Portfolio Management - an introduction

Section 1

Introduction & Framework

Basic definitions

Definition

Portfolio: a set of financial intruments

Definition

Portfolio management: building a portfolio and making it evolve in order to reach the investor's objectives respecting the investor's contraints.

Basic definitions

Definition

Asset classes: groups of financial instruments.

- Equity
- Bond
- Money market
- Alternatives
- Derivatives

A few milestones

1. Pure *stock picking* based on fundamental analysis (30s; Graham & Dodd)
2. Thanks to the CAPM ('50-'70; mainly Treynor and Sharpe building on the work of Markowitz) the returns of the investments are decomposed into contributions coming from the market and the management. Further insight is provided by multi-factors models ('76, Ross' APT)
3. The rapid growth of the asset management industry leads to more sofisticated techniques of performance analysis (80s; Brinson, Fachler,...)
4. The efficient market hypothesis ('70, Fama) inspires many researches about the capability of the funds managers to systematically beat the market. First Etfs are quoted during the 90s in the US, then spread in EU a decade later.

The investment management process

Two main approaches: *bottom-up* and *top-down*. The first is mainly based on pure *stock picking*. The more recent *top-down* approach follows three main steps:

- Strategic asset allocation
- Tactical asset allocation
- Stock picking

The final step is always the performance analysis: measurement and attribution

Section 2

Portfolio optimization - recap

The general problem - stochastic dominance

We can choose among admissible allocation in order to obtain the best objective $\Psi_\alpha(\omega)$ at the end of the investment period. Problem: $\Psi(\omega)$ is a r.v.: we need a criteria to choose between alternative allocations.

- strong dominance: the allocation α dominates β iff $\Psi_\alpha(\omega) \geq \Psi_\beta(\omega) \quad \forall\omega$
- weak dominance or first order stochastic dominance: the allocation α dominates β iff $Q_{\Psi_\alpha}(u) \geq Q_{\Psi_\beta}(u) \quad \forall u \in [0,1]$
- n-order stochastic dominance: the allocation α dominates β iff $I^n[f_{\Psi_\alpha}](\psi) \leq I^n[f_{\Psi_\beta}](\psi) \quad \forall\psi \in (-\infty, +\infty)$ where $I^n[f](\psi)$ is the iterated integral $\int_{-\infty}^{\psi} I^{n-1}[f](s)ds$

The general problem - satisfaction

Stochastic dominance provides only a partial order. We therefore create a map between allocations and the real numbers. We now have a total order, but there are several ways to define the mapping. The index of satisfaction can be characterized through the following properties:

- money equivalence
- estimability
- sensibility
- consistence with stochastic dominance
- constancy
- positive homogeneity
- translation invariance
- sub/super additivity
- co-monotonic additivity
- concavity/convexity
- risk aversion/propensity/neutrality

The general problem - utility

Among all the possible indexes of satisfaction we focus on the one obtained from the following steps:

1. define an objective $\Psi_\alpha(\omega)$ for the investor depending on the allocation α
2. define a utility function $u(\Psi_\alpha(\omega))$ that describes how the investor likes a particular outcome of the objective
3. define an index of satisfaction as $E[u(\Psi_\alpha(\omega))]$.

In particular, we assume the portfolio return as the investor objective.

From the general problem to mean variance optimization

The fundamental single period problem:

$$\max_{\mathbf{w}|\mathbf{w}'\mathbf{1}=1} \mathrm{E}[u(1+\mathbf{w}'\mathbf{r})]$$

Where u is growing and concave. The problem is stated for a unit of initial wealth.
If $\mathbf{r}$ is multivariate normal then $\mathrm{E}[u(1+\mathbf{w}'\mathbf{r})] = g(\mu_{\mathbf{w}}, \sigma^2_{\mathbf{w}})$, where g is increasing in its first argument and decreasing in the second. The same result can be obtained if $u(1+\mathbf{w}'\mathbf{r}) = (1+\mathbf{w}'\mathbf{r}) - \frac{b}{2}(1+\mathbf{w}'\mathbf{r})^2$ (on the growing part).

From the general problem to mean variance optimization

One more relevant family: the exponential utility function. Let's assume $u(x) = -\exp(-\lambda x)$ and normal returns. Then:

$$\mathrm{E}[u(1 + \boldsymbol{w}'\boldsymbol{r})] = -\exp(\lambda\mu_w - \frac{1}{2}\lambda^2\sigma_w^2).$$

Then:

$$\max_{\mathbf{w}|\mathbf{w}'\mathbf{1}=1} \mathrm{E}[u(1 + \boldsymbol{w}'\boldsymbol{r})] = \max_{\mathbf{w}|\mathbf{w}'\mathbf{1}=1} \boldsymbol{w}'\boldsymbol{\mu} - \frac{1}{2}\lambda\boldsymbol{w}'\boldsymbol{V}\boldsymbol{w}$$

From the general problem to mean variance optimization

The problem can be restated in three ways:

- $\max\limits_{\mathbf{w}|\mathbf{w}'\mathbf{1}=1,\mathbf{w}'\mathbf{V}\mathbf{w}=\bar{\sigma}^2} \mathbf{w}'\boldsymbol{\mu}$
- $\min\limits_{\mathbf{w}|\mathbf{w}'\mathbf{1}=1,\mathbf{w}'\boldsymbol{\mu}=\bar{\mu}} \mathbf{w}'\mathbf{V}\mathbf{w}$
- $\max\limits_{\mathbf{w}|\mathbf{w}'\mathbf{1}=1} \mathbf{w}'\boldsymbol{\mu} - \frac{\lambda^*}{2}\mathbf{w}'\mathbf{V}\mathbf{w}$ if g is smooth and concave, for a certain scalar λ^*.

What is λ^*?

$$\lambda^* = -2\frac{\partial g(\mu_w, \sigma_w^2)}{\partial \mu_w} \Bigg/ \frac{\partial g(\mu_w, \sigma_w^2)}{\partial \sigma_w^2}$$

Section 3

Basic results

The optimal solution - n risky assets

Let's solve the following problem:

$$\min_{\mathbf{w}|\mathbf{w}'\mathbf{1}=1,\mathbf{w}'\boldsymbol{\mu}=\bar{\mu}} \mathbf{w}'\mathbf{V}\mathbf{w}$$

Define:

$$\begin{aligned}
A &= \mathbf{1}'\boldsymbol{V}^{-1}\boldsymbol{\mu} \\
B &= \boldsymbol{\mu}'\boldsymbol{V}^{-1}\boldsymbol{\mu} \\
C &= \mathbf{1}'\boldsymbol{V}^{-1}\mathbf{1} \\
D &= BC - A^2 \\
\boldsymbol{w}_1 &= \frac{1}{D}(B\boldsymbol{V}^{-1}\mathbf{1} - A\boldsymbol{V}^{-1}\boldsymbol{\mu}) \\
\boldsymbol{w}_2 &= \frac{1}{D}(C\boldsymbol{V}^{-1}\boldsymbol{\mu} - A\boldsymbol{V}^{-1}\mathbf{1})
\end{aligned}$$

Then the solution is:

$$\boldsymbol{w} = \boldsymbol{w}_1 + \bar{\mu}\boldsymbol{w}_2$$

n risky assets - main properties

Two mutual funds separation

For any given return expectation level $\bar{\mu}$ the optimal portfolio can be decomposed as a combination of two basic portfolios. Any two distinct optimal portfolios generate the set of optimal portfolios.

Geometrical interpretation

The variance of the optimal solution is given by:

$$\sigma^2 = \frac{C}{D}\bar{\mu}^2 - \frac{2A}{D}\bar{\mu} + \frac{B}{D}$$

which draws a parabola on the $(\sigma^2; \mu)$ plane and a hyperbola on the $(\sigma; \mu)$ plane.

n risky assets - main properties

Minimum variance portfolio

The vertex of the curve is the minimum variance portfolio which has expectation and variance respectively equal to $\frac{A}{C}$ and $\frac{1}{C}$. The weights are therefore $w = \frac{\boldsymbol{V}^{-1}\mathbf{1}}{\mathbf{1}'\boldsymbol{V}^{-1}\mathbf{1}}$.

The optimal solution - one risk free asset

Let $w_0 = 1 - \mathbf{w}'\mathbf{1}$ the percentage of wealth invested in the risk free asset. The problem can be restated as:

$$\min \quad \mathbf{w}'\mathbf{V}\mathbf{w}$$
$$\text{s.t.}$$
$$\mathbf{w}'\boldsymbol{\mu} + (1 - \mathbf{w}'\mathbf{1})r_f = \bar{\mu}$$

which has solution equal to:

$$\boldsymbol{w} = (\bar{\mu} - r_f)\frac{\boldsymbol{V}^{-1}(\boldsymbol{\mu} - \mathbf{1}r_f)}{(\boldsymbol{\mu} - \mathbf{1}r_f)'\boldsymbol{V}^{-1}(\boldsymbol{\mu} - \mathbf{1}r_f)}$$

One risk free asset - main properties

Geometrical interpretation

The variance of the optimal solution is given by:

$$\sigma^2 = \frac{(\hat{\mu} - r_f)^2}{J}$$

where $J = B - 2Ar_f + Cr_f^2$. Hence the frontier draws a couple of half lines on the $(\sigma; \mu)$ plane starting from $(0; r_f)$ with slope $\pm\sqrt{J}$.

One risk free asset - main properties

Tangency portfolio

If $r_f \neq \frac{A}{C}$ the two efficient frontiers with and without risk free asset have one intersection point corresponding to

$$\boldsymbol{w}_t = \frac{\boldsymbol{V}^{-1}(\boldsymbol{\mu} - \mathbf{1} r_f)}{A - C r_f}$$

with

$$\mu_t = \frac{B - A r_f}{A - C r_f}$$
$$\sigma_t^2 = \frac{J}{(A - C r_f)^2}$$

If $r_f = \frac{A}{C}$ the two half lines are equal to the asymptotes and the intersection is empty.

Section 4

Variations & Practical tips

Additional constraints and class of the problem

- Linear constraints allow an explicit solution
- Since the variance is a convex function with regard to the weights, constraining the weights to be element of the intersection of a hyperplane and a convex set leads to convex optimization, which can be solved numerically for reasonable problems.
- A subset of the previous class, namely cone programming, is easier to solve. The functional must be linear or quadratic, the feasible set must be the intersection of a hyperplane and a cone.
- A more detailed model can include a constraint on the number of assets, lower bounds etc.: the problem usually becomes a mixed integer one.

Alternatives

- Different measures of risk: semi variance, mean absolute deviation, VaR, ES, ...
- Expected utility maximization
- Multi period optimization
- Linear programming with clever constraints

From theory to practice

- Main fields of application: asset allocation, equity optimization (both absolute and vs benchmark), index tracking
- The model has really bad out of sample performances, mostly due to statistical error
- When used to perform myopic optimization it often requires abrupt changes in the composition between two periods
- Optimization (absolute) without a positivity constraint can lead to highly leveraged positions
- Sometimes return premiums respect to the risk free rate can provide slightly more stable results

From theory to practice

- Sign constraints (and constraints in general) can improve the out of sample performance
- Further improvements can be achieved by grouping/constraining assets with return/risk ratio small (in absolute value)
- Relative optimization can lead to significantly dominated (in a mean variance way) portfolio. A possible alternative is an absolute optimization with bilateral constraints on the active weights
- Long short 120/20 or 130/30 strategies can exploit negative infos even on stocks with small weights. If short selling is not an option one can resort to a two stage optimization

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