Exercise 6

6.1

When two bodies move along a line, there is a special system of coordinates in which the momentum of one body is equal and opposite to that of the other. That is, the total momentum of the two bodies is zero. This frame of reference is called the center-of-mass system (abbreviated CM). If the bodies have masses m_1 and m_2 and are moving at speed v_1 and v_2, show that the CM system is moving at speed

$$v_{\mathrm{CM}} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

Ans:
By the definition of the center-of-mass system, the origin of this system is the center of mass. Suppose that the bodies are at x_1 and x_2, respectively, then the position of the origin is at

$$x_{\mathrm{CM}} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

Differentiate both side of the equation with respect to time, we have

$$v_{\mathrm{CM}} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}$$

6.2

Generalize Ex. 6.1 to any number of masses moving along a line, i.e., show that the speed of the coordinate system, in which the

total momentum is zero, is given by

$$v_{\text{CM}} = \frac{\sum m_i v_i}{\sum m_i}$$

Ans:

By the definition, the position of the center of mass is given by

$$x_{\text{CM}} = \frac{\sum m_i x_i}{\sum m_i}$$

Differentiate both side of the equation with respect to t, we have

$$v_{\text{CM}} = \frac{\sum m_i v_i}{\sum m_i}$$

6.3

If T is the total kinetic energy of the two masses in Ex. 6.1, and T_{CM} is their total kinetic energy in the CM system, show that

$$T = T_{\text{CM}} + \left(\frac{m_1 + m_2}{2}\right) v_{\text{CM}}^2$$

Ans:

Assume that m_1 and m_2 have velocity v_1 and v_2, respectively. The total kinetic energy in the CM system is given by

$$\begin{aligned}
T_{\text{CM}} &= \frac{1}{2} m_1 (v_1 - v_{\text{CM}})^2 + \frac{1}{2} m_2 (v_2 - v_{\text{CM}})^2 \\
&= \left(\frac{1}{2} m_1 v_1^2 - m_1 v_1 v_{\text{CM}} + \frac{1}{2} m_1 v_{\text{CM}}^2\right) + \left(\frac{1}{2} m_2 v_2^2 - m_2 v_2 v_{\text{CM}} + \frac{1}{2} m_2 v_{\text{CM}}^2\right) \\
&= \left(\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2\right) + \left(-(m_1 + m_2) v_{\text{CM}}^2 + \frac{1}{2} (m_1 + m_2) v_{\text{CM}}^2\right) \\
&= T - \frac{1}{2} (m_1 + m_2) v_{\text{CM}}^2 \quad \left[(m_1 v_1 + m_2 v_2) v_{\text{CM}} = (m_1 + m_2) v_{\text{CM}}^2\right]
\end{aligned}$$

Therefore,

$$T = T_{\mathrm{CM}} + \frac{1}{2}(m_1 + m_2)v_{CM}^2$$

6.4

Generalize the result of Ex. 6.3 to any number of masses. Show that

$$T = T_{\mathrm{CM}} + \frac{\sum m_i}{2} v_{\mathrm{CM}}^2$$

Ans:

The kinetic energy in the CM system is

$$\begin{aligned}
T_{\mathrm{CM}} &= \sum \frac{1}{2} m_i (v_i - v_{\mathrm{CM}})^2 \\
&= \sum \frac{1}{2} m_i (v_i^2 - 2v_i v_{\mathrm{CM}} + v_{\mathrm{CM}}^2) \\
&= \sum \frac{1}{2} m_i v_i^2 + \sum \frac{1}{2} m_i v_{\mathrm{CM}}^2 - \sum m_i v_i v_{\mathrm{CM}} \\
&= T + \sum \frac{1}{2} m_i v_{\mathrm{CM}}^2 - \sum m_i v_{\mathrm{CM}}^2 \\
&= T - \sum \frac{1}{2} m_i v_{\mathrm{CM}}^2
\end{aligned}$$

Therefore,

$$T = T_{\mathrm{CM}} + \frac{\sum m_i}{2} v_{\mathrm{CM}}^2$$

6.5

Two gliders with masses m_1 and m_2 are free to move on a horizontal air track. m_2 is stationary and m_1 collides with it perfectly elastically. They rebound with equal and opposite velocities. What is the ratio m_2/m_1 of their masses?

Ans:
Assume that m_1 collides with velocity v_i and ends up with velocity v_f. By the conservation of momentum,

$$m_1 v_i = m_1 v_f - m_2 v_f$$

Plus, by the conservation of energy, we have

$$\frac{1}{2}m_1 v_i^2 = \frac{1}{2}m_1 v_f^2 + \frac{1}{2}m_2 v_f^2$$

Sorting the system equations,

$$\begin{aligned} m_1(v_i - v_f) &= -m_2 v_f \\ m_1(v_i - v_f)(v_i + v_f) &= m_2 v_f^2 \\ v_i + v_f &= -v_f \end{aligned}$$

We have $v_i = -2v_f$. And $m_1(-2)v_f = m_1 v_f - m_2 v_f \Rightarrow m_2/m_1 = 3$.

6.6

A neutron having a kinetic energy E collides head-on with a stationary nucleus of C^{12} and rebounds perfectly elastically in the direction from which it came. What is its final kinetic energy E'?

Ans:
Assume that the mass of the neutron is m, then the mass of the nucleus of C^{12} is $12m$. The initial velocity of the neutron is $\sqrt{\frac{2E}{m}}$. The velocity after the bounding is $-\sqrt{\frac{2E'}{m}}$. By the conservation of momentum,

$$m\sqrt{\frac{2E}{m}} = -m\sqrt{\frac{2E'}{m}} + 12mv'$$

where v' is the velocity of the nucleus after the collision. By the conservation of Energy, $E-E'=\frac{1}{2}12m(v')^2 \Rightarrow v'=\sqrt{\frac{E-E'}{6m}}$. Hence,

$$\begin{aligned}
\sqrt{2E} &= -\sqrt{2E'}+2\sqrt{6(E-E')} \\
2E+2E'+4\sqrt{EE'} &= 24(E-E') \\
13E'+2\sqrt{EE'}-11E &= 0 \\
(13\sqrt{E'}-11\sqrt{E})(\sqrt{E'}+\sqrt{E}) &= 0
\end{aligned}$$

Therefore, $E'=\left(\frac{11}{13}\right)^2 E \approx 0.71E$.

6.7

A projectile of mass $m=10\,\mathrm{kg}$ is shot vertically upward from the earth with an initial velocity $v_p=500\,\mathrm{m\,s^{-1}}$.

1. **Calculate the recoil velocity of the earth v_E.**
2. **Calculate the ratio of the kinetic energy of the earth T_E to that of the projectile T_p at the moment of their separation.**
3. **Sketch qualitatively the velocity and kinetic energy of the projectile and of the earth versus time.**

Neglect air resistance and the orbital motion of the earth.

Ans:
Assume that the mass of the earth is M.

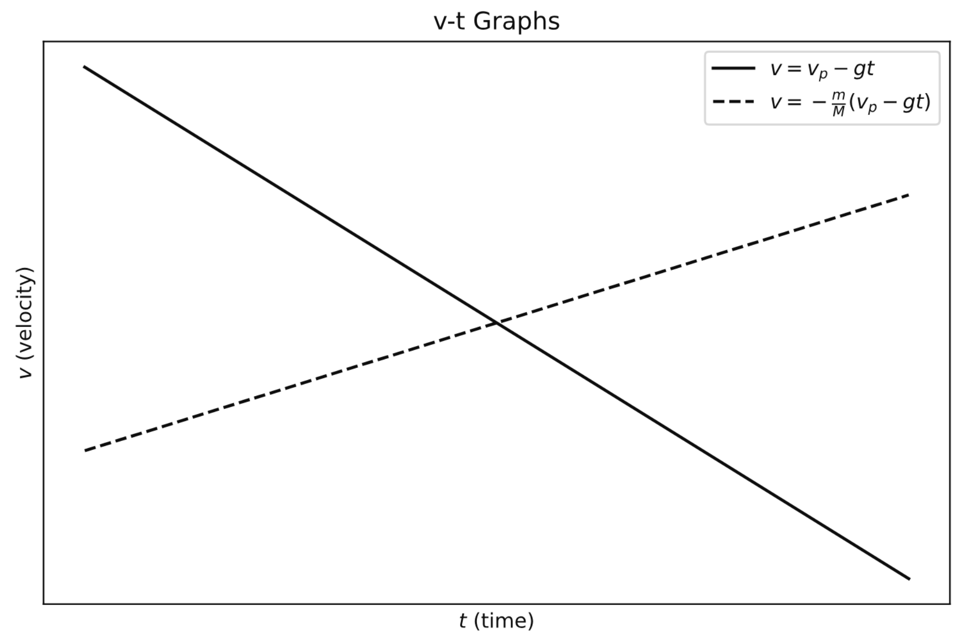
1. By the conservation of momentum,

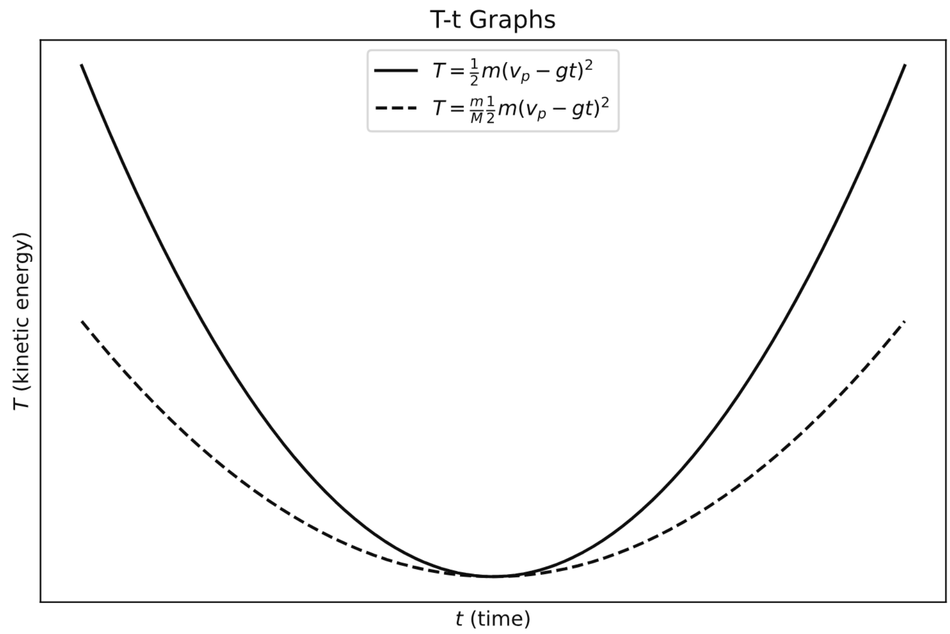
$$mv_p+Mv_E=0 \Rightarrow v_E=\frac{m}{M}v_p=\frac{10}{5.97\times10^{24}}\times500\approx 8.4\times10^{-22}\,\mathrm{m\,s^{-1}}$$

2. The ratio is

$$\begin{aligned}
\frac{T_E}{T_p} &= \frac{Mv_E^2/2}{mv_p^2/2} \\
&= \frac{m^2v_p^2/M}{mv_p^2} \\
&= \frac{m}{M} \\
&= \frac{10}{= 5.97 \times 10^{24}} \\
&\approx 1.7 \times 10^{-24}
\end{aligned}$$

3. By kinematic, with initial upward velocity v_p, $v = v_p - \int_0^t \frac{GMm}{(R_E + x(\sigma))^2} d\sigma \approx v_p - gt$. Hence the kinetic energy of the projectile is about $\frac{1}{2}m(v_p - gt)^2$. Consider the projectile and the earth as a system, since there is no external forces action on the system, the momentum is conserved in the system. Hence, $v_E = -\frac{m}{M}v_p$ and $\frac{T_E}{T_p} = \frac{m}{M}$. Below are the sketches.





6.8

A particle of mass $m = 1.0\,\mathrm{kg}$, traveling at a speed $V = 10\,\mathrm{m\,s^{-1}}$, strikes a particle at rest of mass $M = 4.0\,\mathrm{kg}$ and rebounds in the direction from which it came, with a speed V_F. If an amount of heat $h = 20\,\mathrm{joules}$ is produced in the collision, what is V_F? (Define all introduced quantities and state clearly from what physical laws your initial equations are derived.)

Ans:
Assume that M gained velocity v after the striking. Then by conservation of momentum,

$$mV = -mV_F + Mv$$

By conservation of energy,

$$\frac{1}{2}mV^2 = \frac{1}{2}mV_F^2 + \frac{1}{2}Mv^2 + h$$

Combining the two equations, we have

$$\begin{array}{c} M^2v^2 = m^2(V+V_F)^2 \\ Mv^2 = m(V+V_F)(V-V_F) - 2h \\ \hline \dfrac{1}{M} = \dfrac{1}{m}\dfrac{V-V_F}{V+V_F} - \dfrac{2h}{m^2(V+V_F)^2} \end{array}$$

Hence,

$$m^2(V+V_F)^2 = mM(V^2 - V_F^2) - 2Mh$$

and

$$\begin{aligned} 0 &= m(m+M)V_F^2 + 2m^2VV_F + m(m-M)V^2 + 2Mh \\ V_F &= \frac{1}{2m(m+M)}\left(-2m^2V \pm \sqrt{4m^4V^2 - 4m(m+M)(m(m-M)V^2 + 2Mh)}\right) \\ &= \frac{1}{10}(-20 \pm \sqrt{400 - 20(-300+160)}) \\ &\approx 3.66\ \mathrm{m\,s^{-1}} \end{aligned}$$

6.9

A machine gun mounted on the north end of a $10,000\ \mathrm{kg}$, $5\ \mathrm{m}$ long platform, free to move on a horizontal air-bearing, fires bullets into a thick target mounted on the south end of the platform. The gun fires 10 bullets of mass $100\ \mathrm{g}$ each every second at a muzzle velocity of $500\ \mathrm{m\,s^{-1}}$. Does the platform move? If so, in what direction at what speed v?

Ans:
Consider the whole dynamics to be a system. Since there is no external forces acting upon it, the momentum is conserved and the center of mass stands still. As bullets flies from north to south, for the sake of balancing the mass, the platform moves toward north.

Assume that the platform is put along on a one-dimensional space with the origin at center of mass of the platform. Then the center of mass of the system is given by 6.1, namely

$$x_{\mathrm{CM}} = \frac{10 \times 0.1 \times 2.5}{10 \times 0.1 + 10,000} = \frac{2.5}{10,001}$$

After the bullets shot and hit, by conservation of momentum, the system stood still and x_{CM} remained the same. Assume that the platform moves x, then

$$x_{\mathrm{CM}} = \frac{2.5}{10,001} = \frac{10 \times 0.1 \times (x - 2.5) + 10,000x}{10,001}$$

that is, $x = \frac{5}{10{,}001}$. It takes 10 seconds to finished the shots, therefore, on average, the speed v is equal to $\frac{5}{10{,}001}/10 \approx 5 \times 10^{-5}\,\mathrm{m\,s^{-1}}$. ***However*** the movement is discrete, since the time the bullet passed from north to south is $5/500 = 0.01\,\mathrm{s}$, which is much smaller than the period $1\,\mathrm{s}$ of the shooting. Hence, by conservation of momentum, during the bullet flying, the speed of the platform is

$$500 \times 0.1 = 10,000v \Rightarrow v = 5 \times 10^{-4}\,\mathrm{m\,s^{-1}}$$

6.10

A mass m_1 is connected by a cable over a pulley to a container of water, which initially has mass $m_2(t=0) = m_0$, as shown in Fig. 6-1. the system is then released and m_2 (with help of an internal pump) ejects water in the downward direction at a constant rate $dm/dt = r_0$ with velocity v_0 relative to the container. Find the acceleration a of m_1, as a function of time. Neglect the masses of cabler and pulley.

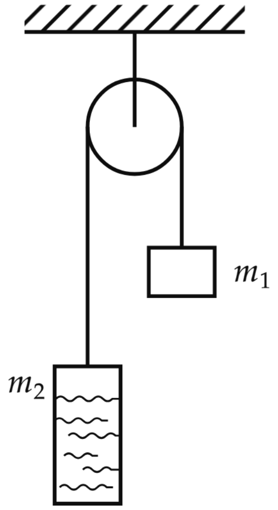


Figure 6-1

Ans:

Assume that the tension of the cable is $T(t)$, then

$$T(t) - (m_2 - r_0 t)g + \frac{d}{dt}\int_0^t \frac{d}{d\tau} m v_0 d\tau = -m_2 \boldsymbol{a}(t) \ldots (1)$$

Note that the $\frac{d}{dt}\int_0^t \frac{d}{d\tau} m v_0 d\tau$ part is derived by conservation of momentum. We write the change of the momentum as a function $\int_0^t \frac{d}{d\tau} m v_0 d\tau$ and take its derivative, which is the rate of change of momentum, i.e., the force.

For m_1, we have

$$T(t) - m_1 g = m_1 \boldsymbol{a}(t) \ldots (2)$$

Combining equation (1) and equation (2), and cancel out $T(t)$. Then

$$\begin{array}{r} T(t) - (m_2 - r_0 t)g + \frac{d}{dt}\int_0^t \frac{d}{d\tau} m v_0 d\tau = -m_2 \boldsymbol{a}(t) \\ T(t) - m_1 g = m_1 \mathbf{a}(t) \\ \hline (m_2 - m_1 - r_0 t)g - v_0 r_0 = (m_1 + m_2)\boldsymbol{a}(t) \end{array}$$

Hence,

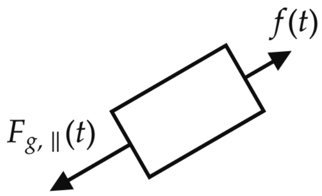
$$\boldsymbol{a}(t) = \frac{(m_2 - m_1)g - r_0(gt + v_0)}{m_1 + m_2}$$

6.11

A toboggan slides down an essentially frictionless, snow covered slope, scooping up snow along the path. If the slope is 30° and the toboggan picks up $0.50\,\mathrm{kg}$ of snow per meter of travel, calculate its acceleration a at an instant when its speed is $4.0\,\mathrm{m\,s^{-1}}$ and its mass (including content) is $9.0\,\mathrm{kg}$.

Ans:

We draw the free-body diagram of the slides including content.



Here, $F_{g,\parallel}(t) = M_{All}g\sin 30^\circ$ and $f(t) = mv^2(t)$ where M_{All} is the total mass, m is the mass of the snow the toboggan picks up per meter of travel, and $v(t)$ is the speed of the toboggan. $f(t)$ is so in a sense that $mv(t)\,\mathrm{kg\,s^{-1}}$ indicates how much mass of snow the toboggan picks up at the moment t, and $mv^2(t)$ indicates the quantity of the rate of change in momentum for the new picked up snow. Therefore,

$$\begin{aligned} F_{g,\parallel}(t) - f(t) &= M_{All}a(t) \\ M_{All}g\sin 30^\circ - mv^2(t) &= M_{All}a(t) \end{aligned}$$

Substitute with the given quantity, we have a

$$\begin{aligned} a(t) &= 9.81 \times \frac{1}{2} - \frac{1}{9} \times 0.50 \times 4.0^2 \\ &\approx 4.0\,\mathrm{m\,s^{-1}} \end{aligned}$$

6.12

The end of a chain, of mass per unit length μ, at rest on a table top at $t = 0$, is lifted vertically at a constant speed v, as shown in Fig. 6-2. Evaluate the upward lifting force F as a function of

time.

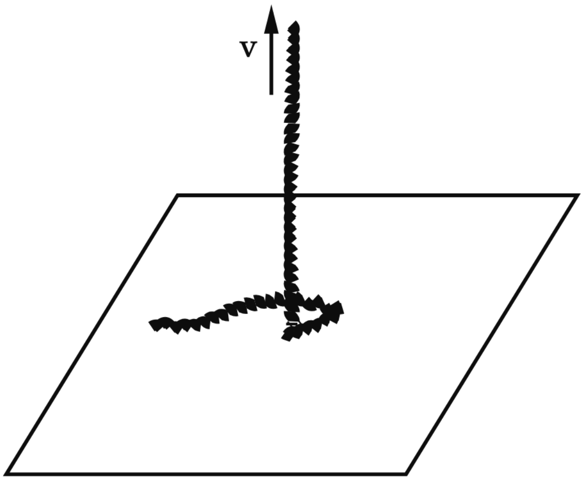


Figure 6-2

Ans:

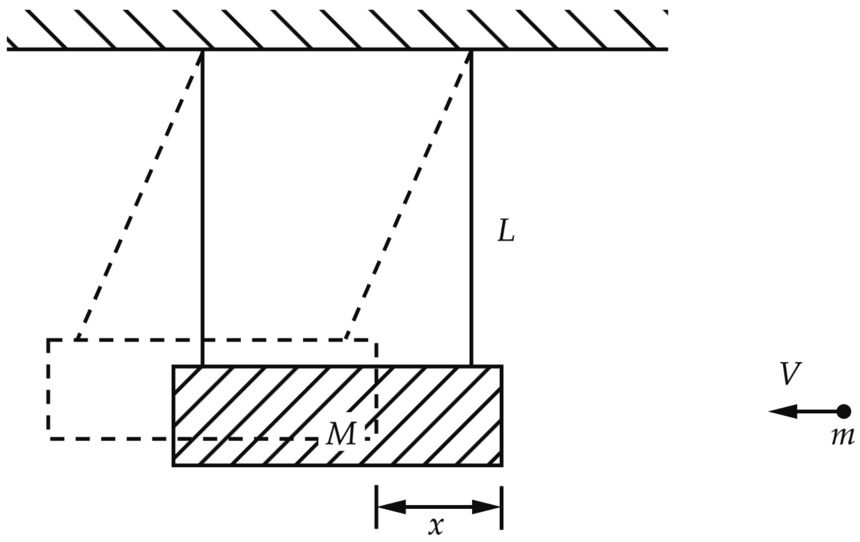
The rate of change in mass of the part chain being lifted up is μv, and thus the rate of change of momentum is μv^2. Since the chain is lifted up in a constant speed, for the part chain that is lifted up, the net force acting upon it is zero. Hence,

$$F - \mu v^2 - g \int_0^t \mu v = 0 \Rightarrow F = \mu v(v + gt)$$

6.13

The speed of a rifle bullet may be measured by means of a ballistic pendulum: The bullet, of known mass m and unknown speed V, embeds itself in a stationary wooden block of mass M, suspended as a pendulum of length L, as shown in Fig. 6-3. This sets the block to swinging. The amplitude x of swing may be measured and, using conservation of energy, the velocity of the block immediately after impact may be found. Derive an expression for the speed of the

bullet in terms of m, M, L, and x.



Ans:

By conservation of energy, assume that the speed after embedding is v,

$$
\begin{aligned}
(M+m)g(L-\sqrt{L^2-x^2}) &= \frac{1}{2}(M+m)v^2 \\
2g(L-\sqrt{L^2-x^2}) &= v^2 \\
\sqrt{2g(L-\sqrt{L^2-x^2})} &= v
\end{aligned}
$$

By conservation of momentum,

$$
\begin{aligned}
mV &= (M+m)v \\
V &= \frac{M+m}{m}\sqrt{2g(L-\sqrt{L^2-x^2})}
\end{aligned}
$$

We use binomial expansion to approximate the vertical change in distance of the swing,

$$
\begin{aligned}
L-\sqrt{L^2-x^2} &= L-(L^2-x^2)^{1/2} \\
&= L-\left(L-\frac{1}{2}(L^2)^{-1/2}x^2-\frac{1}{8}(L^2)^{-3/2}(x^2)^2+\ldots\right) \\
&\approx L-\left(L-\frac{1}{2}(L^2)^{-1/2}x^2\right) \quad \text{for small } x. \\
&= \frac{x^2}{2L}
\end{aligned}
$$

Hence,

$$v \approx \sqrt{\frac{g}{L}}x \quad \text{and} \quad V \approx x\left(\frac{M+m}{m}\right)\sqrt{\frac{g}{L}}$$

6.14

Two gliders A and A' are connected rigidly together and have a combined mass M and are separated by a distance $2L$. Another glider B of mass m, length L, is constrained to move between A and A', as shown in Fig. 6-4. All gliders move on a very long linear air track without friction. All collisions between (A, A') and B are perfectly elastic. Originally the whole system is at rest and glider B is in contact with glider A. A cap between A and B then is exploded, giving a total kinetic energy T to the system.

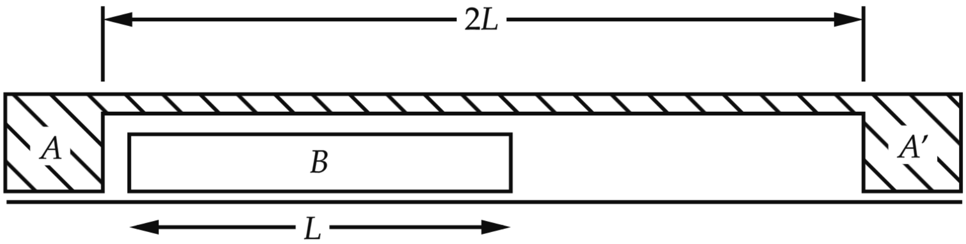


Figure 6-4

1. **Show the *qualitative* features of B's motion, i.e., position x on the track, velocity v with respect to the track, by sketching x and v as functions of time. Use the *same* time scale for both sketches.**
2. **Calculate the period τ_0 in terms of T, L, m, and M.**

Hint: **The relative velocity of B with respect to (A, A') is**

$$\boldsymbol{v}_{rel} = \boldsymbol{v}_B - \boldsymbol{v}_{(A,A')}.$$

Ans:

Assume that the velocities of (A, A') and B is v_i and u_i, respectively, after the ith collision. v_0 and u_0 is the original velocities after the explosion. By conservation of momentum and energy,

$$\begin{array}{c} \frac{1}{2}Mv_0^2 + \frac{1}{2}mu_0^2 = T \\ Mv_0 + mu_0 = 0 \\ \hline M^2v_0^2 + Mmu_0^2 = 2TM \\ M^2v_0^2 - m^2u_0^2 = 0 \end{array}$$

We have $v_0 = -\frac{m}{M}\sqrt{\frac{2M}{m(m+M)}T} = -\sqrt{\frac{2m}{M(m+M)}T}$ and $u_0 = \sqrt{\frac{2M}{m(m+M)}T}$.

Since the momentum and energy is conserved, there are two different equations for two variables, which the variables will be determined. Hence, there exists one and only one solution for the quantity of v_0 and u_0, which have been calculated above. Note that they would change sign by every collision. Therefore,

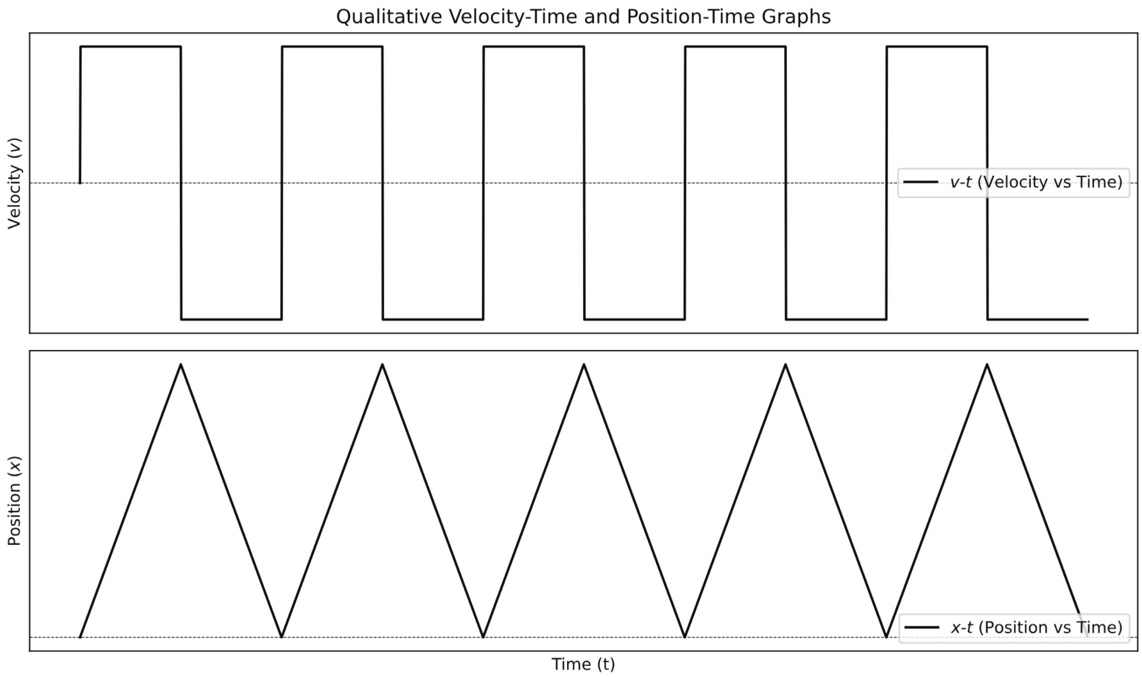
$$v_i = \begin{cases} \sqrt{\frac{2m}{M(m+M)}T} & \text{if } i \text{ is odd.} \\ -\sqrt{\frac{2m}{M(m+M)}T} & \text{if } i \text{ is even.} \end{cases}$$

$$u_i = \begin{cases} -\sqrt{\frac{2M}{m(m+M)}T} & \text{if } i \text{ is odd.} \\ \sqrt{\frac{2M}{m(m+M)}T} & \text{if } i \text{ is even.} \end{cases}$$

We can compute the time it would take for one collision:

$$
\begin{aligned}
(u-v)t &= L \\
t &= \frac{L}{(u-v)} \\
&= \frac{L}{\left(\left(1+\frac{m}{M}\right)\sqrt{\frac{2M}{m(m+M)}T}\right)} \\
&= \frac{L}{\left(\frac{M+m}{M}\right)\sqrt{\frac{2M}{m(m+M)}T}} \\
&= L\sqrt{\frac{Mm}{2T(M+m)}}
\end{aligned}
$$

1. The graphs are



2. A period is the time the system spends colliding twice, namely,

$$
2L\sqrt{\frac{Mm}{2T(M+m)}}
$$

6.15

Two equally massive gliders, moving on a level air track at equal and opposite, velocities, v and $-v$, collide almost elastically, and rebound with slightly smaller speeds. They lose a fraction $f \ll 1$ of their kinetic energy in the collision. If these same gliders collide

with one of them initially at rest, with what speed will the second glider move after the collision? (This small residual speed Δv may easily be measured in terms of the final speed v of the originally stationary glider, and thus the elasticity of the spring bumpers may be determined.)
Note: **If $x \ll 1$, $\sqrt{1-x} \approx 1-\frac{x}{2}$.**

Ans:
In the first situation, when the gliders both move, they lost fraction f of their kinetic energy in the collision. In the second situation, one glider stay still. So the relative velocity between them is $\boldsymbol{v}$ instead of $2\boldsymbol{v}$. From atomic perspective, the atoms collides more mildly. One way to think this is that with less velocity, right after the collision, the atoms strike less deeply into the atoms of the other gliders, causing a much more shallow layer of atoms to vibrate. If the relation between the depth of vibration, vibration and the velocity is linear, it is reasonable to guess that with relative velocity $\boldsymbol{v}$, we have the fraction of energy loss half with that with relative velocity $2\boldsymbol{v}$, namely $\frac{f}{2}$.
Assume that the mass of the gliders are m. After the collision, say the second glider have velocity $\boldsymbol{u}$ and the first glider $\boldsymbol{w}$, by conservation of momentum and energy,

$$
\begin{aligned}
m\boldsymbol{v} &= m\boldsymbol{u} + m\boldsymbol{w} \\
\frac{1}{2}m\boldsymbol{v}^2 &= \frac{1}{2}m\boldsymbol{u}^2 + \frac{1}{2}m\boldsymbol{w}^2 + \frac{1}{2}m\boldsymbol{v}^2\frac{f}{2}
\end{aligned}
$$

Then we have,

$$
\begin{aligned}
\boldsymbol{v}^2(1-f) &= \boldsymbol{u}^2 + \boldsymbol{v}^2 - 2\boldsymbol{v}\boldsymbol{u} + \boldsymbol{u}^2 \\
0 &= 2\boldsymbol{u}^2 - 2\boldsymbol{v}\boldsymbol{u} + \frac{f}{2}\boldsymbol{v}^2 \\
\boldsymbol{u} &= \frac{1}{4}(2\boldsymbol{v} \pm \sqrt{4\boldsymbol{v}^2 - 4f\boldsymbol{v}^2}) \\
&= \frac{\boldsymbol{v}}{2}(1 \pm \sqrt{1-f}) \\
&\approx \frac{\boldsymbol{v}}{2}\left(1 \pm \left(1 - \frac{f}{2}\right)\right)
\end{aligned}
$$

Here, $\boldsymbol{u}$ can only be $\frac{v}{2}\left(1+1-\frac{f}{2}\right)=\boldsymbol{v}-\frac{vf}{4}$ since it indicates the velocity of the originally stationary gliders and

$\boldsymbol{w}=\frac{v}{2}\left(1-\left(1-\frac{f}{2}\right)\right)=\frac{vf}{4}$ indicates the originally moving gliders. The velocity left for the originally moving gliders is the residual speed, which is $\Delta v=\frac{vf}{4}$.

6.16

A rocket of initial mass $m=M_0$ ejects its burnt fuel at a constant rate $dm/dt=-r_0$ and at a velocity V_0 (relative to the rocket).

1. **Calculate the initial acceleration a of the rocket (neglect gravity).**
2. **If $V_0=2.0\,\mathrm{km\,s^{-1}}$, at what rate r_0 must fuel be ejected to develop $10^5\,\mathrm{kg\text{-}wt}$ of thrust?**
3. **Write a differential equation which connects the speed v of the rocket with its residual mass $m=M$, and solve the equation, if you can.**

Ans:

1. The rate of change in momentum of the fuel is r_0V_0. By Newton's third law of dynamics (Conservation of Momentum), the rocket get a lift up force r_0V_0. By Newton's second law of dynamics, the initial acceleration is given by

$$\begin{aligned} M_0a &= r_0V_0 \\ a &= \frac{r_0V_0}{M_0} \end{aligned}$$

2. To get lifted up, the lift-up force need to overcome the force of gravity. Hence,

$$r_0\times 2.0\times 10^3\geq 10^5\times 9.8\Rightarrow r_0\geq 490\,\mathrm{kg\,s^{-1}}$$

3. The mass at time t is $M = M_0 - r_0 t$. And the differential equation is

$$r_0 V_0 - (M_0 - r_0 t)g = (M_0 - r_0 t)v'(t)$$

It is easy to solve,

$$\begin{aligned} v(t) &= \int_0^t \frac{r_0 V_0}{M_0 - r_0 \tau} d\tau - gt = -V_0 \ln(M_0 - r_0 t) - gt \\ v(M) &= -V_0 \ln(M) - gt \end{aligned}$$

If we neglect gravity, they the differential equation is

$$r_0 V_0 = (M_0 - r_0 t)v'(t)$$

and

$$v(M(t)) = \int_{M(0)}^{M(t)} \frac{-V_0}{M(t)} dM = -V_0 \ln \left| \frac{M}{M_0} \right|$$

Note that we substituted variable $M = M_0 - r_0 t$.

6.17

An earth satellite of mass $10\,\mathrm{kg}$ and average cross-sectional area $0.50\,\mathrm{m}^2$ is moving in a circular orbit at $200\,\mathrm{km}$ altitude where the molecular mean free paths are many meters and the air density is about $1.6 \times 10^{-10}\,\mathrm{kg\,m^{-3}}$. Under the crude assumption that the molecular impacts with the satellite are effectively inelastic (but that the molecules do not literally stick to the satellite but drop away from it at low relative velocity),

1. **Calculate the retarding force F_R that the satellite would experience due to air friction.**
2. **How should such a frictional force vary with the satellite's velocity v? Would speed of a circular satellite orbit vs. height.**

Ans:

With altitude 200 km, the speed of the satellite is computed,

$$\frac{v^2}{(6380+200)\times 10^3} = \frac{G\times 5.98\times 10^{24}}{((6380+200)\times 10^3)^2}$$
$$v \approx 7.784\times 10^3\ \mathrm{m\,s^{-1}}$$

Let the cross-sectional area be A, the density of the air be ρ. By definitions, $v=\frac{dx}{dt}$ and $\rho=\frac{dm}{dV}$. Thus $vA\rho=\frac{dx}{dt}A\frac{dm}{dV}=\frac{dm}{dt}$. Since the collision of the molecules is effectively inelastic, the rate of change of momentum is $\frac{dmv}{dt}=v\frac{dm}{dt}=v^2A\rho$.

1. The retarding force is given by the rate of change of momentum of the molecules colliding inelastically on the satellite, which is

$$F_R = v^2A\rho \approx 4.8\times 10^{-3}\ \mathrm{N}$$

2. From above $F_R \propto v^2$.