# Views

Section 1

# Standard scenario analysis

# Linear views

Given a vector of N market variables $\boldsymbol{r} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ a linear view can be expressed as:

$$\boldsymbol{V}\boldsymbol{r} = \boldsymbol{\mu}_v$$

conditioning on the statement leads to the distribution $(r|\boldsymbol{V}\boldsymbol{r} = \boldsymbol{\mu}_v) \sim \mathcal{N}(\bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\Sigma}})$, where:

$$\bar{\boldsymbol{\mu}} = \boldsymbol{\mu} + \boldsymbol{\Sigma}\boldsymbol{V}'(\boldsymbol{V}\boldsymbol{\Sigma}\boldsymbol{V}')^{-1}(\boldsymbol{\mu}_v - \boldsymbol{V}\boldsymbol{\mu})$$
$$\bar{\boldsymbol{\Sigma}} = \boldsymbol{\Sigma} - \boldsymbol{\Sigma}\boldsymbol{V}'(\boldsymbol{V}\boldsymbol{\Sigma}\boldsymbol{V}')^{-1}\boldsymbol{V}\boldsymbol{\Sigma}$$

# Proof - sketch

One can show that the conditional variable is a linear combination of a jointly normal random vector. First, build a N by N matrix $\boldsymbol{S}$ composed by a full rank arbitrary matrix $\boldsymbol{Q}$ and $\boldsymbol{V}$. Define $\boldsymbol{y} = \boldsymbol{Sr}$, and $\boldsymbol{y}_q = \boldsymbol{Qr}$, $\boldsymbol{y}_v = \boldsymbol{Vr}$. Perform an orthogonal projection of $\boldsymbol{y}_q$ over the span of $\boldsymbol{y}_v$ thus obtaining $\boldsymbol{y}_q = \alpha + \beta \boldsymbol{y}_v + \boldsymbol{u}$. Exploiting the properties of the residuals you can prove the normality and compute the moments.

Section 2

# Black & Litterman

# Implied views

The way to compute the implied market expected returns is the most popular (and misunderstood) result of the work by Black and Litterman.
Let's assume that all the investors maximize an unconstrained mean-variance utility function $\boldsymbol{w'\mu} - \lambda \boldsymbol{w'\Sigma w}$. Differentiating and setting to 0 yields:

$$\bar{\boldsymbol{\mu}} = 2\lambda \boldsymbol{\Sigma w}$$

The weights of the investment universe are unknown: usually a broad enough index is employed, but other choices can be used in practice. $\boldsymbol{\Sigma}$ is estimated from the data; Black & Litterman set $\lambda \approx 1.2$.
Note that the main result by Black & Litterman can be applied even with other values of $\bar{\boldsymbol{\mu}}$.

# General settings

The market variables are assumed conditionally normal:

$$\boldsymbol{r}|\boldsymbol{\mu} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$$

where $\boldsymbol{\Sigma}$ is the market covariance matrix and $\boldsymbol{\mu}$ is the realization of a random variable $\boldsymbol{\mu} \sim \mathcal{N}(\bar{\boldsymbol{\mu}}, \bar{\boldsymbol{\Sigma}})$. Here $\bar{\boldsymbol{\Sigma}} = \frac{1}{t}\boldsymbol{\Sigma}$ where $t$ is used to model the uncertainty level of the guess $\bar{\boldsymbol{\mu}}$. The unconditional distribution (called "Black and Litterman's prior") then reads:

$$\boldsymbol{r} \sim \mathcal{N}\left(\bar{\boldsymbol{\mu}}, \left(1 + \frac{1}{t}\right)\boldsymbol{\Sigma}\right)$$

It is a common choice to set $t = T$.

# Views

According to the approach by Black and Litterman, views are expressed as a (linear) statement on the expectations.

$$\boldsymbol{v}\boldsymbol{\mu}|\boldsymbol{\mu}_v \sim \mathcal{N}(\boldsymbol{\mu}_v, \boldsymbol{\Sigma}_v)$$

where the conditional notation highlights that $\boldsymbol{\mu}_v$ is considered as the realization of a properly defined random variable and $\boldsymbol{v}$ is a matrix whose $k$ rows correspond to $k$ different statements. $\boldsymbol{\Sigma}_v$ represents the uncertainty on the views, and it is usually set so to inherit the structure of the market covariance matrix, rescaled by a constant $c$:

$$\boldsymbol{\Sigma}_v = \frac{1}{c}\boldsymbol{v}\boldsymbol{\Sigma}\boldsymbol{v}'$$

# Other topics

To describe qualitative views one can resort to the quantiles of the distribution induced by the market, setting the $i$-th element of the views' vector as:

$$\boldsymbol{\mu}_v^{(i)} = (\boldsymbol{v}\bar{\boldsymbol{\mu}})^{(i)} + q_i \sqrt{(\boldsymbol{v}\boldsymbol{\Sigma}\boldsymbol{v}')^{(i,i)}}$$

where usually $q_i \in \{-2, -1, 1, 2\}$.
In order to assign different confidence levels to each view, one can resort to the formula:

$$\boldsymbol{\Sigma}_v = \mathrm{diag}(\boldsymbol{c})^{1/2} \boldsymbol{v}\boldsymbol{\Sigma}\boldsymbol{v}' \mathrm{diag}(\boldsymbol{c})^{1/2}$$

where $\boldsymbol{c} = (c_1, ..., c_k)$ is a vector of positive entries. The qualitative views are obtained from the same formula as above.

# Posterior distribution

One can prove the following statement on the so called Black and Litterman's posterior distribution:

$$\boldsymbol{r}|\boldsymbol{\mu}_v \sim \mathcal{N}(\boldsymbol{\mu}_p, \boldsymbol{\Sigma}_p)$$

where:

$$\boldsymbol{\mu}_p = \bar{\boldsymbol{\mu}} + \frac{1}{t}\boldsymbol{\Sigma}\boldsymbol{v}' \left(\frac{1}{t}\boldsymbol{v}\boldsymbol{\Sigma}\boldsymbol{v}' + \boldsymbol{\Sigma}_v\right)^{-1} (\boldsymbol{\mu}_v - \boldsymbol{v}\bar{\boldsymbol{\mu}})$$

$$\boldsymbol{\Sigma}_p = \left(1 + \frac{1}{t}\right)\boldsymbol{\Sigma} - \frac{1}{t^2}\boldsymbol{\Sigma}\boldsymbol{v}' \left(\frac{1}{t}\boldsymbol{v}\boldsymbol{\Sigma}\boldsymbol{v}' + \boldsymbol{\Sigma}_v\right)^{-1} \boldsymbol{v}\boldsymbol{\Sigma}$$

# Proof - sketch

Most of the proof is based on the following statement on conditioning of normal r.v.: let $\boldsymbol{Y}$ an n-variate normal random variable, $\boldsymbol{a}$ a full rank $k$ x $n$ matrix, $\boldsymbol{X}$ an arbitrary k-variate random variable. Then $\boldsymbol{X}|\boldsymbol{y}$ is normal if and only if $\boldsymbol{X} \stackrel{d}{=} \boldsymbol{a}\boldsymbol{Y} + \boldsymbol{Z}$ where $\boldsymbol{Z}$ is a k-variate normal random variable independent from $\boldsymbol{Y}$ with zero mean and covariance matrix equal to that of $\boldsymbol{X}|\boldsymbol{y}$. The unconditional distribution of $\boldsymbol{X}$ is also normal, and $\boldsymbol{Y}|\boldsymbol{x}$ is normal, too. If the distributions of $\boldsymbol{Y}$ and $\boldsymbol{X}|\boldsymbol{y}$ are known, then the parameters of all the other distributions mentioned can be computed. According to the previous statement one can write $\boldsymbol{v}\boldsymbol{\mu} \stackrel{d}{=} \boldsymbol{M} + \boldsymbol{W}$ where $\boldsymbol{W}$ is normal and independent from $\boldsymbol{M}$. Rearranging the terms it is easy to see that $\boldsymbol{M}|\boldsymbol{\mu}$ is also normal with known parameters. Inverting the conditioning one obtains that $\boldsymbol{\mu}|\boldsymbol{\mu}_v$ is normal. Finally $\boldsymbol{r}|\boldsymbol{\mu}_v \stackrel{d}{=} \boldsymbol{\mu}|\boldsymbol{\mu}_v + \boldsymbol{U}$ where $\boldsymbol{U} \sim \mathcal{N}(\boldsymbol{0}, \boldsymbol{\Sigma})$ is independent from $\boldsymbol{\mu}$.

# References


- Jorge Mina and Jerry Yi Xiao, Return to RiskMetrics: The Evolution of a Standard, 2001
- Attilio Meucci, The Black-Litterman Approach: Original Model and Extensions, 2008
- Black, F. and Litterman, R., Asset allocation: combining investor views with market equilibrium. Goldman Sachs Fixed Income Research, 1990