# Exercise 7

## 7.1

**Generalize Exs. 6.1 through 6.4 to three dimensional motion using vector notation.**
**If two bodies have masses $m_1$ and $m_2$ and are moving at velocities $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, show that the CM system is moving at velocity**

$$\boldsymbol{v}_{\text{cm}} = \frac{m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2}{m_1 + m_2}$$

Ans:
By the definition of CM system, the center of mass is the coordinate such that the net momentum in that frame is zero.
Assume that the frame of reference is moving in velocity $\boldsymbol{v}_{\text{cm}}$, then $m_1(\boldsymbol{v}_1 - \boldsymbol{v}_{cm}) + m_2(\boldsymbol{v}_2 - \boldsymbol{v}_{\text{cm}}) = 0$, which implies that

$$\boldsymbol{v}_{\text{cm}} = \frac{m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2}{m_1 + m_2}$$

## 7.2

**Show that for $N$ bodies of masses $m_i$ and velocities $\boldsymbol{v}_i$ the velocity of the coordinate system, in which the total momentum is zero, is given by**

$$\boldsymbol{v}_{\text{cm}} = \frac{\sum\limits_{i=1}^{N} m_i\boldsymbol{v}_i}{\sum\limits_{i=1}^{N} m_i}$$

Ans:
Assume that the coordinate system moves in velocity $\boldsymbol{v}_{\text{cm}}$, then

$\sum_{i=1}^{N} m_i(\boldsymbol{v}_i - \boldsymbol{v}_{\text{cm}}) = 0$, which implies that

$$\boldsymbol{v}_{\text{cm}} = \frac{\sum_{i=1}^{N} m_i \boldsymbol{v}_i}{\sum_{i=1}^{N} m_i}$$

## 7.3

**If $T$ is the total kinetic energy of the two masses in Ex. 7.1, and $T_{\text{CM}}$ their total kinetic energy in the CM system, show that**

$$T = T_{\text{CM}} + \left(\frac{m_1 + m_2}{2}\right)|\boldsymbol{v}_{\text{cm}}|^2$$

Ans:
By the definition of kinetic energy, the total kinetic energy in the CM system is

$$\begin{aligned} T_{\text{CM}} &= \frac{1}{2}m_1(\boldsymbol{v}_1 - \boldsymbol{v}_{\text{cm}}) \cdot (\boldsymbol{v}_1 - \boldsymbol{v}_{\text{cm}}) + \frac{1}{2}m_2(\boldsymbol{v}_2 - \boldsymbol{v}_{\text{cm}}) \cdot (\boldsymbol{v}_2 - \boldsymbol{v}_{\text{cm}}) \\ &= \frac{1}{2}m_1|\boldsymbol{v}_1|^2 + \frac{1}{2}m_2|\boldsymbol{v}_2|^2 - (m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2) \cdot \boldsymbol{v}_{\text{cm}} + \frac{1}{2}(m_1 + m_2)|\boldsymbol{v}_{\text{cm}}|^2 \end{aligned}$$

Ex. 7.1 states that $\boldsymbol{v}_{\text{cm}} = \frac{m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2}{m_1 + m_2}$, that is
$m_1\boldsymbol{v}_1 + m_2\boldsymbol{v}_2 = (m_1 + m_2)\boldsymbol{v}_{\text{cm}}$. Hence,

$$\begin{aligned} T_{\text{CM}} &= \frac{1}{2}m_1|\boldsymbol{v}_1|^2 + \frac{1}{2}m_2|\boldsymbol{v}_2|^2 - (m_1 + m_2)\boldsymbol{v}_{\text{cm}} \cdot \boldsymbol{v}_{\text{cm}} + \frac{1}{2}(m_1 + m_2)|\boldsymbol{v}_{\text{cm}}|^2 \\ &= T - \frac{1}{2}(m_1 + m_2)|\boldsymbol{v}_{\text{cm}}|^2 \end{aligned}$$

Therefore,

$$T = T_{\text{CM}} + \left(\frac{m_1 + m_2}{2}\right)|\boldsymbol{v}_{\text{cm}}|^2$$

## 7.4

**Generalize the result of Ex. 7.3 to $N$ masses. Show that**

$$T = T_{\mathrm{CM}} + \frac{\sum\limits_{i=1}^{N} m_i}{2} |\boldsymbol{v}_{\mathrm{cm}}|^2$$

Ans:

By the definition of kinetic energy, the total kinetic energy in the CM system is

$$\begin{aligned} T_{\mathrm{CM}} &= \sum_{i=1}^{N} \frac{1}{2} m_i (\boldsymbol{v}_i - \boldsymbol{v}_{\mathrm{cm}}) \cdot (\boldsymbol{v}_i - \boldsymbol{v}_{\mathrm{cm}}) \\ &= \sum_{i=1}^{N} \frac{1}{2} m_i |\boldsymbol{v}_i|^2 - \sum_{i=1}^{N} m_i \boldsymbol{v}_i \cdot \boldsymbol{v}_{\mathrm{cm}} + \sum_{i=1}^{N} \frac{1}{2} m_i |\boldsymbol{v}_{\mathrm{cm}}|^2 \end{aligned}$$

Ex. 7.2 states that

$$\boldsymbol{v}_{\mathrm{cm}} = \frac{\sum\limits_{i=1}^{N} m_i \boldsymbol{v}_i}{\sum\limits_{i=1}^{N} m_i}$$

that is, $\sum\limits_{i=1}^{N} m_i \boldsymbol{v}_i = \sum\limits_{i=1}^{N} m_i \boldsymbol{v}_{\mathrm{cm}}$. Hence,

$$\begin{aligned} T_{\mathrm{CM}} &= \sum_{i=1}^{N} \frac{1}{2} m_i |\boldsymbol{v}_i|^2 - \sum_{i=1}^{N} m_i \boldsymbol{v}_{\mathrm{cm}} \cdot \boldsymbol{v}_{\mathrm{cm}} + \sum_{i=1}^{N} \frac{1}{2} m_i |\boldsymbol{v}_{\mathrm{cm}}|^2 \\ &= T - \sum_{i=1}^{N} \frac{1}{2} m_i |\boldsymbol{v}_{\mathrm{cm}}|^2 \end{aligned}$$

Therefore,

$$T = T_{\mathrm{CM}} + \frac{\sum\limits_{i=1}^{N} m_i}{2} |\boldsymbol{v}_{\mathrm{cm}}|^2$$

## 7.5

**A particle is initially at point $\boldsymbol{r}_0$, and is moving under gravity with an initial velocity $\boldsymbol{v}_0$. Find the subsequent motion $\boldsymbol{r}(t)$.**

Ans:
Since kinematics and dynamics are independent on axes, if the acceleration of gravity is $\boldsymbol{g}$, for each axis, $r(t) = r_0 + v_0 t + \frac{1}{2}at^2$. Thus,

$$\boldsymbol{r}(t) = \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{g}t^2$$

## 7.6

**You are given three vectors,**

$$\begin{aligned} \boldsymbol{a} &= 3\boldsymbol{i} + 2\boldsymbol{j} - \boldsymbol{k} \\ \boldsymbol{b} &= 2\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k} \\ \boldsymbol{c} &= \boldsymbol{i} + 3\boldsymbol{j} \end{aligned}$$

**Find**

1. $\boldsymbol{a} + \boldsymbol{b}$
2. $\boldsymbol{a} - \boldsymbol{b}$
3. $\boldsymbol{a}_x$
4. $\boldsymbol{a} \cdot \boldsymbol{i}$
5. $\boldsymbol{a} \cdot \boldsymbol{b}$
6. $(\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}$

Ans:

1. $\boldsymbol{a} + \boldsymbol{b} = 5\boldsymbol{i} + \boldsymbol{j}$
2. $\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{i} + 3\boldsymbol{j} - 2\boldsymbol{k}$
3. $\boldsymbol{a}_x = 3$
4. $\boldsymbol{a} \cdot \boldsymbol{i} = 3$
5. $\boldsymbol{a} \cdot \boldsymbol{b} = 6 - 2 - 1 = 3$

6. $(\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c} = (3+6)\boldsymbol{b} - (6-2-1)\boldsymbol{c} = 15\boldsymbol{i} - 18\boldsymbol{j} + 9\boldsymbol{k}$

## 7.7

**A particle of mass $1\,\mathrm{kg}$ is moving in such a way that its position is described by the vector**

$$\boldsymbol{r}(t) = t\boldsymbol{i} + (t + t^2/2)\boldsymbol{j} - (4/\pi^2)\sin(\pi t/2)\boldsymbol{k}$$

1. **Find the position, velocity $\boldsymbol{v}(t)$, acceleration $\boldsymbol{a}(t)$, and kinetic energy $T(t)$ of the particle at $t = 0$ and $t = 1$ second.**
2. **Find the force $\boldsymbol{F}(t)$ that will produce this motion.**
3. **Find the radius of curvature $R(t)$ of the particle's path at $t = 1$ second.**

Ans:

1. The position function is already given, and $\boldsymbol{r}(0) = 0$, $\boldsymbol{r}(1) = \boldsymbol{i} + \frac{3}{2}\boldsymbol{j} - \frac{4}{\pi^2}\boldsymbol{k}$. Velocity is the rate of change in position with respect to time, i.e., the derivative of the position function.

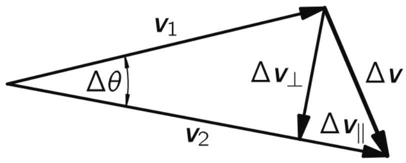
$$\boldsymbol{v}(t) = \dot{\boldsymbol{r}}(t) = \boldsymbol{i} + (1+t)\boldsymbol{j} - \frac{2}{\pi}\cos\left(\frac{\pi t}{2}\right)\boldsymbol{k}$$

   Thus, $\boldsymbol{v}(0) = \boldsymbol{i} + \boldsymbol{j} - \frac{2}{\pi}\boldsymbol{k}$ and $\boldsymbol{v}(1) = \boldsymbol{i} + 2\boldsymbol{j}$. Acceleration is the rate of change in velocity with respect to time.

$$\boldsymbol{a}(t) = \dot{\boldsymbol{v}}(t) = \boldsymbol{j} + \sin\left(\frac{\pi t}{2}\right)\boldsymbol{k}$$

   Thus, $\boldsymbol{a}(0) = \boldsymbol{j}$ and $\boldsymbol{a}(1) = \boldsymbol{j} + \boldsymbol{k}$.
2. By Newtons law of dynamics, $\boldsymbol{F}(t) = m\boldsymbol{a}(t)$. Here, $m = 1$. $\boldsymbol{F}(t)$ is the same as $\boldsymbol{a}(t)$, which is $\boldsymbol{F}(t) = \boldsymbol{j} + \sin(\pi t/2)\boldsymbol{k}$.

3. From the figure below,




it can be seen that $\Delta v = v_2 - v_1$. As $\Delta t \to 0$, $v_1 = v_2 = v$, $\left|\frac{dv}{dt}\right| = \frac{\Delta v_\parallel}{\Delta t} = a_\parallel$ and $v\Delta\theta = \Delta v_\perp$. Assume that the radius of curvature at some time $t$ is $R$, then $\Delta\theta = \frac{v}{2\pi R} 2\pi \Delta t = \frac{v}{R}\Delta t$. Hence, $v\Delta\theta = \frac{v^2}{R}\Delta t = \Delta v_\perp$, that is, $a_\perp = \Delta v_\perp / \Delta t = \frac{v^2}{R}$. Here, $a_\perp$ can be derived by $\sqrt{a^2 - a_\parallel^2}$, and $a_\parallel = \frac{a \cdot v}{|v|^2} v$ Therefore in our problem,

$$
\begin{aligned}
R &= \frac{|\boldsymbol{v}|^2}{|\boldsymbol{a}_\perp|} \\
&= \frac{|\boldsymbol{v}|^2}{\sqrt{\boldsymbol{a}^2 - \boldsymbol{a}_\parallel^2}} \\
&= \frac{|\boldsymbol{v}|^2}{\sqrt{\boldsymbol{a}^2 - ((\boldsymbol{a} \cdot \boldsymbol{v})\boldsymbol{v}/|\boldsymbol{v}|^2)^2}}
\end{aligned}
$$

Specifically, at $t = 1$,

$$
R(1) = \frac{5}{\sqrt{2 - \frac{4}{5}}} \approx 4.6\ \mathrm{m}
$$

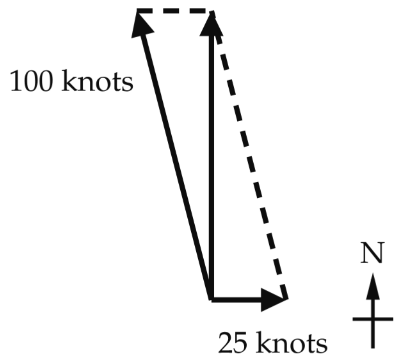
## 7.8

**A pilot flying at an air speed of 100 knots wishes to travel duo north. He knows, from talking to the airport meteorologist, that there is a 20 knot wind from west to east at his flight altitude.**

1. **In what direction should he head his plane?**
2. **What will be the duration $T$ of his flight, if his destination is 100 land miles away? (Neglect the time for landing and take-off, and note that 1 knot = 1.15 miles per hour.)**

Ans:

1. Since the pilot want to fly toward north, the sum of the vector the pilot heading and the vector the wind blows should point toward north, i.e., the sum the west-east component of the two vectors is zero.




Assume that the pilot flies toward a direction at an angle of $\theta$ west of north, then $100 \sin \theta = 25$, which is $\theta = \arcsin 0.25 \approx 0.25 \text{ rad} \approx 14.5^\circ$. Hence the pilot should head his plane $14.5^\circ$ west of north.

2. The speed for the pilot heading north is $100 \cos \theta \approx 96.8 \text{ knots}$. Thus it will take $100/(96.8 \times 1.15/60) \approx 53.89 \text{ min}$.

## 7.9

**A cyclist rides at $10 \text{ mi h}^{-1}$ due north and the wind, which is blowing at $6 \text{ mi h}^{-1}$ from a point between $N$ and $E$, appears to the cyclist to come from a point $15^\circ$ E of N.**

1. **Find the true direction of the wind.**
2. **Find the direction in which the wind will appear to meet the cyclist on his return if he rides at the same speed.**

Ans:

1. In the view from the cyclist, the wind comes from the point $15^\circ$ $E$ of $N$. Assume that the speed of wind to the cyclist is $v$

and the true wind is at an angle $\theta$ $E$ of $N$. Thus, $(-6\sin\theta, -6\cos\theta) - (0, 10) = (-v\sin 15^\circ, -v\cos 15^\circ)$.

$$\begin{aligned} -6\sin\theta &= \tan 15^\circ(-6\cos\theta - 10) \\ 6.212\cos(\theta + \arcsin 0.966) &= -2.679 \\ \theta + 75.016 &\approx 115.548 \\ \theta &\approx 115.548 - 75.016 \\ &\approx 40.5^\circ \end{aligned}$$

Hence, the true direction of the wind is $40.5^\circ$ $E$ of $N$.

2. The wind from the view of cyclist on his return is $(-6\sin\theta, -6\cos\theta) - (0, -10) = (-3.897, 5.438)$, which from $35.6^\circ$ $E$ of $S$.

## 7.10

**A man standing on the bank of a river $1.0\,\mathrm{mi}$ wide wishes to get to a point directly opposite him on the other band. He can do this in two ways:**

1. **head somewhat upstream, so that his resultant motion is straight across,**
2. **head toward the opposite band and then walk up along the bank from the point downstream to which the current has carried him.**

**If he can swim $2.5\,\mathrm{mi\,h^{-1}}$ and walk $4.0\,\mathrm{mi\,h^{-1}}$, and if the current is $2.0\,\mathrm{mi\,h^{-1}}$ which is the faster way to cross, and by how much?**

Ans:
Assume that the river is $d$ wide, the man can swim with speed $v$ and can walk with speed $u$, and that the current is at speed $w$.

In the first case, the man hoped to move straight across the river, which demands the sum of the swimming vector and the current vector is zero in the current direction. If that requires him to swim at an angle $\theta$ up with the straight cross, then $(v\cos\theta, v\sin\theta) - (0, w) = (v\cos\theta, 0)$. Hence, its crossing speed is $v\cos(\arcsin(w/v))$, and it would take him $d/v\cos(\arcsin(w/v))$ to finish the cross. Plug in the quantity of the variables, we have the time is $1/2.5\cos(\arcsin(2.0/2.5)) = 2/3 \approx 0.67\,\mathrm{h} = 40\,\mathrm{min}$.

In the second case, since the man swam right ahead the bank, it would take him $d/v$ to get across the river. At the mean time, he will be carried down $w \times (d/v)$. He had to spend $(w \times (d/v))/u$ walking to the point. Thus the total time he spent is $d/v + (w \times (d/v))/u$. Plug in the quantity of the variables, we have the time is $1/2.5 + (2 \times (1/2.5))/4.0 = 3/5 \approx 0.6\,\mathrm{h} = 36\,\mathrm{min}$.

Therefore, method 2 is faster by $4\,\mathrm{min}$.

## 7.11

**A motorboat that runs at a constant speed $V$ relative to the water is operated in a straight river channel where the water is flowing smoothly with a constant speed $R$. The boat is first sent on a round trip from its anchor point to a point a distance $d$ directly upstream. It is then sent on a round trip from its anchor point to a point a distance $d$ away directly across the stream. For simplicity assume that the boat runs the entire distance in each case at full speed and that no time is lost in reversing course at the end of the outward lap. if $t_V$ is the time the boat took to make the round trip on line with the stream flow, $t_A$ the time the boat took to make the round trip across the stream, and $t_L$ the time the boat would take to go a distance $2d$ on a lake.**

1. **What is the ratio $t_V/t_A$?**
2. **What is the ratio $t_A/t_L$?**

Ans:
When the boat is running upstream, the net velocity would be $V - R$ upstream. When it runs downstream, the net velocity would be $V + R$. Thus it would take $t_V = d(1/(V-R) + 1/(V+R)) = 2dV/(V^2 - R^2)$ to finish the round trip. When the boat is running across the stream, the net velocity in the stream direction should be zero. Assume that the boat heads in an angle of $\theta$ up with the moving direction, then$V \sin\theta = R$ and $V\cos\theta = V\cos(\arcsin(R/V))$.
$t_A = d/V\cos(\arcsin(R/V)) = 2d/\sqrt{V^2 - R^2}$. If the boat runs on a lake, there is no stream affecting the boat's movement. So $t_L = 2d/V$. Consequently,

1. $t_V/t_A = V\sqrt{V^2 - R^2}/(V^2 - R^2) = V/\sqrt{V^2 - R^2}$.
2. $t_A/t_L = V/\sqrt{V^2 - R^2} = t_V/t_A$.

## 7.12

**Use vectors to find the great circle distance $D$ between two points on the earth ($\text{radius} = r_{♁}$), whose latitudes and longitudes are $(\lambda_1, \phi_1)$ and $(\lambda_2, \phi_2)$.**

***Note:* Use a system of rectangular coordinates with origin at the center of the earth, one axis along the earth's axis, another pointed toward $\lambda = 0$, $\phi = 0$, and the third axis pointed toward $\lambda = 0$, $\phi = 90^\circ$ W. Let longitudes vary from $0^\circ$ westward to $360^\circ$.**

Ans:
Consider the center of the earth the origin. We use spherical coordinate to represent points in Euclid space. Hence, the vector pointing from origin toward a point $(x, y, z)$ at the surface of the earth with latitude and longitude $(\lambda, \phi)$ where $\lambda \in [0, 180^\circ]$ and $\phi \in [0, 360^\circ)$ is equal to $r_{♁}(\cos\lambda\cos\phi, \cos\lambda\sin\phi, \sin\lambda)$. For two points $(\lambda_1, \phi_1)$ and $(\lambda_2, \phi_2)$ on the earth, the angle $\theta$ between the

corresponding vectors $v_1$ and $v_2$ is given by

$$v_1 \cdot v_2 = |v_1||v_2| \cos\theta$$

where $v_1 = r_♁(\cos\lambda_1 \cos\phi_1, \cos\lambda_1 \sin\phi_1, \sin\lambda_1)$ and $v_2 = r_♁(\cos\lambda_2 \cos\phi_2, \cos\lambda_2 \sin\phi_2, \sin\lambda_2)$. Thus,

$$\begin{aligned}\cos\theta &= \cos\lambda_1 \cos\lambda_2(\cos\phi_1 \cos\phi_2 + \sin\phi_1 \sin\phi_2) + \sin\lambda_1 \sin\lambda_2 \\ \theta &= \arccos(\cos\lambda_1 \cos\lambda_2(\cos(\phi_1 - \phi_2)) + \sin\lambda_1 \sin\lambda_2)\end{aligned}$$

Therefore, the great circle distance $D$ is

$$r_♁\theta = r_♁ \arccos(\cos\lambda_1 \cos\lambda_2(\cos(\phi_1 - \phi_2)) + \sin\lambda_1 \sin\lambda_2)$$

## 7.13

**What is the magnitude and direction of the acceleration $a$ of the moon at**

1. **New moon?**
2. **Quarter moon?**
3. **Full moon?**

***Note:***

$$\begin{aligned}R_{♁☼} &= 1.50 \times 10^8 \text{ km} \\ R_{♁☾} &= 3.85 \times 10^5 \text{ km} \\ M_☼ &= 3.33 \times 10^5 M_♁\end{aligned}$$

Ans:
At New moon, the moon is between the earth and the sun. Thus,

its acceleration is

$$\boldsymbol{a} \approx \frac{GM_{☼}}{(R_{♁☼} - R_{♁☾})^2} - \frac{GM_{♁}}{R_{♁☾}^2}$$
$$= GM_{♁}\left(\frac{3.33 \times 10^5}{(1.50 \times 10^8 \times 10^3 - 3.85 \times 10^5 \times 10^3)^2} - \frac{1}{(3.85 \times 10^5 \times 10^3)^2}\right)$$
$$\approx 3.2 \times 10^{-3}\ \mathrm{m\ s^{-1}}\ \text{toward sun}$$

At Quarter moon, the earth position and the sun position viewed from the moon are approximately at an right angle. Hence the magnitude of the acceleration is

$$\boldsymbol{a} \approx \sqrt{\left(\frac{GM_{☼}}{R_{♁☼}^2}\right)^2 + \left(\frac{GM_{♁}}{R_{♁☾}^2}\right)^2}$$
$$= GM_{♁}\sqrt{\left(\frac{3.33 \times 10^5}{\left(1,.50 \times 10^8 \times 10^3\right)^2}\right)^2 + \left(\frac{1}{(3.85 \times 10^5 \times 10^3)^2}\right)^2}$$
$$= GM_{♁}\sqrt{(1.48 \times 10^{-17})^2 + (6.75 \times 10^{-18})^2}$$
$$= GM_{♁}\sqrt{2.19 \times 10^{-34} + 4.55 \times 10^{-35}}$$
$$\approx 3.8 \times 10^{-3}\ \mathrm{m\ s^{-1}}\ \text{bearing} \approx 24^{\circ}(\tan(\text{bearing}) \approx 0.45)$$

At Full moon the earth is between the moon and the sun. The acceleration of the moon is

$$\boldsymbol{a} \approx \frac{GM_{☼}}{(R_{♁☼} + R_{♁☾})^2} + \frac{GM_{♁}}{R_{♁☾}^2}$$
$$= GM_{♁}\left(\frac{3.33 \times 10^5}{(1.50 \times 10^8 \times 10^3 + 3.85 \times 10^5 \times 10^3)^2} + \frac{1}{(3.85 \times 10^5 \times 10^3)^2}\right)$$
$$\approx 8.5 \times 10^{-3}\ \mathrm{m\ s^{-1}}\ \text{toward sun}$$

## 7.14

**Two identical $45^{\circ}$ wedges $M_1$ and $M_2$, with smooth faces and $M_1 = M_2 = 8\,\mathrm{kg}$, are used to move a smooth-faced mass $M = 384\,\mathrm{kg}$ , as shown in Fig. 7-1. Both wedges rest upon a smooth horizontal**

**plane; one wedge is butted against a vertical wall, and to the other wedge a force $F = 592\,\text{kg-wt}$ is applied horizontally.**

1. **What is the magnitude and direction of the acceleration $\boldsymbol{a}_1$ of the movable wedge $M_1$?**
2. **What is the magnitude and direction of the acceleration $\boldsymbol{a}$ of the larger wedge $M$?**
3. **What force $F_2$ does the stationary wedge $M_2$ exert on the heavy mass $M$?**

**Neglect friction.**

Ans:
Assume that the acceleration of $M_1$ is $\boldsymbol{a}_1 = (a, 0)$. Then the acceleration of $M$ will be $\boldsymbol{a} = (a/2, a/2)$. Suppose that the force between $M_1$ and $M$ and the force between $M_2$ and $M$ are $F_1 = (f_1, f_1)$ and $F_2 = (f_2, f_2)$, then $F - f_1 = M_1 a$, $f_1 + f_2 - Mg = Ma/2$ and $f_1 - f_2 = Ma/2$. Thus, $a = (F - Mg/2)/(M_1 + M/2) = 2g$.

1. $\boldsymbol{a}_1 = (a, 0) = 2g$, rightward.
2. $\boldsymbol{a} = (a/2, a/2) = (g, g) = \sqrt{2}g$, bearing $45^\circ$.
3. $F_2 = (f_2, f_2) = (Mg/2, Mg/2) \approx 272\,\text{kg-wt}$, bearing $45^\circ$.

## 7.15

**A mass $m$ is suspended from a frictionless pivot at the end of a string of arbitrary length, and is set to whirling in a horizontal circular path whose plane is a distance $H$ below the pivot point, as shown in Fig. 7-2. Find the period of revolution $T$ fo the mass in its orbit.**

Ans:
Assume that the circular path has radius $R$, then the magnitude of acceleration of $m$ is $R(2\pi/T)^2$. If the tension of the string is $T$, then $T \times H/\sqrt{H^2+R^2} = mg$ and
$T \times R/\sqrt{H^2+R^2} = ma = mR(2\pi/T)^2$. Hence, $T = 2\pi\sqrt{H/g}$.

## 7.16

**Two small, sticky, putty balls, $a$ and $b$, each of mass $1\,\mathrm{gram}$, travel under the influence of gravity with acceleration $-9.8\boldsymbol{k}\,\mathrm{m\,s^{-2}}$. Given the initial conditions: at $t=0$,**

$$
\begin{aligned}
r_a(0) &= 7\boldsymbol{i} + 4.9\boldsymbol{k},\\
v_a(0) &= 7\boldsymbol{i} + 3\boldsymbol{j},\\
r_b(0) &= 49\boldsymbol{i} + 4.9\boldsymbol{k},\\
v_b(0) &= -7\boldsymbol{i} + 3\boldsymbol{j},
\end{aligned}
$$

**find $r_a(t)$ and $r_b(t)$ for all times $t > 0$.**

Ans:
Since there are sticky, if they collides with each other, they will stick together. $r_a(t) = (7+7t)\boldsymbol{i} + 3t\boldsymbol{j} + (4.9-9.8t^2)\boldsymbol{k}$.
$r_b = (49-7t)\boldsymbol{i} + 3t\boldsymbol{j} + (4.9-9.8t^2)\boldsymbol{k}$. Note that when $t=3$, $r_a = r_b$, which mean they intersect at $t=3$ and will stick together, moving with velocity of their center of mass. Hence,

$$
r_a = \begin{cases} (7+7t)\boldsymbol{i} + 3t\boldsymbol{j} + 4.9(1-t^2)\boldsymbol{k} \\ 28\boldsymbol{i} + 3t\boldsymbol{j} + 4.9(1-t^2)\boldsymbol{k} \end{cases}
$$

and

$$
r_b = \begin{cases} (49-7t)\boldsymbol{i} + 3t\boldsymbol{j} + 4.9(1-t^2)\boldsymbol{k} \\ 28\boldsymbol{i} + 3t\boldsymbol{j} + 4.9(1-t^2)\boldsymbol{k} \end{cases}
$$

## 7.17

**You are on a ship traveling steadily east at** $15 \, \text{knots}$**. A ship on a steady course whose speed is known to be** $26 \, \text{knots}$ **is observed** $6.0 \, \text{miles}$ **due south of you; it is later observed to pass behind you, its distance of closest approach being** $3.0 \, \text{miles}$**.**

1. **What was the course of the other ship?**
2. **What was the time** $T$ **between its position south of you and its position of closest approach?**

Ans:

1 knot = 1.15 miles per hour.

We represent the position and velocity of the ships in terms of the basis of cartesian coordinate whose origin is fixed at where our ship observed the other ship at the first place, and whose unit is in mile. Thus, at $t = 0$, our ship is at $(0, 0)$ with velocity $(15, 0)$, and the other ship is at $(0, -6.0)$ with velocity $\boldsymbol{v} = (v_1, v_2)$ where $|\boldsymbol{v}| = 26$. After some time $T$, we have the closest distance between us and the other ship equals $3.0 \, \text{miles}$, namely, $|(15T - v_1 T, -6.0 + v_2 T)| = 3.0$. This happens when $d|(15t - v_1 t, -6.0 + v_2 t)|/dt = 0$. That is,

$$
\begin{aligned}
\frac{d}{dt}\left(225t^2 - 30t^2 v_1 + 36 - 12v_2 t + (v_1^2 + v_2^2)t^2\right)\Big|_T &= 0 \\
\frac{d}{dt}\left(225t^2 - 30t^2 v_1 + 36 - 12v_2 t + 676t^2\right)\Big|_T &= 0 \\
\frac{d}{dt}\left((901 - 30v_1)t^2 - 12v_2 t + 36\right)\Big|_T &= 0 \\
(1802 - 60v_1)t - 12v_2\Big|_T &= 0 \\
(1802 - 60v_1)T - 12v_2 &= 0
\end{aligned}
$$

Thus, $T = 12v_2/(1802 - 60v_1)$. Substitute T back to $|(15T - v_1 T, -6.0 + v_2 T)| = 3.0$, we have

$$
\frac{144v_2^2}{4 \times (901 - 30v_1)} - \frac{144v_2^2}{2 \times (901 - 30v_1)} + 36 = 9
$$

and $v_2^2 = 3(901 - 30v_1)/4$. Since $v_1^2 + v_2^2 = 26^2$, $4v_1^2 - 90v_1 - 1 = 0$, which implies that $v_1 \approx 18.77$ or $v_1 \approx 3.73$. Because we had later found that the other ship passed behind our ship, its velocity parallel to our velocity can't be greater than 15 knots. So $v_1 \approx 3.73$. Again, by $v_1^2 + v_2^2 = 26^2$ we have $\boldsymbol{v} = (3.73, 25.73)$. And $T \approx 0.19\,\mathrm{h}$. Therefore,

1. The course of the other ship was due North tilted a little bit east.
2. $T \approx 0.19\,\mathrm{h}$.