# Algorithms & Data Structures II

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## Introduction to Dynamic Programming

### Weighted Interval Scheduling

*The idea:* Sort by finish time and select
`OPT(j) = max(v[j]) + OPT(p[j]), OPT(j-1))`. Either you take job $j$, and skip conflicts, or don't take it. The algorithm runs in $O(n \log n)$ time, where $n$ is the number of jobs.

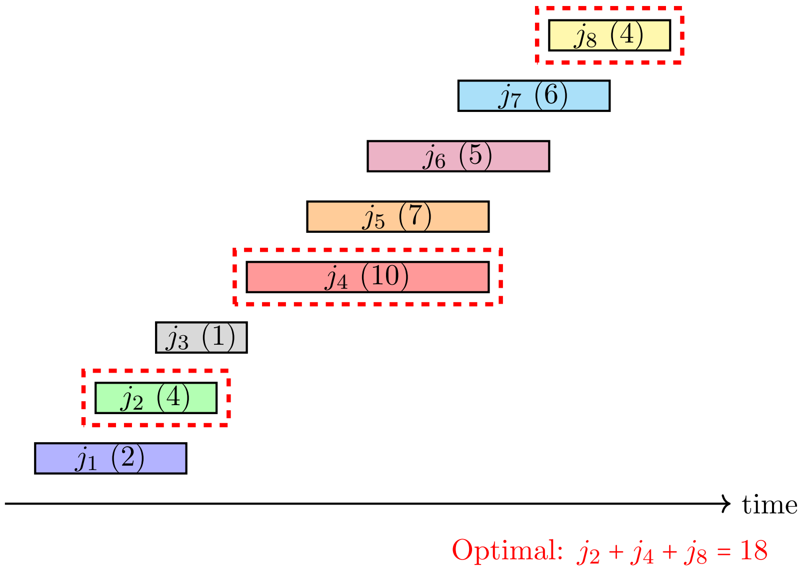



Figure 1: Weighted Interval Scheduling Example: Jobs are sorted by finish time. The optimal solution selects non-overlapping jobs with maximum total value.

### Knapsack

*The problem:* Fill a backpack with as many books as possible. `OPT(i,w) = max(OPT(i-1,w), v[i] + OPT(i-1`
The option is to take an item $i$ or not and the algorithm runs in $O(nW)$ time, where $n$ is the number of items and $W$ is knapsack capacity. The algorithm creates a 2D table consisting of $n \times W$

### Sequence Alignment

*The problem:* Match characters, gap in $X$, or gap in $Y$. The running time of this algorithm is

## Max Flow & Min Cut

### Flow Algorithms

| Name | Running Time | Explanation | Notes |
|---|---|---|---|
| Ford-Fulkerson | $O(\|f^{\cdot}\|m)$ | $\|f^{\cdot}\| =$ max flow value<br>$m = \#$ edges | Pseudopolynomial;<br>can fail to terminate |
| Scaling | $O(m^2 \log C)$ | $C =$ max edge capacity | Weakly polynomial;<br>assumes integer capacities |
| Edmonds-Karp | $O(m^2 n)$ | $n = \#$ vertices<br>$m = \#$ edges | Uses BFS;<br>strongly polynomial |

### Running the algorithms

When running the algorithms by hand, just use a basic approach—it's manageable when the graph is reasonably simple. For min-cut, find the saturated edges (at capacity) along the residual graph boundary between reachable and unreachable vertices from the source.

### Flow Problem Recipe

#### Build the Graph

- Vertices (source + sink)
- Edges (capacities + direction)

#### Describe the Algorithm

- Find max flow ($m$) and INSERT ALGORITHM
- Return an answer

#### Analyse Runtime

- Build Parameters from question
- Remember the time it takes to build the graph

# Randomised Algorithms

### Expected Value

Expected value is the average outcome over many trials. For a discrete random variable $X$:

$$\mathbb{E}[X] = \sum_i x_i \cdot \mathbb{P}[X = x_i]$$

**Common trick:** Use indicator random variables. If $X = X_1 + X_2 + \cdots + X_n$ where each $X_i$ is an indicator (0 or 1), then:

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X_i] = \sum_{i=1}^{n} \mathbb{P}[X_i = 1]$$

**Geometric distribution:** If success probability is $p$, then expected number of trials until first success is $\frac{1}{p}$.

**Example - QuickSort:** Let $X_{ij}$ be an indicator that equals 1 if elements $i$ and $j$ are compared, 0 otherwise. Total comparisons:

$$X = \sum_{i<j} X_{ij} \implies \mathbb{E}[X] = \sum_{i<j} \mathbb{E}[X_{ij}] = \sum_{i<j} \mathbb{P}[i \text{ and } j \text{ compared}]$$

Elements $i$ and $j$ are compared if one is chosen as pivot before any element between them, so $\mathbb{P}[i \text{ and } j \text{ compared}] = \frac{2}{j-i+1}$. Summing gives $\mathbb{E}[X] = O(n \log n)$.

### Useful Sums

$$\begin{aligned}
&\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, \quad |x| < 1\\
&\sum_{n=0}^{\infty} n \cdot x^n = \frac{x}{(1-x)^2}, \quad |x| < 1\\
&\sum_{k=1}^{n} \frac{1}{k} = H(n), \quad \ln(n) < H(n) < \ln(n) + 1\\
&\mathbb{P}[\text{first success in round } j] = (1-p)^{j-1} p\\
&\mathbb{E}[X] = \sum x_i \cdot \mathbb{P}[X = x_i] \quad \text{where } x_i \in \text{possible outcomes}\\
&\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]\\
&\text{if } \mathbb{P}[\text{success}] = p \text{ then } \mathbb{E}[\# \text{ tries to success}] = \frac{1}{p}
\end{aligned}$$

## Hashing

### Chained Hashing

Store elements in array of linked lists. Each element $x$ goes to list at $A[h(x)]$. Operations take $O(|A[h(x)]| + 1)$ time. With universal hashing, expected $O(1)$ time per operation.

### Linear Probing

Store elements directly in array. If $A[h(x)]$ is occupied, place $x$ in next empty slot (wrapping around). Forms clusters of consecutive elements. Operations take $O(C(h(x)) + 1)$ time where $C(i)$ is cluster size. Cache-efficient but DELETE is $O(n^2)$.

### Hash Functions

**Universal hashing:** Family $H$ is universal if for any $x \neq y$, $\Pr[h(x) = h(y)] \leq \frac{1}{m}$ for random $h \in H$.

**Dot product hashing:** For $x = (x_1, x_2, \ldots, x_r)$ in base $m$ (prime), define

$$h_a(x) = (a_1 x_1 + a_2 x_2 + \cdots + a_r x_r) \bmod m$$

where $a = (a_1, \ldots, a_r)$ chosen randomly. This family is universal.

### Performance Comparison

| Data Structure | SEARCH | INSERT | DELETE | Space |
|---|---|---|---|---|
| Linked List | $O(n)$ | $O(n)$ | $O(n)$ | $O(n)$ |
| BBST | $O(\log n)$ | $O(\log n)$ | $O(\log n)$ | $O(n)$ |
| Direct Addressing | $O(1)$ | $O(1)$ | $O(1)$ | $O(\|U\|)$ |
| Chained Hashing (universal) | $O(1)^{\cdot}$ | $O(1)^{\cdot}$ | $O(1)^{\cdot}$ | $O(n)$ |
| Linear Probing | $O(C(h(x)) + 1)$ | $O(C(h(x)) + 1)$ | $O(n^2)$ | $O(n)$ |

$^{\cdot}$ Expected time (when using universal hash function)

## Amortised Analysis

### Aggregate Method

The idea: Analyze the total cost $T(m)$ of a worst-case sequence of $m$ operations, then compute the amortized cost as $T(m)/m$ per operation.

### Accounting Method

Assign a *cost* $\hat{c}_i$ to each operation. When $\hat{c}_i > c_i$, store the difference as credits. When $\hat{c}_i < c_i$, use stored credits to pay. Ensure $\sum_{i=1}^{m} \hat{c}_i \geq \sum_{i=1}^{m} c_i$ always holds.

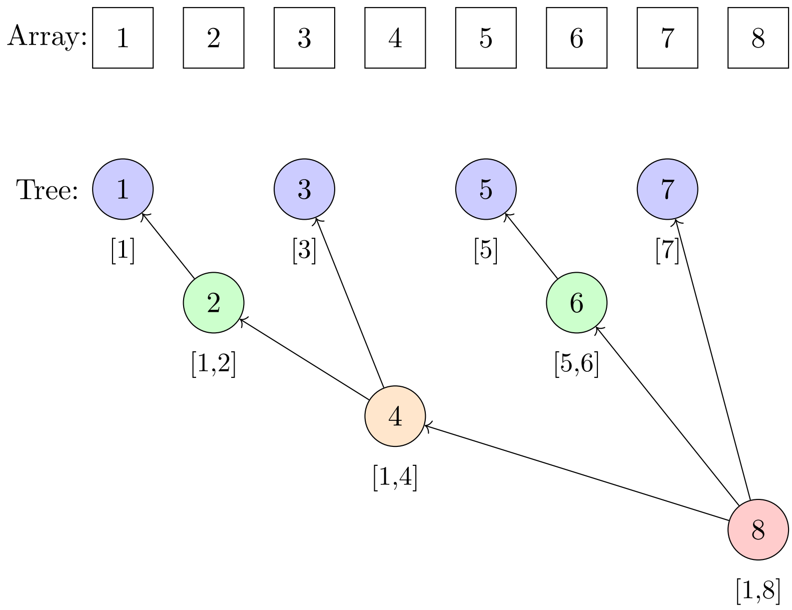
### Potential Method

Define a potential function $\Phi(D)$ that maps the data structure state to a real value. Set amortized cost $\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1})$. Require $\Phi(D_i) \geq 0$ for all $i$ and $\Phi(D_0) = 0$.

## Partial Sums & Dynamic Arrays

### Partial Sums (Fenwick Trees)

A fenwick tree is effectively binary tree consisting of only the right child. This allows for an in-place data structure to compute the partial sum.




### Example

here is an example on how to calculate a fenwick tree.

```
Index:  1   2   3   4   5   6   7   8
Value:  3   2   5   1   4   6   2   3

Index:  1   2   3   4   5   6   7   8
BIT:    3   2   5   1   4   6   2   3

Index:  1   2   3   4   5   6   7   8
BIT:    3   2   5   1   4   6   2   3

Index:  1   2   3   4   5   6   7   8
BIT:    3   5   5   1   4   6   2   3

Index:  1   2   3   4   5   6   7   8
BIT:    3   5   5   1   4   6   2   3

Index:  1   2   3   4   5   6   7   8
BIT:    3   5   5  11   4   6   2   3

Index:  1   2   3   4   5   6   7   8
BIT:    3   5   5  11   4   6   2   3

Index:  1   2   3   4   5   6   7   8
BIT:    3   5   5  11   4  10   2   3

Index:  1   2   3   4   5   6   7   8
BIT:    3   5   5  11   4  10   2   3

Index:  1   2   3   4   5   6   7   8
BIT:    3   5   5  11   4  10   2  26
```

```
Final Fenwick Tree:
Index:  1    2    3    4    5    6    7    8
BIT:    3    5    5   11    4   10    2   26
Range: [1][1,2][3][1,4][5][5,6][7][1,8]
```

## Dynamic Arrays

### Rotated Array

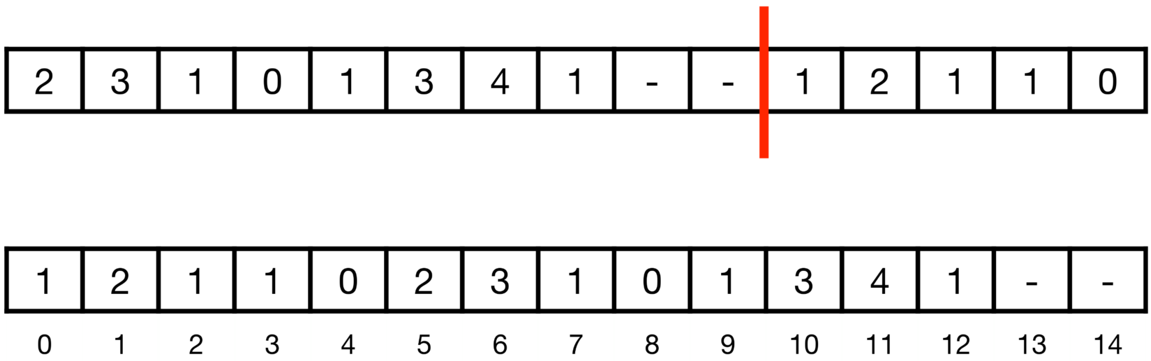



Figure 2: 1 level rotated array

**Key idea:** Store offset to mark the start of the array.

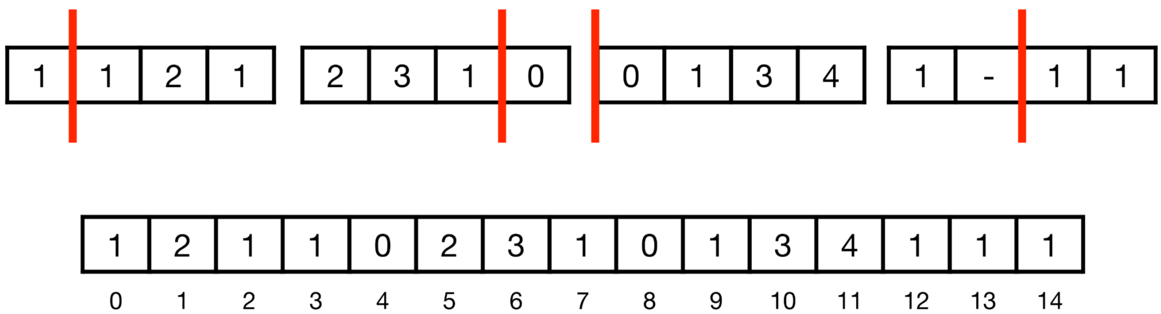



Figure 3: 2 level rotated array

**Key idea:** Split into $\sqrt{n}$ rotated buckets of size $\sqrt{n}$. When inserting/deleting, rebuild one bucket in $O(\sqrt{n})$ time, then propagate overflow/underflow.

| Data Structure | Access | Insert | Delete | space |
|---|---|---|---|---|
| 1 level rotated array | O(1) | O($\sqrt{n}$) | O($\sqrt{n}$) | O($n$) |
| 2 level rotated array | O(1) | O($n^{\varepsilon}$) | O($n^{\varepsilon}$ | O($n$) |

In a $k$-level rotated array, $\varepsilon = \frac{1}{k}$ determines that you partition $n$ elements into $\sqrt[k]{n}$ buckets of size $\sqrt[k]{n}$ each, giving $O(n^{1/k}) = O(n^{\varepsilon})$ time for INSERT/DELETE operations.

# NP Completeness

Boils down to if a problem is solvable in polynomial time or non-deterministic polynomial time. The mindset I have for this is best describes via excercise 2 from week *NP Completeness*.

## Example - Customer Uniqueness (KT 8.2)

A store trying to analyse the behavior of its customers will often main a two-dimensional array $A$, where the rots correspond to its customers and the clumns correspond to the products it sells. The antry $A[i, j]$ specifies the quatity of project $j$ that has been purchaed by customer $i$.

Here's a tiny example of such an array $A$

| | luquid detergent | beer | diapers | cat litter |
|---|---|---|---|---|
| Raj | 0 | 6 | 0 | 3 |
| Alanis | 2 | 0 | 0 | 7 |
| Chelsea | 0 | 0 | 0 | 7 |

One thing that a store might want to do with this data is the following. LLet us say that a subset $S$ off the customers is *diverse* if no two of the customers in $S$ have ever bought the same product (i.e., for each product, at most one of the customers in $s$ has ever bought it). A diverse set of customers can be useful, for example, as a target pool for market research.

We can now define the Diverse Subset Problem as follows: Given an $m \times n$ array $A$ as defined abve, and a number $k \leq m$, is there a subset of at least $k$ of customers that is *diverse*?

Show that Diverse subset in NP-complete.

### Solution

We show that $IS \leq_p DS$, where $IS$ is Independent Set and $DS$ is Diverse Subset.

Given a graph $G = (V, E)$ with $|V| = m$ vertices and $|E| = n$ edges, construct an $m \times n$ matrix $A$ where vertices are customers (rows) and edges are products (columns). For each edge $(u, v) \in E$, set $A[u, (u, v)] = 1$ and $A[v, (u, v)] = 1$ (all other entries are 0).

Now $G$ has an independent set of size $k$ if and only if $A$ has a diverse subset of size $k$. Since the construction takes polynomial time, $DS$ is NP-complete.

## List of NP-complete Problems

- Subset Sum
- Independent Set ⊛
- Vertex Cover ⊛
- Set Cover
- Longest Path ⊛
- Max Cut ⊛
- 3-coloring ⊛

- Hamiltonian Cycle ⊛
- Travelling Salesperson ⊛