Confidence Intervals in the Normal Model and the Binomial Model

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Confidence Interval - Normal Distribution

- **Example:** In connection with the behaviour of children in road traffic, the reaction time of 10-year-old children is investigated. Experience has shown that reaction times are normally distributed, but the paramaters μ and σ are unknown.
- Sample of n=51 observations:

```
##  [1] 0.53 1.29 0.65 0.56 0.15 1.47 0.26 0.46 1.12 0.63 1.80 0.59 0.89 0.87 0.39
## [16] 0.40 1.25 1.47 1.19 1.13 0.30 1.16 0.25 0.23 0.88 1.00 1.25 0.95 1.28 0.69
## [31] 0.83 0.36 1.25 1.24 1.03 0.65 0.70 1.26 0.85 0.12 0.80 0.92 0.74 0.45 1.52
## [46] 1.21 0.82 1.46 0.62 0.80 0.73
```

Confidence Interval for μ, $1-\alpha = 0.95$

We assume that $\sigma = 0.4$.

- $$\left[\bar{X}_{(n)} - u_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}, \bar{X}_{(n)} + u_{1-\alpha/2}\frac{\sigma}{\sqrt{n}}\right]$$

```
alpha <- 0.05
q <- qnorm(1-alpha/2,0,1)
mv <- mean(values)
lb1 <- mv-q*sigma/(n^0.5)
ub1 <- mv+q*sigma/(n^0.5)
```

- From $n = 51$, $u_{1-\alpha/2} = 1.959964$, $\bar{x}_{(n)} = 0.8519608$ we get as the confidence interval for μ: [**0.7421808, 0.9617407**]
- If we increase the confidence level to 99% we get a wider confidence interval

```
alpha <- 0.01
q <- qnorm(1-alpha/2,0,1)
mv <- mean(values)
lb2 <- mv-q*sigma/(n^0.5)
ub2 <- mv+q*sigma/(n^0.5)
```

- 99% confidence interval for μ: [**0.7076855, 0.9962361**]
- The package TeachingDemos includes a function z.test which can be used to evaluate the confidence interval directly.

```
library(TeachingDemos)
```

```
## Warning: package 'TeachingDemos' was built under R version 4.0.5
```

```r
z.test(x = values, stdev = sigma, alternative = "two.sided", conf.level = 0.95)$conf.int
```

```
## [1] 0.7421808 0.9617407
## attr(,"conf.level")
## [1] 0.95
```

```r
z.test(x = values, stdev = sigma, alternative = "two.sided", conf.level = 0.99)$conf.int
```

```
## [1] 0.7076855 0.9962361
## attr(,"conf.level")
## [1] 0.99
```

σ is unknown

- $$\left[\bar{X}_{(n)} + t_{n-1,1-\alpha/2}\frac{S_{(n)}}{\sqrt{n}}, \bar{X}_{(n)} + t_{n-1,1-\alpha/2}\frac{S_{(n)}}{\sqrt{n}}\right]$$

```r
qt <- qt(1-alpha/2,n-1)
mv <- mean(values)
s <- sd(values)
lb3 <- mv-qt*s/(n^0.5)
ub3 <- mv+qt*s/(n^0.5)
```

- From $n = 51$, $t_{n-1,1-\alpha/2} = 2.6777933$, $\bar{x}_{(n)} = 0.8519608$, $s_{(n)} = 0.4034006$ we get as the confidence interval for μ: [**0.7006992, 1.0032223**]
- Mention that the length of the interval is bigger as in the case if σ is known.
- With the function t.test the confidence interval can be evaluated directly.

```r
t.test(x = values, alternative = "two.sided", conf.level = 1-alpha)$conf.int
```

```
## [1] 0.7006992 1.0032223
## attr(,"conf.level")
## [1] 0.99
```

Confidence interval for σ^2, $1 - \alpha = 0.95$

- μ is known

$$\left[\frac{Q_n}{\chi^2_{n,1-\alpha/2}}, \frac{Q_n}{\chi^2_{n,\alpha/2}}\right]$$

```r
mu <- 0.8
Qn <- sum((values - mu)^2)
qchi1 <-qchisq(alpha/2,n)
qchi2 <-qchisq(1-alpha/2,n)
lb3 <- Qn/qchi2
ub3 <- Qn/qchi1
```

- From $n = 51$, $\chi^2_{n,1-\alpha/2} = 80.746659$, $\chi^2_{n,\alpha/2} = 28.734712$, $s_{(n)} = 0.4034006$ we get as the confidence interval for σ^2: [**0.1024724, 0.2879549**]
- Taking the square root of the bounds we get as a confidence interval for the standard deviation σ: [**0.320113, 0.5366143**]

- μ is unknown

$$\left[\frac{(n-1)S_{(n)}^2}{\chi^2_{n-1,1-\alpha/2}}, \frac{(n-1)S_{(n)}^2}{\chi^2_{n-1,\alpha/2}}\right]$$

```
s <- sd(values)
qchi3 <-qchisq(alpha/2,n-1)
qchi4 <-qchisq(1-alpha/2,n-1)
lb4 <- (n-1)*s^2/qchi4
ub4 <- (n-1)*s^2/qchi3
```

- From $n = 51$, $\chi^2_{n-1,1-\alpha/2} = 27.9907489$, $\chi^2_{n-1,\alpha/2} = 79.4899785$, $s_{(n)} = 0.4034006$ we get as the confidence interval for σ^2: [**0.1023601, 0.290689**]
- Taking the square root of the bounds we get as a confidence interval for the standard deviation σ: [**0.3199377, 0.5391559**]
- Applying the function sigma.test of the package TeachingDemos the confidence interval can be evaluated directly.

```
library(TeachingDemos)
sigma.test(x = values, alternative = "two.sided", conf.level = 1-alpha)$conf.int
```

```
## [1] 0.1023601 0.2906890
## attr(,"conf.level")
## [1] 0.99
```

One sided confidence intervals

$1-\alpha = 0.95$ **upper bound for** μ **if** σ **is unknown**

- $$\bar{X}_{(n)} + t_{n-1,1-\alpha}\frac{S_{(n)}}{\sqrt{n}}$$
- From $n = 51$, $t_{n-1,1-\alpha} = 2.4032719$, $\bar{x}_{(n)} = 0.8519608$, $s_{(n)} = 0.4034006$ we get as the upper confidence bound for μ: **0.9877153**
- Applying the t.test function we get directly

```
t.test(x = values, alternative = "less", conf.level = 1-alpha)$conf.int
```

```
## [1]      -Inf 0.9877153
## attr(,"conf.level")
## [1] 0.99
```

$1-\alpha = 0.95$ **lower bound for** σ **if** μ **is unknown**

- $$\sqrt{\frac{(n-1)S_{(n)}^2}{\chi^2_{n-1,1-\alpha}}}$$
- From $n = 51$, $\chi^2_{n-1,1-\alpha} = 76.1538912$, $s_{(n)} = 0.4034006$ we get as the lower confidence bound for σ: **0.3268704**
- Applying the function sigma.test of the package TeachingDemos we get

```
library(TeachingDemos)
(sigma.test(x = values, alternative = "greater", conf.level = 1-alpha)$conf.int)^0.5
```

```
## [1] 0.3268704       Inf
## attr(,"conf.level")
## [1] 0.99
```

What is the minimum sample size so that the length of the $1-\alpha = 0.95$ confidence interval for μ is 0.1 or less?

für Normalverteilungen

$\sigma = 0.4$ is known

- length of the confidence interval

$$l = 2u_{1-\alpha/2}\frac{\sigma}{\sqrt{n}} \leq 0.1 \Rightarrow n \geq \boxed{\left(\frac{2u_{1-\alpha/2}\sigma}{0.1}\right)^2}$$

- From $u_{1-\alpha/2} = 1.959964$ we get as the lower bound for the sample size: $n \geq 424.6333825$

σ is unknown

- length of the confidence interval

$$l = 2t_{n-1,1-\alpha/2}\frac{S_{(n)}}{\sqrt{n}} \leq 0.1$$

- Mention that the length l depends on the sample values, i.e. it is a random variable. Since $S_{(n)}$ is an unbiased estimator for σ and $t_{n-1,1-\alpha/2}$ is decreasing with respect to n we get in the average for $n \geq 51$

$$E(l) = 2t_{n-1,1-\alpha/2}\frac{\sigma}{\sqrt{n}} \leq 2t_{51-1,1-\alpha/2}\frac{\sigma}{\sqrt{n}} \leq 0.1 \Rightarrow n \geq \boxed{\left(\frac{2t_{50,1-\alpha/2}\sigma}{0.1}\right)^2}$$

- From $t_{50,1-\alpha/2} = 2.0085591$ we get in the average as the lower bound for the sample size: $n \geq 458.9169153$

Confidence Interval for an unknown proportion

- **Example:** In a survey, 75 of the 500 respondents voted for Party XY.

Two sided confidence interval

- approximative confidence interval for the unkonwn proportion p

$$\left[\hat{p} - u_{1-\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}, \hat{p} + u_{1-\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}\right]$$

- $1-\alpha = 0.95$: From n=500, $\hat{p} = \frac{75}{500} = 0.15$, $u_{1-\alpha/2} = 1.959964$ we get as the confidence for the unknown proportion **[0.1187019, 0.1812981]**
- Checking the rule of thumb: $n\hat{p} \geq 10$ and $n(1-\hat{p}) \geq 10$ we see that the approximative calculation is ok.
- Applying the function binom.test() we can evaluate the exact confidence interval.

```
binom.test(x=75, n=500, alternative = "two.sided", conf.level = 1-alpha)$conf.int
```

```
## [1] 0.1198486 0.1843596
## attr(,"conf.level")
## [1] 0.95
```

One sided confidence interval

- approximative upper confidence bound for the unkonwn proportion p

$$\hat{p} + u_{1-\alpha}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

- $1-\alpha = 0.95$: From n=500, $\hat{p} = \frac{75}{500} = 0.15$, $u_{1-\alpha} = 1.6448536$ we get as the upper confidence bound for the unknown proportion **0.1812981**]
- Applying the function binom.test() we can evaluate the exact confidence interval.

```
binom.test(x=75, n=500, alternative = "less", conf.level = 1-alpha)$conf.int
```

```
## [1] 0.0000000 0.1788032
## attr(,"conf.level")
## [1] 0.95
```

What is the minimum sample size so that the length of the $1-\alpha = 0.95$ confidence interval for p is 0.05 or less?

unknown proportion

- length of the confidence interval

$$l = 2u_{1-\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}} \leq 0.05$$

- Mention that the length l depends on the sample values, i.e. it is a random variable. Since we have that for every $p \in [0,1] : p(1-p) \leq 0.25$ we get

$$l \leq 2u_{1-\alpha/2}\sqrt{\frac{0.25}{n}} \leq 0.05 \Rightarrow n \geq \boxed{\left(\frac{u_{1-\alpha/2}}{0.05}\right)^2}$$

- From $u_{1-\alpha/2} = 1.959964$ we get that if $n \geq 1536.5835283$ the length of the confidence interval will be less than 0.05.
- We have a better estimation if we know an upper bound for p. If for example $p \leq 0.2$ we get

$$l \leq \left(\frac{2 * u_{1-\alpha/2}\sqrt{0.2 \cdot 0.8}}{0.05}\right)^2.$$

- Thus we have $n \geq 983.4134581$ if $p \leq 0.2$.