Algorithms & Data Structures II

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December 6, 2025

Dynamic Programming

Weighted Interval Scheduling

The idea: Sort by finish time and select:

$$\text{OPT}(j) = \begin{cases} \max(v[j]) + \text{OPT}(p[j]) \\ \text{OPT}(j-1) \end{cases}$$

Either you take job j, and skip conflicts, or don't take it. The algorithm runs in $O(n \log n)$ time, where n is the number of jobs.

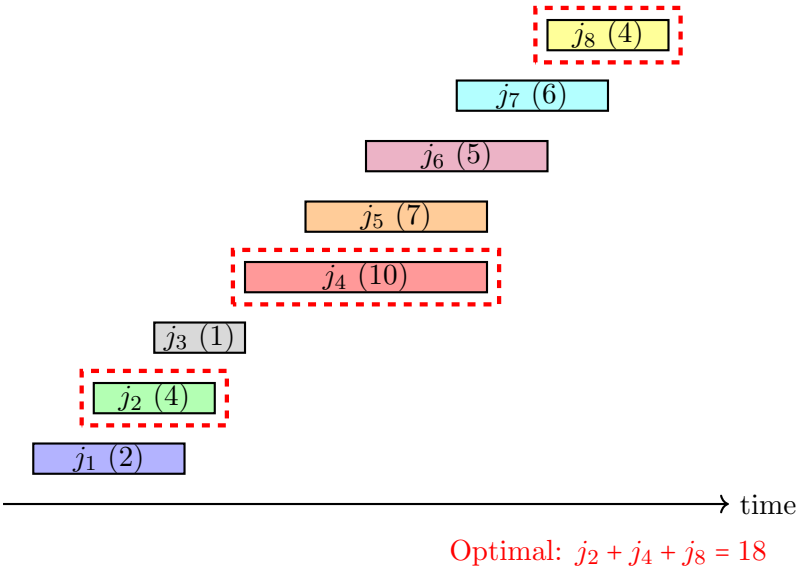


Figure 1: Weighted Interval Scheduling Example: Jobs are sorted by finish time. The optimal solution selects non-overlapping jobs with maximum total value.

Subset Sum TODO

Knapsack

The problem: Fill a backpack with as many books as possible.

$$\text{OPT}(i, w) = \begin{cases} \text{OPT}(i-1, w) \\ v[i] + \text{OPT}(i-1, w - w[i]) \end{cases}$$

The option is to take an item i or not and the algorithm runs in $O(nW)$ time, where n is the number of items and W is knapsack capacity. The algorithm creates a 2D table consisting of $n \times W$

Sequence Alignment TODO

Match characters, gap in X, or gap *The problem:* Match characters, gap in X, or gap in Y. The running time of this algorithm is

Max Flow & Min Cut

Flow Algorithms

Name	Running Time	Explanation	Notes
Ford-Fulkerson	$\mathrm{O}(\lvert f^* \rvert m)$	$\lvert f^* \rvert$ = max flow value m = # edges	Pseudopolynomial; can fail to terminate
Scaling	$\mathrm{O}(m^2 \log C)$	C = max edge capacity	Weakly polynomial; assumes integer capacities
Edmonds-Karp	$\mathrm{O}(m^2 n)$	n = # vertices m = # edges	Uses BFS; strongly polynomial

Running the algorithms

When running the algorithms by hand, just use a basic approach—it's manageable when the graph is reasonably simple. For min-cut, find the saturated edges (at capacity) along the residual graph boundary between reachable and unreachable vertices from the source.

Flow Problem Recipe

Build the Graph

- Vertices (source + sink)
- Edges (capacities + direction)

Describe the Algorithm

- Find max flow (m) and use the fastest flow algorithm for your given case
- Return an answer

Analyse Runtime

- Build Parameters from question
- Remember the time it takes to build the graph

Edmonds-Karp real running time: $\mathrm{O}(\min(m\lvert f^* \rvert, n \cdot m^2))$

Randomised Algorithms

Expected Value

Expected value is the average outcome over many trials. For a discrete random variable X:

$$\mathbb{E}[X] = \sum_i x_i \cdot \mathbb{P}[X = x_i]$$

Common trick: Use indicator random variables. If $X = X_1 + X_2 + \cdots + X_n$ where each X_i is an indicator (0 or 1), then:

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X_i] = \sum_{i=1}^{n} \mathbb{P}[X_i = 1]$$

Geometric distribution: If success probability is p, then expected number of trials until first success is $\frac{1}{p}$.

Example - QuickSort: Let X_{ij} be an indicator that equals 1 if elements i and j are compared, 0 otherwise. Total comparisons:

$$X = \sum_{i<j} X_{ij} \implies \mathbb{E}[X] = \sum_{i<j} \mathbb{E}[X_{ij}] = \sum_{i<j} \mathbb{P}[i \text{ and } j \text{ compared}]$$

Elements i and j are compared if one is chosen as pivot before any element between them, so $\mathbb{P}[i \text{ and } j \text{ compared}] = \frac{2}{j-i+1}$. Summing gives $\mathbb{E}[X] = \mathrm{O}(n \log n)$.

Useful Sums

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x| < 1$$

$$\sum_{n=0}^{\infty} n \cdot x^n = \frac{x}{(1-x)^2}, \quad |x| < 1$$

$$\sum_{k=1}^{n} \frac{1}{k} = H(n), \quad \ln(n) < H(n) < \ln(n) + 1$$

$$\mathbb{P}[\text{first success in round } j] = (1-p)^{j-1} p$$

$$\mathbb{E}[X] = \sum x_i \cdot \mathbb{P}[X = x_i] \quad \text{where } x_i \in \text{possible outcomes}$$

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y]$$

$$\text{if } \mathbb{P}[\text{success}] = p \text{ then } \mathbb{E}[\#\text{ tries to success}] = \frac{1}{p}$$

Randomised Algorithms

- Contention solution (??)
- Minimum cut
- Coupon Collector
- Selection
- QuickSort
- Hashing

Hashing

Chained Hashing

Store elements in array of linked lists. Each element x goes to list at $A[h(x)]$. Operations take $O(|A[h(x)]| + 1)$ time. With universal hashing, expected $O(1)$ time per operation.

Linear Probing

Store elements directly in array. If $A[h(x)]$ is occupied, place x in next empty slot (wrapping around). Forms clusters of consecutive elements. Operations take $O(C(h(x)) + 1)$ time where $C(i)$ is cluster size. Cache-efficient but DELETE is $O(n^2)$.

Hash Functions

Universal hashing: Family H is universal if for any $x \neq y$, $\Pr[h(x) = h(y)] \leq \frac{1}{m}$ for random $h \in H$.

Dot product hashing: For $x = (x_1, x_2, \ldots, x_r)$ in base m (prime), define

$$h_a(x) = (a_1x_1 + a_2x_2 + \cdots + a_rx_r) \bmod m$$

where $a = (a_1, \ldots, a_r)$ chosen randomly. This family is universal.

Performance Comparison

Data Structure	SEARCH	INSERT	DELETE	Space
Linked List	$O(n)$	$O(n)$	$O(n)$	$O(n)$
BBST	$O(\log n)$	$O(\log n)$	$O(\log n)$	$O(n)$
Direct Addressing	$O(1)$	$O(1)$	$O(1)$	$O(\|U\|)$
Chained Hashing (universal)	$O(1)^*$	$O(1)^*$	$O(1)^*$	$O(n)$
Linear Probing	$O(C(h(x)) + 1)$	$O(C(h(x)) + 1)$	$O(n^2)$	$O(n)$

* Expected time (when using universal hash function)

Amortised Analysis

Aggregate Method

The idea: Analyze the total cost $T(m)$ of a worst-case sequence of m operations, then compute the amortized cost as $T(m)/m$ per operation.

Accounting Method

Assign a *cost* $\hat{c}_i$ to each operation. When $\hat{c}_i > c_i$, store the difference as credits. When $\hat{c}_i < c_i$, use stored credits to pay. Ensure $\sum_{i=1}^{m} \hat{c}_i \geq \sum_{i=1}^{m} c_i$ always holds.

Potential Method

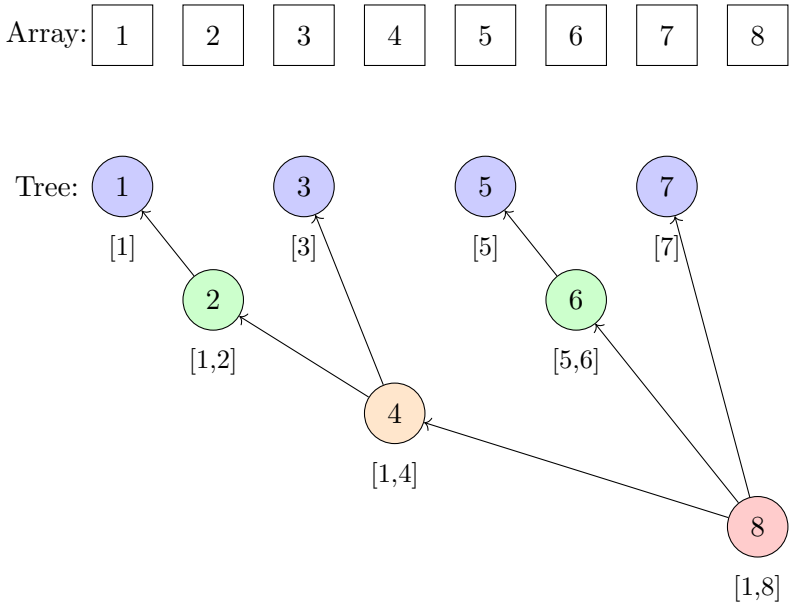
Define a potential function $\Phi(D)$ that maps the data structure state to a real value. Set amortized cost $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1})$. Require $\Phi(D_i) \geq 0$ for all i and $\Phi(D_0) = 0$.

Partial Sums & Dynamic Arrays

Partial Sums (Fenwick Trees)

A fenwick tree is effectively binary tree consisting of only the right child. This allows for an in-place data structure to compute the partial sum.

	Sum	Update	Space	Create
Fenwick Tree	$O(\log n)$	$O(\log n)$	In-place	$O(n \log n)$



Example

here is an example on how to calculate a fenwick tree.

```
Index:  1   2   3   4   5   6   7   8
Value:  3   2   5   1   4   6   2   3

Index:  1   2   3   4   5   6   7   8
BIT:    3   2   5   1   4   6   2   3

Index:  1   2   3   4   5   6   7   8
BIT:    3   2   5   1   4   6   2   3

Index:  1   2   3   4   5   6   7   8
BIT:    3   5   5   1   4   6   2   3
```

```
Index:  1   2   3   4   5   6   7   8
BIT:    3   5   5   1   4   6   2   3

Index:  1   2   3   4   5   6   7   8
BIT:    3   5   5  11   4   6   2   3

Index:  1   2   3   4   5   6   7   8
BIT:    3   5   5  11   4   6   2   3

Index:  1   2   3   4   5   6   7   8
BIT:    3   5   5  11   4  10   2   3

Index:  1   2   3   4   5   6   7   8
BIT:    3   5   5  11   4  10   2   3

Index:  1   2   3   4   5   6   7   8
BIT:    3   5   5  11   4  10   2  26

Final Fenwick Tree:
Index:  1   2   3   4   5   6   7   8
BIT:    3   5   5  11   4  10   2  26
Range: [1][1,2][3][1,4][5][5,6][7][1,8]
```

Dynamic Arrays

Rotated Array

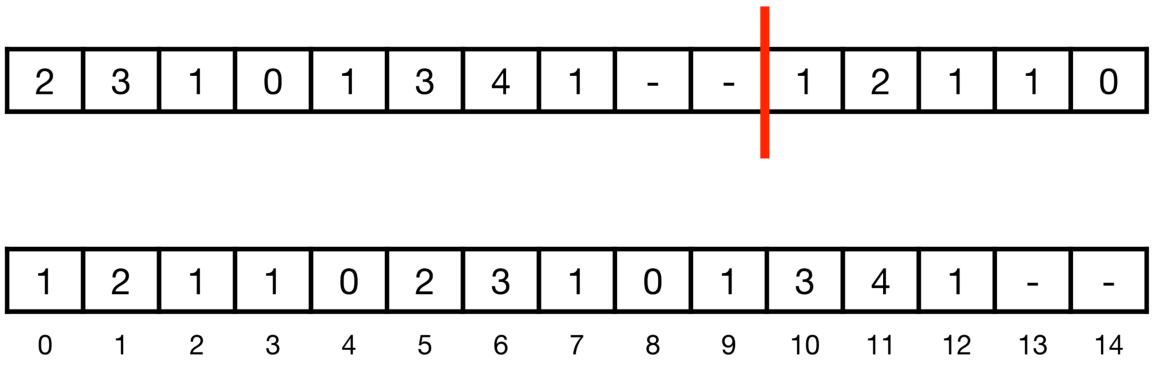


Figure 2: 1 level rotated array

Key idea: Store offset to mark the start of the array.

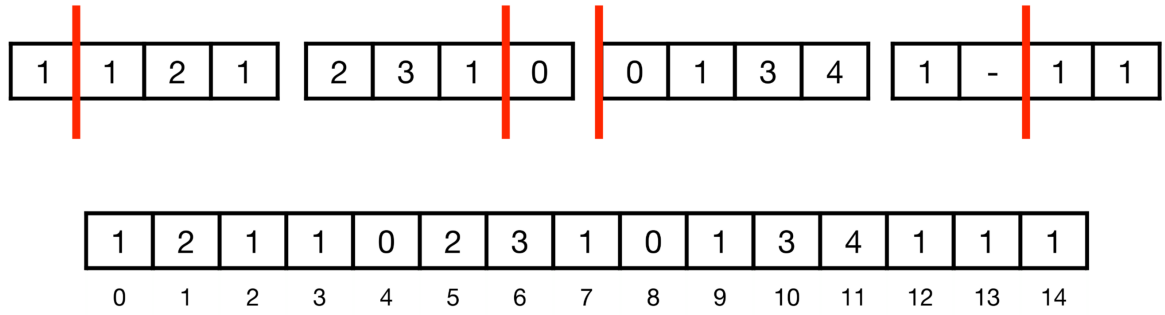


Figure 3: 2 level rotated array

Key idea: Split into $\sqrt{n}$ rotated buckets of size $\sqrt{n}$. When inserting/deleting, rebuild one bucket in $O(\sqrt{n})$ time, then propagate overflow/underflow.

Data Structure	Access	Insert	Delete	space
1 level rotated array	O(1)	$O(\sqrt{n})$	$O(\sqrt{n})$	$O(n)$
2 level rotated array	O(1)	$O(n^{\varepsilon})$	$O(n^{\varepsilon}$	$O(n)$

In a k-level rotated array, $\varepsilon = \frac{1}{k}$ determines that you partition n elements into $\sqrt[k]{n}$ buckets of size $\sqrt[k]{n}$ each, giving $O(n^{1/k}) = O(n^{\varepsilon})$ time for INSERT/DELETE operations.

NP Completeness

Boils down to if a problem is solvable in polynomial time or non-deterministic polynomial time. The mindset I have for this is best describes via exercise 2 from week *NP Completeness*.

Recipe

I want to find a NP problem, and then show that it is polynomially reducable to my problem (convert it to my problem) and argue correctness.

Example - Customer Uniqueness (KT 8.2)

A store trying to analyse the behavior of its customers will often main a two-dimensional array A, where the rows correspond to its customers and the columns correspond to the products it sells. The entry $A[i, j]$ specifies the quantity of project j that has been purchased by customer i.

Here's a tiny example of such an array A

	liquid detergent	beer	diapers	cat litter
Raj	0	6	0	3
Alanis	2	0	0	7
Chelsea	0	0	0	7

One thing that a store might want to do with this data is the following. Let us say that a subset S off the customers is *diverse* if no two of the customers in S have ever bought the same

product (i.e., for each product, at most one of the customers in s has ever bought it). A diverse set of customers can be useful, for example, as a target pool for market research.

We can now define the Diverse Subset Problem as follows: Given an $m \times n$ array A as defined above, and a number $k \leq m$, is there a subset of at least k of customers that is *diverse*?

Show that Diverse subset in NP-complete.

Solution

We show that $IS \leq_p DS$, where IS is Independent Set and DS is Diverse Subset.

Given a graph $G = (V, E)$ with $|V| = m$ vertices and $|E| = n$ edges, construct an $m \times n$ matrix A where vertices are customers (rows) and edges are products (columns). For each edge $(u, v) \in E$, set $A[u, (u, v)] = 1$ and $A[v, (u, v)] = 1$ (all other entries are 0).

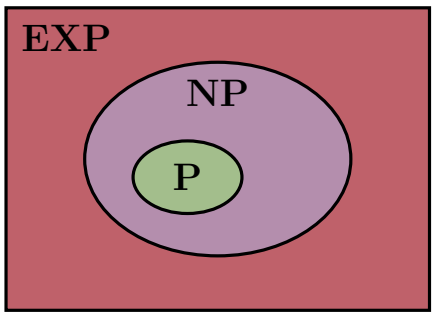
Now G has an independent set of size k if and only if A has a diverse subset of size k. Since the construction takes polynomial time, DS is NP-complete.

Describe the argument a little better

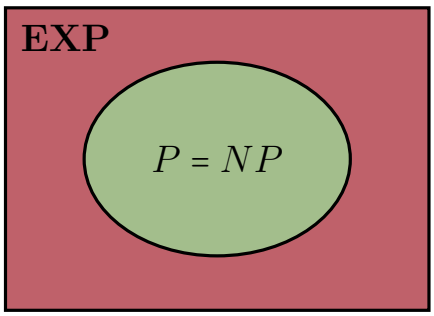
List of NP-complete Problems

- Subset Sum
- Independent Set (Graph Algorithm)
- Vertex Cover (Graph Algorithm)
- Set Cover
- Longest Path (Graph Algorithm)
- Max Cut (Graph Algorithm)
- 3-Coloring (Graph Algorithm)
- Hamiltonian Cycle (Graph Algorithm)
- Travelling Salesperson (Graph Algorithm)

Is P = NP?



If $P \neq NP$



If $P = NP$

i	1	2	3	4
$\pi[i]$	0	0	1	2
`P[i]`	a	b	a	b

Figure 4: Prefix function table

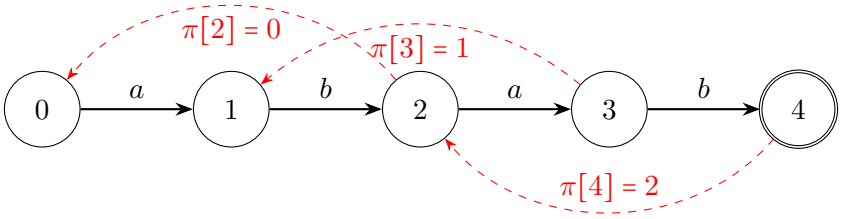


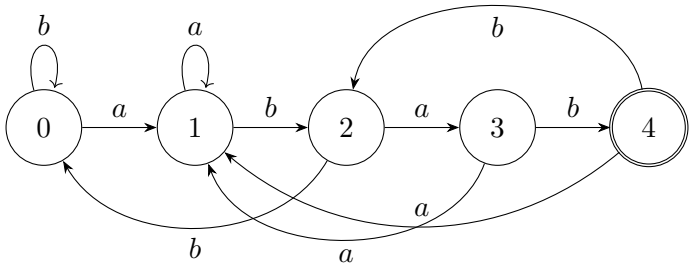
Figure 5: Automaton for KMP

String Matching

Finite Automata

This is a way to match strings (similar to FSM). When building the graph, just remember that each nodes needs a edge pointing out for each letter in the alphabet.

Example: Finite Automaton for pattern "abab". *Note:* if a character path you want to take isn't shown, go back to the beginning.



Knuth–Morris–Pratt (KMP)

Given a prefix π for a string with length i, then build a graph from that pattern.

Example: KMP for pattern "abab"

The solid arrows show character transitions, while dashed red arrows show failure links (π values).

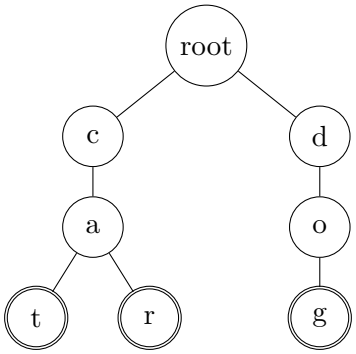
String Matching Overview

	Construction Time	Matching Time	Total Time	Explanation
Finite Automaton	$O(m \cdot \|\Sigma\|)$	$O(n)$	$O(m \cdot \|\Sigma\| + n)$	Σ is the size of the alphabet, m is the size of the pattern
KMP	$O(m)$	$O(n)$	$O(n+m)$	n is the size of the string, m is the size of the pattern

Tries

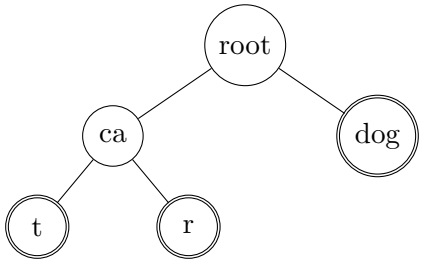
Regular Tries

A trie is a tree-based data structure for storing strings where each path from the root represents a sequence of characters. Words sharing common prefixes share the same path through the tree, making tries space-efficient for storing related strings like dictionaries. Each node can be marked to indicate the end of a valid word, allowing the structure to distinguish between prefixes and complete words. Operations like insertion, deletion, and lookup run in $O(m)$ time where m is the length of the string, independent of how many words are stored. This makes tries particularly useful for tasks like autocomplete, spell checking, and IP routing where prefix matching is essential.



Compact Tries

A compact trie improves upon the standard trie by compressing chains of nodes with single children into single edges labeled with strings rather than individual characters. This significantly reduces space usage from $O(n)$ to $O(s)$ where s is the number of strings stored, making it more practical for large datasets. The time complexity remains the same as standard tries since we can still traverse character-by-character along compressed edges.



Tries Running Time

	Search	Prefix Search	Preprocessing Time	Space	Explanation
Trie	$O(m)$	$O(m + \text{occ})$	$O(n)$	$O(n)$	Standard trie stores each character in separate nodes
Compact Trie	$O(m)$	$O(m + \text{occ})$	$O(n)$	$O(s)$	Compresses chains Space: $s = \#\text{strings}$ instead of $n =$ total length