} g \\ &= -\frac{1}{9} g \end{aligned}$$

Therefore, the acceleration of M_2 is $\frac{g}{9}$ upward. And $T = M_2 \times \frac{10}{9} g = 222.2g$ g-wt.

5.7

A simple crane is made of two parts, "A" with mass M_A, length D, height H, and distance $D/2$ between wheels of radius r; and part "

B", a uniform rod or boom of length L and mass M_B. The crane is shown assembled in Fig. 5-6, with the pivot point P at midpoint of top of A. The center of gravity of A is midway between the wheels.



1. With the rod or boom B set at angle θ with the horizontal, what is the maximum mass M_{max} that the crane can lift without tipping over?
2. If there is a mass $M' = (4/5)M_{max}$ at the end of the rope, what is the minimum time t necessary to raise this load M' a distance $(L\sin\theta)$ from the ground? (The angle θ remains fixed, and the mass of the rope may be neglected.)

Ans:

1. Assume that the normal force the ground acting on the left wheel is W, then the normal force action on the other wheel is $(M_A + M_B + M')g - W$ Where $W \in [0,\, M_A + M_B + M']$. Set a fulcrum at P. Since the system is balanced, the net torque in

P's point of view should equal zero. That is,

$$W\frac{D}{4} + M_B g\frac{L}{2}\cos\theta + M'gL\cos\theta = ((M_A + M_B + M')g - W)\frac{D}{4}$$

And

$$\begin{aligned} M' &= \frac{\left(M_A g\frac{D}{4} + M_B g\left(\frac{D}{4} - \frac{L}{2}\cos\theta\right) - W\frac{D}{2}\right)}{g\left(L\cos\theta - \frac{D}{4}\right)} \\ &= \frac{(M_A + M_B)D - 2M_B L\cos\theta}{4L\cos\theta - D} - \frac{W\frac{D}{w}}{g\left(L\cos\theta - \frac{D}{4}\right)} \end{aligned}$$

We can see that M' has its maximum when $W = 0$. Therefore,

$$M'_{max} = \frac{(M_A + M_B)D - 2M_B L\cos\theta}{4L\cos\theta - D}$$

2. M' has its maximum mass M'_{max} can be interpreted as the rope have its maximum tension $M'_{max}g$. For having the minimum time t, we want to raise the load with the maximum tension $M'_{max}g$. Thus the net force acting on M' is $M'_{max}g - \frac{4}{5}M'_{max}g = \frac{1}{5}M'_{max}g$. By Newton's second law of dynamics, $\frac{1}{5}M'_{max}g = M'_{max}a$, which implies that the acceleration $a = \frac{g}{5}$. By kinematics, $L\sin\theta = \frac{1}{2}at^2$. So $t = \sqrt{\frac{10L\sin\theta}{g}}$

5.8

**An early arrangement for measuring the acceleration of gravity, called At-wood's Machine, is shown in Fig.5-7. The pulley P and cord C have negligible mass and fiction. The system is balanced with equal masses M on each side as shown (solid line), and then a small rider m is added to one side. The combined masses accelerate through a certain distance h, the rider is caught on a ring and the two equal masses then move on with constant speed, v . find the value of g that correspoonds to the measured values of m

, M, h, and v.

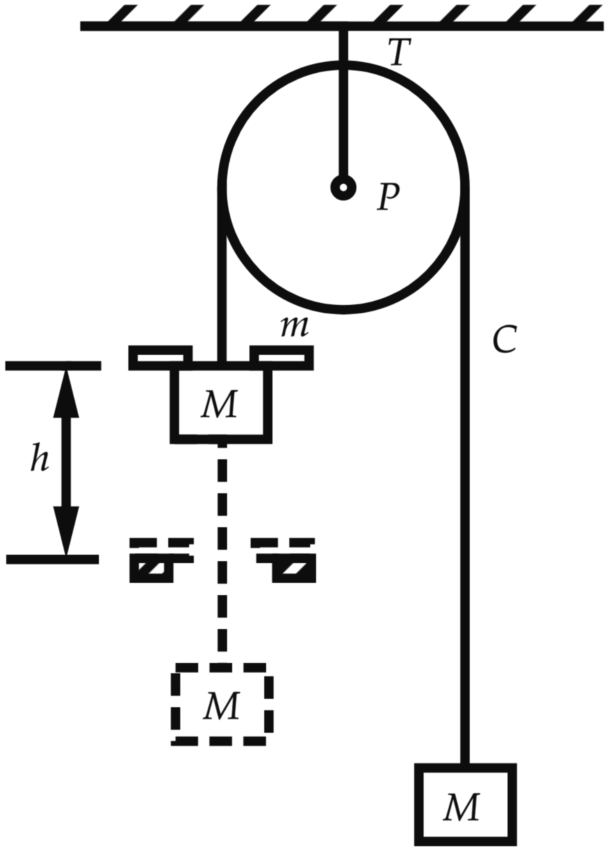


Fig. 5-7

Ans:

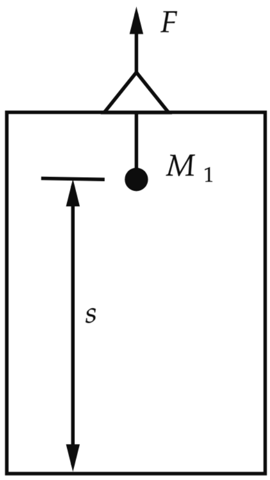
After adding the rider m, the net force is mg downward with respect to the left M. By Newton's second law of dynamics, acceleration is $a = \frac{m}{2M+m} g$. Using kinematics, we know that $v^2 = 0^2 + 2\frac{m}{2M+m} gh$. That is, $g = \frac{(2M+m)v^2}{2mh}$.

5.9

An elevator of mass M_2 has hanging from its ceiling a mass M_1, as shown in Fig. 5-8. The elevator is being accelerated upward by a constant force F. (F is greater than $(M_1 + M_2)$g.) The mass M_1 is initially a distance s above the elevator floor.

1. **Find the acceleration a_0 of the elevator.**
2. **What is the tension T in the string connecting the mass M_1 to the elevator?**

3. **If the string suddenly breaks, what is the acceleration a of the elevator immediately after, and what is the acceleration a' of mass M_1?**
4. **How much time t does it take for M_1 to hit the bottom of the elevator?**



Ans:

1. By Newton's second law of dynamics, $(M_1+M_2)a_0 = F-(M_1+M_2)g$. That is, $a_0 = \frac{F}{M_1+M_2} - g$.
2. The tension $T - M_1 g = M_1 a_0 = \frac{M_1 F}{M_1+M_2} - M_1 g$. So $T = \frac{M_1 F}{M_1+M_2}$.
3. By Newton's second law of dynamics, $a = \left(\frac{F}{M_2} - g\right)$. And $a' = -g$.
4. Set our view inside the elevator, then the net acceleration is $a' - a = -\frac{F}{M_2}$. Hence, applying formula from kinematics, we have $-s = \frac{1}{2}\left(-\frac{F}{M_2}t^2\right)$, which is $t = \sqrt{\frac{2M_2 s}{F}}$.

5.10

Given the system shown in Fig. 5-9, consider all surfaces frictionless. If $m = 150\,\mathrm{g}$ is released when it is $d = 1\,\mathrm{m}$ above the base of $M = 1650\,\mathrm{g}$, how long after release, Δt, will m strike the base of M?

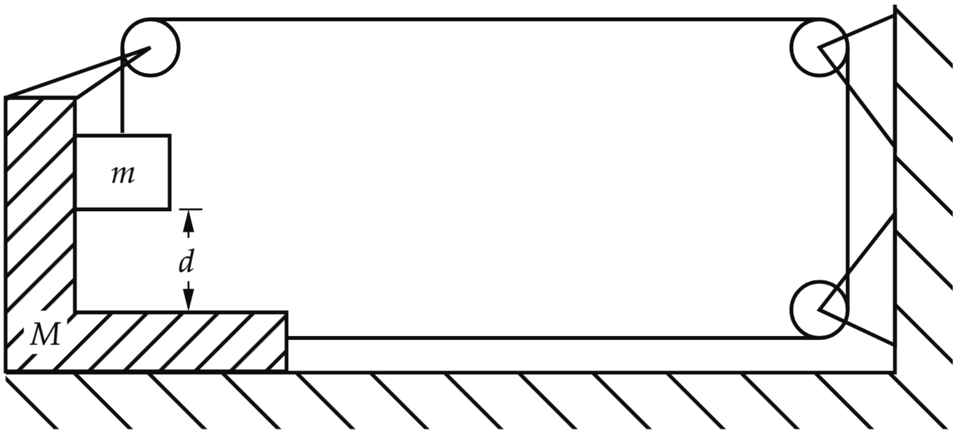


Figure 5-9

(unsolved)

Ans:

We analyzing the free-body diagram of m. In the vertical direction of m, it has

$$mg - T = ma$$

In the horizontal direction, $F = m\frac{a}{2}$. For M, $T - F = M\frac{a}{2}$. Hence

$$mg - \left(M\frac{a}{2} + m\frac{a}{2}\right) = ma$$

That is,

$$mg = \frac{1}{2}(M + 3m)a$$

and

$$a = \frac{2mg}{M + 3m}$$

Applying kinematics, we have

$$\frac{1}{2}a(\Delta t)^2 = d$$

Therefore, $\Delta t = \sqrt{\frac{2d}{a}} = \sqrt{\frac{(M+3m)d}{mg}} \approx 1.19$.

5.11

None of the identical gondolas on the Martian canal Rimini is quite

able to support the load of both Paolo and Francesca, two affectionate marsupials who refuse to go in separate boats. The enterprising gondolier, Giuseppe, collects their fare by rigging them up from the mast as shown in Fig. 5-10, using the massless ropes and massless, frictionless pulleys characteristic of Martian construction. Giuseppe ferries them across them before they hit either the mast of the deck. Assuming Paulo's mass is $90\,\mathrm{kg}$ and Francesca's is $60\,\mathrm{kg}$, how much load W does Giuseppe save?

hint: Remember that the tension in a massless cord that passes over a massless, frictionless pulley is the same on both sides of the pulley.

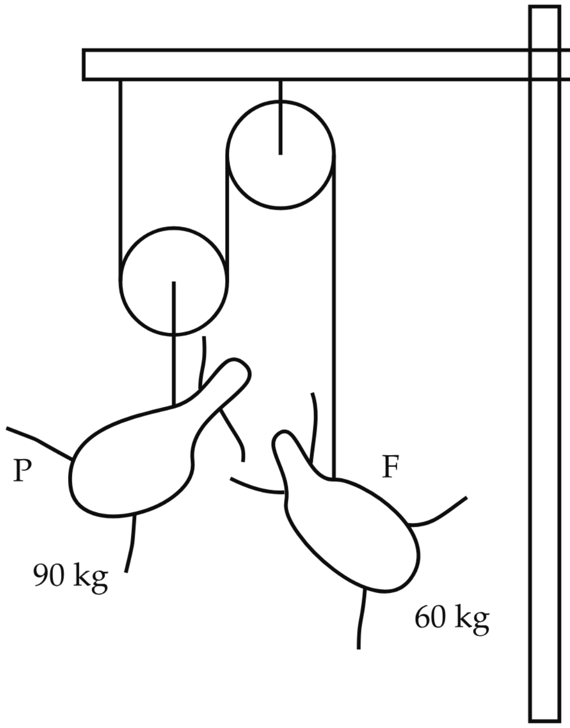


Figure 5 – 10

Ans:

Assume that the tension in the cord is T. Analyzing the free-body diagram of Paolo and Francesca, we have

$$2T - 90g = 90\frac{a}{2}$$
$$T - 60g = 60(-a)$$

Therefore, $a = \frac{2g}{11}$. And $T = \frac{540}{11}g$. The the weight of the system is $3T = \frac{1620}{11}g$. Thus, Giuseppe had saved $90 + 60 - \frac{1620}{11} \approx 2.7\,\mathrm{kg} - \mathrm{wt}$.

5.12

A painter working from a "bosun's" chair is hung down the side of a tall building, as shown in Fig. 5-11. Wishing to move in a hurry, the $180\,\mathrm{lb}$ painter pulls down on the fall rope so hard that he presses against the chair with a force of only $100\,\mathrm{lb}$. The chair itself weighs $30.0\,\mathrm{lb}$.

1. **What is the acceleration a of the painter and the chair?**
2. **What is the total force F supported by the pulley?**

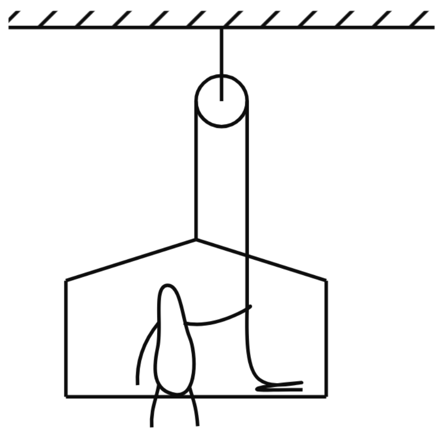


Figure 5 - 11

Ans:

Assume that the painter pull the rope with a force T.

3. By analyzing the free-body diagram of the painter, we have

$$180g - 100g - T = 180a$$

For the chair,

$$30g + 100g - T = 30a$$

Combining the two equation, we have

$$a = -\frac{g}{3}$$

That is, **a** is equal to $\frac{g}{3}$ upward.

4. From the equation above, $T = 140g$. The pulley supports $2T = 280g$.

5.13

A space traveler about to leave for the moon has a spring balance and a $1.0\,\mathrm{kg}$ mass A, which when hung on the balance on the Earth gives the reading of $9.8\,\mathrm{N}$. Arriving at the moon at a place where the acceleration of gravity is not known exactly but has a value of about one sixth the acceleration of gravity at the Earth's surface, he picks up a stone B which gives a reading of $9.8\,\mathrm{N}$ when weighed on the spring balance. He then hangs A and B over a pulley as shown in Fig. 5-12 and observes that B falls with an acceleration of $1.2\,\mathrm{ms}^{-2}$. What is the mass m_B of the stone B?

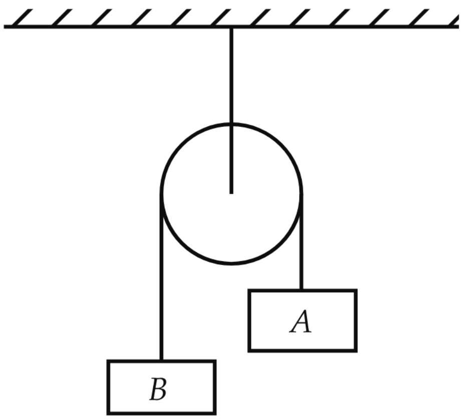


Figure 5 - 12

Ans:
Let the gravity on the moon be g'.
From the things the space traveler had done with the spring, we know that $1.0g = 9.8\,\mathrm{N}$ and $m_B g' = 9.8\,\mathrm{N}$, which is, $m_B = \frac{g}{g'}\,\mathrm{kg}$.

From Figure 5-12, by Newton's law of dynamics, we have

$$m_B g' - m_A g' = 1.2(m_A + m_B)$$

That is,

$$
\begin{aligned}
g' &= \frac{1.2\left(1+\frac{g}{g'}\right)}{\frac{g}{g'}-1} \\
\Rightarrow \qquad g-g' &= 1.2+\frac{1.2g}{g'} \\
\Rightarrow \qquad 0 &= (g')^2-(g-1.2)g'+1.2g
\end{aligned}
$$

And

$$
\begin{aligned}
g' &= \frac{g-1.2\pm\sqrt{(g-1.2)^2-4.8g}}{2} \\
&\approx 6.9 \text{ or } 1.7\,\mathrm{ms}^{-2}
\end{aligned}
$$

Since g' is about one sixth of g, $g'=1.7\,\mathrm{ms}^{-2}$.
Therefore, $m_B=\frac{g}{g'}=\frac{9.8}{1.7}\approx 5.8\,\mathrm{kg}$.

5.14

Use numerical methods to solve the following exercises.

A mass suspended from a spring hangs motionless, and is then given an upward blow such that it moves initially at unit speed. If the mass and spring constant are such that the equation of motion is $\ddot{x}=-x$, find the maximum height $x_{\max}$ attained by numerical integration of the equation of motion.

Ans:
It is a two-dimensional phase space with initial states

$$
\begin{aligned}
x(0) &= 0 \\
\dot{x}(0) &= 1
\end{aligned}
$$

The motion of the system is given by

$$
\begin{aligned}
x_{i+1} &= x_i+\dot{x}_i\Delta t \\
\dot{x}_{i+1} &= \dot{x}_i-x_i\Delta t
\end{aligned}
$$

Let $X_i = \begin{bmatrix} x_i \\ \dot{x}_i \end{bmatrix}$. Then $X_0 = \begin{bmatrix} x_0 \\ \dot{x}_0 \end{bmatrix} = \begin{bmatrix} x(0) \\ \dot{x}(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$X_{i+1} = \begin{bmatrix} x_{i+1} \\ \dot{x}_{i+1} \end{bmatrix} = \begin{bmatrix} 1 & \Delta t \\ -\Delta t & 1 \end{bmatrix} \begin{bmatrix} x_i \\ \dot{x}_i \end{bmatrix} = \begin{bmatrix} 1 & \Delta t \\ -\Delta t & 1 \end{bmatrix} X_i$$

It is easy to see that

$$X_n = \begin{bmatrix} 1 & \Delta t \\ -\Delta t & 1 \end{bmatrix}^n X_0$$

We diagonalize the matrix.

The characteristic polynomial of $\begin{bmatrix} 1 & \Delta t \\ -\Delta t & 1 \end{bmatrix}$ is

$$\det(\begin{bmatrix} 1-\lambda & \Delta t \\ -\Delta t & 1-\lambda \end{bmatrix}) = (1-\lambda)^2 + (\Delta t)^2$$

And

$$\begin{aligned} & (1-\lambda)^2 + (\Delta t)^2 = 0 \\ \Rightarrow \quad & \lambda^2 - 2\lambda + 1 + (\Delta t)^2 = 0 \\ \Rightarrow \quad & \lambda = \frac{2 \pm \sqrt{4 - 4 - 4(\Delta t)^2}}{2} \\ \Rightarrow \quad & \lambda = 1 \pm (\Delta t)i \end{aligned}$$

If $\lambda = 1 - (\Delta t)i$, then

$$\begin{aligned} \ker(\begin{bmatrix} 1 & \Delta t \\ -\Delta t & 1 \end{bmatrix} - \lambda I) &= \ker(\begin{bmatrix} 1-(1-(\Delta t)i) & \Delta t \\ -\Delta t & 1-(1-(\Delta t)i) \end{bmatrix}) \\ &= \ker(\begin{bmatrix} (\Delta t)i & \Delta t \\ -\Delta t & (\Delta t)i \end{bmatrix}) \\ &= \ker(\begin{bmatrix} (\Delta t)i & \Delta t \\ 0 & 0 \end{bmatrix}) \\ &= \operatorname{span}(\begin{bmatrix} i \\ 1 \end{bmatrix}) \end{aligned}$$

If $\lambda = 1 + (\Delta t)i$, then

$$\begin{aligned}
\ker(\begin{bmatrix} 1 & \Delta t \\ -\Delta t & 1 \end{bmatrix} - \lambda I) &= \ker(\begin{bmatrix} 1-(1+(\Delta t)i) & \Delta t \\ -\Delta t & 1-(1+(\Delta t)i) \end{bmatrix}) \\
&= \ker(\begin{bmatrix} -(\Delta t)i & \Delta t \\ -\Delta t & -(\Delta t)i \end{bmatrix}) \\
&= \ker(\begin{bmatrix} -(\Delta t)i & \Delta t \\ 0 & 0 \end{bmatrix}) \\
&= \operatorname{span}(\begin{bmatrix} -i \\ 1 \end{bmatrix})
\end{aligned}$$

Let $P = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} -\frac{i}{2} & \frac{1}{2} \\ \frac{i}{2} & \frac{1}{2} \end{bmatrix}$.

Hence,

$$\begin{bmatrix} 1-(\Delta t)i & 0 \\ 0 & 1+(\Delta t)i \end{bmatrix} = P^{-1} \begin{bmatrix} 1 & \Delta t \\ -\Delta t & 1 \end{bmatrix} P$$

And

$$\begin{aligned}
X_n &= \begin{bmatrix} 1 & \Delta t \\ -\Delta t & 1 \end{bmatrix}^n X_i \\
&= \left(P \begin{bmatrix} 1-(\Delta t)i & 0 \\ 0 & 1+(\Delta t)i \end{bmatrix} P^{-1} \right)^n X_0 \\
&= P \begin{bmatrix} 1-(\Delta t)i & 0 \\ 0 & 1+(\Delta t)i \end{bmatrix}^n P^{-1} X_0 \\
&= P \begin{bmatrix} (1-(\Delta t)i)^n & 0 \\ 0 & (1+(\Delta t)i)^n \end{bmatrix} P^{-1} X_0 \\
&= \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} (1-(\Delta t)i)^n & 0 \\ 0 & (1+(\Delta t)i)^n \end{bmatrix} \begin{bmatrix} -\frac{i}{2} & \frac{1}{2} \\ \frac{i}{2} & \frac{1}{2} \end{bmatrix} X_0
\end{aligned}$$

We want to calculate $X(t)$. Using the sum over $[t, 0]$ with partition $0 = t_0 < t_1 < t_2 < \cdots < t_n = t$. We set $t_{i+1} - t_i = \Delta t$ for simplicity. Then $t_n = n\Delta t$ and $\Delta t = \frac{t_n}{n}$. Hence,

$$X_n(t_n) = \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \left(1-\left(\frac{t_n}{n}\right)i\right)^n & 0 \\ 0 & \left(1+\left(\frac{t_n}{n}\right)i\right)^n \end{bmatrix} \begin{bmatrix} -\frac{i}{2} & \frac{1}{2} \\ \frac{i}{2} & \frac{1}{2} \end{bmatrix} X_0$$

We expect that $\lim_{n\to\infty} X_n(t_n) = X(t)$. And

$$\lim_{n\to\infty} X_n(t_n) = \lim_{n\to\infty} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \left(1-\left(\frac{t_n}{n}\right)i\right)^n & 0 \\ 0 & \left(1+\left(\frac{t_n}{n}\right)i\right)^n \end{bmatrix} \begin{bmatrix} -\frac{i}{2} & \frac{1}{2} \\ \frac{i}{2} & \frac{1}{2} \end{bmatrix} X_0$$

$$= \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lim_{n\to\infty} \left(1-\left(\frac{t_n}{n}\right)i\right)^n & 0 \\ 0 & \lim_{n\to\infty} \left(1+\left(\frac{t_n}{n}\right)i\right)^n \end{bmatrix} \begin{bmatrix} -\frac{i}{2} & \frac{1}{2} \\ \frac{i}{2} & \frac{1}{2} \end{bmatrix} X_0$$

$$= \lim_{n\to\infty} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} e^{-it_n} & 0 \\ 0 & e^{it_n} \end{bmatrix} \begin{bmatrix} -\frac{i}{2} & \frac{1}{2} \\ \frac{i}{2} & \frac{1}{2} \end{bmatrix} X_0$$

(Applying Euler's Formula)

$$= \lim_{n\to\infty} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \cos t_n - i\sin t_n & 0 \\ 0 & \cos t_n + i\sin t_n \end{bmatrix} \begin{bmatrix} -\frac{i}{2} & \frac{1}{2} \\ \frac{i}{2} & \frac{1}{2} \end{bmatrix} X_0$$

$$\left(X_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right)$$

$$= \lim_{n\to\infty} \begin{bmatrix} i & -i \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \cos t_n - i\sin t_n & 0 \\ 0 & \cos t_n + i\sin t_n \end{bmatrix} \begin{bmatrix} -\frac{i}{2} & \frac{1}{2} \\ \frac{i}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \lim_{n\to\infty} \begin{bmatrix} i\cos t_n + \sin t_n & -i\cos t_n + \sin t_n \\ \cos t_n - i\sin t_n & \cos t_n + i\sin t_n \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$= \lim_{n\to\infty} \begin{bmatrix} \sin t_n \\ \cos t_n \end{bmatrix}$$

Finally, we have

$$\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} = X(t) = \lim_{n\to\infty} X_n(t_n) = \lim_{n\to\infty} \begin{bmatrix} \sin t_n \\ \cos t_n \end{bmatrix} = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$$

Hence, the maximum height $x_{\max}$ happens when $t = \frac{\pi(1+4k)}{2}$ where $k \in \mathbb{Z}$. Which is $x_{\max} = 1$.

5.15

A particle of mass m moves along a straight line. Its motion is resisted by a force proportional to its velocity, $F = -kv$. It starts with speed $v = v_0$ at $x = 0$ and $t = 0$.

1. **Find x as a function of t by numerical integration.**

2. **Find the time $t_{\frac{1}{2}}$ required to lose half its speed, and the maximum distance $x_{\max}$ attained.**

Notes:

1. **Adjust the scales of x and t so that the equation of motion has simple numerical coefficients.**
2. **Invent a scheme to attain good accuracy with a relatively coarse interval for Δt**
3. **Use dimensional analysis to deduce how $t_{\frac{1}{2}}$ and $x_{\max}$ should depend upon v_0, k and m, and solve for the actual motion only for a single convenient value of v_0, say $v_0 = 1.00$ (in the modified x and t units)**

Ans:
This is a second order ordinary differential equation $m\ddot{x} = -k\dot{x}$ with two-dimensional phase space.
We rescale t and x by substitute

$$\tilde{t} = c_1 t, \quad \tilde{x} = c_2 x$$

Then

$$\frac{d^2\tilde{x}}{d\tilde{t}^2} = \frac{d}{dt}\left(\frac{d\tilde{x}}{dx}\frac{dx}{dt}\frac{dt}{d\tilde{t}}\right)\frac{dt}{d\tilde{t}} = \frac{c_2}{c_1^2}\left(\frac{d^2x}{dt^2}\right) = \frac{c_2}{c_1^2}\left(-k\frac{dx}{dt}\right) = \frac{c_2}{c_1^2}(-k)\frac{c_1}{c_2}\frac{d\tilde{x}}{d\tilde{t}} =$$

Note that chain rule was used.
Let $c_1 = k$, $c_2 = 1$. Then we have a rescaled differential equation.

$$\ddot{\tilde{x}} = -\dot{\tilde{x}}$$

The initial states of this system is

$$\tilde{X}_0 = \begin{bmatrix}\tilde{x}_0\\ \dot{\tilde{x}}_0\end{bmatrix} = \begin{bmatrix}\tilde{x}(0)\\ \dot{\tilde{x}}(0)\end{bmatrix} = \begin{bmatrix}0\\ \tilde{v}_0\end{bmatrix}$$

We apply the numerical method using numerical integration.

$$\tilde{X}_{i+1} = \begin{bmatrix}\tilde{x}_{i+1} \dot{\tilde{x}}_{i+1}\end{bmatrix} = \begin{bmatrix}\tilde{x}_i + \dot{\tilde{x}}_i \Delta t \\ \dot{\tilde{x}}_i - \dot{\tilde{x}} \Delta t\end{bmatrix} = \begin{bmatrix}1 & \Delta t \\ 0 & (1-\Delta t)\end{bmatrix} \begin{bmatrix}\tilde{x}_i \\ \dot{\tilde{x}}_i\end{bmatrix} = \begin{bmatrix}1 & \Delta t \\ 0 & (1-\Delta t)\end{bmatrix} \tilde{X}_i$$

It is easy to see that

$$\tilde{X}_n = \begin{bmatrix}1 & \Delta t \\ 0 & (1-\Delta t)\end{bmatrix}^n \tilde{X}_0$$

Hence, it is reasonable for us to diagonalize $A := \begin{bmatrix}1 & \Delta t \\ 0 & (1-\Delta t)\end{bmatrix}$.

The characteristic polynomial of A is

$$P_A(\lambda) = (1-\lambda)(1-\Delta t - \lambda)$$

$P_A(\lambda) = 0$ when $\lambda = 1$ or $\lambda = 1 - \Delta t$.

When $\lambda = 1$:

The eigenspace with respect to λ is $E_\lambda = \text{span}(\begin{bmatrix}1 \\ 0\end{bmatrix})$

When $\lambda = 1 - \Delta t$:

The eigenspace with respect to λ is $E_\lambda = \text{span}(\begin{bmatrix}1 \\ -1\end{bmatrix})$

Let $P = \begin{bmatrix}1 & 1 \\ 0 & -1\end{bmatrix}$. Then $P^{-1} = \begin{bmatrix}1 & 1 \\ 0 & -1\end{bmatrix}$. And

$$P^{-1}AP = \begin{bmatrix}1 & 0 \\ 0 & 1-\Delta t\end{bmatrix}$$

Which is,

$$A = P \begin{bmatrix}1 & 0 \\ 0 & 1-\Delta t\end{bmatrix} P^{-1}$$

Consider the partition $0 = t_0 < t_1 < t_2 < \cdots < t_n = t$ with $t_{i+1} - t_i = \Delta t$, $\forall i$ for simplicity. So $\Delta t = \frac{t_n}{n}$ We expect that

$\tilde{X}(\tilde{t}) = \lim_{n\to\infty} \tilde{X}_n(t_n)$. That is

$$\begin{aligned}
\lim_{n\to\infty} \tilde{X}_n(t_n) &= \lim_{n\to\infty} \begin{bmatrix} 1 & \Delta t \\ 0 & 1-\Delta t \end{bmatrix}^n \tilde{X}_0 \\
&= \lim_{n\to\infty} \left(P \begin{bmatrix} 1 & 0 \\ 0 & 1-\Delta t \end{bmatrix} P^{-1} \right)^n \begin{bmatrix} \tilde{x}_0 \\ \dot{\tilde{x}}_0 \end{bmatrix} \\
&= \lim_{n\to\infty} P \begin{bmatrix} 1^n & 0 \\ 0 & (1-\Delta t)^n \end{bmatrix} P^{-1} \begin{bmatrix} 0 \\ \tilde{v}_0 \end{bmatrix} \\
&= P \begin{bmatrix} 1 & 0 \\ 0 & \lim_{n\to\infty} \left(1 - \frac{t_n}{n}\right)^n \end{bmatrix} P^{-1} \begin{bmatrix} 0 \\ \tilde{v}_0 \end{bmatrix} \\
&= \lim_{n\to\infty} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^{-t_n} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ \tilde{v}_0 \end{bmatrix} \\
&= \lim_{n\to\infty} \begin{bmatrix} 1 & e^{-t_n} \\ 0 & -e^{-t_n} \end{bmatrix} \begin{bmatrix} \tilde{v}_0 \\ -\tilde{v}_0 \end{bmatrix} \\
&= \lim_{n\to\infty} \begin{bmatrix} \tilde{v}_0 - \tilde{v}_0 e^{-t_n} \\ \tilde{v}_0 e^{-t_n} \end{bmatrix}
\end{aligned}$$

Therefore,

$$\tilde{X}(t) = \begin{bmatrix} \tilde{x}(t) \\ \dot{\tilde{x}}(t) \end{bmatrix} = \lim_{n\to\infty} \tilde{X}_n(t_n) = \lim_{n\to\infty} \begin{bmatrix} \tilde{v}_0 - \tilde{v}_0 e^{-t_n} \\ \tilde{v}_0 e^{-t_n} \end{bmatrix} = \begin{bmatrix} \tilde{v}_0 - \tilde{v}_0 e^{-\tilde{t}} \\ \tilde{v}_0 e^{-\tilde{t}} \end{bmatrix}$$

1. For $x(t)$, we have

$$\begin{aligned}
& \tilde{x}(t) = \tilde{v}_0 - \tilde{v}_0 e^{-\tilde{t}} \\
\Rightarrow \quad & c_2 x(t) = \frac{c_2}{c_1} v_0 \left(1 - e^{-\tilde{t}/c_1}\right) \\
\Rightarrow \quad & x(t) = \frac{v_0}{c_1} \left(1 - e^{-\tilde{t}/c_1}\right) \\
& = \frac{v_0}{k} \left(1 - e^{-t/k}\right)
\end{aligned}$$

2. For $v(t) = \dot{x}(t)$, we have

$$\begin{aligned}
& \dot{\tilde{x}}(t) = \tilde{v}_0 e^{-\tilde{t}} \\
\Rightarrow \quad & \frac{c_2}{c_1} \dot{x}(t) = \frac{c_2}{c_1} v_0 e^{-\tilde{t}/c_1} \\
\Rightarrow \quad & \dot{x}(t) = v_0 e^{-t/k}
\end{aligned}$$

$\frac{v_0}{2} = v_0 e^{-t/k} \Rightarrow t = k \ln 2$. Hence, it requires $t_{\frac{1}{2}} = k \ln 2$ to lose half its speed. From 1., it is easy to see that $x_{\max}$ happens as $t \to \infty$, which is, $\lim_{t\to\infty} \frac{v_0}{k}(1 - e^{-t/k}) = \frac{v_0}{k}$.

We can use the half-way(middle point) of x_i and $\dot{x}_i$, i.e., $\dot{x}_{i+1_{\text{mid}}} = \dot{x}_i - \dot{x}_i \frac{\Delta t}{2}$, $x_{i+1} = x_i + \dot{x}_{i+1_{\text{mid}}} \Delta t$, to attain a better accuracy.

For the units-modified value $\tilde{X}$ and $\tilde{t}$, we take $\tilde{v}_0 = 1.00$, then we have the equation of the motion is

$$\tilde{x}(t) = 1 - e^{-t}$$
$$\dot{\tilde{x}}(t) = e^{-t}$$

5.16

A certain charged particle moves in an electric and a magnetic filed according to the equations,

$$\frac{dv_x}{dt} = -2v_y,$$
$$\frac{dv_y}{dt} = 1 + 2v_x.$$

At $t = 0$ the particle starts at $x = 0$, $y = 0$ with velocity $v_x = 1.00$, $v_y = 0$. Determine the nature of the motion by numerical integration.

Ans:
This is a second order ordinary differential equations system

$$\ddot{x} = -2\dot{y}$$
$$\ddot{y} = 1 + 2\dot{x}$$

with a four-dimensional phase space $(x, y, \dot{x}, \dot{y})$. We rescale and substitute

$$\tilde{t} = c_1 t, \quad \tilde{x} = c_2 x, \quad \tilde{y} = c_3 y$$

So

$$\frac{d^2\tilde{x}}{d\tilde{t}^2} = \frac{d}{dt}\left(\frac{d\tilde{x}}{dx}\frac{dx}{dt}\frac{dt}{d\tilde{t}}\right)\frac{dt}{d\tilde{t}} = \frac{d\tilde{x}}{dx}\left(\frac{dt}{d\tilde{t}}\right)^2\left(-2\frac{d\tilde{y}}{d\tilde{t}}\frac{c_1}{c_3}\right) = -2\frac{c_2}{c_1c_3}\frac{d\tilde{y}}{d\tilde{t}}$$
$$\frac{d^2\tilde{y}}{d\tilde{t}^2} = \frac{d}{dt}\left(\frac{d\tilde{y}}{dy}\frac{dy}{dt}\frac{dt}{d\tilde{t}}\right)\frac{dt}{d\tilde{t}} = \frac{d\tilde{y}}{dy}\left(\frac{dt}{d\tilde{t}}\right)^2\left(1+2\frac{d\tilde{x}}{d\tilde{t}}\frac{c_1}{c_2}\right) = \frac{c_3}{c_1^2} + 2\frac{c_3}{c_1c_2}\frac{d\tilde{x}}{d\tilde{t}}$$

Let $\frac{c_2}{c_1c_3} = \frac{1}{2}$, $\frac{c_3}{c_1c_2} = \frac{1}{2}$ and $\frac{c_3}{c_1^2} = 1$. Then $c_1 = 2,\ c_2 = 4\ c_3 = 4$.
The new differential equations system is

$$\ddot{\tilde{x}} = -\dot{\tilde{y}}$$
$$\ddot{\tilde{y}} = 1 + \dot{\tilde{x}}$$

And

$$\begin{bmatrix} \tilde{x}_0 \\ \tilde{y}_0 \\ \dot{\tilde{x}}(0) \\ \dot{\tilde{y}}(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

The numerical integration is given by

$$\begin{bmatrix} \tilde{x}_{i+1} \\ \tilde{y}_{i+1} \\ \dot{\tilde{x}}_{i+1} \\ \dot{\tilde{y}}_{i+1} \end{bmatrix} = \begin{bmatrix} \tilde{x}_i + \dot{\tilde{x}}_i\Delta t \\ \tilde{y}_i + \dot{\tilde{y}}_i\Delta t \\ \dot{\tilde{x}}_i - \dot{\tilde{y}}_i\Delta t \\ \dot{\tilde{y}}_i + (1+\dot{\tilde{x}}_i)\Delta t \end{bmatrix} = \begin{bmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & -\Delta t \\ 0 & 0 & \Delta t & 1 \end{bmatrix} \begin{bmatrix} \tilde{x}_i \\ \tilde{y}_i \\ \dot{\tilde{x}}_i \\ \dot{\tilde{y}}_i \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \Delta t \end{bmatrix}$$

It is easy to see that

$$\begin{bmatrix} \tilde{x}_n \\ \tilde{y}_n \\ \dot{\tilde{x}}_n \\ \dot{\tilde{y}}_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & -\Delta t \\ 0 & 0 & \Delta t & 1 \end{bmatrix}^n \begin{bmatrix} \tilde{x}_0 \\ \tilde{y}_0 \\ \dot{\tilde{x}}_0 \\ \dot{\tilde{y}}_0 \end{bmatrix} + \sum_{k=0}^{n} \begin{bmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & -\Delta t \\ 0 & 0 & \Delta t & 1 \end{bmatrix}^k \begin{bmatrix} 0 \\ 0 \\ 0 \\ \Delta t \end{bmatrix}$$

Let $A := \begin{bmatrix} 1 & 0 & \Delta t & 0 \\ 0 & 1 & 0 & \Delta t \\ 0 & 0 & 1 & -\Delta t \\ 0 & 0 & \Delta t & 1 \end{bmatrix}$. We diagonalize A. The characteristic polynomial of A is given by
$P_A(\lambda) = \det(A - \lambda I) = (1-\lambda)^2((1-\lambda)^2 + \Delta t^2) = (1-\lambda)^2(\lambda - (1 \pm (\Delta t)i))$
. $P_A(\lambda) = 0$ when $\lambda = 1$ or $1 - (\Delta t)i$ or $1 + (\Delta t)i$.

If $\lambda = 1,$ then

$$\ker(A - I) = \ker(\begin{bmatrix} 0 & 0 & \Delta t & 0 \\ 0 & 0 & 0 & \Delta t \\ 0 & 0 & 0 & -\Delta t \\ 0 & 0 & \Delta t & 0 \end{bmatrix})$$
$$= \operatorname{span}(\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix})$$

If $\lambda = 1 - (\Delta t)i,$ then

$$\ker(A - (1 - (\Delta t)i)I) = \ker(\begin{bmatrix} (\Delta t)i & 0 & \Delta t & 0 \\ 0 & (\Delta t)i & 0 & \Delta t \\ 0 & 0 & (\Delta t)i & -\Delta t \\ 0 & 0 & \Delta t & (\Delta t)i \end{bmatrix})$$
$$= \ker(\begin{bmatrix} i & 0 & 1 & 0 \\ 0 & i & 0 & 1 \\ 0 & 0 & i & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix})$$
$$= \operatorname{span}(\begin{bmatrix} 1 \\ i \\ -i \\ 1 \end{bmatrix})$$

If $\lambda = 1 + (\Delta t)i,$ then

$$\ker(A-(1+(\Delta t)i)I) = \ker(\begin{bmatrix} -(\Delta t)i & 0 & \Delta t & 0 \\ 0 & -(\Delta t)i & 0 & \Delta t \\ 0 & 0 & -(\Delta t)i & -\Delta t \\ 0 & 0 & \Delta t & -(\Delta t)i \end{bmatrix})$$

$$= \ker(\begin{bmatrix} -i & 0 & 1 & 0 \\ 0 & -i & 0 & 1 \\ 0 & 0 & i & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix})$$

$$= \operatorname{span}(\begin{bmatrix} 1 \\ -i \\ i \\ 1 \end{bmatrix})$$

Let $P = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & i & -i \\ 0 & 0 & -i & i \\ 0 & 0 & 1 & 1 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & \frac{i}{2} & \frac{1}{2} \\ 0 & 0 & -\frac{i}{2} & \frac{1}{2} \end{bmatrix}$.

Hence $P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1-(\Delta t)i & 0 \\ 0 & 0 & 0 & 1+(\Delta t)i \end{bmatrix}$.

And $A = P \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1-(\Delta t)i & 0 \\ 0 & 0 & 0 & 1+(\Delta t)i \end{bmatrix} P^{-1}$

Now we calculate the numerical integration.

$$\lim_{n\to\infty}\begin{bmatrix}\tilde{x}_n\\ \tilde{y}_n\\ \dot{\tilde{x}}_n\\ \dot{\tilde{y}}_n\end{bmatrix} = \lim_{n\to\infty}\left(\begin{bmatrix}1&0&\Delta t&0\\0&1&0&\Delta t\\0&0&1&-\Delta t\\0&0&\Delta t&1\end{bmatrix}^n\begin{bmatrix}\tilde{x}_0\\ \tilde{y}_0\\ \dot{\tilde{x}}_0\\ \dot{\tilde{y}}_0\end{bmatrix} + \sum_{k=0}^{n}\begin{bmatrix}1&0&\Delta t&0\\0&1&0&\Delta t\\0&0&1&-\Delta t\\0&0&\Delta t&1\end{bmatrix}^k\begin{bmatrix}0\\0\\0\\ \Delta\end{bmatrix}\right.$$

$$= \lim_{n\to\infty}\left(P\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1-(\Delta t)i&0\\0&0&0&1+(\Delta t)i\end{bmatrix}P^{-1}\right)^n\begin{bmatrix}\tilde{x}_0\\ \tilde{y}_0\\ \dot{\tilde{x}}_0\\ \dot{\tilde{y}}_0\end{bmatrix}+$$

$$\lim_{n\to\infty}\sum_{k=0}^{n}\left(P\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1-(\Delta t)i&0\\0&0&0&1+(\Delta t)i\end{bmatrix}P^{-1}\right)^k\begin{bmatrix}0\\0\\0\\ \Delta t\end{bmatrix}$$

$$= P\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&\lim_{n\to\infty}(1-(\Delta t)i)^n&0\\0&0&0&\lim_{n\to\infty}(1+(\Delta t)i)^n\end{bmatrix}P^{-1}\begin{bmatrix}\tilde{x}_0\\ \tilde{y}_0\\ \dot{\tilde{x}}_0\\ \dot{\tilde{y}}_0\end{bmatrix}+$$

$$\lim_{n\to\infty}\sum_{k=0}^{n}P\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&(1-(\Delta t)i)^k&0\\0&0&0&(1+(\Delta t)i)^k\end{bmatrix}P^{-1}\begin{bmatrix}0\\0\\0\\ \Delta t\end{bmatrix}$$

Consider the partition $0 = t_0 < t_1 < t_2 < \cdots < t_n = t$. Let $t_{i+1} - t_i = \Delta t$ for simplicity. So $\Delta t = \frac{t_n}{t}$. We expect that

$$\lim_{n\to\infty}\begin{bmatrix}\tilde{x}_n(t_n)\\ \tilde{y}_n(t_n)\\ \dot{\tilde{x}}_n(t_n)\\ \dot{\tilde{y}}_n(t_n)\end{bmatrix} = \begin{bmatrix}\tilde{x}(t)\\ \tilde{y}(t)\\ \dot{\tilde{x}}(t)\\ \dot{\tilde{y}}(t)\end{bmatrix}$$

Hence,

$$\lim_{n\to\infty}\begin{bmatrix}\tilde{x}_n(t_n)\\ \tilde{y}_n(t_n)\\ \dot{\tilde{x}}_n(t_n)\\ \dot{\tilde{y}}_n(t_n)\end{bmatrix} = P\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \lim\limits_{n\to\infty}(1-(\Delta t)i)^n & 0\\ 0 & 0 & 0 & \lim\limits_{n\to\infty}(1+(\Delta t)i)^n\end{bmatrix}P^{-1}\begin{bmatrix}\tilde{x}_0\\ \tilde{y}_0\\ \dot{\tilde{x}}_0\\ \dot{\tilde{y}}_0\end{bmatrix} +$$

$$\lim_{n\to\infty}\sum_{k=0}^{n} P\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & (1-(\Delta t)i)^k & 0\\ 0 & 0 & 0 & (1+(\Delta t)i)^k\end{bmatrix}P^{-1}\begin{bmatrix}0\\ 0\\ 0\\ \Delta t\end{bmatrix}$$

$$= P\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \lim\limits_{n\to\infty}\left(1-\left(\frac{t_n}{n}\right)i\right)^n & 0\\ 0 & 0 & 0 & \lim\limits_{n\to\infty}\left(1+\left(\frac{t_n}{n}\right)i\right)^n\end{bmatrix}P^{-1}\begin{bmatrix}\tilde{x}_0\\ \tilde{y}_0\\ \dot{\tilde{x}}_0\\ \dot{\tilde{y}}_0\end{bmatrix} +$$

$$\lim_{n\to\infty}\sum_{k=0}^{n} P\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \left(1-\left(\frac{t_n}{n}\right)i\right)^k & 0\\ 0 & 0 & 0 & \left(1+\left(\frac{t_n}{n}\right)i\right)^k\end{bmatrix}P^{-1}\begin{bmatrix}0\\ 0\\ 0\\ \frac{t_n}{n}\end{bmatrix}$$

$$= P\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \lim\limits_{n\to\infty} e^{-it_n} & 0\\ 0 & 0 & 0 & \lim\limits_{n\to\infty} e^{it_n}\end{bmatrix}P^{-1}\begin{bmatrix}\tilde{x}_0\\ \tilde{y}_0\\ \dot{\tilde{x}}_0\\ \dot{\tilde{y}}_0\end{bmatrix} +$$

$$\lim_{n\to\infty}\sum_{k=0}^{n} P\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & \left(1-\left(\frac{t_n}{n}\right)i\right)^k & 0\\ 0 & 0 & 0 & \left(1+\left(\frac{t_n}{n}\right)i\right)^k\end{bmatrix}P^{-1}\begin{bmatrix}0\\ 0\\ 0\\ \frac{t_n}{n}\end{bmatrix}$$

We first find $P^{-1}\begin{bmatrix}0\\ 0\\ 0\\ \frac{t_n}{n}\end{bmatrix}$.

$$P^{-1}\begin{bmatrix}0\\0\\0\\\frac{t_n}{n}\end{bmatrix}=\begin{bmatrix}1&0&0&1\\0&1&1&0\\0&0&\frac{i}{2}&\frac{1}{2}\\0&0&-\frac{i}{2}&\frac{1}{2}\end{bmatrix}\begin{bmatrix}0\\0\\0\\\frac{t_n}{n}\end{bmatrix}$$
$$=\begin{bmatrix}\frac{t_n}{n}\\0\\\frac{t_n}{2n}\\\frac{t_n}{2n}\end{bmatrix}$$

Next, $\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&\left(1-\left(\frac{t_n}{n}\right)i\right)^k&0\\0&0&0&\left(1+\left(\frac{t_n}{n}\right)i\right)^k\end{bmatrix}\begin{bmatrix}\frac{t_n}{n}\\0\\\frac{t_n}{2n}\\\frac{t_n}{2n}\end{bmatrix}$.

$$\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&\left(1-\left(\frac{t_n}{n}\right)i\right)^k&0\\0&0&0&\left(1+\left(\frac{t_n}{n}\right)i\right)^k\end{bmatrix}\begin{bmatrix}\frac{t_n}{n}\\0\\\frac{t_n}{2n}\\\frac{t_n}{2n}\end{bmatrix}=\begin{bmatrix}\frac{t_n}{n}\\0\\\left(1-\left(\frac{t_n}{n}\right)i\right)^k\frac{t_n}{2n}\\\left(1+\left(\frac{t_n}{n}\right)i\right)^k\frac{t_n}{2n}\end{bmatrix}$$

Thus, we have

$$\lim_{n\to\infty}\sum_{k=0}^{n} P\begin{bmatrix}\frac{t_n}{n}\\ 0\\ \left(1-\left(\frac{t_n}{n}\right)i\right)^k\frac{t_n}{2n}\\ \left(1+\left(\frac{t_n}{n}\right)i\right)^k\frac{t_n}{2n}\end{bmatrix}$$

$$=\lim_{n\to\infty} P\begin{bmatrix}t_n\\ 0\\ \frac{1-\left(1-\left(\frac{t_n}{n}\right)i\right)^{n+1}}{1-\left(1-\left(\frac{t_n}{n}\right)i\right)}\frac{t_n}{2n}\\ \frac{1-\left(1+\left(\frac{t_n}{n}\right)i\right)^{n+1}}{1-\left(1+\left(\frac{t_n}{n}\right)i\right)}\frac{t_n}{2n}\end{bmatrix}$$

$$=\lim_{n\to\infty} P\begin{bmatrix}t_n\\ 0\\ \frac{1-e^{-itn}}{2i}\\ \frac{1-e^{itn}}{-2i}\end{bmatrix}$$

$$=\lim_{n\to\infty}\begin{bmatrix}1&0&1&1\\0&1&i&-i\\0&0&-i&i\\0&0&1&1\end{bmatrix}\begin{bmatrix}t_n\\ 0\\ \frac{1-e^{-itn}}{2i}\\ \frac{1-e^{itn}}{-2i}\end{bmatrix}$$

$$=\lim_{n\to\infty}\begin{bmatrix}t_n+\frac{e^{itn}-e^{-itn}}{2i}\\ 1-\frac{e^{itn}+e^{-itn}}{2}\\ -1+\frac{e^{itn}+e^{-itn}}{2}\\ t_n+\frac{e^{itn}-e^{-itn}}{2i}\end{bmatrix}$$

$$=\lim_{n\to\infty}\begin{bmatrix}t_n+\sin t_n\\ 1-\cos t_n\\ -1+\cos t_n\\ t_n+\sin t_n\end{bmatrix}$$

Back to $\lim_{n\to\infty}\begin{bmatrix}\tilde{x}_n(t_n)\\ \tilde{y}_n(t_n)\\ \dot{\tilde{x}}_n(t_n)\\ \dot{\tilde{y}}_n(t_n)\end{bmatrix}$.

We have

$$
\begin{aligned}
\lim_{n\to\infty}\begin{bmatrix}\tilde{x}_n(t_n)\\ \tilde{y}_n(t_n)\\ \dot{\tilde{x}}_n(t_n)\\ \dot{\tilde{y}}_n(t_n)\end{bmatrix} &= P\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&\lim\limits_{n\to\infty}e^{-it_n}&0\\0&0&0&\lim\limits_{n\to\infty}e^{it_n}\end{bmatrix}P^{-1}\begin{bmatrix}\tilde{x}_0\\ \tilde{y}_0\\ \dot{\tilde{x}}_0\\ \dot{\tilde{y}}_0\end{bmatrix}+\lim_{n\to\infty}\begin{bmatrix}t_n+\sin t_n\\ 1-\cos t_n\\ -1+\cos t_n\\ t_n+\sin t_n\end{bmatrix}\\
&= P\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&\lim\limits_{n\to\infty}e^{-it_n}&0\\0&0&0&\lim\limits_{n\to\infty}e^{it_n}\end{bmatrix}\begin{bmatrix}1&0&0&1\\0&1&1&0\\0&0&\frac{i}{2}&\frac{1}{2}\\0&0&-\frac{i}{2}&\frac{1}{2}\end{bmatrix}\begin{bmatrix}0\\0\\2\\0\end{bmatrix}\\
&\quad+\lim_{n\to\infty}\begin{bmatrix}t_n+\sin t_n\\ 1-\cos t_n\\ -1+\cos t_n\\ t_n+\sin t_n\end{bmatrix}\\
&= P\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&\lim\limits_{n\to\infty}e^{-it_n}&0\\0&0&0&\lim\limits_{n\to\infty}e^{it_n}\end{bmatrix}\begin{bmatrix}0\\2\\i\\-i\end{bmatrix}+\lim_{n\to\infty}\begin{bmatrix}t_n+\sin t_n\\ 1-\cos t_n\\ -1+\cos t_n\\ t_n+\sin t_n\end{bmatrix}\\
&= P\begin{bmatrix}0\\2\\ \lim\limits_{n\to\infty}ie^{-it_n}\\ \lim\limits_{n\to\infty}-ie^{it_n}\end{bmatrix}+\lim_{n\to\infty}\begin{bmatrix}t_n+\sin t_n\\ 1-\cos t_n\\ -1+\cos t_n\\ t_n+\sin t_n\end{bmatrix}\\
&= \begin{bmatrix}1&0&1&1\\0&1&i&-i\\0&0&-i&i\\0&0&1&1\end{bmatrix}\begin{bmatrix}0\\2\\ \lim\limits_{n\to\infty}ie^{-it_n}\\ \lim\limits_{n\to\infty}-ie^{it_n}\end{bmatrix}+\begin{bmatrix}t_n+\sin t_n\\ 1-\cos t_n\\ -1+\cos t_n\\ t_n+\sin t_n\end{bmatrix}\\
&= \begin{bmatrix}i(e^{-it_n}-e^{it_n})\\ 2-(e^{-it_n}+e^{it_n})\\ e^{-it}+e^{it}\\ i(e^{-it_n}-e^{it_n})\end{bmatrix}+\begin{bmatrix}t_n+\sin t_n\\ 1-\cos t_n\\ -1+\cos t_n\\ t_n+\sin t_n\end{bmatrix}\\
&= \begin{bmatrix}2\sin t_n\\ 2-2\cos t_n\\ 2\cos t_n\\ 2\sin t_n\end{bmatrix}+\begin{bmatrix}t_n+\sin t_n\\ 1-\cos t_n\\ -1+\cos t_n\\ t_n+\sin t_n\end{bmatrix}\\
&= \begin{bmatrix}t_n+3\sin t_n\\ 3-3\cos t_n\\ \end{bmatrix}
\end{aligned}
$$

$$\begin{bmatrix} -1 + 3\cos t_n \\ t_n + 3\sin t_n \end{bmatrix}$$

Therefore, substitute t, x and y back to $\tilde{t}$, $\tilde{x}$ and $\tilde{y}$, we have.

$$\begin{bmatrix} x(t) \\ y(t) \\ \dot{x}(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{8}t + \frac{3}{4}\sin\frac{t}{2} \\ \frac{3}{4} - \frac{3}{4}\cos\frac{t}{2} \\ -\frac{1}{2} + \frac{3}{2}\cos\frac{t}{2} \\ \frac{1}{4}t + \frac{3}{2}\sin\frac{t}{2} \end{bmatrix}$$

5.17

A shell is fired with a muzzle velocity $v = 1000\,\mathrm{ft\,s^{-1}}$ at an angle of 45° with the horizontal. Its motion is resisted by a force proportional to the cube of its velocity ($F = -kv^3$). The coefficient k is such that the resisting force is equal to twice the weight of the shell when $v = 1000\,\mathrm{ft\,s^{-1}}$. find the approximate maximum height attained $h_{\max}$, and the horizontal range R by numerical integration, and compare these with the values expected in the absence of resistance.

(unsolved)

Ans:

Instead of using numerical approach. We use analytic approach at this moment.

The initial velocity in horizontal is $500\sqrt{2}\,\mathrm{ft\,s^{-1}}$, and $500\sqrt{2}\,\mathrm{ft\,s^{-1}}$ as well in vertical.

For horizontal direction, this is a first order differential equation

$$m\dot{v}_x = -kv_x^3$$

Applying separation of variables, we have

$$
\begin{aligned}
m\dot{v_x} &= -kv_x^3 \\
\Rightarrow \qquad \int \frac{dv_x}{v_x^3} &= \int -\frac{k}{m}dt \\
\Rightarrow \qquad -\frac{1}{2}v_x^{-2} &= -\frac{k}{m}t + C_1
\end{aligned}
$$

Hence, $v_x = \pm\frac{1}{\frac{m}{2kt}+C_2}$. Since $v(0) > 0,\ v_x = \frac{1}{\frac{m}{2kt}+C_2}$.

$v_x(0) = \frac{1}{C_2} = 500\sqrt{2} \Rightarrow C_2 = \frac{1}{500\sqrt{2}}$. So $v_x = \frac{1}{\frac{m}{2kt}+\frac{1}{500\sqrt{2}}}$.

For horizontal direction, the differential equation is
$m\dot{v}_y = -mg - kv_y^3 \Rightarrow \ddot{y} = -g - \frac{k}{m}\dot{y}^3$.
Let $\tilde{t} = c_1 t, \quad \tilde{y} = c_2 y$. Then

$$
\frac{d^2\tilde{y}}{d\tilde{t}^2} = \frac{d}{dt}\left(\frac{d\tilde{y}}{dy}\frac{dy}{dt}\frac{dt}{d\tilde{t}}\right)\frac{dt}{d\tilde{t}} = \frac{c_2}{c_1^2}\left(-g - \frac{k}{m}\frac{d\tilde{y}}{d\tilde{t}}\frac{c_1}{c_2}\right) = -\frac{c_2 g}{c_1^2} - \frac{k}{mc_1}\frac{d\tilde{y}}{d\tilde{t}}
$$

Let $c_1 = \frac{k}{m}$ and $c_2 = \frac{k^2}{gm^2}$, then $\frac{d^2\tilde{y}}{d\tilde{t}^2} = -1 - \frac{d\tilde{y}}{d\tilde{t}}$.
We apply the numerical approach,

$$
\begin{bmatrix} y_{i+1} \\ \dot{y}_{i+1} \end{bmatrix} = \begin{bmatrix} y_i + \dot{y}_i \Delta t \\ \dot{y}_i - (1+\dot{y}_i)\Delta t \end{bmatrix} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1-\Delta t \end{bmatrix}\begin{bmatrix} y_i \\ \dot{y}_i \end{bmatrix} + \begin{bmatrix} 0 \\ -\Delta t \end{bmatrix}
$$

It is easy to see that

$$
\begin{bmatrix} y_n \\ \dot{y}_n \end{bmatrix} = \begin{bmatrix} 1 & \Delta t \\ 0 & 1-\Delta t \end{bmatrix}^n \begin{bmatrix} y_0 \\ \dot{y}_0 \end{bmatrix} + \sum_{k=0}^{n} \begin{bmatrix} 1 & \Delta t \\ 0 & 1-\Delta t \end{bmatrix}^k \begin{bmatrix} 0 \\ -\Delta t \end{bmatrix}
$$

Let $A := \begin{bmatrix} 1 & \Delta t \\ 0 & 1-\Delta t \end{bmatrix}$. We diagonalize A. The characteristic polynomial of A is

$$
P_A(\lambda) = (1-\lambda)(1-\Delta t - \lambda)
$$

$P_A(\lambda) = 0$ when $\lambda = 1$ or $\lambda = 1 - \Delta t$.

If $\lambda = 1,\ E_\lambda = \text{span}(\begin{bmatrix}1\\0\end{bmatrix})$

If $\lambda = 1-\Delta t,\ E_\lambda = \text{span}(\begin{bmatrix}1\\-1\end{bmatrix})$.

Take $P = \begin{bmatrix}1 & 1\\0 & -1\end{bmatrix}$, $P^{-1} = \begin{bmatrix}1 & 1\\0 & -1\end{bmatrix}$. Then $A = P\begin{bmatrix}1 & 0\\0 & 1-\Delta t\end{bmatrix}P^{-1}$.

Hence

$$\begin{bmatrix}y_n\\\dot{y}_n\end{bmatrix} = \left(P\begin{bmatrix}1 & 0\\0 & 1-\Delta t\end{bmatrix}P^{-1}\right)^n\begin{bmatrix}y_0\\\dot{y}_0\end{bmatrix} + \sum_{k=0}^{n}\left(P\begin{bmatrix}1 & 0\\0 & 1-\Delta t\end{bmatrix}P^{-1}\right)^k\begin{bmatrix}0\\-\Delta t\end{bmatrix}$$

Consider the partition $0 < t_0 < t_1 < t_2 < \cdots < t_n = t$. Let $t_{i+1} - t_i = \Delta t$, $\forall i$ for simplicity. $\Delta t = \frac{t_n}{n}$.

Substitute $\Delta t = \frac{t_n}{n}$ and $\begin{bmatrix}y_0\\\dot{y}_0\end{bmatrix} = \begin{bmatrix}0\\v_y(0)\end{bmatrix}$ to the equation.

$$\begin{aligned}
\begin{bmatrix}y_n\\\dot{y}_n\end{bmatrix} &= P\begin{bmatrix}1 & 0\\0 & (1-\frac{t_n}{n})^n\end{bmatrix}P^{-1}\begin{bmatrix}0\\v_y(0)\end{bmatrix} + \sum_{k=0}^{n}P\begin{bmatrix}1 & 0\\0 & (1-\frac{t_n}{n})^k\end{bmatrix}P^{-1}\begin{bmatrix}0\\-\frac{t_n}{n}\end{bmatrix}\\
&= P\begin{bmatrix}1 & 0\\0 & (1-\frac{t_n}{n})^n\end{bmatrix}P^{-1}\begin{bmatrix}0\\v_y(0)\end{bmatrix} + \sum_{k=0}^{n}P\begin{bmatrix}1 & 0\\0 & (1-\frac{t_n}{n})^k\end{bmatrix}\begin{bmatrix}1 & 1\\0 & -1\end{bmatrix}\begin{bmatrix}0\\-\frac{t_n}{n}\end{bmatrix}\\
&= P\begin{bmatrix}1 & 0\\0 & (1-\frac{t_n}{n})^n\end{bmatrix}P^{-1}\begin{bmatrix}0\\v_y(0)\end{bmatrix} + \sum_{k=0}^{n}P\begin{bmatrix}1 & 0\\0 & (1-\frac{t_n}{n})^k\end{bmatrix}\begin{bmatrix}-\frac{t_n}{n}\\\frac{t_n}{n}\end{bmatrix}\\
&= P\begin{bmatrix}1 & 0\\0 & (1-\frac{t_n}{n})^n\end{bmatrix}P^{-1}\begin{bmatrix}0\\v_y(0)\end{bmatrix} + P\begin{bmatrix}-t_n\\\frac{1-(1-\frac{t_n}{n})^{n+1}}{1-(1-\frac{t_n}{n})}\frac{t_n}{n}\end{bmatrix}
\end{aligned}$$

We expect that

$$\lim_{n\to\infty}\begin{bmatrix}y_n(t)\\\dot{y}_n(t)\end{bmatrix} = \begin{bmatrix}\tilde{y}(t)\\\dot{\tilde{y}}(t)\end{bmatrix}$$

And

$$\lim_{n\to\infty}\begin{bmatrix} y_n(t) \\ \dot{y}_n(t) \end{bmatrix} = \lim_{n\to\infty}\left(P\begin{bmatrix} 1 & 0 \\ 0 & (1-\frac{t_n}{n})^n \end{bmatrix}P^{-1}\begin{bmatrix} 0 \\ v_y(0) \end{bmatrix} + P\begin{bmatrix} -t_n \\ \frac{1-(1-\frac{t_n}{n})^{n+1}}{1-(1-\frac{t_n}{n})}\frac{t_n}{n} \end{bmatrix}\right)$$

$$= P\begin{bmatrix} 1 & 0 \\ 0 & \lim_{n\to\infty}(1-\frac{t_n}{n})^n \end{bmatrix}P^{-1}\begin{bmatrix} 0 \\ v_y(0) \end{bmatrix} + P\begin{bmatrix} -t_n \\ \lim_{n\to\infty}\left(\frac{1-(1-\frac{t_n}{n})^{n+1}}{1-(1-\frac{t_n}{n})}\frac{t_n}{n}\right) \end{bmatrix}$$

$$= P\begin{bmatrix} 1 & 0 \\ 0 & e^{-t_n} \end{bmatrix}P^{-1}\begin{bmatrix} 0 \\ v_y(0) \end{bmatrix} + P\begin{bmatrix} -t_n \\ 1-e^{-t_n} \end{bmatrix}$$

Further sorting the equation, we have

$$\begin{aligned}
\lim_{n\to\infty}\begin{bmatrix} y_n(t) \\ \dot{y}_n(t) \end{bmatrix} &= P\begin{bmatrix} 1 & 0 \\ 0 & e^{-t_n} \end{bmatrix}P^{-1}\begin{bmatrix} 0 \\ v_y(0) \end{bmatrix} + P\begin{bmatrix} -t_n \\ 1-e^{-t_n} \end{bmatrix} \\
&= P\begin{bmatrix} 1 & 0 \\ 0 & e^{-t_n} \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ v_y(0) \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -t_n \\ 1-e^{-t_n} \end{bmatrix} \\
&= \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & e^{-t_n} \end{bmatrix}\begin{bmatrix} v_y(0) \\ -v_y(0) \end{bmatrix} + \begin{bmatrix} -t_n+1-e^{-t_n} \\ e^{-t_n}-1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & e^{-t_n} \\ 0 & -e^{-t_n} \end{bmatrix}\begin{bmatrix} v_y(0) \\ -v_y(0) \end{bmatrix} + \begin{bmatrix} -t_n+1-e^{-t_n} \\ e^{-t_n}-1 \end{bmatrix} \\
&= \begin{bmatrix} v_y(0)-v_y(0)e^{-t_n}-t_n+1-e^{-t_n} \\ v_y(0)e^{-t_n}+e^{-t_n}-1 \end{bmatrix} \\
&= \begin{bmatrix} v_y(0)-t_n+1-(v_y(0)+1)e^{-t_n} \\ e^{-t_n}(v_y(0)+1)-1 \end{bmatrix}
\end{aligned}$$

Hence,

$$\begin{bmatrix} \tilde{y}(t) \\ \dot{\tilde{y}}(t) \end{bmatrix} = \begin{bmatrix} \tilde{v}_y(0)-t_n+1-(\tilde{v}_y(0)+1)e^{-t_n} \\ e^{-t_n}(\tilde{v}_y(0)+1)-1 \end{bmatrix}$$

And

$$\begin{bmatrix} y(t) \\ \dot{y}(t) \end{bmatrix} = \begin{bmatrix} \left(\frac{mg}{k}v_y(0)-\frac{m}{k}t+1-\left(\frac{mg}{k}v_y(0)+1\right)e^{-\frac{m}{k}t}\right)\frac{gm^2}{k^2} \\ \left(e^{\frac{m}{k}t}\left(\frac{mg}{k}v_y(0)+1\right)-1\right)\frac{mg}{k} \end{bmatrix}$$

Since the coefficient k is such that the resisting force is equal to twice the weight of the shell when $v = 1000\,\mathrm{ft\,s^{-1}}$, we can calculate

$$k(1000)^3 = 2mg \Rightarrow k = \frac{2}{(1000)^3}mg$$

With $v_y(0) = 500\sqrt{2}$. $$
\begin{bmatrix}
y(t) \\
\dot{y}(t)
\end{bmatrix}=\begin{bmatrix}
\left(\frac{(1000)^{3}}{2}\times 500\sqrt{ 2 } -
\frac{(1000)^{3}}{2g}t + 1 - \left(\frac{(1000)^{3}}{2}\times
500\sqrt{ 2 } + 1 \right)e^{-\frac{(1000)^{3}}{2g}t} \right)
g\left(\frac{(1000)^{3}}{2} \right)^{2}\
\left(e^{-\frac{(1000)^{3}}{2g}t}\left(\frac{(1000)^{3}}
{2}\times 500\sqrt{ 2 } + 1 \right) - 1
\right)\frac{(1000)^{3}}{2}
\end{bmatrix}