UPPSALA
UNIVERSITET

U.U.D.M. Project Report 2017:17

Portfolio Performance Analysis

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Examensarbete i matematik, 30 hp
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Juni 2017

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Contents

Abstract

This thesis shows several analyses within portfolio performance. It is of great importance for asset managers to be able to show clients and to truly understand the performance of the portfolio. To measure the performance of a portfolio the rate of return is considered. Different definitions of returns exist with different aims and meanings. The discrete compounded return and the continuous compounded return are described and analyzed. Moreover the money-weighted return and time-weighted return are defined and described. The thesis shows similarities and the differences of the two return methodologies. The measurement frequency of the return is studied and furthermore, the thesis gives a brief introduction to the Brinson methodology, which performance attribution models of today are build upon. Performance attribution is an important tool that measures the excess return which arise between the portfolio return and its benchmark. Performance attribution quantifies the active decisions that are being made in the investment decision process. It is a fundamental tool for the analyst in understanding the sources of the portfolio return. The excess return studied in performance attribution can be defined both arithmetic and geometric, which leads to different approaches and models, and hence different advantages and problems. An arithmetic and a geometric multi-level attribution model are presented and analyzed, where the models have a top-down investment decision process. In conclusion three geometric approaches for cross-currency portfolios are presented. All models presented here are used for equity portfolios.

Acknowledgement

I sincerely want to thank my supervisor professor Maciej Klimek for the support and guidance. I also want to thank Markus, Marcus and Mikael for interesting discussions and contribution to my extended knowledge within the field.

I would also like to take the opportunity to thank my loving parents for the great encouragement.

1 Mathematical Framework for Portfolio Return

1.1 Simple Return

When analyzing a collection of asset investments also known as a portfolio there is an interest in measuring the performance. In order to measure the performance the value of the assets in the portfolio over a specific time period is considered. The rate of return r is one way to estimate performance and is defined as the profit or loss of the portfolio's value relative to the starting value of the portfolio

$$r = \frac{V_T - V_0}{V_0}, \tag{1}$$

where V_0 is the value of the portfolio at the beginning of the time period and V_T is the value of the portfolio at the end of the time period. It is not always an easy task to estimate the value of the portfolio. In order to obtain a reasonable value one must use the asset's current economic value, that is the traded market value. From the moment that one buys a security the portfolio is economically exposed to the price changes, even though the trade might not yet have been settled, i.e paid for. Dividend announcement should also be included in the portfolio valuation as well as interest received from fixed income assets [1].

During a time period with no external cash flows, the total time period can be compounded into n subperiods in the following way

$$1 + r = \frac{V_1}{V_0}\frac{V_2}{V_1} \cdots \frac{V_{n-1}}{V_{n-2}}\frac{V_T}{V_{n-1}} = \frac{V_T}{V_0},$$

which can also be written as

$$1 + r = (1 + r_1)(1 + r_2) \cdot \ldots \cdot (1 + r_{n-1})(1 + r_n),$$

for which the rates of return are discrete compounded [1].

1.2 Discrete Compounded Return

Normally, when banks pay interest on an account, one receives interest on the interest payments [1]. There are several different types of rates that are encountered within finance. Nominal rate, effective rate, periodic rate and annualized rate are some of the most common rates. They are needed in order to fully understand the different concepts of returns.

The *nominal interest rate* is the stated interest rate of loan or bonds. The issuer guarantees the interest rate and it is the monetary price that lender receives from borrowers. The nominal interest rate do not compound periods. One can not compare nominal interest rates if they have different compounding periods [2].

The *effective rate* do include compounding, in contrast to the nominal interest rate. The effective rate translates the nominal rate into a rate with annual compounding, so that the rates are comparable. The effective return r for n periods in the year is defined as

$$r = \left(1 + \frac{\bar{r}}{n}\right)^n - 1,$$

where $\bar{r}$ is the nominal rate of return or the annual interest rate [2].

The *periodic interest rate* is the interest rate charged over a specific time period. Hence, it is equal to the annual interest rate divided by the frequency of compounding

$$r_p = \frac{\bar{r}}{n}.$$

Thus the effective return r depends on the periodic rate as

$$r = (1 + r_p)^n$$

[2]. The *annualized return* is a return that is scaled to a period of 1 year. The general formula for the annualized return is given by

$$r_{Ann} = \left(\prod_{i=1}^{N} (1 + r_i) \right)^{\frac{1}{N}} - 1,$$

where N is the holding period in years and r_i is the corresponding return. The "1" in the exponent is due to that the return is measured in the unit 1 year [3].

1.3 Continuous Compounded Return

The continuously compounded return $\tilde{r}$ is defined as

$$\tilde{r} = \ln \left(\frac{V_T}{V_0} \right),$$

where V_0 is the value of the portfolio at the beginning of the time period and V_T is the value of the portfolio at the end of the time period. Thus, the continuous compounded return is connected to the simple return r in the following way

$$\tilde{r} = \ln(1 + r),$$
$$\Leftrightarrow r = e^{\tilde{r}} - 1.$$

The connection between the discrete and continuous compounded return is easily seen, for discrete compounded returns

$$r = \left(1 + \frac{\bar{r}}{n}\right)^n - 1.$$

Furthermore when n approaches infinity we have

$$r = \lim_{n \to \infty} \left(1 + \frac{\bar{r}}{n}\right)^n - 1 = e^{\bar{r}} - 1,$$

which is the continuous compounded return [1].

The continuous compounded returns are easily calculated over several time periods since

$$1 + r_1 = e^{\bar{r}_1}$$

and

$$1 + r_2 = e^{\bar{r}_2}.$$

Hence,

$$1 + r = (1 + r_1)(1 + r_2) = e^{\bar{r}_1 + \bar{r}_2},$$

and

$$\ln(1 + r) = \ln(1 + r_1) + \ln(1 + r_2)$$

[1]. The annualized return for the continuous compounded return is given by

$$
\begin{aligned}
\tilde{r}_A &= \ln(1 + r_A) \\
&= \ln\left[\left(\prod_{i=1}^{N}(1 + r_i)\right)^{\frac{1}{N}}\right] \\
&= \frac{1}{N}\sum_{i=1}^{N}\tilde{r}_i,
\end{aligned}
$$

where N is the holding period in years and $\tilde{r}_i$ is the corresponding return [4].

1.4 Numerical Results for Compounded Returns

Assume that the annual interest rate is 12 %. For discrete compounding when the frequency of compounding increases the effective rate increases but at a slower and slower rate which is seen in Figure 1

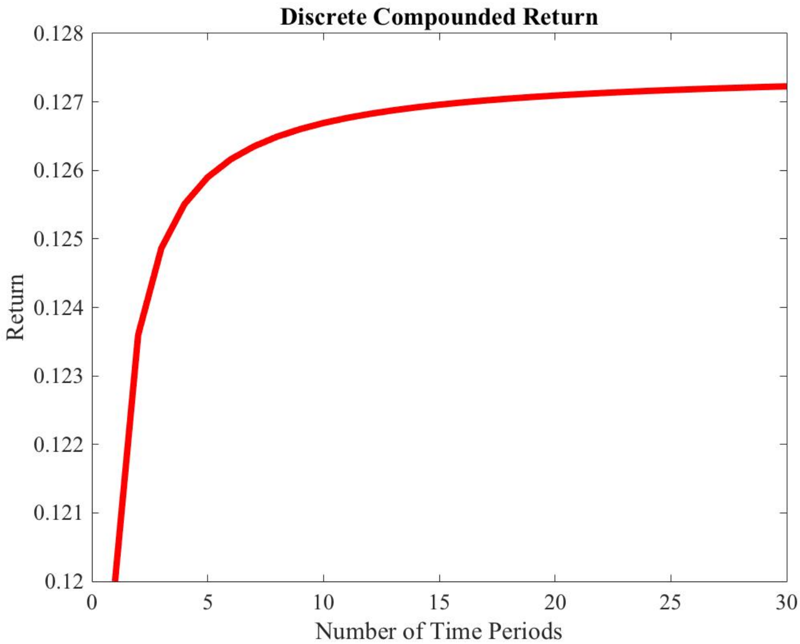

Figure 1.

Moreover,

	Frequency	$(1 + 0.12/n)^n - 1$
Annually:	$n = 1$	$(1.12)^1 - 1 = 12\%$
Quarterly:	$n = 4$	$(1.03)^4 - 1 = 12.55\%$
Monthly:	$n = 12$	$(1.01)^{12} - 1 = 12.68\%$
Continuous Compounded Return:	$n \to \infty$	$\exp(0.12) - 1 = 12.75\ \%$

As seen the more frequent payments the higher compounded return. Hence, the difference increases between the effective and nominal return as the frequency of compounding increases.

For the case when the security is considered to return 12% annually the nominal rate is $\bar{r} = \ln(1.12) = 11.3329\%$. Below is the result for the discrete compounded returns

	Frequency	$\bar{r} = n \cdot ((1+0.12)^{1/n} - 1)$	$n \cdot \bar{r}$
Annual	$n = 1:$	12%	$1 \cdot 12 = 12$
Quarterly	$n = 4:$	2.8737%	$4 \cdot 2.8737 = 11.4949\%$
Monthly	$n = 12:$	0.948879%	$12 \cdot 0.948879 = 11.3866\%$

As seen in the table, $n \cdot \bar{r}$ is strictly greater than the nominal rate. This always holds for $\bar{r} > 0$ and for every integer n where $n > 1$ which is shown below.

The Bernoulli's inequality states that for $x > -1$, $x \neq 0$ and for integers $m > 1$, then

$$(1+x)^m > (1+xm)$$

[11].

For the sequence $\{a_n\} = \left(1 + \frac{r}{n}\right)^n$ it follows that

$$\lim_{n\to\infty} \{a_n\} = \lim_{n\to\infty} \left(1 + \frac{r}{n}\right) = e^r$$

[12].

Assume that the number of time periods are $n > 1$ and consider the discrete compounded rate of return

$$(1+r) = (1+r_1)^n,$$

and the continuously compounded rate of return

$$(1+r) = e^{r_2},$$

where $r_1, r_2 > 0$. Then, the following holds

$$e^{r_2} = (1+r_1)^n.$$

Furthermore the sequence $\{a_n\} = \left(1 + \frac{r}{n}\right)^n$ is monotonically increasing for $r > 0$ and $n > 0$, since

$$\begin{aligned}
\frac{\left(1 + \frac{r}{n+1}\right)^{n+1}}{\left(1 + \frac{r}{n}\right)^n} &= \left(1 + \frac{r}{n}\right)\left(\frac{1 + \frac{r}{n+1}}{1 + \frac{r}{n}}\right)^{n+1} \\
&= \left(1 + \frac{r}{n}\right)\left(\frac{n}{n+r} \cdot \frac{n+r+1}{n+1}\right)^{n+1} \\
&= \left(\frac{n+r}{n}\right)\left(1 - \frac{r}{(n+1)(n+r)}\right)^{n+1}.
\end{aligned}$$

Then, since $\frac{r}{(n+1)(n+r)} \in (0,1)$ the Bernoulli's inequality holds, and

$$\begin{aligned}
\left(\frac{n+r}{n}\right)\left(1 - \frac{r}{(n+1)(n+r)}\right)^{n+1} &> \left(\frac{n+r}{n}\right)\left(1 - \frac{r}{n+r}\right) \\
&= 1.
\end{aligned}$$

Thus the sequence is monotonically increasing. Hence

$$\begin{aligned}
& e^{r_2} = \lim_{n\to\infty} \left(1 + \frac{r_2}{n}\right)^n > \left(1 + \frac{r_2}{n}\right)^n \\
& \Leftrightarrow (1+r_1)^n > \left(1 + \frac{r_2}{n}\right)^n \\
& \Leftrightarrow r_1 > \frac{r_2}{n} \\
& \Leftrightarrow nr_1 > r_2
\end{aligned}$$

Thus, $r_2 < nr_1$ and the discrete compounded returns are positively biased.

1.5 Adjustments for External Cash Flows

External cash flow is here considered to be any new money that is inserted or withdrawn from the portfolio. New money can be of any form such as cash, securities or other instruments. Income from the investments in the portfolio such as dividend, coupon payments and so on is not considered to be external cash flow [1].

The most commonly used calculations for the return are based on (1). For portfolios with external cash flows one must adjust the calculations of the returns, since the cash flow will affect the value of the portfolio [5].

Two different methodologies can be applied for the calculation of the return when external cash flows occur, known as time-weighted and money-weighted approaches. When considering the money-weighted return each amount that is invested is supposed to contribute equally in the return calculation, no matter when it is invested. Due to this methodology it becomes significant to preform well when the largest amount of money is invested. In contrast to the money-weighted approach is the time-weighted methodology in which all the time periods are equally weighted irrespective of the amount invested, thereof the name time-weighted [1].

1.5.1 Money-Weighted Return

Peter Dietz suggested a method that adjusts for external cash flow where the return r is given by

$$r = \frac{V_T - V_0 - C}{V_0 + \frac{C}{2}},$$

with the assumption that the cash flow has been invested in the middle of the time period. It is known as the *simple Dietz method*.

In the beginning Dietz described that in the method it is assumed that half of the contributions are made at the end of the time period and the other half at the beginning of the period. Hence, $\frac{C}{2}$ is withdrawn from the numerator, corresponding to the contribution at the end of the period. Furthermore, $\frac{C}{2}$ is added in the denominator, corresponding to the contribution at the beginning of the period

$$r = \frac{V_T - \frac{C}{2}}{V_0 + \frac{C}{2}} - 1.$$

This is equivalent to

$$r = \frac{V_T - V_0 - C}{V_0 + \frac{C}{2}},$$

which is the most commonly used expression for the simple Dietz method.

The assumption that all cash flows occur in the middle of the time period is a rather strict simplification. Thus the simple Dietz method may be extended by weighting each cash flow so that the actual timing of the cash flow is reflected in the calculation. The *Modified Dietz method* is therefore defined as

$$r = \frac{V_T - V_0 - C}{V_0 + \sum_t C_t W_t},$$

where C is the the total external cash flow within the period, C_t is the external cash flow that occur at time t and W_t is the weight corresponding to day t. The weight W_t is given by

$$W_t = \frac{TD - D_t}{TD},$$

where TD is the number of days within the period and D_t is the number of days after the beginning of the period which includes holidays and weekends. Thus the weight W_t reflects the time that remains in the period after the cash flow took place. The number of days D_t should reflect whether the cash flow was received in the beginning or towards the end of the day. Hence, if the money was received in the beginning of the day and the investor had the opportunity to invest them, then the day should be included in D_t. Then again if the money was received at the end of the day, the investor might not have had the chance to respond towards the cash flow. Thus, the day should not be included in D_t.

For the modified Dietz formula the following holds

$$\begin{aligned} r &= \frac{V_T - V_0 - C}{V_0 + \sum_t C_t W_t} \\ &= \frac{V_T - \sum_t C_t + \sum_t C_t W_t}{V_0 + \sum_t C_t W_t} - 1 \\ &= \frac{V_T - \sum_t C_t(1 - W_t)}{V_0 + \sum_t C_t W_t} - 1. \end{aligned}$$

As seen the cash flow C_t with the weight W_t that is added in the denominator, has a corresponding withdrawal in the numerator $C_t(1 - W_t)$. The weight W_t represents the amount of time that is left within the period after that the cash flow took place. Thus the corresponding weight $(1 - W_t)$ represents the time length for which the cash flow is not available for investment and hence $C_t(1 - W_t)$ must then be withdrawn in the numerator [1].

1.5.2 Time-Weighted Return

Consider the definition of the true time-weighted rate of return that adjusts for external cash flows. The subperiods are chain-linked and defined as follows

$$1 + r = \frac{V_1 - C_1}{V_0} \frac{V_2 - C_2}{V_1} \cdots \frac{V_{n-1} - C_{n-1}}{V_{n-2}} \frac{V_T - C_n}{V_{n-1}},$$

where V_t is the value of the portfolio at the end of the period and immediately after the cash flow C_t. Hence, the cash flow is here assumed to be inserted at the end of the time period.

Alternatively, if it is assumed that the cash flow is inserted at the beginning of the subperiods, thus the money is available for investment throughout the day and hence the rates of return are given by

$$1 + r = \frac{V_1}{V_0 + C_1} \frac{V_2}{V_1 + C_2} \cdots \frac{V_{n-1}}{V_{n-2} + C_{n-1}} \frac{V_T}{V_{n-1} + C_n}.$$

Furthermore, if it is assumed that the cash flows were inserted in the middle of the subperiods, the rates of return are then given by

$$1 + r = \frac{V_1 - \frac{C_1}{2}}{V_0 + \frac{C_1}{2}} \frac{V_2 - \frac{C_2}{2}}{V_1 + \frac{C_2}{2}} \cdots \frac{V_{n-1} - \frac{C_{n-1}}{2}}{V_{n-2} + \frac{C_{n-1}}{2}} \frac{V_T - \frac{C_n}{2}}{V_{n-1} + \frac{C_n}{2}}.$$

As seen below the time-weighted return equals the simple Dietz return in this case

$$r_t = \frac{V_t - V_{t-1} - C_t}{V_{t-1} + \frac{C_t}{2}} = \frac{V_t - \frac{C_t}{2}}{V_{t-1} + \frac{C_t}{2}} - 1.$$

This is a hybrid methodology when the simple Dietz return is used for each individual day. Due to the daily weighting it is considered to be a time-weighted return [1].

1.6 Interpretation of the Money- and Time-Weighted Return

The difference between the two methodologies is the way of handling external cash flow, without any cash flow the methods are equivalent and give the same return.

1.6.1 Example

Assume that the starting value of the portfolio for period 1 is $V_0 = 200$ and the end value of period 1 is 400. Furthermore assume that a cash flow occured at the end of period 1, $C_1 = 1000$. Hence, the starting value of period 2 is $V_1 = 1400$ and the end value of the period is $V_2 = 800$. Then the time-weighted return is given by,

$$\frac{1400 - 1000}{200} \cdot \frac{800}{1400} - 1 \approx 14.29\%,$$

while the money-weighted return is given by

$$\frac{800 - 200 - 1000}{200 + \frac{1000}{2}} \approx -57.14\%.$$

The time-weighted return can at first sight seem counterintuitive since it is positive, even though the value at the end of the period is smaller than the total amount invested. Despite this, most performance analysts do prefer to measure the returns by using the time-weighted method. This is due to the definition of the time-weighting, that each time period is weighted equally regardless of the amount invested. Hence, the timing of the external cash flows does not affect the return, which is to be preferred since the portfolio managers usually do not determine the timing of the cash flow, thus it should not affect the performance. The time-weighted rate of return is the preferred industry standard as it is not sensitive to contributions or withdrawals [1].

1.6.2 Numerical Results

The numerical results that follows are based on a portfolio containing 11 different stocks. The stocks are simulated by geometric Brownian motions, each simulated 5,000,000 times in the figures below. It is assumed that a cash flow occur in the middle of the time period. The money-weighted return is defined as

$$r_{MWR} = \frac{V_2 - V_0 - C}{V_0 + \frac{C}{2}},$$

and the time-weighted return is defined as

$$r_{TWR} = \frac{V_1 - C}{V_0} \cdot \frac{V_2}{V_1} - 1.$$

Figure 2 shows the mean of the time-weighted and money-weighted returns as a function of the cash flow, where the cash flow is given by the percentage shown in the x-axis times the initial value of the portfolio. The time-weighted return has a greater mean than the money-weighted return for every cash flow.

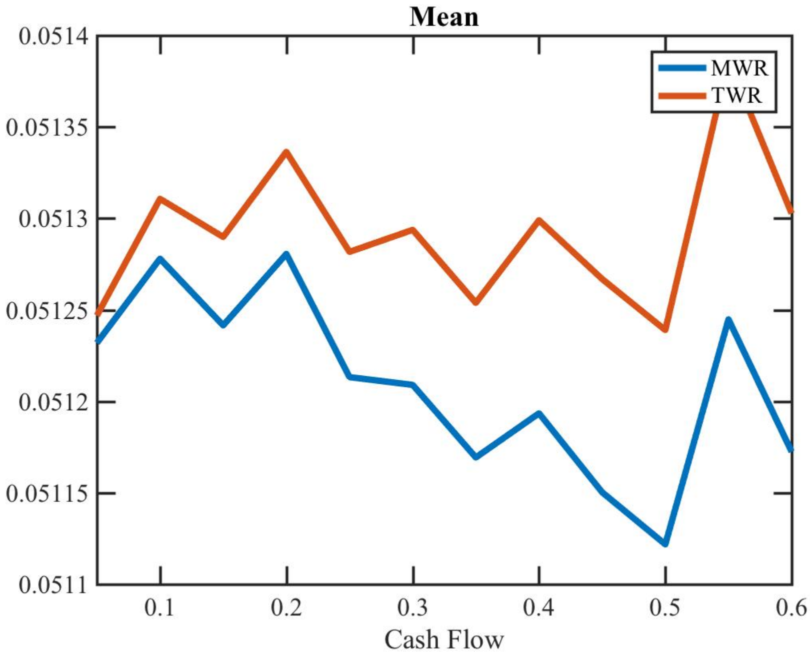

Figure 2.

Figure 3 shows the variance of the time-weighted and money-weighted returns as a function of the cash flow, where the cash flow is given by the percentage shown in the x-axis times the initial value of the portfolio. The money-weighted return and the time-weighted return have almost equal variance for small cash flows, but as the cash flows increase the variances increase at different rates. Due to larger cash flows the variance of the money-weighted return is greater than the variance of the time-weighted return. The differences between the returns are more easily seen for larger cash flows. Hence, the differences are as seen reflected in the variance as well.

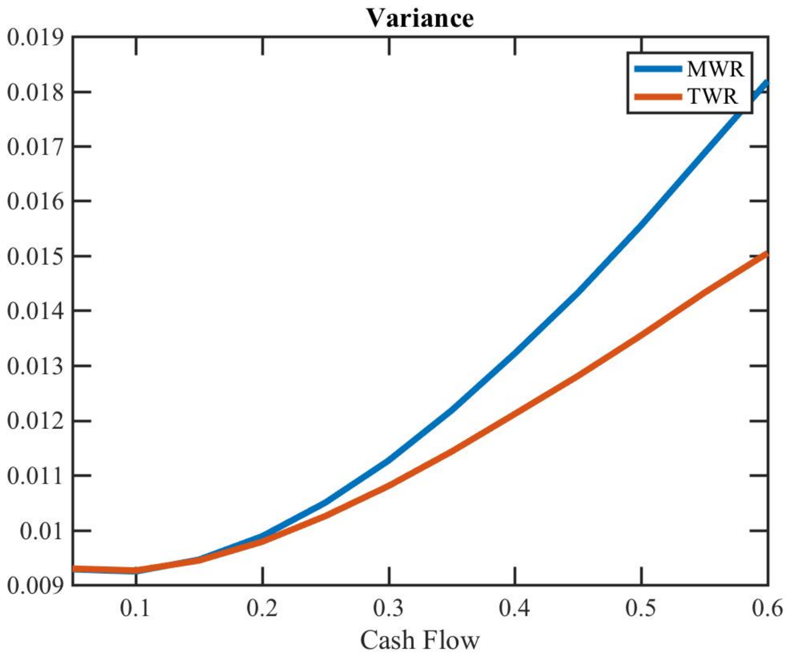

Figure 3.

Figure 4 shows the fraction for which the money-weighted return is lesser than the time-weighted return, given that the end value of the portfolio is lesser than the total amount

invested. The cash flow is given by the percentage shown in the x-axis times the initial value of the portfolio.

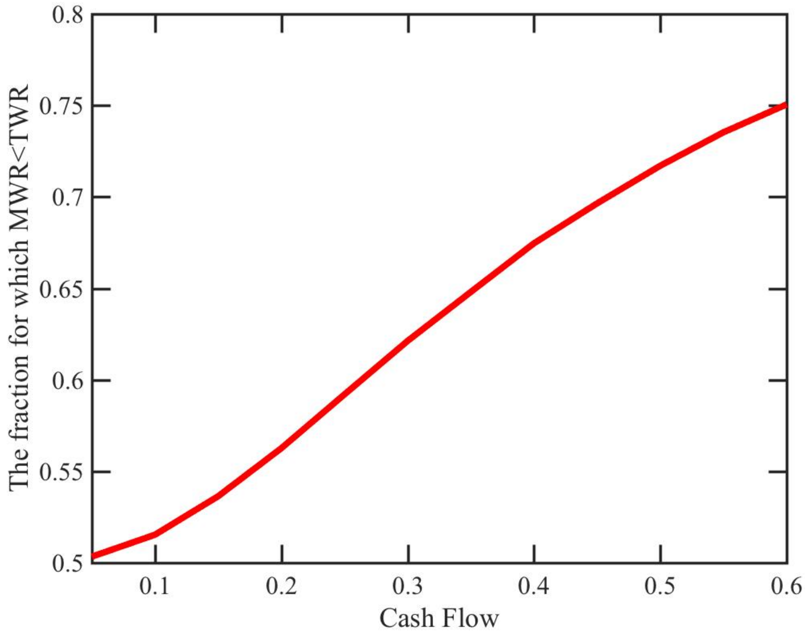

Figure 4.

As seen in the figure, when the amount of cash flow increases the fraction increases as well. The greater the cash flow, the more significant is the performance of the portfolio in the second period, since the second period is the period with the greatest amount invested. Thus the portfolio value has an increasing dependence towards the result of the second period as the cash flow increases. The figure shows that the money-weighted return responds more negatively than the time-weighted return towards an increasing cash flow when end value of the portfolio is lesser than the total amount invested.

Figure 5 shows the mean of the money-weighted return and the time-weighted return, given that the end value of the portfolio is lesser than the total amount invested. The cash flow is given by the percentage shown in the x-axis times the initial value of the portfolio. The money-weighted return and the time-weighted return have almost equal means for small cash flows, but as the cash flow increases the means decrease at different rates. Due to large cash flows the mean of time-weighted return is larger than the mean of the money-weighted return. As the cash flow increases the difference between the two means of the returns increases as well.

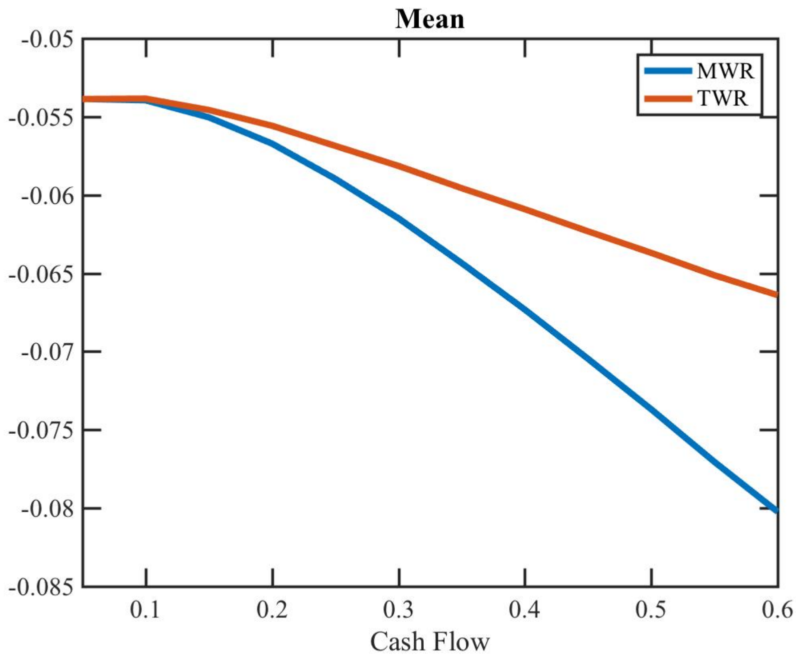

Figure 5.

Figure 6 shows the variances of the money-weighted return and the time-weighted return, given that the end value of the portfolio is lesser than the total amount invested. The cash flow is given by the percentage shown in the x-axis times the initial value of the portfolio. The money-weighted return and the time-weighted return have almost equal variance for small cash flows, but as the cash flow increases the variances increase at different rates. Thus the difference in variance between the money-weighted return and the time-weighted return increases with the cash flow.

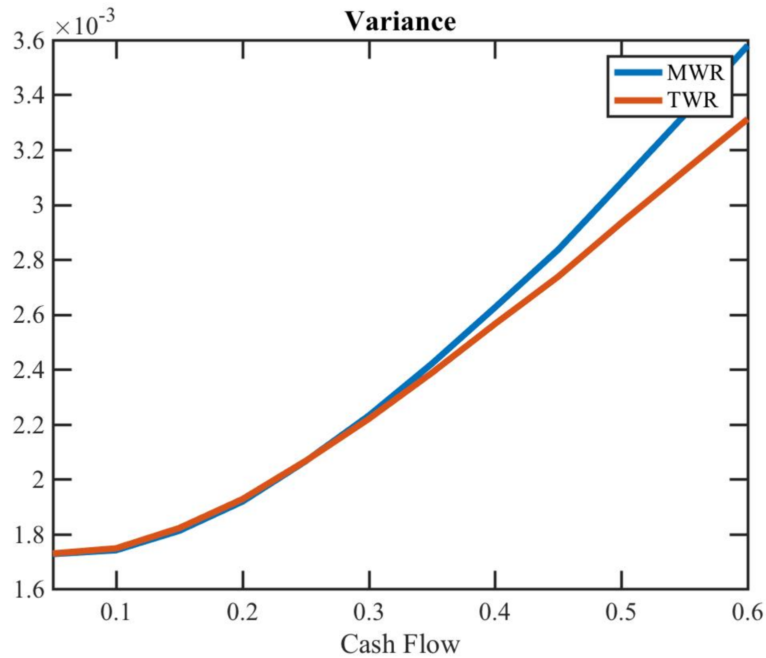

Figure 6.

Figure 7 shows the fraction for which the money-weighted return is lesser than the time-weighted return, given that the end value of the portfolio is lesser than the total amount invested and that the end value of period 1 is greater than the amount initially

invested. The cash flow is given by the percentage shown in the x-axis times the initial value of the portfolio.

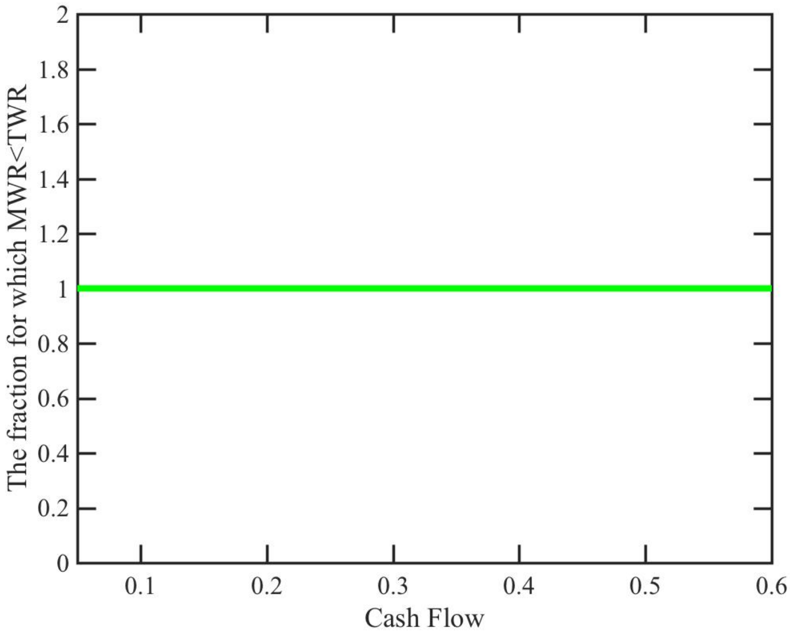

Figure 7.

The fraction is equal to 1 for every cash flow in the plot. Since the end value is lesser than the invested amount and the portfolio made a profit in the first period, the portfolio must have in the end lost a greater amount of money in the second period than was earned in the first period. The results of Figure 4 combined with Figure 7 conclude that the money-weighted return is only greater than the time-weighted return when the portfolio has lost money in the first period. Hence the time-weighted return is affected more positively of a profit in the first period than the money-weighted return. The money-weighted return gives greater significance to the loss in the second period compared to the time-weighted return.

Figure 8 shows the fraction for which the money-weighted return is greater than the time-weighted return, given that the end value of the portfolio is greater than the total amount invested. The cash flow is given by the percentage shown in the x-axis times the initial value of the portfolio.

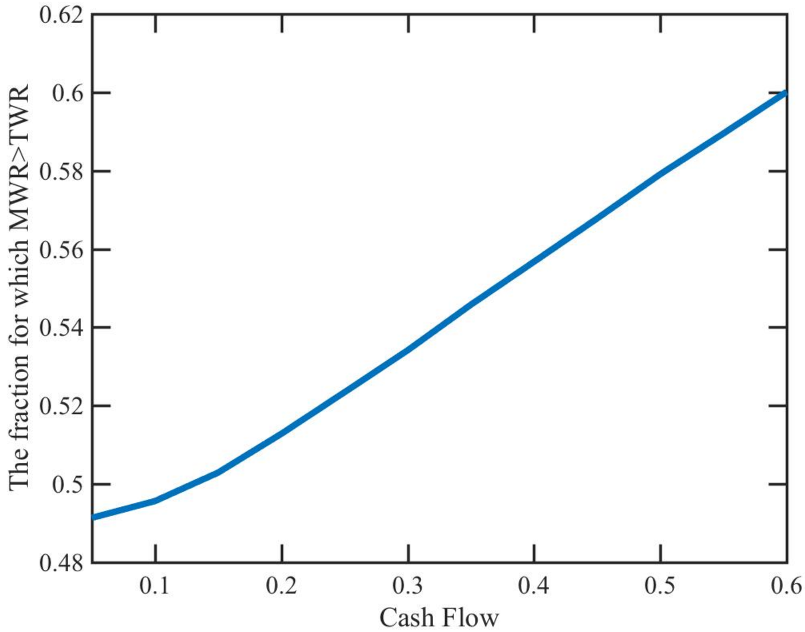

Figure 8.

As seen in the figure when the amount of cash flow increases the fraction increases as well. The greater the cash flow, the more significant it gets for the portfolio to perform well in the second period in order for the portfolio to have a greater value in the end of the time-period than the total amount invested. The money-weighted return responds more positively towards an increasing cash flow when end value of the portfolio is greater than the total amount invested compared to the time-weighted return.

Figure 9 shows the means of the money-weighted return and the time-weighted return, given that the end value of the portfolio is greater than the total amount invested. The cash flow is given by the percentage shown in the x-axis times the initial value of the portfolio. The mean of the money-weighted return is very close to the mean of the time-weighted return for small cash flows. When the cash flow increases the difference of the means increases. The money-weighted return has for larger cash flows a greater mean than the time-weighted return.

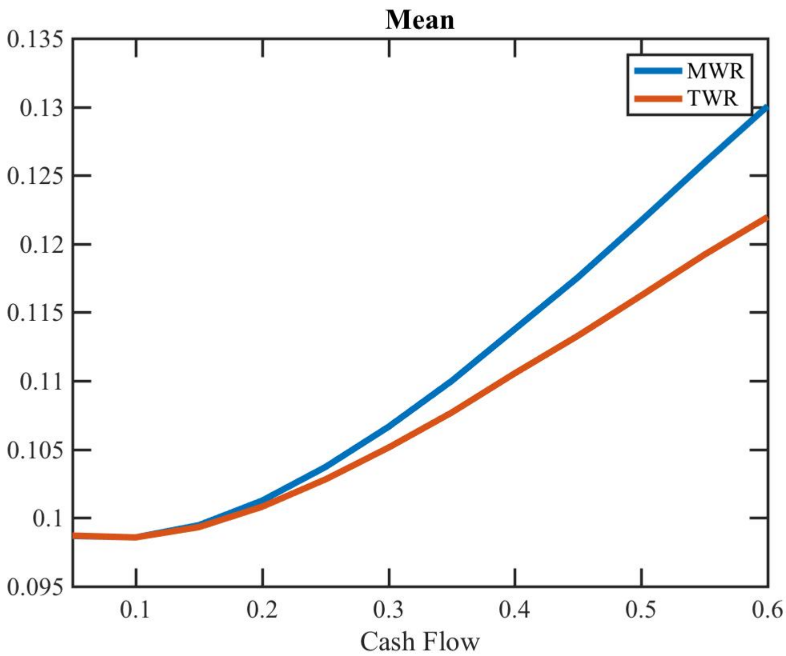

Figure 9.

Figure 10 shows the variance of the money-weighted return and the time-weighted return, given that the end value of the portfolio is greater than the total amount invested. The cash flow is given by the percentage shown in the x-axis times the initial value of the portfolio. The variance of the money-weighted return is very close to the variance of the time-weighted return for small cash flows, but when the cash flows increase, the difference of the variances increases. The money-weighted return has for larger cash flows a greater variance than the time-weighted return.

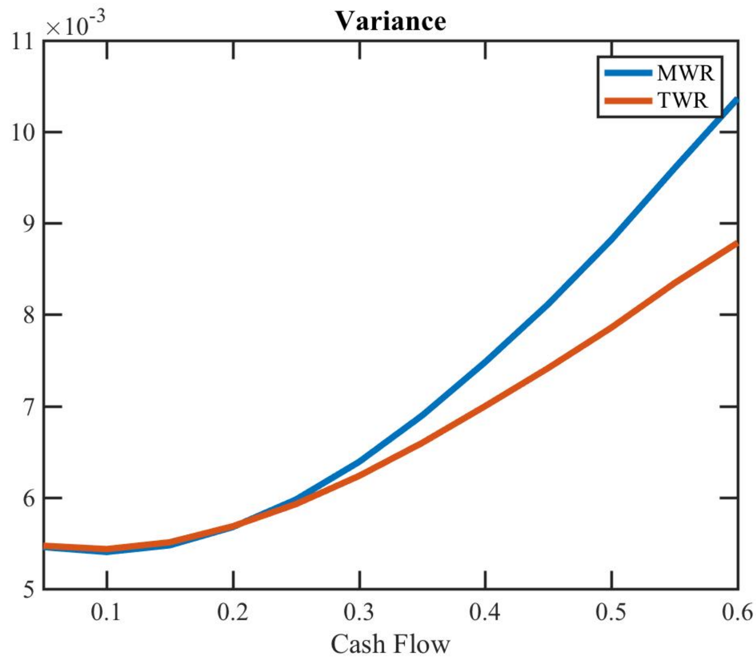

Figure 10.

Figure 11 shows the fraction for which the money-weighted return is greater than the time-weighted return, given that the end value of the portfolio is greater than the total amount invested and that the end value of period 1 is lesser than the amount initially

invested. The cash flow is given by the percentage shown in the x-axis times the initial value of the portfolio.

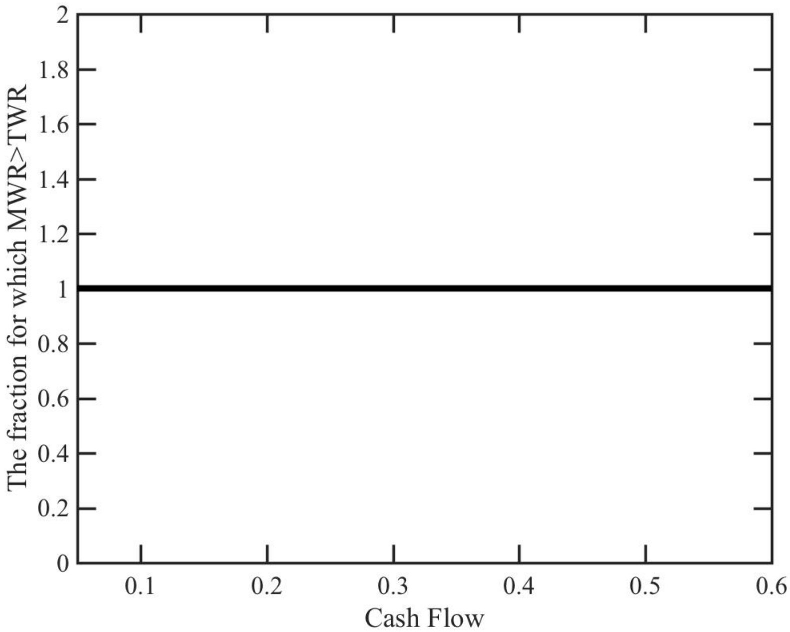

Figure 11.

The fraction is equal to 1 for every cash flow in the plot. Since the end value is greater than the invested amount and the portfolio made a loss in the first period, the portfolio must have in the end earned more money in the second period than was lost in the first period. The results of Figure 8 combined with the results of Figure 11 conclude that the money-weighted return is only lesser than the time-weighted return when the portfolio has made a profit in the first period. Hence the time-weighted return gives a greater significance to a profit in the first period than the money-weighted return. Moreover, the money-weighted return responds more positively to a profit in the last period in contrast to the time-weighted return.

Figure 12 shows the fraction for which the money-weighted return is lesser than the time-weighted return, given that the end value of period 1 is greater than the amount initially invested and the end value of period 2 is lesser than the value of the beginning of period 2. The cash flow is given by the percentage shown in the x-axis times the initial value of the portfolio.

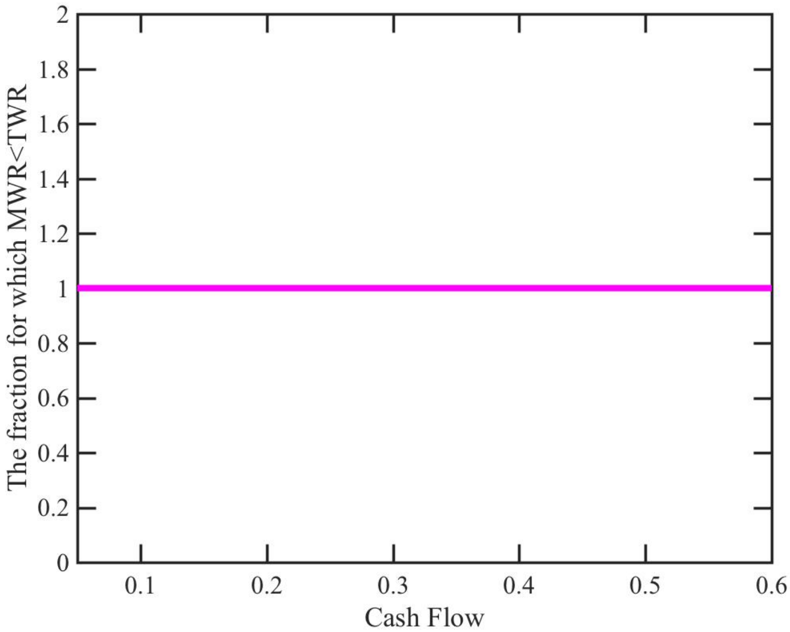

Figure 12.

The fraction is equal to 1 for every cash flow in the plot.

Figure 13 shows the fraction for which the money-weighted return is greater than the time-weighted return, given that the end value of period 1 is lesser than the amount initially invested and the end value of period 2 is greater than the value of the beginning of period 2. The cash flow is given by the percentage shown in the x-axis times the initial value of the portfolio.

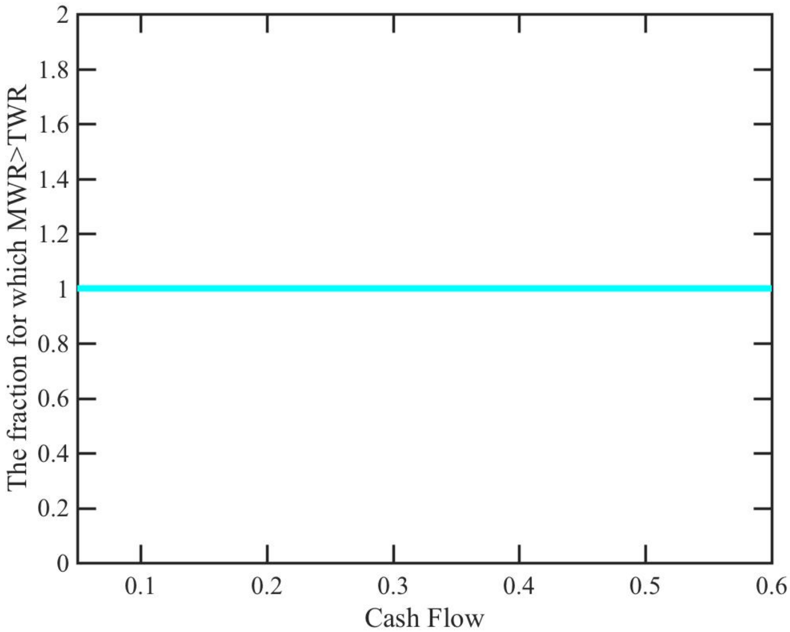

Figure 13.

The fraction is equal to 1 for every cash flow in the plot. As seen in Figure 12 and Figure 13, the money-weighted return gives greater significance to the result in the second period than the first period compared to the time-weighted return.

Figure 14 shows the fractions for which the money-weighted return and time-weighted return are lesser than zero, given that the portfolio value at the end of period 2 is lesser than the total invested amount. The cash flow is given by the percentage shown in the x-axis times the initial value of the portfolio.

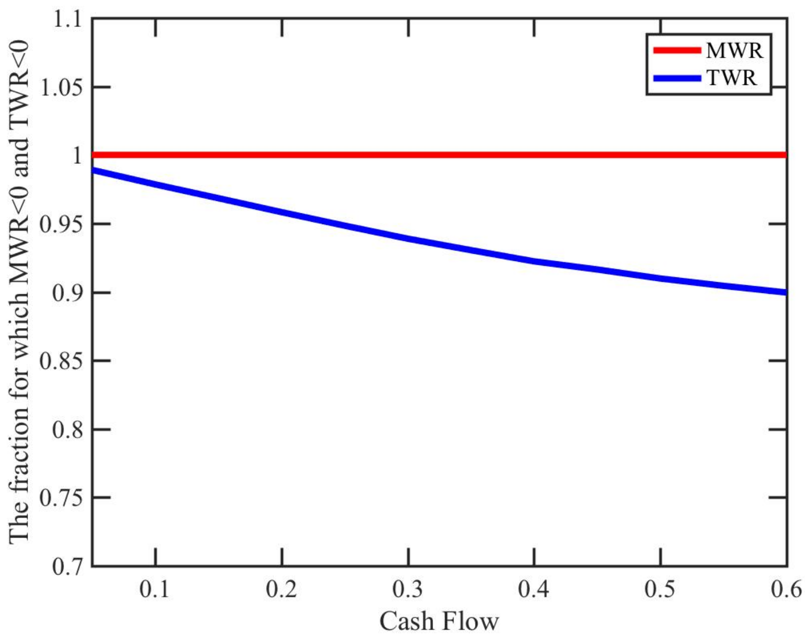

Figure 14.

The fraction of the money-weighted return is equal to one, regardless of the cash flow, while the fraction of the time-weighted return decreases as the amount of cash flow increases. Thus the time-weighted return can be positive even though the end value of the portfolio is lesser than the amount invested. Greater profits in the first period are included when greater cash flows occur, since the result of the first period have a smaller impact on the value of the portfolio at the end of the last time period. Larger profits of the first period increase the time-weighted return and may contribute to a positive return. Due to that the time-weighted case weights both periods equally, the fraction decreases as the cash flow increases.

Figure 15 shows the fractions for which the money-weighted return and time-weighted return are lesser than zero, given that the portfolio value at the end of period 2 is lesser than the total invested amount and the portfolio's value at the end of period 1 is greater than the initial invested value times a constant. The four considered constants are c_1 =1.01, c_2 =1.05, c_3 =1.1, and c_4 =1.15, for which the money-weighted return is constantly equal to one, while the time-weighted return decreases as the constant is chosen greater and greater. The cash flow is given by the percentage shown in the x-axis times the initial value of the portfolio.

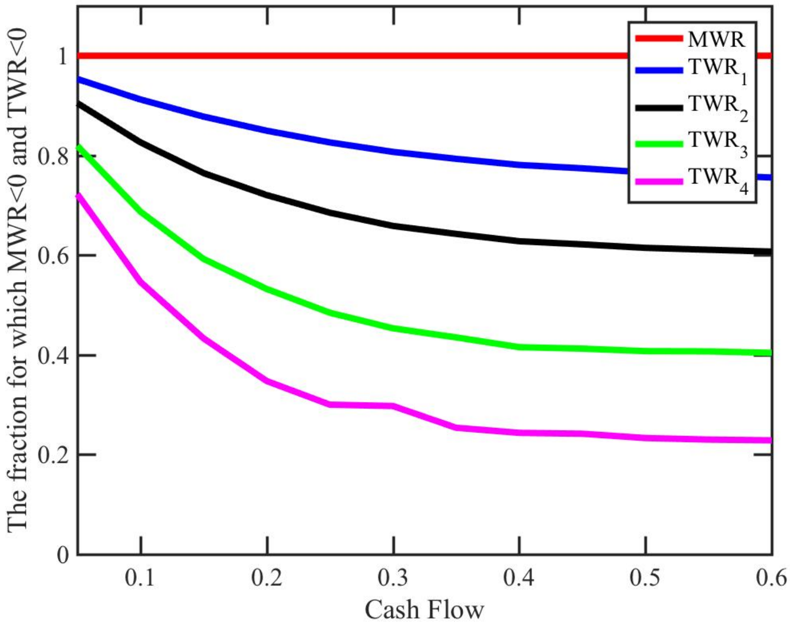

Figure 15.

When the constant increases the profits in the first period also increase, which contributes to more positive returns, hence the lines decrease as the constant increases.

Figure 16 shows the means given that the portfolio's value at the end of period 2 is lesser than the total invested amount and the portfolio's value at the end of period 1 is greater than the initial invested amount times a constant. The four considered constants are once again c_1 =1.01, c_2 =1.05, c_3 =1.1, and c_4 =1.15, corresponding to TWR_1 and MWR_1, TWR_2 and MWR_2, TWR_3 and MWR_3, TWR_4 and MWR_4.

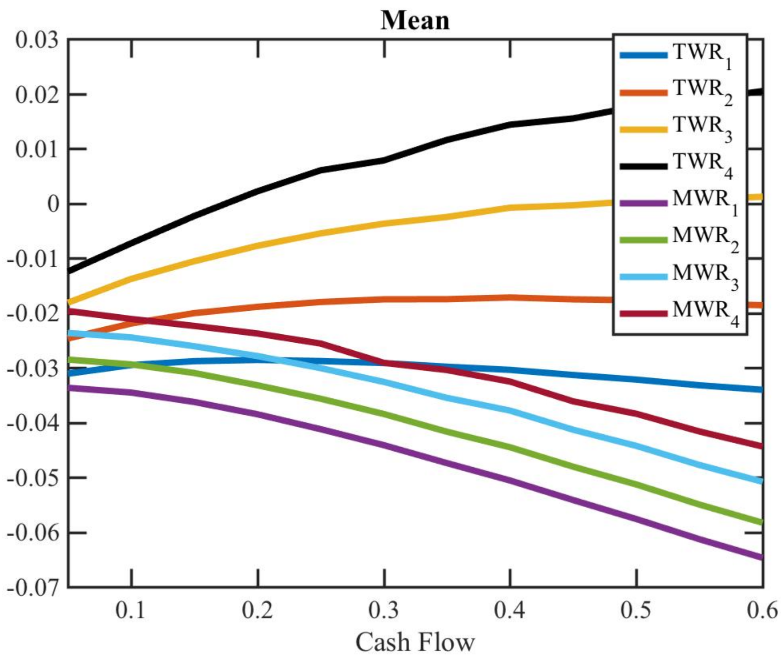

Figure 16.

As seen in the figure the greater the constant the greater the mean. The mean of the time-weighted return is greater than the corresponding money-weighted return for every cash flow.

Figure 17 shows the variances given that the portfolio value at the end of period 2 is lesser than the total invested amount and the portfolio's value at the end of period 1 is greater than the initial invested value times a constant. The constants c_1 =1.01, c_2 =1.05, c_3 =1.1, and c_4 =1.15, corresponding to TWR_1 and MWR_1, TWR_2 and MWR_2, TWR_3 and MWR_3, TWR_4 and MWR_4.

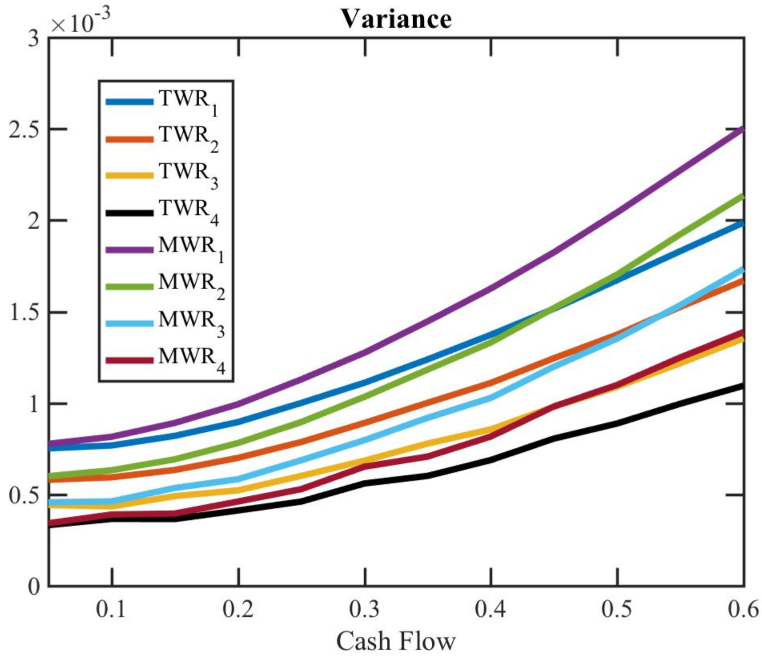

Figure 17.

As seen in the figure the smaller the constant the greater the variance. The variance of the money-weighted return is larger than the corresponding variance of the time-weighted return for each cash flow.

The differences between the time-weighted and money-weighted return increase as the cash flow increases. The results show that the money-weighted return gives the most significance to the time period where the invested amount is the greatest. The time-weighted return weights both periods equally and is therefore not as affected as the money-weighted return during the time with the greatest invested amount.

1.7 Frequency of Measurement

The frequency of measurement affects the return. Below follows return calculations for one year with different number of time periods. The different frequencies that are considered are daily, monthly, quarterly, half-year and annual. It is assumed that a cash flow occurs at the end of day 157. Furthermore there are considered to be 252 trading days in one year. The portfolio that is considered contains 11 stocks, which have been simulated 2,000,000 times each by using geometric Brownian motions. The modified Dietz method is used for the return calculations. When modified Dietz is measured daily it is considered to be a time-weighted return as above mentioned. The more frequently measured the more is the return time-weighted. Hereafter in the figures the following notion holds D = Measurement with daily frequency, M = Measurement with monthly frequency, Q = Measurement with quarterly frequency, H = Measurement for every half-year and A = Measurement with annual frequency.

For the 2,000,000 simulations, Figure 18 shows the means of the returns for the different frequencies.

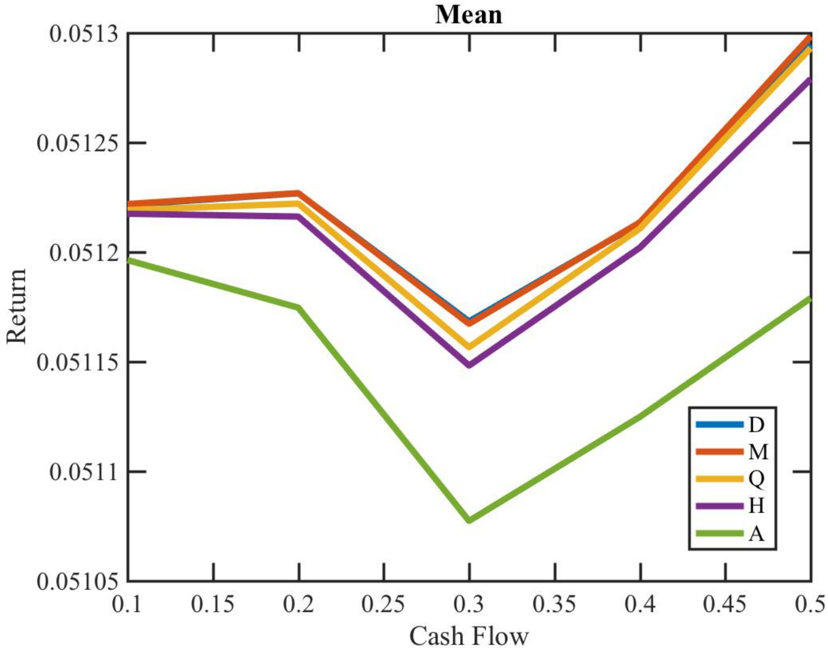

Figure 18.

The means of the returns vary with the frequency. The greatest means have the daily and monthly returns which are almost equal for every cash flow, followed by the quarterly return, half-year return and the smallest mean has the return with the annual frequency.

Figure 19 shows the variances of the returns for the different frequencies, for the 2,000,000 simulations.

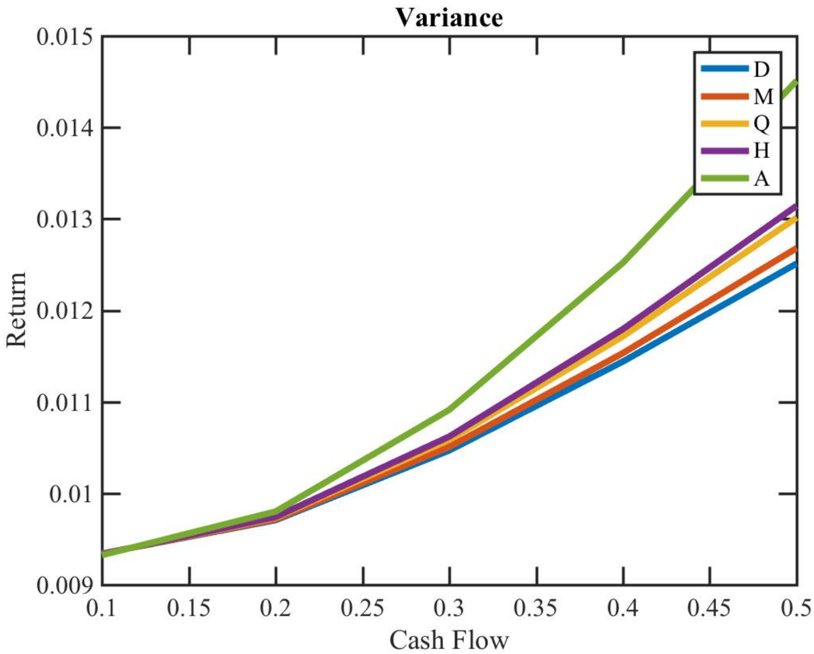

Figure 19.

The variances of the returns have a strong dependence with the frequency. When the cash flow increases the figure shows that the lesser the frequency the greater the variance.

Figure 20 shows the fraction for which the return is lesser than 0, given that the end value of the portfolio is lesser than the value of the portfolio right after the cash flow at day 157.

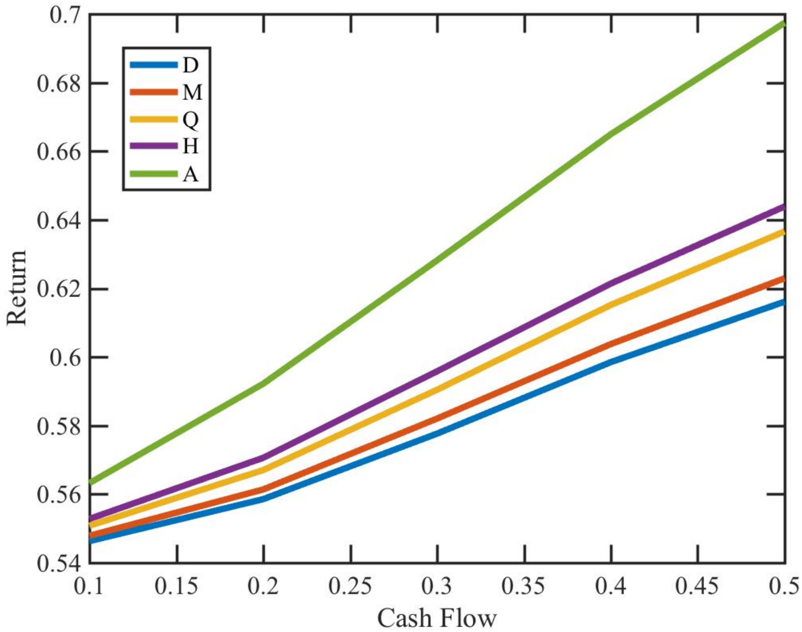

Figure 20.

The fractions of the returns increase as the cash flow increases. The figure also shows that the more frequently measured returns, the lesser the fraction. The fraction for the respective frequency increases with the cash flow, since the difference between the portfolio value at the end of the period and the amount invested increases.

Figure 21 shows the means of the returns for the different frequencies, given that the end value of the portfolio is lesser than the value of the portfolio right after the cash flow at day 157.

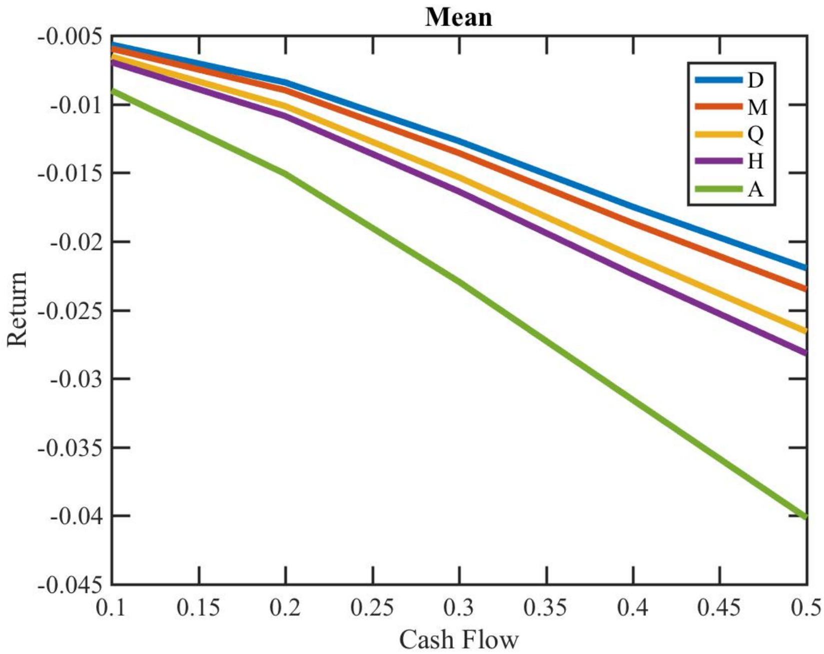

Figure 21.

The greater the measurement frequency of the return the greater the mean value. For every frequency the figure shows that the mean value decreases as the cash flow increases.

Figure 22 shows the variances of the returns for the different frequencies, given that the end value of the portfolio is lesser than the value of the portfolio right after the cash flow at day 157.

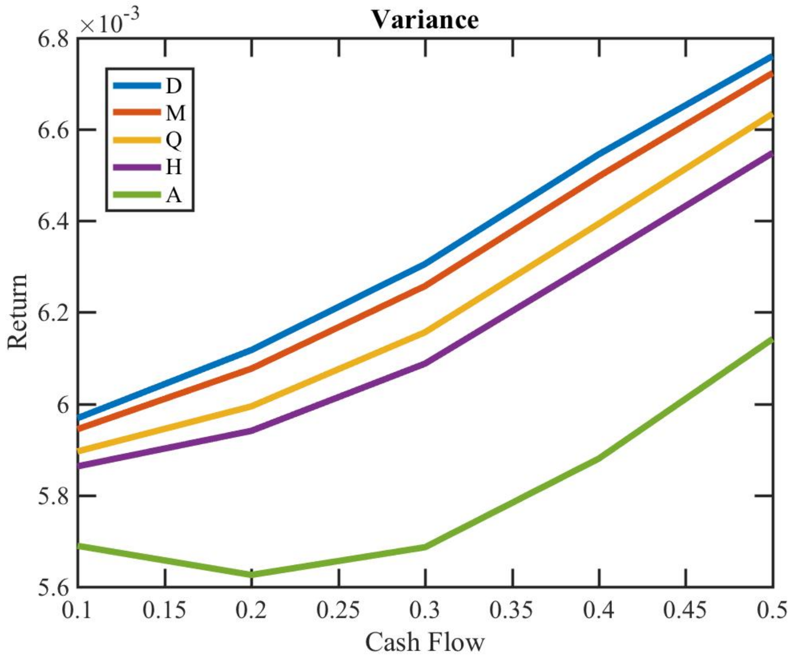

Figure 22.

The greater the measurement frequency of the return the greater the variance. This is not something that is seen in the other situations, normally the greater the measurement frequency the smaller the variance.

Figure 23 shows the fraction for which the return is lesser than 0, given that the end value of the portfolio is lesser than the invested amount.

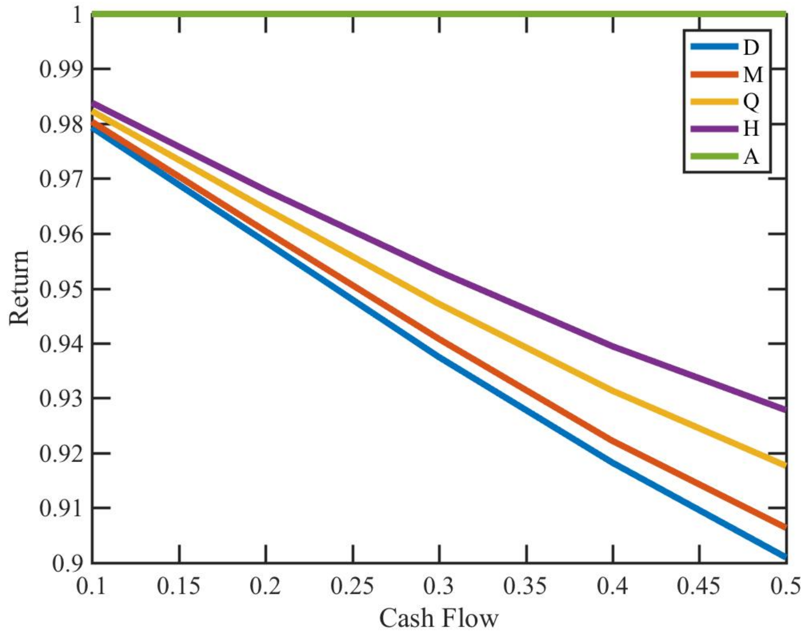

Figure 23.

The fraction for the annual return is constantly equal to one for every cash flow, hence the return is never positive when the portfolio has lost more money than invested. The other returns however may be positive even though the portfolio has lost more money than invested, and the fractions for the other returns decrease as the cash flow increases. Greater profits in the first periods are included when greater cash flows occur, since the result of the periods before the cash flow have a smaller impact on the value of the portfolio at the end. Larger profits of the first periods increase the more time-weighted returns and may contribute to a positive return. Due to that the time-weighted case weights the periods equally, the fraction decreases as the cash flow increases. The lower the frequency the fewer time periods, and the greater significance is given to the poor performance after that the cash flow occurred.

Figure 24 shows the means of the returns for the different frequencies, given that the end value of the portfolio is lesser than the invested amount.

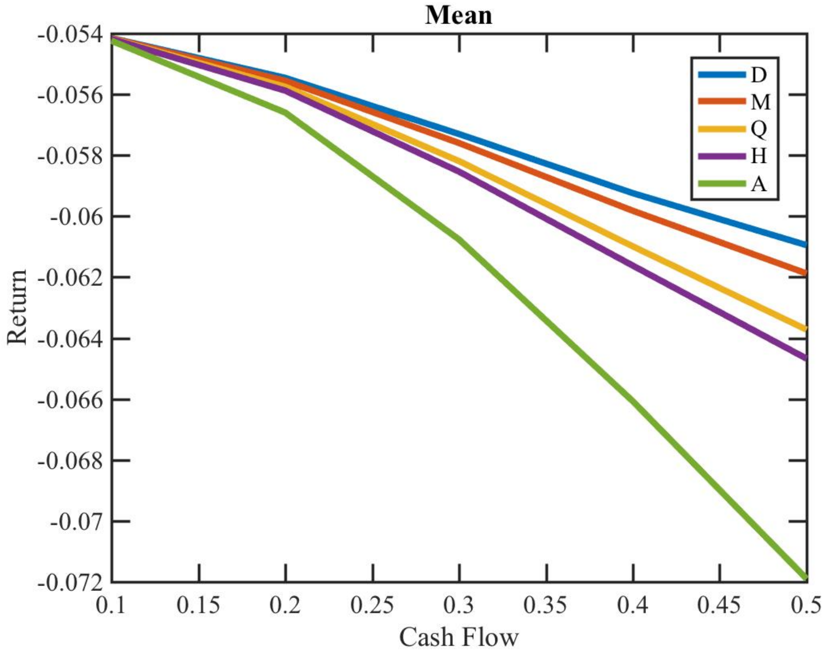

Figure 24.

The higher the measurement frequency the greater the mean value. The figure shows that the returns with lower frequency react more negatively towards an increasing cash flow when end value of the portfolio is lesser than the total amount invested. The lower the frequency the fewer time periods, and the more does the loss of the portfolio affect the overall return negatively.

Figure 25 shows the variance of the returns for the different frequencies, given that the end value of the portfolio is lesser than the invested amount.

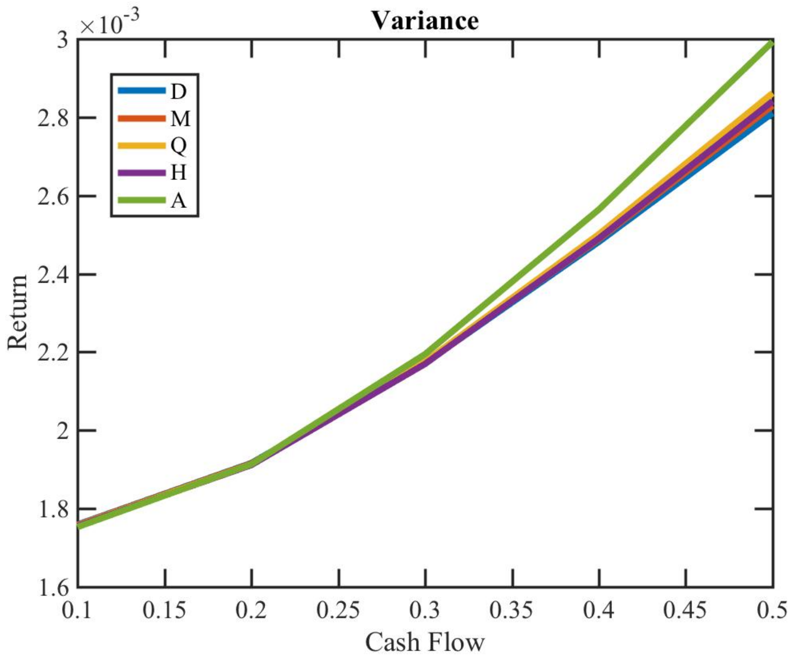

Figure 25.

The greater the cash flow the greater is the difference between the variance of the annual return and the other returns. The variance of the other returns are almost equal while the variance of the annual return increases faster than the others as the cash flow increases.

Figure 26 shows the fraction for which the returns are greater than 0, given that the end value of the portfolio is greater than the value of the portfolio right after the cash flow at day 157 and that the value of portfolio at day 156 is lesser than the initial value.

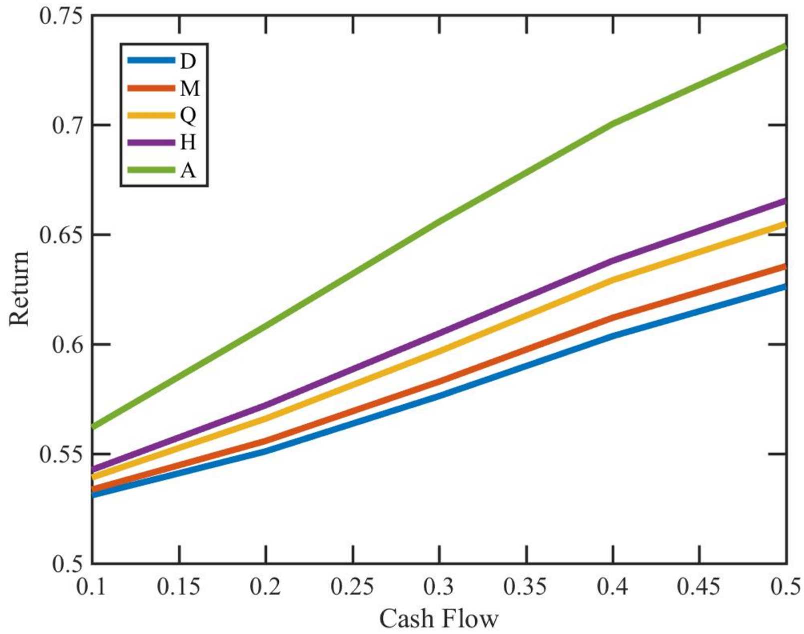

Figure 26.

The portfolio has lost money during the 156 first days, but made a profit after the cash flow. The fraction increases for the respective return when the cash flow increases. The greater the measurement frequency the smaller the fraction is for the returns. The lower the frequency the fewer time periods, and the more does the profit during the days 158 to 252 affect the overall return positively, due to that the profit occur when the invested amount is the greatest.

Figure 27 shows the means of the returns for the different frequencies, given that the end value of the portfolio is greater than the value of the portfolio right after the cash flow at day 157 and that the value of portfolio at day 156 is lesser than the initial value.

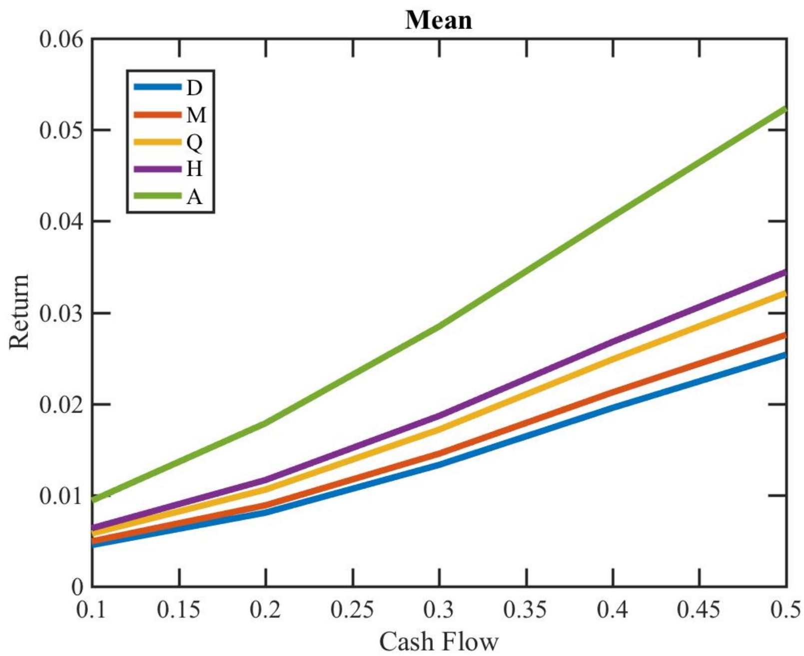

Figure 27.

The greater the measurement frequency the smaller the mean value. The figure shows that the profit during the days 158 to 252 affects the overall return more positively, for the returns with lower frequencies due to that the profits occur when the greatest amount of money is invested.

Figure 28 shows the variance of the returns for the different frequencies, given that the end value of the portfolio is greater than the value of the portfolio right after the cash flow at day 157 and that the value of portfolio at day 156 is lesser than the initial value.

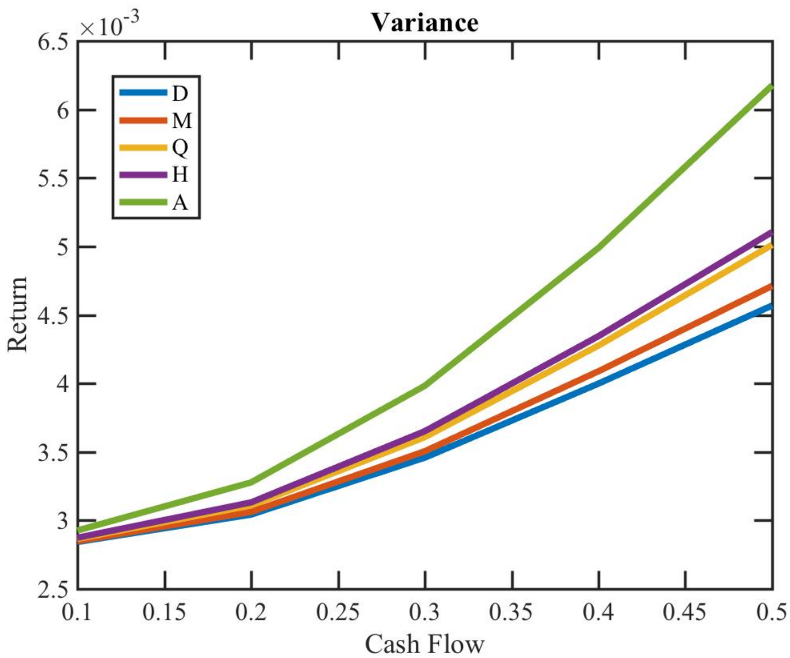

Figure 28.

The variances of the returns are almost equal for small cash flows, but as the cash flow increases the variances of the returns increase with different rates, hence the differences increase. The greater the measurement frequency the lesser the variance.

Figure 29 shows the fraction for which the returns are greater than 0, given that the end value of the portfolio is greater than the invested amount.

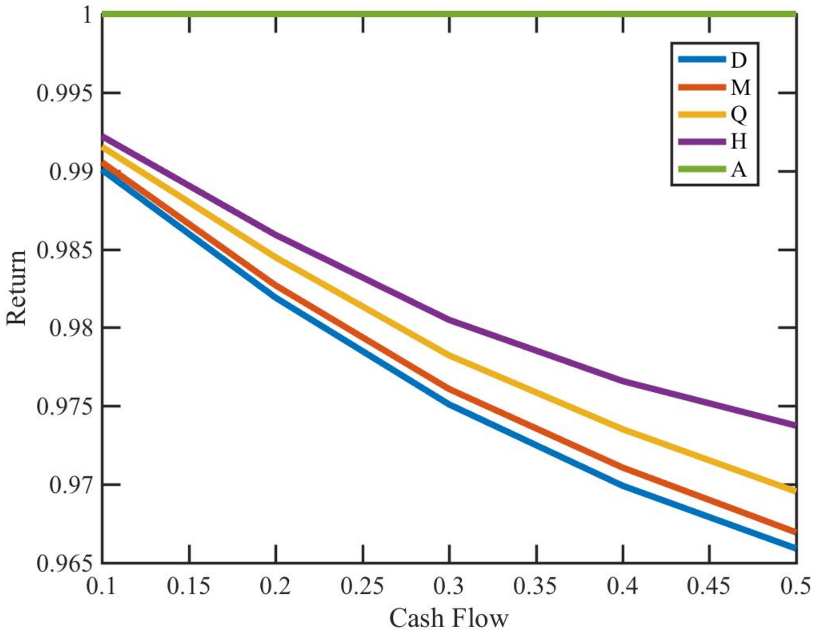

Figure 29.

The fraction for the annual return is constantly equal to one for every cash flow. The fractions for the other returns decrease as the cash flow increases. The figure also shows that the more frequently measured returns, the lesser the fraction. The returns with lower frequency react more positively towards an increasing cash flow when the end value of the portfolio is greater than the total amount invested.

Figure 30 shows the means of the returns for the different frequencies, given that the end value of the portfolio is greater than the invested amount.

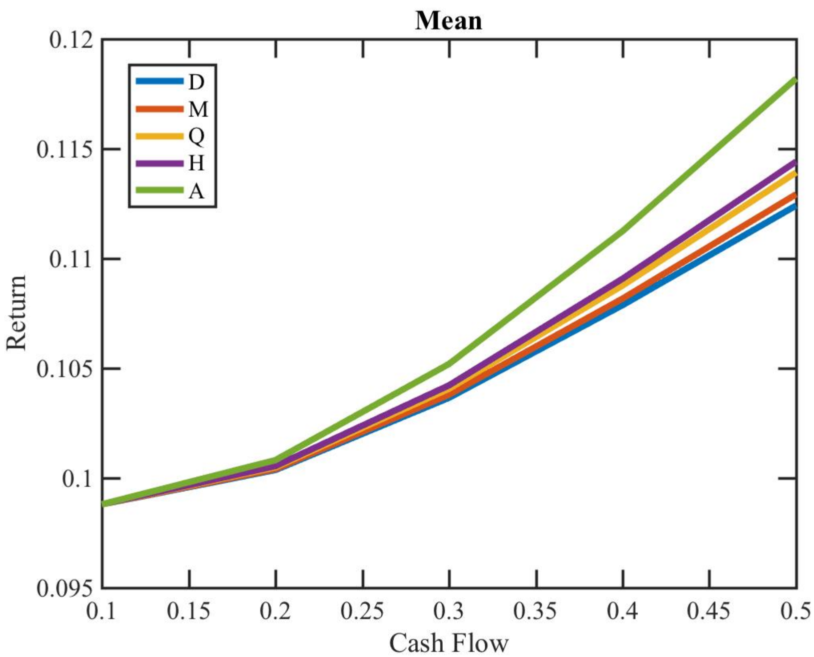

Figure 30.

The means are very similar for small cash flows but as the cash flow increases the returns increase at different rates. The greater the measurement frequency, the smaller the mean value.

Figure 31 shows the variances of the returns for the different frequencies, given that the end value of the portfolio is greater than the invested amount.

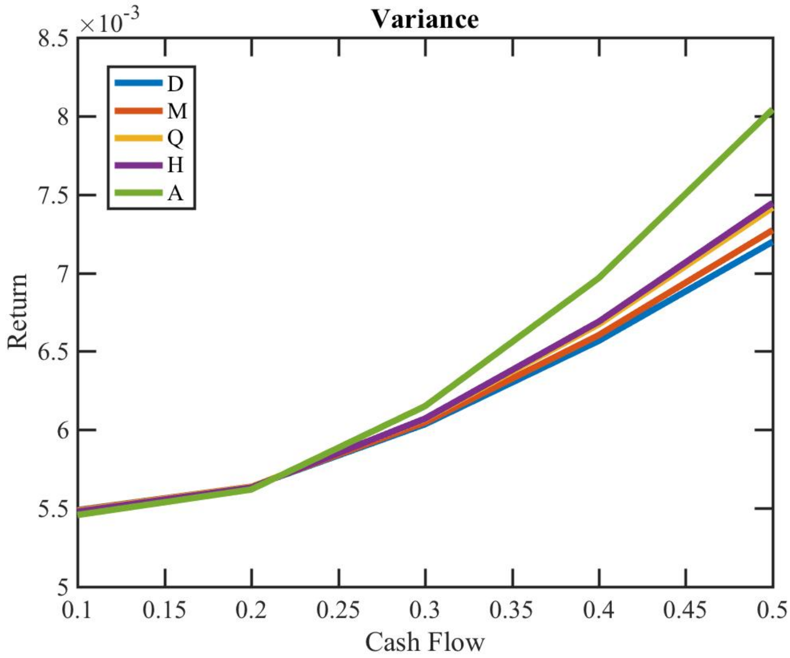

Figure 31.

The variance of the returns are again almost equal for small cash flows and increase at different rates as the cash flow increases. The greater the measurement frequency the smaller the variance.

The industry has successively been driven towards daily calculations as standard, due to the request and need for more accurate results and the existing performance presentation standards. To obtain accurate return calculations a frequent valuation such as daily valuation is needed in order to be able to provide exact returns for long time periods [1].

1.8 Average Annual Return

For long time periods it is sometimes preferred to use standardized periods for return comparisons. The annual return is the most established one. The annual return can be computed arithmetically or geometrically where the geometric average is defined as

$$r_G = \left(\prod_{i=1}^{m} (1 + r_i) \right)^{\frac{n}{m}} - 1,$$

and the arithmetic average is defined as

$$r_A = \frac{n}{m} \sum_{i=1}^{m} r_i,$$

where m is the number of periods considered and n is the number of periods within the year, for example if the returns are measured with a monthly frequency, $n = 12$. Furthermore, r_i is the periodic rate [1].

The arithmetic average does not compound to the actual return

$$(1+r_A)^{\frac{m}{n}} = \left(1+\frac{n}{m}\sum_{i=1}^{m} r_i\right)^{\frac{m}{n}} \neq \prod_{i=1}^{m}(1+r_i)$$

while the geometric average compounds to the actual return

$$(1+r_G)^{\frac{m}{n}} = \prod_{i=1}^{m}(1+r_i).$$

Moreover the geometric average and the effective return are as seen below closely related

$$\begin{aligned} r_G &= \left(\prod_{i=1}^{m}(1+r_i)\right)^{\frac{n}{m}} - 1 \\ &= \left(\prod_{i=1}^{m}\left(1+\frac{\bar{r}_i}{n}\right)\right)^{\frac{n}{m}} - 1, \end{aligned}$$

since r_i is the periodic rate. If it is assumed that $\bar{r}_i = \bar{r}_j$ for $\forall\ i, j$, then

$$\left(\prod_{i=1}^{m}\left(1+\frac{\bar{r}_i}{n}\right)\right)^{\frac{n}{m}} - 1 = \left(1+\frac{\bar{r}_i}{n}\right)^{n} - 1,$$

which is the effective rate of return. Furthermore, the geometric average return is also closely related to the annualized return. For $n = 1$, that is the frequency of measurement is one per year, the geometric average return is equal to the annualized return

$$r_G = \left(\prod_{i=1}^{m}(1+r_i)\right)^{\frac{1}{m}} - 1,$$

since m is number of years.

1.8.1 Numerical Results

The returns are simulated by using a portfolio that contains 10 stocks, each simulated 5,000,000 times by geometric Brownian motions. The value of the portfolio is measured at an annual frequency and during a time period of 10 years. Table 1 shows the means and the variances of the computed returns,

Table 1: Annual Return

	r_A	r_G	r_{cum}^A	r_{cum}^G	r_{cum}
Mean	0.0305	0.0252	0.4290	0.3501	0.3501
Variance	0.0012	0.0011	0.3784	0.2750	0.2750

where r_A is the arithmetic average return, r_G is the geometric average return, r_{cum}^A is the arithmetic cumulative return, r_{cum}^G is the geometric cumulative return and r_{cum} is the actual cumulative return.

Figure 32 shows the cumulative actual return, where outliers are observed.

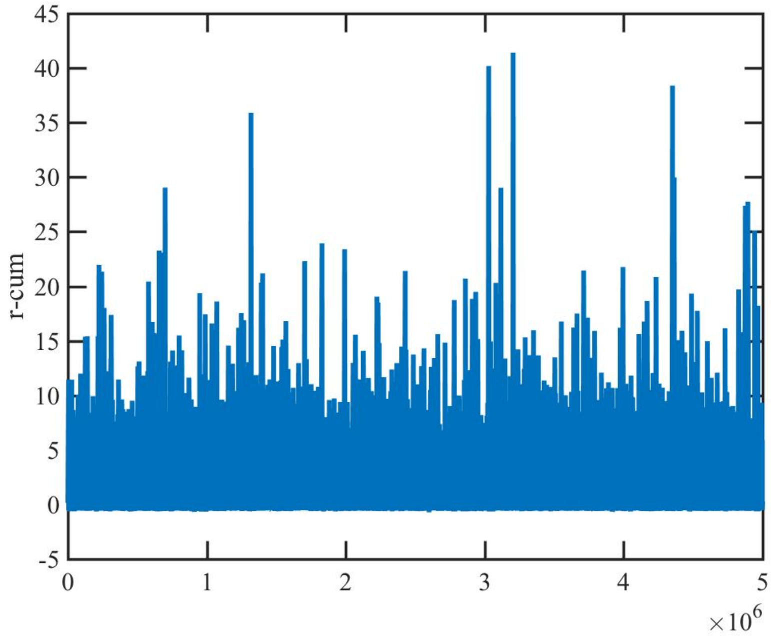

Figure 32.

Figure 33 shows one simulation of the portfolio return over the 10 time periods. The arithmetic average is greater than the geometric average, thus the arithmetic cumulative return will be greater than the actual return.

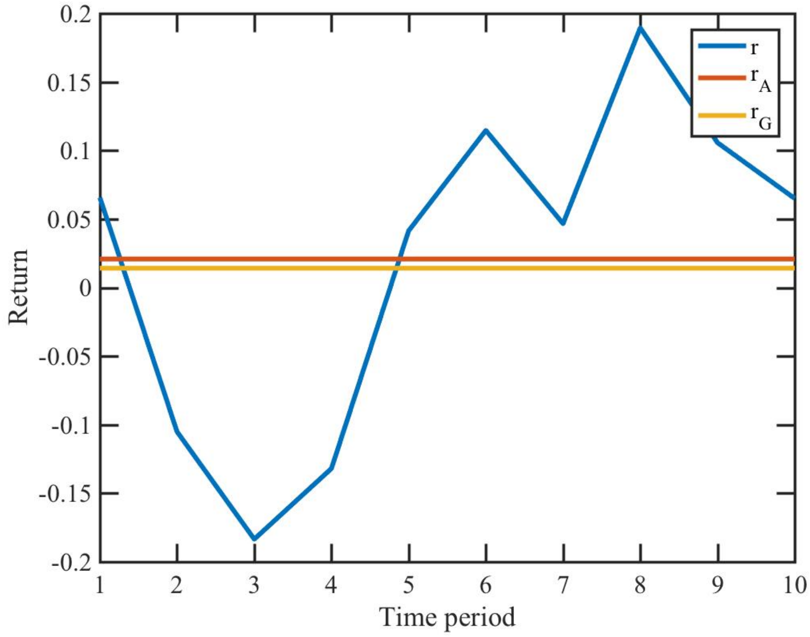

Figure 33.

As seen in Figure 34 the arithmetic average is greater than the geometric average for all 5,000,000 simulations. The figure shows the difference between the arithmetic and geometric averages, which is non-negative for every simulation.

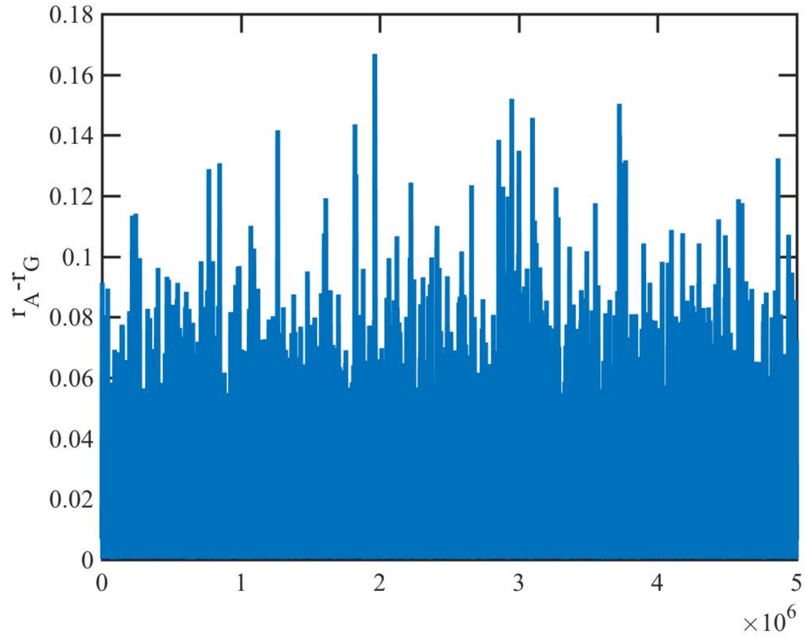

Figure 34.

Since $r_A \geq r_G$ holds for every simulation it follows that $r^A_{cum} \geq r^G_{cum}$ holds for every simulation. Thus it follows that the differences between the arithmetic and the geometric cumulative returns are non-negative, which is seen in Figure 35.

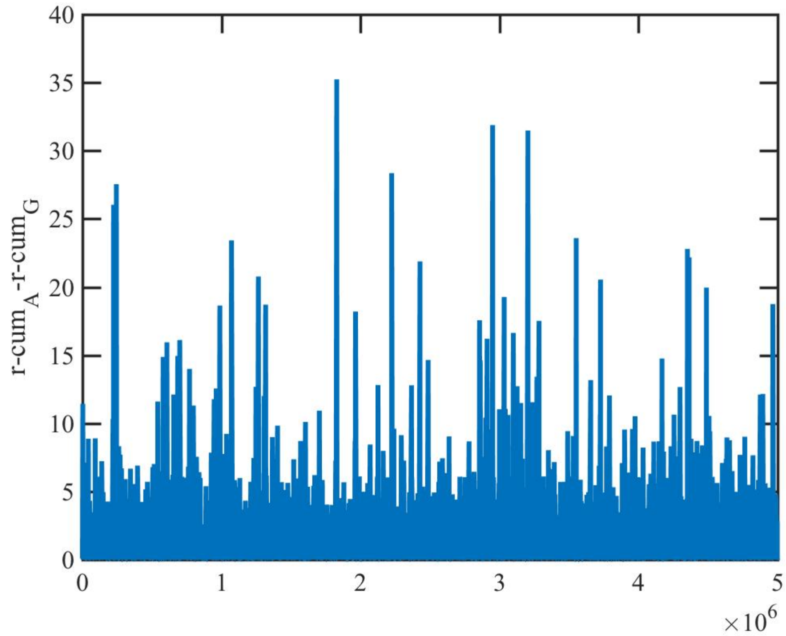

Figure 35.

Figure 36 shows the first 100 simulations of the cumulative returns. The arithmetic cumulative return follows the actual and geometric cumulative return closely, but is greater than the geometric return. The 75th simulation is an outlier among the first 100 simulations. Among the arithmetic cumulative returns it is even a more extreme outlier and it is very misleading, due to the skewed value.

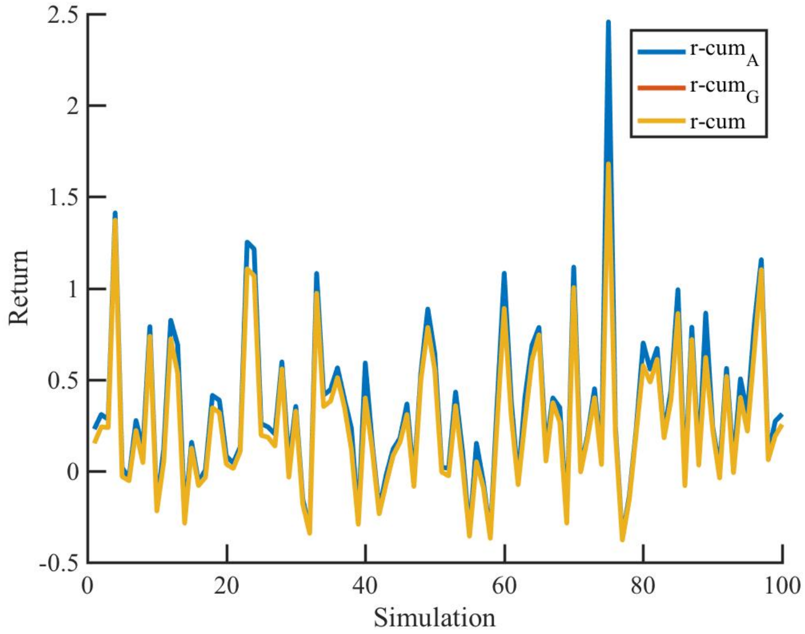

Figure 36.

The phenomena in Figure 36 that the outlier becomes even more extreme with the arithmetic average is seen in Figure 37 as well. The large outliers seen in Figure 32 are even more extreme outliers here, due to the fact that the arithmetic average gives greater significance to large outliers than the geometric average.

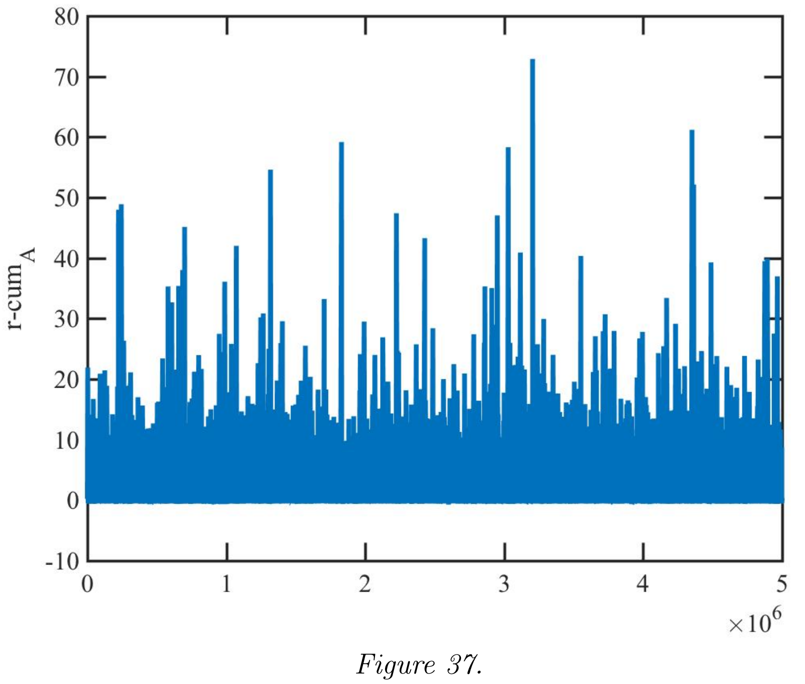

Figure 37.

1.9 Gross-of-fee Return and Net-of-fee Return

When considering investment performance one must not forget about the prescribed fees. It is of great importance that the performance of the portfolio manager is appropriately measured due to fees, since some fees are within while others are outside of the portfolio manager's authority. Hence, the valuation of the investment managers performance should depend upon the fees that is within the manager's control. There are three different types of fees and costs that rise with portfolio management

- Transaction fees - the costs that rise when selling and buying securities, bid/offer spread, broker's commission and other costs, taxes and fees related to transactions.
- Portfolio management - the fees that are charged for management of the account
- Custody and other administrative fees - fees such as legal fees, audit fees and any other fee.

Since administration fees and custody are outside of the portfolio manager's authority they should not be included in the evaluation of the performance of the manager. The manager determines whether or not to sell and buy assets, therefore the transaction fees should be withdrawn before the performance calculations, known as net of transaction costs.
Generally, the portfolio management fee is drawn immediately from the account, if not the return is known as gross-of-fees. In the calculation of the gross of fee return, the management fee is considered as external cash flow. If the management fee is taken from the account the return is known as net of fees. To obtain accurate results, the net of fee and gross of fee returns should be calculated separately. The gross return can nevertheless be estimated from the net return and the other way around. One can "gross-up" the net return in the following way

$$r_g = (1 + r_n)(1 + f) - 1,$$

and "net-down" the gross return

$$r_n = \frac{1 + r_g}{1 + f} - 1,$$

where r_g is the gross return of portfolio management fee, r_n is the net return of the portfolio management fee and f is the portfolio management fee rate of the nominal period [1].

1.10 Portfolio Component Return

To consider the total portfolio return is one part of the analysis of the performance, but to get a deeper understanding and in order to evaluate the decisions of the investment process, one should also study returns of sectors or components, that together equal the total portfolio return. The component returns are calculated in the same way as the return of the portfolio. Internal cash flows between different components should be considered in the return calculations as external cash flow. Coupon payments and dividend are considered as cash flows out of the current asset sector and into a cash sector, if these sectors are separated [1].

The transactions that occur may lead to a situation when the sector weight is zero but the sector can still contribute with a profit or loss. This may arise when one buys or sells within a sector which has no current holdings and assumes that the cash flow occurred at the end of the day. One can handle these situations by allowing the sector weight to depend on the size of the cash flow and then adjust the cash flow in the cash sector so that they cancel.

Due to these problems some firms have started to use the assumption that transactions occur midday, thus the denominator will depend on the cash flow and not equal zero. Another solution that some firms have implemented is that they assume that transaction inflows occur at the beginning of the day and transaction outflows at the end of the day. Thereby avoiding that the denominator will be equal to zero, this is a system-inspired solution [1].

2 Benchmark

The portfolio return tells a lot about the outcome of the investments, but often it is as important to compare the investments with an appropriate benchmark. The comparison clarifies the advantages and the weaknesses of the investment decisions. A benchmark is a reference portfolio used for comparison of a portfolio's performance.

The selected benchmark should have the following properties in order to be considered appropriate

- The chosen benchmark should have similar aims and investment strategy as the portfolio.
- It should be possible to invest in all the securities that the benchmark consists of.
- It should be possible to have access not only to the returns of the benchmark but also to the weights.

Within performance the benchmarks are either indexes, peer groups or random portfolios. Peer groups consist of several competitor portfolios with similar strategies, while random portfolios are a mix of indexes and peer groups [1].

3 Performance Attribution

The aim of performance attribution is to measure the excess return which arises between the portfolio return and its benchmark, and to quantify the differences which are due to the active decisions that are being made in the investment decision process. Hence, the analyst is assigned to quantify the investment manager's decisions. Performance attribution is a fundamental tool for the analyst in understanding the sources of the portfolio return [1].

The excess return studied in performance attribution can be defined both arithmetic and geometric, which leads to different approaches and models, and hence different advantages and problems.

The *arithmetic excess return* is defined as

$$R^P - R^B,$$

where R^P is the portfolio return and R^B is the benchmark return. The *geometric excess return* is defined as

$$\frac{1+R^P}{1+R^B} - 1$$

Both definitions of the excess return are used in the field of performance attribution, but as mentioned above one distinguish between the arithmetic and the geometric approaches and models. Regardless of approach, the excess return is divided into different effects that reflect and measure the active decisions of the investment decision process. The key when analyzing the excess return is to truly understand the investment decision process [1].

3.1 The Brinson Methodology

The foundation of performance attribution was established by Brinson, Hood, and Beebower (1986) and Brinson and Fachler (1985). The articles now known as the Brinson model, consist of the theories which the performance attribution methodologies of today are founded upon [1].

The articles assume that the portfolio return R^P is defined as

$$R^P = \sum_i w_i^P R_i^P,$$

where w_i^P is the weight of the ith asset class in the portfolio and R_i^P is the return of the ith asset class in the portfolio.

Similarly, the benchmark return R^B is defined as

$$R^B = \sum_i w_i^B R_i^B,$$

where w_i^B is the weight of the ith asset class in the benchmark and R_i^B is the return of the ith asset class in the benchmark.

The weights of the portfolio and benchmark sum to one respectively,

$$\sum_i w_i^P = \sum_i w_i^B = 1$$

[1]. Besides the weights and returns, the models are build upon the decisions that are taken when the returns and weights of the portfolio differ from those of the benchmark. Hence the models introduce notional portfolios which are constructed with active or passive returns and weights that illustrate the added value caused by each decision. The notional portfolios I, II, III and IV are defined in the following chart [6].

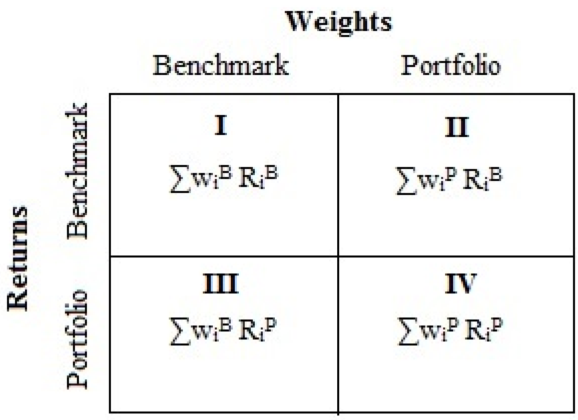

Figure 38.

3.1.1 Arithmetic Excess Return

Brinson, Hood and Beebower as well as Brinson and Fachler presented models that break down the arithmetic excess return $R^P - R^B$. These articles assume that the portfolio manager aims to increase value through security selection and asset allocation [1].

The arithmetic excess return in the article Brinson and Fachler (1985) is divided into three different parts; asset allocation, security selection and interaction.

In asset allocation the portfolio manager aims to overweight the categories for which the benchmark return of asset class i has outperformed the total benchmark return. Overweighting is when the portfolio manager has invested a greater weight in the ith asset class than the benchmark. Thus, underweighting is when the weight of the portfolio in the ith asset class is lesser than the benchmark's weight.

In security selection the portfolio manager aims to select those individual securities within the asset class that outperform the benchmark's securities within the corresponding asset class.

Interaction on the other hand is not included in the investment decision process. Thus, there is likely no individual responsible for adding value through interaction within investment management. Hence, the existence of the interaction term is considered to be a drawback of the Brinson model [1].

The return of the semi notional fund, notional portfolio II, is defined as

$$\sum_i w_i^P R_i^B,$$

and is used in the definition of the asset allocation. The contribution to asset allocation A_i for the ith asset class depends on the semi notional portfolio in the following way

$$A_i = (w_i^P - w_i^B)(R_i^B - R^B).$$

Thus, it reflects the investment manager's asset allocation decision but excludes security selection, since only the index returns are being used [1]. The total attribution effect A is then given by $A = \sum_i A_i$. The sign of the contribution A_i depends on the difference between the weights as well as the difference between the return of the benchmark's ith asset class and the total benchmark return. Hence when the portfolio manager is overweighted in a negative market that has outperformed the overall benchmark, the effect is positive. The different combinations are illustrated in the chart below

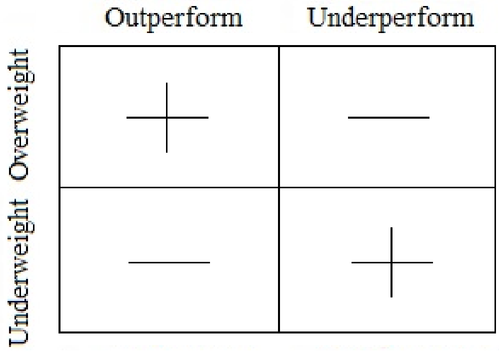

Figure 39.

The selection notional fund, notional portfolio III, is defined as

$$\sum_i w_i^B R_i^P,$$

which excludes asset allocation decisions since the benchmark weight is held fix and the portfolio return is applied. The return of the semi notional portfolio is included in the definition of the security selection for asset class i which is defined as

$$S_i = w_i^B (R_i^P - R_i^B).$$

It excludes allocation since the benchmark weight is held fix, and moreover it calculates the difference between the portfolio's and benchmark's returns. The total security effect is given by $S = \sum_i S_i$. The definitions of security selection and asset allocation do not sum to the excess return

$$A + S \neq R^P - R^B.$$

Thus another term must be added in order to avoid residuals, hence the interaction term is introduced. The contribution to interaction for asset class i is defined as

$$I_i = (w_i^P - w_i^B)(R_i^P - R_i^B).$$

The total interaction effect is given by $I = \sum_i I_i$, and the excess return is now completely described

$$\begin{aligned}\sum_i A_i + \sum_i S_i + \sum_i I_i &= \sum_i (w_i^P - w_i^B)(R_i^B - R^B) + \sum_i w_i^B (R_i^P - R_i^B) + \\ &\sum_i (w_i^P - w_i^B)(R_i^P - R_i^B) \\ &= R^P - R^B,\end{aligned}$$

since $\sum_i (w_i^P - w_i^B) R^B = 0$ [1].

The notional portfolios defined above illustrate how the excess return is divided into the different effects

Asset Allocation	II-I	$\sum (w_i^P - w_i^B)(R_i^B - R^B)$
Security Selection	III-I	$\sum w_i^B (R_i^P - R_i^B)$
Interaction	IV-III-II+I	$\sum (w_i^P - w_i^B)(R_i^P - R_i^B)$
Total Value Added	IV-I	$\sum w_i^P R_i^P - w_i^B R_i^B$

The difference between the notional portfolios II-I describes the allocation effect, while the selection effect is the difference between notional portfolios III and I. The interaction effect depends on all of the notional portfolios. Since the interaction effect is not reflected in the investment decision process, analysts prefer to include the interaction term into the allocation or selection effect [1]. Some argue that the interaction term should be combined with the secondary decision, so that only the allocation and security selection effects are considered [6].

3.1.2 Geometric Excess Return

Several geometric attribution models have been published during the years and are mainly similar. The Brinson model defined above can be modified so that it instead breaks down the geometric excess return

$$\frac{1+R^P}{1+R^B} - 1.$$

The asset allocation contribution can as in the original Brinson model depend upon the semi notional portfolio. Now instead of considering the arithmetic difference, the geometric difference is measured. Define the contribution to the asset allocation A_i for the ith asset class as

$$A_i = (w_i^P - w_i^B)\left(\frac{1+R_i^B}{1+R^B} - 1\right),$$

where w_i^P, w_i^B, R_i^B and R^B are defined as above. Thus, summing over all asset classes gives us the total attribution effect A

$$\begin{aligned} A &= \sum_i (w_i^P - w_i^B)\left(\frac{1+R_i^B}{1+R^B} - 1\right) \\ &= \sum_i \left(\frac{w_i^P + w_i^P R_i^B}{1+R^B} - w_i^P\right) - \sum_i \left(\frac{w_i^B + w_i^B R_i^B}{1+R^B} - w_i^B\right) \\ &= \frac{1+R_S^B}{1+R^B} - 1, \end{aligned}$$

where $R_S^B = \sum_i w_i^P R_i^B$ is the semi notional portfolio. It remains to define the security selection contributions S_i

$$S_i = w_i^P\left(\frac{1+R_i^P}{1+R_i^B} - 1\right)\left(\frac{1+R_i^B}{1+R_S}\right).$$

The term $\frac{1+R_i^B}{1+R_S^B}$ is not expected when considering the original Brinson-Fachler model. It can be argued that it should be included, because of the fact that the term adds more value when the benchmark is outperformed when it is performing well and adds less value when the benchmark is outperformed when it is performing poorly. The sum over all security selection contributions gives the total security selection effect S

$$\begin{aligned} S &= \sum_i w_i^P\left(\frac{1+R_i^P}{1+R_i^B} - 1\right)\left(\frac{1+R_i^B}{1+R_S}\right) \\ &= \sum_i w_i^P\left(\frac{1+R_i^P}{1+R_i^B}\right)\left(\frac{1+R_i^B}{1+R_S}\right) - w_i^P\left(\frac{1+R_i^B}{1+R_S}\right) \\ &= \frac{1+R^P}{1+R_S} - 1. \end{aligned}$$

Hence, the asset allocation effect and the security selection effect compound to the geometric excess return

$$(1+S)(1+A)-1=\frac{1+R^P}{1+R^B}-1,$$

[1].

4 Holding-based and Transaction-based Model

The greater difference between the transaction-based and the holding-based attribution models are the difference in managing the transactions. The holding-based model assumes that during the period that is analyzed no transactions occur and thus the portfolio holdings remain unchanged. If in fact transactions have occurred during the period, they will lead to a difference between the actual return and the computed return. Thus the model will cause residuals. The closest the holding-based attribution model gets to a transaction-based model is by considering daily periods, but if intra day trading occurs the model will still create residuals.

The transaction-based attribution model on the other hand introduces a transaction effect that is caused by the transactions of the current day, due to that the transaction price and the end-of-day price usually differs. The transaction information is needed in order to be able to perform a transaction-based attribution. Similarly, if securities are withdrawn from the portfolio the withdrawal price is used and the difference between the prices contributes to a transaction effect [6].

5 Multilevel Attribution

There are several performance attribution models. When choosing an appropriate model it is of great importance that the model reflects the investment decision process, in order for the computed values to be meaningful.

The model considered here is known as the top-down model, hence the model should be used for top-down strategies, that are strategies where allocation decisions are prior to selection decisions. It is a more complex model compared to the original Brinson model, since it consists of hierarchical decisions which lead to attribution performed on different levels. The model embeds the interaction term in the secondary effect, the selection effect. The top-down investment strategy may for example first decide on regional weighting, then sector weighting and then the decision on security selection is made. Thus it is a hierarchical process since the weighting of every decision depends on the prior decision [6].

Furthermore, a transaction-based attribution model will be considered. In order to measure the transaction effect the weights and returns for the holding-based model are required, as will be seen later. Therefore the weights and returns for the holding-based model are firstly introduced. The models are applicable for equity attribution of portfolios consisting of one single currency. Further extensions are required when multi-currency portfolios are considered in performance attribution.

5.1 Top-Down Holding-based Attribution Model

The definitions of returns and weights for the holding-based top-down model, that is when it is assumed that no transactions occur during the time period considered, are given by

$$w_g^P = \begin{cases} \dfrac{BMV_g^P}{BMV_\phi^P}, & \text{if } |g| = N \\ \sum\limits_{i \in \Omega_g} w_i^P, & \text{if } |g| < N \end{cases}$$

$$w_g^B = \begin{cases} \dfrac{BMV_g^B}{BMV_\phi^B}, & \text{if } |g| = N \\ \sum\limits_{i \in \Omega_g} w_i^B, & \text{if } |g| < N \end{cases}$$

where w_g^P is the weight of group g of the portfolio and w_g^B is the weight of group g of the benchmark. Furthermore the return of group g of the portfolio R_g^P and the return of group g of the benchmark R_g^B are given by

$$R_g^P = \begin{cases} \dfrac{EMV_g^P - BMV_g^P + Income}{BMV_g^P}, & \text{if } |g| = N \\ \dfrac{\sum\limits_{i \in \Omega_g} w_i^P R_i^P}{w_g^P}, & \text{if } |g| < N \end{cases}$$

$$R_g^B = \begin{cases} \dfrac{EMV_g^B - BMV_g^B + Income}{BMV_g^B}, & \text{if } |g| = N \\ \dfrac{\sum\limits_{i \in \Omega_g} w_i^B R_i^B}{w_g^B}, & \text{if } |g| < N \end{cases}$$

where

ϕ = Represents the total level, that is the total portfolio/benchmark of equities
g = The vector representing the group
$|g|$ = The hierarchical level of group g, that is the number of elements of g
N = The lowest level in the hierarchy, that is the security level
Ω_g = All subgroups of g at one level below in the hierarchy.
BMV_g^P = The beginning value of the portfolio for group g
BMV_g^B = The beginning value of the benchmark for group g
BMV_ϕ^P = The beginning value of the total portfolio
BMV_ϕ^B = The beginning value of the total benchmark
EMV_g^P = The end value of the portfolio for group g
EMV_g^B = The end value of the benchmark for group g
$Income$ = Any income related to the invested asset

The group g can be both a collection of assets or one single asset. When $g = \phi$, the group is equal to the total portfolio or benchmark. Furthermore, $|\phi| = 0$ and $w_\phi^P = w_\phi^B = 1$. Moreover for an arbitrary group (i, j, k) it follows that $\forall\ m,\ (i, j, k, m) \in \Omega_{(i,j,k)}$ [6].

The top-down methodology is applicable for both arithmetic and geometric attribution models, both presented below.

5.1.1 Arithmetic Approach

The arithmetic effects and component calculations defined in the holding-based model are used in the transaction-based model and are defined as follows

$$CA_g = \left(w_g^P - \frac{w_{\bar{g}}^P}{w_{\bar{g}}^B} w_g^B \right) (R_g^B - R_{\bar{g}}^B)$$

$$EA_{g,n} = \begin{cases} \sum\limits_{i \in \Omega_g} CA_i, & \text{if } n = |g| + 1 \\ \sum\limits_{i \in \Omega_g} EA_{i,n}, & \text{if } n > |g| + 1 \end{cases}$$

and the arithmetic excess return RA_ϕ is given by

$$RA_\phi = R_\phi^P - R_\phi^B = \sum_{n=1}^{N} EA_{\phi,n}$$

where

CA_g = Arithmetic component contribution to group g
$EA_{g,n}$ = Arithmetic effect of group g at level n

The calculation for the component is similar to the original Brinson-Fachler model. The dissimilarities however are that the benchmark weight w_g^B is scaled and the difference in the returns is between group g and its parent, that is the prior decision in the hierarchy. Thus the component effect is reflected by the hierarchical structure. The weights are scaled in order for the component calculation to provide a fair result, so that the proportion of the weight does not depend on the prior decision. For example, assume that the benchmark chooses at the first decision level to allocate $w_g^B = 0.4$ and the portfolio $w_g^P = 0.2$ to the group g. At the next decision level the benchmark and the portfolio both want to invest half of the amount into the same subgroup g_s that is $w_{g_s}^B = 0.2$ and $w_{g_s}^P = 0.1$. Then the difference is $(0.2 - 0.1)$ and it is not equal to zero even though the portfolio followed the benchmark strategy, hence the benchmark weight must be scaled. Due to the scaling the computed difference is instead $(0.1 - \frac{0.2}{0.4} \cdot 0.2) = 0$.

The effect of a decision is as seen in the formula stated above, equal to the sum of the components of the subgroup one hierarchal level below if they are components of the decision. Moreover the effect of a group is equal to the sum of the effects of the subgroups at one hierarchical level below, if their subgroups are the components of the decision.

The formulas for effects and components hold for every level n in the hierarchy. The effects for which $n < N$ are known as allocation effects and for $n = N$ selection effects.

The arithmetic excess return is the difference between the benchmark's return and the portfolio's return. The difference can be described by the effects, since the excess return is equal to the sum of all total effects at all decision levels.

The effects and components are interpreted differently. Effects reflect the impact of a decision, while components are parts of effects. Thus, components increase the knowledge and details about the decisions, but do not alone represent an investment decision [6].

5.1.2 Geometric Approach

The formulas for the arithmetic holding-based attribution model can be used for the definitions of the geometric model, which are defined as follows

$$R^{HL} = \begin{cases} R_\phi^B, & \text{if } L = 0, \\ EA_{\phi,L} + R^{H(L-1)}, & \text{if } L > 0 \end{cases}$$

$$CG_g = \frac{CA_g}{1 + R^{H|\bar{g}|}}$$

$$EG_{g,n} = \begin{cases} \sum\limits_{i \in \Omega_g} CG_i, & \text{if } n = |g| + 1, \\ \sum\limits_{i \in \Omega_g} EG_{i,n}, & \text{if } n > |g| + 1 \end{cases}$$

and the geometric excess return RG_ϕ is given by

$$RG_\phi = \frac{1 + R_\phi^P}{1 + R_\phi^B} - 1 = \prod_{n=1}^{N} (1 + EG_{\phi,n}) - 1,$$

where

R^{HL}	=	The hybrid portfolio's return at level L
CG_g	=	Geometric component contribution to group g
$EG_{g,n}$	=	Geometric effect of group g at level n

That the geometric component calculations are equal to the arithmetic components divided by 1 plus the return of the hybrid portfolio is not surprising, recall that the geometric model presented above defined the contribution to the allocation effect for asset class i as

$$A_i = (w_i^P - w_i^B) \left(\frac{1 + R_i^B}{1 + R^B} - 1 \right).$$

Compare to

$$
\begin{aligned}
CG_g &= \frac{CA_g}{1+R^{H|\bar{g}|}} \\
&= \left(w_g^P - \frac{w_{\bar{g}}^P}{w_{\bar{g}}^B} w_g^B\right)\left(\frac{R_g^B - R_{\bar{g}}^B}{1+R^{H|\bar{g}|}}\right) \\
&= \left(w_g^P - \frac{w_{\bar{g}}^P}{w_{\bar{g}}^B} w_g^B\right)\left(\frac{R_g^B - R_{\bar{g}}^B}{1+R_{\bar{g}}^B}\right)\left(\frac{1+R_{\bar{g}}^B}{1+R^{H|\bar{g}|}}\right) \\
&= \left(w_g^P - \frac{w_{\bar{g}}^P}{w_{\bar{g}}^B} w_g^B\right)\left(\frac{1+R_g^B}{1+R_{\bar{g}}^B} - 1\right)\left(\frac{1+R_{\bar{g}}^B}{1+R^{H|\bar{g}|}}\right) \\
&= \left(w_g^P - \frac{w_{\bar{g}}^P}{w_{\bar{g}}^B} w_g^B\right)\left(\frac{1+R_g^B}{1+R_{\bar{g}}^B} - 1\right)\left(\frac{1+R_{\bar{g}}^B}{1+R_\phi^B + EA_{\phi,1} + \cdots + EA_{\phi,|\bar{g}|}}\right) \\
&= \left(w_g^P - \frac{w_{\bar{g}}^P}{w_{\bar{g}}^B} w_g^B\right)\left(\frac{1+R_g^B}{1+R_{\bar{g}}^B} - 1\right)\left(\frac{1+R_{\bar{g}}^B}{(1+R_\phi^B)(1+EG_{\phi,1})\cdot \cdots \cdot (1+EG_{\phi,|\bar{g}|})}\right)
\end{aligned}
$$

Due to the hierarchical decision process the weights are not only scaled, also the components are scaled where the prior return and decisions effects are used for the scaling. The hybrid return is constructed to be included in the scaling. These are the formulas that are needed for the geometric holding-based attribution model. Now let's introduce the transaction-based attribution model [6].

5.2 Top-Down Transaction-based Attribution Models

For the transaction-based model, the days that the transactions occur at have to be single periods. The transactions are assumed to occur at the end of the day. Moreover the transaction-based weights and returns are defined as follows

$$
\bar{w}_g^P = \begin{cases} \dfrac{BMV_g^P}{BMV_\phi^P}, & \text{if } |g| = N \\ \sum\limits_{i\in\Omega_i} \bar{w}_i^P, & \text{if } 0 < |g| < N \end{cases}
$$

$$
\bar{R}_g^P = \begin{cases} R_g^P & \text{if } |g| = N \text{ and } TI_g = 0, \\ \dfrac{EMV_g^P - BMV_g^P - TI_g^P}{BMV_g^P}, & \text{if } |g| = N \text{ and } TI_g \neq 0 \\ \dfrac{\sum\limits_{i\in\Omega} \bar{w}_i^P \bar{R}_i^P}{\bar{w}_g^P}, & \text{if } 0 < |g| < N \\ \sum\limits_{i\in\Omega} \bar{w}_i^P \bar{R}_i^P, & \text{if } |g| = 0 \end{cases}
$$

where

$\bar{w}_g^P$	=	The transaction-based weight of the portfolio for group g
$\bar{R}_g^P$	=	The transaction-based return of the portfolio for group g
TI_g	=	The transactions occurred during the day within group g, inflows as well as outflows
BMV_g^P	=	The beginning value of the portfolio for group g
EMV_g^P	=	The end value of the portfolio for group g

The transaction effect for the arithmetic model is defined as follows

$$TA_g = \begin{cases} \bar{w}_g^P \bar{R}_g^P - w_g^P R_g^P, & \text{if } |g| = N \\ \sum\limits_{i \in \Omega_g} TA_i, & \text{if } |g| < N \end{cases}$$

The arithmetic excess return is then given by

$$RA_\phi = \sum_{n=1}^{N} EA_{\phi,n} + TA_\phi,$$

since,

$$\begin{aligned} RA_\phi &= \sum_{n=1}^{N} EA_{\phi,n} + TA_\phi \\ &= R_\phi^P - R_\phi^B + \bar{R}_\phi^P - R_\phi^P \\ &= \bar{R}_\phi^P - R_\phi^B. \end{aligned}$$

The transaction effect for the geometric attribution model is defined as

$$TG_g = \begin{cases} 0, & \text{if } TA_\phi = 0 \\ \dfrac{TA_g}{TA_\phi}\left(\dfrac{1+\bar{R}_\phi^P}{1+R_\phi^P} - 1\right), & \text{if } |g| = N \\ \sum\limits_{i \in \Omega_g} TG_i, & \text{if } |g| < N \end{cases}$$

and the geometric excess return is then given by

$$RG_\phi = \frac{1+\bar{R}_\phi^P}{1+R_\phi^B} - 1 = \prod_{n=1}^{N}(1+EG_{\phi,n})(1+TG_\phi) - 1.$$

since,

$$\begin{aligned} RG_\phi &= \prod_{n=1}^{N}(1+EG_{\phi,n})(1+TG_\phi) - 1 \\ &= \frac{1+R_\phi^P}{1+R_\phi^B} \cdot \frac{1+\bar{R}_\phi^P}{1+R_\phi^P} - 1 \\ &= \frac{1+\bar{R}_\phi^P}{1+R_\phi^B} - 1, \end{aligned}$$

[6].

5.3 Multi-Period Geometric Attribution

When attribution is performed for different time periods and the single-periods are used together to present the total attribution effects, the effects for the single-periods are compounded. The components can not be compounded, since the multiplication is only applicable for effects. The cumulative effects are given by compounding the effects for every period considered

$$EG^{cum}_{g,n,T} = \prod_{t=1}^{T}(1 + EG_{g,n,t}) - 1.$$

Thus the geometric excess return is given by

$$\begin{aligned}
RG_{\phi,T,Cum} &= \frac{1 + R^P_{\phi,T,Cum}}{1 + R^B_{\phi,T,Cum}} - 1 \\
&= \prod_{t=1}^{T}\frac{1 + R^P_{\phi,t}}{1 + R^B_{\phi,t}} - 1 \\
&= \prod_{t=1}^{T}(1 + RG_{\phi,t}) - 1 \\
&= \prod_{t=1}^{T}\prod_{n=1}^{N}(1 + EG_{\phi,n,t}) - 1 \\
&= \prod_{n=1}^{N}(1 + EG^{cum}_{\phi,n,T}) - 1,
\end{aligned}$$

[6].

5.4 Multi-Period Arithmetic Attribution

The formulas do not work as nicely for the arithmetic model as for the geometric model for situations with multi-periods. The cumulative arithmetic excess return $RA_{\phi,T,Cum}$ is given by

$$\begin{aligned}
RA_{\phi,T,Cum} &= \prod_{t=1}^{T}(1 + R^P_{\phi,t}) - 1 - \left(\prod_{t=1}^{T}(1 + R^B_{\phi,t}) - 1\right) \\
&\neq \sum_{t=1}^{T}(R^P_{\phi,t} - R^B_{\phi,t}),
\end{aligned}$$

nor

$$RA_{\phi,T,Cum} \neq \prod_{t=1}^{T}(1 + (R^P_{\phi,t} - R^B_{\phi,t})) - 1.$$

Thus there is no trivial way to easily link periods for the arithmetic model. To solve this problem smoothing-algorithms are being used so that the model completely describes the excess return. Several different approaches of smoothing exist. Carino, GRAP, Frongello are some used smoothing methods [1].

5.5 Short Positions

Morningstar proposes that for portfolios or benchmarks with both long and short positions, the positions should be parted. Hence the number of groups may then be doubled since the partition takes place already at the fist decision level. The long positions have positive weights and the short positions have negative weights. Thus for a short position a negative weight multiplied with a negative return will give a positive contribution, which is reasonable since short positions are taken for securities for which the market values are believed to fall [6].

5.6 Numerical Result for the Arithmetic Attribution Model

A portfolio containing American stocks is here considered. The portfolio is observed 2016-01-11 to 2016-01-13. The attribution models consider the portfolio at two time periods 2016-01-11 to 2016-01-12 and 2016-01-12 to 2016-01-13, where the closing and transaction prices are observed. The investments of the portfolio are assumed to follow a top-down investment decision process where decisions AAA, BBB and CCC were made firstly follow by decisions EEE, FFF and GGG. Then were security selection decisions made. Due to that both long and short positions were taken, the number of groups considered are almost doubled.

During the first time period 2016-01-11 to 2016-01-12, three transactions occurred. The portfolio manager invested in AA US, HAL US and AET US. Table 2 shows the result of the portfolio's performance computed by the arithmetic transaction-based top-down attribution model presented above.

Table 2: Arithmetic 2016-01-11/2016-01-12.

Total	$EA_{\phi,1} = 0,005363850$	$EA_{\phi,2} = 0,000393359$	$EA_{\phi,3} = -0,000507925$
AAA: Long	$CA_{(1)} = -0,009697022$	$EA_{(1),2} = -0,000854302$	$EA_{(1),3} = -0,003064193$
EEE: Long		$CA_{(1,1)} = -0,000753288$	$EA_{(1,1),3} = 0,000442764$
AA US			$CA_{(1,1,1)} = 0,000491506$
AAPL US			$CA_{(1,1,2)} = 0,000145082$
ACE US			$CA_{(1,1,3)} = -0,000350650$
AFL US			$CA_{(1,1,4)} = 0,000584161$
ALL US			$CA_{(1,1,5)} = -0,000079345$
BA US			$CA_{(1,1,6)} = -0,000098701$
BHI US			$CA_{(1,1,7)} = -0,000249290$
FFF: Long		$CA_{(1,2)} = -0,000307163$	$EA_{(1,2),3} = -0,004397317$
CNP US			$CA_{(1,2,1)} = -0,000616587$
CTXS US			$CA_{(1,2,2)} = -0,000380741$
ED US			$CG_{(1,2,3)} = -0,000742147$
EVER US			$CA_{(1,2,4)} = -0,000490481$
GOOG US			$CA_{(1,2,5)} = -0,001532652$
IBM US			$CA_{(1,2,6)} = -0,000609511$
KLAC US			$CA_{(1,2,7)} = -0,000025197$
GGG: Long		$CA_{(1,3)} = 0,000206149$	$EA_{(1,3),3} = 0,000890360$
LEN US			$CA_{(1,3,1)} = -0,000383500$
LYB US			$CA_{(1,3,2)} = 0,000010343$
NVDA US			$CA_{(1,3,3)} = -0,000005586$
OMC US			$CA_{(1,3,4)} = 0,000151872$
PFE US			$CA_{(1,3,5)} = -0,000320257$
PFIS US			$CA_{(1,3,6)} = 0,001234537$
PX US			$CA_{(1,3,7)} = -0,000004429$
UNM US			$CA_{(1,3,8)} = 0,000158084$
VLO US			$CA_{(1,3,9)} = 0,000533132$
XOM US			$CA_{(1,3,10)} = -0,000483838$

AAA: Short	$CA_{(2)} = 0,016303079$	$EA_{(2),2} = -0,001318510$	$EA_{(2),3} = 0,002072523$
EEE: Short		$CA_{(2,1)} = -0,000406199$	$EA_{(2,1),3} = 0,000607503$
AAL US			$CA_{(2,1,1)} = 0,000135687$
AIG US			$CA_{(2,1,2)} = -0,000058980$
APOL US			$CA_{(2,1,3)} = 0,000479644$
BLL US			$CA_{(2,1,4)} = 0,000057079$
BMS US			$CA_{(2,1,5)} = -0,000005927$
FFF: Short		$CA_{(2,2)} = -0,000594438$	$EA_{(2,2),3} = -0,001077366$
CI US			$CA_{(2,2,1)} = -0,000356790$
CMA US			$CA_{(2,2,2)} = -0,000387043$
CMG US			$CA_{(2,2,3)} = 0,001157478$
CVS US			$CA_{(2,2,4)} = -0,000951296$
HAL US			$CA_{(2,2,5)} = -0,000017425$
HES US			$CA_{(2,2,6)} = -0,000173583$
KMX US			$CA_{(2,2,7)} = -0,000206904$
KSS US			$CA_{(2,2,8)} = -0,000141802$
GGG: Short		$CA_{(2,3)} = -0,000317873$	$EA_{(2,3),3} = 0,002542386$
LNC US			$CA_{(2,3,1)} = 0,001540515$
MON US			$CA_{(2,3,2)} = 0,000146832$
MRK US			$CA_{(2,3,3)} = 0,000511648$
NFLX US			$CA_{(2,3,4)} = -0,000222656$
TWTR US			$CA_{(2,3,5)} = 0,000566046$
BBB: Long	$CA_{(3)} = 0,000061684$	$EA_{(3),2} = 0,000489714$	$EA_{(3),3} = -0,000717630$
EEE Long		$CA_{(3,1)} = 0,000362957$	$EA_{(3,1),3} = -0,000144190$
CI US			$CA_{(3,1,1)} = -0,000030622$
CNL US			$CA_{(3,1,2)} = -0,000113567$
FFF: Long		$CA_{(3,2)} = 0,000126756$	$EA_{(3,2),3} = -0,000573440$
HUM US			$CA_{(3,2,1)} = -0,000072120$
NKA US			$CA_{(3,2,2)} = -0,000682765$
ODP US			$CA_{(3,2,3)} = -0,000000158$
PCL US			$CA_{(3,2,4)} = 0,000202561$
RAD US			$CA_{(3,2,5)} = -0,000020957$
BBB: Short	$CA_{(4)} = -0,000555697$	$EA_{(4),2} = 0,002076458$	$EA_{(4),3} = 0,001201374$
EEE: Short		$CA_{(4,1)} = 0,000375267$	$EA_{(4,1),3} = 0$
AET US			$CA_{(4,1,1)} = 0$
FFF: Short		$CA_{(4,2)} = 0,001701191$	$EA_{(4,2),3} = 0,001201374$
SPLS US			$CA_{(4,2,1)} = 0,000598755$
WY US			$CA_{(4,2,2)} = 0,000602618$
CCC	$CA_{(5)} = -0,000748194$	$EA_{(5),2} = 0$	$EA_{(5),3} = 0$
EEE		$CA_{(5,1)} = 0$	$EA_{(5,1),3} = 0$
CAT US			$CA_{(5,1,1)} = 0$

The transaction effects are given by

$$TA_{(1,1,1)} = -0,007715981,$$
$$TA_{(2,2,5)} = 0,007453985,$$
$$TA_{(4,1,1)} = 0,021368436,$$

and $TA_{(i,j,k)} = 0$ for all other (i, j, k). Hence,

$$\begin{aligned} TA_{\phi} &= TA_{(1,1,1)} + TA_{(2,2,5)} + TA_{(4,1,1)} \\ &= 0,021106441. \end{aligned}$$

Thus the arithmetic excess return is given by

$$
\begin{aligned}
RA_{\phi} &= \sum_{n=1}^{3} EA_{\phi,n} + TA_{\phi} \\
&= 0,026355725.
\end{aligned}
$$

The second period that is considered here is from 2016-01-12 to 2016-01-13. One transaction occurred at the end of the day. The investor bought the stock AA US, which causes the transaction effect seen below. The attribution result from the second period is shown below in Table 3.

Table 3: Arithmetic 2016-01-12/2016-01-13

Total	$EA_{\phi,1} = 0,000111256$	$EA_{\phi,2} = 0,001750793$	$EA_{\phi,3} = 0,002628574$
AAA: Long	$CA_{(1)} = 0,000580241$	$EA_{(1),2} = 0,000706198$	$EA_{(1),3} = -0,001550654$
EEE: Long		$CA_{(1,1)} = 0,000346836$	$EA_{(1,1),3} = -0,000716654$
AA US			$CA_{(1,1,1)} = -0,000045064$
AAPL US			$CA_{(1,1,2)} = 0,000031013$
ACE US			$CA_{(1,1,3)} = -0,000288804$
AFL US			$CA_{(1,1,4)} = 0,000298504$
ALL US			$CA_{(1,1,5)} = -0,000134012$
BA US			$CA_{(1,1,6)} = 0,000414215$
BHI US			$CA_{(1,1,7)} = -0,000992506$
FFF: Long		$CA_{(1,2)} = 0,000370133$	$EA_{(1,2),3} = 0,001879340$
CNP US			$CA_{(1,2,1)} = 0,000243559$
CTXS US			$CA_{(1,2,2)} = 0,000662553$
ED US			$CG_{(1,2,3)} = 0,000624991$
EVER US			$CA_{(1,2,4)} = -0,001036789$
GOOG US			$CA_{(1,2,5)} = 0,000652808$
IBM US			$CA_{(1,2,6)} = 0,000555760$
KLAC US			$CA_{(1,2,7)} = 0,000176458$
GGG: Long		$CA_{(1,3)} = -0,000010771$	$EA_{(1,3),3} = -0,002713340$
LEN US			$CA_{(1,3,1)} = -0,000277468$
LYB US			$CA_{(1,3,2)} = 0,000064770$
NVDA US			$CA_{(1,3,3)} = -0,000000124$
OMC US			$CA_{(1,3,4)} = 0,000053015$
PFE US			$CA_{(1,3,5)} = -0,000125407$
PFIS US			$CA_{(1,3,6)} = -0,000840407$
PX US			$CA_{(1,3,7)} = -0,000013793$
UNM US			$CA_{(1,3,8)} = 0,000169601$
VLO US			$CA_{(1,3,9)} = -0,001363984$
XOM US			$CA_{(1,3,10)} = -0,000379544$
AAA: Short	$CA_{(2)} = -0,001234414$	$EA_{(2),2} = 0,002337182$	$EA_{(2),3} = 0,004492133$
EEE: Short		$CA_{(2,1)} = 0,000657225$	$EA_{(2,1),3} = -0,000470744$
AAL US			$CA_{(2,1,1)} = -0,000140711$
AIG US			$CA_{(2,1,2)} = 0,000027639$
APOL US			$CA_{(2,1,3)} = -0,000160847$
BLL US			$CA_{(2,1,4)} = -0,000197827$
BMS US			$CA_{(2,1,5)} = 0,000001002$

FFF: Short		$CA_{(2,2)} = 0,001100431$	$EA_{(2,2),3} = 0,005544401$
CI US CMA US CMG US CVS US HAL US HES US KMX US KSS US			$CA_{(2,2,1)} = 0,001513571$ $CA_{(2,2,2)} = -0,000717443$ $CA_{(2,2,3)} = 0,004032340$ $CA_{(2,2,4)} = 0,000656904$ $CA_{(2,2,5)} = -0,000000713$ $CA_{(2,2,6)} = -0,000141490$ $CA_{(2,2,7)} = 0,000688856$ $CA_{(2,2,8)} = -0,000487624$
GGG: Short		$CA_{(2,3)} = 0,000579527$	$EA_{(2,3),3} = -0,000581524$
LNC US MON US MRK US NFLX US TWTR US			$CA_{(2,3,1)} = 0,000152987$ $CA_{(2,3,2)} = -0,000150171$ $CA_{(2,3,3)} = -0,000462186$ $CA_{(2,3,4)} = -0,000749020$ $CA_{(2,3,5)} = 0,000626867$
BBB: Long	$CA_{(3)} = 0,000074390$	$EA_{(3),2} = -0,000274004$	$EA_{(3),3} = 0,000231749$
EEE: Long		$CA_{(3,1)} = -0,000201452$	$EA_{(3,1),3} = 0,000467755$
CI US CNL US			$CA_{(3,1,1)} = 0,000098784$ $CA_{(3,1,2)} = 0,000368971$
FFF: Long		$CA_{(3,2)} = -0,000072552$	$EA_{(3,2),3} = -0,000236005$
HUM US NKA US ODP US PCL US RAD US			$CA_{(3,2,1)} = 0,000067351$ $CA_{(3,2,2)} = 0,000017732$ $CA_{(3,2,3)} = -0,000002977$ $CA_{(3,2,4)} = -0,000327775$ $CA_{(3,2,5)} = 0,000009664$
BBB: Short	$CA_{(4)} = 0,000288267$	$EA_{(4),2} = -0,001018583$	$EA_{(4),3} = -0,000544654$
EEE: Short		$CA_{(4,1)} = -0,000178070$	$EA_{(4,1),3} = 0$
AET US			$CA_{(4,1,1)} = 0$
FFF: Short		$CA_{(4,2)} = -0,000840513$	$EA_{(4,2),3} = -0,000544654$
SPLS US WY US			$CA_{(4,2,1)} = -0,000265944$ $CA_{(4,2,2)} = -0,000278709$
CCC	$CA_{(5)} = 0,000402772$	$EA_{(5),2} = 0$	$EA_{(5),3} = 0$
EEE		$CA_{(5,1)} = 0$	$EA_{(5,1),3} = 0$
CAT US			$CA_{(5,1,1)} = 0$

The transaction effect is given by

$$TA_{(1,1,1)} = -0,007101036,$$

and $TA_{(i,j,k)} = 0$ for all other (i,j,k). Hence,

$$TA_\phi = TA_{(1,1,1)} = -0,007101036.$$

Thus the geometric excess return is given by

$$\begin{aligned} RA_\phi &= \sum_{n=1}^{3} EA_{\phi,n} + TA_\phi \\ &= -0,002610412. \end{aligned}$$

If one tries to compound the two single periods they do not add up to the total arithmetic excess return.

$$\begin{aligned} (1+\bar{R}^P_{\phi,1}) \cdot (1+\bar{R}^P_{\phi,2}) - 1 - \big((1+R^B_{\phi,1}) \cdot (1+R^B_{\phi,2}) - 1\big) &= 1.012298064 \cdot (1-0.003921690) - 1 \\ &\quad - ((1-0.014057662) \cdot (1-0.001311278) - 1) \\ &= 0.023678651 \\ &\neq RA_{\phi,1} + RA_{\phi,2} = 0.023745313. \end{aligned}$$

Hence a smoothing algorithm must be used in order for the mathematical formulas to completely describe the excess return.

5.7 Numerical Result for the Geometric Attribution Model

Below follows the results of the geometric top-down attribution model presented above for the portfolio with American stocks. The same data was used as for the arithmetic approach, hence the same transactions occurred.

Table 4: Geometric 2016-01-11/2016-01-12

Total	$EG_{\phi,1} = 0,005440328$	$EG_{\phi,2} = 0,000396809$	$EG_{\phi,3} = -0,000512176$
AAA: Long	$CG_{(1)} = -0,009835283$	$EG_{(1),2} = -0,000861795$	$EG_{(1),3} = -0,003089840$
EEE: Long		$CG_{(1,1)} = -0,000759894$	$EG_{(1,1),3} = 0,000446470$
AA US			$CG_{(1,1,1)} = 0,000495620$
AAPL US			$CG_{(1,1,2)} = 0,000146297$
ACE US			$CG_{(1,1,3)} = -0,000353585$
AFL US			$CG_{(1,1,4)} = 0,000589051$
ALL US			$CG_{(1,1,5)} = -0,000080009$
BA US			$CG_{(1,1,6)} = -0,000099527$
BHI US			$CG_{(1,1,7)} = -0,000251377$
FFF: Long		$CG_{(1,2)} = -0,000309857$	$EG_{(1,2),3} = -0,004434122$
CNP US			$CG_{(1,2,1)} = -0,000621747$
CTXS US			$CG_{(1,2,2)} = -0,000383928$
ED US			$CG_{(1,2,3)} = -0,000748359$
EVER US			$CG_{(1,2,4)} = -0,000494586$
GOOG US			$CG_{(1,2,5)} = -0,001545481$
IBM US			$CG_{(1,2,6)} = -0,000614613$
KLAC US			$CG_{(1,2,7)} = -0,000025408$
GGG: Long		$CG_{(1,3)} = 0,000207957$	$EG_{(1,3),3} = 0,000897812$
LEN US			$CG_{(1,3,1)} = -0,000386710$
LYB US			$CG_{(1,3,2)} = 0,000010430$
NVDA US			$CG_{(1,3,3)} = -0,000005633$
OMC US			$CG_{(1,3,4)} = 0,000153143$
PFE US			$CG_{(1,3,5)} = -0,000322937$
PFIS US			$CG_{(1,3,6)} = 0,001244870$
PX US			$CG_{(1,3,7)} = -0,000004466$
UNM US			$CG_{(1,3,8)} = 0,000159407$
VLO US			$CG_{(1,3,9)} = 0,000537595$
XOM US			$CG_{(1,3,10)} = -0,000487887$
AAA: Short	$CG_{(2)} = 0,016535530$	$EG_{(2),2} = -0,001330073$	$EG_{(2),3} = 0,002089870$
EEE: Short		$CG_{(2,1)} = -0,000409762$	$EG_{(2,1),3} = 0,000612588$
AAL US			$CG_{(2,1,1)} = 0,000136823$
AIG US			$CG_{(2,1,2)} = -0,000059473$
APOL US			$CG_{(2,1,3)} = 0,000483658$
BLL US			$CG_{(2,1,4)} = 0,000057556$
BMS US			$CG_{(2,1,5)} = -0,000005976$

FFF: Short		$CG_{(2,2)} = -0,000599651$	$EG_{(2,2),3} = -0,001086384$
CI US			$CG_{(2,2,1)} = -0,000359776$
CMA US			$CG_{(2,2,2)} = -0,000390282$
CMG US			$CG_{(2,2,3)} = 0,001167166$
CVS US			$CG_{(2,2,4)} = -0,000959258$
HAL US			$CG_{(2,2,5)} = -0,000017571$
HES US			$CG_{(2,2,6)} = -0,000175036$
KMX US			$CG_{(2,2,7)} = -0,000208636$
KSS US			$CG_{(2,2,8)} = -0,000142989$
GGG: Short		$CG_{(2,3)} = -0,000320661$	$EG_{(2,3),3} = 0,002563666$
LNC US			$CG_{(2,3,1)} = 0,001553409$
MON US			$CG_{(2,3,2)} = 0,000148061$
MRK US			$CG_{(2,3,3)} = 0,000515931$
NFLX US			$CG_{(2,3,4)} = -0,000224519$
TWTR US			$CG_{(2,3,5)} = 0,000570784$
BBB: Long	$CG_{(3)} = 0,000062564$	$EG_{(3),2} = 0,000494008$	$EG_{(3),3} = -0,000723636$
EEE: Long		$CG_{(3,1)} = 0,000366140$	$EG_{(3,1),3} = -0,000145397$
CI US			$CG_{(3,1,1)} = -0,000030879$
CNL US			$CG_{(3,1,2)} = -0,000114518$
FFF: Long		$CG_{(3,2)} = 0,000127868$	$EG_{(3,2),3} = -0,000578240$
HUM US			$CG_{(3,2,1)} = -0,000072724$
NKA US			$CG_{(3,2,2)} = -0,000688480$
ODP US			$CG_{(3,2,3)} = -0,000000160$
PCL US			$CG_{(3,2,4)} = 0,000204256$
RAD US			$CG_{(3,2,5)} = -0,000021133$
BBB: Short	$CG_{(4)} = -0,000563620$	$EG_{(4),2} = 0,002094669$	$EG_{(4),3} = 0,001211429$
EEE: Short		$CG_{(4,1)} = 0,000378558$	$EG_{(4,1),3} = 0$
AET US			$CG_{(4,1,1)} = 0$
FFF: Short		$CG_{(4,2)} = 0,001716110$	$EG_{(4,2),3} = 0,001211429$
SPLS US			$CG_{(4,2,1)} = 0,000603767$
WY US			$CG_{(4,2,2)} = 0,000607662$
CCC	$CG_{(5)} = -0,000758862$	$EG_{(5),2} = 0$	$EG_{(5),3} = 0$
EEE		$CG_{(5,1)} = 0$	$EG_{(5,1),3} = 0$
CAT US			$CG_{(5,1,1)} = 0$

The transaction effects are given by

$$\begin{aligned} TG_{(1,1,1)} &= -0,007784550, \\ TG_{(2,2,5)} &= 0,007520226, \\ TG_{(4,1,1)} &= 0,021558330, \end{aligned}$$

and $TG_{(i,j,k)} = 0$ for all other (i,j,k). Hence,

$$\begin{aligned} TG_{\phi} &= TG_{(1,1,1)} + TG_{(2,2,5)} + TG_{(4,1,1)} \\ &= 0,021294006. \end{aligned}$$

Hence the geometric excess return is given by

$$\begin{aligned} RG_{\phi} &= \prod_{n=1}^{3}(1 + EG_{\phi,n}) \cdot (1 + TG_{\phi}) - 1 \\ &= 0.026731508. \end{aligned}$$

The second period considered is from 2016-01-12 to 2016-01-13. One transaction occurred at the end of the day, AA US, which causes the transaction effect seen below. The results of the second period for the geometric attribution model is shown in Table 5.

Table 5: Geometric 2016-01-12/2016-01-13

Total	$EG_{\phi,1} = 0,000111402$	$EG_{\phi,2} = 0,001752897$	$EG_{\phi,3} = 0,002627127$
AAA: Long	$CG_{(1)} = 0,000581003$	$EG_{(1),2} = 0,000707046$	$EG_{(1),3} = -0,001549801$
EEE: Long		$CG_{(1,1)} = 0,000347253$	$EG_{(1,1),3} = -0,000716260$
AA US			$CG_{(1,1,1)} = -0,000045039$
AAPL US			$CG_{(1,1,2)} = 0,000030996$
ACE US			$CG_{(1,1,3)} = -0,000288645$
AFL US			$CG_{(1,1,4)} = 0,000298339$
ALL US			$CG_{(1,1,5)} = -0,000133938$
BA US			$CG_{(1,1,6)} = 0,000413987$
BHI US			$CG_{(1,1,7)} = -0,000991960$
FFF: Long		$CG_{(1,2)} = 0,000370578$	$EG_{(1,2),3} = 0,001878305$
CNP US			$CG_{(1,2,1)} = 0,000243425$
CTXS US			$CG_{(1,2,2)} = 0,000662189$
ED US			$CG_{(1,2,3)} = 0,000624647$
EVER US			$CG_{(1,2,4)} = -0,001036218$
GOOG US			$CG_{(1,2,5)} = 0,000652448$
IBM US			$CG_{(1,2,6)} = 0,000555454$
KLAC US			$CG_{(1,2,7)} = 0,000176360$
GGG: Long		$CG_{(1,3)} = -0,000010784$	$EG_{(1,3),3} = -0,002711846$
LEN US			$CG_{(1,3,1)} = -0,000277315$
LYB US			$CG_{(1,3,2)} = 0,000064735$
NVDA US			$CG_{(1,3,3)} = -0,000000124$
OMC US			$CG_{(1,3,4)} = 0,000052986$
PFE US			$CG_{(1,3,5)} = -0,000125338$
PFIS US			$CG_{(1,3,6)} = -0,000839944$
PX US			$CG_{(1,3,7)} = -0,000013785$
UNM US			$CG_{(1,3,8)} = 0,000169507$
VLO US			$CG_{(1,3,9)} = -0,001363233$
XOM US			$CG_{(1,3,10)} = -0,000379335$
AAA: Short	$CG_{(2)} = -0,001236035$	$EG_{(2),2} = 0,002339990$	$EG_{(2),3} = 0,004489660$
EEE: Short		$CG_{(2,1)} = 0,000658015$	$EG_{(2,1),3} = -0,000470485$
AAL US			$CG_{(2,1,1)} = -0,000140633$
AIG US			$CG_{(2,1,2)} = 0,000027623$
APOL US			$CG_{(2,1,3)} = -0,000160759$
BLL US			$CG_{(2,1,4)} = -0,000197718$
BMS US			$CG_{(2,1,5)} = 0,000001002$

FFF: Short		$CG_{(2,2)} = 0,001101753$	$EG_{(2,2),3} = 0,005541349$
CI US			$CG_{(2,2,1)} = 0,001512737$
CMA US			$CG_{(2,2,2)} = -0,000717048$
CMG US			$CG_{(2,2,3)} = 0,004030121$
CVS US			$CG_{(2,2,4)} = 0,000656542$
HAL US			$CG_{(2,2,5)} = -0,000000712$
HES US			$CG_{(2,2,6)} = -0,000141412$
KMX US			$CG_{(2,2,7)} = 0,000688476$
KSS US			$CG_{(2,2,8)} = -0,000487356$
GGG: Short		$CG_{(2,3)} = 0,000580223$	$EG_{(2,3),3} = -0,000581204$
LNC US			$CG_{(2,3,1)} = 0,000152902$
MON US			$CG_{(2,3,2)} = -0,000150089$
MRK US			$CG_{(2,3,3)} = -0,000461932$
NFLX US			$CG_{(2,3,4)} = -0,000748608$
TWTR US			$CG_{(2,3,5)} = 0,000626522$
BBB: Long	$CG_{(3)} = 0,000074487$	$EG_{(3),2} = -0,000274333$	$EG_{(3),3} = 0,000231622$
EEE Long		$CG_{(3,1)} = -0,000201694$	$EG_{(3,1),3} = 0,000467497$
CI US			$CG_{(3,1,1)} = 0,000098729$
CNL US			$CG_{(3,1,2)} = 0,000368768$
FFF: Long		$CG_{(3,2)} = -0,000072639$	$EG_{(3,2),3} = -0,000235875$
HUM US			$CG_{(3,2,1)} = 0,000067314$
NKA US			$CG_{(3,2,2)} = 0,000017722$
ODP US			$CG_{(3,2,3)} = -0,000002975$
PCL US			$CG_{(3,2,4)} = -0,000327595$
RAD US			$CG_{(3,2,5)} = 0,000009658$
BBB: Short	$CG_{(4)} = 0,000288645$	$EG_{(4),2} = -0,001019807$	$EG_{(4),3} = -0,000544354$
EEE: Short		$CG_{(4,1)} = -0,000178284$	$EG_{(4,1),3} = 0$
AET US			$CG_{(4,1,1)} = 0$
FFF: Short		$CG_{(4,2)} = -0,000841522$	$EG_{(4,2),3} = -0,000544354$
SPLS US			$CG_{(4,2,1)} = -0,000265798$
WY US			$CG_{(4,2,2)} = -0,000278556$
CCC	$CG_{(5)} = 0,000403301$	$EG_{(5),2} = 0$	$EG_{(5),3} = 0$
EEE		$CG_{(5,1)} = 0$	$EG_{(5,1),3} = 0$
CAT US			$CG_{(5,1,1)} = 0$

The transaction effect is given by

$$TG_{(1,1,1)} = -0,007078531,$$

and since $TA_{(i,j,k)} = 0$ for all other (i, j, k) it follows that

$$TG_{\phi} = TG_{(1,1,1)} = -0,007078531.$$

Thus the geometric excess return is given by

$$\begin{aligned} RG_{\phi} &= \prod_{n=1}^{3}(1 + EG_{\phi,n}) \cdot (1 + TG_{\phi}) - 1 \\ &= -0,002613839. \end{aligned}$$

The two periods can be compounded and thus together describe the entire time period, since the effects can be multiplied as above mentioned. Table 7 shows the results for the compounded effects of the two time periods.

Table 7: Geometric Compounded 2016-01-11/2016-01-13

Total	$EG^{cum}_{\phi,1} = 0,005552337$	$EG^{cum}_{\phi,2} = 0,002150401$	$EG^{cum}_{\phi,3} = 0,002113606$
AAA: Long		$EG^{cum}_{(1),2} = -0,000155358$	$EG^{cum}_{(1),3} = -0,004634852$
EEE: Long			$EG^{cum}_{(1,1),3} = -0,000270109$
FFF: Long			$EG^{cum}_{(1,2),3} = -0,002564146$
GGG: Long			$EG^{cum}_{(1,3),3} = -0,001816468$
AAA: Short		$EG^{cum}_{(2),2} = 0,001006805$	$EG^{cum}_{(2),3} = 0,006588913$
EEE: Short			$EG^{cum}_{(2,1),3} = 0,000141815$
FFF: Short			$EG^{cum}_{(2,2),3} = 0,004448945$
GGG: Short			$EG^{cum}_{(2,3),3} = 0,001980972$
BBB: Long		$EG^{cum}_{(3),2} = 0,000219540$	$EG^{cum}_{(3),3} = -0,000492182$
EEE: Long			$EG^{cum}_{(3,1),3} = 0,000322033$
FFF: Long			$EG^{cum}_{(3,2),3} = -0,000813979$
BBB: Short		$EG^{cum}_{(4),2} = 0,001072726$	$EG^{cum}_{(4),3} = 0,000666416$
EEE: Short			$EG^{cum}_{(4,1),3} = 0$
FFF: Short			$EG^{cum}_{(4,2),3} = 0,000666416$
CCC		$EG^{cum}_{(5),2} = 0$	$EG^{cum}_{(5),3} = 0$
EEE			$EG^{cum}_{(5,1),3} = 0$

The cumulative transaction effects are given by

$$\begin{aligned} TG^{cum}_{(1,1,1)} &= -0,014807978, \\ TG^{cum}_{(2,2,5)} &= 0,007520226, \\ TG^{cum}_{(4,1,1)} &= 0,021558330, \end{aligned}$$

and $TG^{cum}_{(i,j,k)} = 0$ for all other (i, j, k). Hence,

$$\begin{aligned} TG^{cum}_{\phi} &= TG^{cum}_{(1,1,1)} + TG^{cum}_{(2,2,5)} + TG^{cum}_{(4,1,1)} \\ &= 0,014064746. \end{aligned}$$

Thus the geometric cumulative excess return is given by

$$\begin{aligned} RG^{cum}_{\phi} &= \prod_{n=1}^{3} (1 + EG^{cum}_{\phi,n}) \cdot (1 + TG^{cum}_{\phi}) - 1 \\ &= 0,024047797. \end{aligned}$$

5.7.1 Arithmetic vs Geometric Approach

The arithmetic excess return RA_ϕ and geometric excess return RG_ϕ are related by

$$RG_\phi = \frac{RA_\phi}{1 + R^B_\phi},$$

since

$$\begin{aligned}
RG_\phi &= \frac{1+R_\phi^P}{1+R_\phi^B} - 1 \\
&= \frac{1+R_\phi^P}{1+R_\phi^B} - \frac{1+R_\phi^B}{1+R_\phi^B} \\
&= \frac{R_\phi^P - R_\phi^B}{1+R_\phi^B}.
\end{aligned}$$

Thus for well performing markets, that is when $R_\phi^B > 0$, the arithmetic return is always greater than the geometric excess return. However, for falling markets, that is when $R_\phi^B < 0$, the arithmetic excess return is always lesser than the geometric excess return.

For these two periods studied the benchmark returns were negative, hence the geometric return was greater than the arithmetic excess return.

The arithmetic model is known to be more intuitive and easier understood than the geometric approach. It works well for the single-period but when several periods are considered, the single-periods do not add up to the arithmetic excess return, and smoothing algorithms are therefore introduced, that may distort the results. The geometric approach however may appear less intuitive for some, but when considering multi-periods they compound to the geometric excess return, which is very preferable.

6 Multi-Currency Attribution

The frameworks of the attribution models have to be extended and modified, when multi-currency portfolios are analyzed. Three different multi-currency approaches are here considered for the extension of the geometric attribution model. The approach of a naïve currency approach is firstly presented.

6.1 Naïve Currency Approach

The return of the portfolio in the base currency, that is the domestic currency of the portfolio, is given by

$$R^P = \sum_{i=1}^{n} w_i^P R_i^P,$$

where w_i^P is the portfolio weight of the ith asset class and R_i^P is the portfolio return in the base currency of the ith asset class. The local currency return of the portfolio is given by

$$R_L^P = \sum_{i=1}^{n} w_i^P R_{Li}^P,$$

where R_{Li}^P is the return in the local currency of the ith asset class. The benchmark return in the base currency is defined as

$$R^B = \sum_{i=1}^{n} w_i^B R_i^B,$$

where w_i^B is the benchmark weight of the ith asset class and R_i^B is the benchmark return in the base currency of the ith asset class. The local return of the benchmark is given by

$$R_L^B = \sum_{i=1}^{n} w_i^B R_{Li}^B,$$

where R_{Li}^B is the return of the benchmark in local currency of the ith asset class. Furthermore the local semi-notional return of the portfolio in local currency is given by

$$R_{SL} = \sum_{i=1}^{n} w_i^P R_{Li}^B.$$

The currency performance of the portfolio is defined as

$$R_C^P = \frac{1 + R^P}{1 + R_L^P} - 1$$

which measures the relative difference between the total portfolio return in the base currency and the total portfolio return in the local currency. The currency performance of the benchmark is similarly defined and hence is given by

$$R_C^B = \frac{1 + R^B}{1 + R_L^B} - 1,$$

which measures the relative difference between the total benchmark return in the base currency and the total benchmark return in the local currency. The relative difference between the currency performance of the portfolio and the benchmark is known as the naïve currency attribution

$$\frac{\frac{1+R^P}{1+R_L^P}}{\frac{1+R^B}{1+R_L^B}} - 1,$$

which is equivalent to

$$\frac{1+R^P}{1+R_L^P} \cdot \frac{1+R_L^B}{1+R^B} - 1.$$

The name naïve refers to the fact that the model is rather simple since it does not consider interest rate differentials. The allocation effect for the naïve approach is defined in terms of the local returns, namely as the relative difference between the local semi-notional return and the local benchmark return

$$A = \frac{1+R_{SL}}{1+R_L^B} - 1.$$

The contribution of asset class i is given by

$$A_i = (w_i^P - w_i^B)\left(\frac{1+R_{Li}^B}{1+R_L^B} - 1\right).$$

The sum of the contributions over all asset classes then gives the total allocation effect

$$\begin{aligned}\sum_{i=1}^n A_i &= \sum_{i=1}^n (w_i^P - w_i^B)\left(\frac{1+R_{Li}^B}{1+R_L^B} - 1\right)\\ &= \sum_{i=1}^n \frac{(w_i^P - w_i^B)R_{Li}^B}{1+R_L^B}\\ &= \frac{1+R_{SL}}{1+R_L^B} - 1.\end{aligned}$$

The total selection effect is defined in terms of local returns and is given by

$$S = \frac{1+R_L^P}{1+R_{SL}} - 1.$$

Thus the total selection effect is the relative difference between the local return of the portfolio and the local semi-notional return. The contribution of asset class i for the selection effect is given by

$$S_i = w_i^P\left(\frac{1+R_{Li}^P}{1+R_{Li}^B} - 1\right)\frac{1+R_{Li}^B}{1+R_{SL}}.$$

Hence, they sum to the total selection effect

$$\begin{aligned}\sum_{i=1}^n S_i &= \sum_{i=1}^n w_i^P\left(\frac{1+R_{Li}^P}{1+R_{Li}^B} - 1\right)\frac{1+R_{Li}^B}{1+R_{SL}}\\ &= \sum_{i=1}^n w_i^P\left(\frac{R_{Li}^P - R_{Li}^B}{1+R_{Li}^B}\right)\frac{1+R_{Li}^B}{1+R_{SL}}\\ &= \sum_{i=1}^n \left(\frac{R_L^P - R_{SL}}{1+R_{SL}}\right)\\ &= \frac{1+R_L^P}{1+R_{SL}} - 1.\end{aligned}$$

Thus the allocation effect and the selection effect, together with the result of the naïve currency attribution compound to the geometric excess return

$$\frac{1+R^P}{1+R^B}-1=\left(\frac{1+R_{SL}}{1+R_L^B}\right)\left(\frac{1+R_L^P}{1+R_{SL}}\right)\left(\frac{1+R^P}{1+R_L^P}\right)\left(\frac{1+R_L^B}{1+R^B}\right)-1.$$

The naïve currency effect can be further analyzed. Moreover, the currency return of the benchmark is defined as the relative difference between the base currency and the local currency for each currency i

$$c_i=\frac{1+R_i^B}{1+R_{Li}^B}-1$$

[1].

Assume that the currency return of the portfolio is equal to the currency return of the benchmark. The implied currency return of the benchmark is then equal to

$$R_{BC}=\sum_{i=1} w_i^B c_i,$$

and the implied currency return of the portfolio is equal to

$$R_{SC}=\sum_{i=1} w_i^P c_i.$$

The implied currency return is not in general equal to the currency performance stated above

$$\begin{aligned}
R_{BC}&=\sum_{i=1} w_i^B c_i\\
&=\sum_{i=1} w_i^B\left(\frac{1+R_i^B}{1+R_{Li}^B}-1\right)\\
&=\sum_{i=1} w_i^B\left(\frac{R_i^B-R_{Li}^B}{1+R_{Li}^B}\right)\\
&\neq\sum_{i=1}^{n} w_i^B\left(\frac{R_i^B-R_{Li}^B}{1+R_L^B}\right)\\
&=\frac{1+R^B}{1+R_L^B}-1\\
&=R_C^B,
\end{aligned}$$

and furthermore

$$\begin{aligned}
R_{SC}&=\sum_{i=1} w_i^P c_i\\
&=\sum_{i=1} w_i^P\left(\frac{1+R_i^B}{1+R_{Li}^B}-1\right)\\
&=\sum_{i=1} w_i^P\left(\frac{R_i^B-R_{Li}^B}{1+R_{Li}^B}\right)\\
&\neq\sum_{i=1}^{n} w_i^P\left(\frac{R_i^P-R_{Li}^P}{1+R_L^P}\right)\\
&=\frac{1+R^P}{1+R_L^P}-1\\
&=R_C^P.
\end{aligned}$$

Hence the implied portfolio and benchmark currency return are not equal to the real currency return, which is due to that the market values of the underlying securities change during the time period considered. Thus the currency exposure changes and the effects that rise are known as the compounded effects. If the market values did not change for the underlying assets it follows that $R_{Li} = 0$ and $R_L = 0$, and then $R_{BC} = R_C^B$. If the currency decisions are independently taken from the market usually by currency overlay managers, the compounding effects should be separated from the currency overlay. The compounding effects rise due to the changing market values of the underlying securities, hence the currency overlay managers should not be evaluated depending on these effects, since they are not informed of these market values during the time between measurements. The compounding effect of the benchmark is defined as

$$\frac{1 + R_{BC}}{1 + R_C^B} - 1,$$

while the compounding effect of the portfolio is defined

$$\frac{1 + R_C^P}{1 + R_{SC}} - 1.$$

The effects are named compounding due to that the difference between the real currency return and the implied currency return is known as compounding. Furthermore the naïve currency effect can be separated into the two compounding effects and a currency allocation effect that evaluates the currency overlay decision. The currency allocation is given by

$$(w_i^P - w_i^B)\left(\frac{1 + c_i}{1 + R_{BC}} - 1\right), \tag{2}$$

which is constructed similar to the previous definitions of allocations. Thus the total currency allocation effect is given by

$$\frac{1 + R_{SC}}{1 + R_{BC}} - 1,$$

since

$$\begin{aligned}
\sum_{i=1}^{n}(w_i^P - w_i^B)\left(\frac{1 + c_i}{1 + R_{BC}} - 1\right) &= \sum_{i=1}^{n}\frac{(w_i^P - w_i^B)c_i}{1 + R_{BC}} \\
&= \frac{1 + R_{SC}}{1 + R_{BC}} - 1.
\end{aligned}$$

Thus, the naïve currency effect is now fully described

$$\left(\frac{1 + R_C^P}{1 + R_{SC}}\right)\left(\frac{1 + R_{BC}}{1 + R_C^B}\right)\left(\frac{1 + R_{SC}}{1 + R_{BC}}\right) - 1 = \frac{1 + R_C^P}{1 + R_C^B} - 1 \tag{3}$$

[1].

The transactions of the portfolio usually occur at different exchange rates than those of the benchmark. Hence the currency return of the benchmark and the portfolio generally differs. The currency return of the portfolio in each market is given by

$$\tilde{c}_i = \frac{1 + R_i^P}{1 + R_{Li}^P} - 1.$$

The currency return of the portfolio is hereafter not assumed to be equal to the benchmark currency return. The total currency return of the portfolio is defined as

$$R_{PC} = \sum_{i=1}^{n} w_i^P \tilde{c}_i.$$

A currency timing or currency selection effect is then defined for the currency overlay manager. The contribution to the currency selection effect in currency i is defined as

$$w_i^P \left(\frac{1+\tilde{c}_i}{1+c_i} - 1 \right) \left(\frac{1+c_i}{1+R_{SC}} \right).$$

Thus the definition is similar to the definition of the stock selection effect. The total currency selection effect is then given by

$$\frac{1+R_{PC}}{1+R_{SC}} - 1.$$

Since,

$$\begin{aligned} \sum_{i=1}^{n} w_i^P \left(\frac{1+\tilde{c}_i}{1+c_i} - 1 \right) \left(\frac{1+c_i}{1+R_{SC}} \right) &= \sum_{i=1}^{n} w_i^P \left(\frac{\tilde{c}_i - c_i}{1+R_{SC}} \right) \\ &= \frac{1+R_{PC}}{1+R_{SC}} - 1. \end{aligned}$$

Local currency portfolio returns are for most performance measurement systems calculated from the benchmark currency returns, hence the currency selection effect is lost [1].

The naïve currency effect defined in (3) must be modified in order to be fully described when the currency selection effect is included. The naïve currency effect is given by the portfolio and benchmark compounding together with the currency allocation and selection effect in the following way

$$\begin{aligned} \left(\frac{1+R^P}{1+R_L^P} \right) \left(\frac{1+R_L^B}{1+R^B} \right) - 1 &= \left(\frac{1+R_C^P}{1+R_C^B} \right) - 1 \\ &= \underbrace{\left(\frac{1+R_C^P}{1+R_{PC}} \right)}_{\substack{\text{Portfolio} \\ \text{Compounding}}} \underbrace{\left(\frac{1+R_{BC}}{1+R_C^B} \right)}_{\substack{\text{Benchmark} \\ \text{Compounding}}} \underbrace{\left(\frac{1+R_{PC}}{1+R_{SC}} \right)}_{\substack{\text{Currency} \\ \text{Selection}}} \underbrace{\left(\frac{1+R_{SC}}{1+R_{BC}} \right)}_{\substack{\text{Currency} \\ \text{Allocation}}} -1. \end{aligned}$$

The portfolio compounding effect has been modified so that the naïve currency attribution effect is fully described. The naïve currency effect that includes currency selection compounds together with the stock selection effect and the asset allocation effect to the excess return

$$\left(\frac{1+R^P}{1+R^B} \right) - 1 = \left(\frac{1+R_L^P}{1+R_L^B} \right) \left(\frac{1+R_C^P}{1+R_C^B} \right) - 1$$

$$= \underbrace{\left(\frac{1+R_L^P}{1+R_{SL}} \right)}_{\substack{\text{Stock} \\ \text{Selection}}} \underbrace{\left(\frac{1+R_{SL}}{1+R_L^B} \right)}_{\substack{\text{Asset} \\ \text{Allocation}}} \underbrace{\left(\frac{1+R_{PC}}{1+R_{SC}} \right) \left(\frac{1+R_{SC}}{1+R_{BC}} \right)}_{\substack{\text{Currency} \\ \text{Overlay}}} \underbrace{\left(\frac{1+R_C^P}{1+R_{PC}} \right) \left(\frac{1+R_{BC}}{1+R_C^B} \right)}_{\substack{\text{Compounding} \\ \text{Effects}}} -1$$

[1].

6.1.1 Numerical Results

The naïve currency attribution is here illustrated by an example where the portfolio have investments in four different countries and where the Swedish Krona is the domestic currency. Table 8 shows the numerical values that are used for this performance example.

Table 8

	Portfolio Weight w_i^P	Benchmark Weight w_i^B	Portfolio Local Return R_{Li}^P	Benchmark Local Return R_{Li}^B	Portfolio Base Return R_i^P	Benchmark Base Return R_i^B	Portfolio Currency Return $\tilde{C}_i$	Benchmark Currency Return C_i
Swedish equities	0.30	0.30	0.03	0.04	0.03	0.04	0	0
Japanese equities	0.15	0.20	0.04	0.06	-0.0484	-0.03116	-0.085	-0.086
US equities	0.35	0.25	0.05	0.02	0.03425	0.00266	-0.015	-0.017
UK equities	0.20	0.25	-0.02	-0.04	0.08682	0.06752	0.109	0.112
Total	1	1	0.0285	0.019	0.0310915	0.023313		

By using the values in Table 8 the calculation for the local semi-notional return is given by

$$R_{SL} = \sum_{i=1}^{4} w_i^P R_{Li}^B = 0.02.$$

The total stock selection is given by

$$\frac{1+R_L^P}{1+R_{SL}} - 1 = \frac{1+0.0285}{1+0.02} - 1 = 0.00833333,$$

and the total allocation effect is given by

$$\frac{1+R_{SL}}{1+R_L^B} = \frac{1+0.02}{1+0.019} - 1 = 0.00098135.$$

Table 9 presents the stock selection and asset allocation of each country and the naïve currency effect.

Table 9

	Stock Selection $w_i^P\left(\frac{1+R_{Li}^P}{1+R_{Li}^B}-1\right)\left(\frac{1+R_{Li}^B}{1+R_{SL}}\right)$	Asset Allocation $(w_i^P - w_i^B)\left(\frac{1+R_{Li}^B}{1+R_L^B}-1\right)$	Naïve Currency Attribution $\left(\frac{1+R^P}{1+R_L^P}\right)\left(\frac{1+R_L^B}{1+R^B}\right)-1$
Swedish equities	-0.00294118	0	
Japanese equities	-0.00294118	-0.00201178	
US equities	0.01029441	0.00009814	
UK equities	0.00392157	0.00289500	
Total	0.00833333	0.00098135	-0.00170567

The geometric excess return is given by

$$\begin{aligned} RG &= \left(\frac{1+R_L^P}{1+R_{SL}}\right)\left(\frac{1+R_{SL}}{1+R_L^B}\right)\left(\frac{1+R^P}{1+R_L^P}\right)\left(\frac{1+R_L^B}{1+R^B}\right) - 1 \\ &= (1+0.00833333)(1+0.00098135)(1+(-0.00170567)) - 1 = 0.0076013. \end{aligned}$$

The naïve currency attribution can be further analyzed by introducing compounding and currency allocation and selection as stated above. The total currency selection effect is given by

$$\frac{1+R_{PC}}{1+R_{SC}}-1=\frac{1+0.0038}{1+0.00355}-1=0.00024912,$$

and the total currency allocation effect is given by

$$\frac{1+R_{SC}}{1+R_{BC}}-1=\frac{1+0.00355}{1+0.00655}-1=-0.00298048.$$

Table 10 shows further details of the currency attribution effects.

Table 10

	Currency Selection $w_i^P\left(\frac{1+\tilde{c}_i}{1+c_i}-1\right)\left(\frac{1+c_i}{1+R_{SC}}\right)$	Currency Allocation $(w_i^P-w_i^B)\left(\frac{1+c_i}{1+R_{BC}}-1\right)$	Portfolio Compounding $\frac{1+R_C^P}{1+R_{PC}}-1$	Benchmark Compounding $\frac{1+R_{BC}}{1+R_C^B}-1$
Swedish equities	0	0		
Japanese equities	0.00014947	0.00459739		
US equities	0.00069752	-0.00233968		
UK equities	-0.00059788	-0.00523819		
Total	0.00024912	-0.00298048	-0.00127546	0.00230765

The currency effects compound to the same result stated in Table 9

$$\begin{aligned}&\left(\frac{1+R^P}{1+R_L^P}\right)\left(\frac{1+R_L^B}{1+R^B}\right)-1=\left(\frac{1+R_{PC}}{1+R_{SC}}\right)\left(\frac{1+R_{SC}}{1+R_{BC}}\right)\left(\frac{1+R_C^P}{1+R_{PC}}\right)\left(\frac{1+R_{BC}}{1+R_C^B}\right)-1\\&=(1+0.00024912)(1+(-0.00298048))(1+(-0.00127546))(1+0.00230765)-1\\&=-0.00170567.\end{aligned}$$

6.2 Multi-Currency Model with Interest Rate Differentials

The naïve currency attribution is as above mentioned named naïve due to that the approach does not include interest rate differentials. Currency managers are exposed to interest rate differentials when they invest in different currencies [1]. Karnosky and Singer consider interest rate differentials and introduced the return premium which is the local market return minus the local interest rate differential in their article published 1994 [9].

The importance of the interest rate differentials is seen both for allocation and currency decisions. A country allocator that invests in a foreign market automatically takes a long position in that currency. In order to hedge the currency position by using a forward contract, the currency manager has to sell the foreign currency and buy the domestic currency. That is the manager borrows the foreign currency to be able to buy the domestic currency. The hedging strategy results in either a cost or profit and depends upon the interest rate differentials. Currency managers that take active currency positions use forward contracts or other derivatives that implicitly depend on interest rate differentials. The price of the forward currency contract depends upon the interest rates between the currencies. Thus the exposure of the interest rate differentials results in profits or costs. Due to the country decision taken by the country manager, the cost or profit should belong to him or her, and not the currency manager and the forward currency return rather than the spot currency return should quantify the currency allocation effects [1].

The forward exchange rate of a forward contract and the dependence of the interest rate differentials are shown here. Let F^t be the forward exchange rate at time 0 for a forward contract expiring at time t, and let S^0 be the spot rate at time 0. Then the following holds

$$F^t = S^0(1 + t\delta) = S^0 e^{t(i_d - i_f)},$$

where i_f is the foreign interest rate and i_d is the domestic interest rate, and $\delta = \frac{F^t - S^0}{S^0 t}$. Hence for a first-order approximation it follows that

$$\delta = i_d - i_f$$

[10]. The forward currency return and the spot currency return are defined by using the spot rates and forward exchange rate. S_i^t is at time t the spot rate of currency i and F_i^{t+1} is the forward exchange rate at time t for a forward contract expiring at time $t + 1$. The currency return between spot rates is given by

$$c_i = \frac{S_i^{t+1}}{S_i^t} - 1.$$

The return on a benchmark forward contract is given by

$$f_i = \frac{S_i^{t+1}}{F_i^{t+1}} - 1$$

and the forward premium or the interest rate differential is given by

$$d_i = \frac{F_i^{t+1}}{S_i^t} - 1.$$

Thus,

$$(1 + c_i) = (1 + d_i)(1 + f_i).$$

Furthermore, the hedged benchmark return for currency i is given by

$$R_{Hi}^B = \frac{1 + R_i^B}{1 + f_i} - 1.$$

Moreover

$$(1 + R_{Hi}^B) = (1 + d_i)(1 + R_{Li}^B),$$

since

$$\begin{aligned}
(1 + R_{Hi}^B) &= \frac{1 + R_i^B}{1 + f_i} \\
&= \frac{(1 + c_i)(1 + R_{Li}^B)}{1 + f_i} \\
&= (1 + d_i)(1 + R_{Li}^B).
\end{aligned}$$

Thus the hedged benchmark return is equal to the product of the interest rate differential and the local benchmark return, or the quotient of the base return and the return of the forward contract [1].

The formulas for the naïve currency model need to be modified when interest rate differentials are included. In the model presented here one only hedge currency positions that deviate from the benchmark. The currency manager hedge currency positions with currency derivatives, thus the performance of the currency decisions should be quantified with the forward currency rate, and not the spot rate. The semi-notional currency return or the implied currency return is defined so that any deviations from the benchmark are added or withdrawn

$$\bar{R}_{SC} = \sum_{i=1}^{n} w_i^B c_i + \sum_{i=1}^{n} (w_i^P - w_i^B) f_i.$$

The currency allocation formula for the naïve approach (2) needs to be modified, forward rates f_i should be used instead of spot rates c_i. Thus the contribution for currency allocation is defined as

$$(w_i^P - w_i^B)\left(\frac{1+f_i}{1+R_{BC}} - 1\right), \tag{4}$$

where R_{BC} is defined as previously. The total currency allocation effect is given by

$$\begin{aligned}
\sum_{i=1}^{n}(w_i^P - w_i^B)\left(\frac{1+f_i}{1+R_{BC}} - 1\right) &= \frac{\sum_{i=1}^{n}(w_i^P - w_i^B) f_i}{1+R_{BC}} \\
&= \frac{1+\sum_{i=1}^{n}(w_i^P - w_i^B) f_i + R_{BC}}{1+R_{BC}} - \frac{1+R_{BC}}{1+R_{BC}} \\
&= \frac{1+\bar{R}_{SC}}{1+R_{BC}} - 1.
\end{aligned}$$

The semi-notional return that includes cost of hedge to neutral is given by

$$R_{SH} = \sum_{i=1}^{n} w_i^B R_{Li}^B + (w_i^P - w_i^B) R_{Hi}^B.$$

Once again deviations from the benchmark currency positions are added or withdrawn. The profits or costs of the currency positions are due to the country allocator, hence it must be included in the asset allocation effect. Therefore hedged benchmark returns are used instead of local returns. The asset allocation effect that uses hedged benchmarks is defined as

$$(w_i^P - w_i^B)\left(\frac{1+R_{Hi}^B}{1+R_L^B} - 1\right),$$

and the interest rate differential is then included

$$(w_i^P - w_i^B)\left(\frac{(1+R_{Li}^B)(1+d_i)}{1+R_L^B} - 1\right).$$

Thus the total asset allocation is given by

$$\begin{aligned}
\sum_{i=1}^{n}(w_i^P - w_i^B)\left(\frac{1+R_{Hi}^B}{1+R_L^B} - 1\right) &= \sum_{i=1}^{n} \frac{(w_i^P - w_i^B) R_{Hi}^B}{1+R_L^B} \\
&= \frac{1+\sum_{i=1}^{n}(w_i^P - w_i^B) R_{Hi}^B + R_L^B}{1+R_L^B} - \frac{1+R_L^B}{1+R_L^B} \\
&= \frac{1+R_{SH}}{1+R_L^B} - 1.
\end{aligned}$$

The excess return is given by

$$\frac{1+R^P}{1+R^B}-1=\underbrace{\left(\frac{1+R_L^P}{1+R_{SL}}\right)}_{\substack{\text{Stock}\\ \text{Selection}}}\underbrace{\left(\frac{1+R_{SH}}{1+R_L^B}\right)}_{\substack{\text{Asset}\\ \text{Allocation}}}\underbrace{\left(\frac{1+R_{SL}}{1+R_{SH}}\right)}_{\substack{\text{Hedging}\\ \text{Cost}}}\underbrace{\left(\frac{1+R^P}{1+R_L^P}\right)\left(\frac{1+R_L^B}{1+R^B}\right)}_{\substack{\text{Naïve Currency}\\ \text{Approach}}}-1.$$

For this model only deviations from the benchmark currency positions are hedge, which differs from some other approaches where any deviation from the base currency is hedged [1].

6.2.1 Numerical Results

The numerical values used in Table 8 are here reused. Furthermore Table 11 presents the values for the forward return of the different currencies.

Table 11

	Benchmark Hedge Return $R_{Hi}=\frac{1+R_i^B}{1+f_i}$	Forward Currency	Forward Return f_i
Swedish equities	0.04	SEK	0
Japanese equities	0.059	JPY	-0.0855
US equities	0.029	USD	-0.0253
UK equities	-0.032	GBP	0.1026

By using the formulas stated above it follows that

$$\begin{aligned}R_{SH}&=\sum_{i=1}^{n}w_i^BR_{Li}^B+(w_i^P-w_i^B)R_{Hi}^B\\&=0.02055,\end{aligned}$$

and

$$\begin{aligned}\bar{R}_{SC}&=\sum_{i=1}^{n}w_i^Bc_i+\sum_{i=1}^{n}(w_i^P-w_i^B)f_i\\&=0.003165.\end{aligned}$$

The total currency allocation effect is then given by

$$\begin{aligned}\frac{1+\bar{R}_{SC}}{1+R_{BC}}-1&=\frac{1+0.003165}{1+0.00655}-1\\&=-0.00336297.\end{aligned}$$

The total asset allocation effect is given by

$$\begin{aligned}\frac{1+R_{SH}}{1+R_L^B}-1&=\frac{1+0.02055}{1+0.019}-1\\&=0.00152100.\end{aligned}$$

The total stock selection effect remains unchanged

$$\frac{1+R_L^P}{1+R_{SL}} - 1 = \frac{1+0.0285}{1+0.02} - 1$$
$$= 0.00833333.$$

The total cost of hedging is given by

$$\frac{1+R_{SL}}{1+R_{SH}} = \frac{1+0.02}{1+0.02055} - 1$$
$$= -0.000538925.$$

The attribution effects are summarized and presented in Table 12, where the stock selection and asset allocation are shown for each currency.

Table 12

	Stock Selection $w_i^P\left(\frac{1+R_{Li}^P}{1+R_{Li}^B}-1\right)\left(\frac{1+R_{Li}^B}{1+R_{SL}}\right)$	Asset Allocation $(w_i^P - w_i^B)\left(\frac{1+R_{Hi}}{1+R_L^B}-1\right)$	Cost of Hedging $\frac{1+R_{SL}}{1+R_{SH}}-1$	Naïve Currency Attribution $\left(\frac{1+R^P}{1+R_L^P}\right)\left(\frac{1+R_L^B}{1+R^B}\right)-1$
Swedish equities	-0.00294118	0		
Japanese equities	-0.00294118	-0.00196271		
US equities	0.01029441	0.00098135		
UK equities	0.00392157	0.00250245		
Total	0.00833333	0.00152100	-0.000538925	-0.00170567

The geometric excess return is then fully described

$$\begin{aligned} RG &= (1+0.00833333)(1+0.00152100)(1-0.000538925)(1-0.00170567)-1 \\ &= 0.0076012. \end{aligned}$$

The currency attribution can be further studied and the result is shown in Table 13.

Table 13

	Currency Allocation $(w_i^P - w_i^B)\left(\frac{1+f_i}{1+R_{BC}}-1\right)$
Swedish equities	0
Japanese equities	0.00457255
US equities	-0.00316427
UK equities	-0.00477125
Total	-0.00336297

6.3 Multi-Currency Model with Interest Rate Differentials Including Forward Contracts

The framework of the multi-currency model with interest rate differentials can be further extended to include forward contracts. The returns of the portfolio and benchmark then need to be modified in order to include the forward contracts. The portfolio return is then defined as

$$R^P = \sum_{i=1}^{n} w_i^P R_i^P + \sum_{i=1}^{n} \tilde{w}_i^P \tilde{f}_i$$

and the definition of the benchmark return is given by

$$R^B = \sum_{i=1}^{n} w_i^B R_i^B + \sum_{i=1}^{n} \tilde{w}_i^B f_i,$$

where in currency i, $\tilde{f}_i$ is the return of the portfolio forward currency contract, f_i is the return of the benchmark forward currency contract, $\tilde{w}_i^P$ is the portfolio weight of the forward currency contract and $\tilde{w}_i^B$ is the benchmark weight of the forward currency contract [1].

Furthermore the implied currency returns R_{BC} and R_{PC}, and the semi-notional currency return R_{SC} are extended to include the forward currency contracts, hence

$$R_{BC} = \sum_{i=1}^{n} w_i^B c_i + \sum_{i=1}^{n} \tilde{w}_i^B f_i$$

and

$$R_{PC} = \sum_{i=1}^{n} w_i^P \tilde{c}_i + \sum_{i=1}^{n} \tilde{w}_i^P \tilde{f}_i$$

and

$$R_{SC} = \sum_{i=1}^{n} w_i^B c_i + \sum_{i=1}^{n} (w_i^P - w_i^B + \tilde{w}_i^P) f_i.$$

The market exposures for the equities are described by the spot rate returns, since the assets create the exposure and not the forward contracts. The forward currency rates are only used for hedging [1].

Equation (4) is adjusted so that the weights of the forward contracts are included in the currency allocation effects

$$[(w_i^P + \tilde{w}_i^P) - (w_i^B + \tilde{w}_i^B)] \left(\frac{1 + f_i}{1 + R_{BC}} - 1 \right).$$

It then follows that

$$\begin{aligned}
&\sum_{i=1}^{n} [(w_i^P + \tilde{w}_i^P) - (w_i^B + \tilde{w}_i^B)] \left(\frac{1 + f_i}{1 + R_{BC}} - 1 \right) \\
&= \frac{1 + \sum_{i=1}^{n} [(w_i^P + \tilde{w}_i^P) - (w_i^B + \tilde{w}_i^B)] f_i + R_{BC}}{1 + R_{BC}} - 1 \\
&= \frac{1 + \sum_{i=1}^{n} [(w_i^P + \tilde{w}_i^P) - (w_i^B + \tilde{w}_i^B)] f_i + \sum_{i=1}^{n} w_i^B c_i + \sum_{i=1}^{n} \tilde{w}_i^B f_i}{1 + R_{BC}} - 1 \\
&= \frac{1 + \sum_{i=1}^{n} w_i^B c_i + \sum_{i=1}^{n} (w_i^P - w_i^B + \tilde{w}_i^P) f_i - 1}{1 + R_{BC}} - 1 \\
&= \frac{1 + R_{SC}}{1 + R_{BC}} - 1.
\end{aligned}$$

The return R_{SC}^P includes the forward contracts, and is used for measuring the currency selection effect

$$R_{SC}^P = \sum_{i=1}^n w_i^P c_i + \sum_{i=1}^n \tilde{w}_i^P f_i.$$

The contribution to the selection effect for forward currency contracts is given by

$$\tilde{w}_i^P \left(\frac{1+\tilde{f}_i}{1+f_i} - 1\right)\left(\frac{1+f_i}{1+R_{SC}^P}\right),$$

and the contribution to the currency selection effect for the assets is defined as

$$w_i^P \left(\frac{1+\tilde{c}_i}{1+c_i} - 1\right)\left(\frac{1+c_i}{1+R_{SC}^P}\right).$$

Then it follows that

$$\begin{aligned}
&\sum_{i=1}^n w_i^P \left(\frac{1+\tilde{c}_i}{1+c_i} - 1\right)\left(\frac{1+c_i}{1+R_{SC}^P}\right) + \sum_{i=1}^n \tilde{w}_i^P \left(\frac{1+\tilde{f}_i}{1+f_i} - 1\right)\left(\frac{1+f_i}{1+R_{SC}^P}\right) \\
&= \frac{\sum_{i=1}^n w_i^P(\tilde{c}_i - c_i)}{1+R_{SC}^P} + \frac{\sum_{i=1}^n \tilde{w}_i^P(\tilde{f}_i - f_i)}{1+R_{SC}^P} \\
&= \frac{1+\sum_{i=1}^n w_i^P(\tilde{c}_i - c_i) + \sum_{i=1}^n \tilde{w}_i^P(\tilde{f}_i - f_i) + \sum_{i=1}^n w_i^P c_i + \sum_{i=1}^n \tilde{w}_i^P f_i}{1+R_{SC}^P} - 1 \\
&= \frac{1+\sum_{i=1}^n w_i^P \tilde{c}_i + \sum_{i=1}^n \tilde{w}_i^P \tilde{f}_i}{1+R_{SC}^P} - 1 \\
&= \frac{1+R_{PC}}{1+R_{SC}^P} - 1.
\end{aligned}$$

The effects stated above compounds to the excess return in the following way

$$\frac{1+R^P}{1+R^B} - 1 = \underbrace{\left(\frac{1+R_L^P}{1+R_{SL}}\right)}_{\substack{\text{Selection}\\\text{Effect}}}\underbrace{\left(\frac{1+R_{SH}}{1+R_L^B}\right)}_{\substack{\text{Asset}\\\text{Allocation}}}\underbrace{\left(\frac{1+R_{SL}}{1+R_{SH}}\right)}_{\substack{\text{Hedging}\\\text{Cost}}}\underbrace{\left(\frac{1+R^P}{1+R_L^P}\right)\left(\frac{1+R_L^B}{1+R^B}\right)}_{\substack{\text{Naïve Currency}\\\text{Attribution}}} -1,$$

where the total currency effects can be further analyzed

$$\begin{aligned}
&\underbrace{\left(\frac{1+R_{PC}}{1+R_{SC}^P}\right)\left(\frac{1+R_{SC}}{1+R_{BC}}\right)}_{\substack{\text{Currency}\\\text{Overlay}}}\underbrace{\left(\frac{1+R_{SH}}{1+R_{SL}}\right)\left(\frac{1+R_{SC}^P}{1+R_{SC}}\right)}_{\substack{\text{Hedging}\\\text{Mismatch}}}\underbrace{\left(\frac{1+R_C^P}{1+R_{PC}}\right)\left(\frac{1+R_{BC}}{1+R_C^B}\right)}_{\text{Compounding}} -1 \\
&= \left(\frac{1+R_C^P}{1+R_C^B}\right)\left(\frac{1+R_{SH}}{1+R_{SL}}\right) - 1.
\end{aligned}$$

The term $\left(\frac{1+R_{SC}^P}{1+R_{SC}}\right)$ is introduced so that no residuals appear in the calculation for the excess return [1].

6.3.1 Numerical Results

The numerical values stated in Table 8 and Table 11 are here used once more. The including of forward contracts lead to the introduction of the forward weights seen in Table 15.

Table 15

	Portfolio Forward weight $\tilde{w}_i^P$	Benchmark Forward weight $\tilde{w}_i^B$	Portfolio Forward Return $\tilde{f}_i$	Benchmark Forward Return f_i
SEK forward contract	0.20	0.25	0	0
JPY forward contract	-0.05	-0.10	-0.0700	-0.0855
USD forward contract	-0.10	-0.05	-0.0300	-0.0253
GBP forward contract	-0.05	-0.10	0.1100	0.1026

The base returns of the portfolio and benchmark have to be updated in order to include the result of the forward contracts. The base return of the portfolio is given by

$$\begin{aligned} R^P &= \sum_{i=1}^{n} w_i^P R_i^P + \sum_{i=1}^{n} \tilde{w}_i^P \tilde{f}_i \\ &= 0.0320915 \end{aligned}$$

and the benchmark return is given by

$$\begin{aligned} R^B &= \sum_{i=1}^{n} w_i^B R_i^B + \sum_{i=1}^{n} \tilde{w}_i^B f_i \\ &= 0.022868. \end{aligned}$$

Furthermore the forward contracts extend R_{BC} and R_{SC} as follows

$$\begin{aligned} R_{BC} &= \sum_{i=1}^{n} w_i^B c_i + \sum_{i=1}^{n} \tilde{w}_i^B f_i \\ &= 0.006105. \end{aligned}$$

and

$$\begin{aligned} R_{SC} &= \sum_{i=1}^{n} w_i^B c_i + \sum_{i=1}^{n} (w_i^P - w_i^B + \tilde{w}_i^P) f_i \\ &= 0.00484. \end{aligned}$$

Moreover, R_{PC} and R_{SC}^P are computed as follows

$$\begin{aligned} R_{PC} &= \sum_{i=1}^{n} w_i^P \tilde{c}_i + \sum_{i=1}^{n} \tilde{w}_i^P \tilde{f}_i \\ &= 0.0048. \end{aligned}$$

and

$$\begin{aligned} R_{SC}^P &= \sum_{i=1}^{n} w_i^P c_i + \sum_{i=1}^{n} \tilde{w}_i^P f_i \\ &= 0.005225. \end{aligned}$$

The total currency allocation effect is given by

$$\frac{1+R_{SC}}{1+R_{BC}}-1=\frac{1+0.00484}{1+0.006105}-1$$
$$=-0.00125732,$$

while the total currency selection effect is given by

$$\frac{1+R_{PC}}{1+R^P_{SC}}=\frac{1+0.0048}{1+0.005225}-1$$
$$=-0.00042279.$$

The total stock selection effect is given by

$$\frac{1+R^P_L}{1+R_{SL}}-1=0.00833333,$$

and the asset allocation effect is given by

$$\frac{1+R_{SH}}{1+R^B_L}-1=0.00152100.$$

Furthermore,

$$R^P_C=\frac{1+R^P}{1+R^P_L}-1=\frac{1+0.0320915}{1.0285}-1$$
$$=0.00349198.$$

and

$$R^B_C=\frac{1+R^B}{1+R^B_L}-1=\frac{1+0.022868}{1+0.019}-1$$
$$=0.00379588.$$

The naïve currency effect is given by

$$\left(\frac{1+R^P}{1+R^P_L}\right)\left(\frac{1+R^B_L}{1+R^B}\right)-1=\left(\frac{1+0.0320915}{1+0.0285}\right)\left(\frac{1+0.0190}{1+0.022868}\right)-1$$
$$=-0.00030275.$$

The attribution effects are now summarized in Table 16 where the stock selection and asset allocation are shown for each currency.

Table 16

	Stock Selection	Asset Allocation	Hedging Cost	Naïve Currency Attribution
	$w^P_i\left(\frac{1+R^P_{Li}}{1+R^B_{Li}}-1\right)\left(\frac{1+R^B_{Li}}{1+R_{SL}}\right)$	$(w^P_i-w^B_i)\left(\frac{1+R_{Hi}}{1+R^B_L}-1\right)$	$\frac{1+R_{SL}}{1+R_{SH}}-1$	$\left(\frac{1+R^P}{1+R^P_L}\right)\left(\frac{1+R^B_L}{1+R^B}\right)-1$
Swedish equities	-0.00294118	0		
Japanese equities	-0.00294118	-0.00196271		
US equities	0.01029441	0.00098135		
UK equities	0.00392157	0.00250245		
Total	0.00833333	0.00152100	-0.000538925	-0.00030275

The different effects compound to the excess return as follows

$(1 + 0.00833333)(1 + 0.00152100)(1 + (-0.000538925))(1 + (-0.00030275)) - 1 = 0.00901720.$

The total currency effect is compounded by the hedging cost and the naïve currency attribution, hence is given by

$$(1 + (-0.000538925))(1 + (-0.00030275)) - 1 = -0.00084151.$$

The currency effects can be further studied and a summary is seen in Table 17.

Table 17

Equities	Currency Allocation $\frac{1+R_{SC}}{1+R_{BC}} - 1$	Currency Selection $w_i^P \left(\frac{1+\tilde{c}_i}{1+c_i} - 1\right)\left(\frac{1+c_i}{1+R_{SC}^P}\right)$	Hedging Mismatch $\left(\frac{1+R_{SL}}{1+R_{SH}}\right)\left(\frac{1+R_{SC}^P}{1+R_{SC}}\right) - 1$	Compounding $\left(\frac{1+R_C^P}{1+R_{PC}}\right)\left(\frac{1+R_{BC}}{1+R_C^B}\right) - 1$
Swedish equities		0		
Japanese equities		0.00014922		
US equities		0.00069636		
UK equities		-0.00059688		
Forwards		Currency Selection $\tilde{w}_i^P \left(\frac{1+\tilde{f}_i}{1+f_i} - 1\right)\left(\frac{1+f_i}{1+R_{SC}^P}\right)$		
SEK Forwards		0		
JPY Forwards		-0.00077097		
USD Forward		0.00046758		
GBP Forwards		-0.00036808		
Total	-0.00125732	-0.00042279	-0.00015599	0.00099562

Further analysis of the currency allocation is seen in Table 18.

Table 18

	$[(w_i^P + \tilde{w}_i^P) - (w_i^B + \tilde{w}_i^B)] \cdot \left(\frac{1+f_i}{1+R_{BC}} - 1\right)$
SEK	0.00030340
JPY	0
USD	-0.00156072
GBP	0
Total	-0.00125732

The currency effects presented in Table 17 compound to the total currency effect

$(1 + (-0.00125732)(1 + (-0.00042279)(1 + (-0.00015599))(1 + 0.00099562) = -0.00084142.$

6.4 Comparison of Different Multi-Currency Models

For the multi-currency models presented above the stock selection effects are equal. The stock selection effect is for all models given by local returns, hence interest rate differentials do not effect the result, and the stock selection effects are defined equally.

The asset allocation effects differ for the naïve model and for the models when interest rate differentials are included, since the interest rate differentials are included in the asset allocation effects. As argued above the interest rate differentials should be profits or costs to the country allocator and not to the currency overlay manager. The asset allocation effects are equal for the approaches that include interest rate differentials. The investments of forward currency contracts performed by the currency overlay manager should as expected not influence the asset allocation effect that measures the performance of the country allocator.

The naïve currency attribution effects are equal for the model including interest rate differentials without forward currency contracts and the naïve currency attribution model, but the currency allocation effects for the two models differ due to that the interest rate differentials are not included in the currency allocation for the naïve attribution model.

The two approaches that include interest rate differentials do not have equal currency effects. They have the same stock selection and asset allocation effects but since the forward contracts changes the currency exposure the excess returns differ, hence so does the total currency effects.

7 Conclusion

This thesis shows that the discrete compounded return is positively biased compared to the continuous compounded return, and the higher frequency of compounding the greater the effective return.

The money-weighted return and the time-weighted return are equal when no cash flow occur. When cash flows do occur the differences of the returns show. The differences may be so great that they even show result with different sign, that is one is positive and the other is negative. The money-weighted return is never positive if the end value of the portfolio is lesser than the total invested amount, while the time-weighted return may be positive. The time-weighted return weights all periods equally no matter amount invested, while the money-weighted return is most effected during the periods that the invested amounts are the greatest. The result above show that the greater the cash flow the greater the difference between the two returns. For all 5,000,000 simulations, the estimated mean value was for the time-weighted return greater than the money-weighted return, while the estimated variance of the money-weighted return was greater than variance of the time-weighted return.

The different measurement frequencies of the returns showed strong dependence of the frequency for the mean value and the variance. The dependence was easily seen as the cash flow increased. The mean and variance of the returns were always increasing or decreasing with the measurement frequency.

The Brinson methodology was described and both an arithmetic approach and a geometric approach were shown. The relation between the different excess returns showed that the arithmetic excess return is always greater than the geometric excess return when the total benchmark return is greater than zero and always lesser than the geometric excess return when the benchmark return is lesser than zero. The single-periods in the geometric attribution approach together compound to the geometric excess return. The single-periods in the arithmetic attribution approach do not add or compound to the arithmetic excess return. Hence one usually introduces a smoothing algorithm in order to fully describe the total excess return, even though the results may be distorted. This is also true for the arithmetic and geometric multi-level attribution models presented above. The single-periods in the multi-level arithmetic attribution model do not add or compound to the excess return.

In conclusion were three approaches for cross-currency portfolios presented, where the naïve currency attribution model was first described. Then were two approaches that includes interest rate differentials presented, one that did not consider forward currency contracts and one that did. The three stock selection effects are defined equally for all three models. The asset allocation effects were equally defined for the two models that included interest rate differentials, and different for the naïve currency model. The naïve currency attribution effects are equal for the model including interest rate differentials without forward currency contracts and the naïve currency attribution model, but the currency allocation effects differ due to that the interest rate differentials are not included in the naïve attribution model.

The two approaches that included interest rate differentials do not have equal total currency effect. They have the same stock selection and asset allocation effects but since the forward contracts changes the currency exposure the excess returns differ, hence so do the total currency effects.

The code is not included in the document but feel free to contact the author if the code is of interest.

8 References

1. Bacon, Carl R. *Practical Portfolio Performance Measurement and Attribution* (2nd ed). England: John Wiley & Sons, Incorporated, 2008.

2. G. Elias. *What is the difference between effective interest rates and nominal interest rates?* Available at http://www.csun.edu/ ghe59995/docs/Interpreting%20Nominal%20&%20Effective%20Interest%20Rates.pdf. Accessed: 5/31/17.

3. Piasecki, K .*Rates of Return on Open-End Debt Investment Funds and Bank Deposits in Poland in the years 1995–2015 – A Comparative Analysis.* The Journal of University of Szczecin, Volume 16, Issue 2, Dec 2016.

4. *Return Calculations.* Available at: https://faculty.washington.edu/ezivot/econ424/returnCalculations.pdf. Accessed: 5/31/17.

5. Colin, A. *Mastering Attribution in Finance.* Harlow, England: Pearson Education, 2016.

6. Morningstar. *Equity Performance Attribution Methodology.* May 31, 2011.

7. Brinson, G. and Fachler, N. (1985) *Measuring non-US equity portfolio performance.* Journal of Portfolio Management Spring, 73– 76.

8. Brinson, G., Hood, R. and Beebower, G. (1986) *Determinants of portfolio performance.* Financial Analyst Journal July–August, 39– 44.

9. Karnosky, D. and Singer, B. (1994) *Global asset management and performance attribution.* The Research Foundation of the Institute of Chartered Financial Analysts, February.

10. Sohmen, E. *The Theory of Forward Exchange.* Available at: https://www.princeton.edu/~ies/IES_Studies/S17.pdf. Accessed: 6/13/17.

11. Ben-Ari, I and Conrad, K. *Maclaurin's Inequality and a Generalized Bernoulli Inequality.* Available at: http://www.math.uconn.edu/ kconrad/articles/maclaurin.pdf. Accessed: 6/19/17.

12. Rudin, W. *Principles of Mathematical Analysis* (3rd ed). United States of America: McGraw-Hill, 1976.