Portfolio optimization with fuzzy models

Section 1

Fuzzy set theory at a glance

Introduction

Fuzzy sets were introduced by Zadeh in 1965 in order to model concept in a way closer to the human representation. The set identified by the statement "integers greater than 5" is perfectly represented in the usual way. The set identified by "young males", instead, is not as easily represented, as "young" is not well determined.
We can describe a set by a membership function, e.g. a function which takes as input an element and tells us if it is a member of the set or not. Classical, or "crisp" sets have a membership function that assumes only two values: 0 if the input is not an element of the set, 1 otherwise. Fuzzy sets can be represented by a function which takes values in $[0, 1]$.

Modelling with membership functions

To define a membership function:

- identify a variable which you want to model
- determine the domain of interest
- use "labels" to describe the variable when its value lies in a certain set
- for each label you need a membership function: identify some characteristic values in which the membership function corresponding to each label assumes the value of 1 or 0
- finally choose the function shape

Membership functions: example

Example: We want to model the age of a person. We choose as range the interval [0,110]. "young", "adult" and "old" are the labels employed. The membership function related to "young" is at her maximum when the age is less than 14, is 0 when the age is greater than 25; "adult" is 0 before 15 and after 60, while is 1 between 18 and 50; "old" is 0 before 50 and 1 after 70. Finally we choose a piecewise linear function.

α-sets, convexity and basic operations

- An α set is the crisp set composed by the elements for which the membership function is greater than or equal to α.

$$\alpha_\mu(x) = \{x|\mu(x) \geq \alpha\}$$

  The α set with $\alpha = 0$ is the support of the function.

- A set is said to be convex iif

$$\mu(\lambda x_1 + (1-\lambda)x_2) \geq \min(\mu(x_1), \mu(x_2))$$

  for any x_1, x_2 elements of the support and $\lambda \in [0,1]$.

- Complement: $\mu_{\bar{f}}(x) = 1 - \mu_f(x)$
- Union: $\mu_{f_1 \cup f_2}(x) = \max(\mu_{f_1}, \mu_{f_2})$
- Intersection: $\mu_{f_1 \cap f_2}(x) = \min(\mu_{f_1}, \mu_{f_2})$

Fuzzy measures

Let $\mathcal{B}$ the Borel field of the arbitrary set X. A fuzzy measure is a function g defined on $\mathcal{B}$ with the following properties:

- $g(\emptyset) = 0, g(X) = 1$
- if $A, B \in \mathcal{B}$ and $A \subseteq B$ then $g(A) \leq g(B)$
- if $A_n \in \mathcal{B}$, $A_1 \subseteq A_2 \subseteq$ ... then $\lim_{n\to\infty} g(A_n) = g(\lim_{n\to\infty} A_n)$

With respect to a probability measure this definition relaxes the countable additivity property, which holds true for countable $\{A_n\}$ s.t. $A_n \cap A_m = \emptyset \quad \forall n, m$ s.t. $n \neq m$:

$$\mathbb{P}\left(\bigcup_n A_n\right) = \sum_n \mathbb{P}(A_n)$$

The extension principle

Let X be the Cartesian product of universes $X_1 x ... x X_r$ and $\tilde{A}_1, ..., \tilde{A}_r$ be r fuzzy sets in $X_1, ..., X_r$ respectively; f is a mapping from X to a universe Y, $y = f(x_1, ..., x_r)$. Then the extension principle allows us to define a fuzzy set $\tilde{B}$ in Y by:

$$\tilde{B} = \{(y, \mu_{\tilde{B}}(y)) | y = f(x), x \in X\}$$

where

$$\mu_{\tilde{B}}(y) = \begin{cases} \sup_{(x_1,...,x_r) \in f^{-1}(y)} \min\{\mu_{\tilde{A}_1}(x_1), ..., \mu_{\tilde{A}_r}(x_r)\} \text{ if } f^{-1}(x) \neq \emptyset \\ 0 \text{ otherwise} \end{cases}$$

The extension principle allows an easy translation of many crisp tool, e.g. defining the arithmetic operations on fuzzy numbers

Fuzzy optimization

In 1970 Bellman and Zadeh proposed the basic framework for fuzzy optimization, where both the objective function and the constraints are fuzzy. Both the objective function and the constraints are defined by their respective membership functions. Then we want to find a solution which satisfies both the objective and the constraints. That is equivalent to an intersection between all the involved sets.
In a fuzzy environment there is no difference between objectives and constraints, and the solution of the problem is a fuzzy set itself.
More formally, we can have several goals $\tilde{G}_1, ..., \tilde{G}_n$ and constraints $\tilde{C}_1, ..., \tilde{C}_m$: a decision is defined as $\tilde{D} = \tilde{G}_1 \cap ... \cap \tilde{G}_n \cap \tilde{C}_1 \cap ... \cap \tilde{C}_m$.
Consequently we have $\mu_{\tilde{D}} = \min_{i,j}(\mu_{\tilde{G}_i}, \mu_{\tilde{C}_j}) = \min(\mu_k)$

Some applications in portfolio optimization

- Scenario uncertainty. Sometimes a probabilistic distribution represents something much far from our knowledge. Fuzzy theory helps with dealing with less information, exploiting e.g. expert panel evaluations as input to build a *possibilistic* scenario
- Parameters uncertainty. In some cases providing estimates is very difficult, even impossible. This is the case, for example, of an estimate of the price of a new listed company. Fuzzy numbers are sometimes used to build more robust models.
- Fuzzy objectives and constraints. A fuzzy objective can better cope with the real aspirations of the investor, while fuzzy constraints can be used to find a solution of an infeasible problem when small breaches of the constraints are actually tolerated.
- Multiple objectives. The fuzzy optimization theory provides a natural way to model multi objective problems which are very common in practice, allowing to enrich the mean variance representation of the investor utility function.

Section 2

The index tracking problem

The index tracking problem

Let's assume we want to build a portfolio which tracks a reference index realizing an excess return. We choose the mean downside absolute deviation as risk measure. The problem can be stated, substituting the expectations with the corresponding sample estimators, as follow:

$$\max E(\boldsymbol{w}) = \sum_t (R_t - I_t)/T$$

$$\min S(\boldsymbol{w}) = \sum_t |(R_t - I_t)^-|/T$$

subject to

$$\mathbf{w}'\mathbf{1} = 1$$

$$w_i \geq 0 \quad \forall i$$

where $\mathbf{w}$ is a vector of weights, $\mathbf{r}_t$ is a vector of stock returns, $R_t = \mathbf{w}'\mathbf{r}_t$ is the portfolio return, I_t is the index return.

The index tracking in a fuzzy framework

We are now moving to a fuzzy framework. To express aspiration levels like "not too different from the benchmark" and "a quite higher return" we employ the S shape membership functions:

$$\mu_E(\boldsymbol{w}) = \frac{1}{1 + \exp(-\alpha_E(E(\boldsymbol{w}) - E_M))}$$

$$\mu_S(\boldsymbol{w}) = \frac{1}{1 + \exp(\alpha_S(S(\boldsymbol{w}) - S_M))}$$

where $\alpha_E, \alpha_S, E_M, S_M$ are parameters that determine the location and shape of the function.
According to the fuzzy optimization framework our objective is $\max_{\boldsymbol{w}} \min(\mu_E, \mu_S)$, subject to the (crisp) set of constraints above.

Membership function

The max/min problem above can be expressed as:

$$
\begin{aligned}
\max\ & \lambda \\
\text{s.t.}\ & \mu_E \geq \lambda \\
& \mu_S \geq \lambda
\end{aligned}
$$

and letting $\eta = \log(\lambda/(1-\lambda))$ one can write:

$$
\begin{aligned}
\max\ & \eta \\
\text{s.t.}\ & \alpha_E(E(\boldsymbol{w}) - E_M) - \eta \geq 0 \\
& \alpha_S(S(\boldsymbol{w}) - S_M) + \eta \leq 0
\end{aligned}
$$

Mean downside absolute deviation

Consider that $|(x)^-| = |x|/2 - x/2$. Moreover, introduce the auxiliary variables b_t^+, b_t^- s.t.

$$
\begin{aligned}
b_t^+ + b_t^- &= \frac{|R_t - I_t|}{2} \\
b_t^+ - b_t^- &= \frac{R_t - I_t}{2} \\
b_t^+ &\geq 0 \quad \forall t \\
b_t^- &\geq 0 \quad \forall t
\end{aligned}
$$

The MDAD can then be reformulated in the following way:

$$
S(\boldsymbol{w}) = \sum_t \frac{2b_t^-}{T}
$$

Final formulation

The problem turns out to be linear:

$$
\begin{aligned}
&\max \ \eta \\
&\text{s.t. } \ \alpha_E \left(\left(\sum_t (R_t - I_t)/T \right) - E_M \right) - \eta \geq 0 \\
&\qquad \alpha_S \left(\left(\sum_t \frac{2b_t^-}{T} \right) - S_M \right) + \eta \leq 0 \\
&\qquad w_i \geq 0 \quad \forall i \\
&\qquad b_t^- \geq 0 \quad \forall t \\
&\qquad b_t^+ \geq 0 \quad \forall t \\
&\qquad \mathbf{w}'\mathbf{1} = 1 \\
&\qquad b_t^+ - b_t^- = \frac{R_t - I_t}{2} \quad \forall t
\end{aligned}
$$

References

Fang, Y., Lai, K.K., Wang, S., Fuzzy portfolio optimization, Springer (2008)

Zimmermann, H.J., Fuzzy set theory and its application, Springer (2001)