1 Question 2

1.1 Subquestion 1

From the slides, we get the following conditions for the variables on the balanced growth path

$$s_k f(k^*, h^*) - (\delta_k + g + n + ng) k^* = 0 \tag{1}$$

$$s_h f(k^*, h^*) - (\delta_h + g + n + ng) h^* = 0. \tag{2}$$

solving for k^* and h^*, respectively, gives

$$k^* = \frac{s_k f(k^*, h^*)}{(\delta_k + g + n + ng)} \tag{3}$$

$$h^* = \frac{s_h f(k^*, h^*)}{(\delta_h + g + n + ng)}. \tag{4}$$

However, since $f(k^*, h^*)$ depends on the variables that we try to solve for, this is not the final form yet.
Since we are assuming $F(K_t, H_t, A_t L_t) = K_t^\alpha H_t^\beta (A_t L_t)^{1-\alpha-\beta}$, we can further solve the expression

$$f(k^*, h^*) = \frac{F(K_t, H_t, A_t L_t)}{A_t L_t} = \frac{K_t^\alpha H_t^\beta (A_t L_t)^{1-\alpha-\beta}}{A_t L_t} = \left(\frac{K_t}{A_t L_t}\right)^\alpha \left(\frac{H_t}{A_t L_t}\right)^\beta. \tag{5}$$

By definition of k^* and h^*, this is gives

$$f(k^*, h^*) = k_t^\alpha h_t^\beta. \tag{6}$$

Now, inserting this into 3 and 4

$$k^* = \frac{s_k k_t^\alpha h_t^\beta}{(\delta_k + g + n + ng)} \Leftrightarrow k^* = \left(\frac{s_k h_t^\beta}{(\delta_k + g + n + ng)}\right)^{\frac{1}{1-\alpha}} \tag{7}$$

$$h^* = \frac{s_h k_t^\alpha h_t^\beta}{(\delta_h + g + n + ng)} \Leftrightarrow h^* = \left(\frac{s_h k_t^\alpha}{(\delta_h + g + n + ng)}\right)^{\frac{1}{1-\beta}}. \tag{8}$$

It becomes evident, that k^* and h^* are a function of each other. However, we can simply substitute and then solve for the expression depending on parameters only

$$k^* = \left(\frac{s_k \left(\left(\frac{s_h k_t^\alpha}{(\delta_h+g+n+ng)}\right)^{\frac{1}{1-\beta}}\right)^\beta}{(\delta_k + g + n + ng)}\right)^{\frac{1}{1-\alpha}} \tag{9}$$

$$h^* = \left(\frac{s_h \left(\left(\frac{s_k h_t^\beta}{(\delta_k+g+n+ng)}\right)^{\frac{1}{1-\alpha}}\right)^\alpha}{(\delta_h + g + n + ng)}\right)^{\frac{1}{1-\beta}}. \tag{10}$$

which can now be solved for the respective variable. For k^*:

$$
\begin{aligned}
k^* &= \left(\frac{s_k \left(s_h k_t^{\alpha}\right)^{\frac{\beta}{1-\beta}}}{(\delta_h + g + n + ng)^{\frac{\beta}{1-\beta}} (\delta_k + g + n + ng)} \right)^{\frac{1}{1-\alpha}} \\
&= \left(\frac{s_k s_h^{\frac{\beta}{1-\beta}}}{(\delta_h + g + n + ng)^{\frac{\beta}{1-\beta}} (\delta_k + g + n + ng)} \right)^{\frac{1}{1-\alpha}} k_t^{\frac{\alpha\beta}{1-\beta-\alpha+\alpha\beta}} \\
&= \left(\frac{s_k s_h^{\frac{\beta}{1-\beta}}}{(\delta_h + g + n + ng)^{\frac{\beta}{1-\beta}} (\delta_k + g + n + ng)} \right)^{\frac{1-\beta-\alpha+\alpha\beta}{(1-\alpha)^2+\alpha\beta-\beta}}
\end{aligned}
$$

For h^*:

$$
\begin{aligned}
h^* &= \left(\frac{s_h \left(s_k h_t^{\beta}\right)^{\frac{\alpha}{1-\alpha}}}{(\delta_k + g + n + ng)^{\frac{\alpha}{1-\alpha}} (\delta_h + g + n + ng)} \right)^{\frac{1}{1-\beta}} \\
&= \left(\frac{s_h s_k^{\frac{\alpha}{1-\alpha}}}{(\delta_k + g + n + ng)^{\frac{\alpha}{1-\alpha}} (\delta_h + g + n + ng)} \right)^{\frac{1}{1-\beta}} h_t^{\frac{\alpha\beta}{1-\beta-\alpha+\alpha\beta}} \\
&= \left(\frac{s_h s_k^{\frac{\alpha}{1-\alpha}}}{(\delta_k + g + n + ng)^{\frac{\alpha}{1-\alpha}} (\delta_h + g + n + ng)} \right)^{\frac{1-\beta-\alpha+\alpha\beta}{(1-\beta)^2+\alpha\beta-\alpha}} .
\end{aligned}
$$

Thus, the solution vector becomes

$$
\begin{bmatrix} k^* \\ h^* \end{bmatrix}' = \begin{bmatrix} \left(\frac{s_k s_h^{\frac{\beta}{1-\beta}}}{(\delta_h+g+n+ng)^{\frac{\beta}{1-\beta}} (\delta_k+g+n+ng)} \right)^{\frac{1-\beta-\alpha+\alpha\beta}{(1-\alpha)^2+\alpha\beta-\beta}} \\ \left(\frac{s_h s_k^{\frac{\alpha}{1-\alpha}}}{(\delta_k+g+n+ng)^{\frac{\alpha}{1-\alpha}} (\delta_h+g+n+ng)} \right)^{\frac{1-\beta-\alpha+\alpha\beta}{(1-\beta)^2+\alpha\beta-\alpha}} \end{bmatrix}'
$$